Geometric derivations of quantum corrections for gauge coupling functions

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Universiteit Utrecht

Utrecht 2017

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Contents

Abstract

This thesis discusses several aspects of computing the nonperturbative corrections to the gauge coupling functions in $\mathcal{N} = 2$ supersymmetric field theories and type II superstring theories. We first reviewed the Seiberg-Witten theory for the $\mathcal{N} = 2$ gauge field theory with gauge group SU(2) and no matter. Then we derived the 4-dimensional supergravity action resulting from compactification of type IIB superstring on a Calabi-Yau threefold. With these two ingredients, we further geometrically engineered the $\mathcal{N} = 2$ gauge field theory in type IIA superstring and computed the instanton corrections with local mirror symmetry. In the end we discussed how to fix an integral basis in the mirror symmetry computation for compact Calabi-Yau threefolds and fourfolds, discussed how the corresponding monodromies can be used to determine whether a singularity in the complex structure moduli space of a Calabi-Yau space is of finite Weil-Petersson distance, and show these ideas in some examples. We always focus on the explicit computation in the examples and give references for more general results.

Chapter 1

Introduction

String theory (and its various nonperturbative relatives, namely the M- and F-theory) is one of the most promising candidate for the physical theory of everything. In string theory one replaces point particles by 1-dimensional strings and higher dimensional extended objects. With supersymmetry and the cancellation of Weyl anomaly, the dimension of the spacetime in string theory is constrained to be exactly 10. The movement and interaction of strings and branes in the 10-dimensional spacetime can then generate interesting physics, including the unification of gravity and quantum mechanics. In addition to its physical success, it also serves as an important connection between physics and mathematics.

The successful application of mirror symmetry to solve a long-standing problem in enumerative geometry [13] serves as an example among the various success of applying string theory into mathematics. Inspired by such achievements, more and more mathematicians are becoming increasingly interested in physical problems, and more and more physicists are participating in research projects in modern mathematics. An excellent example is the cooperation between Atiyah and Witten, who both won Fields medals because of their works that enhanced the connection between mathematics and physics.

In this thesis, we focus on such connections. In particular, our aim is to understand what physicists contributed to mathematics in the last 20 years. This thesis serves as a record of my personal journey along the boundary between physics and mathematics, to be more precise, between string theory and geometry.

In a standard introduction to quantum field theory, perturbative physics is the main topic and marvellous techniques such as the method of Feynman diagram are developed. However, the physical quantities under considerations do not only receive corrections from perturbative region but also receive nonperturbative corrections. Nonperturbative effects in quantum field theories are invisible to perturbative computations. In a quantum field theory with coupling constant g, the partition function of the field theory is schematically of the form

$$Z(g) = \sum_n a_n g^n + e^{-A/g} \sum_n a_n^{(1)} g^n + \mathcal{O}(e^{-2A/g}), \tag{1.1}$$

where the coefficients a_n in the first sum come from perturbative computation by, e.g., summing over Feynman diagrams. The second and further terms are nonperturbative corrections. They are typically accompanied with weight $e^{-A/g}$, which is a well-known example of a function that is smooth but not analytic. To be more precise, recall that the function defined on $\mathbb{R}$

$$f(g) = \begin{cases} e^{-1/g} & \text{for } g > 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

is infinitely differentiable at $g = 0$ with $f^{(n)}(0) = 0$ for all n, while f is not analytic at $g = 0$ since its power series expansion, which is identically 0 in a neighbourhood of $g = 0$, does not coincide with f in this neighbourhood. As a result, the nonperturbative part that weighs with $e^{-A/g}$ is not viable in perturbative computation since during such a computation we always assume that the physical quantity is a power series in g. Such a characterisation cannot capture the nonperturbative information.

But nonperturbative effects are indeed present and controlling the world around us. A well-known yet to be understood example is the colour confinement in quantum chromodynamics which is crucial for our understanding of the low energy behaviour of quarks and gluons. As a result of such importance and difficulty of nonperturbative effects, every development of a technique that enables us to delve deeper in the nonperturbative realm is considered as a big success in theoretical physics.

According to Standard Model, our world is governed by gauge theories without supersymmetry. Systematically investigation into such theories for nonperturbative effects is in general too hard, and in such cases one usually first look for a simpler model, possibly with more symmetry. Supersymmetry is a very nice constraint on the system for such purposes. In four dimensions, a quantum field theory can has $\mathcal{N} = 1, 2, 4$ group of supersymmetry generators (there are also evidences on the existence of nontrivial $\mathcal{N} = 3$ theories, see, e.g., [26]). A theory with $\mathcal{N} = 4$ supersymmetry is too constrained for investigation into the nonperturbative effects. On the other hand, a theory with $\mathcal{N} = 1$ supersymmetry is possibly not restricted enough for a determination of nonperturbative corrections. An $\mathcal{N} = 2$ theory is the best. Seiberg and Witten proposed a beautiful programme known as Seiberg-Witten theory that determines all nonperturbative corrections to such a theory at once.

In Seiberg-Witten theory, the instanton corrections are computed from a geometrical object called the Seiberg-Witten curve. Various group of researchers later also found solutions to $\mathcal{N} = 2$ theory that bypass the need for such a curve. This fact makes one wonder if the Seiberg-Witten curve is really important in solving $\mathcal{N} = 2$ theories. String theory turned out to give the correct explanation. To talk about it, we have to first set up a link between string theory and geometry.

As mentioned above, string theory is consistent only in certain spacetime dimensions. For bosonic string the dimension is 26, and for superstring the dimension is 10. In order to make contact with our physical 4-dimensional world, one needs to make the surplus 6 dimensions compact such that they are hidden in our low energy world. Such a process is called string compactification. It is the main bridge that connects string theory with geometry since many interesting geometrical properties of the 6-dimensional compactification space is reflected in the 4-dimensional physics and can be computed physically.

The 4D theory after compactification is a supergravity theory. For the type II superstring on a 3-dimensional smooth Calabi-Yau manifold (from now on we always use the complex dimension for convenience), the 4D theory will be an $\mathcal{N}=2$ supergravity with abelian gauge fields. When D-branes are present in string theory, one can further realise non-abelian gauge groups in 4 dimensions. The $\mathcal{N}=2$ non-abelian field theory that Seiberg and Witten solved can thus be embedded into type IIA superstring theory such that they appear in the 4D theory after compactification. Further one can push the Planck scale to infinity to decouple gravity in the 4D theory and recover the original 4D supersymmetric field theory. Hence we have another way to solve the $\mathcal{N}=2$ field theory by solving the corresponding type IIA string theory on a suitably chosen compactification space.

There is already a powerful method to compute the nonperturbative corrections for type IIA compactification, namely mirror symmetry. From the worldsheet σ-model point of view, it is a consequence of a trivial sign flipping. But from the spacetime point of view, the implication is huge. Mirror symmetry means for every Calabi-Yau threefold Y_3, there exists another Calabi-Yau threefold $\hat{Y}_3$ such that the 4D theory resulting from IIA/Y_3 is equivalent to the 4D theory resulting from IIB/$\hat{Y}_3$. Such an equivalence is good for computation of nonperturbative corrections because a quantity that receives quantum corrections in the IIA/Y_3 theory is mapped under mirror symmetry to a quantity in the IIB/$\hat{Y}_3$ theory that does not receive any quantum correction! One can thus use the geometrical properties of the mirror $\hat{Y}_3$ to compute the quantity in IIB/$\hat{Y}_3$ and then maps it to IIA/Y_3 for a completely solution to the nonperturbative corrections. For the $\mathcal{N}=2$ field theory realised in IIA/Y_3, people found that the mirror geometry $\hat{Y}_3$ contains exactly the Seiberg-Witten curve! This explains the physical significance of the Seiberg-Witten curve.

In mirror symmetry computation, the geometric quantity on the mirror $\hat{Y}_3$ is computed by solving a system of linear differential equations. As we all know, the solution space of a system of linear differential equations is a vector space, and after finding a basis of linear independent solutions, we need some extra data to fix a particular solution for our needs. In mirror symmetry, such a process is called finding an integral basis and is completely solved for the mirror symmetry of Calabi-Yau threefolds.

In addition to Calabi-Yau threefolds, there are good reasons to consider a string theory compactified on Calabi-Yau spaces of dimension 4. For example, when the conjectured 12-dimensional F-theory is compactified on a Calabi-Yau fourfold, the resulting 4-dimensional theory will be an $\mathcal{N}=1$ supergravity [34]. The geometry and mirror symmetry for Calabi-Yau fourfolds are more complicated than threefolds as expected. Most importantly there was no good systematic formula to fix an integral basis for them until the recent paper [16].

There is one comment about the overall principle of this thesis. The author believes that the existing literatures are more suitable for general knowledge. But for a starter in this field, there is often a lack of concrete computation. Moreover, the general story is often too complicated to be written in a short master thesis. Hence we will focus on providing enough details of the computations. For more general information we will refer the reader to existing papers. The author hopes that these explicitly computations could one day help a confused amateur entering this beautiful field of research.

The structure of this thesis is as follows: In chapter 2 we review the Seiberg-Witten theory for $\mathcal{N} = 2$, SU(2) gauge theory without matter, focusing on the geometric structures. In chapter 3 we explicitly compute the 4D supergravity action of type IIB superstring compactified on a Calabi-Yau threefold. The emphasis is on its vector multiplet. In chapter 4 we embed the $\mathcal{N} = 2$, SU(2) gauge theory without matter considered in chapter 2 into a type IIA superstring theory compactified on a noncompact Calabi-Yau threefold with singularity, and then explicitly solve the theory via local mirror symmetry which is a modified version of mirror symmetry for noncompact Calabi-Yau spaces. In the final chapter 5, we go through the mirror symmetry for compact Calabi-Yau threefolds and fourfolds, focusing on the determination of an integral basis. As an application, we use the integral basis to determine whether a singularity in the complex structure moduli space is at finite distance or not.

1.1 Literatures on the basic mathematics

In this thesis we will not list the basics about complex geometry and toric geometry. The author realised that at the time of writing his own explanation will not be better than the existing professional literatures. Instead of repeating on the formulae, we give a suggestion of relevant literatures.

In general, it is always assumed that the reader has some background in real differential geometry as what an introductory course in general relativity provides. Then one can start learning about complex differential geometry. For physicists, there are lecture notes [11], [9], [28], and books [65], [43]. The latter book is also a good reference on Calabi-Yau manifolds. For the mathematically inclined reader, there are books [44], [3], [77], [80] and [33]. Furthermore, what we actually need in complex differential geometry is really about Kähler manifolds, hence a search with the keyword “Kähler ” can return some unexpected nice lecture notes. Two dedicated books on Kähler manifolds are [4] and [64].

Then one can study toric geometry. This is a particularly nice class of algebraic varieties such that all the interesting topological properties can be relatively easily (although tedious) computed from some combinatorial toric data defining the toric variety. Main examples of Calabi-Yau spaces in physics belong to this class. There notes [9], [28] also contain short introduction to toric geometry. Furthermore for physicists, there are two short reviews [72], [58] and a longer one [15]. If the reader is interested in rigorous mathematical treatments, there are introductory notes, e.g. [17], written by Cox, and also books [23], [19], [25] and [66].

In the end, one has to mention the encyclopaedic book [41] which contains reviews about everything one needs in this field of research.

Chapter 2

The Seiberg-Witten theory

In this chapter, we review the Seiberg-Witten solution to $\mathcal{N} = 2$, D $= 4$, SU(2) supersymmetric gauge field theory. We will not focus on the details of various field theory constructions and solutions but merely aim to get familiar with several common features of field and string theories that will appear again and again throughout this thesis, namely that

1. A family of theories (instead of one theory) are under consideration;
2. There is a *moduli space* which classifies the theories in such a family;
3. The physical quantities that we want to understand can be expressed in terms of certain geometric quantities of some nice geometric spaces with well-behaved singularities;
4. The changes of the geometric quantities with respect to a variation in the moduli space are determined by the so-called Picard-Fuchs differential equations;
5. The physical quantities coming from the solution of the Picard-Fuchs equations often solve our family of theories completely, i.e. both perturbatively and non-perturbatively.

We will meet all these features in the Seiberg-Witten theory.

We temporarily adopt the convention $(\eta_{\mu\nu}) = \mathrm{diag}(+1, -1, -1, -1)$ in this chapter and mainly follow [5] and [61]. For more details on Seiberg-Witten theory, the reader should consult many of the existing nice reviews, including [21], [76], [5], [1], [50], and of course the excellent originals [70], [71].

The goal of Seiberg-Witten theory is to determine the low-energy, i.e. macroscopic, Wilsonian effective action from the high-energy, i.e. microscopic, pure SU(2) gauge theory with $\mathcal{N} = 2$ supersymmetry.

2.1 The microscopic $\mathcal{N}=2$ SUSY action

For our purpose, the $\mathcal{N}=2$ SUSY action is conveniently constructed with superspace.

Required by $\mathcal{N}=2$ supersymmetry, a theory with such supersymmetry is completely determined by a holomorphic function $\mathcal{F}$ called the prepotential. Then in the language of $\mathcal{N}=2$ superspace, the action for a general $\mathcal{N}=2$ SUSY Yang-Mills theory is constrained to be

$$S=\frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\tilde{\theta}\,\mathcal{F}(\Psi), \tag{2.1}$$

where θ^α and $\tilde{\theta}^\alpha$ are two sets of Grassmann coordinates, and Ψ is the $\mathcal{N}=2$ chiral superfield. The holomorphic prepotential $\mathcal{F}$ only depends on Ψ, not on $\Psi^\dagger$. Such property is referred to as *holomorphy*, which is one of the key properties that govern supersymmetric theories [69].

It is convenient to express an $\mathcal{N}=2$ action in the $\mathcal{N}=1$ superspace language. Perform the integration over the second set of Grassmann coordinates $\tilde{\theta}^\alpha$, and we get the general $\mathcal{N}=2$ action in terms of $\mathcal{N}=1$ superfields

$$S=\frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\left\{\int \mathrm{d}^2\theta\,\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha+\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\left(\Phi^\dagger e^{-2gV}\right)^a\mathcal{F}_a(\Phi)\right\}, \tag{2.2}$$

where $\mathcal{F}_a := \frac{\partial}{\partial\Phi^a}\mathcal{F}(\Phi), \mathcal{F}_{ab} := \frac{\partial}{\partial\Phi^a}\frac{\partial}{\partial\Phi^b}\mathcal{F}(\Phi)$. All the superfields V, Φ, W live in the adjoint representation of the gauge group SU(2). Then in terms of the $\mathcal{N}=1$ language, an $\mathcal{N}=2$ vector multiplet contains an $\mathcal{N}=1$ vector multiplet $W\supset\{A_\mu,\lambda_\alpha\}$ and an $\mathcal{N}=1$ chiral multiplet $\Phi\supset\{\phi,\psi^\beta\}$, where A_μ and ϕ are bosonic fields with spin 1 and 0, respectively.

Our starting point is the microscopic $\mathcal{N}=2$ super Yang-Mills action. In addition to the above requirement by the $\mathcal{N}=2$ supersymmetry, the microscopic prepotential $\mathcal{F}$ is further restricted by renormalisability, so that it depends on Ψ quadratically

$$\mathcal{F}_{\mathrm{cl}}(\Psi):=\frac{1}{2}\mathrm{Tr}(\tau\Psi^2), \tag{2.3}$$

where τ is the complexified coupling constant

$$\tau=\frac{\theta}{2\pi}+\frac{4\pi\mathbf{i}}{g^2}, \tag{2.4}$$

with g the gauge coupling constant and θ the instanton angle. When we integrate out the high-energy fluctuations to get the low-energy effective action, the action is no longer required to be renormalisable and the macroscopic prepotential $\mathcal{F}$ will receive quantum corrections.

Plug the $\mathcal{F}_{\mathrm{cl}}$ into the general action (2.2) and we have the microscopic lagrangian

$$S=\mathrm{Im}\,\mathrm{Tr}\int \mathrm{d}^4x\,\frac{\tau}{16\pi}\left\{\int \mathrm{d}^2\theta\,W^\alpha W_\alpha+\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi^\dagger e^{-2gV}\Phi\right\}. \tag{2.5}$$

We can further expand all the superfields to write down the bosonic part of the action in component fields

$$S = \int \mathrm{d}^4x \, \mathrm{Tr} \left\{ -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{g^2}|\nabla_\mu \phi|^2 \right\} + S_{\mathrm{aux}}, \tag{2.6}$$

where the dynamical fields are the complex scalar ϕ (with its superpartner) in the $\mathcal{N} = 1$ chiral multiplet, and the massless gauge vector field A_μ (with its superpartner) in the $\mathcal{N} = 1$ vector multiplet. The covariant derivative is $\nabla_\mu \phi = \partial_\mu \phi - \mathbf{i}[A_\mu, \phi]$ as expected. The fields D and F are "auxiliary", with

$$S_{\mathrm{aux}} = \frac{1}{g^2} \int \mathrm{d}^4x \, \mathrm{Tr} \left\{ \frac{1}{2}D^2 - g\phi^\dagger[D, \phi] + F^\dagger F \right\}, \tag{2.7}$$

which implies that they are not dynamical and can be eliminated from the action using their equations of motion. This procedure in turn generates a potential for the scalar field

$$V(\phi) = \frac{1}{2}\mathrm{Tr}\left([\phi^\dagger, \phi]^2\right). \tag{2.8}$$

We can see that the potential V has a flat direction, hence this scalar ϕ is conventionally called the Higgs field. We will discuss its implication in the following section.

It is also known that the $\mathcal{N} = 2$ theory with SU(2) gauge group and no matter is asymptotically free.

2.2 The general form of the effective action

We now determine the general form of the $\mathcal{N} = 2$ low-energy effective action. First let us recall that for the $\mathcal{N} = 2$ supersymmetry to be unbroken, the vacuum field configuration should satisfy $V(\phi) = 0$. Classically, from the potential (2.8) we can make the following SU(2) gauge choice

$$\phi = \frac{1}{2}a\sigma^3 \tag{2.9}$$

$$= \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \tag{2.10}$$

where a is a constant complex number. In mathematical words, we are restricting ourselves to only consider the Cartan subalgebra of the gauge Lie algebra, which for the SU(2) group is generated by σ^3.

But there is still a residual gauge transformation (a transformation in the Weyl group) $a \mapsto -a$ that maps a vacuum field configuration to a gauge equivalent one. So we label gauge inequivalent vacua by the variable

$$u := \left\langle \mathrm{Tr}(\phi^2) \right\rangle, \tag{2.11}$$

with

$$\langle\phi\rangle = \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{2.12}$$

Then we put all u together to form the *moduli space* $\mathcal{M}$, the space of all gauge inequivalent vacua, of the SUSY gauge field theory. The variable u functions as a coordinate on the moduli space. The moduli space $\mathcal{M}$ after being compactified by adding a point at infinity is roughly speaking just the Riemann sphere (the complex projective line) $\mathbb{P}^1$ with several singularities removed.

If the scalar field ϕ has a nonvanishing vacuum configuration $u \neq 0$, then the SU(2) gauge symmetry is broken down to U(1) gauge symmetry by the Higgs mechanism. The gauge fields A_μ^b with $b = 1, 2$ (and their superpartners) become massive with mass $m \sim |a|$. Hence when we integrate out the high-energy fluctuations, the fields left are the massless field A_μ^3 (and its superpartner) and the nonvanishing constant scalar ϕ. From the general action (2.2) with the superfields in the adjoint representation of U(1) gauge group, we have the following form of the general low-energy effective action

$$S = \frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x \left\{ \int \mathrm{d}^2\theta\, \mathcal{F}''(\Phi) W^\alpha W_\alpha + \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \Phi^\dagger \mathcal{F}'(\Phi) \right\}. \tag{2.13}$$

Written in component form and it reads

$$S = \frac{1}{4\pi}\int \mathrm{d}^4x \left\{ -\frac{1}{4}\mathrm{Im}(\tau) F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\mathrm{Re}(\tau) F_{\mu\nu}\tilde{F}^{\mu\nu} + \mathrm{Im}(\tau)|\nabla_\mu \phi|^2 \right\}, \tag{2.14}$$

where $\tau(\phi) := \mathcal{F}''(\phi)$ is the effective complexified coupling function.

Our goal is to determine the prepotential $\mathcal{F}$ in this low-energy effective action (2.13).

2.3 Moduli space, dual variable and monodromy

We can tell from equation (2.14) that $\mathrm{Im}(\tau)$ is a metric for the sigma model, hence a metric on the moduli space $\mathcal{M}$ where it is given by $\mathrm{d}s^2 = \mathrm{Im}(\tau(a))\mathrm{d}a\mathrm{d}\bar{a}$. Unitarity requires this metric to be positive $\mathrm{Im}(\tau) > 0$ on the entire moduli space. At the same time τ is a holomorphic function on the (compactified) moduli space so we immediately realise that there must be singularities in the moduli space, otherwise by Liouville's theorem the coupling $\tau = \mathcal{F}''$ as a holomorphic function on the compact space $\mathcal{M}$ will be constant.

Seiberg and Witten argued that there are exactly three singularities in the moduli space $\mathcal{M}$: a singularity in the asymptotically free region at $u = \infty$ and two more singularities in the strong-coupling region at $u = \pm\Lambda^2$, where Λ is the dynamically generated scale at which the gauge coupling becomes strong. Then in order to obtain a nontrivial coupling function τ, we cover the entire moduli space $\mathcal{M}$ by three patches each around one of the three singularities. The effective coupling τ will be determined on each of these three patches together constrained by a natural consistency condition. We can roughly compare the idea here with what mathematicians do when they define manifolds via local charts and then require functions on a manifold to behave coherently in different charts (or in more physical words, require the functions to be covariant under general coordinate transformations).

Recall that in the asymptotically free region near $u = \infty$, the Higgs field VEV (vacuum expectation value) is given by a. We can do reliable perturbative calculations in this region, and the prepotential has contributions from the classical part, the perturbative 1-loop NSVZ part, and the non-perturbative instanton corrected part[68]:

$$\mathcal{F}(a) = \frac{1}{2}\tau_{\rm cl}a^2 + \frac{\mathbf{i}}{2\pi}a^2\log\frac{a^2}{\Lambda^2} + \frac{a^2}{2\pi\mathbf{i}}\sum_{k=1}^{\infty} c_k\left(\frac{\Lambda}{a}\right)^{4k}, \tag{2.15}$$

in which all the non-perturbative contributions c_l are hard to compute directly and are determined indirectly by Seiberg-Witten theory. Note that the perturbative part does not receive any contribution from diagrams containing more than one loops. This kind of results are called non-renormalisation theorems and is crucial for supersymmetric theories. The gauge coupling can then be derived

$$\begin{aligned}\tau(a) &= \mathcal{F}''(a) &(2.16)\\ &= \tau_{\rm cl} + \frac{\mathbf{i}}{\pi}\left[2\log\frac{a}{\Lambda} + 3 - \frac{1}{2}\sum_{l=1}^{\infty}(4l-1)(4l-2)c_l\left(\frac{\Lambda}{a}\right)^{4l}\right]. &(2.17)\end{aligned}$$

In the strong-coupling regions, the above non-perturbative sum will diverge. It turns out that to describe the theory properly in such regions, we need to take into consideration the dual field $a_{\rm D}$ as well as a into the coordinates. The dual field $a_{\rm D}$ is defined by

$$a_{\rm D} := \mathcal{F}'(a). \tag{2.18}$$

We also define other dual field quantities

$$\begin{aligned}\Phi_{\rm D} &:= \mathcal{F}'(\Phi), &(2.19)\\ \mathcal{F}'_{\rm D}(\Phi_{\rm D}) &:= -\Phi. &(2.20)\end{aligned}$$

And then written purely in dual quantities the effective action (2.13) becomes

$$S = \frac{1}{16\pi}\mathrm{Im}\int \mathrm{d}^4x\left\{\int \mathrm{d}^2\theta\,\mathcal{F}''_{\rm D}(\Phi_{\rm D})W^{\alpha}_{\rm D}W_{{\rm D},\alpha} + \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,\Phi^{\dagger}_{\rm D}\mathcal{F}'_{\rm D}(\Phi_{\rm D})\right\}. \tag{2.21}$$

Note especially from the definition of the dual $\Phi_{\rm D}$ and $\mathcal{F}_{\rm D}$ that there is a relation

$$\mathcal{F}''_{\rm D}(\Phi_{\rm D}) = -\frac{\mathrm{d}\Phi}{\mathrm{d}\Phi_{\rm D}} = -\frac{1}{\mathcal{F}''(\Phi)}, \tag{2.22}$$

which, after defining the dual coupling $\tau_{\mathrm{D}}(a_{\mathrm{D}}) := \mathcal{F}''_{\mathrm{D}}(a_{\mathrm{D}})$, actually means

$$\tau_{\mathrm{D}}(a_{\mathrm{D}}) = -\frac{1}{\tau(a)}, \tag{2.23}$$

i.e., via the duality transformation, the role of strong and weak couplings are exchanged.

There are of course other duality transformations. In fact the full group of duality transformations for the theory is $\mathrm{SL}_2(\mathbb{Z})$ acting on $(a_{\mathrm{D}}(u), a(u))$. We will not discuss the details but merely point out that a consequence of this is that the field variables $(a_{\mathrm{D}}(u), a(u))$ form an $\mathrm{SL}_2(\mathbb{Z})$ vector bundle over the moduli space $\mathcal{M}$, and the monodromy group (which will be introduced shortly) is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

To determine the prepotential, the behaviours of the field variables $(a_{\mathrm{D}}(u), a(u))$ around closed contours encircling the singularities provide particularly valuable information. In the moduli space $\mathcal{M}$, we start from u, move around a closed contour encircling a singularity u_0. When we return to our start point u, the vector $(a_{\mathrm{D}}(u), a(u))$ will change linearly

$$\begin{pmatrix} a_{\mathrm{D}}(u) \\ a(u) \end{pmatrix} \mapsto M_{u_0} \begin{pmatrix} a_{\mathrm{D}}(u) \\ a(u) \end{pmatrix}, \tag{2.24}$$

where the 2×2 matrix M_{u_0} is called the monodromy around u_0. Note that if we go around a loop that does not contain any singularity in $\mathcal{M}$, then the vector $(a_{\mathrm{D}}(u), a(u))$ will of course go back to itself because it is holomorphic in the region that the loop encircles. Our next step is to determine the three monodromies M_∞, $M_{+\Lambda^2}$ and $M_{-\Lambda^2}$.

2.4 Determining the monodromies

2.4.1 M_∞ in the asymptotic free region

In this region, the perturbative 1-loop term in the prepotential (2.15) dominates and we have the perturbative dependence $a(u) = \sqrt{2u}$. Then we compute the dual field variable

$$a_{\mathrm{D}} = \frac{\partial}{\partial a}\mathcal{F}(a) = \frac{\mathbf{i}}{\pi}\sqrt{2u}\left(\log\frac{2u}{\Lambda^2} + 1\right). \tag{2.25}$$

We move around a circle counterclockwise with very large radius in the moduli space and write schematically $u \mapsto e^{2\pi\mathbf{i}}u$ where $|u|$ is very large. Therefore

$$a \mapsto -a \tag{2.26}$$

$$a_{\mathrm{D}} \mapsto -a_{\mathrm{D}} + 2a, \tag{2.27}$$

i.e., the monodromy M_∞ is given by

$$M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}. \tag{2.28}$$

2.4.2 $M_{+\Lambda^2}$ in the strong coupling region

Recall that the SU(2) gauge symmetry is broken by the nonvanishing vev a of the Higgs field ϕ and we have chosen the remaining massless U(1) fields as the only dynamical fields in our low-energy effective action (2.14). Seiberg and Witten argued that at the strong coupling singularities, it is the masses of the solitionic objects, i.e. monopoles and dyons, that vanish such that our low-energy effective action becomes singular since it does not contain enough fields.

At the singularity $u = +\Lambda^2$, the mass of the magnetic monopole vanishes. The BPS mass formula tells us that the mass of a magnetic monopole is determined by the dual field a_{D}:

$$m^2 = 2|a_{\mathrm{D}}|^2. \tag{2.29}$$

Hence we know that at the singularity $u = +\Lambda^2$, the dual field $a_{\mathrm{D}} = 0$ vanishes.

In the dual description near the strong coupling singularity $u = +\Lambda^2$, the theory is an $\mathcal{N} = 2$ SUSY QED with light electrons whose β-function for the gauge coupling is given by

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_{\mathrm{D}} = \frac{g_{\mathrm{D}}^3}{8\pi}, \tag{2.30}$$

where g_{D} is the coupling constant related to τ_{D} by $\tau_{\mathrm{D}}(a_{\mathrm{D}}) = \frac{4\pi\mathbf{i}}{g_{\mathrm{D}}^2(a_{\mathrm{D}})}$, where we have set $\theta_{\mathrm{D}} = 0$ (a requirement by super QED). The renormalisation scale μ is proportional to a_{D} and hence we have

$$a_{\mathrm{D}} \frac{\mathrm{d}}{\mathrm{d}a_{\mathrm{D}}} \tau_{\mathrm{D}} = -\frac{\mathbf{i}}{\pi}, \tag{2.31}$$

for $u \sim +\Lambda^2$, which is being integrated once to get

$$\tau_{\mathrm{D}}(a_{\mathrm{D}}) = -\frac{\mathbf{i}}{\pi} \log a_{\mathrm{D}}. \tag{2.32}$$

Recall from the relation (2.22) that $\tau_{\mathrm{D}}(a_{\mathrm{D}}) = -\frac{da}{da_{\mathrm{D}}}$. Integrate once again and we have

$$a(a_{\mathrm{D}}) = a_0 + \frac{\mathbf{i}}{\pi} a_{\mathrm{D}} \log a_{\mathrm{D}}, \tag{2.33}$$

for $u \sim +\Lambda^2$, where a_0 is a constant.

Recall that in the patch of the moduli space near $u = +\Lambda^2$, the field a_{D} is a good coordinate. One can further show that a_{D} depends linearly on u in this patch, and we have

$$a_D(u) \sim c_0(u - \Lambda^2), \tag{2.34}$$

$$a(u) \sim a_0 + \frac{\mathbf{i}}{\pi} c_0(u - \Lambda^2) \log(u - \Lambda^2), \tag{2.35}$$

where c_0 is a constant. The monodromy as we move around $u = +\Lambda^2$, schematically $u - \Lambda^2 \mapsto e^{2\pi \mathbf{i}}(u - \Lambda^2)$, can be straightforwardly read out

$$M_{+\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}. \tag{2.36}$$

2.4.3 $M_{-\Lambda^2}$ from the consistency condition

The solitonic object that becomes massless at the singularity $u = -\Lambda^2$ can be shown to be a dyon with -1 unit of electric charge and 1 unit of magnetic charge and we can use the same technique to find the monodromy $M_{-\Lambda^2}$ around this singularity.

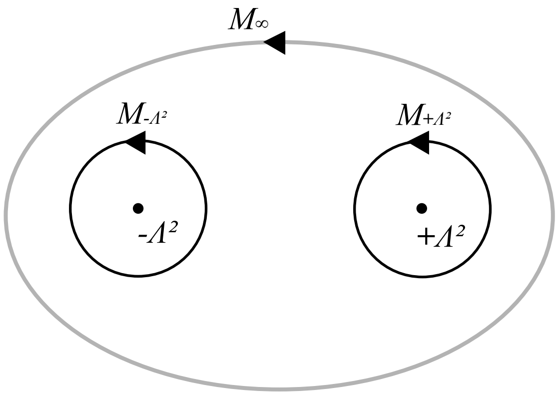

Figure 2.1: The relation $M_\infty = M_{+\Lambda^2} M_{-\Lambda^2}$.

But there is a simpler way to get $M_{-\Lambda^2}$: To have a consistent theory, when we first loop around $+\Lambda^2$ and then $-\Lambda^2$, the resulting loop can be deformed to be a circle around ∞ (see figure 2.1). Taking into account the orientations of the loops, the monodromies must satisfy

$$M_\infty = M_{+\Lambda^2} M_{-\Lambda^2}. \tag{2.37}$$

So we immediately get

$$M_{-\Lambda^2} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}. \tag{2.38}$$

As a side note, there are of course other ways to determine the monodromies. For example, in the moduli space, there is also an unbroken $u \mapsto -u$ symmetry. Such a symmetry corresponds to a 2π shift in the vacuum angle $\theta \mapsto \theta + 2\pi$. And the corresponding transformation on the vector $(a_{\mathrm{D}}(u), a(u))$ is given by the matrix

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.39}$$

The symmetry $u \mapsto -u$ relates the monodromy around $+\Lambda^2$ and $-\Lambda^2$

$$M_{-\Lambda^2} = T M_{+\Lambda^2} T^{-1}. \tag{2.40}$$

And we can use this relation as a check on the monodromies. Or with this relation we even do not need to consider the β-function around $+\Lambda^2$ and simply use M_∞ and the constraint (2.37) to solve the three monodromies.

Now we have transformed the physical problem of determining the low-energy effective action to a mathematical problem which is to find multi-valued functions $(a_{\mathrm{D}}(u), a(u))$ defined on the Riemann sphere $\mathbb{P}^1$ with three singularities at $\infty, \pm\Lambda^2$ that have the monodromies $M_\infty, M_{\pm\Lambda^2}$, respectively. Seiberg and Witten proposed a geometric approach which is proven to be extremely enlightening in the sense that it can be easily generalised in the string theory framework.

2.5 The "auxiliary" Seiberg-Witten elliptic curve

Seiberg and Witten realised the desired functions (a_{D}, a) as periods of certain meromorphic 1-form on an "auxiliary" elliptic curve, which is an alias for a 2-dimensional torus (possibly degenerate) in algebraic geometry. We have added the quotation marks to the adjective "auxiliary" because in the SUSY field theory setting, the physical meaning of such an elliptic curve is unclear. One can actually bypass all the geometries and merely use techniques from differential equations to determine the functions with the desired monodromies. Hence one may tempt to conclude that this curve is purely auxiliary. But we will see in future chapters that the Seiberg-Witten elliptic curve naturally appears in the string theory setting and hence it is really an object with physical significance.

The crucial insight is that any two of the three monodromy matrices $M_\infty, M_{\pm\Lambda^2}$ generate the principal congruence subgroup

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv \pm 1 \mod 2, \text{ and } b, c \equiv 0 \mod 2 \right\} \tag{2.41}$$

of the modular group $\mathrm{SL}_2(\mathbb{Z})$. While the modular group parametrises all elliptic curves, the subgroup $\Gamma(2)$ parametrises a particular family of them, namely the Seiberg-Witten curves. There are many physically equivalent forms of the equation of the Seiberg-Witten curves. Here we use the form that appeared in the original paper

$$y^2 = (x + \Lambda^2)(x - \Lambda^2)(x - u), \tag{2.42}$$

where $(x, y) \in \mathbb{C}^2$. After adding two points at the infinity to compactify the two $\mathbb{C}$'s, we get a compact 2-torus which is the Seiberg-Witten curve Σ_{SW}. This curve is also called the modular curve of the group $\Gamma(2)$.

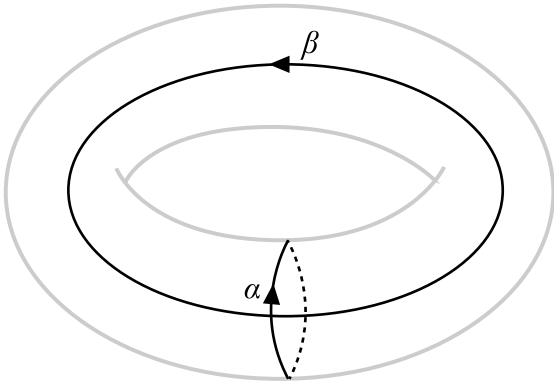

Figure 2.2: The basis for $H_1(\Sigma_{\mathrm{SW}}, \mathbb{Z})$ with $\alpha \cdot \beta = 1$.

The Riemann surface Σ_{SW} is of genus 1, hence its Dolbeault cohomology $H^{1,0}(\Sigma_{\mathrm{SW}})$ is of dimension 1 and one can pick an element ω to be the unique (up to normalisation) holomorphic 1-form on Σ_{SW}. Also, the first homology $H_1(\Sigma_{\mathrm{SW}}, \mathbb{Z})$ is of rank 2, and we pick a basis (α, β) such that they intersect as $\alpha \cdot \beta = 1$ (see figure 2.2). Then from the theory of Riemann surfaces, we can tell that the period vector (Π, Π_{D}) with

$$\Pi(u) = \int_\alpha \omega, \tag{2.43}$$

$$\Pi_{\mathrm{D}}(u) = \int_\beta \omega \tag{2.44}$$

enjoys also the desired monodromy properties given by $M_\infty, M_{\pm\Lambda^2}$. Furthermore, Riemann's Bilinear Relations guarentee that the modulus of the curve Σ_{SW}

$$\tau(u) = \frac{\Pi_{\mathrm{D}}(u)}{\Pi(u)} \tag{2.45}$$

is automatically positive $\mathrm{Im}\tau > 0$. All these properties are resembling the physical data that we want to compute in the SUSY field theory. Especially, note that the coupling function

$$\tau = \frac{\mathrm{d}a_{\mathrm{D}}}{\mathrm{d}a} = \frac{\mathrm{d}a_{\mathrm{D}}}{\mathrm{d}u} \bigg/ \frac{\mathrm{d}a}{\mathrm{d}u}\,, \tag{2.46}$$

so we should find a meromorphic 1-form λ_{SW} called *the Seiberg-Witten 1-form* such that

$$\frac{\mathrm{d}}{\mathrm{d}u}\lambda_{\mathrm{SW}} = \omega, \tag{2.47}$$

and then the physical data are nicely presented

$$a = \int_\alpha \lambda_{\mathrm{SW}}, \tag{2.48}$$

$$a_{\mathrm{D}} = \int_\beta \lambda_{\mathrm{SW}}, \tag{2.49}$$

$$\tau = \frac{\mathrm{d}a_{\mathrm{D}}}{\mathrm{d}u} \bigg/ \frac{\mathrm{d}a}{\mathrm{d}u}\,. \tag{2.50}$$

Indeed here the Seiberg-Witten 1-form λ_{SW} is given by

$$\lambda_{\mathrm{SW}}(u) := \frac{1}{\sqrt{2}\pi}\frac{(x-u)\,\mathrm{d}x}{y} = \frac{1}{\sqrt{2}\pi}\sqrt{\frac{x-u}{x^2-\Lambda^4}}\,\mathrm{d}x. \tag{2.51}$$

And one can check that the holomorphic differential in this case is

$$\omega(u) = \frac{\mathrm{d}}{\mathrm{d}u}\lambda_{\mathrm{SW}} = -\frac{1}{2\sqrt{2}\pi}\frac{\mathrm{d}x}{y}, \tag{2.52}$$

containing the standard holomorphic differential $\frac{\mathrm{d}x}{y}$ as expected.

Then the fields are

$$a_D(u) = \frac{\sqrt{2}}{\pi}\int_{+\Lambda^2}^{u}\sqrt{\frac{x-u}{x^2-\Lambda^4}}\,\mathrm{d}x, \tag{2.53}$$

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{+\Lambda^2}\sqrt{\frac{x-u}{x^2-\Lambda^4}}\,\mathrm{d}x, \tag{2.54}$$

and the coupling is simply $\tau(u) = \frac{a'_{\mathrm{D}}(u)}{a'(u)}$.

This completes our discussion on the geometrical solution of Seiberg-Witten theory.

2.6 Computation of the prepotential via Picard-Fuchs equations

Although the low energy effective coupling is expressed in terms of certain period integrals on the Seiberg-Witten curve, it is still not practical to compute all the instanton corrected terms directly from (2.53) and (2.54). If one is interested in finding the exact numbers coming from the instanton correction, then one better opt for other techniques. Luckily, there is a ubiquitous method to deal with such problems, namely the Picard-Fuchs differential equations.

Picard-Fuchs equations arise in the context of variation of complex structure of complex manifolds. In terms of what we are discussing, we see that the family of Seiberg-Witten curves are parametrised by a complex parameter u. Then the PF equations describe the variation of the periods Π and Π_D with respect to a small change in the parameter u. We will discuss this method later in more details. Now we merely list the PF operator for a and a_D:

$$\mathcal{L} = (u^2 - \Lambda^4)\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{1}{4}. \tag{2.55}$$

The PF operators annihilate the fields a and a_D:

$$\mathcal{L}a = 0, \tag{2.56}$$

$$\mathcal{L}a_\mathrm{D} = 0. \tag{2.57}$$

Then we can find a basis of solutions for the PF equations by noticing its relation with certain special functions, or obtain a series solution by Frobenius method. Then the correct solution is a linear combination of the basis, whose coefficients are determined by evaluating the period integrals (2.53) and (2.54) to the lowest order.

In our example one can express the fields in terms of the hypergeometric functions

$$a_\mathrm{D}(u) = \mathbf{i}\frac{u-\Lambda^2}{2\Lambda}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};2;\frac{\Lambda^2-u}{2\Lambda^2}\right), \tag{2.58}$$

$$a(u) = \sqrt{2(u+\Lambda^2)}\,{}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\frac{2\Lambda^2}{u+\Lambda^2}\right). \tag{2.59}$$

We can use various identities of hypergeometric functions to bring the solutions to the neighbourhood around other singularities.

To determine the instanton corrections one can invert the series of $a(u)$ perturbatively, then plug it into the expansion of $a_\mathrm{D}(u)$ to obtain the expansion of $a_\mathrm{D}(a)$. Finally one integrate a_D with respect to a to obtain the desired prepotential and read out the instanton corrections. We do not compute the instanton corrections. For details we refer the reader to [52]. We will come back to this at the end of chapter 4.

This completes our discussion on the Seiberg-Witten solution to SU(2), $\mathcal{N} = 2$ SUSY gauge theory without matter. Let us note again that we have seen all the characteristics in the beginning of this chapter which will repeatedly appear in our discussion on mirror symmetry.

Chapter 3

Calabi-Yau compactification

In this chapter we discuss the Calabi-Yau compactification of type IIA and IIB superstrings. The first physical motivation for compactification is to resolve the apparent inconsistency between our 4D world and the 10D requirement of superstring theories. Furthermore, the compactification provides a valuable bridge connecting the fancy superstring physics and the beautiful geometry of Calabi-Yau manifolds. In fact, after factoring (at least locally, in the fibre bundle sense) the 10D vacuum X of superstring into a product $X = M_4 \times Y$ where M_4 is the Minkowski $\mathbb{R}^4$ and Y is a Calabi-Yau threefold, many important mathematical properties of the Calabi-Yau manifold will be encoded in the resulting 4D effective theory. This not only enables us to study the effective theory using theory of Calabi-Yau geometry (just like what Einstein did in general relativity by introducing differential geometry into the physics community) but also helps mathematicians to find their way in solving many hard mathematical problems involving Calabi-Yau manifolds by introducing physical intuitions. One famous example is mirror symmetry and related mathematical physics, which will be our main topic in the following chapters.

The goal of this chapter is to get familiar with the structure of the 4D effective action. We will compute the effective action explicitly. Our focus is type IIB superstring and we will merely list the effective action for type IIA superstring at the end of this chapter. In addition, we will focus only on the bosonic sector and 4D vector multiplet since they are most relevant for our purpose. The curious reader looking for broader discussions regarding string compactification can refer to many existing good reviews, including [27], [38], [7] and [39].

In this chapter, we adopt the "correct" signature of the metric $(-, +, \ldots, +)$. All quantities with a hat live in 10 dimensions, in contrast all quantities without a hat live in the 4 dimensions after compactification.

3.1 The 10D type IIB supergravity action

Our starting point is the 10D type IIB supergravity action, which is already a low energy effective action for type IIB superstring theory. Its bosonic fields are listed in Table 3.1, where the various field strengths are defined by

$$\hat{H}_3 := \mathrm{d}\hat{B}_2, \tag{3.1}$$
$$\hat{F}_3 := \mathrm{d}\hat{C}_2 - \hat{a}\hat{H}_3, \tag{3.2}$$
$$\hat{F}_5 := \mathrm{d}\hat{A}_4 - \hat{H}_3 \wedge \hat{C}_2. \tag{3.3}$$

NS-NS sector	dilaton $\hat{\phi}$
	metric $\hat{g}$
	2-form $\hat{B}_2$
R-R sector	axion $\hat{a}$
	2-form $\hat{C}_2$
	4-form $\hat{A}_4$

Table 3.1: Bosonic field contents for 10D type IIB supergravity, in addition the 4-form has a self dual field strength $\hat{F}_5 = \star\hat{F}_5$.

The 10D type IIB supergravity action in the string frame is given by

$$\begin{aligned} S = & \int e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}\star 1 + 2\mathrm{d}\hat{\phi}\wedge\star\mathrm{d}\hat{\phi} - \frac{1}{4}\hat{H}_3\wedge\star\hat{H}_3\right) \\ & -\frac{1}{2}\int\left(\mathrm{d}\hat{a}\wedge\star\mathrm{d}\hat{a} + \hat{F}_3\wedge\star\hat{F}_3 + \frac{1}{2}\hat{F}_5\wedge\star\hat{F}_5\right) \\ & -\frac{1}{2}\int \hat{A}_4\wedge\hat{H}_3\wedge\mathrm{d}\hat{C}_2, \end{aligned} \tag{3.4}$$

where the self-duality condition $\hat{F}_5 = \star\hat{F}_5$ is not implied from the action but imposed by hand in the equation of motion.

3.2 The factorisation of the metric $\hat{g}_{MN}$

Consider the 10-dimensional spacetime X splitting into a 4-dimensional Minkowski spacetime M_4 and a 6-dimensional compact internal space Y:

$$X = M_4 \times Y. \tag{3.5}$$

In this chapter we choose Y to be a smooth Calabi-Yau threefold, i.e. a smooth Calabi-Yau manifold of 3 complex dimensions. On a Calabi-Yau manifold, it is of course natural to use complex coordinate systems. Hence we work in mixed coordinate systems $(x^\mu, y^\alpha, y^{\bar{\alpha}})$, for $\mu = 1, \dots, 4$, $\alpha = 1, 2, 3$ and $y^{\bar{\alpha}} := \bar{y}^\alpha$ is a shorthand for the conjugate coordinates.

Although there could be more general Ansatz for the factorisation of the 10D metric like including cetrain "warp factors", we do not consider these complicated cases and merely decompose the metric as

$$\hat{g}_{MN}(x, y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{\alpha\bar{\beta}}(x, y) \end{pmatrix}, \tag{3.6}$$

where we write a function $f(x, y)$ as a shorthand for $f(x, y, \bar{y})$. We can thus think of the entire 10D spacetime X as a fibration of Calabi-Yau threefolds over the base spacetime M_4. Note also that the metric $\hat{g}_{MN}$ and $g_{\mu\nu}$ are Lorentzian with signature $(-, +, \dots, +)$ but the metric $g_{\alpha\bar{\beta}}$ on the Calabi-Yau threefold is Riemannian with metric $(+, \dots, +)$.

General features of Calabi-Yau threefolds are summarised in appendix A and the beginning of chapter 5.

3.3 The field contents of the effective 4D theory

What are the field contents of the effective 4D theory and how do we represent them? One should remember that in general relativity and of course superstring theory the spacetime itself is dynamic, meaning that we should consider the variation of a solution around a background. Calabi-Yau threefolds also come in family in the sense that given a Calabi-Yau threefold, we can deform its complex and Kähler structures in controlled way such that the manifold after deformation is again a Calabi-Yau threefold and hence also valid for our use in compactification. Thus the field contents of our effective 4D theory not only come from the fields in 10D (table 3.1), but also encodes possible deformations of the background Calabi-Yau threefold. These fields coming from deformations of Calabi-Yau threefolds are called *moduli fields*.

Just like the original Kaluza-Klein compactification of pure Einstein gravity from 5D to 4D generates an infinite tower of massive and massless moduli, the compactification of superstring on Calabi-Yau threefolds also generates an infinite tower of massive and massless moduli fields with the lightest mass proportional to the inverse of the characteristic length of Calabi-Yau threefold. When we compute the resulting low energy effective action, we only keep the massless moduli. This will sometimes imply inconsistencies, but we will not bother with this issue. Interested reader can consult, e.g. appendix C.3 in [39].

The mass of a moduli of Calabi-Yau threefolds is the eigenvalue of the Laplace operator of the background Calabi-Yau manifold. So when we consider massless moduli, we only need to worry about the harmonic differential forms. We choose a basis for the harmonic forms on Y as in Table 3.2.

$H^{1,1}$	ω_i	$i = 1, \ldots, h^{1,1}$
$H^{2,2}$	$\tilde{\omega}^i$	$i = 1, \ldots, h^{1,1}$
$H^{2,1}$	η_a	$a = 1, \ldots, h^{2,1}$
H^3	(α_A, β^A)	$A = 0, \ldots, h^{2,1}$

Table 3.2: Basis of harmonic forms on Y. The (α_A, β^A) part intersects symplectically, see the beginning of chapter 5.

Fix a background Calabi-Yau threefold and denote all background fields by adding a circle, e.g. $\mathring{g}$. As a Kähler manifold, the background metric $\mathring{g}_{\alpha\bar{\beta}}$ only has nonvanishing mixed components. When we perturb the background metric, we have to discriminate perturbations with mixed indices $\delta g_{\alpha\bar{\beta}}$ and pure indices $\delta g_{\alpha\beta}$. The former represents deformations for Kähler form and the latter actually labels deformations of complex structure of the background Calabi-Yau threefold. When we expand the pure $\delta g_{\alpha\beta}$ in terms of harmonic differential forms on the Calabi-Yau threefold, we need to further define a basis $(0,2)$-form

$$(b_a)_{\bar{\alpha}\bar{\beta}} := -\frac{\mathbf{i}}{\|\Omega\|^2}(\eta_a)_{\bar{\alpha}\gamma\delta}\Omega^{\gamma\delta}{}_{\bar{\beta}}, \tag{3.7}$$

and its complex conjugate $\bar{b}_a$, where $\Omega \in H^{3,0}$ is the (unique up to normalisation) holomorphic 3-form on Y and $\eta_a \in H^{2,1}$ for $a = 1, \ldots, h^{2,1}$. This definition is understood to be the isomorphism between $H^{0,2}(Y, TY)$ and $H^{2,1}(Y)$ under the assumption of the existence of a holomorphic volume form in $H^{3,0}$. The cohomology $H^{0,2}(Y, TY)$ characterises the deformation of complex structure mathematically.

Then we also perturb the other 10D fields and expand them in terms of the harmonic forms on Y as in Table 3.3, where the relevant 4D fields are arranged in $\mathcal{N} = 2$ supergravity multiplets as in Table 3.4.

NS-NS sector	$\hat{\phi}(x,y) = \phi(x)$
	$\delta g_{\alpha\bar{\beta}}(x,y) = \mathbf{i}v^i(x)\,(\omega_i)_{\alpha\bar{\beta}}(y)$ $\delta g_{\alpha\beta}(x,y) = \bar{z}^a(x)\,(\bar{b}_a)(y)$
	$\hat{B}_2(x,y) = B_2(x) + b^i(x)\,\omega_i(y)$
R-R sector	$\hat{a}(x,y) = a(x)$
	$\hat{C}_2(x,y) = C_2(x) + c^i(x)\,\omega_i(y)$
	$\hat{A}_4(x,y) = D_2^i(x) \wedge \omega_i(y) + \rho_i(x) \wedge \tilde{\omega}^i(y)$ $+V^A(x) \wedge \alpha_A(y) - U_A(x) \wedge \beta^A(y)$

Table 3.3: Expansion of the 10D fields in harmonic forms

gravity multiplet	1	$(g_{\mu\nu}, V^0)$
vector multiplets	$h^{2,1}$	(V^A, z^A)
hypermultiplets	$h^{1,1}$	(v^i, b^i, c^i, ρ_i)
tensor multiplet	1	(B_2, C_2, ϕ, a)

Table 3.4: 4D fields arranged in $\mathcal{N} = 2$ supergravity multiplets

3.4 Reduction of the gravitational term

First we do the reduction for the general Einstein-Hilbert term

$$S_{\text{EH}} = \int \mathrm{d}^{10}x \sqrt{-\hat{g}}\, \hat{R}, \tag{3.8}$$

where $\hat{g} = \det(\hat{g}_{MN})$.

From our expansion in Table 3.3 the metric and its inverse on Y is now

$$g_{\alpha\beta} = \bar{z}^a (\bar{b}_a)_{\alpha\beta}, \tag{3.9}$$

$$g^{\alpha\beta} = -z^a (b_a)_{\bar{\alpha}\bar{\beta}}\, \mathring{g}^{\alpha\bar{\alpha}}\, \mathring{g}^{\beta\bar{\beta}}, \tag{3.10}$$

$$g_{\alpha\bar{\alpha}} = \mathring{g}_{\alpha\bar{\alpha}} - \mathbf{i} v^i (\omega_i)_{\alpha\bar{\alpha}}, \tag{3.11}$$

$$g^{\alpha\bar{\alpha}} = \mathring{g}^{\alpha\bar{\alpha}} + \mathbf{i} v^i (\omega_i)_{\beta\bar{\beta}}\, \mathring{g}^{\alpha\bar{\beta}}\, \mathring{g}^{\beta\bar{\alpha}}, \tag{3.12}$$

where the metric with a circle represents the ground state metric on Y. The fields z^a parametrise the deformation of complex structure and v^i parametrise the deformation of Kähler structure.

Now we compute the Ricci scalar $\hat{R}$ up to second order in the moduli fields. Expand the summation in $\hat{R}$:

$$\begin{aligned} \hat{R} = {} & R + \Big[g^{\mu\nu} R_{\mu\alpha\nu}{}^{\alpha} + g^{\alpha\bar{\beta}} (R_{\alpha\mu\bar{\beta}}{}^{\mu} + R_{\alpha\gamma\bar{\beta}}{}^{\gamma} + R_{\alpha\bar{\gamma}\bar{\beta}}{}^{\bar{\gamma}}) \\ & + g^{\alpha\beta} R_{\alpha\mu\beta}{}^{\mu} + g^{\alpha\beta} (R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}) + \text{c.c} \Big], \end{aligned} \tag{3.13}$$

where R is the Ricci scalar in 4 dimensions.

For convenience we also define some shorthands following [38]:

$$(\omega_i)_{\alpha}^{\beta} := (\omega_i)_{\alpha\bar{\gamma}}\, \mathring{g}^{\beta\bar{\gamma}}, \tag{3.14}$$

$$(\omega_i)^{\bar{\beta}\beta} := (\omega_i)_{\alpha\bar{\alpha}}\, \mathring{g}^{\alpha\bar{\beta}}\, \mathring{g}^{\beta\bar{\alpha}}, \tag{3.15}$$

$$(b_a)^{\gamma\beta} := (b_a)_{\bar{\beta}\bar{\gamma}}\, \mathring{g}^{\beta\bar{\beta}}\, \mathring{g}^{\gamma\bar{\gamma}}, \tag{3.16}$$

$$(\omega_i g) := (\omega_i)_{\alpha\bar{\beta}}\, \mathring{g}^{\alpha\bar{\beta}}, \tag{3.17}$$

$$(\omega_i \omega_j) := (\omega_i)_{\alpha\bar{\alpha}} (\omega_j)_{\beta\bar{\beta}}\, \mathring{g}^{\alpha\bar{\beta}}\, \mathring{g}^{\beta\bar{\alpha}}, \tag{3.18}$$

$$(b_a \bar{b}_b) := (b_a)_{\bar{\alpha}\bar{\beta}} (\bar{b}_b)_{\alpha\beta}\, \mathring{g}^{\alpha\bar{\beta}}\, \mathring{g}^{\beta\bar{\alpha}}, \tag{3.19}$$

and we stick to these index contraction structures because ω_i are $(1,1)$-forms which are antisymmetric in the two indices. This will bring in ambiguities just like in the 2-component spinor formalism if one is not careful when writing down different index structures.

To compute the Riemann curvatures, we first list all nonvanishing Levi-Civita connections containing at least one index in Y:

$$\Gamma^{\beta}_{\mu\alpha} = \frac{1}{2}g^{\beta\gamma}\partial_\mu g_{\alpha\gamma} + \frac{1}{2}g^{\beta\bar{\gamma}}\partial_\mu g_{\alpha\bar{\gamma}} \tag{3.20}$$

$$= -\frac{\mathbf{i}}{2}(\omega_i)_{\alpha}^{\beta}\partial_\mu v^i + \frac{1}{2}(\omega_i)^{\bar{\gamma}\beta}(\omega_j)_{\alpha\bar{\gamma}}v^i\partial_\mu v^j - \frac{1}{2}(b_a)^{\gamma\beta}(\bar{b}_b)_{\alpha\gamma}z^a\partial_\mu\bar{z}^b, \tag{3.21}$$

$$\Gamma^{\bar{\beta}}_{\mu\alpha} = \frac{1}{2}g^{\bar{\beta}\gamma}\partial_\mu g_{\alpha\gamma} + \frac{1}{2}g^{\beta\gamma}\partial_\mu g_{\alpha\bar{\gamma}} \tag{3.22}$$

$$= \frac{1}{2}(b_a)_{\alpha}^{\bar{\beta}}\partial_\mu\bar{z}^a - \frac{\mathbf{i}}{2}(\omega_i)^{\gamma\bar{\beta}}(\bar{b}_a)_{\alpha\gamma}v^i\partial_\mu\bar{z}^a + \frac{\mathbf{i}}{2}(\bar{b}_a)^{\bar{\beta}\bar{\gamma}}(\omega_i)_{\alpha\bar{\gamma}}\bar{z}^a\partial_\mu v^i, \tag{3.23}$$

$$\Gamma^{\mu}_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{\alpha\beta} \tag{3.24}$$

$$= -\frac{1}{2}(\bar{b}_a)_{\alpha\beta}\partial^\mu\bar{z}^a, \tag{3.25}$$

$$\Gamma^{\mu}_{\alpha\bar{\beta}} = -\frac{1}{2}g^{\mu\nu}\partial_\nu g_{\bar{\alpha}\beta} \tag{3.26}$$

$$= \frac{\mathbf{i}}{2}(\omega_i)_{\alpha\bar{\beta}}\partial^\mu v^i. \tag{3.27}$$

We now compute the Ricci scalar (3.13) term by term.

The second term is reduced to

$$\begin{aligned} g^{\mu\nu}R_{\mu\alpha\nu}{}^{\alpha} &= g^{\mu\nu}\left(\partial_\mu\Gamma^{\alpha}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\bar{\gamma}}\Gamma^{\bar{\gamma}}_{\alpha\nu} - \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu}\right) \\ &= -\frac{\mathbf{i}}{2}(\omega_i g)\nabla_\mu\partial^\mu v^i + \frac{1}{2}(\omega_i\omega_j)v^i\nabla_\mu\partial^\mu v^j - \frac{1}{2}(b_a\bar{b}_b)z^a\nabla_\mu\partial^\mu\bar{z}^b \\ &\quad + \frac{1}{4}(\omega_i\omega_j)\partial_\mu v^i\partial^\mu v^j - \frac{1}{4}(b_a\bar{b}_b)\partial_\mu z^a\partial_\mu\bar{z}^b. \end{aligned} \tag{3.28}$$

The terms containing 4D covariant derivative ∇_μ can be transformed into terms with only partial derivatives. This is a trick that we will use frequently in the dimension reduction of the Ricci scalar. It is important to note that

$$\nabla_\mu\sqrt{-\hat{g}} = \sqrt{-g_4}\partial_\mu\sqrt{g_6} \tag{3.29}$$

$$= \frac{1}{2}\sqrt{-\hat{g}}(g^{\alpha\beta}\partial_\mu g_{\alpha\beta} + g^{\bar{\alpha}\bar{\beta}}\partial_\mu g_{\bar{\alpha}\bar{\beta}} + 2g^{\alpha\bar{\beta}}\partial_\mu g_{\alpha\bar{\beta}}). \tag{3.30}$$

As an example we perform a partial integration to the third term in (3.28):

$$\int \mathrm{d}^{10}x\,\sqrt{-\hat{g}}\left\{-\frac{1}{2}(b_a\bar{b}_b)z^a\nabla_\mu\partial^\mu\bar{z}^b\right\} \sim \int \mathrm{d}^{10}x\,\left\{\frac{1}{2}(b_a\bar{b}_b)\nabla_\mu(\sqrt{-\hat{g}}z^a)\partial^\mu\bar{z}^b\right\} \quad (3.31)$$

$$= \int \mathrm{d}^{10}x\,\sqrt{-\hat{g}}\left\{\frac{1}{2}(b_a\bar{b}_b)\partial_\mu z^a\partial^\mu\bar{z}^b\right\}. \quad (3.32)$$

And the entire contribution (3.28) becomes

$$g^{\mu\nu}R_{\mu\alpha\nu}{}^{\alpha} \sim \frac{1}{2}\left((\omega_i g)(\omega_j g)-\frac{1}{2}(\omega_i\omega_j)\right)\partial_\mu v^i\partial^\mu v^j + \frac{1}{4}(b_a\bar{b}_b)\partial_\mu z^a\partial^\mu\bar{z}^b. \quad (3.33)$$

The results for other terms in the Ricci scalar (3.13) are listed below

$$\begin{aligned}
g^{\alpha\bar{\beta}}R_{\alpha\mu\bar{\beta}}{}^{\mu} &= g^{\alpha\bar{\beta}}\left(-\partial_\mu\Gamma^\mu_{\alpha\bar{\beta}}+\Gamma^\mu_{\alpha\gamma}\Gamma^\gamma_{\mu\bar{\beta}}+\Gamma^\mu_{\alpha\bar{\gamma}}\Gamma^{\bar{\gamma}}_{\mu\bar{\beta}}-\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{\alpha\bar{\beta}}\right)\\
&= \frac{\mathbf{i}}{2}(\omega_i g)\nabla_\mu\partial^\mu v^i+\frac{1}{2}(\omega_i\omega_j)v^i\nabla_\mu\partial^\mu v^j\\
&\quad -\frac{1}{4}(\omega_i\omega_j)\partial_\mu v^i\partial^\mu v^j-\frac{1}{4}(b_a\bar{b}_b)\partial_\mu z^a\partial^m\bar{z}^b\\
&\sim \frac{1}{2}\left((\omega_i g)(\omega_j g)-\frac{1}{2}(\omega_i\omega_j)\right)\partial_\mu v^i\partial^\mu v^j-\frac{1}{4}(b_a\bar{b}_b)\partial_\mu z^a\partial^\mu\bar{z}^b, && (3.34)\\
g^{\alpha\bar{\beta}}R_{\alpha\gamma\bar{\beta}}{}^{\gamma} &= g^{\alpha\bar{\beta}}\left(\Gamma^\gamma_{\alpha\mu}\Gamma^\mu_{\gamma\bar{\beta}}-\Gamma^\gamma_{\gamma\mu}\Gamma^\mu_{\alpha\bar{\beta}}\right)\\
&= \frac{1}{4}\left((\omega_i\omega_j)-(\omega_i g)(\omega_j g)\right)\partial_\mu v^i\partial^\mu v^j, && (3.35)\\
g^{\alpha\bar{\beta}}R_{\alpha\bar{\gamma}\bar{\beta}}{}^{\bar{\gamma}} &= g^{\alpha\bar{\beta}}\left(\Gamma^{\bar{\gamma}}_{\alpha\mu}\Gamma^\mu_{\bar{\gamma}\bar{\beta}}-\Gamma^{\bar{\gamma}}_{\bar{\gamma}\mu}\Gamma^\mu_{\alpha\bar{\beta}}\right)\\
&= -\frac{1}{4}(\omega_i g)(\omega_j g)\partial_\mu v^i\partial^\mu v^j-\frac{1}{4}(b_a\bar{b}_b)\partial_\mu z^a\partial^\mu\bar{z}^b, && (3.36)\\
g^{\alpha\beta}R_{\alpha\mu\beta}{}^{\mu} &= g^{\alpha\beta}\left(-\partial_\mu\Gamma^\mu_{\alpha\beta}-\Gamma^\mu_{\mu\nu}\Gamma^\nu_{\alpha\beta}\right)\\
&= -\frac{1}{2}(\beta_a\bar{b}_b)z^a\nabla_\mu\partial^\mu\bar{z}^b\\
&\sim \frac{1}{2}(b_a\bar{b}_b)\partial_\mu z^a\partial^\mu\bar{z}^b. && (3.37)
\end{aligned}$$

And the last two terms actually do not contribute to the result in order lower than 3,

$$g^{\alpha\beta}(R_{\alpha\gamma\beta}{}^{\gamma}+R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}) \sim 0. \quad (3.38)$$

Hence we have derived

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x\sqrt{-g_4}\left(\mathcal{K}R+P_{ij}\partial_\mu v^i\partial^\mu v^j+Q_{ab}\partial_\mu z^a\partial^\mu\bar{z}^b\right), \quad (3.39)$$

where

$$P_{ij} := \int_Y \mathrm{d}^6y\sqrt{g_6}\left\{(\omega_i g)(\omega_j g) - \frac{1}{2}(\omega_i\omega_j)\right\}, \tag{3.40}$$

$$Q_{ab} := \frac{1}{2}\int_Y \mathrm{d}^6y\sqrt{g_6}\,(b_a\bar{b}_b). \tag{3.41}$$

3.5 Reduction of the remaining terms

In this section we do the reduction for other terms. Using our expansion, we first derive the field strengths

$$\hat{H}_3 = H_3 + \mathrm{d}b^i \wedge \omega_i, \tag{3.42}$$

$$d\hat{C}_2 = \mathrm{d}C_2 + \mathrm{d}c^i \wedge \omega_i, \tag{3.43}$$

$$d\hat{A}_4 = \mathrm{d}D_2^i \wedge \omega_i + \mathrm{d}\rho_i \wedge \tilde{\omega}^i + F^A \wedge \alpha_A - G_A \wedge \beta^A, \tag{3.44}$$

$$\hat{F}_3 = \mathrm{d}C_2 + \mathrm{d}c^i \wedge \omega_i - a(H_3 + \mathrm{d}b^i \wedge \omega_i), \tag{3.45}$$

$$\begin{aligned}\hat{F}_5 = {} & F^A \wedge \alpha_A - G_A \wedge \beta^A \\ & +(\mathrm{d}D_2^i - \mathrm{d}b^i \wedge C_2 - c^i H_3) \wedge \omega_i + \mathrm{d}\rho_i \wedge \tilde{\omega}^i - c^i \mathrm{d}b^j \wedge \omega_i \wedge \omega_j,\end{aligned} \tag{3.46}$$

where $F^A := \mathrm{d}V^A$ and $G_A := \mathrm{d}U_A$.

A formula that will be repeatedly used in the following is, schematically,

$$\star_{10}(E_n \wedge I_p) = (-1)^{np}(\star_4 E_n) \wedge (\star_6 I_p), \tag{3.47}$$

for $E_n \in \Omega^n(M_4)$ and $I_p \in \Omega^p(Y)$.

Then straightforwardly we have the following expansions:

$$
\begin{aligned}
2\int_Y \mathrm{d}\hat{\phi}\wedge\star\mathrm{d}\hat{\phi} &= 2\mathcal{K}\mathrm{d}\phi\wedge\star\mathrm{d}\phi, && (3.48)\\
-\frac{1}{4}\int_Y \hat{H}_3\wedge\star\hat{H}_3 &= -\frac{\mathcal{K}}{4}H_3\wedge\star H_3-\mathcal{K}g_{ij}\mathrm{d}b^i\wedge\star\mathrm{d}b^j, && (3.49)\\
-\frac{1}{2}\int_Y \mathrm{d}\hat{a}\wedge\star\mathrm{d}\hat{a} &= -\frac{\mathcal{K}}{2}\mathrm{d}a\wedge\star\mathrm{d}a, && (3.50)\\
-\frac{1}{2}\int_Y \hat{F}_3\wedge\star\hat{F}_3 &= -\frac{\mathcal{K}}{2}(\mathrm{d}C_2-aH_3)\wedge\star(\mathrm{d}C_2-aH_3)\\
&\quad -2\mathcal{K}g_{ij}(\mathrm{d}c^i-a\mathrm{d}b^i)\wedge\star(\mathrm{d}c^j-a\mathrm{d}b^j), && (3.51)\\
-\frac{1}{4}\int_Y \hat{F}_5\wedge\star\hat{F}_5 &= \frac{1}{4}(\mathrm{Im}\mathcal{M}^{-1})^{AB}\left(G_A-\mathcal{M}_{AC}F^C\right)\wedge\star\left(G_B-\overline{\mathcal{M}}_{BD}F^D\right)\\
&\quad -\mathcal{K}g_{ij}(\mathrm{d}D_2^i-\mathrm{d}b^i\wedge C_2-c^i\mathrm{d}B_2)\wedge\star(\mathrm{d}D_2^j-\mathrm{d}b^j\wedge C_2-c^j\mathrm{d}B_2)\\
&\quad -\frac{1}{16\mathcal{K}}g^{ij}(\mathrm{d}\rho_i-\mathcal{K}_{ikl}c^k\mathrm{d}b^l)\wedge\star(\mathrm{d}\rho_j-\mathcal{K}_{jmn}c^m\mathrm{d}b^n), && (3.52)\\
-\frac{1}{2}\int_Y \hat{A}_4\wedge\hat{H}_3\wedge\mathrm{d}\hat{C}_2 &= -\frac{1}{2}\mathcal{K}_{ijk}D_2^i\wedge\mathrm{d}b^j\wedge\mathrm{d}c^k-\frac{1}{2}\rho_i(\mathrm{d}B_2\wedge\mathrm{d}c^i+\mathrm{d}b^i\wedge\mathrm{d}C_2), && (3.53)
\end{aligned}
$$

where the definitions of the field couplings are in appendix A.

3.6 The 4D effective action for type IIB superstring

Note that the terms in the original action (3.4) are sightly different from what we have done. We can remedy this by Weyl rescaling.

First we consider the following two terms in the action

$$\int_X e^{-2\hat{\phi}}\left(-\frac{1}{2}\hat{R}\star 1+2\mathrm{d}\hat{\phi}\wedge\star\mathrm{d}\hat{\phi}\right). \tag{3.54}$$

We perform a Weyl scaling $\hat{g}_{MN}\mapsto\Omega^{-2}\hat{g}_{MN}$ with $\Omega=e^{-\hat{\phi}/4}$ to transform the above terms to Einstein's frame

$$\int_X\left(-\frac{1}{2}\hat{R}\star 1-\frac{1}{4}\mathrm{d}\hat{\phi}\wedge\star\mathrm{d}\hat{\phi}\right). \tag{3.55}$$

We can then plug in the equations (3.39) and (3.48):

$$\int_{M_4}\left(-\frac{1}{2}\mathcal{K}R\star 1-\frac{1}{4}\mathcal{K}\mathrm{d}\phi\wedge\star\mathrm{d}\phi-\frac{1}{2}P_{ij}\mathrm{d}v^i\wedge\star\mathrm{d}v^j-\frac{1}{2}Q_{ab}\mathrm{d}z^a\wedge\star\mathrm{d}\bar{z}^b\right), \tag{3.56}$$

and perform another Weyl rescaling with $\Omega = \mathcal{K}^{1/2}$ to obtain

$$\int_{M_4} \left(-\frac{1}{2} R \star 1 - \frac{1}{4} \mathrm{d}\phi \wedge \star \mathrm{d}\phi - \frac{1}{2\mathcal{K}} P_{ij} \mathrm{d}v^i \wedge \star \mathrm{d}v^j - \frac{1}{2\mathcal{K}} Q_{ab} \mathrm{d}z^a \wedge \star \mathrm{d}\bar{z}^b - \frac{3}{4} \mathrm{d} \ln \mathcal{K} \wedge \star \mathrm{d} \ln \mathcal{K} \right). \tag{3.57}$$

We can further perform a field redefinition for v^i to eliminate the last annoying term and the whole Weyl+redefinition procedure can be applied to all the remaining terms. But from now on we only focus on the vector multiplets hence those redefinitions are not relevant. We collect the relevant terms:

$$\begin{aligned} S &= \int_{M_4} \left\{ -\frac{1}{2} R \star 1 - g_{ab} \mathrm{d}z^a \wedge \star \mathrm{d}\bar{z}^b \right\} \\ &\quad + \int_{M_4} \frac{1}{4} (\mathrm{Im}\mathcal{M}^{-1})^{AB} \left(G_A - \mathcal{M}_{AC} F^C \right) \wedge \star \left(G_B - \overline{\mathcal{M}}_{BD} F^D \right) + \cdots , \end{aligned} \tag{3.58}$$

where $g_{ab} := Q_{ab}/2\mathcal{K}$. Note that all the Hodge stars $\star$ are now defined on the 4D spacetime M_4.

We still need to impose the self-duality condition $\hat{F}_5 = \star \hat{F}_5$. The implication on the relevant fields of the self-duality condition reads

$$\star G_A = (\mathrm{Re}\mathcal{M})_{AB} \star F^B - (\mathrm{Im}\mathcal{M})_{AB} F^B. \tag{3.59}$$

This equation means that we can only keep one of the two fields F^A and G_A as dynamical field. This constraint can be derived as an equation of motion for the field G_A once we add a total derivative term

$$\frac{1}{2} F^A \wedge G_A \tag{3.60}$$

in the action. We choose to keep F^A and eliminate G_A to obtain the action for the field F^A

$$\int_{M_4} \left\{ \frac{1}{2} (\mathrm{Im}\mathcal{M})_{AB} F^A \wedge \star F^B + \frac{1}{2} (\mathrm{Re}\mathcal{M})_{AB} F^A \wedge F^B \right\}. \tag{3.61}$$

Finally, we collect all relevant terms and present the 4D effective action for type IIB superstring compactified on a smooth Calabi-Yau threefold Y:

$$\begin{aligned} S_{\mathrm{IIB}/Y} &= \int_{M_4} \left\{ -\frac{1}{2} R \star 1 - g_{ab} \mathrm{d}z^a \wedge \star \mathrm{d}\bar{z}^b \right. \\ &\quad \left. + \frac{1}{2} (\mathrm{Im}\mathcal{M})_{AB} F^A \wedge \star F^B + \frac{1}{2} (\mathrm{Re}\mathcal{M})_{AB} F^A \wedge F^B + \cdots \right\}. \end{aligned} \tag{3.62}$$

The above 4D effective action $S_{\mathrm{IIB}/Y}$ looks quite familiar. Indeed it is an action for a 4D, $\mathcal{N} = 2$ ungauged supergravity theory. We can further compare (3.62) with the component action in Seiberg-Witten theory (2.14) and see some of the similarities. The different signs on the gauge fields are merely a consequence of using differential form notations in this chapter. Our 4D, $\mathcal{N} = 2$ supergravity effective action is more general than the one in Seiberg-Witten theory in the sense that it includes the coupling with gravity and now the coupling between the scalars z^a are different from the couplings matrix between gauge fields. One can roughly say that a supergravity theory to a supersymmetry theory is like general relativity to special relativity in the sense that supergravity theories localise the supersymmetry. For more general discussion on $\mathcal{N} = 2$ supergravity theories, see [2].

3.7 The 4D effective action for type IIA superstring

Similarly, the moduli fields for type IIA strings are organised into $\mathcal{N} = 2$ supergravity multiplets as shown in Table 3.5.

gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vector multiplets	$h^{1,1}$	(A^i, v^i, b^i)
hypermultiplets	$h^{2,1}$	$(z^a, \xi^A, \tilde{\xi}_A)$
tensor multiplet	1	$(B_2, \phi, \xi^0, \tilde{\xi}_0)$

Table 3.5: 4D fields arranged in $\mathcal{N} = 2$ supergravity multiplets

The resulting vector multiplet sector of 4D effective action for type IIA string is

$$\begin{aligned} S_{\mathrm{IIA}/Y} &= \int_{M_4} \Big\{ -\frac{1}{2} R \star 1 - g_{ij} \mathrm{d}t^i \wedge \star \mathrm{d}\bar{t}^j \\ &\quad + \frac{1}{2} (\mathrm{Im}\mathcal{N})_{IJ} F^I \wedge \star F^J + \frac{1}{2} (\mathrm{Re}\mathcal{N})_{IJ} F^I \wedge F^J + \cdots \Big\}, \end{aligned} \tag{3.63}$$

where $F^I := \mathrm{d}A^I$ and $t^i := b^i + \mathbf{i}v^i$ is the complexified Kähler moduli.

One interesting thing to notice is that the role of the complex structure moduli z^a in $S_{\mathrm{IIB}/Y}$ and the Kähler moduli v^i in $S_{\mathrm{IIA}/Y}$ looks similar. This is a related to mirror symmetry which physically is a valuable tool to compute all the nonperturbative couplings in the effective 4D theory exactly. Mirror symmetry and its application will be our main topic in the remaining chapters.

Chapter 4

Geometric Engineering of Quantum Field Theories

In this chapter, we will show that by properly choosing a *local*, i.e. non-compact, Calabi-Yau threefold X and putting type IIA superstring theory on it, the gauge group of the 4D effective supergravity will be enhanced to the non-abelian SU(2) group (from of the abelian U(1) group). Furthermore, by pushing the Planck mass M_{Pl} to infinity while carefully rescaling other quantities, the gravity sector will decouple from the 4D theory leaving a SUSY gauge field theory. This procedure is called the geometric engineering of quantum field theories [46].

Just like 4D SUSY gauge field theory, the 4D effective supergravity receives worldsheet instanton corrections which is hard to compute directly. Fortunately, for the type II superstrings, there is already a powerful technique called *mirror symmetry* that computes all the worldsheet instanton contributions exactly. Thus we have another way to solve a 4D SUSY quantum field theory: first putting type IIA on a suitable local Calabi-Yau threefold, then compute the exact solution via mirror symmetry, and finally decouple the gravity from the effective 4D supergravity to obtain the exact effective theory for the original 4D SUSY field theory.

Interestingly, although at first no obvious, this procedure naturally generalises the Seiberg-Witten solution to the 4D SUSY gauge theories in the sense that the previously "auxiliary" Seiberg-Witten elliptic curve Σ_{SW} is now actually part of the mirror of the local Calabi-Yau threefold! Furthermore the instanton corrections to the 4D SUSY gauge theory has nicer geometric significance as they are coming from string worldsheet wrapping multiple times on certain 2-cycle in the local Calabi-Yau threefold.

We will do the computation explicitly for the SU(2) case. Some references on geometric engineering and local mirror symmetry are [46], [14], [22], [55] and of course the big book [41]. The first part of the paper [67] contains a nice review on toric geometry and its usage in local mirror symmetry. The paper [47] is a generalisation of the original work [46]. For an overall picture of embedding SUSY field theories into string theory, [63] is a pleasure to read. Lastly, if you read Japanese, the lecture notes by Konishi [56] are very nice and explains all the basics for local mirror symmetry.

4.1 Singular Calabi-Yau threefolds and enhancement of gauge symmetry

In the previous chapter, we learnt that after being compactified on a smooth Calabi-Yau threefold, the resulting 4-dimensional effective field theory has an abelian U(1) gauge symmetry. How should one realise non-abelian gauge symmetries? The answer is not obvious and it crucially depends on the existing of extended solitonic objects in superstring theories, namely the *D-branes* [74].

Consider putting a type IIA superstring on a K3 surface, i.e. a Calabi-Yau twofold. The resulting theory will be 6-dimensional. Denote a basis of $(1,1)$-forms on K3 by ω_i as usual and denote their Poincaré dual holomorphic 2-cycles by C_2^i. Just like what we have done in the Calabi-Yau threefold compactification, the 6D theory has neutral vector bosons A^i coming from the dimensional reduction of the type IIA Ramond-Ramond 3-from $\hat{C}_3$

$$\hat{C}_3 = A^i \wedge \omega_i. \tag{4.1}$$

But now D2-branes bring us more vector bosons: When a D2-brane wraps around a holomorphic 2-cycle C_2^i (we assume C_2^i to be a 2-sphere S^2 for the moment) in the K3-surface, a charged BPS particle state in 6 dimensions appears. (Note that the worldvolume of a Dp-brane is of $p+1$ spacetime dimensions) Moreover, a D2-brane can wrap around such a holomorphic 2-cycle in two opposite orientations, which leads to a pair of oppositely charged $W_i^\pm$ vector bosons in 6 dimensions. The mass of $W_i^\pm$ is proportional to the volume of the 2-cycle

$$\frac{M_{W_i^\pm}}{M_{\mathrm{Pl}}} \sim \mathrm{Vol}(C_2^i). \tag{4.2}$$

Now we see that, when the volume of such a 2-cycle C_2^i approaches zero such that the K3 surface develops a singularity, the mass of the $W_i^\pm$ bosons vanishes. Together with the neutral vector boson A^i coming directly from the dimensional reduction, all SU(2) gauge bosons are recovered! Of course one can show more rigorously in group-theoretical arguments that indeed one gets a SU(2) gauge theory. One can find more details in [48].

In general, a singularity of a K3 surface can be more complicated than just being the result of one shrinking S^2. The possible singularities of K3 surfaces were classified by Kodaira in terms of simple-laces ADE Lie groups. The ADE classification roughly states that the blow-up of a singularity in a K3 surface, e.g. replacing a singularity by an S^2 like above, looks like a sequence of spheres S^2 intersecting according to the Dynkin diagrams of simple-laced ADE Lie groups. Interestingly, if we wrap a D2-brane around a singularity, the corresponding gauge group will be exactly the classification group of the singularity! Indeed, above we have considered only one S^2, which corresponds the A_1 Lie group SU(2).

Note also from the mass formula (4.2) that when we push the Planck mass M_{Pl} to infinity to decouple gravity, D2-branes wrapping around non-singular 2-cycles will generate very heavy vector bosons which will not affect our low energy effective theory at all. Hence we can focus on a neighbourhood of a singularity. Such a neighbourhood is non-compact and is called an ALE (asymptotically locally Euclidean) space and is an example of *local Calabi-Yau threefolds.*

The above story is in 6 dimensions. To obtain a 4-dimensional theory, we should further compactify on a 2-dimensional space. In other words we will fibre a non-compact ALE space over a 2-dimensional base space. One can show that to reach an $\mathcal{N} = 2$ theory in 4D, the 2 dimensional base should have non-vanishing curvature. Furthermore, the singularities of the base space controls the matter fields in the the 4D, $\mathcal{N} = 2$ theory. We consider here the theory without any matter field. This means that the base geometry is simply a 2-sphere $\mathbb{P}^1$.

There is an important relation between the gauge coupling constant g_6 in 6 dimensions to g_4 in 4 dimensions, namely

$$\frac{1}{g_6^2}\text{Vol(Base)} = \frac{1}{g_4^2}. \tag{4.3}$$

This relation will be used to rescale the volume of the base and the fibre such that gravity is properly decouples as $M_{\text{Pl}} \to \infty$.

4.2 Local Mirror symmetry

In the last section, we understood why one only needs to consider local Calabi-Yau threefolds in geometric engineering. Local mirror symmetry is the mirror symmetry tuned for local Calabi-Yau threefolds. It considers less data than the full mirror symmetry of Calabi-Yau threefolds hence is a good start for our next step in mirror symmetry in the following chapters.

Before we list the local mirror data, let us note that local mirror symmetry, although simpler than the full mirror symmetry, contains important issues regarding its basis. There is no way to recover a full prepotential from the local mirror symmetry computation, see [59], section 3. In particular, we expect that an ideal theory of local mirror symmetry should start with merely a local Calabi-Yau space and then “derives everything” from it without further assumptions. But what was actually done in [14] is that the authors embed some of the local Calabi-Yau threefolds into some compact Calabi-Yau threefolds which admits an elliptic fibration. Then the authors send the volume of the elliptic fibre to infinity to “de-compactify” and create the original local Calabi-Yau threefold. The authors derived all the data and solutions from the full mirror symmetry with the large elliptic fibre limit, and conjectured that for those local Calabi-Yau threefolds which cannot be embed into an elliptically-fibred compact Calabi-Yau space, the data and solutions should still work, see also [67], section 3. To the knowledge of the author, there is no solution to this conjecture yet.

We now state the recipe proposed in [14] for the local mirror symmetry computation.

Starting with a 2-dimensional reflexive polyhedra Δ, its integral points are denoted by $\nu^{(i)}$ and the origin has always index $i = 0$. We embed Δ into a 3-dimensional lattice by setting $\bar{\nu}^{(i)} = (1, \nu^{(i)})$ and denote the embedded polytope by $\bar{\Delta}$.

The polyhedron Δ determines a 2D toric variety $\mathbb{P}_\Delta$. Then the local Calabi-Yau threefold on which we compactify the type IIA superstring will be the non-compact Calabi-Yau threefold corresponding to $\bar{\Delta}$, which is the canonical bundle $\mathcal{K}_{\mathbb{P}_\Delta}$. For each integral point $\bar{\nu}^{(i)}$, we assign a complex coordinate Y_i. These Y_i satisfies the usual scaling property of the toric variety and will be combined into the homogeneous coordinates of $\mathbb{P}_\Delta$, which is denoted by (X_0, X_1, X_2).

We also need the Newton polynomial

$$P_\Delta = \sum_{i=0}^{l(\Delta)-1} a_i Y_i, \tag{4.4}$$

where $l(\Delta)$ denotes the number of integer points in the polyhedron Δ. The local mirror of the local Calabi-Yau threefold $\mathcal{K}_{\mathbb{P}_\Delta}$ is then the locus of the Newton polynomial P_Δ inside $\mathbb{P}_\Delta$. Note that the local mirror is actually 1-dimensional. Instead of a full 3-dimensional mirror geometry, this mirror curve contains all the information we need.

Let $\{l^{(i)}\}$ be a basis of the Mori cone of $\mathbb{P}_\Delta$, which satisfies

$$\sum_{i=0}^{l(\Delta)-1} l_i^{(j)} \bar{\nu}^{(i)} = 0. \tag{4.5}$$

Then the coordinate of the complex structure moduli space of the mirror determined by P_Δ is defined by

$$z_i = \prod_{j=1}^{l(\Delta)-1} a_j^{l_j^{(i)}}. \tag{4.6}$$

In these coordinates, the point of large complex structure corresponds to $z_i = 0$.

We also define a meromorphic 1-form Ω on the mirror by a residue construction

$$\Omega = \int_{P_\Delta = 0} \log(P_\Delta) \frac{\mathrm{d}X_1}{X_1} \wedge \frac{\mathrm{d}X_2}{X_2}, \tag{4.7}$$

where we work in the patch where $X_0 = 1$. Then the main object of local mirror symmetry that we are computing are the period integrals of this meromorphic 1-form Ω. We denote these by $\Pi_i(z)$, which clearly depends on the complex structure moduli z^i. Just like what we did at the end of the Seiberg-Witten theory, these periods are annihilated by Picard-Fuchs operators. In terms of the coefficients a_i, the Picard-Fuchs operator here is given by the GKZ-hypergeometric operator

$$\mathcal{L}_k = \prod_{l_i^{(k)}>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i^{(k)}} - \prod_{l_i^{(k)}<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i^{(k)}}, \tag{4.8}$$

which acts on the periods $\Pi(z)$ in the z variables. And the periods can be solved using the Frobenius method. The way to determine the particular solution are spelled out in chapter 5.

Instead of elaborating on this recipe, let us work out an example now.

4.3 Example: pure SU(2) field theory

We do the explicit calculation here for the pure SU(2) case and derive the perturbative and 1-instanton contribution to the field theory gauge coupling function.

4.3.1 The local A-model

To obtain an SU(2) gauge group enhancement without matter field, we need an A_1-singularity ALE space as the fibre and a $\mathbb{P}^1$ base. In other words, we need to consider a fibration of $\mathbb{P}^1$ over $\mathbb{P}^1$. These fibre bundles are called the Hirzebruch surfaces F_n where $n \in \mathbb{N}$. The toric surfaces F_0, F_1 and F_2 come from reflexive polyhedra hence are suitable for local mirror symmetry and they will all yield the same result in 4D. We choose the F_2 surface as our example, and the local A-model geometry is the canonical bundle $\mathcal{K}_{\mathsf{F}_2}$.

The local Calabi-Yau threefold $\mathcal{K}_{\mathsf{F}_2}$ is given by the following toric data (note that the toric data of F_2 is embedded in it):

				$l^{(1)}$	$l^{(2)}$
D_0	1	0	0	-2	0
D_1	1	1	0	1	0
D_2	1	0	1	0	1
D_3	1	-1	0	1	-2
D_4	1	-2	-1	0	1

The Mori generators $l^{(1)}, l^{(2)}$ satisfies

$$\sum_{k=0}^{4} l_k^{(i)} \bar{\nu}^{(k)} = 0. \tag{4.9}$$

And the D_k corresponding to the vertex $\bar{\nu}^{(k)}$ is the toric divisor $D_k = \{Y_k = 0\}$.

The remaining discussions of this subsection are all about the surface F_2.

From the toric data, we can determine the intersection numbers of the toric divisors in the base surface F_2. First we have the following two linear equivalence relations between the divisors:

$$\begin{aligned} D_1 &= D_3 + 2D_4, && (4.10) \\ D_2 &= D_4, && (4.11) \end{aligned}$$

which are coming from the standard relation $\sum_{i=1}^{2} \nu_i^{(k)} D_i = 0$.

Then from the standard intersection products

$$D_1 \cdot D_2 = D_2 \cdot D_3 = D_3 \cdot D_4 = D_4 \cdot D_1 = 1, \tag{4.12}$$

with other $D_i \cdot D_j = 0$ for $i \neq j$, we derive the following self-intersection products in the surface F_2:

$$\begin{aligned} D_1 \cdot D_1 &= 2, && (4.13) \\ D_3 \cdot D_3 &= -2, && (4.14) \\ D_2 \cdot D_2 = D_4 \cdot D_4 &= 0. && (4.15) \end{aligned}$$

By the formula for the total Chern class of a toric variety $c = \prod(1 + D_i)$, we have the first Chern class

$$c_1(\mathsf{F}_2) = 2D_1, \tag{4.16}$$

where the Poincaré duality is automatically imposed.

The Mori cone vectors encode the intersections between the corresponding basis of effective curves with the Kähler cone divisors. Hence we can choose D_1 and D_2 as a basis for the Kähler cone of F_2, since $l^{(i)} \cdot D_j = \delta_j^i$ for $i, j = 1, 2$. Let J_1 be the Poincaré dual of D_1 and J_2 be the Poincaré dual of D_2. Then we have the following useful data

$$\begin{aligned} J_1 \cdot J_1 &= 2, && (4.17) \\ J_1 \cdot J_2 &= 1, && (4.18) \\ J_2 \cdot J_2 &= 0, && (4.19) \\ c_1(\mathsf{F}_2) \cdot J_1 &= 4, && (4.20) \\ c_1(\mathsf{F}_2) \cdot J_2 &= 2, && (4.21) \end{aligned}$$

where we define the notation $J_i \cdot J_j = \int_{\mathsf{F}_2} J_i \wedge J_j$ and $c_1 \cdot J_i = \int_{\mathsf{F}_2} c_1 \wedge J_i$.

4.3.2 The local B-model

The local mirror curve for the local A-model is given by the locus of the Newton polynomial inside F_2

$$P = a_0 Y_0 + a_1 Y_1 + a_2 Y_2 + a_3 Y_3 + a_4 Y_4, \tag{4.22}$$

where Y_i is the variable corresponding to $\bar{\nu}^{(i)}$.

Furthermore, there are the following constraints between the coordinates

$$Y_0^2 = Y_1 Y_3, \tag{4.23}$$

$$Y_2 Y_4 = Y_3^3, \tag{4.24}$$

such that the toric A-model geometry is embeded in a projective space. The general constriant is $\prod Y_i^{l_i^{(k)}} = 1$. These constraints can be solved by setting $Y_1 = x^2$, $Y_2 = \zeta$ and $Y_3 = s^2$, with the coordinates (x, ζ, s) projective. Then the curve can now be rewritten as (in the patch $s = 1$)

$$a_0 x + a_1 x^2 + a_2 \zeta + a_3 + a_4 \frac{1}{\zeta} = 0. \tag{4.25}$$

Since we can rescale the coordinates Y_i in the toric variety $\mathcal{K}_{\mathsf{F}_2}$, we can also rescale the coefficients a_i such that the defining equation $P = 0$ is invariant. Thus we define the complex structure moduli

$$z_{\mathrm{f}} = \frac{a_1 a_3}{a_0^2}, \tag{4.26}$$

$$z_{\mathrm{b}} = \frac{a_2 a_4}{a_3^2}. \tag{4.27}$$

The GKZ-operators in the variable a_i are simply

$$\tilde{\mathcal{L}}_1 = \partial_{a_1}\partial_{a_3} - \partial_{a_0}^2, \tag{4.28}$$

$$\tilde{\mathcal{L}}_2 = \partial_{a_2}\partial_{a_4} - \partial_{a_3}^2. \tag{4.29}$$

And these two operators annihilate two periods Π_1, Π_2 of the meromorphic 1-form Ω.

Due to the rescaling symmetry, the a_i coefficients parametrise the complex deformation of the B-model curve redundantly, and the good complex structure moduli are $z_{\mathrm{f}}, z_{\mathrm{b}}$. So we now transform the above GKZ operators into the expression in $z_{\mathrm{f}}, z_{\mathrm{b}}$. It is always useful to rewrite the derivatives in terms of logarithmic derivatives $\theta_{a_i} = a_i \partial_{a_i}$, and similarly $\theta_{\mathrm{f}} = z_{\mathrm{f}}\partial_{z_{\mathrm{f}}}, \theta_{\mathrm{b}} = z_{\mathrm{b}}\partial_{z_{\mathrm{b}}}$. We only show the computation for $\tilde{\mathcal{L}}_1$.

There are some useful identities between the logarithmic derivatives

$$\theta_0 = -2\theta_{\rm f}, \tag{4.30}$$
$$\theta_1 = \theta_{\rm f}, \tag{4.31}$$
$$\theta_2 = \theta_{\rm b}, \tag{4.32}$$
$$\theta_3 = \theta_{\rm f} - 2\theta_{\rm b}, \tag{4.33}$$
$$\theta_4 = \theta_{\rm b}. \tag{4.34}$$

These are understood to hold as identities between differential operators acting on the periods $\Pi_i(z_{\rm f}, z_{\rm b})$.

And with another obvious relation $z\theta_z = (\theta_z - 1)z$, we have

$$\begin{aligned} a_1 a_3 \tilde{\mathcal{L}}_1 &= a_1 a_3(\partial_{a_1}\partial_{a_3} - \partial_{a_0}^2) \\ &= \theta_{a_1}\theta_{a_3} - z_{\rm f} a_0 \partial_{a_0}^2 \\ &= \theta_{a_1}\theta_{a_3} z_{\rm f}(\theta_{a_0} - 1)\theta_{a_0} \\ &= \theta_{\rm f}(\theta_{\rm f} - 2\theta_{\rm b}) + 2z_{\rm f}(2\theta_{\rm f} + 1)\theta_{\rm f}. \end{aligned} \tag{4.35}$$

In the derivation we have freely multiplied $a_1 a_3$ in front of the operator $\tilde{\mathcal{L}}_1$ because we are only interested in the differential equations $\tilde{\mathcal{L}}_i \Pi_j = 0$.

Hence the GKZ-hypergeometric operators are given by

$$\mathcal{L}_1 = 2z_{\rm f}\theta_{\rm f}(2\theta_{\rm f} + 1) + \theta_{\rm f}(2\theta_{\rm b} - \theta_{\rm f}), \tag{4.36}$$
$$\mathcal{L}_2 = z_{\rm b}(2\theta_{\rm b} - \theta_{\rm f} + 1)(2\theta_{\rm b} - \theta_{\rm f}) - \theta_{\rm b}^2. \tag{4.37}$$

4.3.3 Solution to the GKZ system: mirror maps and worldsheet instanton numbers

Observe that $z_{\rm f} = z_{\rm b} = 0$ is a regular singular point of the Picard-Fuchs differential equations. In fact, it corresponds to the so-called *large complex structure point* in the complex structure moduli space in the coordinates $z_{\rm f}, z_{\rm b}$. Around this point the index of the Frobenius Ansatz are maximally degenerate. Mirror symmetry then tells that the *mirror map* which relates type IIB complex structure deformations with type IIA Kähler deformations are exactly the single-logarithmic solutions around the large complex structure point. Furthermore there is a double-logarithmic solution which encodes the information of gauge coupling functions of the 4D theory.

Now, we carefully derive a series representation of the solution to the Picard-Fuchs system around $z_{\rm b} = z_{\rm f} = 0$ by Frobenius method. An introduction to this method can be found in the book [81]. First we write down a power series Ansatz

$$f(z_{\rm f}, z_{\rm b}; \rho_1, \rho_2) = \sum_{n,m\geq 0} a_{n,m}(\rho_1, \rho_2) z_{\rm f}^{n+\rho_1} z_{\rm b}^{m+\rho_2}, \tag{4.38}$$

and hit it with our Picard-Fuchs operators $\mathcal{L}_1$ and $\mathcal{L}_2$ to obtain the following recurrence relations between the coefficients:
For $n = 0, m \geq 0$,

$$\rho_1(2\rho_2 - \rho_1 + 2m)a_{0,m} = 0. \tag{4.39}$$

For $n \geq 1, m \geq 0$,

$$(n + \rho_1)(2\rho_2 - \rho_1 + 2m - n)a_{n,m} + 2(\rho_1 + n - 1)(2\rho_1 + 2n - 1)a_{n-1,m} = 0. \tag{4.40}$$

For $n \geq 0, m = 0$,

$$\rho_2^2 a_{n,0} = 0. \tag{4.41}$$

For $n \geq 0, m \geq 1$,

$$(m + \rho_2)^2 a_{n,m} - (\rho_1 - 2\rho_2 + n - 2m + 1)(\rho_1 - 2\rho_2 + n - 2m + 2)a_{n,m-1} = 0. \tag{4.42}$$

Note that, in particular, for $n = m = 0$, we have the indicial equations

$$\begin{aligned} \rho_1(2\rho_2 - \rho_1)a_{0,0} &= 0, & (4.43)\\ \rho_2^2 a_{0,0} &= 0, & (4.44) \end{aligned}$$

meaning that the indices ρ_1, ρ_2 are indeed maximally degenerate around $z_\mathrm{b} = z_\mathrm{f} = 0$.

Leaving the initial $a_{0,0}$ undetermined, we can solve these recurrence relations

$$a_{n,m}(\rho_1,\rho_2) = \begin{cases} \frac{\prod_{j=0}^{2m-1}(2\rho_2-\rho_1+j)}{\prod_{j=1}^{m}(j+\rho_2)^2}\, a_{0,0}(\rho_1,\rho_2) & \text{for } n = 0, m \geq 0,\\ \frac{(-1)^n \prod_{j=0}^{2n-1}(2\rho_1+j)\prod_{j=0}^{2m-n-1}(2\rho_2-\rho_1+j)}{\prod_{j=1}^{n}(\rho_1+j)\prod_{j=1}^{m}(\rho_2+j)^2}\, a_{0,0}(\rho_1,\rho_2) & \text{for } 0 < n < 2m,\\ \frac{\prod_{j=0}^{2n-1}(2\rho_1+j)}{\prod_{j=1}^{n}(\rho_1+j)\prod_{j=1}^{m}(\rho_2+j)^2}\, a_{0,0}(\rho_1,\rho_2) & \text{for } n = 2m,\\ \frac{\prod_{j=0}^{2n-1}(2\rho_1+j)}{\prod_{j=1}^{n}(\rho_1+j)\prod_{j=1}^{m}(\rho_2+j)^2\prod_{j=1}^{n-2m}(\rho_1-2\rho_2+j)}\, a_{0,0}(\rho_1,\rho_2) & \text{for } n > 2m. \end{cases} \tag{4.45}$$

If we act the Picard-Fuchs operators on our Ansatz $f(z_\mathrm{f}, z_\mathrm{b}; \rho_1, \rho_2)$, provided that the above recurrence relations are satisfied, then we have

$$\begin{aligned} \mathcal{L}_1 f &= \rho_1(2\rho_2 - \rho_1)\, a_{0,0}\, z_\mathrm{f}^{\rho_1} z_\mathrm{b}^{\rho_2}, & (4.46)\\ \mathcal{L}_2 f &= \rho_2^2\, a_{0,0}\, z_\mathrm{f}^{\rho_1} z_\mathrm{b}^{\rho_2}. & (4.47) \end{aligned}$$

These two equations imply that when we set $\rho_1 = \rho_2 = 0$, the function

$$f(z_\mathrm{f}, z_\mathrm{b}; 0, 0) = a_{0,0}(0, 0) \tag{4.48}$$

solves the Picard-Fuchs system. Furthermore, if we make linear combinations of the derivatives of f with respect to ρ_1 and ρ_2 and set $\rho_1 = \rho_2 = 0$, then the Picard-Fuchs system also has the following solutions

$$\begin{aligned}
\text{constant} &= 1, &(4.49)\\
-t_{\rm f} &:= \frac{\partial}{\partial\rho_1} f(z_{\rm f}, z_{\rm b}; 0, 0) = \log z_{\rm f} + \cdots , &(4.50)\\
-t_{\rm b} &:= \frac{\partial}{\partial\rho_2} f(z_{\rm f}, z_{\rm b}; 0, 0) = \log z_{\rm b} + \cdots , &(4.51)\\
\Omega &:= \left(\frac{\partial}{\partial\rho_1}\frac{\partial}{\partial\rho_1} + \frac{\partial}{\partial\rho_1}\frac{\partial}{\partial\rho_2}\right) f(z_{\rm f}, z_{\rm b}; 0, 0)\\
&= (\log z_{\rm f})^2 + (\log z_{\rm f})(\log z_{\rm b}) + \cdots , &(4.52)
\end{aligned}$$

where the two single-log solutions (4.50) and (4.51) are called *the mirror maps* and the double-log solution (4.63) with the inverse series of (4.50) and (4.51) plugged in will be used to determine *the worldsheet instanton numbers* of our model.

We can specify the condition on $a_{0,0}$ now. The choice for the normalisation is

$$\begin{aligned}
a_{0,0}(0,0) &= 1, &(4.53)\\
\frac{\partial}{\partial\rho_1}a_{0,0}(0,0) = \frac{\partial}{\partial\rho_2}a_{0,0}(0,0) &= 0, &(4.54)\\
\frac{\partial}{\partial\rho_1}\frac{\partial}{\partial\rho_1}a_{0,0}(0,0) = \frac{\partial}{\partial\rho_1}\frac{\partial}{\partial\rho_2}a_{0,0}(0,0) &= 0, &(4.55)
\end{aligned}$$

and the derivatives of $a_{n,m}$ for $(n,m) \neq (0,0)$ are calculated to be

$$\frac{\partial}{\partial\rho_1}a_{n,m}(0,0) = \begin{cases} 0 & \text{for } 0 < n < 2m, \\ -\frac{(2m-1)!}{(m!)^2} & \text{for } n = 0, m \geq 0, \\ \frac{2(2n-1)!}{n!(m!)^2(n-2m)!} & \text{for } n \geq 2m. \end{cases} \qquad (4.56)$$

$$\frac{\partial}{\partial\rho_2}a_{n,m}(0,0) = \begin{cases} 0 & \text{for } n > 0, \\ \frac{2(2m-1)!}{(m!)^2} & \text{for } n = 0, m \geq 0. \end{cases} \qquad (4.57)$$

Now we can write down the mirror maps completely

$$\begin{aligned}
-t_{\rm f} &= \log z_{\rm f} - \sum_{m=1}^{\infty}\frac{(2m-1)!}{(m!)^2} z_{\rm b}^m + \sum_{n=1}^{\infty}\frac{2(2n-1)!}{(n!)^2} z_f^n\\
&\quad + \sum_{m=0}^{\infty}\sum_{n=2m}^{\infty}\frac{2(2n-1)!}{n!(m!)^2(n-2m)!} z_{\rm f}^n z_{\rm b}^m &(4.58)\\
&= \log z_{\rm f} + \Sigma_{\rm f}(z_{\rm f}, z_{\rm b}), &(4.59)\\
-t_{\rm b} &= \log z_{\rm b} + \sum_{m=0}^{\infty}\frac{2(2m-1)!}{(m!)^2} z_{\rm b}^m &(4.60)\\
&= \log z_{\rm b} + \Sigma_{\rm b}(z_{\rm f}, z_{\rm b}), &(4.61)
\end{aligned}$$

where we have defined the regular parts Σ_f and Σ_b. The convention is that $k! = 0$ for all negative k.

Furthermore, the double derivatives of $a_{n,m}$ for $(n,m) \neq (0,0)$ can also be computed

$$\left(\partial^2_{\rho_1} + \partial_{\rho_1}\partial_{\rho_2}\right) a_{n,m}(0,0) = \begin{cases} 0 & \text{for } 0 < n < 2m, \\ -\frac{2(2m-1)!}{(m!)^2}\left(\sum_{k=m+1}^{2m-1}\frac{1}{k}\right) & \text{for } n = 0, m \geq 1, \\ \frac{4(2n-1)!}{n!(m!)^2(n-2m)!}\left(\sum_{k=1}^{2n-1}\frac{2}{k} - \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{m}\frac{1}{k}\right) & \text{for } n \geq 2m. \end{cases} \tag{4.62}$$

Then the double logarithmic solution Ω is

$$\begin{aligned} \Omega &= (\log z_\text{f})^2 + (\log z_\text{f})(\log z_\text{b}) + (2\Sigma_\text{f} + \Sigma_\text{b})\log z_\text{f} + \Sigma_\text{f}\log z_\text{b} \\ &\quad - \sum_{m=2}^{\infty}\frac{2(2m-1)!}{(m!)^2}\left(\sum_{k=m+1}^{2m-1}\frac{1}{k}\right) z_\text{b}^m \\ &\quad + \sum_{m=0}^{\infty}\sum_{n=2m}^{\infty}\frac{4(2n-1)!}{n!(m!)^2(n-2m)!}\left(\sum_{k=1}^{2n-1}\frac{2}{k} - \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{m}\frac{1}{k}\right) z_\text{f}^n z_\text{b}^m. \end{aligned} \tag{4.63}$$

Denote the last two big sums by Σ_2, and it can be written in terms of the mirror map

$$\Omega = t_\text{f}^2 + t_\text{f} t_\text{b} + (\Sigma_2 - \Sigma_\text{f}^2 - \Sigma_\text{f}\Sigma_\text{b}), \tag{4.64}$$

where the terms in the parenthesis contains the information of the worldsheet instanton correction and the terms with quadratic t_f, t_b dependence are directly related to the intersection properties of the A-model geometry.

Now we are ready to compute the worldsheet instanton numbers. Define two variables:

$$q_\text{f} := e^{-t_\text{f}}, \tag{4.65}$$

$$q_\text{b} := e^{-t_\text{b}}, \tag{4.66}$$

and then invert the mirror maps perturbatively around $z_f = z_b = 0$, i.e., write down the functions of z_f and z_b in terms of q_f and q_b. This can be done on a computer algebra system. The inverse series up to order 4 are given by

$$z_\text{f} = q_\text{f} - 2q_\text{f}^2 + 3q_\text{f}^3 - 4q_\text{f}^4 + q_\text{f}q_\text{b} - 4q_\text{f}^2 q_\text{b} + 3q_\text{f}^3 q_\text{b} - 2q_\text{f}^2 q_\text{b}^2 + \cdots, \tag{4.67}$$

$$z_\text{b} = q_\text{b} - 2q_\text{b}^2 + 3q_\text{b}^3 - 4q_\text{b}^4 + \cdots. \tag{4.68}$$

After plugging the inverse mirror maps (4.67) and (4.68) into the double-log solution (4.63), we get, up to order 4,

$$\Omega = t_\text{f}^2 + t_\text{f}t_\text{b} + 4q_\text{f} + q_\text{f}^2 + \frac{4}{9}q_\text{f}^3 + \frac{1}{4}q_\text{f}^4 + (4q_\text{f} + 16q_\text{f}^2 + 36q_\text{f}^3)q_\text{b} + q_\text{f}^2 q_\text{b}^2 + \cdots. \tag{4.69}$$

As a side note, it is interesting to observe that the coefficient of $q_{\rm b}^m$ in the inverse mirror map for $z_{\rm b}$ seems to be $(-1)^{m-1}m$. This guess is actually true. Since the series of the mirror map (4.60) actually converges to the function

$$-t_{\rm b} = -2\log(1+\sqrt{1-4z_{\rm b}}) + \log(4z_{\rm b}). \tag{4.70}$$

Then with the variable $q_{\rm b}$, the inverse mirror map can be computed

$$z_{\rm b} = \frac{q_{\rm b}}{(1+q_{\rm b})^2} \tag{4.71}$$

$$= \sum_{m=1}^{\infty}(-1)^{m-1}m\, q_{\rm b}^m. \tag{4.72}$$

This fact can be used to see a more detailed structure of the double-log solution and it will allow a direct computation of perturbative 1-loop logarithmic correction to the Seiberg-Witten prepotential and its 1-instanton correction.

According to the local mirror symmetry paper [14] and the standard result on the integral basis on Calabi-Yau threefolds (see also chapter 5), the double-log solution has the following generic form (or this can be read as a definition of *the worldsheet instanton numbers* $d_{n,m}$)

$$\Omega = \frac{1}{2}J_1\cdot J_1\, t_{\rm f}^2 + J_1\cdot J_2\, t_{\rm f}t_{\rm b} + \sum_{n,m\geq 0} d_{n,m}\left(2\frac{\partial}{\partial t_{\rm f}}\right)\mathrm{Li}_3(q_{\rm f}^n q_{\rm b}^m) \tag{4.73}$$

$$= t_{\rm f}^2 + t_{\rm f}t_{\rm b} - 2\sum_{n,m\geq 0}\sum_{k=1}^{\infty} n\, d_{n,m}\frac{q_{\rm f}^{kn}q_{\rm b}^{km}}{k^2}, \tag{4.74}$$

where $\mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$ is the polylogarithm function.

Finally, compare (4.69) with (4.74) and we have computed the worldsheet instanton numbers up to order 4:

$d_{n,m}$	$m=0$	1	2	3
$n=1$	-2	-2	0	0
2	0	-4	0	
3	0	-6		
4	0			

Table 4.1: Worldsheet instanton number for local F_2 model up to order 4.

It takes longer time to computer higher order corrections and they agree with the existing results.

It is important to notice the following two distribution pattern of the worldsheet instanton numbers

$$d_{n,0} = \begin{cases} -2 & \text{for } m = 1, \\ 0 & \text{for } m > 1, \end{cases} \tag{4.75}$$

$$d_{n,1} = -2n. \tag{4.76}$$

This is a naive guess from the table of the worldsheet instanton numbers. I do not have a good reason for these results but I believe that these can be checked from the A-model localisation computations. These two group of instanton numbers for $m = 0$ and $m = 1$ will be crucial for our determination of the perturbative and 1-instanton part the field theory coupling function.

4.3.4 The field theory limit

Our effective IIA theory in four dimensions is a supergravity theory, while the Seiberg-Witten solution deals with a supersymmetric gauge theory. In order to recover the result of Seiberg-Witten theory, we need to decouple gravity from our construction above. This is done by taking the so-called rigid limit in which we send the Planck scale $M_{\rm Pl}$ to infinity. We will only show this limiting process in our example. For the general theory, see [37] and [2].

Pushing the Planck scale to infinity is conceptually clear, but we also have to specify clearly what it really does on the parameters $t_{\rm f}, t_{\rm b}$ that describe the volume of the fibre $\mathbb{P}^1$ and the base $\mathbb{P}^1$ in our local F_2 model. In other words, how does the local A-model geometry change under the decoupling limit $M_{\rm Pl} \to \infty$?

According to the relation (4.3), the size $t_{\rm b}$ of the base in our F_2 surface corresponds to the 4-dimensional gauge coupling $1/g^2$ at the string scale, which by asymptotic freedom should be pushed to zero when we take $M_{\rm Pl} \to \infty$. Hence we should send $t_{\rm b} \to \infty$.

According to the relation (4.2), the size $t_{\rm f}$ of the fibre in our F_2 surface corresponds to the masses $\frac{M_{W^\pm}}{M_{\rm Pl}}$ of the charged $W^\pm$ gauge bosons, and we want to keep $M_{W^\pm}$ finite in the limit. Hence we should send $t_{\rm f} \to 0$. Note that this exactly implies a singular fibre $\mathbb{P}^1$ that enhances the gauge symmetry to SU(2).

Now apparently there is a problem: In the Seiberg-Witten field theory, there is only one Kähler modulus. But now we have two Kähler moduli $t_{\rm f}$ and $t_{\rm b}$. How should we resolve this mismatch?

The answer lies in the renormalisation group behaviour for our SU(2) supersymmetric gauge theory: Recall from chapter 2 that the running of the coupling g in the perturbative region is dominated by the 1-loop NSVZ beta function

$$\frac{1}{g^2} \sim \log \frac{M_{W^\pm}}{\Lambda}, \tag{4.77}$$

where we have $t_{\rm f} \sim M_{W^\pm} \sim a$, the vev of the Higgs boson. So we should match

$$t_{\rm b} \sim -\log t_{\rm f}. \tag{4.78}$$

While in gauge theories, an n-instanton background weights with e^{-n/g^2}. At the same time, recall from equation (2.17) that an n-instanton background will contribute a $\left(\frac{\Lambda}{a}\right)^{4n}$ to the effective coupling τ. Which means during the limit we have to keep $e^{-t_{\rm b}}/t_{\rm f}^4$ fixed. Taking all these into account, we assign

$$t_{\rm f} = -4Ra, \tag{4.79}$$
$$q_{\rm b} = 4R^4\Lambda^4, \tag{4.80}$$

where Λ is the Seiberg-Witten field scale and the numerical factors are chosen for later convenience. The rigid limit is then $R \to 0$ while keeping a and Λ fixed. The variable R is actually the radius of the compactification circle of a 5D gauge theory, which will be briefly discussed later.

4.3.5 Identification of the gauge coupling function

After taking the rigid limit, our type IIA prepotential will reduce to the Seiberg-Witten prepotential. In the local mirror symmetry, the definition of a prepotential purely in terms of the local A-model geometry is tricky, so instead of integrating once to obtain a local type IIA prepotential-like object, we directly work with the double-log solution Ω. Recall that the gauge coupling functions are twice derivative of the prepotential, hence we identify the Seiberg-Witten coupling function with the derivative of our double-log solution. We will recover the perturbative and 1-instanton correction directly, hence we need the terms in the instanton summation of Ω up to order 2 in $q_{\rm b}$. With the general form (4.74) and the worldsheet instanton numbers (4.75), (4.76), we have

$$\Omega = t_{\rm f}^2 + t_{\rm f}t_{\rm b} + 4\sum_{n=1}^{\infty}\frac{q_{\rm f}^n}{n^2} + 4q_{\rm b}\sum_{n=1}^{\infty} n^2 q_{\rm f}^n + \cdots . \tag{4.81}$$

Then the gauge coupling function is identified with the first derivative of Ω with respect to $t_{\rm f}$

$$\tau = \frac{\mathbf{i}}{2\pi}\frac{\partial\Omega}{\partial t_{\rm f}} \tag{4.82}$$
$$= \frac{\mathbf{i}}{2\pi}\left(t_{\rm f} + \frac{1}{2}t_{\rm b} - 2\sum_{n=1}^{\infty}\frac{q_{\rm f}^n}{n} - 2q_{\rm b}\sum_{n=1}^{\infty} n^3 q_{\rm f}^n + \cdots\right), \tag{4.83}$$

where the overall normalisation is chosen for convenience.

Now we are ready to take the field theory limit. Insert the relations (4.79) and (4.80), and taking the limit $R \to 0$, we have

$$\begin{aligned}\tau &= \frac{\mathbf{i}}{\pi}\left[8Ra - \left(\frac{1}{2}\log(2R^4\Lambda^4) + 2\sum_{i=1}^{\infty}\frac{q_{\mathrm{f}}^n}{n}\right) - 8R^4\Lambda^4\sum_{n=1}^{\infty}n^3 q_{\mathrm{f}}^n + \cdots\right] &(4.84)\\ &\to \frac{\mathbf{i}}{\pi}\left[2\log\left(\frac{a}{\Lambda}\right) - \frac{3}{16}\left(\frac{\Lambda}{a}\right)^4 + \cdots\right]. &(4.85)\end{aligned}$$

In the expression we have omitted higher order terms and constant terms.

The result agrees very well with the general form (2.17)! And we can read out the 1-instanton contribution $c_1 = \frac{1}{16}$, which agrees well with the field theory computation in [52]. It is important to note that one has to be careful with normalisations. Our normalisation of the field theory prepotential (2.15) is different from the one in the paper [52]. After properly rescaling the variable a by a factor of $\sqrt{2}$, we find the relation

$$c_k = 2^{2k-1}\mathcal{F}_k. \tag{4.86}$$

This explains the difference of a factor of 2 between our stringy result and the field theoretical result in [52].

The higher instanton contributions are not easy to compute with our method here unless one has a general formula for higher worldsheet instanton numbers $d_{n,m}$. In the original paper [46], the authors used another method which essentially compares the string theoretical result and the field theoretical result and derived a formula between all worldsheet instanton numbers $d_{n,m}$ and field theoretical instanton corrections c_k.

4.3.6 Identification with the Seiberg-Witten curve

One can not help to ask if string theory can really reproduce all the instanton corrections in Seiberg-Witten theory. The answer is of course positive and the reason is that actually under the field theory limit and a suitable coordinate transformation, the local B-model curve (4.25) becomes the Seiberg-Witten elliptic curve! We repeat the B-model curve here for convenience:

$$a_0 x + a_1 x^2 + a_2\zeta + a_3 + a_4\frac{1}{\zeta} = 0. \tag{4.87}$$

We set $a_1 = a_2 = 1, a_4 = \Lambda^4$ by the scaling symmetry. Then redefine

$$\begin{aligned} t &= x + \frac{a_0}{4}, &(4.88)\\ u &= \frac{1}{2}\left(\frac{a_0^2}{8} - a_3\right) &(4.89)\end{aligned}$$

and the curve becomes

$$\zeta + \frac{\Lambda^4}{\zeta} + 2(t^2 - u) = 0, \tag{4.90}$$

which is already the Seiberg-Witten curve in a physical equivalent form. This form of the curve is often used when the gauge group is more complicated. To transform it into the familiar form, we first substitute

$$y = \zeta + (t^2 - u) \tag{4.91}$$

and get another physically equivalent Seiberg-Witten curve

$$y^2 = (t^2 - u)^2 - \Lambda^4. \tag{4.92}$$

This form of the curve is often used when the matter contents are present. Finally, with a subtler substitution (called an isogeny, see [71])

$$\hat{x} = u - t^2, \tag{4.93}$$

$$\hat{y} = \mathbf{i} t y, \tag{4.94}$$

we finally get

$$\hat{y}^2 = (\hat{x}^2 - \Lambda^4)(\hat{x} - u). \tag{4.95}$$

4.3.7 Last remark: F_0, F_1 or F_2?

The complex surfaces F_0, F_1 and F_2 are all $\mathbb{P}^1$ fibrations over $\mathbb{P}^1$. In the geometric engineering paper [46], the author showed that they will give the same Seiberg-Witten results under the rigid limit. So it seems that we can choose these surfaces arbitrarily. But this turns out to be not true.

Instead of the compactification of 10D type IIA superstring theory on our local threefold $\mathcal{K}_{\mathsf{F}_n}$, we can also compactify the 11D M-theory on the same local threefold $\mathcal{K}_{\mathsf{F}_n}$. This will give a five-dimensional gauge theory on $M_4 \times S^1$, where the radius R of the S^1 is exactly the variable in the field theory limit (4.79), (4.80). When we take the field theory limit $R \to 0$, we recover our 4D Seiberg-Witten gauge theory. But in the decompactification limit $R \to \infty$, we expect to recover the SU(2) gauge theory in 5 dimensions. Although the model F_0 and F_1 can actually reproduce the 4-dimensional results, they fail to give the correct 5-dimensional behaviour. Hence we have chosen the F_2 model in our example. For further information, see [22].

Chapter 5

Periods for Calabi-Yau spaces and singularities

In this chapter, we deal with the mirror symmetry problem more systematically. In the previous chapters, we have seen that the periods

$$\Pi_{i,j} = \int_{\Gamma_j} \Omega_i \tag{5.1}$$

of certain differential k-forms $\Omega_i \in H^k$ (paired with k-cycles $\Gamma_j \in H_k$) satisfy the Picard-Fuchs differential equations. Although it is straightforward to find a basis of the solution space of the Picard-Fuchs system with Frobenius method, we still need to fix the proper linear combinations of these basis solutions to really get the periods $\Pi_{i,j}$. Fixing such a linear combination is also called fixing an integral basis for the H_k since these integral homology basis are of physically importance in superstring theory.

The focus of this chapter is fixing an integral basis for the middle cohomology for a Calabi-Yau space. The techniques for Calabi-Yau threefolds and their holomorphic volume form $\Omega \in H^3$ are well-known and will be our first thing to recall. The solution to the renowned Fermat quintic threefold will shown as an example. The method for Calabi-Yau fourfolds is less understood and it is discussed recently in a paper by [16]. We will describe their method and show its application in the Fermat sextic fourfold and another two-moduli example. We will not compute the worldsheet instanton numbers since they are not the main topic here. These worldsheet instanton numbers can be derived with the method discussed in the last chapter.

In the end we will discuss the Hodge-theoretic criteria determining whether a singular locus in a Calabi-Yau manifold is of finite Weil-Petersson distance or not. This is done by considering the monodromy properties of the period vector around the singular locus. These monodromies are exactly coming from the logarithmic parts of the periods that we fixed using the methods discussed in the first two sections of this chapter. As examples, we will check on three examples. Such criteria are of physical significance. For instance, it can provide some insight to the swampland conjecture.

We give a summary of the basics of Hodge theory we need in appendix B. For more details on mirror symmetry, there are reviews [40], [51] [42] and books [18], [41].

5.1 The middle cohomology H^3 of a Calabi-Yau threefold Y_3

5.1.1 Some geometrical facts of H^3

First we recall the Hodge diamond for a Calabi-Yau threefold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{5.2}$$

The $h^{1,1}$ parametrises deformations of the Kähler structure on Y_3 and $h^{2,1}$ parametrises deformations of the complex structure on Y_3.

The middle cohomology is $H^3(Y_3, \mathbb{C})$. In the context of variation of Hodge structure, one needs to consider the primitive subspace of $H^3(Y_3, \mathbb{C})$. By definition the primitive subspace of H^3 is given by the kernel of the map

$$L : H^3(Y_3, \mathbb{C}) \to H^5(Y_3, \mathbb{C}), \tag{5.3}$$

where L is the Lefschetz operator. From the Hodge diamond we see that the entire $H^3(Y_3, \mathbb{C})$ is already primitive so in the following we can freely work with $H^3(Y_3, \mathbb{C})$.

The middle cohomology $H^3(Y_3, \mathbb{C})$ has a Hodge structure

$$H^3(Y_3, \mathbb{C}) = H^{3,0}(Y_3) \oplus H^{2,1}(Y_3) \oplus H^{1,2}(Y_3) \oplus H^{0,3}(Y_3), \tag{5.4}$$

with $H^{j,i}(Y_3) = \overline{H^{i,j}(Y_3)}$. In terms of the decreasing Hodge filtration $F^0 \supset F^1 \supset F^2 \supset F^3$, we have

$$\begin{aligned} F^0 &= H^{3,0}(Y_3) \oplus H^{2,1}(Y_3) \oplus H^{1,2}(Y_3) \oplus H^{0,3}(Y_3), && (5.5)\\ F^1 &= H^{3,0}(Y_3) \oplus H^{2,1}(Y_3) \oplus H^{1,2}(Y_3), && (5.6)\\ F^2 &= H^{3,0}(Y_3) \oplus H^{2,1}(Y_3), && (5.7)\\ F^3 &= H^{3,0}(Y_3). && (5.8) \end{aligned}$$

The natural bilinear pairing on $H^3(Y_3, \mathbb{C})$ is given by

$$Q(\alpha, \beta) = \int_{Y_3} \alpha \wedge \beta, \tag{5.9}$$

for $\alpha, \beta \in H^3(Y_3, \mathbb{C})$. This bilinear form is skew-symmetric and obviously satisfies

$$Q(H^{p,q}, H^{r,s}) = 0, \tag{5.10}$$

for $p + q = r + s = 3$, unless $p = s$ and $q = r$. Hence the real-valued symmetric positive R is thus defined to be

$$R(\alpha, \alpha) = \mathbf{i}Q(\alpha, \bar{\alpha}), \tag{5.11}$$

and it defines a polarisation of the Hodge structure on $H^3(Y_3, \mathbb{C})$.

Since the pairing Q is skew-symmetric, we can choose a so-called symplectic basis α_I, β^I for $H^3(Y_3, \mathbb{Z})$ and their Poincaré dual A^I, B_I for $H_3(Y_3, \mathbb{Z})$ such that their intersection products satisfy

$$A^I \cdot B_J = \delta^I_J, \tag{5.12}$$

$$A^I \cdot A^J = 0, \tag{5.13}$$

$$B_I \cdot B_J = 0. \tag{5.14}$$

And we expand the holomorphic volume form $\Omega \in H^{3,0}(Y_3)$ in such a symplectic basis

$$\Omega = X^I \alpha_I - F_I \beta^I, \tag{5.15}$$

where the expansion coefficients X^I, F_I are the periods of Ω. As we always do in linear algebra, we write these coefficients in a vector

$$\Pi = \begin{pmatrix} X^I \\ F_J \end{pmatrix} = \begin{pmatrix} \int_{A^I} \Omega \\ \int_{B_J} \Omega \end{pmatrix}. \tag{5.16}$$

Writing in terms of the periods, the Kähler potential for the Weil-Petersson metric on the complex structure moduli space is given by

$$K = -\log R(\Omega, \Omega) \tag{5.17}$$

$$= -\log(\mathbf{i}\Pi^\dagger \Sigma \Pi), \tag{5.18}$$

where

$$\Sigma = \begin{pmatrix} 0 & 1_{k\times k} \\ -1_{k\times k} & 0 \end{pmatrix} \tag{5.19}$$

is the standard symplectic matrix and $k = \frac{1}{2}(1 + h^{2,1})$.

5.1.2 Integral basis from special geometry

The complex structure moduli space of a Calabi-Yau threefold is well-known to be a special Kähler manifold. This fact fixes an integral basis completely.

From special geometry, we know that the periods X^I are projective coordinates on the complex structure moduli space and we define the local coordinates to be $t^i = \frac{X^I}{X^0}$. The periods F_I are then degree-1 homogeneous functions on X^I. In terms of a holomorphic prepotential F, which is degree-2 homogeneous on X^I, we have $F_I = \frac{\partial F}{\partial X^I}$. Note that we denote the conplex structure deformations by t because in the string context, we are working in the IIB side and the complex structure deformations will be mapped to the Kähler deformations on the IIA side.

It is natural to redefine the prepotential $F = (X^0)^2\mathcal{F}$, and then it was found that the prepotential has the following form

$$\mathcal{F} = -\frac{\mathcal{K}_{ijk}}{3!}t^it^jt^k + \frac{n_{ij}}{2}t^it^j + b_it^i - \frac{\mathbf{i}\chi\zeta(3)}{16\pi^3} + \mathcal{F}_{\text{Inst.}}(q), \tag{5.20}$$

where $\mathcal{K}_{ijk}$ is the classical intersection of the basis of the $(1,1)$-forms, n_{ij} are not relevant for physical quantities, $b_i = \frac{1}{24}\int_{Y_3} c_2(Y_2) \wedge \omega_i$, χ is the Euler characteristic of Y_3, and $\mathcal{F}_{\text{Inst.}}(q)$ encodes the nonperturbative corrections.

With such a form of prepotential, one can write down the solution to the Picard-Fuchs equations for a general Calabi-Yau hypersurface in toric ambient space. First we have the Frobenius series Ansatz

$$f(z_1,\ldots,z_{h^{2,1}};\rho_1,\ldots,\rho_{h^{2,1}}) = \sum_{n_1,\ldots,n_{h^{2,1}}} c_{n_1,\ldots,n_{h^{2,1}}}(\rho_1,\ldots,\rho_{h^{2,1}}) z_1^{n_1+\rho_1}\cdots z_{h^{2,1}}^{n_{h^{2,1}}+\rho_{h^{2,1}}}, \tag{5.21}$$

with the coefficients fixed to be

$$c_{n_1,\ldots,n_{h^{2,1}}}(\rho_1,\ldots,\rho_{h^{2,1}}) = \frac{\Gamma\left(1-\sum_k l_0^{(k)}(n_k+\rho_k)\right)}{\prod_{i>0}\Gamma\left(1+\sum_k l_i^{(k)}(n_k+\rho_k)\right)}. \tag{5.22}$$

Then the logarithmic solutions are derived from repeatedly derivation with respect to ρ. Define the operators

$$D_i^{(1)} = \frac{1}{2\pi\mathbf{i}}\frac{\partial}{\partial\rho_i}, \tag{5.23}$$

$$D_i^{(2)} = \frac{1}{2}\frac{1}{(2\pi\mathbf{i})^2}\sum\mathcal{K}_{ijk}\frac{\partial}{\partial\rho_j}\frac{\partial}{\partial\rho_k}, \tag{5.24}$$

$$D^{(3)} = -\frac{1}{3!}\frac{1}{(2\pi\mathbf{i})^3}\sum\mathcal{K}_{ijk}\frac{\partial}{\partial r_i}\frac{\partial}{\partial\rho_j}\frac{\partial}{\partial\rho_k}, \tag{5.25}$$

and we have finally arrived at the correct solution

$$\Pi(z) = \begin{pmatrix} f(z;0) \\ D_i^{(1)} f(z;0) \\ D_i^{(2)} f(z;0) \\ D^{(3)} f(z;0) \end{pmatrix}. \tag{5.26}$$

Once we have identified the mirror map as the single-log solution

$$t^i = \frac{D_i^{(1)} f(z;0)}{f(z;0)} = \frac{1}{2\pi\mathbf{i}} \left(\log z_i + \Sigma_i\right), \tag{5.27}$$

where Σ_i encodes the regular part, we can rewrite z in terms of t and insert back to obtain a period vector in terms of Kähler parameters t

$$\Pi(t) = \begin{pmatrix} X^0 \\ X^i \\ F_i \\ F_0 \end{pmatrix} \tag{5.28}$$

$$= X^0 \begin{pmatrix} 1 \\ t^i \\ \frac{\partial \mathcal{F}}{\partial t^i} \\ 2\mathcal{F} - t^i \frac{\partial \mathcal{F}}{\partial t^i} \end{pmatrix} \tag{5.29}$$

$$= X^0 \begin{pmatrix} 1 \\ t^i \\ -\frac{\mathcal{K}_{ijk}}{2} t^i t^j + n_{ij} t^j + b_i + \frac{\partial \mathcal{F}_{\text{Inst.}}(q)}{\partial t^i} \\ \frac{\mathcal{K}_{ijk}}{3!} t^i t^j t^k + b_i t^i - \frac{\mathbf{i}\chi\zeta(3)}{8\pi^3} + \mathcal{F}_{\text{Inst.}}(q) \end{pmatrix}. \tag{5.30}$$

The general form of the solution is thus fixed. Note how each entry depends on the prepotential $\mathcal{F}$ and mirror maps t^i.

5.1.3 Example: The Fermat quintic X_5 inside $\mathbb{P}^4$

Now we carry out the explicit computation for the famous quintic threefold.

The toric data for $\mathbb{P}^4$ is given by

						$l^{(1)}$
D_0	1	0	0	0	0	-5
D_1	1	1	0	0	0	1
D_2	1	0	1	0	0	1
D_3	1	0	0	1	0	1
D_4	1	0	0	0	1	1
D_5	1	-1	-1	-1	-1	1

For convenience, an extra 1-dimensional cone D_0 is added.

For X_5 the locus of a generic degree-5 (hence the name "quintic") polynomial inside $\mathbb{P}^4$. The mirror geometry can be described as follows. For each cone D_i we assign a homogeneous coordinate Y_i. Due to the Mori vector $l^{(1)} = (-5; 1, 1, 1, 1, 1)$, these coordinates satisfies the following relation

$$Y_1 Y_2 Y_3 Y_4 Y_5 = Y_0^5. \tag{5.31}$$

And the mirror space is given by the locus of the following polynomial

$$P = \sum_{i=0}^{5} a_i Y_i. \tag{5.32}$$

The complex structure is parametrised by

$$z = -\frac{a_1 a_2 a_3 a_4 a_5}{a_0^5}. \tag{5.33}$$

Then the PF operator can be computed to be

$$\mathcal{L} = \theta^4 - 5^5 z \prod_{k=1}^{4} (\theta + \frac{k}{5}). \tag{5.34}$$

From $\mathcal{L}$ we conclude that the generalised hypergeometric function

$$X_0(z) = {}_4F_3\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; 1, 1, 1; 5^5 z\right) \tag{5.35}$$

is a holomorphic solution to the PF equation. This kind of holomorphic solution is checked explicitly for the sextic fourfold X_6 below.

To use the general form of the integral basis, we evaluate the topological quantities. We denote the hyperplane class of $\mathbb{P}^4$ by $H = D_1 = \cdots = D_5$. Then inside $\mathbb{P}^4$ the hyperplane class intersects with itself with $H^4 = 1$.

The divisor class representing the Calabi-Yau X_5 inside $\mathbb{P}^4$ is $D_0 = D_1 + \cdots + D_5 = 5H$. Hence the total Chern class of the quintic is given by

$$c(X_5) = \frac{(1+H)^5}{1+5H} \tag{5.36}$$

$$= 1 + 10H^2 - 40H^3, \tag{5.37}$$

i.e. $c_2(X_5) = 10H^2$, $c_3(X_5) = -40H^3$. Note that $c_1(X_5) = 0$ simply means that the quintic X_5 is Calabi-Yau. The homology classes here are understood to be the restriction of H on X_5.

The Euler characteristic of the quintic is given by the integration of its top Chern class, which, by Poincaré duality, is the intersection between the class representing the quintic and its top Chern class $\chi(X_5) = c_3(X_5) \cdot D_0 = -200$.

And the intersection products inside the quintic X_5 are simply

$$H^3 = H^3 \cdot D_0 = 5 \tag{5.38}$$

$$c_2(X_5) \cdot H = 50. \tag{5.39}$$

The homology classes here are understood to be the restriction on X_5.

Then from the general formula, we have the following leading terms in the period vector

$$\Pi = X^0 \begin{pmatrix} 1 \\ t \\ -\frac{5}{2}t^2 + \frac{11}{2}t + \frac{25}{12} + \cdots \\ \frac{5}{6}t^3 + \frac{25}{12}t + \frac{25\mathbf{i}\zeta(3)}{\pi^3} + \cdots \end{pmatrix}. \tag{5.40}$$

5.2 The middle cohomology H^4 of a Calabi-Yau fourfold Y_4

5.2.1 Some geometrical facts of H^4

The geometry of Calabi-Yau fourfolds is more complicated than Calabi-Yau threefolds as expected. One can find the detailed information on fourfolds in [29], [53], [54], [35], [6] and [16].

First we recall the Hodge diamond for a Calabi-Yau fourfold

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} \tag{5.41}$$

Analogous to the threefold, the $h^{1,1}$ parametrises deformations of the Kähler structure on Y_4 and $h^{3,1}$ parametrises deformations of the complex structure on Y_4. Furthermore, there is one more free deformation parameter that is parametrised by $h^{2,1}$ whose physical and mathematical properties are far from being fully understood. The reader can consult [30] and [31] for more details on this $h^{2,1}$. In the following we assume $h^{2,1} = 0$. Finally the remaining parameter $h^{2,2}$ is not free, but is restricted by

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}). \tag{5.42}$$

The middle cohomology $H^4(Y_4, \mathbb{C})$ is more complicated than the middle cohomology of a Calabi-Yau threefold. First it has the Hodge decomposition

$$H^4(Y_4, \mathbb{C}) = H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}(Y_4) \oplus H^{1,3}(Y_4) \oplus H^{0,4}(Y_4), \tag{5.43}$$

with $H^{j,i}(Y_4) = \overline{H^{i,j}(Y_4)}$. In terms of the decreasing Hodge filtration $F^0 \supset F^1 \supset F^2 \supset F^3 \supset F^4$, we have

$$\begin{aligned}
F^0 &= H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}(Y_4) \oplus H^{1,3}(Y_4) \oplus H^{0,4}(Y_4), && (5.44)\\
F^1 &= H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}(Y_4) \oplus H^{1,3}(Y_4), && (5.45)\\
F^2 &= H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}(Y_4), && (5.46)\\
F^3 &= H^{4,0}(Y_4) \oplus H^{3,1}(Y_4), && (5.47)\\
F^4 &= H^{4,0}(Y_4). && (5.48)
\end{aligned}$$

In contrast to the fact that the middle cohomology H^3 is already primitive for a Calabi-Yau threefold, the H^4 of the fourfold is not. In particular, it is the $H^{2,2}(Y_4)$ that makes things more complicated. The primitive subspace of $H^{2,2}(Y_4)$ is the kernel of the map

$$L : H^{2,2}(Y_4) \to H^{3,3}(Y_4). \tag{5.49}$$

Now from the Hodge diamond we see that $H^{3,3}$ is not trivial but has dimension $h^{1,1}$. So we define the primitive subspace

$$P^{2,2}(Y_4) = \left\{\alpha \in H^{2,2}(Y_4) \middle| J \wedge \alpha = 0\right\}, \tag{5.50}$$

where $J \in H^{1,1}(Y_4)$ is the Kähler form. Then by Lefschetz decomposition, we have

$$H^{2,2}(Y_4) = P^{2,2}(Y_4) \oplus LP^{1,1}(Y_4), \tag{5.51}$$

where the primitive subspace $P^{1,1}(Y_4)$ is the kernel of the map

$$L^3 : H^{1,1}(Y_4) \to H^{4,4}(Y_4), \tag{5.52}$$

and the $LP^{1,1}(Y_4)$ part is called the *primary vertical cohomology* and denoted by $H^{2,2}_{\mathrm{V}}(Y_4)$ in physical literature.

But the primitive subspace $P^{2,2}(Y_4)$ is still not what physicists are actually looking at since they are not all accessible by studying period integrals. Let $\Omega(z) \in H^{4,0}(Y_4)$ be the unique up to rescaling holomorphic volume form of the Calabi-Yau fourfold, where we have explicitly labelled its dependence on the complex structure parameters z. Then the space of interest is the subspace of $P^{2,2}(Y_4)$ that is generated by repeated variation of $\Omega(z)$ with respect to complex structure, i.e. the subspace of that is generated by $\partial_{z_{i_1}} \cdots \partial_{z_{i_k}} \Omega(z)$. This subspace is called *the primitive horizontal subspace* in physical literature and is denoted by $H^{2,2}_{\mathrm{H}}(Y_4)$. For Calabi-Yau threefolds we do not have to worry about the horizontal subspace because we have a better control over the variation of the holomorphic volume form $\Omega(z)$ such that the entire middle cohomology is generated by repeated variations.

To summarise, the cohomology $H^{2,2}(Y_4)$ has the following decomposition

$$\begin{aligned} H^{2,2}(Y_4) &= P^{2,2}(Y_4) \oplus H^{2,2}_{\mathrm{V}}(Y_4) && (5.53)\\ &= \left(H^{2,2}_{\mathrm{H}}(Y_4) \oplus H^{2,2}_{\mathrm{RM}}(Y_4)\right) \oplus H^{2,2}_{\mathrm{V}}(Y_4), && (5.54)\end{aligned}$$

where the "remaining" part $H^{2,2}_{\mathrm{RM}}(Y_4)$ exists in some particularly constructed examples [10] but they do bother us in the examples in the next sections. The *primary horizontal cohomology* is then the sumation

$$H^4_{\mathrm{H}}(Y_4, \mathbb{C}) = H^{4,0}(Y_4) \oplus H^{3,1}(Y_4) \oplus H^{2,2}_{\mathrm{H}}(Y_4) \oplus H^{1,3}(Y_4) \oplus H^{0,4}(Y_4) \tag{5.55}$$

The natural bilinear pairing on $H^4(Y_4, \mathbb{C})$ is given by

$$Q(\alpha, \beta) = \int_{Y_4} \alpha \wedge \beta, \tag{5.56}$$

for $\alpha, \beta \in H^4(Y_4, \mathbb{C})$. This bilinear form is symmetric and obviously satisfies

$$Q(H^{p,q}, H^{r,s}) = 0, \tag{5.57}$$

for $p + q = r + s = 4$, unless $p = s$ and $q = r$. Hence the real-valued symmetric positive R is thus defined to be

$$R(\alpha, \alpha) = Q(\alpha, \bar{\alpha}), \tag{5.58}$$

and it defines a polarisation of the Hodge structure on $H^4(Y_4, \mathbb{C})$.

Similarly to the threefold, we can then choose a set of basis $\alpha^{(p)}$ for $H^4_{\mathrm{H}}(Y_4, \mathbb{Z})$. Then the holomorphic volume form $\Omega \in H^{4,0}(Y_4)$ has the following expansion

$$\Omega = X^0 \alpha^{(0)} + X^i \alpha^{(1)}_i + H_p \alpha^{(2),p} + F_i \alpha^{(3),i} + F_0 \alpha^{(4)}, \tag{5.59}$$

where the indices run over $i = 1, \ldots, h^{3,1}$, $j = 1, \ldots, h^{2,2}_{\mathrm{H}}$. In terms of such a basis, we have an analogy to the symplectic pairing Σ for the fourfold which can be chosen to be

$$\eta^{-1} = \begin{pmatrix} * & * & * & * & 1 \\ * & * & * & \eta^{(1)} & 0 \\ * & * & \eta^{(2)} & 0 & 0 \\ * & \eta^{(1)} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.60}$$

where $\eta^{(1)}$ is a $h^{3,1} \times h^{3,1}$ real matrix, and $\eta^{(2)}$ is a $h^{2,2}_{\mathrm{H}} \times h^{2,2}_{\mathrm{H}}$ real matrix. The name η reminds us the Frobenius algebra structure. We will not discuss about it in this thesis and interested reader can consult [53].

5.2.2 Integral basis from homological mirror symmetry

In the realm of Calabi-Yau fourfolds, we do not have nice geometrical structure like the special Kähler geometry on the Kähler and complex structure moduli spaces. Instead there is still Frobenius structures on the algebra of physical observables that could be used to fix an integral basis. Such a method is used in [6] to fix integral basis for 1-modulus fourfolds like the sextic inside $\mathbb{P}^5$. But in general it could be very hard to be carried out.

In the paper [16], the authors considered instead using the homological mirror symmetry conjecture to fix an integral basis and it worked much better. We describe this method and show it in two examples.

Homological mirror symmetry is a conjectured mathematical treatment of the mirror symmetry phenomenon in physics proposed by Kontsevich [57]. Along with the SYZ-mirror symmetry conjecture [75] (and its descendent, the Gross-Siebert programme [36]), they are two mainstream mathematical theories that try to put the mathematics of the mirror symmetry in a firmer grounding.

Homological mirror symmetry is useful when one wants to describe how D-branes behave under mirror symmetry and it works in the context of topological string theory. In mathematical world, topological A-branes are described by the Fukaya category $\mathsf{Fuk}(Y_4)$ of a fourfold Y_4 in the symplectic viewpoint, and topological B-branes are described by the bounded derived category of coherent sheaves $\mathsf{D}^b(Y_4)$ of the fourfold Y_4 in the algebro-geometrical viewpoint. Homological mirror symmetry conjectures the existence of the mirror $\hat{Y}_4$ such that the two categories are equivalent (in the categorical sense)

$$\mathsf{Fuk}(Y_4) \simeq \mathsf{D}^b(\hat{Y}_4). \tag{5.61}$$

For our purpose, the idea is that via the homological mirror symmetry, the central charges of topological A-branes and B-branes are identified.

On the A-side, a topological A-brane L can wrap around a special Lagrangian cycle Σ, which is Poincaré dual to an element in $H^4_{\mathrm{H}}(Y_4, \mathbb{C}) \cap H^4(Y_4, \mathbb{Z})$ and satisfies

$$\mathrm{Re}(e^{i\theta}\Omega)|_\Sigma = 0, \tag{5.62}$$
$$J|_\Sigma = 0, \tag{5.63}$$

where Ω is the holomorphic volume form and J is the Kähler form on Y_4. The central charge of such an A-brane L is exactly given by

$$Z_A(L) = \int_\Sigma \Omega, \tag{5.64}$$

which exactly constitutes the integral period vector that we want to determine!

While on the B-side, a B-brane represented by a complex $\mathcal{E}^\bullet$ in $\mathsf{D}^b(\hat{Y}_4)$. What $\mathcal{E}^\bullet$ exactly looks like is not important for the moment, we need its central charge formula.

More precisely, the asymptotic (i.e., near the large complex structure point) central charge for $\mathcal{E}^\bullet$ is

$$Z_B^{\text{asy}}(\mathcal{E}^\bullet) = \int_{\hat{Y}_4} e^J\,\Gamma_{\mathbb{C}}(\hat{Y}_4)(\text{ch}\mathcal{E}^\bullet)^\vee, \tag{5.65}$$

where $\Gamma_{\mathbb{C}}(\hat{Y}_4)$ is called the Γ-class (originally defined in [62], [45] and [49]), which for the Calabi-Yau fourfold $\hat{Y}_4$ is given by

$$\Gamma_{\mathbb{C}}(\hat{Y}_4) = 1 + \frac{1}{24}c_2 + \frac{\mathbf{i}\zeta(3)}{8\pi^3}c_3 + \frac{1}{5760}(7c_2^2 - 4c_4). \tag{5.66}$$

As a side note, we see the familiar term $\frac{\mathbf{i}\zeta(3)}{8\pi^3}c_3$ that appears in the threefold solution (5.30). Gauss-Bonnet theorem tells us that $\int_{Y_3} c_3 = \chi$, hence these two terms are actually identical! This is not a coincidence, as the solution to the Picard-Fuchs equations are encoded in the Γ-class. See the paper [62] for more information.

In the paper [16], the authors constructed a basis for the B-branes and then carried out the asymptotic central charge formula explicitly to obtain an integral basis for the fourfold. We list their results below.

Let ω_i be a basis for the $(1,1)$-forms, its Poincaré dual divisor D_i, and the corresponding Mori curves be $\mathcal{C}^i$. We also choose a basis for the 4-cycles in $\hat{Y}_4$ to be of the form $D_a \cap D_b$ and there are $h_{\text{H}}^{2,2}$ of them. Then the leading term (i.e., logarithmic term) in the integral period *near large volume point* is given by

$$\Pi^{\text{asy}} = \begin{pmatrix} 1 \\ -t^a \\ \frac{1}{2}h_{ij}t^it^j + h_it^i + h \\ -\frac{1}{3!}C^0_{aijk}t^it^jt^k - \frac{1}{4}C^0_{aaij}t^it^j - \left(\frac{1}{6}C^0_{aaij} + \frac{1}{24}c^a_i\right)t^i - \left(\frac{1}{24}C^0_{aaaa} + c^a_0\right) \\ \frac{1}{4!}C^0_{ijkl}t^it^jt^kt^l + \frac{1}{2}c_{ij}t^it^j + c_it^i + c_0 \end{pmatrix}, \tag{5.67}$$

where $a = 1, \ldots, h^{1,1}$. The topological coefficients are given by

$$\begin{aligned}
h_{ij} &= D_a \cdot D_b \cdot D_i \cdot D_j, & (5.68)\\
h_i &= \frac{1}{2} D_a \cdot D_b \cdot (D_a + D_b) \cdot D_i, & (5.69)\\
h &= \frac{1}{12} D_a \cdot D_b \cdot (2D_a^2 + 3D_a \cdot D_b + 2D_b^2) + \frac{1}{24} c_2(\hat{Y}_4) \cdot D_a \cdot D_b, & (5.70)\\
C^0_{ijkl} &= D_i \cdot D_j \cdot D_k \cdot D_l, & (5.71)\\
c_i^a &= c_2(\hat{Y}_4) \cdot D_a \cdot D_i, & (5.72)\\
c_0^a &= \frac{1}{48} c_2(\hat{Y}_4) \cdot D_a^2 + \frac{\mathbf{i}\zeta(3)}{8\pi^3} c_3(\hat{Y}_4) \cdot D_a & (5.73)\\
c_{ij} &= \frac{1}{24} c_2(\hat{Y}_4) \cdot D_i \cdot D_j, & (5.74)\\
c_i &= \frac{\mathbf{i}\zeta(3)}{8\pi^3} c_3(\hat{Y}_4) \cdot D_i, & (5.75)\\
c_0 &= \frac{1}{5760} \int_{\hat{Y}_4} (7c_2(\hat{Y}_4)^2 - 4c_4(\hat{Y}_4)). & (5.76)
\end{aligned}$$

Thus, with all the topological data, one can easily determine the integral basis for a considerable number of Calabi-Yau fourfold models. We have to point out a concern on this method: In fact, one has to make sure that the basis for 4-cycles defined by $D_a \cap D_b$ really lead to an integral basis. In the paper [16], the authors used additional properties of elliptic fibrations to ensure that one indeed get an integral basis for elliptically-fibred Calabi-Yau fourfolds, e.g. for the example X_{24}. But for a generic Calabi-Yau fourfold which does not admit an elliptic fibration, how to choose a basis of the 4-cycles properly is still unclear.

Finally, there is an extra piece of information, which is the pairing matrix η^{-1} on Y_4 which is analogous to the symplectic Σ for the threefold. This is important when one wants to compute superpotential, etc. In the paper [16], the authors argue that it is not given by the classical intersection numbers on Y_4 but by the open string index

$$\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \int_{\hat{Y}_4} \mathrm{Td}(\hat{Y}_4)(\mathrm{ch}\mathcal{E}^\bullet)^\vee \mathrm{ch}\mathcal{F}^\bullet, \qquad (5.77)$$

where the Todd genus $\mathrm{Td}(\hat{Y}_4)$ for the Calabi-Yau fourfold is given by

$$\mathrm{Td}(\hat{Y}_4) = 1 + \frac{1}{12} c_2(\hat{Y}_4) + 2\mathrm{Vol}, \qquad (5.78)$$

where Vol is the volume form. With a basis $v = (\mathcal{E}_1^\bullet, \ldots, \mathcal{E}_n^\bullet)$for the topological B-branes, define the matrix $\eta_{ij} = \chi(v_i, v_j)$. Finally the intersection matrix between period vectors in the A-model will be the inverse η^{-1}. We will see it in the following two examples.

5.2.3 Example: The Fermat sextic X_6 inside $\mathbb{P}^5$

Now we carry out the explicit computation for the sextic threefold.

The toric data for $\mathbb{P}^5$ is given by

							$l^{(1)}$
D_0	1	0	0	0	0	0	−6
D_1	1	1	0	0	0	0	1
D_2	1	0	1	0	0	0	1
D_3	1	0	0	1	0	0	1
D_4	1	0	0	0	1	0	1
D_5	1	0	0	0	0	1	1
D_6	1	−1	−1	−1	−1	−1	1

Similar to the quintic, we assign variables $Y_0, \ldots, Y_6$ satisfying

$$Y_1 Y_2 Y_3 Y_4 Y_5 Y_6 = Y_0^6. \tag{5.79}$$

Then the mirror of X_6 is given by the equation

$$P = \sum_{i=0}^{6} a_i Y_i, \tag{5.80}$$

and the good variable for complex structure is

$$z = \frac{a_1 a_2 a_3 a_4 a_5 a_6}{a_0^6}. \tag{5.81}$$

Then the Picard-Fuchs operator is

$$\mathcal{L} = \theta^5 - 6^6 z \prod_{k=1}^{5} \left(\theta + \frac{k}{6} \right). \tag{5.82}$$

This time we derive the holomorphic solution more carefully. From the general form of the generalised hypergeometric differential operator annihilating ${}_pF_q(\mathbf{a}; \mathbf{b}; z)$

$$\theta \prod_{k=1}^{q} (\theta + b_k - 1) - z \prod_{k=1}^{p} (\theta + a_k), \tag{5.83}$$

we conclude that the holomorphic function

$${}_5F_4 \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}; 1, 1, 1, 1; 6^6 z \right), \tag{5.84}$$

is a solution to the PF equation. As we check now, this is indeed the case. We can solve the equation with Frobenius method. Plug in the following Ansatz

$$X^0(z;\rho) = \sum_{k=0}^{\infty} c_k(\rho) z^{k+\rho}, \tag{5.85}$$

and then we have the following indicial equation

$$\rho^5 c_0(\rho) = 0, \tag{5.86}$$

and the recursive relations

$$c_k(\rho) = \frac{6[6(\rho+k-1)+1]\cdots[6(\rho+k-1)+5]}{(\rho+k)^5} c_{k-1}(\rho), \qquad \text{for } k \geq 1. \tag{5.87}$$

After a bit of manipulation, mainly using the analytic continuation of continued product

$$(z)_n := z(z+1)\cdots(z+n-1) \tag{5.88}$$

$$= \frac{\Gamma(z+n)}{\Gamma(z)}, \tag{5.89}$$

we obtain the result

$$X^0(z;\rho) = \sum_{k=0}^{\infty} \frac{\Gamma(6(k+\rho)+1)}{\Gamma(k+\rho+1)^6} z^{k+\rho}. \tag{5.90}$$

Note that $\rho = 0$ satisfies the indicial equation. And it is also maximally degenerate, meaning that $z = 0$ corresponds to the large complex structure point. The holomorphic solution ($\rho = 0$) is then given by the series which really describe the generalised hypergeometric function

$$X^0(z) = \sum_{k=0}^{\infty} \frac{(6k)!}{(k!)^6} z^k \tag{5.91}$$

$$= {}_5F_4\left(\frac{1}{6},\frac{2}{6},\frac{3}{6},\frac{4}{6},\frac{5}{6};1,1,1,1;6^6 z\right). \tag{5.92}$$

The topological data for the sextic can be computed similarly to the quintic. Denote the hyperplane class of $\mathbb{P}^5$ by H which intersects with itself with $H^5 = 1$. The X_6 is then represented by the class $D_0 = 6H$ inside $\mathbb{P}^5$. It follows the following intersection data inside X_6

$$H^4 = 6, \tag{5.93}$$

$$c_2 = 15H^2, \tag{5.94}$$

$$c_3 = -70H^3, \tag{5.95}$$

$$c_4 = 435H^4, \tag{5.96}$$

where the class H is understood to be its restriction on X_6. The Euler characteristic is given by $\chi = c_4 \cdot D_0 = 2610$ (intersection product evaluated in $\mathbb{P}^5$).

Then we apply the formula in the new paper to get the integral period

$$\Pi = X^0 \begin{pmatrix} 1 \\ -t \\ 3t^2 + 6t + \frac{29}{4} + \cdots \\ -t^3 - \frac{3}{2}t^2 - \frac{19}{4}t - \frac{17}{8} + \frac{105\mathbf{i}\zeta(3)}{2\pi^3} + \cdots \\ \frac{1}{4}t^4 + \frac{15}{8}t^2 - \frac{105\mathbf{i}\zeta(3)}{2\pi^3} - \frac{11}{64} + \cdots \end{pmatrix}. \tag{5.97}$$

It is worth mentioning that in the paper [6], the authors claim to have first fixed an integral basis for the sextic by solving the constraints of Griffiths transversality and Frobenius structures. Although their period vector Π' is not the same as Π above

$$\Pi' = X^0 \begin{pmatrix} 1 \\ t \\ -\frac{1}{2}t^2 + \frac{1}{2}t + \frac{5}{8} + \cdots \\ -t^3 + \frac{3}{2}t^2 - \frac{3}{4}t - \frac{15}{8} + \frac{105\mathbf{i}\zeta(3)}{2\pi^3} + \cdots \\ \frac{1}{4}t^4 + \frac{15}{8}t^2 - \frac{105\mathbf{i}\zeta(3)}{2\pi^3}t - \frac{75}{64} + \cdots \end{pmatrix}, \tag{5.98}$$

they are related by an integral coordinate transformation $\Pi = T\Pi'$ with

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 11 & 9 & -6 & 0 & 0 \\ -4 & -7 & 6 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.99}$$

In fact, as we have mentioned, it is not clear that our $H \cap H$ can really induce an integral basis Π. Meanwhile, Π' can be shown to be an integral basis. We will nevertheless use Π in the computation later, and its results regarding to the finiteness of singularities agree with the results from Π'.

5.2.4 Example: The hypersurface X_{24} inside $\mathbb{P}^{1,1,1,1,8,12}$

This is the example in the paper [16]. Since the period vector around large complex structure locus is not completely printed in the paper, we carry out the computation explicitly here for further reference.

The fourfold X_{24} under consideration is an elliptic fibration. To obtain such a fibration structure, its ambient space should also admit a fibration.

The toric data for the polytope Δ^* is given by

$$
\begin{array}{cc|cccccc|cc}
 & & & & & & & & l^{(e)} & l^{(b)} \\
K_M & x_0 & 1 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\
D_1 & x & 1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\
D_2 & y & 1 & 0 & -1 & 0 & 0 & 0 & 3 & 0 \\
E & z & 1 & 2 & 3 & 0 & 0 & 0 & 1 & -4 \\
L & u_1 & 1 & 2 & 3 & 1 & 1 & 1 & 0 & 1 \\
L & u_2 & 1 & 2 & 3 & -1 & 0 & 0 & 0 & 1 \\
L & u_3 & 1 & 2 & 3 & 0 & -1 & 0 & 0 & 1 \\
L & u_4 & 1 & 2 & 3 & 0 & 0 & -1 & 0 & 1
\end{array}
$$

In the first column are the toric divisors and in the second column are the coordinates.

This toric ambient space $\mathbb{P}_{\Delta^*}$ is a blown-up weighted projective space $\mathbb{P}^{1,1,1,1,8,12}$. It describes a fibration of $\mathbb{P}^{1,2,3}$ over the base $\mathbb{P}^3$, where the fibre provides the ambient space for the elliptic fibre of X_{24}. We can also work out its polar polytope Δ, which has the following toric data

$$
\text{Polyhedron } \Delta : \quad
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 6 & 6 & 6 \\
1 & 1 & 1 & 6 & 6 & -18 \\
1 & 1 & 1 & 6 & -18 & 6 \\
1 & 1 & 1 & -18 & 6 & 6
\end{array}
$$

We have only listed the vertices. Then the mirror $\hat{X}_{24}$ is given by locus of $P_{\Delta^*} = 0$ inside $\mathbb{P}_\Delta$ where

$$
\begin{aligned}
P_{\Delta^*} &= x_0 \left(\alpha_0 u_1 u_2 u_3 u_4 xyz + z^6(\alpha_1 u_1^{24} + \alpha_2 u_2^{24} \alpha_3 u_3^{24} + \alpha_4 u^{24}) \right. && (5.100) \\
&\quad \left. + \alpha_5 (u_1 u_2 u_3 u_4 z)^6 + \alpha_6 x^3 + \alpha_7 y^2 \right), && (5.101)
\end{aligned}
$$

and the corresponding complex structure variables are defined as

$$
\begin{aligned}
z_{\mathrm{e}} &= \frac{\alpha_5 \alpha_6^2 \alpha_7^3}{\alpha_0^6}, && (5.102) \\
z_{\mathrm{b}} &= \frac{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{\alpha_5^4}. && (5.103)
\end{aligned}
$$

So that the Picard-Fuchs operators are given by

$$
\begin{aligned}
\mathcal{L}_1 &= \theta_{\mathrm{e}}(\theta_{\mathrm{e}} - 4\theta_{\mathrm{b}}) - 12 z_{\mathrm{e}}(6\theta_{\mathrm{e}} + 5)(6\theta_{\mathrm{e}} + 1), && (5.104) \\
\mathcal{L}_2 &= \theta_{\mathrm{b}}^4 - z_{\mathrm{b}}(4\theta_{\mathrm{b}} - \theta_{\mathrm{e}})(4\theta_{\mathrm{b}} - \theta_{\mathrm{e}} + 1)(4\theta_{\mathrm{b}} - \theta_{\mathrm{e}} + 2)(4\theta_{\mathrm{b}} - \theta_{\mathrm{e}} + 3). && (5.105)
\end{aligned}
$$

Now we use the result of [16]. First we have to identify a basis for the Kähler cone of X_{24}. From the Mori vector we can choose them to be

$$D_{\rm b} = L, \tag{5.106}$$
$$D_{\rm e} = 4L + E. \tag{5.107}$$

This time the dimension is too high so we use the Sage code provided by the author of [16] to compute the intersections on X_{24}. They are

$$D_{\rm e}^4 = 64, \tag{5.108}$$
$$D_{\rm e}^3 \cdot D_{\rm b} = 16, \tag{5.109}$$
$$D_{\rm e}^2 \cdot D_{\rm b}^2 = 4, \tag{5.110}$$
$$D_{\rm e} \cdot D_{\rm b}^3 = 1, \tag{5.111}$$
$$D_{\rm b}^4 = 0. \tag{5.112}$$

We also have the following topological data for X_{24}

$$c_2 = 11D_{\rm e}^2 + 4D_{\rm b}D_{\rm e} - 10D_{\rm b}^2, \tag{5.113}$$
$$c_4 = 24D_{\rm b}^2 \cdot D_{\rm e}^2 + 82D_{\rm b}^3 \cdot D_{\rm e} - 93312D_{\rm b}^4, \tag{5.114}$$

hence

$$c_2 \cdot D_{\rm e}^2 = 728, \tag{5.115}$$
$$c_2 \cdot D_{\rm b} \cdot D_{\rm e} = 182, \tag{5.116}$$
$$c_2 \cdot D_{\rm b}^2 = 48, \tag{5.117}$$
$$c_3 \cdot D_{\rm e} = -3860, \tag{5.118}$$
$$c_3 \cdot D_{\rm b} = -960, \tag{5.119}$$
$$\chi = 23328. \tag{5.120}$$

With these data we can use the formula for the integral basis. The 4-cycle basis are give by

$$H^{\rm b} = D_{\rm b}^2, \tag{5.121}$$
$$H_{\rm b} = E \cdot D_{\rm b} = D_{\rm e} \cdot D_{\rm b} - 4D_{\rm b}^2. \tag{5.122}$$

Then corresponding to the basis

$$(\mathcal{O}_M, \mathcal{O}_E, \mathcal{O}_{\tilde{D}_b}, \mathcal{O}_{H^b}, \mathcal{O}_{H_b}, \tilde{\mathcal{C}}^b, \tilde{\mathcal{C}}^e, \mathcal{O}_{\rm pt.}), \tag{5.123}$$

the integral period vector is given by

$$\Pi = X^0 \begin{pmatrix} 1 \\ -\tau \\ -t \\ 2\tau^2 + \tau t + \tau + 2 + \cdots \\ \frac{1}{2}t^2 - \frac{3}{2}t + \frac{17}{12} + \cdots \\ -\frac{8}{3}\tau^3 - 2\tau^2 t - \frac{1}{2}\tau t^2 - \tau^2 - \frac{1}{2}\tau t - \frac{31}{4}\tau - 2t - 1 + \frac{120\mathbf{i}\zeta(3)}{\pi^3} + \cdots \\ -\frac{1}{6}t^3 + t^2 - \frac{9}{4}t + \frac{5\mathbf{i}\zeta(3)}{2\pi^3} + \frac{11}{6} + \cdots \\ \frac{8}{3}\tau^4 + \frac{8}{3}\tau^3 t + \tau^2 t^2 + \frac{1}{6}\tau t^3 + \frac{91}{6}\tau^2 + \frac{91}{12}\tau t + t^2 - \frac{965\mathbf{i}\zeta(3)}{2\pi^3}\tau - \frac{120\mathbf{i}\zeta(3)}{\pi^3}t - \frac{37}{6} + \cdots \end{pmatrix}, \tag{5.124}$$

where the holomorphic solution is now [54]

$$X^0(z_{\mathrm{e}}, z_{\mathrm{b}}) = \sum_{k_{\mathrm{e}}=0}^{\infty}\sum_{k_{\mathrm{b}}=0}^{\infty} \frac{(6k_{\mathrm{e}})!(4k_{\mathrm{b}})!}{(2k_{\mathrm{e}})!(3k_{\mathrm{e}})!k_{\mathrm{e}}!(k_{\mathrm{b}}!)^4} z_{\mathrm{e}}^{k_{\mathrm{e}}} z_{\mathrm{b}}^{k_{\mathrm{b}}}. \tag{5.125}$$

And the intersection matrix η^{-1} is given by

$$\eta^{-1} = \begin{pmatrix} 2 & 1 & -2 & 1 & 4 & 1 & 0 & 1 \\ 1 & 2 & -3 & -2 & 3 & 4 & -1 & 0 \\ -2 & -3 & -4 & -2 & 0 & -1 & 0 & 0 \\ 1 & -2 & -2 & -4 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.126}$$

5.3 Criteria of finite distance

Recall in Seiberg-Witten theory, the moduli space of the $\mathcal{N} = 2$ SUSY field theory contains several singularity points (e.g., $u = \pm\Lambda^2, \infty$) such that at these points, the effective action behaves “singularly”. This is generally true for Calabi-Yau compactifications. For example, the quintic threefold X_5 moduli space contains three special points, namely the large complex structure point $z = 0$, the conifold point $z = 1/5^5$, and the $\mathbb{Z}_5$-orbifold point $z = \infty$. At the singular points, something physically important happens. In some cases, one is interested in the following question: For a singularity z_0 and any non-singular point z inside the complex structure moduli space, is the Weil-Petersson distance (i.e., the distance coming from the Kähler potential $-\log \int [\Omega \wedge \bar{\Omega}]$) between z_0 and z finite or infinite? This is not an easy problem. Naïvely one needs an explicit form of the Weil-Petersson metric and then try to find a geodesic between z_0 and all regular points inside the complex structure moduli space.

Fortunately, in the paper [78], the author derived a equivalence condition for the finiteness of a singular point in a Calabi-Yau manifold of 1-moduli and conjectured its correctness for Calabi-Yau manifolds that has higher-dimensional complex structure moduli space. The conjecture for Calabi-Yau threefolds with 2-moduli is proved in [60]. We will discuss about the condition and apply it to the quintic threefold and sextic fourfold.

5.3.1 Condition for finiteness of a singularity

The condition relies on the asymptotic behaviour of the period vector Π near a singularity point. The first piece we need is the nilpotent orbit theorem of Schmid. We work in a neighbourhood of a singularity in the complex structure moduli space. Such a neighbourhood is represented as a punctured disk $\Delta^\times = \{0 < |z| < 1\}$, where the removed 0 corresponds to the singularity in this coordinate patch.

Let $\Pi(z)$ be the period vector for the holomorphic volume form Ω. When we loop around the singularity 0, the period vector acquires a monodromy $\Pi \mapsto M\Pi$. Such a monodromy is known to be quasi-unipotent, i.e., there are some integers r, s such that

$$(M^s - 1)^r = 0. \tag{5.127}$$

After redefinition $M^s \mapsto M$ we can assume that the monodromy is unipotent. As an example, the large complex structure point is characterised by its maximal unipotent monodromy, which simply means that for a Calabi-Yau n-fold, the monodromy M around large complex structure satisfies $(M-1)^{n+1} = 0$.

A unipotent matrix M has a logarithm. It is defined by

$$N = \log M \tag{5.128}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(M-1)^k. \tag{5.129}$$

Note that the sum is finite since M is unipotent.

Let $t = \frac{1}{2\pi \mathbf{i}} \log z$. Then the nilpotent orbit theorem states that near $z = 0$, the period vector has the following expansion

$$\Pi = \exp(tN)(a_0 + a_1 z + a_2 z^2 + \cdots), \tag{5.130}$$

with a_0 nonzero.

It is important to notice that we are also using the symbol t to denote the full single-log solution in the period. This is not problematic since we will be only care about a_0, and when working in the limit $z \to 0$, the infinite sum part of the single-log solution is suppressed.

Now we are able to describe the finiteness criteria. The scenario we are considering is the complex structure moduli space of a Calabi-Yau space. Around one singularity, the neighbourhood is of the form

$$S = (\Delta^\times)^r \times \Delta^m, \tag{5.131}$$

where $0 \in \bar{S} = \Delta^{r+m}$ is the singularity. The form of S simply means that there are r singular complex structure variables among all $r+m$ complex structure moduli. The singular divisors are assumed to be resolved to normal crossings $\bar{S} - S = D_1 \cup \cdots \cup D_r$. Denote the nilpotent part in the Jordan decomposition of monodromy around D_j by M_j and the corresponding logarithm by $N_j = \log M_j$.

Then it is conjectured by Wang [79] that the singularity 0 is of finite Weil-Petersson distance if and only if

$$N_j a_0 = 0 \tag{5.132}$$

for all $j = 1, \ldots, r$. This conjecture is proven for Calabi-Yau space with arbitrary dimension with $r = 1, m = 0$ by Wang [78], and for Calabi-Yau threefolds with $r = 2$ by Lee [60].

Our discussion above is admittedly vague and we are going to show two $r = 1, m = 0$ examples below. For mathematical precise statements and proof, we refer the reader to the talk [79].

5.3.2 Example: The quintic mirror $\hat{X}_5$

The large complex structure point

Around the large complex structure point $z = 0$, we have already solved the period. We reproduce the asymptotic parts below for convenience

$$\Pi_0 = X^0 \begin{pmatrix} 1 \\ t \\ -\frac{5}{2}t^2 + \frac{11}{2}t + \frac{25}{12} + \cdots \\ \frac{5}{6}t^3 + \frac{25}{12}t + \frac{25\mathbf{i}\zeta(3)}{\pi^3} + \cdots \end{pmatrix}. \tag{5.133}$$

It is straightforward to compute the monodromy M_0 induced by the loop $t \mapsto t+1$

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & -5 & 1 & 0 \\ 5 & 8 & -1 & 0 \end{pmatrix}. \tag{5.134}$$

This monodromy is unipotent $(M_0 - 1)^4 = 0$. And from its Jordan decomposition we know that its semisimple part is 1. Its logarithm is

$$\begin{aligned} N_0 &= \log M_0 &(5.135)\\ &= (M_0 - 1) - \frac{1}{2}(M_0-1)^2 + \frac{1}{3}(M_0-1)^3 &(5.136)\\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{11}{2} & -5 & 0 & 0 \\ \frac{25}{6} & \frac{11}{2} & -1 & 0 \end{pmatrix}, &(5.137) \end{aligned}$$

and the operator in nilpotent orbit theorem is

$$\exp(tN_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ -\frac{5}{2}t^2 + \frac{11}{2}t & -5t & 1 & 0 \\ \frac{5}{6}t^3 + \frac{25}{6}t & \frac{5}{2}t^2 + \frac{11}{2}t & -t & 1 \end{pmatrix}. \tag{5.138}$$

We can read out a_0 from the period Π_0

$$a_0 = \begin{pmatrix} 1 \\ 0 \\ \frac{25}{12} \\ \frac{25\mathbf{i}\zeta(3)}{\pi^3} \end{pmatrix}. \tag{5.139}$$

As a check, we can compute to see that

$$\exp(tN_0)a_0 = \Pi_0 \tag{5.140}$$

for the asymptotic part.

Then the quantity of interest

$$N_0 a_0 = \begin{pmatrix} 0 \\ 1 \\ \frac{11}{2} \\ \frac{25}{12} \end{pmatrix} \tag{5.141}$$

is nonzero. This implies that the large complex structure point $z = 0$ is of infinite Weil-Petersson distance.

The conifold point

There is another interesting singularity of the quintic at $z = 1/5^5$ which is called the conifold point. This point is argued physically to be of finite Weil-Petersson distance. Let us now use the criteria to check the claim.

We have to instead solve the period around $z = 1/5^5$. Redefine $\Delta = 1 - 5^5 z$ and change the variable to Δ. The Picard-Fuchs operator becomes

$$\mathcal{L}_c = \tilde{\theta}_\Delta^4 - (1-\Delta)\prod_{k=1}^{4}\left(\tilde{\theta}_\Delta + \frac{k}{5}\right), \tag{5.142}$$

where the differential operator is defined as $\tilde{\theta}_\Delta := (\Delta - 1)\partial_\Delta$.

We plug in the Ansatz $\Delta^\rho \sum c_k \Delta^k$ and find that the indicial equation is

$$\rho^4 - 4\rho^3 + 5\rho^2 - 2\rho = 0, \tag{5.143}$$

whose solutions are

$$\rho = 0, 1, 1, 2. \tag{5.144}$$

Note that the indices are not maximally degenerate around the conifold point.

By Frobenius method, we conclude that the leading terms around $\Delta = 0$ in periods are

$$\Pi_{\mathrm{c}} = \begin{pmatrix} 1 + \cdots \\ \Delta + \cdots =: \omega_1(\Delta) \\ \omega_1(\Delta) t_\Delta + \Delta + \cdots \\ \Delta^2 + \cdots \end{pmatrix}, \tag{5.145}$$

where we have rescaled the single-log solution by $\frac{1}{2\pi\mathbf{i}}$ and defined $t_\Delta = \frac{1}{2\pi\mathbf{i}}\log\Delta$.

The monodromy induced by $t_\Delta \mapsto t_\Delta + 1$ is now

$$M_{\mathrm{c}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.146}$$

This form of the monodromy which contains the block

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{5.147}$$

is typical at the conifold points in any Calabi-Yau threefold.

Note that the matrix M_{c} is already in its Jordan normal form. This matrix is of infinite order and unipotent $(M_{\mathrm{c}} - 1)^2 = 0$. The logarithm is

$$N_{\mathrm{c}} = \log M_{\mathrm{c}} \tag{5.148}$$

$$= M_{\mathrm{c}} - 1 \tag{5.149}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.150}$$

For the nilpotent orbit theorem, we can compute the operator

$$\exp(t_\Delta N_c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.151}$$

and check the agreement in the asymptotic part.

The a_0 part of the period vector around $\Delta = 0$ is

$$a_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.152}$$

and now we can compute

$$N_c a_0 = 0, \tag{5.153}$$

which implies that the conifold point in the moduli space is of finite Weil-Petersson distance. This agrees with the physical expectation.

There is a remaining orbifold point at $z = \infty$ but the monodromy around it is already of finite order, which implies that the logarithm of the monodromy is zero. This implies that the oribifold point is of finite distance.

5.3.3 Example: The sextic mirror $\hat{X}_6$

The large complex structure point

The $z = 0$ corresponds to the large complex structure point of the sextic. We reproduce its period here for convenience

$$\Pi = X^0 \begin{pmatrix} 1 \\ -t \\ 3t^2 + 6t + \frac{29}{4} + \cdots \\ -t^3 - \frac{3}{2}t^2 - \frac{19}{4}t - \frac{17}{8} + \frac{105\mathbf{i}\zeta(3)}{2\pi^3} + \cdots \\ \frac{1}{4}t^4 + \frac{15}{8}t^2 - \frac{105\mathbf{i}\zeta(3)}{2\pi^3} - \frac{11}{64} + \cdots \end{pmatrix}. \tag{5.154}$$

The monodromy M_0 induced by $t \mapsto t + 1$ is given by

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 9 & -6 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \tag{5.155}$$

and it is maximally unipotent $(M_0-1)^5=0$.

Its logarithm is

$$
\begin{aligned}
N_0 &= \log M_0 && (5.156)\\
&= (M_0-1)-\frac{1}{2}(M_0-1)^2+\frac{1}{3}(M_0-1)^3-\frac{1}{4}(M_0-1)^4 && (5.157)\\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0\\ 6 & -6 & 0 & 0 & 0\\ \frac{5}{2} & -3 & -1 & 0 & 0\\ \frac{9}{2} & -2 & -\frac{1}{2} & -1 & 0 \end{pmatrix}. && (5.158)
\end{aligned}
$$

From the period Π_0 we read out

$$
a_0=\begin{pmatrix} 1\\ 0\\ \frac{29}{4}\\ -\frac{17}{8}+\frac{105\mathbf{i}\zeta(3)}{2\pi^3}\\ -\frac{105\mathbf{i}\zeta(3)}{2\pi^3}-\frac{11}{64}\end{pmatrix}. \tag{5.159}
$$

And compute

$$
N_0a_0=\begin{pmatrix} 0\\ -1\\ 6\\ -\frac{19}{4}\\ 3-\frac{105\mathbf{i}\zeta(3)}{2\pi^3}\end{pmatrix}, \tag{5.160}
$$

which is nonzero. This confirms the expectation that the large complex structure point for the sextic is of infinite Weil-Petersson distance.

The conifold point

Analogous to the threefolds, a Calabi-Yau fourfold also have a special singularity called the conifold point. But the behaviour is different from threefolds. In the sextic, the conifold is at $z=1/6^6$.

First we solve the PF equation around the sextic conifold. After changing the variable to $\Delta=1-6^6z$, the PF operator becomes

$$
\mathcal{L}_c=\tilde{\theta}_\Delta^5-(1-\Delta)\prod_{k=1}^{5}\left(\tilde{\theta}_\Delta+\frac{k}{6}\right), \tag{5.161}
$$

where the differential operator is defined as $\tilde{\theta}_\Delta := (\Delta-1)\partial_\Delta$.

Plug in the Ansatz $\Delta^\rho \sum c_k \Delta^k$, and we get the indicial equation

$$-\rho^5 + \frac{15}{2}\rho^4 - 20\rho^3 + \frac{45}{2}\rho^2 - 9\rho = 0, \tag{5.162}$$

whose solutions are

$$\rho = 0, 1, 2, 3, \frac{3}{2}. \tag{5.163}$$

The indices are not maximally degenerate as expected. Moreover the indices do not even contain repeated values. The last index $\rho = \frac{3}{2}$ looks very interesting and this solution has a branch cut. We can further compute this solution with a brunch cut. Assuming $c_1 = 1$, and we have

$$\nu = \Delta^{\frac{3}{2}} + \frac{17}{18}\Delta^{\frac{5}{2}} + \cdots, \tag{5.164}$$

which agrees with the result in [35]. This observation that the periods around the fourfold conifold point has no log-solution but instead have a solution with a non-integer power was firstly made in the paper [35].

Then schematically, the period vector around the conifold has the following form

$$\Pi_{\mathrm{c}} = \begin{pmatrix} 1 + \cdots \\ \Delta + \cdots \\ \Delta^2 + \cdots \\ \Delta^3 + \cdots \\ \Delta^{\frac{3}{2}}(1 + \cdots) \end{pmatrix}, \tag{5.165}$$

so that the monodromy M_{c} induced by $\Delta \mapsto e^{2\pi \mathbf{i}}\Delta$ is clearly of order 2

$$M_{\mathrm{c}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.166}$$

According to the criteria, the matrix $N_{\mathrm{c}} = 0$ implies that the sextic conifold is of finite Weil-Petersson distence.

5.3.4 Example: two parameter mirror $\hat{X}_{24}$

With the solution to the fourfold X_{24} and assuming the conjecture for $r = 2, m = 0$ is true for Calabi-Yau fourfolds, we can deduce the finiteness for the mirror of two parameter example X_{24}. We will only do the computation for the large complex structure point $z_{\mathrm{e}} = z_{\mathrm{b}} = 0$.

We reproduce the period vector for convenience

$$\Pi = X^0 \begin{pmatrix} 1 \\ -\tau \\ -t \\ 2\tau^2 + \tau t + \tau + 2 + \cdots \\ \frac{1}{2}t^2 - \frac{3}{2}t + \frac{17}{12} + \cdots \\ -\frac{8}{3}\tau^3 - 2\tau^2 t - \frac{1}{2}\tau t^2 - \tau^2 - \frac{1}{2}\tau t - \frac{31}{4}\tau - 2t - 1 + \frac{120\mathbf{i}\zeta(3)}{\pi^3} + \cdots \\ -\frac{1}{6}t^3 + t^2 - \frac{9}{4}t + \frac{5\mathbf{i}\zeta(3)}{2\pi^3} + \frac{11}{6} + \cdots \\ \frac{8}{3}\tau^4 + \frac{8}{3}\tau^3 t + \tau^2 t^2 + \frac{1}{6}\tau t^3 + \frac{91}{6}\tau^2 + \frac{91}{12}\tau t + t^2 - \frac{965\mathbf{i}\zeta(3)}{2\pi^3}\tau - \frac{120\mathbf{i}\zeta(3)}{\pi^3}t - \frac{37}{6} + \cdots \end{pmatrix}. \tag{5.167}$$

We denote the locus of $z_{\rm e} = 0$ inside the complex structure moduli space by LR_1, and the locus of $z_{\rm b} = 0$ by LR_2.

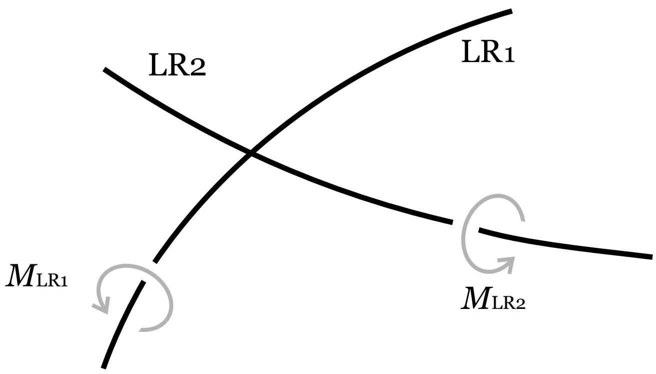

Figure 5.1: The monodromies around LR_1 and LR_2. The intersection $\mathrm{LR}_1 \cap \mathrm{LR}_2$ is the large structure point under consideration.

Note that LR_1 and LR_2 are codimension-1 loci inside the complex structure moduli space. The monodromies are depicted in figure 5.1. Then around LR_1 the monodromy induced by $\tau \mapsto \tau + 1$ is given by

$$M_{\mathrm{LR}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & -4 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 6 & 4 & -4 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -4 & -6 & 6 & 4 & -4 & -1 & 1 \end{pmatrix}. \tag{5.168}$$

And the monodromy around LR_2 induced by $t \mapsto t+1$ is given by

$$M_{\mathrm{LR}_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \tag{5.169}$$

A curious point here is that the monodromy around LR_1 is maximally unipotent $(M_{\mathrm{LR}_1} - 1)^5 = 0$, while the monodromy around LR_2 is not $(M_{\mathrm{LR}_2} - 1)^4 = 0$. This behaviour appears also in certain two parameter threefolds which admit elliptic fibration. And such behaviours indeed come from elliptic fibrations, since the base manifold cannot intersect with itself multiple times.

We can compute the logarithms, which are

$$N_{\mathrm{LR}_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{3} & -2 & 2 & -4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -\frac{4}{3} & -\frac{1}{3} & -2 & 2 & -4 & -1 & 0 \end{pmatrix}, \tag{5.170}$$

$$N_{\mathrm{LR}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\ -\frac{5}{6} & 0 & -\frac{1}{2} & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{2} & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{5.171}$$

And the a_0 is

$$a_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ \frac{17}{12} \\ -1 + \frac{120\mathbf{i}\zeta(3)}{\pi^3} \\ \frac{5\mathbf{i}\zeta(3)}{2\pi^3} + \frac{11}{6} \\ -\frac{37}{6} \end{pmatrix}. \tag{5.172}$$

Compute

$$N_{\mathrm{LR}_1} a_0 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -\frac{31}{4} \\ 0 \\ -\frac{965\mathbf{i}\zeta(3)}{2\pi^3} \end{pmatrix}, \tag{5.173}$$

$$N_{\mathrm{LR}_2} a_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ -\frac{3}{2} \\ -2 \\ -\frac{9}{4} \\ -\frac{120\mathbf{i}\zeta(3)}{\pi^3} \end{pmatrix}. \tag{5.174}$$

We see that none of them is zero, which implies that the complex structure point $\mathrm{LR}_1 \cap \mathrm{LR}_2$ is of infinite Weil-Petersson distence in the complex structure moduli space of X_{24}^*.

Chapter 6

Summary and outlook

In this thesis, we have gone through several solutions of $\mathcal{N} = 2$ theories. We first computed the low-energy effective action for an $\mathcal{N} = 2$ Yang-Mills theory with gauge group SU(2) via Seiberg-Witten theory. Then the Calabi-Yau compactification of type II superstring is performed. It can be checked that the resulting 4-dimensional theory is indeed an $\mathcal{N} = 2$ supergravity. After the compactification, we then embedded the $\mathcal{N} = 2$ Yang-Mills theory with gauge group SU(2) into type IIA superstring by suitably choosing a non-compact Calabi-Yau threefold and pushing the Planck scale M_{Pl} to infinity to decouple gravity. The low-energy effective action can thus be solved via local mirror symmetry. We then worked on a technical aspect of mirror symmetry computation, which is the fixing of an integral basis for Calabi-Yau threefolds and Calabi-Yau fourfolds. Finally we used the solution of the periods and Hodge theory to determine the finite-distance property of a singularity in the complex structure moduli space of Calabi-Yau spaces.

There are of course many more to be done in this subject. Especially, Calabi-Yau fourfolds are still not well-understood as Calabi-Yau threefolds. One can discuss more on the method of fixing an integral basis via homological mirror symmetry, since a good choice of a basis of 4-cycles seems to be crucial for the method to work on an arbitrary Calabi-Yau fourfold. One can also study the $H^{2,1}$ cohomology of a Calabi-Yau fourfold to understand its physical significance and derive its properties, hopefully in a manner similar to the Kähler deformation $H^{1,1}$ and the complex structure deformation $H^{3,1}$. Systematically applying the finite-distance criteria to models with various Calabi-Yau spaces remains to be done. Finally, the finite-distance criteria is still a conjecture for general Calabi-Yau spaces, and it deserve a proof or disproof.

Appendix A

Calabi-Yau threefolds and special geometry

In this appendix we recall some properties of Calabi-Yau threefolds and the special geometry that governs its moduli space. We will also define the quantities that are used in chapter 3. This appendix is mainly to support the material in chapter 3. The Hodge-theoretic properties are reviewed in chapter 5.

A.1 Basics of Calabi-Yau threefold

A Calabi-Yau threefold Y is a compact three (complex) dimensional Kähler manifold with trivial canonical bundle. This definition extends to noncompact and singular Calabi-Yau spaces. In the compact case, there are of course various equivalent definitions of Calabi-Yau threefold and they are of different usage. For example a compact Calabi-Yau threefold is also characterised by its SU(3) holonomy group. This characterisation is useful when one wants to the discuss supersymmetry breaking during compactification and is one of the key properties that let people consider compactification using Calabi-Yau threefolds.

As a Kähler manifold, Y has compatible Riemannian g, complex I and symplectic structures J. Locally, these compatibility conditions are expressed as:

- Written in real coordinates, the Kähler form J is related to the complex structure I by "lowering one index using the Riemannian metric g"

$$J = \frac{1}{2} I_i^k g_{kj} \mathrm{d}y^i \wedge \mathrm{d}y^j. \tag{A.1}$$

- Written in complex coordinates (of course this set of coordinates depend on the complex structure I), the Kähler form J is "the same as the Riemannian metric g"

$$J = \mathbf{i} g_{\mu\bar{\nu}} \mathrm{d}y^\mu \wedge \mathrm{d}y^{\bar{\nu}}, \tag{A.2}$$

where $y^{\bar{\nu}}$ is an alias for $\bar{y}^{\nu}$ and we are sorry about the abuse of the same symbol y for both real and complex coordinates on Y.

And Yau's famous theorem tells that given a Kähler manifold (Y, J) with vanishing first Chern class, there is a unique Ricci-flat Riemannian metric on Y such that its associated Kähler form J' is in the same cohomology class of the original J. This guarantees the existence of Calabi-Yau manifolds.

The Hodge diamond of a Calabi-Yau threefold is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{A.3}$$

where $h^{1,1}$ parametrises Kähler structure deformations and $h^{2,1}$ parametrises complex structure deformations.

We will now discuss the moduli space of Calabi-Yau threefolds. An original and good reference is [12]. We mainly follow [38]. For more details the reader can also consult [51] and [7]. One important property of Calabi-Yau threefold moduli space $\mathcal{M}$ is that it splits into a direct product of Kähler moduli space $\mathcal{M}_{\text{Kähler}}$ and complex structure moduli space $\mathcal{M}_{\text{Complex}}$

$$\mathcal{M} = \mathcal{M}_{\text{Kähler}} \times \mathcal{M}_{\text{Complex}}, \tag{A.4}$$

so that we can describe each of them separately. This splitting also enables the computation of nonperturbative coupling functions via mirror symmetry.

Another important property of the moduli spaces is that they are special Kähler manifolds, which allows one to study them via the periods of the holomorphic volume form Ω in chapter 5. Basically, all properties of the Kähler and complex moduli spaces can be derived form their own prepotentials.

The cohomology basis is chosen as in table 3.2.

A.2 The Kähler structure moduli space

Let $J \in H^{1,1}(Y)$ be the Kähler form and we expand it in the harmonic basis

$$J = v^i \omega_i, \tag{A.5}$$

where the sum is over $i = 1, \ldots, h^{1,1}$.

In string geometry one always consider instead the complexified Kähler class $B_2 + \mathbf{i}J$. Just like in the effective action $S_{\mathrm{IIA}/Y}$ (3.63), the complexified Kähler parameter is $t^i := b^i + \mathbf{i}v^i$.

Then we define the quantities

$$\begin{aligned} \mathcal{K} &:= \frac{1}{3!}\int_Y J\wedge J\wedge J, && \text{(A.6)}\\ \mathcal{K}_i &:= \int_Y \omega_i\wedge J\wedge J, && \text{(A.7)}\\ \mathcal{K}_{ij} &:= \int_Y \omega_i\wedge\omega_j\wedge J, && \text{(A.8)}\\ \mathcal{K}_{ijk} &:= \int_Y \omega_i\wedge\omega_j\wedge\omega_k. && \text{(A.9)} \end{aligned}$$

Obviously the triple intersection $\mathcal{K}_{ijk}$ is more "fundamental" since the other three quantities can be rewritten in terms of $\mathcal{K}_{ijk}$:

$$\begin{aligned} \mathcal{K} &= \frac{1}{3!}\mathcal{K}_{ijk}v^iv^jv^k, && \text{(A.10)}\\ \mathcal{K}_i &= \mathcal{K}_{ijk}v^jv^k, && \text{(A.11)}\\ \mathcal{K}_{ij} &= \mathcal{K}_{ijk}v^k. && \text{(A.12)} \end{aligned}$$

The metric on the complexified Kähler cone is defined as

$$g_{ij} := \frac{1}{4\mathcal{K}}\int_Y \omega_i\wedge\star\omega_j, \tag{A.13}$$

and it can be rewritten as

$$g_{ij} = -\frac{1}{4\mathcal{K}}\left(\mathcal{K}_{ij} - \frac{1}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j\right), \tag{A.14}$$

using (see [73])

$$\star\omega_i = -J\wedge\omega_i + \frac{\mathcal{K}_i}{4\mathcal{K}}\mathcal{K}_i\mathcal{K}_j. \tag{A.15}$$

The Kähler moduli space itself is again a Kähler manifold with Kähler potential K given by

$$K = -\log 8\mathcal{K}. \tag{A.16}$$

Furthermore the Kähler moduli space is a *special Kähler manifold*: there is a prepotential

$$\mathcal{F} = -\frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0}, \tag{A.17}$$

where $X^I = (1, t^i)$ is the special projective coordinate, such that the Kähler potential can be written in terms of the prepotential $\mathcal{F}$ as

$$K = -\log \mathbf{i}(\overline{X}^I \mathcal{F}_I - X^I \overline{\mathcal{F}}_I), \tag{A.18}$$

where $\mathcal{F}_I := \frac{\partial}{\partial X^I}\mathcal{F}$.

The gauge coupling matrix $\mathcal{N}_{IJ}$ is defined as

$$\mathcal{N}_{IJ} := \overline{\mathcal{F}}_{IJ} + \frac{2\mathbf{i}}{(\mathrm{Im}\mathcal{F})_{PQ}X^P X^Q}(\mathrm{Im}\mathcal{F})_{IK}(\mathrm{Im}\mathcal{F})_{JL}X^K X^L, \tag{A.19}$$

where $\mathcal{F}_{IJ} := \frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}\mathcal{F}$. Write in components,

$$\begin{aligned}
\mathrm{Re}\mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^i b^j b^k, && \text{(A.20)}\\
\mathrm{Im}\mathcal{N}_{00} &= -\mathcal{K} + \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right) b^i b^j, && \text{(A.21)}\\
\mathrm{Re}\mathcal{N}_{i0} &= \frac{1}{2}\mathcal{K}_{ijk}b^j b^k, && \text{(A.22)}\\
\mathrm{Im}\mathcal{N}_{i0} &= -\left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right) b^j, && \text{(A.23)}\\
\mathrm{Re}\mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k, && \text{(A.24)}\\
\mathrm{Im}\mathcal{N}_{ij} &= \left(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{\mathcal{K}}\right). && \text{(A.25)}
\end{aligned}$$

A.3 The complex structure moduli space

The Weil-Petersson metric on the complex structure moduli space is given by

$$\begin{aligned}
g_{a\bar{b}} &:= -\frac{\mathbf{i}}{\mathcal{K}\|\Omega\|^2}\int_Y \eta_a \wedge \bar{\eta}_b && \text{(A.26)}\\
&= -\frac{\int_Y \eta_a \wedge \bar{\eta}_b}{\int_Y \Omega \wedge \bar{\Omega}}, && \text{(A.27)}
\end{aligned}$$

where we have used the relation

$$\mathbf{i}\int_Y \Omega \wedge \overline{\Omega} = \mathcal{K}\|\Omega\|^2. \tag{A.28}$$

Be careful with the extra $\mathbf{i}$ appearing in front of the integral. This is due to a change complex coordinate system to real coordinate system. It can also be understand by noticing that the LHS without the $\mathbf{i}$ is purely imaginary while the RHS is a real quantity hence we need an extra $\mathbf{i}$ factor. In general, the $\Omega \in H^{3,0}$ is called a *holomorphic volume form* which is a characteristic of Calabi-Yau manifolds due to its triviality of the canonical bundle. It is a volume form in the sense that $\Omega \wedge \overline{\Omega}$ provides a nowhere vanishing top form on the underlying real manifold. But when the complex dimension of the manifold is odd, e.g. for Calabi-Yau threefolds, one should be careful about an extra **i** factor. Other numerical factors can be determined by a careful computation in local coordinates.

The complex structure moduli space is also a Kähler manifold with Kähler potential

$$K = -\log\left(\mathbf{i}\int_Y \Omega \wedge \overline{\Omega}\right). \tag{A.29}$$

Furthermore, the complex structure moduli space is again a special Kähler manifold with a prepotential $\mathcal{F}$. Written in terms of the prepotential, the Kähler potential is again of the form

$$K = -\log \mathbf{i}(\bar{z}^A \mathcal{F}_A - z^A \bar{\mathcal{F}}_A), \tag{A.30}$$

with $\mathcal{F}_A = \frac{\partial}{\partial z^A}\mathcal{F}$.

The gauge coupling matrix $\mathcal{M}_{AB}$ is then similarly defined as

$$\mathcal{M}_{AB} := \overline{\mathcal{F}}_{AB} + \frac{2\mathbf{i}}{(\mathrm{Im}\mathcal{F})_{PQ}Z^P Z^Q}(\mathrm{Im}\mathcal{F})_{AC}(\mathrm{Im}\mathcal{F})_{BD}Z^C Z^D. \tag{A.31}$$

Appendix B

Summery of Hodge theory

In this appendix we recall some basics on the Hodge theory of Kähler manifolds. For details, we refer the reader to the review article [24], and the books [32], [77], [3] and [33].

For a compact Kähler manifold X of dimension m, its n-th de Rham cohomology with complex coefficients $H^n(X, \mathbb{C})$ admits a Hodge decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \tag{B.1}$$

with $H^{q,p}(X) = \overline{H^{p,q}(X)}$. This decomposition equips the integral cohomology $H^n(X, \mathbb{Z})$ (with the torsion modded out) with a pure Hodge structure of weight n. The hodge number $h^{p,q}$ is defined to be

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X), \tag{B.2}$$

and it obviously satisfies $h^{q,p} = h^{p,q}$.

It is important to point out that the LHS of the Hodge decomposition (B.1) is the complex de Rham cohomology, which is a topological invariant of X not depending on the complex structure on X, while the RHS decomposition really changes with the variation of complex structure on X. One can think about the situation for $\mathbb{R}^2$. Fix a nonzero vector $v \in \mathbb{R}^2$ and we have $\mathbb{R}^2 = v\mathbb{R} \oplus v^{\perp}\mathbb{R}$. It is obvious that the RHS decomposition is not canonically defined and the decomposition varies with the choice of v.

Besides the Hodge decomposition, there is another important decomposition on a complex Kähler manifold, namely the Lefschetz decomposition. Let X be a compact Kähler manifold of dimension m with Kähler form $J \in H^2(X, \mathbb{R})$. Then wedging with J gives the Lefschetz map

$$L : H^n(X, \mathbb{R}) \to H^{n+2}(X, \mathbb{R}). \tag{B.3}$$

Consider the following maps

$$
\begin{aligned}
L^0 : H^n(X,\mathbb{R}) &\rightarrow H^n(X,\mathbb{R}), && \text{(B.4)}\\
L^1 : H^n(X,\mathbb{R}) &\rightarrow H^{n+2}(X,\mathbb{R}), && \text{(B.5)}\\
&\vdots \\
L^{m-n-1} : H^n(X,\mathbb{R}) &\rightarrow H^{2m-n-2}(X,\mathbb{R}), && \text{(B.6)}\\
L^{m-n} : H^n(X,\mathbb{R}) &\rightarrow H^{2m-n}(X,\mathbb{R}). && \text{(B.7)}
\end{aligned}
$$

The hard Lefschetz theorem tells us that all these maps are injective, i.e., $\mathrm{Ker}(L^{m-j}) = 0$ for $j = n, n+1, \ldots, m$. Furthermore, in addition to $L^0 = \mathrm{id}$, the last map is actually an isomorphism. The next power L^{m-n+1} is in general not injective, and one defines its kernel inside $H^n(X,\mathbb{R})$ to be the primitive cohomology

$$P^n(X,\mathbb{R}) := \mathrm{Ker}(L^{m-n+1} : H^n(X,\mathbb{R}) \rightarrow H^{2m-n+2}(X,\mathbb{R})). \tag{B.8}$$

Then one has the Lefschetz decomposition

$$H^n(X,\mathbb{R}) = \bigoplus_{2r\leq n} L^r P^{n-2r}(X,\mathbb{R}). \tag{B.9}$$

Furthermore the Lefschetz decomposition is compatible with the Hodge decomposition. Define $P^n(X,\mathbb{C}) = P^n(X,\mathbb{R}) \otimes \mathbb{C}$, and $P^{p,q}(X) = P^n(X,\mathbb{C}) \cap H^{p,q}(X)$ for $p+q=n$. Then one has

$$P^n(X,\mathbb{C}) = \bigoplus_{p+q=n} P^{p,q}(X), \tag{B.10}$$

for $p = 0, \ldots, n$.

It is often convenient to represent the Hodge structure on $H^n(X,\mathbb{C})$ by a decreasing Hodge filtration $H^n(X,\mathbb{C}) = F^0 \supset F^1 \supset \cdots \supset F^n = 0$ such that $H^n(X,\mathbb{C}) \simeq F^p \oplus \overline{F^{n-p+1}}$. This can be done by defining

$$F^p = H^{n,0} \oplus H^{n-1,1} \oplus \cdots \oplus H^{p,n-p}. \tag{B.11}$$

And the space $H^{p,q}$ in Hodge decomposition can be rewritten as

$$H^{p,q} = F^p \cap \overline{F^q}. \tag{B.12}$$

Let X be a compact Kähler manifold of dimension m with Kähler form J. In the context of variation of complex structure, one also needs a polarisation. On $H^n(X,\mathbb{C})$, this means that we have a nondegenerate bilinear pairing $Q : H^n(X,\mathbb{C}) \times H^n(X,\mathbb{C}) \rightarrow \mathbb{C}$, defined by

$$Q(\alpha,\beta) = \int_X \alpha \wedge \beta \wedge J^{m-n}. \tag{B.13}$$

It is obvious that this pairing is symmetric when n is even and is skew-symmetric when n is odd. Furthermore, it satisfies the following two Hodge-Riemann bilinear relations

$$Q(H^{p,q}, H^{r,s}) = 0, \tag{B.14}$$

for $p + q = r + s = n$, unless $p = s, q = r$.

$$(-1)^{n(n-1)/2}\mathbf{i}^{p-q}Q(\alpha, \bar{\alpha}) > 0, \tag{B.15}$$

for $\alpha \in H^{p,q}$ nonzero with $p + q = n$. This allows us define a positive-definite metric on $H^n(X, \mathbb{C})$.

The Hodge-Riemann bilinear relations can be rewritten in terms of the Hodge filtration $\{F_p\}$. First we define the Weil operator

$$C : H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C}) \tag{B.16}$$

by $C|_{H^{p,q}} = \mathbf{i}^{p-q}$. Then the two relations are

$$Q(F^p, F^{n-p+1}) = 0, \tag{B.17}$$
$$(-1)^{n(n-1)/2}Q(C\alpha, \bar{\alpha}) > 0, \tag{B.18}$$

for all nonzero $\alpha \in H^n(X, \mathbb{C})$.

In mirror symmetry, we often consider the middle cohomology $n = m$. The realisation of these identities are examplified in chapter 5.

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Acknowledgments

First I want to express my deepest gratitude to my supervisor Thomas Grimm who let me explore the boundary between physics and mathematics, which first motivates my study into theoretical physics. I also want to thank Kilian Mayer, Sebastian Greiner, Pierre Corvilain for their help on providing references during my study of the subject, and Thorsten Schimannek for explaining me the technicalities of the paper [16].

During my first days in the Netherlands, Liu Xuewen, Liu Leihua, Wan Jiansong, Leonardo Rocha and Raffaele Silvestri helped me with many problems of living. I sincerely thank you for your kind and valuable helps.

I also want to thank my fellow colleagues in the master programme in theoretical physics at UU, especially Châu Lưu Đại-Thành, Mohammad Abdul Rehman Waris, Domingo Gallegos Pazos and Ward Vleeshouwers.

Thank my parents, who recognise the importance of education. Without your support I cannot continue my dream in theoretical physics.

Finally, I would like to thank my girlfriend Zhao Yiran for her help on preparation of the diagrams in the thesis and my master talk, and for her warmest support during my dark time in my master study. I love you.