UTRECHT UNIVERSITY

MASTER THESIS

Player-Optimization of Payoff in Correlated Coordination Games

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"My greatest concern was what to call it. I thought of calling it 'information,' but the word was overly used, so I decided to call it 'uncertainty.' When I discussed it with John von Neumann, he had a better idea. Von Neumann told me, 'You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy really is, so in a debate you will always have the advantage.'"

C. E. Shannon

"It appears to be a quite general principle that, whenever there is a randomized way of doing something, then there is a nonrandomized way that delivers better performance but requires more thought."

E. T. Jaynes

"The ideas I had about supernatural beings came to me the same way that my mathematical ideas did. So I took them seriously. "

J. F. Nash, Jr.

UTRECHT UNIVERSITY

Abstract

Faculty of Science
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Player-Optimization of Payoff in Correlated Coordination Games

by Adriana CORREIA

In this Thesis we describe two types of two-player games with binary choices as a realization of an Ising system with two spins. We do this for a particular symmetric game, namely "snowdrift/chicken", and for a particular asymmetric game, the "battle of the sexes". By associating the energy of a spin configuration with a set of probabilities, through the Principle of Maximum Entropy, and we express the games as correlated. In the normal Ising model, these probabilities would describe the probabilities of achieving a final state, but here they correspond to the probabilities with which the players receive a certain set of information given by the correlating device, that they can act upon. While this correlation starts off as externally imposed, we develop a way to include the players choice into new correlated probabilities. We show that the payoffs that the players obtain by using their choice is always the same or better when compared to what it would be if they would always follow the initial correlating device while it is in equilibrium, and that it allows for a better payoff than the best uncorrelated solution, the mixed strategy equilibrium, when the initial correlating device is out of equilibrium. Because these are the best payoffs that the players get after they choose, we renormalize the initial correlating device to one that the players always follow. We associate these new probabilities with the energies of a renormalized Ising model, that effectively represents the final statistics of the game, allowing us to treat the problem with standard tools from statistical physics.

Acknowledgements

The people that I would like to acknowledge are the "we" that I use throughout this thesis.

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On a more personal note, I would like to thank the rest of my fellow classmates, in the Netherlands and previously in Portugal, that accompanied me through this Physics journey, and my friends outside of science, who had no idea of what I was talking about but kindly pretended they did.

Finally, I would like to thank my family for always being there and supporting me in every step of the way, even from far away.

Contents

Chapter 1

Introduction

Alice and Bob met each other one day at a cafe in their street, and they fancied each other, although their idea of what makes for good entertainment differed. Alice enjoys going to the **c**inema, while Bob is crazy for **d**inossaur exhibitions. Alice told Bob that that night she was going to watch the new Woody Allen movie in the neighborhood theater, and Bob told her that he was going to the see the opening of the exposition of the biggest T-Rex skeleton ever found, down at the Natural History Museum. They departed a bit sad that they couldn't manage to sync their agendas for that night, because they wished to spend some more time together, but didn't feel comfortable enough to ask the other one to change their plans. So they both thought of surprising the other one by going into the other one's activity, without telling them. However, they still really wanted to go to their own activity, and they didn't want to mismatch with the other person, in case they had the same idea, so it was a risky move. But they wanted very much to make a nice surprise to the other person. How can they increase their chances of finding each other?

Fast-forward in time, Alice and Bob are now married and driving to visit Bob's parents. Alice doesn't get well with her in-laws, so she is really grumpy. Bob cannot really understand why Alice can't make an effort to like them, so he's also not in great spirits. The weather is bad, the road is full of snow, and the mood is heavy inside the car. While they are driving, they are forced to stop when they see a small fallen tree blocking the way. It's freezing cold, so none of them is especially keen to go. Bob first proposes that they both go out of the car to move the tree. Alice doesn't want to **c**ooperate, so she is hellbent in **d**effecting. "This is ridiculous. If you don't want to help me, I will stay here too and we will both freeze to death!", said Bob. "Good, then one day someone will find your bones and put them in a museum like your beloved dinosaurs!", answered Alice angrily. They both sat quietly in the car, expecting the other one to take an attitude. Considering it is only strictly necessary one person to move the tree, that they have equal physical capacity to move it, and that they refuse to talk to each other, how will they solve this situation?

The above are examples of so called *coordination games*. The first game is often called **Battle of the Sexes**, and is an example of a game in which the players don't have the same interest in taking an action - they are asymmetric. The second game is known as **Snowdrift**, or Chicken [1], and they have the win or lose the same if they choose the same thing - they are symmetric. They are called coordination games because their best result comes if they coordinate in a way that is impossible to achieve without communication. However, since the formal formulation of these games assumes that the players don't communicate, it becomes quite a challenge to define a strategy that works for both players.

One of the ways to improve their chances of coordination is with a correlating device, which improves their chances of matching their responses. This will be an external element to the players that will give some information to each of the players about a possible outcome, but the players don't know what information was given to the other player, although they know how it can relate with their own extra information.

Given this extra information, which we will define in more strict terms, the players can act on it or not. They might have a better outcome if they choose carefully how to use this

[1] The backstory for this name involves cars going against each other; in order for them both not to die, one of the drivers has to "chicken out", turning away from the collision path. They win the most when only one of them is the "chicken", but they can never predict what the other one is going to do.

new information.

The players only have two options to play in each game. We can then look at this game configuration as a spin system, each action represented as a spin aligned "up" or "down", and the correlations related with the Ising energy. This defines an initial energy landscape, but it doesn't account for the choice of the players as a reaction to the new information. We want to find a new set of correlated probabilities that, on the game theory side, includes the probabilities with which a player uses the new information and, on the spin side, defines Ising energies (which are related with the actual probabilities of finding a certain final state) that include the players' best interest. If we can achieve this, we can use Statistical Physics methods to describe this interaction with a renormalized energy landscape.

People have been looking at how these coordination games are played on networks, and recent work has been done for asymmetric uncorrelated games (Broere et al., 2017), with extensive numerical work. To extend this analysis to correlated games, we can use an Ising model description and statistical physics tools to provide insight on how the numerical analysis relates with the microscopic behavior of the players, and for that we need to include the actions of the players into the Ising model that describes it.

The structure of this thesis is as follows. In chapter 2, we firstly explain briefly the Ising model. Secondly, we elaborate on the principle of maximum entropy, which allows us to associate to the Ising energies a set of probabilities that can be used in game theory. Then, we introduce in some detail the game theory concepts necessary to establish our problem, and finally we describe more concretely how we will relate, through the principle of maximum entropy, the Ising energies and the game-theoretical structure for our particular games.

In chapter 3, we describe how the players can improve their payoff by introducing probabilities associated with how they use the information that they are given, that we will call the probabilities to *follow* or to *not follow* that information. We show that we can renormalize the probabilities to represent them as originating from Ising energies. Because the players rely heavily on the predictability of the game, the choice through the probabilities is bounded by equilibrium conditions, that delineate how much of a strong response they can have to the external information. We study how we expect the payoffs to behave at the high and low temperature limits.

In chapter 4, we do two case studies for the snowdrift game to illustrate what it means to find the best payoff constricted to our equilibrium conditions, and we do a more succinct analysis for the **BoS** game.

In chapter 5, we present the results for the best payoffs obtained with numerical methods for several values of parameters, for both games.

Finally, in chapter 6 we put the developed work in context. We describe the problems that it solves, what can be improved upon, and how it can be extended and applied. This chapter can also be read before the rest of the work.

Throughout, we analyze the two games described above. In the appendices, we also explore how the players would do if they had different sets of extra information given initially, and why it is necessary to introduce the probabilities of following as we do.

Chapter 2

Theoretical Background

2.1 Ising Model

Statistical physics considers physical systems by the different states that they might be in. To this states, it attributes probabilities of being in one of them, and an energy associated to to each one.

Magnetism is the phenomenon that describes the attraction between certain materials, especially iron [1], and a magnet [2], and the phenomenon of attraction or repulsion between magnets.

The Ising model, developed by Ising's professor Lenz in 1920, is a mathematical model to describe ferromagnetism in statistical physics. This model uses the magnetic dipole moments of atomic *spins*, which can be described in a binary form as being either 1 or -1, to form a lattice where spins are, respectively, either "up" or "down". If enough spins are aligned in in a certain direction, there is a macroscopic magnetic field that is generated and is responsible for the attraction that we see. What this model introduces is that this field is generated by energy interactions between neighboring spins, such that, if enough spins are aligned in a certain direction, all spins will align in that same direction and we are in presence of a *phase transition*.

This phase transition is spontaneous under a certain temperature if there is no external magnetic field (Stoof, Gubbels, and Dickerscheid, 2009), but if we have an external magnetic field the spins will likely stay aligned with that magnetic field.

Let Λ be the collection of lattice sites, $s_k \in \{-1, 1\}$ the value of the spin on site $k \in \Lambda$. Let $s = (s_k)$ be spin configuration of a system that contains the value of each spin in each site. Then the energy of the configuration is given by (Huang, 1987)

$$H(s) = -\sum_{\langle i,j \rangle} J_{ij} s_i s_j - \mu \sum_j h_j s_j, \tag{2.1}$$

where $\langle i, j \rangle$ represents a sum over nearest neighbors, μ is the magnetic moment and h_j is the external magnetic field acting on lattice site j. If h_j is positive, the spins want to align up, if it is negative they want to align down, and if it is zero there is no prefered direction [3]. The matrix J_{ij} represents the interaction between spins. If J_{ij} is positive the spins will want to align with each other, creating a ferromagnetism; if it is negative, they will want to anti-align, creating antiferromagnetism; and if it zero, they don't interact.

The probability that the system is in one of those states is given by

$$P_\beta(s) = \frac{e^{-\beta H(s)}}{Z_\beta}, \tag{2.2}$$

with Z_β called the *partition function* and defined by

$$Z_\beta = \sum_s e^{-\beta H(s)}, \tag{2.3}$$

[1] Iron is *ferrum* in Latin.

[2] Traditionally, the first rocks with magnetic properties were found in Magnesia, Northern Greece, and are named after the region.

[3] The spontaneous phase transition would happen for this case.

for $\beta = \frac{1}{k_B T}$, T the temperature and k_B the Boltzman's constant.

The partition function can be interpreted as the normalization constant of the probabilities. It contains in itself all the statistical information about the system. The relation between the probabilities and the energy configurations will become clearer when we describe one way to achieve that relation through the *principle of maximum entropy* in the next section.

As we can see, because all the probabilities are normalized, we could have, in principle, an overall constant energy common to all spin configurations in the system; we will refer to this energy by E_0.

Let us assume that our spin system only has two lattice sites. Let us also assume that $J_{ij} = J$ and that $\mu h_j = \mu h = B$. Here B represents the energy associated with a magnetic field, and not a magnetic field itself. In this case, we can only have four energy configurations, the result of combinations with each lattice site having its spin up or down:

$$H = -E_0 \mathbb{1} - J \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - B \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.4}$$

where $H_{\mu\nu}$ is the energy associated with the two spins in state α and β, respectively.

The partition function of this system will be given by

$$Z_\beta = e^{\beta(E_0+J+B)} + 2e^{\beta(E_0-J)} + e^{\beta(E_0+J-B)}, \tag{2.5}$$

where, inside the brackets in the exponents we find the energy associated with, respectively, having both spins aligned up, having both spins anti-aligned, and having both spins aligned down. The associated probabilities to be in each state are each exponential function divided by this partition function.

Any symmetric matrix $E_{\mu\nu}$ with $\{\mu, \nu\} \in \{\uparrow, \downarrow\}$, and $E_{\mu\nu} = E_{\uparrow\downarrow}$, can be decomposed as eq. 2.4 using

$$\begin{aligned} E_0 &= \frac{1}{4} \sum_{\mu,\nu} E_{\mu\nu}; \\ J &= -(E_0 - E_{12}); \\ B &= -(E_0 + J - E_{11}) = -2E_0 + E_{12} + E_{11}. \end{aligned} \tag{2.6}$$

2.2 Principle of Maximum Entropy

2.2.1 Physical Motivation

The *Principle of Maximum Entropy* states that the probability distribution that best represents a current state of knowledge about the system is that which maximizes the entropy. This definition was proposed in Jaynes, 1957 and is a very ingenious way to bring together statistical physics and information theory, by stating that the former is a particular case of the latter. The consequence of this insight is that, if we assume a reciprocal relation, we can treat some statistical systems, not necessarily physical, with statistical physics' formalism.

In the Shannon, 1948 the foundation for what would become known as *information theory* is laid down. In it, Shannon postulates that the amount of choice, or uncertainty, associated with a system for which we know the probabilities p of each event should have the following form:

$$S_s(p_1, p_2, ..., p_n) = -K \sum_{i=1}^{n} p_i \ln p_i. \tag{2.7}$$

To arrive at this equation, he assumes three conditions:

- H should be continuous in p_i;
- If all events are equiprobable, such that $p_i = \frac{1}{n}$ and n is the number of events, then S_s should increase monotonically with the number of events n;

- If a choice is broken down in successive choices, then the original S_s should be the weighted sum of each individual S_s.

The first condition is a mathematical convenience. The second condition states that, if there are more elements to be chosen from, then it is natural that the amount of choice, and therefore uncertainty, increases. The third condition implies that the final amount of choice that a system has in the end must be independent from the process that made it so.

To this function S_s that describes the maximum amount of choice, or how disordered the system is, or how much uncertainty we have on a certain state, Shannon called the *entropy* of the system.

This formula of entropy has a striking resemblance with that which was already ubiquitous in statistical physics, namely the *Gibbs entropy* S_G:

$$S_G = -k_B \sum_{i=1}^{n} p_i \ln p_i, \tag{2.8}$$

where p_i is the probability associated with each *microstate*. If all microstates are equally likely and there are Ω microstates in total, then the entropy is given by the famous Boltzamn's formula

$$S_B = k_B \ln \Omega. \tag{2.9}$$

However, the statistical physics description of entropy in eq. 2.8 comes about from an identification with thermodynamic concepts, namely the entropy described in the *second law of thermodynamics*, which can be postulated in different ways but fundamentally demands that (Casquilho and Teixeira, 2011)

- In an isolated system, entropy tends to increases:
$$\Delta S \geq 0; \tag{2.10}$$

- If the system is not isolated and there is an increase in the *heat* Q then
$$\delta Q \leq TdS. \tag{2.11}$$

Statistical physics has two postulates that allow the relation between the thermodynamic and statistical to exist. They are:

- **Ergodic Hypothesis:** The time average, evaluated over a long enough time period, of a physical variable of a thermodynamic system is equal to the ensemble average of that variable in that same system, for the number of particles going to infinity;

- **Indifference Principle:** In an ensemble that represents an isolated system in thermodynamic equilibrium, each possible state is uniformly distributed over all the accessible microstates.

With these postulates we start from the probability of reaching each microstate, associate the energy of each microstate with how likely it is to reach it, and we assume that averages thus obtained coincide with the macroscopic measurements. In the statistical physics' formalism, eq. 2.9's identification as entropy comes as the end point, by realizing that the results from the established thermodynamic relations between entropy and other observables (such as internal energy, pressure, volume, etc) are followed for that particular mathematical description. Hence that these predictions and experiments agree is only an *a posteriori* realization, and is heavily dependent on the acceptance of the postulates.

The principle of maximum entropy casts a new light on the mathematical necessity of defining Boltzman's Principle as entropy, if the concept of thermodynamic entropy and the amount of choice or uncertainty in the system are to be the same concept.

One of the postulates of statistical physics assumes *a priori* probabilities, and Jaynes proposed that that was unnecessary and that the only thing that we could start off with was the measured values of the observables, which represented the average values of microscopic unknown states, and from these try to know how much we do not know about the system.

Starting from the idea that Shannon's entropy (eq. 2.7) represents the uncertainty associated with a system that has the same characteristics that a physics system should (positive, increases with increasing uncertainty and is additive for different sources of uncertainty), we should try to maximize it under the constraints of the average values. If this quantity is maximized in this way, then it is guaranteed that no more information than that which we have is being used.

Jaynes' programme to try to bring the entropy definition to a starting point of the statistical physics formalism is as follows: firstly, it is necessary to demonstrate that the maximization of the entropy predicts a suitable probability distribution; secondly, it is enough to use the previous step with the measured values of the physical variables and arrive at the probability distributions predicted by the postulate that we want to make obsolete. Then we can conclude that the Shannon's entropy definition is equivalent to the Gibbs's entropy.

We describe next the most important steps to arrive at this conclusion, expanding some steps of the calculations in the original paper.

2.2.2 Maximum Entropy Estimates

Let us assume there is a variable x that can only assume discrete value $x_i = 1, 2, ..., n$. We do not know the initial probabilities p_i, only the expectation value of the function $f(x)$:

$$\langle f(x) \rangle = \sum_{i=1}^{n} p_i f(x_i). \tag{2.12}$$

We also assume that all probabilities should add up to one:

$$\sum_{i=1}^{n} p_i = 1. \tag{2.13}$$

To find the probabilities p_i, we try to find the maximum value of the entropy in eq. 2.7, by using Lagrange multipliers. To use this, we add and subtract to the entropy expression the right and left hand sides of eqs. 2.12 and 2.13, multiplied by a constant λ_i:

$$S_s = -K \sum_{i=1}^{n} p_i \ln p_i - \lambda_0' \left(\sum_{i=1}^{n} p_i - 1 \right) - \lambda_1' \left(\sum_{i=1}^{n} p_i f(x_i) - \langle f(x) \rangle \right). \tag{2.14}$$

To find the probability distribution that maximizes H, we derive with relation with p_i and impose that it be zero:

$$\nabla_{p_1, p_2, \dots, p_n} S_s = -K \sum_{i=1}^{n} (dp_i \ln p_i + dp_i) - \lambda_0' \sum_{i=1}^{n} dp_i - \lambda_1' \sum_{i=1}^{n} dp_i f(x_i) = 0 \tag{2.15}$$

$$\Leftrightarrow (\ln p_i + 1) + \frac{\lambda_0'}{K} + \frac{\lambda_1'}{K} f(x_i) = 0 \tag{2.16}$$

$$\Leftrightarrow \ln p_i = -\lambda_0 - \lambda_1 f(x_i) \tag{2.17}$$

$$\Leftrightarrow p_i = e^{-\lambda_0 - \lambda_1 f(x_i)}, \tag{2.18}$$

with $\lambda_0 = \left(\frac{\lambda_0'}{K} + 1 \right)$ and $\lambda_1 = \frac{\lambda_1'}{K}$, which are the Lagrange Multipliers.

Substituting back this expression into eqs. 2.12 and 2.13 we find:

$$\sum_{i=1}^{n} e^{-\lambda_0-\lambda_1 f(x_i)} = 1 \tag{2.19}$$

$$\Leftrightarrow \sum_{i=1}^{n} e^{-\lambda_1 f(x_i)} = e^{\lambda_0} \tag{2.20}$$

$$\Leftrightarrow \lambda_0 = \ln \sum_{i=1}^{n} e^{-\lambda_1 f(x_i)} = \ln Z(\lambda_1); \tag{2.21}$$

$$\langle f(x) \rangle = \sum_{i=1}^{n} e^{-\lambda_0-\lambda_1 f(x_i)} f(x_i) \tag{2.22}$$

$$= -\frac{1}{Z(\lambda_1)} \frac{dZ(\lambda_1)}{d\lambda_1} \tag{2.23}$$

$$= -\frac{d \ln Z(\lambda_1)}{d\lambda_1}. \tag{2.24}$$

$Z(\lambda_1) = \sum_{i=1}^{n} e^{-\lambda_1 f(x_i)}$ is called the *partition function* and we can re-write probabilities as

$$p_i = \frac{e^{-\lambda_1 f(x_i)}}{Z(\lambda_1)}. \tag{2.25}$$

Plugging our newly found expression for the probabilities in the Shannon entropy expression we find

$$S_{max} = \frac{1}{Z(\lambda_1)} \sum_{i=1}^{n} e^{-\lambda_1 f(x_i)} \left(-\ln Z(\lambda_1) - \lambda_1 f(x_i)\right) \tag{2.26}$$

$$= \ln Z(\lambda_1) - \lambda_1 \frac{1}{Z(\lambda_1)} \sum_{i=1}^{n} e^{-\lambda_1 f(x_i)} f(x_i) \tag{2.27}$$

$$= \lambda_0 + \lambda_1 \langle f(x) \rangle. \tag{2.28}$$

This can be generalized to any number of functions on the variable x and their averages $\langle f_r(x) \rangle$ as the new constraints:

$$\langle f_r(x) \rangle = \sum_i p_i f_r(x_i). \tag{2.29}$$

Using these constraints, we find

$$Z(\lambda_1, ..., \lambda_m) = \sum_i e^{-(\lambda_1 f_1(x_i) + \lambda_2 f_2(x_i) + ... + \lambda_m f_m(x_i))}. \tag{2.30}$$

The probability distribution that maximizes entropy is given by

$$p_i = e^{-(\lambda_0 + \lambda_1 f_1(x_i) + ... + \lambda_m f_m(x_i))} \tag{2.31}$$

and we can rewrite the constraints as

$$\lambda_0 = \ln Z(\lambda_1, ..., \lambda_m); \tag{2.32}$$

$$\langle f_r(x) \rangle = -\frac{dZ(\lambda_1, ..., \lambda_m)}{d\lambda_r}. \tag{2.33}$$

A general formula for the maximum entropy can then be found:

$$S_{max} = \lambda_0 + \lambda_1 \langle f_1(x) \rangle + \ldots + \lambda_m \langle f_m(x) \rangle. \tag{2.34}$$

If the functions f_r have other dependencies other than x, namely $\alpha_1, \alpha_2, ...$, then the average rate variation of that function with the parameter can also be calculated:

$$\left\langle \frac{\partial f_r}{\partial \alpha_k} \right\rangle = \sum_i p_i \frac{\partial f_r(x_i)}{\partial \alpha_k} = \tag{2.35}$$

$$\sum_i \frac{e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))}}{Z(\lambda_1, ..., \lambda_m)} \frac{\partial f_r(x_i)}{\partial e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))}} \frac{\partial e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))}}{\partial \alpha_k} \tag{2.36}$$

$$= \frac{1}{Z(\lambda_1, ..., \lambda_m)} \sum_i e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))} \left(-\frac{1}{\lambda_r e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))}} \right) \frac{\partial e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))}}{\partial \alpha_k} \tag{2.37}$$

$$= -\frac{1}{\lambda_r} \frac{1}{Z(\lambda_1, ..., \lambda_m)} \frac{\partial}{\partial \alpha_k} \sum_i e^{-(\lambda_1 f_1(x_i)+...+\lambda_m f_m(x_i))} \tag{2.38}$$

$$= -\frac{1}{\lambda_r} \frac{\partial}{\partial \alpha_k} \ln Z(\lambda_1, ..., \lambda_m), \tag{2.39}$$

where we used the linearity of the differentiation in the second-to-last line.

2.2.3 Application to Statistical Mechanics

What we just obtained is very similar to what we have in statistical mechanics. Supposing that we have different states with energies $E_i(\alpha_1, ...)$, where α_i are parameters on which it might depend, and we only know about the average energy $\langle E \rangle$. Then by maximizing the eq. 2.7 with $K = k_B$ subject to this average we obtain a set of probabilities that can produce the usual relations in statisitcal mechanics with $\lambda_1 = \frac{1}{k_B T}$.

Hence, we conclude that both formulations are equivalent: instead of starting with probabilities and arriving at a formula for the entropy that produces these relations, we start with a formulation for the entropy and end up with the probabilities that produce these same relations, relying on the already measured averages. In this new formalism the temperature is nothing else than the Lagrange multiplier that guarantees the conservation of the average energy, and the partition function the Lagrange multiplier that preserves the sum of the probabilities.

Given this duality, we can use the principle of maximum entropy formalism for any form of generalized energy, because we are now free to interpret the temperature parameter, as well as the partition function, and treat it with the relations known from statistical mechanics, even if it isn't such a system in rigor. We will use this to map the correlations that will exist in game theory, which expressed as a set of probabilities, to the Ising spin system, in its turn expressed as the interaction energies between the spins.

2.3 Game Theory

2.3.1 General Definitions

Game theory is a field in Mathematics that deals with the conflict and cooperation of different intelligent and rational agents in relation with each other. A game is defined by its players, by the options that they can take at each game event, and how much they win given their choice and the other player's choices. We will now formalize these elements and other key concepts, for which we follow the definitions and notation found in Fudenberg and Tirole, 1991.

A player i belongs to the finite set of players $\{1, 2, ..., I\}$. Each player has a pure-strategy space S_i, by which we mean the options of playing that are available to player i, and *payoff* functions u_i for each profile of strategies $s = \{s_1, ..., s_I\}$, with $s_i \in S_i$. We will refer to all the players that are not player i by $-i$.

A two-player zero-sum game happens when $\sum_{i=1}^{2} u_i = 0$, which means that every time one of the players wins, the other looses. This is not a necessary situation, and in particular will not be the situation that we will be dealing with.

A *mixed strategy* σ_i is a probability distribution over pure strategies for player i. These probability distributions are statistically *independent* of those from the other players. The payoffs for each player are the the payoffs of each pure strategy weighted by the mixed strategy probability distribution. The space of mixed strategies is Σ and $\sigma_i(s_i)$ is the probability that σ_i assigns to s_i [4]. The payoff of player i under a certain mixed strategy distribution is then given by

$$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s). \tag{2.40}$$

Now we introduce the notation to represent what happens if we want to change the action of player i while maintaining the actions of all the other players, denoted by $-i$, unchanged. For that we denote $s_{-i} \in S_{-i}$ the pure strategies for all players in the strategy profile s except i and write

$$(s'_i, s_{-i}) = (s_1, s_2, ..., s_{i-1}, s'_i, s_{i+1}, ..., s_I) \tag{2.41}$$

for the resulting pure strategy profile. Similarly, we have for the mixed strategy profiles that

$$(\sigma'_i, \sigma_{-i}) = (\sigma_1, \sigma_2, ..., \sigma_{i-1}, \sigma'_i, \sigma_{i+1}, ..., \sigma_I). \tag{2.42}$$

It is important to note that the mixed strategies σ_i include the pure strategies s_i, because the mixed strategies include degenerate distributions: a pure strategy is degenerate and equal to zero for all pure strategies except for one, which happens with probability 1.

Nash Equilibrium We call the mixed strategy profile σ^* a Nash equilibrium if for all players i

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \tag{2.43}$$

for all $s_i \in S_i$. This means that following the mixed strategy profile σ^* is always better than choosing a certain pure strategy, considering the other players are still following that strategy. Similarly, a pure strategy Nash equilibrium is called s^* and must satisfy for all players that

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \tag{2.44}$$

for all $s_i \in S_i$. This is called a *strict* equilibrium if the inequality becomes strict.

Nash equilibria are important because they are *consistent* predictions of how the game will unfold, in the sense that if all players predict that a certain Nash equilibrium will occur then no player has an incentive to deviate from that equilibrium. Another important characteristic is that it is the only type of equilibrium that players can predict and expect that their opponents will predict, just with the payoff information; each player is independently dealing only with the basic information of the game and can still arrive at that equilibrium. Nonetheless, there might be more than one Nash equilibrium in a game, and other factors might need to be introduced to decide which one will be adopted, for example which of the equilibria gives the best average payoff or, as a different option, is less risky (less variation between possible outcomes).

[4] Since the mixed probabilities are independent among the players, we can define a space Σ_i of all the mixed probabilities of player i, to which σ_i belongs

2.3.2 Symmetric Games

A symmetric game is a game for which the payoffs are symmetric under player exchange. A formal way to define these games is, according to Dasgupta and Maskin, 1986, given by

$$u_i(s_i, s_{-i}) = u_{\pi(i)}(s_{\pi(i)}, s_{\pi(-i)}), \tag{2.45}$$

where $\pi(i)$ is any permutation of players. In this notation, s is the same strategy profile, so this means that the payoffs are the same for all players if they keep to what they are playing.

For a game with two players $i \in \{1, 2\}$, with two pure strategies $s_i \in \{C, D\}$, a general payoff matrix can be seen in table 2.1.

	C	D
C	a, a	b, c
D	c, b	d, d

TABLE 2.1: Symmetric game payoff. The payoffs have the form $u_i(s_i, s_{-i}), u_{-i}(s_{-i}, s_i)$, with s the strategy profile corresponding the row and column.

We can calibrate the payoff matrix by subtracting d to all the payoffs and dividing by $a - d$, and we end up with table 2.2.

	C	D
C	$1, 1$	s, t
D	t, s	$0, 0$

TABLE 2.2: Symmetric game payoff after calibration. The payoffs have the form $u_i(s_i, s_{-i}), u_{-i}(s_{-i}, s_i)$, with s the strategy profile corresponding the row and column. After the transformation, $s = \frac{b-d}{a-d}$ and $t = \frac{c-d}{a-d}$.

According with the values of the parameters t and s, we can define four types of games, with distinct types of Nash equilibria. We will treat the payoffs with the generic label i, knowing that the payoffs are the same for each player. The pure Nash equilibria can be easily assessed using eq. 2.44. For this $2x2$ game, however, we calculate the mixed strategy equilibrium by imposing that the payoff of each player be independent of the play of his choice. From 2.43, we get two conditions:

$$\begin{cases} u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(C, \sigma_{-i}^*) \\ u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(D, \sigma_{-i}^*). \end{cases} \tag{2.46}$$

If A is smaller or equal to B, than it contains B and all the values smaller than B. This means that if C is also smaller or equal than B, then C has to be equal to A. With this we can solve the previous system of inequalities, to obtain

$$u_i(C, \sigma_{-i}^*) = u_i(D, \sigma_{-i}^*) \tag{2.47}$$

$$\Leftrightarrow \sigma_{-i}^*(C)u_i(C, C) + \sigma_{-i}^*(D)u_i(C, D) = \sigma_{-i}^*(C)u_i(C, C) + \sigma_{-i}^*(D)u_i(D, D). \tag{2.48}$$

Assuming that in equilibrium the second player plays C with probability $\sigma_i^*(C) = P_C^i$ and plays D with probability $\sigma_i^*(D) = 1 - P_C^i$, we have that player i has the same average payoff by playing either C or D when

$$P_C^{-i} + (1-p)s = pt \tag{2.49}$$
$$\Leftrightarrow P_C^{-i} u_i(C,C) + (1-P_C^{-i}) u_i(C,D) = P_C^{-i} u_i(C,C) + (1-P_C^{-i}) u_i(D,D) \tag{2.50}$$
$$\Leftrightarrow P_C^{-i} + (1-P_C^{-i})s = P_C^{-i} t \tag{2.51}$$
$$\Leftrightarrow P_C^{-i} = \frac{s}{t+s-1}, \tag{2.52}$$

. We see that the equating of the payoffs of one player gives a condition on the probability of the other player playing C. Because the players are symmetric, this will also be the probability that player $-i$ will predict for player i.

We can calculate the average payoffs of our mixed strategy, following eq. 2.40:

$$\langle u_i(\sigma^*) \rangle \tag{2.53}$$
$$\begin{aligned}=&u_i(C,C)\sigma_i^*(C)\sigma_{-i}^*(C) + u_i(C,D)\sigma_i^*(C)\sigma_{-i}^*(D) + u_i(C,D)\sigma_i^*(D)\sigma_{-i}^*(C)\\ &+ u_i(D,D)\sigma_i^*(D)\sigma_{-i}^*(D)\end{aligned} \tag{2.54}$$
$$= \left(\frac{s}{t+s-1}\right)^2 + (s+t)\left(\frac{s}{t+s-1}\right)\left(1-\frac{s}{t+s-1}\right) \tag{2.55}$$
$$= \frac{st}{t+s-1} \tag{2.56}$$

We now analize the aforementioned four games and their Nash equilibria.

Harmony Game This game is defined by having the parameters $0 \leq s < 1$ **and** $0 \leq t < 1$, s.t. the parameters are always between the diagonal values (respectively, 1 and 0). In this game, both players gain the most if they play C irregardless of what the other player plays, which means that the highest payoff is always found for CC. The best payoff is at the pure Nash equilibrium, which we can verify if we use eq. 2.44:

$$u_i(C,C) \geq u_i(D,C). \tag{2.57}$$

Looking at eqs. 2.52, the mixed strategy only exists if $s+t > 1$, and, given the values of the parameters in this game, the best value of the payoff is for $s = t = 1$, giving a value of 1, and we can conclude that this is not better than the pure Nash equilibrium.

Prisoner's Dilemma This game has parameters $-1 \leq s < 0$ **and** $1 \leq t < 2$, s.t. the parameters are always outside of the interval defined by the diagonal parameters. If we evaluate the Nash inequalities, we see that only one is true:

$$u_i(D,D) \geq u_i(C,D). \tag{2.58}$$

Here we do not have a mixed strategy, because the associated probabilities would be negative.

Stag Hunt This game has parameters $-1 < s < 0$ **and** $0 < t < 1$, s.t. the former parameter is outside the diagonal interval but the latter is inside. In this case we can find two Nash equilibria:

$$u_i(C,C) \geq u_i(D,C); \tag{2.59}$$
$$u_i(D,D) \geq u_i(C,D). \tag{2.60}$$

The first occurs because $t < 1$, so it is always good to play CC when comparing with the neighboring options, and the second because $s < 0$, applying the same reasoning to DD.

How can the players agree on which equilibrium to choose? The Nash equilibrium in DD is *risk dominant*, which means that it is the preferred action if the actions of the other player are very unpredictable. The equilibrium in CC is *payoff dominant*, because, if chosen, is what offers the best payoff; nonetheless, it is the most risky because all other three options represent a lower payoff.

Here again we cannot define a mixed strategy, due to the value of s being negative.

Snowdrift/Chicken This game has parameters $0 < s < 1$ **and** $1 < t < 2$, s.t. the former parameter is inside the diagonal interval but the latter is outside (therefore it represents a shift by one unit in the value of the parameters from the previous game).

This game has the following pure strategy Nash equilibria:

$$u_i(C, D) \geq u_i(D, D); \tag{2.61}$$

$$u_i(D, C) \geq u_i(C, C). \tag{2.62}$$

These Nash equilibria differ from those found in the previous game because they are *off-diagonal*, and they are symmetric, which means that we can't define a pure strategy in C or D for each player, which in practice means that this information alone doesn't allow them to reach the equilibrium states. The best that they can do is play C a number of times, and D another number of times; but since their payoffs for each play will also be different, the percentage of times they play each should be predictable and stable, and this is where the mixed equilibrium comes in.

Here we can always find a value for the mixed Nash equilibrium, with probabilities of playing C for both players given by eq. 2.52. With this distribution, we get an equilibrium in the game, that represents the best payoff that the players can obtain if this is the only information they have on the game.

2.3.3 Asymmetric games

We follow the notation from the symmetric 2×2 games to introduce the asymetric games. We will only treat the two player, two play case. We define again the players as $i, j \in \{1, 2\}$ and now we define that, for each strategy profile $s = s_1, s_2$, we have a complementary strategy profile $\bar{s} = \bar{s}_1, \bar{s}_2$, s.t. if $s_i = C$, then $\bar{s}_i = D$, and if $s_i = D$, then $\bar{s}_i = C$. The bar stands symbolically for negation. The asymmetry condition in the payoffs can then be put as

$$u_i(s_i, s_j) = u_j(\bar{s}_j, \bar{s}_i). \tag{2.63}$$

The general payoff matrix for 2×2 matrix of the games we are interested in has the form of table 2.4.

	C	D
C	a, b	c, c
D	c, c	b, a

TABLE 2.3: Asymmetric game payoff. The payoffs have the form $u_i(s_i, s_{-i}), u_{-i}(s_{-i}, s_i)$, with s the strategy profile corresponding the row and column.

Subtracting c to all the payoffs and renaming, we end up with the payoffs in table

	C	D
C	$1, s$	$0, 0$
D	$0, 0$	$s, 1$

TABLE 2.4: Asymmetric game payoff after calibration. The payoffs have the form $u_i(s_i, s_{-i}), u_{-i}(s_{-i}, s_i)$, with s the strategy profile corresponding the row and column.. After the transformation, $1 = a - c, s = b - s = b - a + 1$.

This game is commonly referred to as **Battle of the Sexes**, and we will work with $s > 0$. Looking at the Nash equilibria, we see that it exists for CC and DD, but, despite them being diagonal, which would allow us to prescribe a pure strategy for each player, unlike the symmetric games here one of the players will do consistently worse than the other, which would put two rational players out of equilibrium. We thus turn to a mixed strategy equilibrium, for which we require that the two payoffs are the same for both choices of action, and they will also be the same for both players.

For $\sigma^*(s)$ the mixed strategy equilibrium profile, we have $\sigma_2^*(C) = P_C^2$, $\sigma_2^*(D) = 1 - P_C^2$, $\sigma_1^*(C) = P_C^1$, $\sigma_1^*(D) = 1 - P_C^1$. Equating the payoff of each of the players as a function of the probability that the other player chooses a strategy, we use eq. 2.48 to calculate the probabilities as functions of the parameters, and we get:

$$P_C^2 = \frac{s}{1+s}; \tag{2.64}$$

$$P_C^1 = \frac{1}{1+s}. \tag{2.65}$$

$$\tag{2.66}$$

The average payoff that they obtain is, according to 2.54,

$$\langle u_i(\sigma^*)\rangle = \frac{s}{1+s}. \tag{2.67}$$

We then see clearly why the "snowdrift" (**SD**) game, in the case of the symmetric games, and the "battle of the sexes" (**BoS**) game are of special interest: the payoffs that they obtain are very sensitive to what the players choose to play and what information they have on what the other player is going to play; while in the first three symmetric games the equilibrium can be found in pure strategies and they can never do better than that, here they can change how much they win by iterating between pure strategies, which gives room for improvement if they are given more information - this will come in the form of correlations.

2.3.4 Representation of the probabilities of final states

We can represent, based on the probabilities that a certain player plays something, the probabilities with which a final state is achieved. In a pure Nash equilibrium, we just have that a certain certain strategy profile will have probability one, while the others will have zero probability. For the games that use a mixed strategy, we can have a probability distribution over all the pure strategies, since the probabilities of the final states are just the products of the the probabilities that each player chooses a play, as shown in tables 2.5 and 2.6 for the mixed strategy equilibrium profile. These probabilities represent the final states *because they are the probabilities that give the players the best possible payoff with the information they have.*

	C	D
C	$\left(\frac{s}{t+s-1}\right)^2$	$\left(\frac{s}{t+s-1}\right)\left(\frac{t-1}{t+s-1}\right)$
D	$\left(\frac{t-1}{t+s-1}\right)\left(\frac{s}{t+s-1}\right)$	$\left(\frac{t-1}{t+s-1}\right)^2$

TABLE 2.5: Probability distribution over strategy profiles (σ^*) for snowdrift game in mixed strategy equilibrium.

	C	D
C	$\left(\frac{1}{s+1}\right)\left(\frac{s}{s+1}\right)$	$\left(\frac{1}{s+1}\right)^2$
D	$\left(\frac{s}{s+1}\right)^2$	$\left(\frac{s}{s+1}\right)\left(\frac{s}{s+1}\right)$

TABLE 2.6: Probability distribution over strategy profiles (σ^*) for battle of the sexes game in mixed strategy equilibrium. The asymmetry of the game shows in the fact that the off diagonal probabilities are not the same.

2.3.5 Correlated Strategies

Until now we considered simultaneous games, where the players decide what each will play simultaneously. We also considered that they do not talk to each other before making this decision, that the only information that they have are the payoffs.

We will break one of these premises by assuming that they can have more information about the game in the form of a *correlation*. A correlation is a set of probabilities attributed to each profile of pure strategies. If they are correlated, the probabilities assigned to each profile of pure strategies cannot be decomposed as a product of probabilities assigned to each player's pure strategy, which is what happens in a mixed strategy.

Usually the correlating device selects one strategy profile and informs each player, independently, what they should play. A very simple example is a traffic light: centrally it is decided that one way will have green and the perpendicular way will have red, but each way only sees either red or green; if they know what the rules of their correlating device is, they can conclude that, if they have a green light, the other way will have red light with probability one, although the inverse relation is not necessarily true.

Formally, a *correlating device* is define by Ω, the space with all possible outcomes from the device; by p, the probability measure on the state space Ω; and by H_i, the *information partition* of player i. If the *true state* is $\omega \in \Omega$, then player i knows that the true state is in $h_i(\omega) \in H_i$. $h_i(\omega)$ represents the outcomes that player i regards as possible if ω is the true state. We will consider that $\omega \in h_i(\omega)$ and subsequently that $h_i(\omega) = h_i$, which means that the true state is always contained in the states that player i assumes are possible with the available information, so that the player is never wrong in the weak sense that the player never regards the true state as impossible.

With this information, the player must adapt his information to the information he receives from the correlating device. Since he only has available a certain information partition $h_i(\omega)$, if he knows the correlating device he must infer what the probabilities are that a certain state is the true state, by using Bayes' Law [5]:

$$p(\omega|h_i) = \frac{p(\omega)}{p(h_i)} \tag{2.68}$$

and $p(\omega|h_i) = 0$ if $\omega \notin h_i$.

We now define a strategy that uses this information structure. We will call $\mathrm{s}(\omega) = \{\mathrm{s}_1(\omega), ..., \mathrm{s}_I(\omega)\}$ [6] to the correlated strategy, and we will drop the reference to ω in the notation. s_i maps the elements of h_i in H_i to the pure strategies s_i in S_i. The new strategies are *adapted* to the information structure.

To be able to use these new strategies, we will require that they follow an expended Nash equilibrium, that we will call *correlated equilibrium*:

Correlated Equilibrium The correlated strategy s^* is a correlated equilibrium if for all players i, all information sets h_i and all pure strategies s_i

$$\sum_{\{\omega|h_i(\omega)=h_i\}} p(\omega|h_i)u_i(\mathrm{s}(\omega)_i^*, \mathrm{s}(\omega)_{-i}^*) \geq \sum_{\{\omega|h_i(\omega)=h_i\}} p(\omega|h_i)u_i(s_i, \mathrm{s}(\omega)_{-i}^*). \tag{2.69}$$

Thus, a strategy is an equilibrium if, given the *a priori* information, the payoff of each players is maximized independently from the other players. Eq. 2.69 is a sum over all "true" states that are possible given the players information, and it means that, weighted by the probability that a state is actually the true state given the information that each player has,

[5] We will assume that use $h_i(\omega) = h_i$. If this was not so, a player might think that a state that is not compatible with the information he just received was the true state.

[6] Note that the correlated strategy is represented by a normal s, while the pure strategy is represented by a mathematical s.

and that the other players play what that eventual true state prescribes, then their best payoff happens when they also play according with what that state prescribes.

It is easy to see why this is called an equilibrium: if the players have a better payoff by playing something else while the rest of the players play as the correlating device tells them to, then the system is not predictable by the information that we have until now. We have not made any assumption about symmetry, so this concept can be used for both symmetric and asymmetric games. If the equilibrium conditions are met, then the average payoff is given by the average of the left-hand side of eq. 2.69 weighted by the information structures:

$$\langle u_i(\sigma^*)\rangle = \sum_{h_i} p(h_i) \sum_{\{\omega|h_i\}} p(\omega|h_i) u_i(\mathrm{s}(\omega)^*_i, \mathrm{s}(\omega)^*_{-i}) \tag{2.70}$$

$$= \sum_{\omega} p(\omega) u_i(\mathrm{s}(\omega)^*_i, \mathrm{s}(\omega)^*_{-i}) \tag{2.71}$$

The mixed strategy profiles can be seen as a particular case of a correlated strategy, in which the probabilities are indeed separable. Although the strategy can be seen as correlated, the key difference is that *the probabilities are not correlated*. In this sense, we can use notation that we just developed here to the probabilities in tables 2.5 and 2.6. It is easy to see, since the mixed strategy is defined by having that the payoffs are the same when the players change strategy, that the mixed equilibria correspond to an equality in ineq. 2.69 for all possible conditions generated.

2.3.6 Traffic Lights: an example about information structure

Let us come back to the traffic light example to make these expressions more clear, exploring a system that is more complicated than the models we will use, but more connected with real life. Let's call G to green, R to red and Y to yellow. The correlating device decides on a pair of colors, for example YR, where each color is the information that each car receives, and we call the traffic light the signaling device. The correlating device has a state space that assigns a color to each of the perpendicular lanes, and so $\Omega = \{GR, YR, RG, RY, RR\}$. There is a probability measure p on these states, that is unknown to us, but in a different system it could be exactly known; in this case, however, we can predict that the highest probabilities are found for GR and RG, somewhat smaller probabilities for YR and RY and even smaller for RR, and we also expect that the probabilities are symmetric under direction exchange ($p(GR) = p(RG)$).

The first car arrives at a crossroad and sees a green light, so he has information $h_1(\omega) = GR$, while the second car sees a red light and therefore has information $h_2(\omega) = \{GR, YR, RR\}$, which means that, if a car sees a green light, he is certain that the other car is seeing a red light, but if he sees a red light he cannot say with certainty what the other car is seeing. From the outside, we can tell that the "true" state is GR, but the players can only know certain states that make what they see possible; nonetheless, the true state is always within the states they deem possible.

Let us say that the mapping from $h_i(\omega)$ to the pure strategy space is that if a player sees G he crosses the street (go), if he sees Y or R he doesn't ($nogo$). Then $s_i, \mathrm{s}_i \in \{go, nogo\}$. For player 2, what the correlating signal tells him to do is $\mathrm{s}_2 = nogo$, but his set of options is still $s_2 \in \{go, nogo\}$. So, for player 2, for $s_2 = go$ and $h_2(\omega) = \{GR, YR, RR\}$ the correlated equilibrium is

$$\frac{p(GR)}{p_2(R)} u_2(nogo, go) + \frac{p(YR)}{p_2(R)} u_2(nogo, nogo) + \frac{p(RR)}{p_2(R)} u_2(nogo, nogo) \geq$$
$$\frac{p(GR)}{p_2(R)} u_2(go, go) + \frac{p(YR)}{p_2(R)} u_2(go, nogo) + \frac{p(RR)}{p_2(R)} u_2(go, nogo),$$

with $p_2(R) = p(GR) + p(YR) + p(RR)$. We don't need to probe for $s_2 = nogo$ because then $s_2 = \mathrm{s}_2$ and we have a strict equality, which is naturally within the equilibrium conditions.

Likewse, for player 1 we would have

$$\frac{p(GR)}{p_1(G)}u_1(go, nogo) \geq \frac{p(GR)}{p_1(G)}u_1(nogo, nogo).$$

Since $p_1(G) = p(GR)$ the condition is that the payoff for player 1 when he goes while the other stays has to be equal or higher compared with that of both staying where they are.

There are more equilibrium conditions, because there are more possible individual information sets. The rest of the information sets would encode more inequalities; therefore, the correlated system is only in equilibrium if all possible conditions are met.

2.4 Ising Model & Game Theory: Contact points

The games that we have seen so far present coordination problems. For the snowdrift, the players are symmetrical and as such don't have incentives to play something different than each other, but would have a better payoff if they would. For the battle of the sexes, the situation is opposite: the players have incentives to play different things, but would have a better payoff if made the same choice. A way of helping these player coordinate is by introducing correlations, that give them a better probability of ending up at the state that is difficult to reach.

So, the players can even better if they have a correlating device. Since this device is defined by a set of probabilities, we can treat our players as follows:

1. Associate each player with a particle with *spin*, and its spin state with what the player chose to play, either "up" $(+1)$ for C, or "down" (-1) for D;

2. Associate an *Ising* energy as given by eq. 2.4 to the system that contains the four possible states;

3. Use the *principle of maximum entropy* (PME) to convert the given energies (and their average) into a probability set, which will be used as our *correlation probabilities.*

Very importantly, because we showed the equivalence between finding the probabilities through the PME and just starting out with those probabilities in a statistical physics context, **this prescription can also be used in its inverse**, meaning that we could just define a set of probabilities, deduce what the corresponding energies are assuming the standard statistical mechanics postulates, and map it onto a spin system. This rationale will be especially relevant later when we introduce the autonomy of the players.

We should state the crucial but subtle difference between the correlations in the Ising model and in game theory: while the probabilities in the Ising model represent the probabilities of finding a certain final state, the probabilities of the correlating device represent the probabilities that a certain final state is chosen, and that the information necessary to achieve that state is communicated to the players. **The probabilities of the correlating device do not necessarily represent the final statistics of the game.** Therefore, we need to find a way to include the players choices in a probabilistic way, and we need to find a new correlating device that the players will always want to follow, and therefore represents the final statistics. If this is achieved, these probabilities do correspond to Ising probabilities, and we would have achieved our goal.

While we deal with symmetric systems, we will be concerned with the snowdrift game. We set the initial energy matrix of this game as a function of the game's parameters, as seen in table 2.7.

	C	D
C	-1	$-t$
D	$-t$	$-(t-s)$

TABLE 2.7: H_{ij} for the snowdrift game.

While the energies in this table are somewhat arbitrary, there are some important properties that it holds:

- If written as a matrix, due to its symmetry it can be decomposed as an Ising system as in eq. 2.4 with $E_0 = -\frac{1+3t-s}{4}$, $J = -\frac{1-t-s}{4}$ and $B = -\frac{2-2t+2s}{4}$;
- The energetic "cost" for one player to go from playing C to playing D is the same in absolute value as the payoff cost: if the other player plays C it is $1-t$, and if the other player plays D it is s;
- The lowest energy states correspond to the Nash equilibrium states.

Using the PME gives the probability distribution corresponding to the correlating device (CD) picking out one of the possible states, given the energy associated with each state, as function of a *temperature-like* parameter, which we will refer to simply as temperature $T = \frac{1}{\beta}$, described in the inverse units of those of our energy and corresponding to the inverse of our Lagrange multiplier, which is β[7]. This temperature is characteristic of the system, and both the players and the correlating device operate under the same temperature at a given moment. Then we have, in practice, a continuum of correlation matrices for each set of energy parameters (s, t), due to the continuous nature of the temperature, which ranges between 0 and $+\infty$.

The correlation matrix (CM) provided by the CD when the PME is applied through equation 2.25 to the energies in table 2.7 is shown in table 2.8.

	C	D
C	e^{β}/Z_B	$e^{\beta t}/Z_B$
D	$e^{\beta t}/Z_B$	$e^{\beta(t-s)}/Z_B$

TABLE 2.8: Correlation probabilities for snowdrift game, given the energies of table 2.7.

The partition function is $Z_B = e^{\beta} + 2e^{\beta t} + e^{\beta(t-s)}$.

Likewise, for the BoS game we define an energy matrix given in table 2.9, which can be transformed into the set of correlated probabilities in table 2.10.

	C	D
C	$-s-1$	0
D	0	$-s-1$

TABLE 2.9: H_{ij} for the battle of the sexes game.

	C	D
C	$e^{\beta(s+1)}/Z_S$	1
D	1	$e^{\beta(s+1)}/Z_S$

TABLE 2.10: Correlation probabilities for battle of the sexes game, given the energies of table 2.9.

Here the partition function is the $Z_S = 2e^{\beta(s+1)} + 2$.

Using the language of game theory, the space of states is $\Omega = \{CC, CD, DC, DD\}$ and the one chosen by the correlating device will be ω with probabilities shown in table 2.8,

[7] In normal statistical physics, this unit normalization role is played by the Boltzman constant, which has units JK^{-1}, corresponding to the inverse of the units of $\frac{E[J]}{T[K]}$. In this situation, it is sufficient to absorb this function in the "temperature" quantity

which can be refered to as $p(\omega)$. This allows us to treat this as game theoretical system. The correlating device tells them to play C or D; in the first case their information set it $h_i = \{CC, CD\}$ and in the second $h_i = \{DC, DD\}$. As such, we must define the correlated equilibrium conditions, following eq. 2.69. We end up with

$$\begin{cases} \frac{p(CC)}{p^1(C)}u_1(C,C) + \frac{p(CD)}{p^1(C)}u_1(C,D) \geq \frac{p(CC)}{p^1(C)}u_1(D,C) + \frac{p(CD)}{p^1(C)}u_1(D,D); \\ \frac{p(DD)}{p^1(D)}u_1(D,D) + \frac{p(DC)}{p^1(D)}u_1(D,C) \geq \frac{p(DD)}{p^1(D)}u_1(C,C) + \frac{p(DC)}{p^1(D)}u_1(C,D). \\ \frac{p(CC)}{p^2(C)}u_2(C,C) + \frac{p(DC)}{p^2(C)}u_2(D,C) \geq \frac{p(CC)}{p^2(C)}u_2(C,D) + \frac{p(DC)}{p^2(C)}u_2(D,D); \\ \frac{p(CD)}{p^2(D)}u_2(C,D) + \frac{p(DD)}{p^2(D)}u_2(D,D) \geq \frac{p(CD)}{p^2(D)}u_2(C,C) + \frac{p(DD)}{p^2(D)}u_2(D,C). \end{cases} \tag{2.72}$$

Here, $p^1(C) = p(CC) + p(CD)$, $p^1(D) = p(DC) + p(DD)$, $p^2(C) = p(CC) + p(DC)$ and $p^2(D) = p(CD) + p(DD)$.

For **SD**, given the symmetry of the system, the equilibrium conditions for player 1 are the same as for player 2, so we only have two conditions:

$$\begin{cases} p(CC) + p(CD)s \geq p(CC)t \\ p(DC)t \geq p(DC) + p(DD)s \end{cases} \tag{2.73}$$

$$\Leftrightarrow \begin{cases} e^{\beta} + e^{\beta t}s \geq e^{\beta}t \\ e^{\beta t}t \geq e^{\beta t} + e^{\beta(t-s)}s \end{cases}. \tag{2.74}$$

The first line makes sure that, if a player is told to play C, that it indeed plays as told. The second line does the same for when the player is told to play D. We will refer to these conditions as **the first correlated quilibrium condition** and **the second correlated quilibrium condition**.

For the **BoS** game, because there is some relation between the players, namely an asymmetric one, we can still reduce the correlated equilibrium conditions from four to two, since the first condition is the same as for the fourth and the second the same as the third:

$$\begin{cases} p(CC) \geq p(CD)s \\ p(DD)s \geq p(DC) \end{cases} \tag{2.75}$$

$$\Leftrightarrow \begin{cases} e^{\beta(s+1)} \geq s \\ e^{\beta(s+1)}s \geq 1 \end{cases}. \tag{2.76}$$

Here, the first condition stands for the first player playing C when it is told to and for the second player playing D when it is told to, while the second condition represents the same if we interchange C and D.

We must make sure that the inequalities in 2.74 and 2.76 are true, when we are analyzing the respective game. If they are not, then one of the players might play something else while the other sticks to the correlating device, which brings the system out of an equilibrium and we stop being able to calculate an average payoff, and therefore the system won't be statistically predictable anymore.

As we will see, these specific equilibrium conditions only be stable up to a certain temperature. Nonetheless, while it is stable, the payoffs are higher than that of the mixed strategy, which is what they obtain if they don't use the correlation matrix at all. When we introduce the freedom for players to deviate from the correlating device, the temperature can be arbitrarily high. Give this freedom, we will see that the mixed strategy corresponds to having an infinitely high temperature.

Chapter 3

Model to Improve Payoffs

We now arrive at the core of this thesis. The question that we are interested in answering is: **can the players have a better payoff than the mixed strategy payoff and than what they get if they just follow the correlating device, while still using it?**

We consider that we have a better payoff if the average payoff of a player using a set of probabilities is higher than that found with a different set. In the cases that we will see, both players will have the same average payoff, so we can compare the payoff of an individual player. In the generalized case that each player has different average payoffs, we would compare the sum of the averages of the payoffs of the two players.

3.1 Previous Approaches

If we want to use the correlating device and treat in the language of Ising spins that leads to an energy $H_{\mu}\nu$, the apparently most natural way to introduce some player freedom is by changing the magnetic field energy B. Using the spin description, the freedom of the players would be encoded in some extra magnetic field that they would apply to their spin particle. One way of achieving this is by allowing the players to make a decision, look at the final state, and encode the players decision in some added or subtracted magnetic field term corresponding to the final state, as shown in table 3.1, where B_1 and B_2 are parameters that correspond, respectively, to players' 1 and 2 final choices. Upon applying again the maximization of entropy, we have a new correlation matrix, shown in table 3.2, where $Z_{new} = e^{\beta(1+B_1+B_2)} + e^{\beta(t+B_1-B_2)} + e^{\beta(t-B_1+B_2)} + e^{\beta(t-s-B_1-B_2)}$.

	C	D
C	$-1 - B_1 - B_2$	$-t - B_1 + B_2$
D	$-t + B_1 - B_2$	$-t + s - B_1 - B_2$

TABLE 3.1: Energy matrix with introduction of magnetic field parameter.

	C	D
C	$e^{\beta(1+B_1+B_2)} / Z_{new}$	$e^{\beta(t+B_1-B_2)} / Z_{new}$
D	$e^{\beta(t-B_1+B_2)} / Z_{new}$	$e^{\beta(t-s-B_1-B_2)} / Z_{new}$

TABLE 3.2: Probability matrix for energies of table 3.1.

This approach was initially developed in (insert mpampis reference). To obtain these new energies we need to add $B_1 - B_2$ to J and $B_1 + B_2$ to B. If the two new magnetic energies are the same, which makes sense in the case of symmetric games, then only the magnetic energy is changed.

If these new probabilities take into account what the players want to play, it is by attributing a plus sign to the parameter if they played C and a minus sign if they played D. However, what they do exactly is *changing the correlating device probabilities so that playing C* *is more likely than playing D*, comparing with the initial correlations.

One problem with this introduction, for our purposes, is that the new magnetic field parameters cannot be introduced independently by the players, meaning that the correlating device is not just used, but it has to be *changed* in the first place to arrive at this form. This change in the correlating device either happens spontaneously, or because the players informed it that they would like subject to this new set of probabilities. In the latter, the players' freedom is obviously compromised.

This is still an interesting problem on its own, and in appendix B we use the optimization tools that we will develop shortly to optimize the payoffs of the players, looking at it as the generalization of an initial correlating device. There we show that it is a better model than the initial correlating device because the magnetic parameter allows for stability through the whole range of temperatures, and it converges to the mixed strategy equilibrium in high temperatures. The payoffs using this correlation will also be better or equal than the previous one. We show in appendix A that this new correlating device is indeed only a quantitative extension of the old one, for the particular case that the correlating device changes the likelihood of CC in relation to DD, but does not allow, in a strict sense, that the players deviate from the original correlating device's recommendation, as we would like them to.

3.2 Follow/Not Follow model

The overall goal is that the players can use the correlation matrix to have better payoffs than those obtained if they just follow the original correlating device, or if they don't use any form of correlation. We must make clear what we mean by "follow".

We define the probability that a player i always plays according to what the correlating device (CD) says, or the probability of **following**, by P^i_F. If this is in response to the CD telling him to play C, we have $P^i_{F_C}$, and if it is in relation with D we have $P^i_{F_D}$. More precisely, it is the probability that a player ends up playing something *given* that that was the instruction, or $P^i_{F_C} = P^i(play_C|C)$ and $P^i_{F_D} = P^i(play_D|D)$. This is an approach that has two main important and useful features: on the one hand, it is introduced independently by the players, while the CD probabilities remain intact; on the other hand, it uses the correlations of the CD by interacting with them, by reacting to its output.

Because there are only two options for the players to play, and not playing is not possible, we can also define the probabilities of **not following** a certain instruction, P_{NF}, which correspond to $P^i_{NF_C} = P^i(play_D|C)$ and $P^i_{NF_D} = P^i(play_C|D)$. Each one of these probabilities is player-dependent, indicated by the label i.

If the original CD is within correlated equilibrium conditions, then we know that the players will have an interest in playing according to the instructions, insofar as they only have a binary choice: either $P_F = 0$ or $P_F = 1$, for both C and D.

Thus, we have in our hands the tools to work out the probability that a certain final configuration will be achieved, by relating what information each player receives from the CD with their response to it.

The reaction of each player to what they are told to play should be independent from the reaction that they would have if they were told to play something else, to account for the fact that that is the only information they have. Also, because the players choice must be independent of each other, we impose that

$$P^i_{F_C} + P^i_{NF_C} = 1; \tag{3.1}$$

$$P^i_{F_D} + P^i_{NF_D} = 1. \tag{3.2}$$

Alternatively, we can have that

$$P^i_{\mu\leftarrow\mu'} = \delta_{\mu\mu'} P^i_{F_{\mu'}} + (1-\delta_{\mu\mu'})(1-P^i_{F_{\mu'}}), \tag{3.3}$$

with player $i \in \{1,2\}$ playing μ after the CD tells him to play $\mu' \in \{C,D\}$. Although the player indices are redundant will prove themselves redundant, we leave them for clarity.

With this definition, we can say that the state in which they end up is a result of how they act in relation to how they were told to play. A player ends up playing C, for example, if he is either told to play C and does so, or is told to play D and does the opposite. This can be represented by

$$P^i_\mu = \sum_{\mu' \in \{C,D\}} P^i_{\mu \leftarrow \mu'} \sum_{\nu' \in \{C,D\}} p(\mu'\nu'). \tag{3.4}$$

This equation considers the four possible "true" states, what they convey to player i and how this player reacts to it. It is easy to see that if $P_{F_{\mu'}} = 1$, then $P^i_{\mu \leftarrow \mu'} = \delta_{\mu\mu'}$, and equation 3.4 reduces to the probability that the CD suggests that player i plays μ. Thus, we can say that this is a *generalization* of the original role of the correlating device, and we have a way to introduce a probabilistic choice from the part of the players.

Because we only have tools to work with a correlating device in which the players always follow, under correlated equilibrium, formally we have to treat the introduction of the new probabilities as the creation of a *new correlating device*. In other words, whatever the correlating device is, it has to be able to tell us information about the final state of the system. If this new correlating device is submited the follow/not follow probabilities again, it is always followed.

The new *renormalized* probabilities of ending up in a certain state, which will have the index R, of our new correlating device (CDR) can therefore be stated as follows [1]:

$$p^R(\mu\nu) = \sum_{\mu',\nu' \in \{C,D\}} C^{12}_{\mu'\nu'} p(\mu'\nu') = \sum_{\mu',\nu' \in \{C,D\}} P^1_{\mu \leftarrow \mu'} P^2_{\nu \leftarrow \nu'} p(\mu'\nu'). \tag{3.5}$$

Assuming the normalization of $p(\mu\nu)$, which is guaranteed by the way it is generated through the PME, we can show that the normalization in eqs. 3.1 and 3.2 proves that $p^R(\mu\nu)$ is normalized:

$$\begin{aligned}
&\sum_{\mu,\nu} p^R(\mu\nu) = \sum_{\mu,\nu} \sum_{\mu',\nu'} P^1_{\mu \leftarrow \mu'} P^2_{\nu \leftarrow \nu'} p(\mu'\nu') \\
&= \sum_{\mu',\nu'} (P^1_{C \leftarrow \mu'} + P^1_{D \leftarrow \mu'})(P^2_{C \leftarrow \nu'} + P^2_{D \leftarrow \nu'}) p(\mu'\nu') \\
&= \sum_{\mu',\nu'} \underbrace{(P^1_{F_{\mu'}} + P^1_{NF_{\mu'}})}_{1} \underbrace{(P^2_{F_{\nu'}} + P^2_{NF_{\nu'}})}_{1} p(\mu'\nu') \\
&\Leftrightarrow \sum_{\mu,\nu} p^R(\mu\nu) = \sum_{\mu',\nu'} p(\mu'\nu') = 1.
\end{aligned} \tag{3.6}$$

This is a very nice result, because it indicates that we have found a transformation that preserves the sum of certain elements. If we want to put this in matrix notation, we can consider that the four probabilities are the components of a vectors,

$$\vec{p} = (p(CC), p(CD), p(DC), p(DD)); \tag{3.7}$$

$$\vec{p}^R = \left(p^R(CC), p^R(CD), p^R(DC), p^R(DD)\right). \tag{3.8}$$

The transformation that acts on the first set of probabilities to give the second is then given by the following transformation matrix:

[1] The sum of 3.5 over ν gives 3.4.

$$\vec{\vec{T}} = \begin{pmatrix} P^1_{F_C}P^2_{F_C} & P^1_{F_C}P^2_{NF_D} & P^1_{NF_D}P^2_{F_C} & P^1_{NF_D}P^2_{NF_D} \\ P^1_{F_C}P^2_{NF_C} & P^1_{F_C}P^2_{F_D} & P^1_{NF_D}P^2_{NF_C} & P^1_{NF_D}P^2_{F_D} \\ P^1_{NF_C}P^2_{F_C} & P^1_{NF_C}P^2_{NF_D} & P^1_{F_D}P^2_{F_C} & P^1_{F_D}P^2_{NF_D} \\ P^1_{NF_C}P^2_{NF_C} & P^1_{NF_C}P^2_{F_D} & P^1_{F_D}P^2_{NF_C} & P^1_{F_D}P^2_{F_D} \end{pmatrix} \tag{3.9}$$

$$= \begin{pmatrix} P^1_{F_C} & P^1_{NF_D} \\ P^1_{NF_C} & P^1_{F_D} \end{pmatrix} \otimes \begin{pmatrix} P^2_{F_C} & P^2_{NF_D} \\ P^2_{NF_C} & P^2_{F_D} \end{pmatrix} \tag{3.10}$$

This matrix has the properties characteristic of transformation matrices that preserve the sum of the elements, and it is called a *stochastic matrix*. This feature is explored in appendix B.

The fact that this transformation can be expressed as eq. 3.10 reflects that we want the decisions of each player to be independent from one another. There is no extra correlation inserted in this transformation, as we will demonstrate shortly.

3.2.1 Follow/Not Follow Probabilities

We now need a form for P_F and P_{NF}. To achieve that, we will again use PME on a set of energies that reflects their choice. Thus, each type of reaction will have an energy associated: either $-B^i_\mu$ if they follow μ, or $+B^i_\mu$ if they do not follow:

$$B^i_{\mu\leftarrow\mu'} = -\delta_{\mu\mu'}B^i_\mu + (1-\delta_{\mu\mu'})B^i_{\mu'} \tag{3.11}$$

With this, we have the following probabilities:

$$P^i_{F_\mu} = \frac{e^{\beta B^i_\mu}}{Z^i_\mu}, \qquad P^i_{NF_\mu} = \frac{e^{-\beta B^i_\mu}}{Z^i_\mu}, \tag{3.12}$$

where $Z^i_\mu = e^{\beta B^i_\mu} + e^{-\beta B^i_\mu}$. These equations obey eq. 3.3, and, being defined in terms of probabilities, we can work easily with them and the original probabilities of the CD in tables 2.8 and 2.10.

We can rewrite eq. 3.3 using eqs. 3.12 and 3.11:

$$P^i_{\mu\leftarrow\mu'} = \frac{\delta_{\mu\mu'}e^{\beta B^i_{\mu'}} + (1-\delta_{\mu\mu'})e^{-\beta B^i_{\mu'}}}{Z^i_{\mu'}} \tag{3.13}$$

$$= \frac{e^{-\beta B^i_{\mu\leftarrow\mu'}}}{Z^i_{\mu'}}, \tag{3.14}$$

such that

$$\sum_\mu e^{-\beta B^i_{\mu\leftarrow\mu'}} = e^{-\beta B^i_{C\leftarrow\mu'}} + e^{-\beta B^i_{D\leftarrow\mu'}} = Z^i_{\mu'} \tag{3.15}$$

We called the new probability parameters B^i_μ but this must not be confused with the B definition in eq. 2.6.

By transforming our probabilities $p(\omega)$ into $p^R(\omega)$, we end up with a set of new probabilities to which we can attribute a set of new energies $H^R_{\mu\nu}$ for which, if applied the PME, yield the exact same probabilities.

This we can express as

$$p^R(\mu\nu) = \frac{e^{-\beta H^R_{\mu\nu}}}{Z_R}. \tag{3.16}$$

Using eq. 3.5, we can find what our renormalized Ising energies are:

$$H^R_{\mu\nu} = -\frac{1}{\beta}\ln\left(Z_R\; p^R(\mu\nu)\right) \tag{3.17}$$

$$= -\frac{1}{\beta}\ln\left(\sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'}P^2_{\nu\leftarrow\nu'}p(\mu'\nu')\right) - \frac{1}{\beta}\ln\left(Z_R\right) \tag{3.18}$$

$$= -\frac{1}{\beta}\ln\left(P^1_{\mu\leftarrow C}P^2_{\nu\leftarrow C}\; p(CC) + P^1_{\mu\leftarrow C}P^2_{\nu\leftarrow D}\; p(CD) + P^1_{\mu\leftarrow D}P^2_{\nu\leftarrow C}\; p(DC)\right. \tag{3.19}$$

$$\left. + P^1_{\mu\leftarrow D}P^2_{\nu\leftarrow D}\; p(DD)\right) - \frac{1}{\beta}\ln\left(Z_R\right).$$

Introducing eq. 3.14 and a generalized version of the correlation probabilities, for which we will relate with the energies $\epsilon_{\mu\nu}$, with associated partition function Z_ϵ, we get a simplified version of our renormalized energies:

$$H^R_{\mu\nu} = -\frac{1}{\beta}\ln\left(\frac{Z_R}{Z_\epsilon}\right) - \frac{1}{\beta}\ln\left(\frac{e^{-\beta\left(B^1_{\mu\leftarrow C}+B^2_{\nu\leftarrow C}+\epsilon_{CC}\right)}}{Z^1_C Z^2_C} + \frac{e^{-\beta\left(B^1_{\mu\leftarrow C}+B^2_{\nu\leftarrow D}+\epsilon_{CD}\right)}}{Z^1_C Z^2_D} + \right.$$

$$\left. \frac{e^{-\beta\left(B^1_{\mu\leftarrow D}+B^2_{\nu\leftarrow C}+\epsilon_{DC}\right)}}{Z^1_D Z^2_C} + \frac{e^{-\beta\left(B^1_{\mu\leftarrow D}+B^2_{\nu\leftarrow D}+\epsilon_{DD}\right)}}{Z^1_D Z^2_D}\right) \tag{3.20}$$

$$= -\frac{1}{\beta}\ln\left(\frac{Z_R}{Z_\epsilon Z^1_C Z^1_D Z^2_C Z^2_D}\right)$$

$$-\frac{1}{\beta}\ln\left(Z^1_D Z^2_D e^{-\beta\left(B^1_{\mu\leftarrow C}+B^2_{\nu\leftarrow C}+\epsilon_{CC}\right)} + Z^1_D Z^2_C e^{-\beta\left(B^1_{\mu\leftarrow C}+B^2_{\nu\leftarrow D}+\epsilon_{CD}\right)} + \right.$$

$$\left. Z^1_C Z^2_D e^{-\beta\left(B^1_{\mu\leftarrow D}+B^2_{\nu\leftarrow C}+\epsilon_{DC}\right)} + Z^1_C Z^2_C e^{-\beta\left(B^1_{\mu\leftarrow D}+B^2_{\nu\leftarrow D}+\epsilon_{DD}\right)}\right). \tag{3.21}$$

$$\tag{3.22}$$

We can absorb the first term in eq. 3.22 by imposing that

$$Z_R = Z_\epsilon Z^1_C Z^1_D Z^2_C Z^2_D, \tag{3.23}$$

and we can rewrite everything in a more compact notation:

$$H^R_{\mu\nu} = -\frac{1}{\beta}\ln\left(\sum_{\mu'\nu'} Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} e^{-\beta\left(B^1_{\mu\leftarrow\mu'}+B^2_{\nu\leftarrow\nu'}+\epsilon_{\mu'\nu'}\right)}\right). \tag{3.24}$$

Here the bar indicates the oposite play, so that $\bar{C} = D$ and $\bar{D} = C$.

We should verify independently that 3.23 is indeed the partition function associated with $H^R_{\mu\nu}$. Plugging eq. 3.24 in eqs. 3.6 and 3.16, we have

$$\sum_{\mu\nu} \frac{e^{-\beta H^R_{\mu\nu}}}{Z_R} = 1 \Leftrightarrow Z_R = \sum_{\mu\nu} e^{-\beta H^R_{\mu\nu}} \tag{3.25}$$

$$\begin{aligned}
\Leftrightarrow Z_R &= \sum_{\mu\nu}\sum_{\mu'\nu'} Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} e^{-\beta\left(B^1_{\mu\leftarrow\mu'} + B^2_{\nu\leftarrow\nu'} + \epsilon_{\mu'\nu'}\right)} \\
&= \sum_{\mu'\nu'} Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} e^{-\beta\epsilon_{\mu'\nu'}} \underbrace{\sum_{\mu} e^{-\beta B^1_{\mu\leftarrow\mu'}}}_{Z^1_{\mu'}} \underbrace{\sum_{\nu} e^{-\beta B^2_{\nu\leftarrow\nu'}}}_{Z^2_{\nu'}} \\
&= \sum_{\mu'\nu'} Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} Z^1_{\mu'} Z^2_{\nu'} e^{-\beta\epsilon_{\mu'\nu'}}.
\end{aligned} \tag{3.26}$$

Because there are only two values for μ' and ν', $Z^1_{\bar{\mu}'} Z^2_{\bar{\nu}'} Z^1_{\mu'} Z^2_{\nu'}$ becomes independent of the indices and equal to $Z^1_C Z^2_C Z^1_D Z^2_D$. Introducing this result in eq. 3.26, we get

$$Z_R = Z^1_C Z^2_C Z^1_D Z^2_D \sum_{\mu'\nu'} e^{-\beta\epsilon_{\mu'\nu'}} = Z^1_C Z^2_C Z^1_D Z^2_D Z_\epsilon. \tag{3.27}$$

In the last step we use the definition of the partition function as generated directly by the set of energies $\epsilon_{\mu\nu}$ through PME.

We then see that the partition function that we obtained by imposing that the constant term vanish in eq. 3.23 is indeed the partition function of our new set of energies.

3.2.2 Follow/Not Follow for several CDs

In what follows we analyze several particular cases of $\epsilon_{\mu\nu}$, to which we make correspond the set of probabilities $p^\epsilon(\omega)$. We will analyze firstly $p^\epsilon(\omega) = p^0(\omega) = \frac{1}{4}$, which corresponds to $\epsilon_{\mu\nu} = C^0$. This energy distribution is equivalent to having an energy distribution that, when represented as an Ising energy, is simply $H^0_{\mu\nu} = E_0$. This is a particular case of an uncorrelated energy distribution.

Next, we study a more general set of uncorrelated probabilities (and energies), for which $p^\epsilon(\omega) = p^{Un}(\mu\nu) = P^1_0(\mu)P^2_0(\nu)$. In this case, the starting point is a separable set of probabilities, also described as a *mixed strategy*. The particular cases of interest will be the mixed strategy equilibria σ^* that we found for the **SD** game and the **BoS** game. We will connect these with the high temperature limits.

Finally, we will analyze what happens for $p^\epsilon(\omega) = p(\omega)$ as defined in tables 2.8 and 2.10, as particular cases of truly correlated sets of energies. Here we will study the effect on the final probabilities and payoffs when the players have a choice on top of a correlated set of probabilities. We will see the extremely correlated case of low temperature.

Equal initial energy for all states For $p^0(\omega) = \frac{1}{4}$, the probability that a player plays in a certain way becomes, from eq. 3.4,

$$P^i_\mu = \frac{1}{2}\sum_{\mu'} P^i_{\mu\leftarrow\mu'} = \frac{1}{2}\left(P^i_{F_\mu} + P^i_{NF_{\bar{\mu}}}\right) = \frac{1}{2}\left(1 + P^i_{F_\mu} - P^i_{F_{\bar{\mu}}}\right) \tag{3.28}$$

since the sum over ν' is just $\frac{1}{2}$.

In eq. 3.28 we see that a homogeneous initial energy distribution still allows for a different final probability distribution.

Using this, we can get a simple expression for the new energies:

$$H^{R_0}_{\mu\nu} = -\frac{1}{\beta}\ln\left(\frac{1}{4}\sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'}\right) - \frac{1}{\beta}\ln\left(Z_R\right) \tag{3.29}$$

$$= -\frac{1}{\beta}\ln\left(P^1_\mu P^2_\nu\right) - \frac{1}{\beta}\ln\left(Z_R\right) \tag{3.30}$$

$$= -\frac{1}{\beta}\ln\left(P^1_\mu\right) - \frac{1}{\beta}\ln\left(P^2_\nu\right) - \frac{1}{\beta}\ln\left(Z_R\right) \tag{3.31}$$

For each particular state we find

$$\begin{cases} H^{R_0}_{CC} = -\frac{1}{\beta}\left(\ln\left(P^1_C\right) + \ln\left(P^2_C\right) + \ln\left(Z_R\right)\right) \\ H^{R_0}_{CD} = -\frac{1}{\beta}\left(\ln\left(P^1_C\right) + \ln\left(P^2_D\right) + \ln\left(Z_R\right)\right) \\ H^{R_0}_{DC} = -\frac{1}{\beta}\left(\ln\left(P^1_D\right) + \ln\left(P^2_C\right) + \ln\left(Z_R\right)\right) \\ H^{R_0}_{DD} = -\frac{1}{\beta}\left(\ln\left(P^1_D\right) + \ln\left(P^2_D\right) + \ln\left(Z_R\right)\right) \end{cases}. \tag{3.32}$$

We see here that in this case the energies add up and are the energies for the uncorrelated game, which is best solved with a mixed stratey. Thus, the best payoff at these energies has to correspond to the mixed strategy equilibrium.

Uncorrelated initial energies For $p^{Un}(\mu\nu) = P^1_0(\mu)P^2_0(\nu)$, we can again calculate the probability that a player chooses a certain play:

$$P^i_\mu = \sum_{\mu'} P^i_{\mu\leftarrow\mu'} \sum_{\nu'} P^i_0(\mu')P^j_0(\nu') = \underbrace{\sum_{\nu'} P^j_0(\nu')}_{1} \sum_{\mu'} P^i_{\mu\leftarrow\mu'} P^i_0(\mu') = \sum_{\mu'} P^i_{\mu\leftarrow\mu'} P^i_0(\mu'). \tag{3.33}$$

In this way, the renormalized probability distribution is also separable:

$$p^{R_{Un}}(\mu\nu) = \sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'} p^{Un}(\mu'\nu') \tag{3.34}$$

$$= \sum_{\mu'} P^1_{\mu\leftarrow\mu'} P^1_0(\mu') \sum_{\nu'} P^2_{\nu\leftarrow\nu'} P^2_0(\nu') \tag{3.35}$$

$$= P^1_\mu P^2_\nu. \tag{3.36}$$

Because the uncorrelated structure is the same, we obtain the renormalized energies with 3.31.

This same result can represent the mixed strategy solutions for the **SD** and **BoS** games. Since the players have the highest payoff when they play those probabilities with probability one (in other words, *they always follow the mixed strategy*), $p^{R_{Un}} = p^{Un}$. This distribution also prescribes the same probability for both players under the same choice, due to the intrinsic symmetry of the game. This means that the renormalized probabilities are symmetric, which should also extend to the probabilities of following and not following. This will be useful as guideline for what form the probabilities should have in a limit when, starting from a correlated initial system, the parameter temperature allows for the initial probabilities to become uncorrelated.

Correlated initial energies For $p(\omega)$ we cannot separate the probabilities of playing C or D in a nice form as before. The form of the probability of a player playing something and of each of the new correlated probabilities are given, respectively, by eqs. 3.4 and 3.5. These are already in an irreducible form. Now, we can't use the separable structure for the renormalized energies, and we get

$$p^R(\mu\nu) = \sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'} p(\mu'\nu') \tag{3.37}$$
$$\neq \sum_{\mu',\nu'} P^1_{\mu\leftarrow\mu'} p(\mu'\nu') \sum_{\alpha',\beta'} P^2_{\alpha\leftarrow\alpha'} p(\alpha'\beta'). \tag{3.38}$$

We then see that *this uncorrelated transformation preserves the correlation properties of the original correlation matrix*.

Because of the temperature, we have a continuum of correlation matrices and follow/not follow probabilities, and therefore renormalized probabilities. Our question then becomes: **how can the players use their uncorrelated free choice, in order to get a better payoff?**

The average payoff function that the players will want to maximize is calculated with the new renormalized probabilities, applied to 2.71.

In our **SD** game, this is simply

$$\langle u_i^R(s^*_{SD})\rangle = p^R(CC) + p^R(CD)s + p^R(DC)t, \tag{3.39}$$

while for the **BoS** it is

$$\langle u_1^R(s^*_{BoS})\rangle = p^R(CC) + p^R(DD)s \tag{3.40}$$

and

$$\langle u_2^R(s^*_{BoS})\rangle = p^R(CC)s + p^R(DD). \tag{3.41}$$

Hence we see the need for the renormalization: we need to find a new set of correlated probabilities that the players always follow in order to have predictable statistics for the payoff. The renormalization does not change the relation of the probabilities with one another, which makes that, for the **SD**, $p^R(CD) = p^R(DC)$, and in the case of the **BoS** game, the payoffs of both players will be the same [2].

The new set of correlated conditions is given by the eqs. **??** for the **SD** by substituting by the renormalized probabilities,

$$\begin{cases} p^R(CC) + p^R(CD)s \geq p^R(CC)t \\ p^R(DC)t \geq p^R(DC) + p^R(DD)s, \end{cases} \tag{3.42}$$

while for the **BoS** they are given by the eqs. **??**, also by susbtituting by its respective normalized probabilities,

$$\begin{cases} p^R(CC) \geq p^R(CD)s \\ p^R(DD)s \geq p^R(DC) \end{cases}. \tag{3.43}$$

3.3 Temperature Limits

One of the features of this model, as has been stated before, is that it has a dependence on a temperature parameter, which conveys an infinite amount of possible correlation matrices. This temperature is imposed by the correlating device, that uses it to decide what the initial average energy of the system is. By playing with the follow/not follow probabilities, the players can change the average energy by changing the probabilities that preserve said average, by trying evaluating what set of probabilities gives them a better average payoff.

[2]The fact that our transformation preserves the symmetry of the probabilities comes from the fact that it does not introduce any permutations between them.

We now analyze what happens at the two most important temperature limits: when $T \to \infty$ and $T \to 0$[3].

3.3.1 High Temperature

When the temperature is high, the initial correlation probabilities converge to an homogeneous uncorrelated system, since $\lim_{\beta\to 0} e^{\beta H_{ij}} = 1$. This means that the players lose any information that they might have got from the CD. At this point, the introduction of the follow/not follow probabilities should predict what we already know that happens for this system: the players will want to play according with the mixed strategy equilibrium. This means that we want to equate eq. 3.28 with the mixed strategy probabilities that we found before for the symmetric and asymmetric games.

Snowdrift

From 2.52, we impose that the renormalized probabilities at high temperature correspond to the mixed strategy equilibrium profile given for this game in table 2.5:

$$\lim_{\beta\to 0} p^R(\mu\nu) = \frac{1}{4}\left(1 + P^1_{F_\mu} - P^1_{F_{\bar\mu}}\right)\left(1 + P^2_{F_\nu} - P^2_{F_{\bar\nu}}\right) \tag{3.44}$$

$$\Rightarrow \lim_{\beta\to 0} p^R(CC) = \frac{1}{4}\left(1 + P^1_{F_C} - P^1_{F_D}\right)\left(1 + P^2_{F_C} - P^2_{F_D}\right)$$

$$= \left(\frac{s}{s+t-1}\right)^2; \tag{3.45}$$

$$\lim_{\beta\to 0} p^R(DD) = \frac{1}{4}\left(1 + P^1_{F_D} - P^1_{F_C}\right)\left(1 + P^2_{F_D} - P^2_{F_C}\right)$$

$$= \left(\frac{t-1}{s+t-1}\right)^2; \tag{3.46}$$

$$\lim_{\beta\to 0} p^R(CD) = \frac{1}{4}\left(1 + P^1_{F_C} - P^1_{F_D}\right)\left(1 + P^2_{F_D} - P^2_{F_C}\right)$$

$$= \left(\frac{s}{s+t-1}\right)\left(\frac{t-1}{s+t-1}\right) \tag{3.47}$$

$$= \lim_{\beta\to 0} p^R(DC) = \frac{1}{4}\left(1 + P^1_{F_D} - P^1_{F_C}\right)\left(1 + P^2_{F_C} - P^2_{F_D}\right); \tag{3.48}$$

If this is the best result that the players can get with a homogeneous distribution, then to achieve it, in our formalism, we see that we must have $P^1_{F_C} - P^1_{F_D} = P^2_{F_C} - P^2_{F_D}$. To maintain the symmetry in between the players, we impose that

$$P^1_{F_\mu} = P^2_{F_\mu}. \tag{3.49}$$

Although this choice is not unique, it does solve the condition and it makes sense from the point of view that, in the original game, the players should be completely interchangeable. We will assume this relation from now on.

To find the values of $P^i_{F_C}$ and $P^i_{F_D}$ that give the payoffs of the mixed equilibrium, we use eq. 2.52 with eq. 3.28, and we get:

$$P^i_{F_C} = \frac{2s}{s+t-1} + P^i_{F_D} - 1. \tag{3.50}$$

There is a continuous range of values of follow C and D that give the mixed equilibrium payoff. However, the convergence to this payoff starting from lower temperatures should

[3] The fact that the follow/not follow probabilities can still range from 0 to 1 for any value of the temperature is due to a temperature-dependent component of B^i_μ. This temperature-dependent component allows for a component in the probability that is independent from it.

be continuous and this will inform us about a specific set of values for this probabilities that will be of interest to us.

Battle of the Sexes

For the **BoS** game we use eqs. 2.65 and 2.64 to obtain

From 2.52, similarly to the symmetric game, we want the renormalized probabilities at high temperature to correspond to the mixed strategy equilibrium, given for this game in table 2.6:

The high temperature effect on the correlations of the symmetric came are the same as for the uncorrelated:

$$\lim_{\beta\to 0} p^R(\mu\nu) = \frac{1}{4}\left(1+P^1_{F_\mu}-P^1_{F_{\bar\mu}}\right)\left(1+P^2_{F_\nu}-P^2_{F_{\bar\nu}}\right) \tag{3.51}$$

$$\begin{aligned}\Rightarrow \lim_{\beta\to 0} p^R(CC) &= \frac{1}{4}\left(1+P^1_{F_C}-P^1_{F_D}\right)\left(1+P^2_{F_C}-P^2_{F_D}\right)\\ &= \left(\frac{1}{s+1}\right)\left(\frac{s}{s+1}\right);\end{aligned} \tag{3.52}$$

$$\begin{aligned}\lim_{\beta\to 0} p^R(DD) &= \frac{1}{4}\left(1+P^1_{F_D}-P^1_{F_C}\right)\left(1+P^2_{F_D}-P^2_{F_C}\right)\\ &= \left(\frac{s}{s+1}\right)\left(\frac{1}{s+1}\right);\end{aligned} \tag{3.53}$$

$$\begin{aligned}\lim_{\beta\to 0} p^R(CD) &= \frac{1}{4}\left(1+P^1_{F_C}-P^1_{F_D}\right)\left(1+P^2_{F_D}-P^2_{F_C}\right)\\ &\left(\frac{1}{s+1}\right)^2\end{aligned} \tag{3.54}$$

$$\begin{aligned}\lim_{\beta\to 0} p^R(DC) &= \frac{1}{4}\left(1+P^1_{F_D}-P^1_{F_C}\right)\left(1+P^2_{F_C}-P^2_{F_D}\right)\\ &= \left(\frac{s}{s+1}\right)^2;\end{aligned} \tag{3.55}$$

To get a consistent result we see that we must have $P^1_{F_C}-P^1_{F_D} = P^2_{F_D}-P^2_{F_C}$ and $P^2_{F_C}-P^2_{F_D} = P^1_{F_D}-P^1_{F_C}$. We choose from symmetry arguments that $P^1_{F_C} = P^2_{F_D}$ and $P^1_{F_D} = P^2_{F_C}$, noting once again that this is not a necessary choice, but we will keep it from now on.

We obtain the values of $P^i_{F_C}$ and $P^i_{F_D}$ that give the payoffs of the mixed equilibrium of this game, we use eq. eqs. 2.65 and 2.64 with eq. 3.28, and we get:

$$P^1_{F_C} = \frac{2}{s+1} + P^1_{F_D} - 1 \tag{3.56}$$

and

$$P^2_{F_C} = \frac{2s}{s+1} + P^2_{F_D} - 1. \tag{3.57}$$

Again, there is a continuous range of values of follow C and D that arrive at the mixed equilibrium payoff, but we will find a specific set of probabilities compatible with it being the convergence limit at high temperatures.

3.3.2 Low Temperature

Snowdrift

Let us look at the shape that the renormalized probabilities of **SD** take for very low temperatures, or when $\beta \to \infty$. Because the exponential terms become very large, we will calculate the limits by dropping in turns the terms with lowest coefficient, which converge to infinity slower. The limits that we are about to calculate rely on the fact that $t > 1$ and $s > 0$. Because all the renormalized probabilities in this limit depend on the original probabilities in this same limit, we first calculate these.

$$\lim_{\beta\to\infty} p(CC) = \lim_{\beta\to\infty} \frac{e^{\beta}}{e^{\beta} + 2e^{\beta t} + e^{\beta(t-s)}} = \lim_{\beta\to\infty} \frac{e^{\beta}}{2e^{\beta t}} = \lim_{\beta\to\infty} \frac{1}{2} e^{\beta(1-t)} = 0; \tag{3.58}$$

$$\lim_{\beta\to\infty} p(DD) = \lim_{\beta\to\infty} \frac{e^{\beta(t-s)}}{e^{\beta} + 2e^{\beta t} + e^{\beta(t-s)}} = \lim_{\beta\to\infty} \frac{e^{\beta(t-s)}}{2e^{\beta t}} = \lim_{\beta\to\infty} \frac{1}{2} e^{\beta(-s)} = 0; \tag{3.59}$$

$$\lim_{\beta\to\infty} p(CD) = \lim_{\beta\to\infty} \frac{e^{\beta t}}{e^{\beta} + 2e^{\beta t} + e^{\beta(t-s)}} = \lim_{\beta\to\infty} \frac{e^{\beta t}}{2e^{\beta t}} = \frac{1}{2} \tag{3.60}$$

$$= \lim_{\beta\to\infty} p(DC); \tag{3.61}$$

We see that in the renormalized probabilities only the terms that depend on $p(CD)$ and $p(DC)$ remain.

$$\lim_{\beta\to\infty} p^R(CC) = \lim_{\beta\to\infty} (P^1_{F_C} P^2_{NF_D} p(CD) + P^1_{NF_D} P^2_{F_C} p(DC)) = \frac{1}{2}(P^1_{F_C} P^2_{NF_D} + P^1_{NF_D} P^2_{F_C}); \tag{3.62}$$

$$\lim_{\beta\to\infty} p^R(DD) = \lim_{\beta\to\infty} (P^1_{NF_C} P^2_{F_D} p(CD) + P^1_{F_D} P^2_{NF_C} p(DC)) = \frac{1}{2}(P^1_{NF_C} P^2_{F_D} + P^1_{F_D} P^2_{NF_C}); \tag{3.63}$$

$$\lim_{\beta\to\infty} p^R(CD) = \lim_{\beta\to\infty} (P^1_{F_C} P^2_{F_D} p(CD) + P^1_{NF_D} P^2_{NF_C} p(DC)) = \frac{1}{2}(P^1_{F_C} P^2_{F_D} + P^1_{NF_D} P^2_{NF_C}) \tag{3.64}$$

$$= \lim_{\beta\to\infty} p^R(DC), \tag{3.65}$$

where we assume in this context that P^i are always in this limit. Because of the symmetry of the game, we can drop the player indices.

The payoff of this game, given by eq. 3.39, reads as follows:

$$\lim_{\beta\to\infty} \langle u_i^R \rangle = P_{F_C} P_{NF_D} + \frac{s+t}{2} \left(P_{F_C} P_{F_D} + P_{NF_D} P_{NF_C} \right) \tag{3.66}$$

$$= P_{F_C} \left(1 - P_{F_D}\right) + \frac{s+t}{2} \left(P_{F_C} P_{F_D} + \left(1 - P_{F_D}\right) \left(1 - P_{F_C}\right) \right). \tag{3.67}$$

There are four limiting cases in this formula, corresponding to the P_{F_C} and P_{F_D} being 0 or 1.

$P_{F_C} = P_{F_D} = 0$:

$$\lim_{\beta\to\infty} \langle u_i \rangle = \frac{s+t}{2} \tag{3.68}$$

$P_{F_C} = P_{F_D} = 1$:

$$\lim_{\beta\to\infty} \langle u_i \rangle = \frac{s+t}{2} \tag{3.69}$$

$P_{F_C} = 1,\ P_{F_D} = 0$:

$$\lim_{\beta\to\infty} \langle u_i \rangle = 1 \tag{3.70}$$

$P_{F_C} = 0,\ P_{F_D} = 1$:

$$\lim_{\beta\to\infty} \langle u_i \rangle = 0 \tag{3.71}$$

We see that, if $s + t < 2$, the best payoff is found in the limits where they always follow or never follow. If $s + t \leq 2$, then the best value is found when the players always follow C, but never follow D. However, this is outside of the first condition of equilibrium, as given by the first equation in 3.42, which would require that $t \leq 1$.

However, it is a good guess that, for $s < 2 - t$, the payoff increases along the line of $P_{F_C} = 1$, while P_{F_D} decreases until the equilibrium line is met, or along the line of $P_{F_D} = 0$, while P_{F_C} increases. The first equilibrium condition, from 3.42, in the low temperature limit, is

$$P_{F_C}(1 - P_{F_D}) + \left(P_{F_C}P_{F_D} + \left(1 - P_{F_D}\right)\left(1 - P_{F_C}\right)\right)\frac{s}{2} \geq P_{F_C}(1 - P_{F_D})t. \tag{3.72}$$

Solving for the equality we get

$$P_{F_C}\left(1 - P_{F_D}\right)(t - 1) - \frac{s}{2}\left(1 - 2P_{F_D}\right) = \left(1 - P_{F_D}\right)\frac{s}{2}. \tag{3.73}$$

For $P_{F_C} = 1$, the solution is given by

$$P_{F_D} = \frac{2t - 2}{2t + s - 2}. \tag{3.74}$$

We can also find a solution along the line of $P_{F_D} = 0$, for which

$$P_{F_C} = \frac{s}{2t + s - 2}. \tag{3.75}$$

The same payoff is obtained by plugging either pair of P_{F_C} and P_{F_D} in eq. 3.67:

$$\lim_{\beta\to\infty} \langle u_i^R \rangle = \frac{t(t + s - 1)}{2t + s - 2} \tag{3.76}$$

To help us visualize these points in the plots of later sections in the $P_{F_C} - P_{F_D}$ plane, we calculate the line that connects them:

$$P_{F_C} = P_{F_D} + \frac{s}{2t + s - 2} \tag{3.77}$$

Battle of the Sexes

We follow the same calculations here as for the **SD**, changing however the correlating device probabilities that we refer to, as the equilibrium condition.

We first calculate the limits when $\beta \to \infty$ for the correting device probabilities:

$$\lim_{\beta\to\infty} p(CC) = \lim_{\beta\to\infty} \frac{e^{\beta(1+s)}}{2e^{\beta(1+s)} + 1} = 1/2 = \lim_{\beta\to\infty} p(DD); \tag{3.78}$$

$$\lim_{\beta\to\infty} p(CD) = \lim_{\beta\to\infty} \frac{1}{2e^{\beta(1+s)} + 1} = 0 = \lim_{\beta\to\infty} p(DC). \tag{3.79}$$

Here we see that only the terms in the renormalized probabilities that depend on the diagonal initial probabilities will be represented.

$$\lim_{\beta\to\infty} p^R(CC) = \lim_{\beta\to\infty}(P^1_{F_C}P^2_{F_C}p(CC) + P^1_{NF_D}P^2_{NF_D}p(DD)) = \frac{1}{2}(P^1_{F_C}P^2_{F_C} + P^1_{NF_D}P^2_{NF_D}); \quad (3.80)$$

$$\lim_{\beta\to\infty} p^R(DD) = \lim_{\beta\to\infty}(P^1_{NF_C}P^2_{NF_C}p(CC) + P^1_{F_D}P^2_{F_D}p(DD)) = \frac{1}{2}(P^1_{NF_C}P^2_{NF_C} + P^1_{F_D}P^2_{F_D}); \quad (3.81)$$

$$\lim_{\beta\to\infty} p^R(CD) = \lim_{\beta\to\infty}(P^1_{F_C}P^2_{NF_C}p(CC) + P^1_{NF_D}P^2_{F_D}p(DD)) = \frac{1}{2}(P^1_{F_C}P^2_{NF_C} + P^1_{NF_D}P^2_{F_D}); \quad (3.82)$$

$$\lim_{\beta\to\infty} p^R(DC) = \lim_{\beta\to\infty}(P^1_{NF_C}P^2_{F_C}p(CC) + P^1_{F_D}P^2_{NF_D}p(DD)) = \frac{1}{2}(P^1_{NF_C}P^2_{F_C} + P^1_{F_D}P^2_{NF_D}); \quad (3.83)$$

where we assume in this context that P^i are always in this limit. Looking at the asymmetry of the game, we can rewrite the probabilities of player 2 as a function of those of player 1. Because the probabilities $p(CC)$ and $p(DD)$ are the same, we see that $p^R(CC)$ will still be equal to $p^R(DD)$, even outside of this limit, as we can see in the first two equations above. This indicates the more general feature that the payoffs of both players will be the same.

Since the average payoffs, given by eqs. 3.40 and 3.41, only depend on $p^R(CC)$ and $p^R(DD)$, we only need to look further at these two:

$$\lim_{\beta\to\infty} p^R(CC) = \frac{1}{2}(P^1_{F_C}P^1_{F_D} + P^1_{NF_D}P^1_{NF_C}); \quad (3.84)$$

$$\lim_{\beta\to\infty} p^R(DD) = \frac{1}{2}(P^1_{NF_C}P^1_{NF_D} + P^1_{F_D}P^1_{F_C}), \quad (3.85)$$

which are indeed equal to each other.

The payoff of the players is then, as a function of the follow and not follow probabilities of player 1,

$$\lim_{\beta\to\infty}\langle u_i^R\rangle = \frac{1}{2}(1+s)\left(P^1_{F_C}P^1_{F_D} + P^1_{NF_C}P^1_{NF_D}\right) = \frac{1}{2}(1+s)\left(2P^1_{F_C}P^1_{F_D} + 1 - P^1_{F_D} - P^1_{F_C}\right). \quad (3.86)$$

Here the best value of the payoff is simply

$$\lim_{\beta\to\infty}\langle u_i^R\rangle = \frac{1}{2}(1+s), \quad (3.87)$$

which happens when

$$P^1_{F_C} = \frac{P^1_{F_D}}{2P^1_{F_D} - 1}. \quad (3.88)$$

In our domain for the probabilities, this condition is only met when player one always follows ($P^1_{F_C} = P^1_{F_D} = 1$) or never follows ($P^1_{F_C} = P^1_{F_D} = 0$); these are exactly the same conditions one would obtain for player 2 given that now we are working with $P^1_{F_\mu} = P^2_{F_{\bar{\mu}}}$, resulting in the same payoff.

We can see that the best payoff here happens in the same way for always follow or never follow because of the symmetry in the energy landscape: since changing these two profiles corresponds to changing $p^R(CC)$ with $p^R(DD)$, the payoffs are the same because these probabilities equal to each other, as were in the original CD.

Thus, contrary to the **SD** game, here we don't run into the equilibrium conditions at zero temperature. However, as we will see, they will be important for higher temperatures.

Chapter 4

Equilibrium Conditions

4.1 Equal probability of following C and D for Snowdrift

We will start by studying a simple set of renormalized probabilities: the probability of following C is the same as the probability of following D. Here, we will pay special attention to the cases where either the players always follow both C and D, or they never follow any of the recommendations. If the players always follow, it means that their renormalized probabilities become equal to the original, $p^{R_F} = p(\omega)$. If they never follow, then $p^{R_{NF}}(\mu\nu) = p(\nu\mu)$.

As a toy model, we do this analysis on the **SD** game, which will help us look at what happens when we give more freedom to the follow/not follow probabilities. Because of the symmetry of the players, we drop the player indices. In this simplified model we will see how we will work with the system in a correlated equilibrium, which will be a necessary condition if we then want to vary the follow/not follow probabilities to obtain the best payoff. In this section we don't look at the **BoS** game, but a similar reasoning would apply.

The equilibrium conditions depend heavily on the relation between the value of the parameters $s+1$ and t. If $s+1=t$, then the first equation becomes $p^R(CD) \geq p^R(CC)$, and the second $p^R(DC) \geq p^R(DD)$. Then we can distinguish between $s+1>t$ and $s+1<t$, as these cases will give more or less weight to either the left- or right-hand sides of the equations. We will look at this situation for $t=1.2$, due to the fact that for this value the critial features happen for low temperatures.

In fig. 4.1 we see the left and right hand sides of the conditions in eqs. 3.42, what we have labeled the *first* and *second* condition. Through the rest of this thesis, we will use the analysis that follows, for these two conditions (naturally considering the game in question). In that figure, and the remaining ones of the same form, the color scheme is as follows: we compare blue with orange (first condition) and green with red (second condition), and for the conditions to be met, blue should be above orange and green above red. In the first row, for $s > t-1$, we see that after a certain temperature the second condition is broken. Since we have to have both conditions being met at the same time, this means that after a certain temperature we can no longer establish a correlated equilibrium for those values of P_F.

When $s = t-1$, in the second row of the figure, we have the conditions being met for the whole temperature distribution. We also see that in this situation the plots are the same for always follow or always not follow: this happens because $p^{R_{NF}}(CC) = p(DD)$, and for these parameters, $p(CC) = p(DD)$. In the third row, $s < t-1$, we see that it is the first condition that starts to be broken a sufficiently high temperature.

If we look at fig. 4.2, we see that, away from the limiting values of P_{F_C}, the condition that was broken only up to a certain temperature is now broken for all temperatures, indicating that we can never find a correlated equilibrium away from the extreme values of P_{F_C} if $P_{F_C} = P_{F_D}$.

We can gather this information in a more compact way by calculating the solutions for P_F at the equality of the conditions. Solving analytically in *Mathematica*, we see that each condition will have two solutions. Some parts of these solutions won't have real values in the domain of P_F, and so don't represent a change in equilibrium. The real parts of these solutions are not conflicting in the P_F domain, so we define a unique real solution $P^*_{F_{1st}}(s,t,T)$

for the first condition, and $P^*_{F_{2nd}}(s,t,T)$ for the second condition, formed by branches. These branches represent the regions in P_F where each one of those solutions is real.

Definition 1. *$P^*_{F_{1st}}(s,t,T)$ is the real valued solution of the first equilibrium condition of the* **SD** *game, given by $p^R(CC)+p^R(CD)s = p^R(CC)$, in the interval $[0,1]$, when $P_{F_C} = P_{F_D}$.*

Definition 2. *$P^*_{F_{2nd}}(s,t,T)$ is the real valued solution of the second equilibrium condition of the* **SD** *game given by $p^R(DC)t = p^R(DC)+p^R(DD)s$ in the interval $[0,1]$, when $P_{F_C} = P_{F_D}$.*

All the plots that we will present of these solutions correspond to the analytic forms obtained by *Mathematica*. We exclude the introduction of the full form for shortness.

Fig. 4.3 gives more insight into the regions for which the equilibrium is stable, by showing the plots of $P^*_{F_{1st}}$ and $P^*_{F_{2nd}}$ as a function of T. For $s < t-1$, only the solutions of the first condition indicate a change in stability, which means that it is always stable under regarding the second condition; between the upper curve and $P_F = 1$ and $P_F = 0$ the first condition is stable as well, and so the game can be played with P_F in these regions. This is compatible with the reading of figs. 4.1 and 4.2. Also compatible are the results for $s > t-1$, in which only the second condition is broken and there is a small range of values between the the upper curve and $P_F = 1$ and between $P_F = 0$ and the lower curve that allow for a correlated equilibrium and that correspond to the region where the second condition is met; here the first condition is always met.

Looking at this system, where they follow C and D with the same probability, we see that, for a given temperature, there won't be always a chance to play something that doesn't bring the system out of equilibrium; for that the probabilities that they follow C will have to be different than t follow D , as we are about to see [1].

4.2 Different probabilities of following C and D

If we allow P_{F_C} to be different from P_{F_D}, we find values that the players could adopt for all the temperature range. We will analyze in detail some specific values for the **SD** game, and once more drop the player indices in some length. After that, we do a shorter analysis for **BoS**.

4.2.1 Snowdrift

Following the analysis in the previous case, we want to see what happens for different parameters s and t, but now for P_{F_C} different than P_{F_D}. We investigate how the two sides of the inequalities of the equilibrium conditions in 3.42 behave, to get a sense of where the equilibrium breaks.

Then we equate both sides of the equilibrium conditions and we see that the curves that we get span over a bigger range of temperatures than before. Similarly to when $P_{F_C} = P_{F_D}$, there will be two solutions for each condition, and only the branches that are inside of the domain and are real valued represent changes in the equilibrium.

Definition 3. *$P^*_{FC_{1st}}(P_{F_D},s,t,T)$ is the real valued solution of the first equilibrium condition of the* **SD** *game, given by $p^R(CC)+p^R(CD)s = p^R(CC)$, in the interval $[0,1]$.*

Definition 4. *$P^*_{FC_{2nd}}(P_{F_D},s,t,T)$ is the real valued solution of the second equilibrium condition of the* **SD** *game given by $p^R(DC)t = p^R(DC)+p^R(DD)s$ in the interval $[0,1]$, when $P_{F_C} = P_{F_D}$.*

We will see how these functions behave depending on value of the parameters s and t; we will do a detailed analysis for particular examples of $s > t-1$, $s < t-1$; for the former we use $s = 0.5$ and $t = 1.6$, and for the latter we use $s = 0.5$ and $t = 1.2$. We change the case study for $s < t-1$ in relation to the previous section because the visualization is more clear, however the conclusions are fully extendable. In the $s > t-1$ case, the results can be looked at back to back.

[1] The fact that the maximum allowed temperature in these conditions is limited is compatible with the mixed strategy limit that we want to arrive at: in high temperatures, we want the probabilities of playing each C or D to be different.

$s > t - 1$

In figs. 4.4 to 4.10 we can see the left- and right-hand sides of the equilibrium conditions for increasing values of P_{F_C}, changing with the values of P_{F_D} for a certain temperature range. Throughout this range, the first condition is true for most of the values of P_{F_C} and P_{F_D}, being broken only when high P_{F_C} low P_{F_D}, at the same time.

The second condition is only true for low temperatures, if at all. While P_{F_C} is small, the second condition is only true for equally small values of P_{F_D}, and the first condition is always true. For high P_{F_C}, the first condition can be broken for low P_{F_C}, but as this variable increases, only the second condition becomes breakable. These results are better summarized in fig. 4.11, where we can see the solutions $P^*_{FC_{1st}}$ and $P^*_{FC_{2nd}}$ as a function of temperature, for several values of P_{F_D}. Comparing with the previous figures, we see that **the equilibrium is stable above the green line an bellow the yellow line**.

Comparing fig. 4.3b with fig. 4.11, we see that they are consistent, for $P_{F_D} = P_{F_C}$: when the probability of following C is the same as that of following D, there is only equilibrium for values very close to either 0 or 1, and this equilibrium is only found for a limited range of temperatures, always broken at the second condition.

In fig. 4.12 we depict $P^*_{FC_{1st}}$ and $P^*_{FC_{2nd}}$ as a function of P_{F_D}, for several temperatures, on top of the payoff function for the same parameters, given by 3.39.

The equilibrium region is bounded from bellow by $P^*_{FC_{1st}}$ and from above by $P^*_{FC_{2nd}}$. The payoff values are higher for high P_{F_C} and low P_{F_D}, and decrease until they reach the minimum for high P_{F_D} and low P_{F_C}. Knowing this, it is expected that the higher payoffs inside the equilibrium are coincident with $P^*_{FC_{1st}}$; especially, we expect this behaviour for low temperatures, because in this case $s < 2 - t$, in accordance with the discussion in 3.3.2.

Because this will a characteristic feature of the parameters, we look now at the point where $P^*_{FC_{1st}} = P^*_{FC_{2nd}}$.

Definition 5. *$P^*_{F_D}(s, t, T)$ is the solution of $P^*_{FC_{1st}} = P^*_{FC_{2nd}}$ for P_{F_D}.*

Definition 6. *$P^*_{F_C}(s, t, T)$ is the value of $P^*_{FC_{1st}}(P^*_{F_D}, s, t, T)$.*

In this case, we see in figure 4.12 that $P^*_{F_C} > P^*_{F_D}$, and that this point seems to be along the line of the mixed strategy solution, given by eq. 3.50 and represented in gray, a fact that is independent from the temperature. Because of this, we note something interesting along $P^*_{FC_{2nd}}$: its two extremes change, as the temperature increases, from being in $P_{F_C} = 0$ and $P_{F_D} = 1$, to $P_{F_D} = 0$ and $P_{F_D} = 1$ and subsequently to $P_{F_D} = 0$ and $P_{F_C} = 1$.

As the temperature increases, we see that both $P^*_{FC_{1st}}$ and $P^*_{FC_{2nd}}$ approach the mixed strategy equilibrium, so they also approach each other. What we are interested in knowing now is what values of P_{F_C} and P_{F_D} give the highest payoff for a given temperature, inside the interval bounded by the equality solutions of the conditions. As a final consistency remark, we see that only for low temperatures the $P_{F_C} = P_{F_D}$ line (in blue) is inside the equilibrium condition, and only at very low or very high P_F values, always crossing $P^*_{FC_{2nd}}$ when it stops being stable, just as in fig. 4.3b.

$s < t - 1$

We will now explore what happens for $s = 0.5$ and $t = 1.6$, as an example of $s < t - 1$. In figs. 4.13 to 4.19 we can see, similarly to the previous analysis, the left- and right-hand sides of the equilibrium conditions. Unlike the previous case, it is the first condition that is broken when we have P_{F_C} very close to P_{F_D}. This is more clear in fig. 4.20, where we can see $P^*_{FC_{1nd}}$ and $P^*_{FC_{2nd}}$ as a function of temperature. Again we see that the equilibrium is bounded from bellow by $P^*_{FC_{1st}}$ and from above by $P^*_{FC_{2nd}}$. This result is qualitatively compatible with fig. **??**, although the specific values of the parameters are not the same, which makes for a stronger claim that this relation in the parameters has universal characteristics.

In fig. 4.21 we see $P^*_{FC_{2nd}}$ and $P^*_{FC_{1nd}}$ plotted as a function of P_{F_D}, for several temperatures, on top of the payoff, given by 3.39 for the same parameters and temperatures.

Contrary to the previous situation, now we have that $P^*_{F_C} < P^*_{F_D}$, but also on top of the mixed strategy line. This means that it will be $P^*_{FC_{1nd}}$ changing characteristics as temperature increases: while it starts by having its extremes in $P_{F_C} = 1$ and $P_{F_D} = 0$, the latter eventually crosses to $P_{F_D} = 0$ and the former crosses to $P_{F_D} = 1$, which means that the $P_{F_C} = P_{F_D}$ line is now out of the equilibrium interval. This is consistent with having the stability along this line breaking the second equilibrium condition, for finite temperature, at different temperatures depending on if the values are closer to $P_F = 0$ or $P_F = 1$, as we see in fig. **??**. Because $P^*_{FC_{1nd}}$ corresponds to higher payoffs, this change in the location of its edges *will be representative of a change of where to find the maximum payoff*. Because in this particular case $s > 2 + t$, we expect the maximum value of the payoff in low temperatures to be in always follow or never follow; in fact, looking at fig. 4.21a, we see that the off-diagonal corners represent a very high payoff.

4.2.2 Battle of the Sexes

For the **BoS** game, we define the solutions at the equalities of the equilibrium conditions in eq. 3.43.

Definition 7. *$P^{*BoS}_{FC_{1st}}(P_{F_D}, s, t, T)$ is the real valued solution of the first equilibrium condition of the **BoS** game, given by $p^R(CC) = p^R(CD)s$, in the interval $[0, 1]$.*

Definition 8. *$P^{*BoS}_{FC_{2nd}}(P_{F_D}, s, t, T)$ is the real valued solution of the first equilibrium condition of the **BoS** game, given by $p^R(DD)s = p^R(DC)s$, in the interval $[0, 1]$.*

Definition 9. *$P^{*BoS}_{F_D}(s, t, T)$ is the solution of $P^{*BoS}_{FC_{1st}} = P^{*BoS}_{FC_{2nd}}$ for P_{F_D}.*

Definition 10. *$P^{*BoS}_{F_C}(s, t, T)$ is the value of $P^*_{FC_{1st}}(P^{*BoS}_{F_D}, s, t, T)$.*

In fig. 4.22 we see $P^{*BoS}_{FC_{1st}}$ and $P^{*BoS}_{FC_{2nd}}$ represented as function of temperature, over the payoff function for the same parameters. For $s < 1$, we see that $P^{*BoS}_{F_C} > P^{*BoS}_{F_D}$; for $s = 1$, we have that $P^{*BoS}_{F_C} = P^{*BoS}_{F_D}$, and for $s = 1$, $P^{*BoS}_{F_C} = P^{*BoS}_{F_D}$. The always follow or never follow corners will represent the low temperature best payoffs (pink line), while the correlated equilibrium curves converge to the mixed strategy equilibrium (gray). In the cases where $s \neq 1$, the always follow and never follow corners become out of equilibrium.

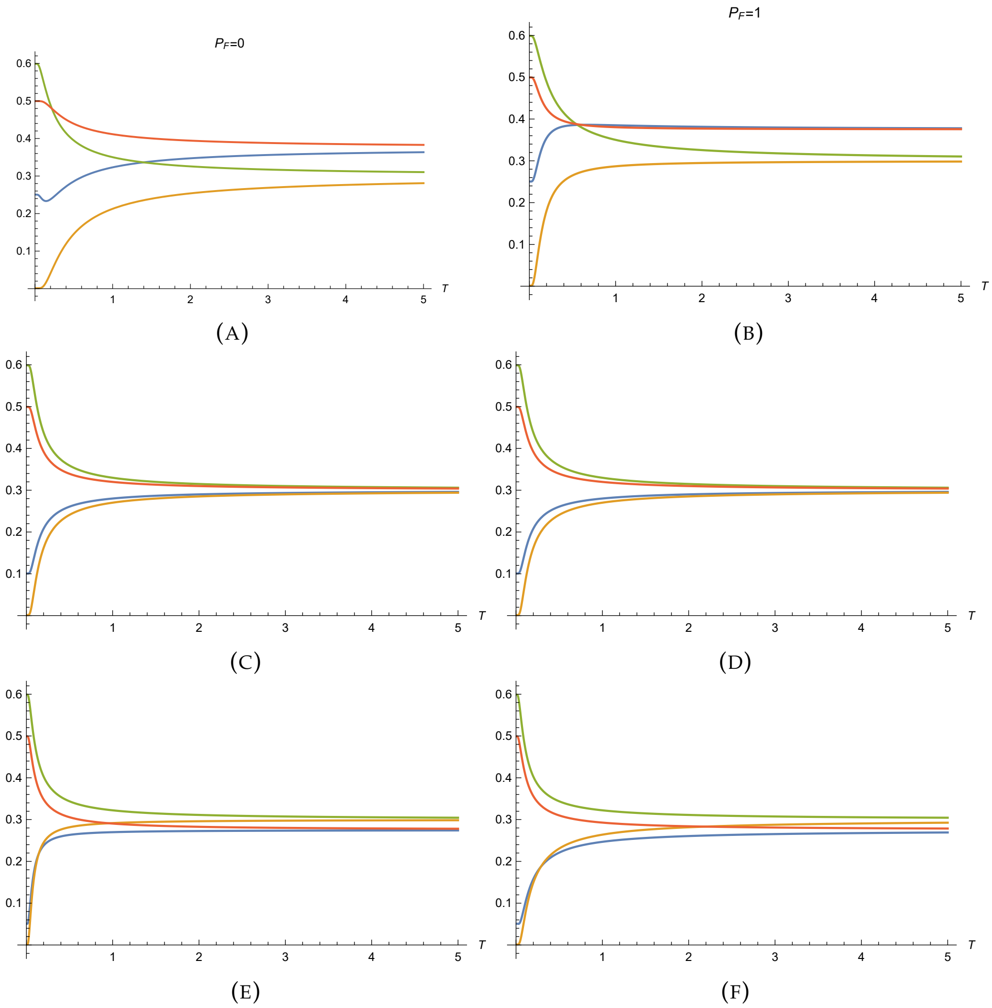


FIGURE 4.1: LHS and RHS of first and second equilibium conditions, given by eq. 3.42, at extreme values of P_F and $t = 1.2$. In the first collumn:$P_{F_C} = P_{F_D} = 0$; in the second column: $P_{F_C} = P_{F_D} = 1$. (A) and (B): $s = 0.5$; (C) and (D): $s = 0.2$; (E) and (F): $s = 0.1$. In blue: LHS of 1st condition; in orange: RHS of 1st condition. In green: LHS of 2nd condition; in red: RHS of 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. For (A) and (B), which represents $s > t - 1$, only the second condition stops being met for a certain temperature, that changes according with the value of P_F. For (C) and (D), or $s = t - 1$, the conditions are met for all temperatures. For (E) and (F), that has $s < t - 1$, it is the first condition that stops being met for a certain temperature, also varying with P_F.

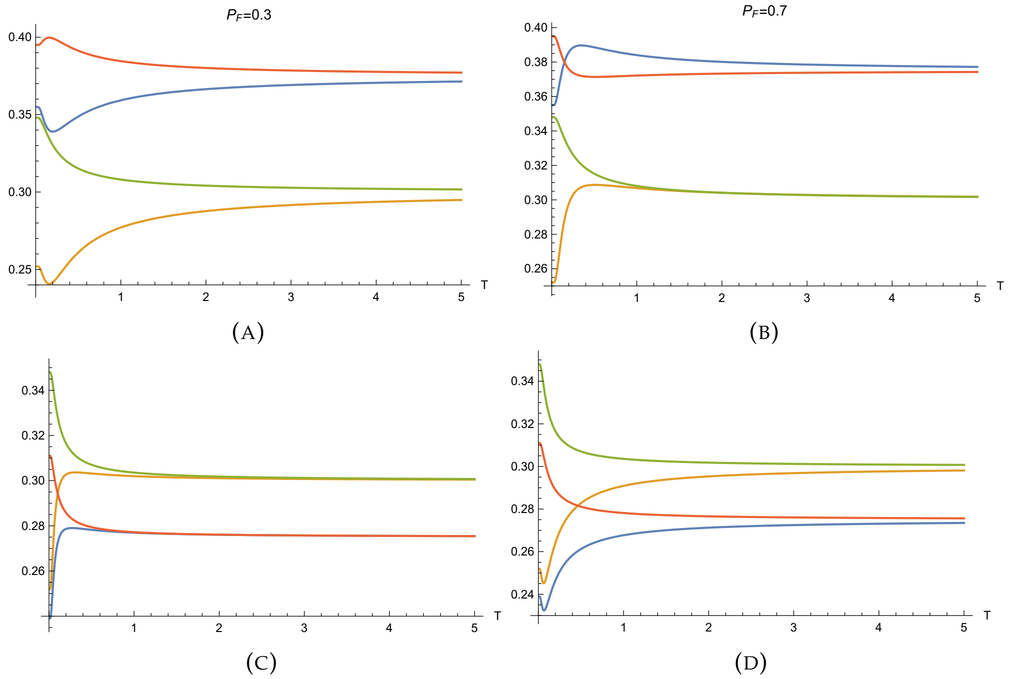


FIGURE 4.2: LHS and RHS of first and second equilibrium conditions, given by eq. 3.42, at intermediate values of P_F and $t = 1.2$. In the first column: $P_{F_C} = P_{F_D} = 0.3$; in the second column: $P_{F_C} = P_{F_D} = 0.7$. (A) and (B): $s = 0.5$; (C) and (D): $s = 0.2$. In blue: LHS of 1st condition; in orange: RHS of 1st condition. In green: LHS of 2nd condition; in red: RHS of 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. For (A) and (B), which represents $s > t - 1$, the first condition is always met, and the second is always broken, throughout the temperature range. For (C) and (D), it is the second condition that is always met instead, whereas the first is always broken.

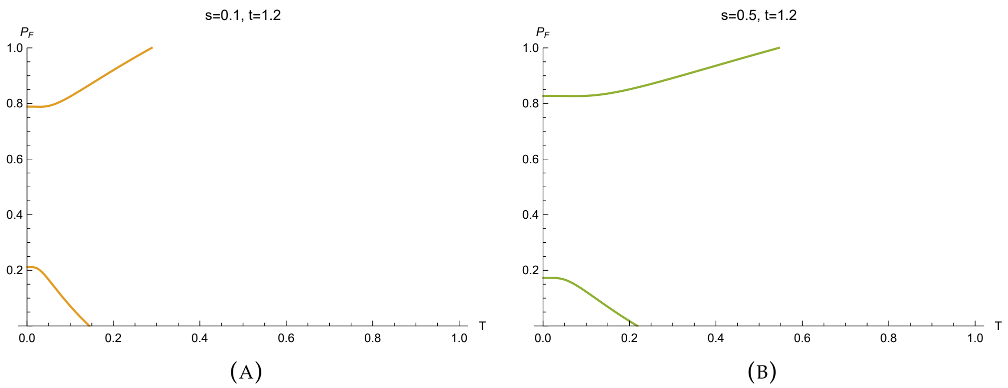


FIGURE 4.3: $P^*_{F_{1st}}$ (orange) and $P^*_{F_{2nd}}$ (green) for (A) $s = 0.1$ and $t = 1.2$ and for (B) $s = 0.5$ and $t = 1.2$, for SD game. For $s > t - 1$, only the second condition has real solutions, while for $s < t - 1$, only the first condition has solutions. The stability region is between the lines and upper curve and $P_F = 1$, and between the lower curve and $P_F = 0$.

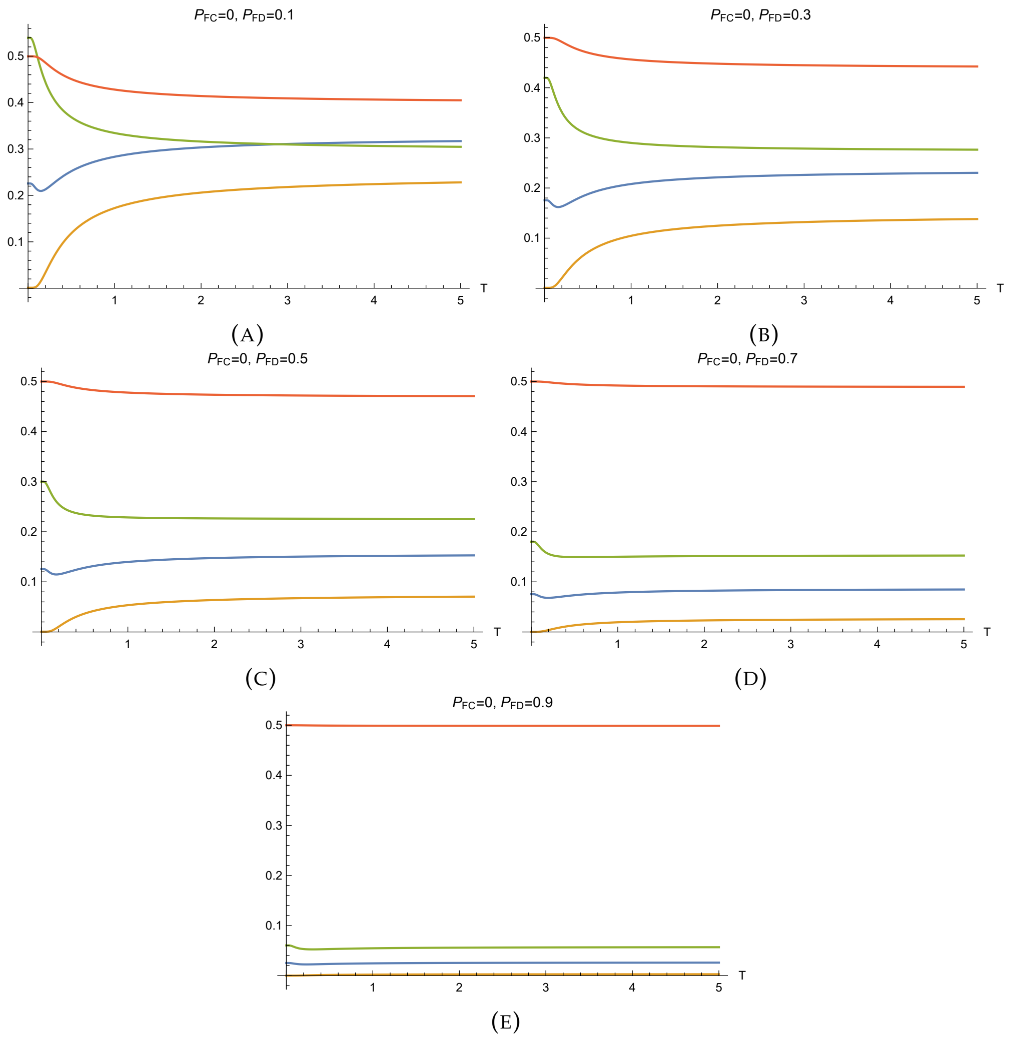


FIGURE 4.4: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

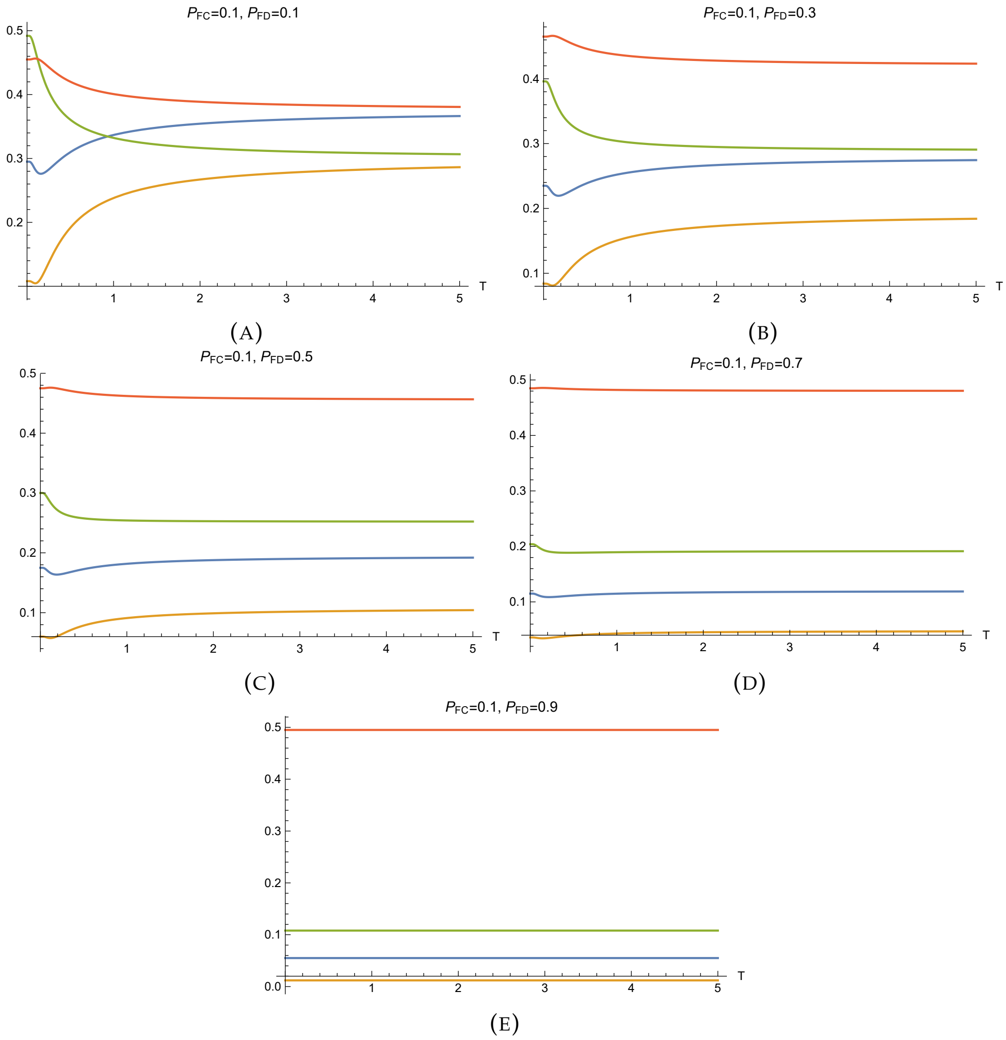


FIGURE 4.5: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.1$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

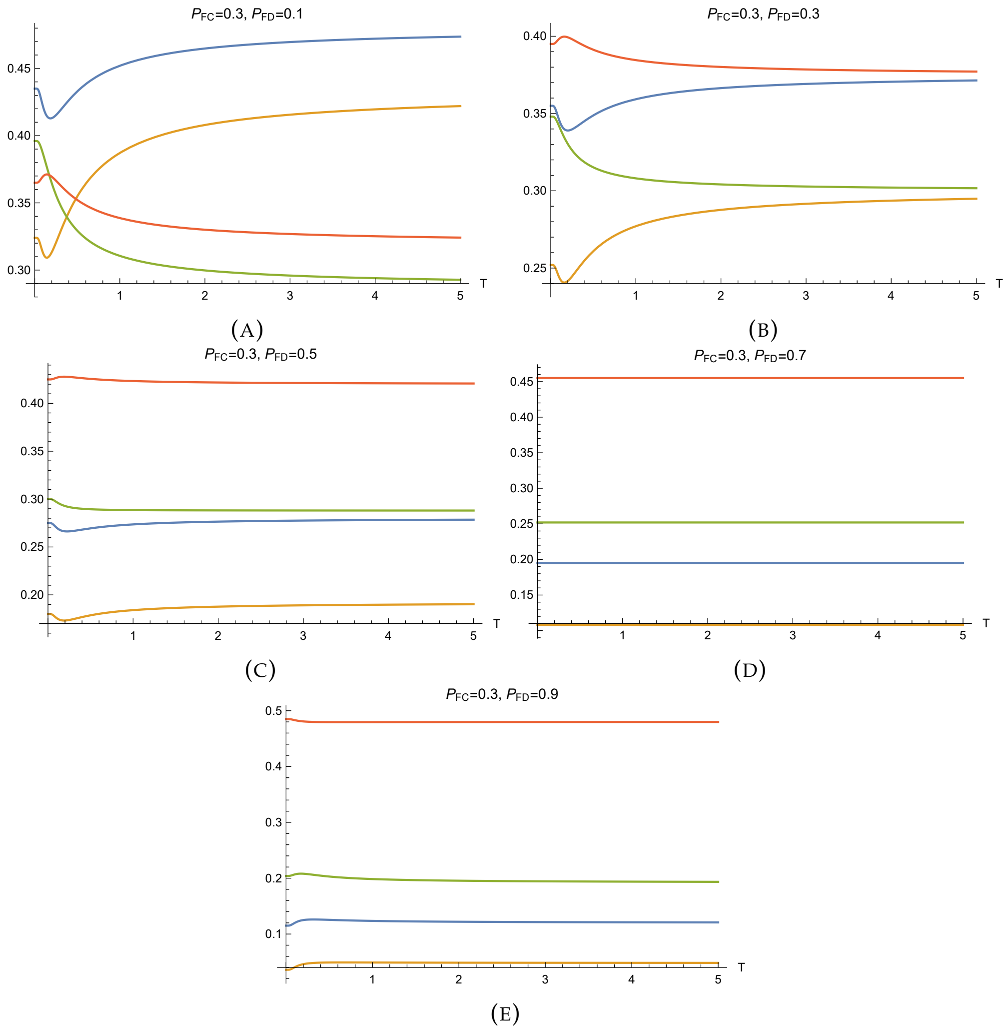


FIGURE 4.6: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.3$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

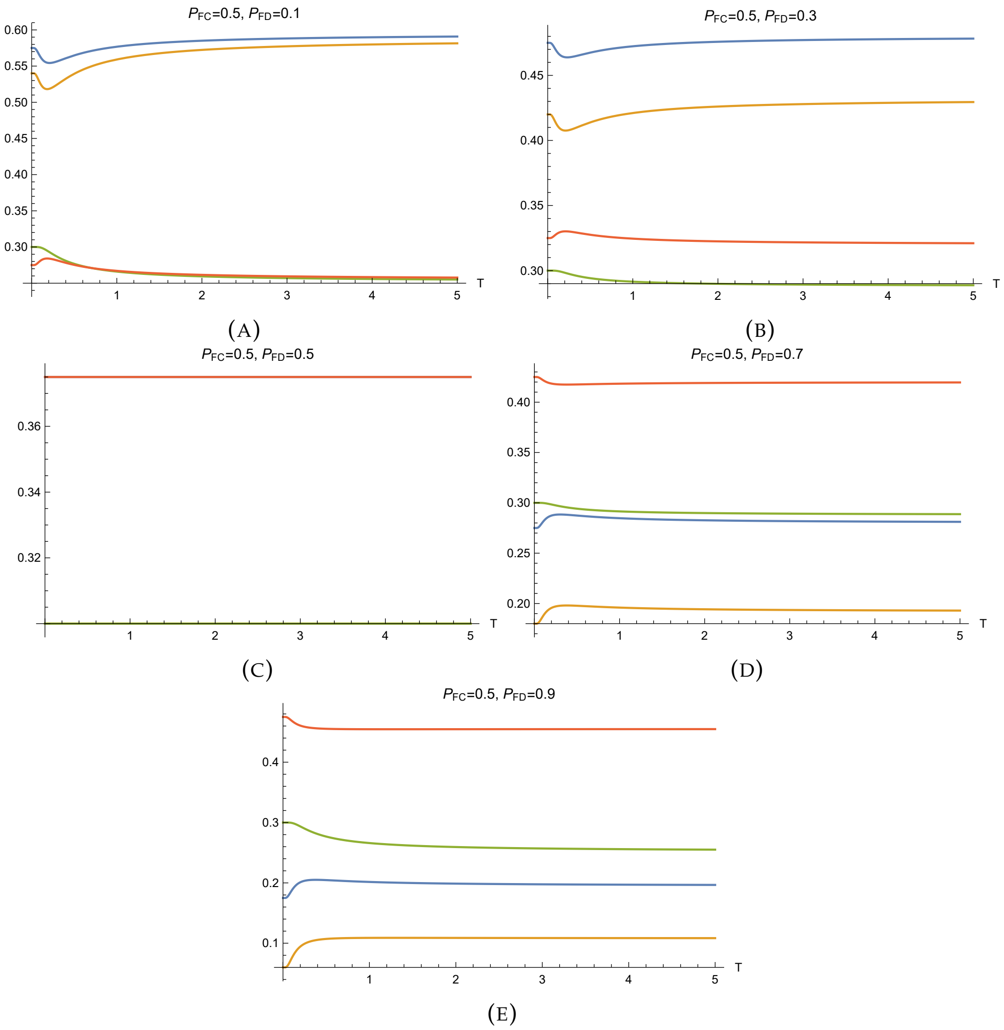


FIGURE 4.7: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.5$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

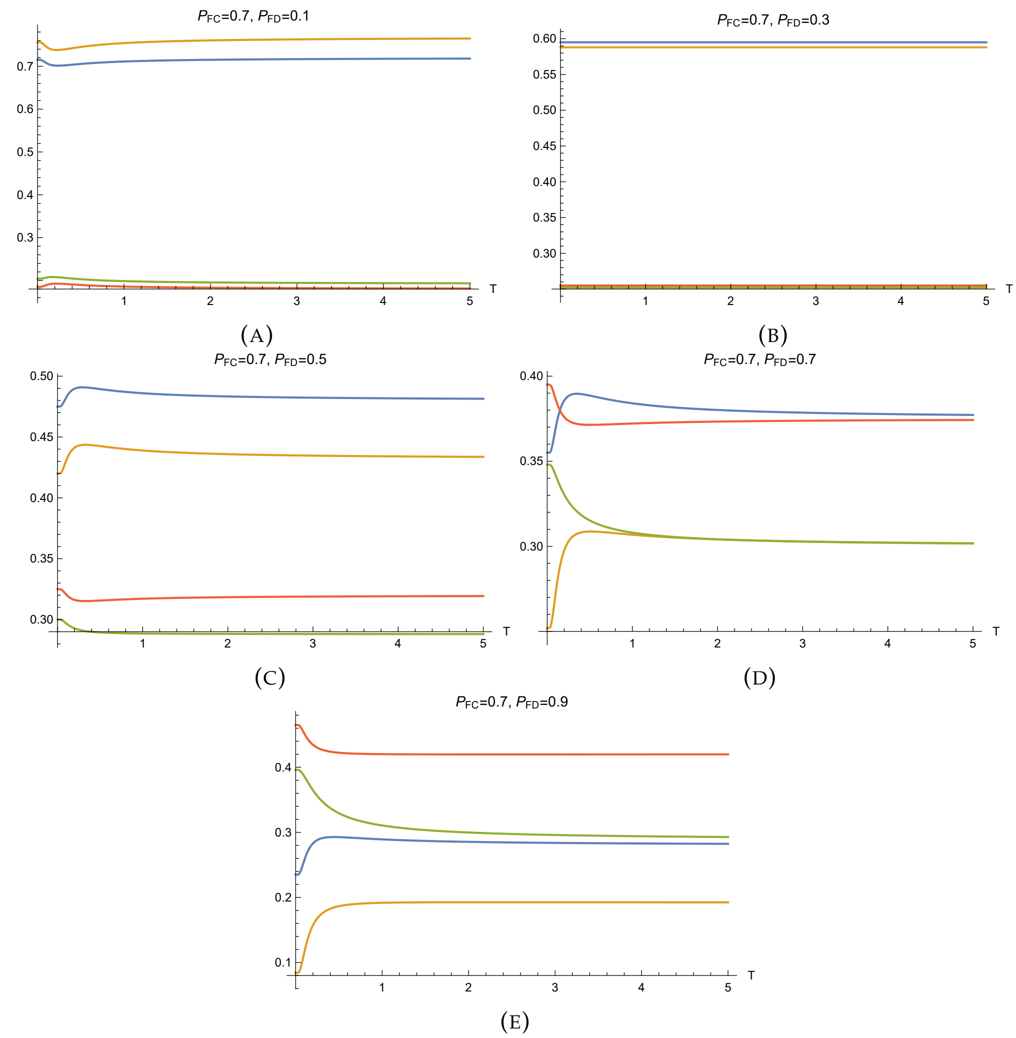


FIGURE 4.8: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.7$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

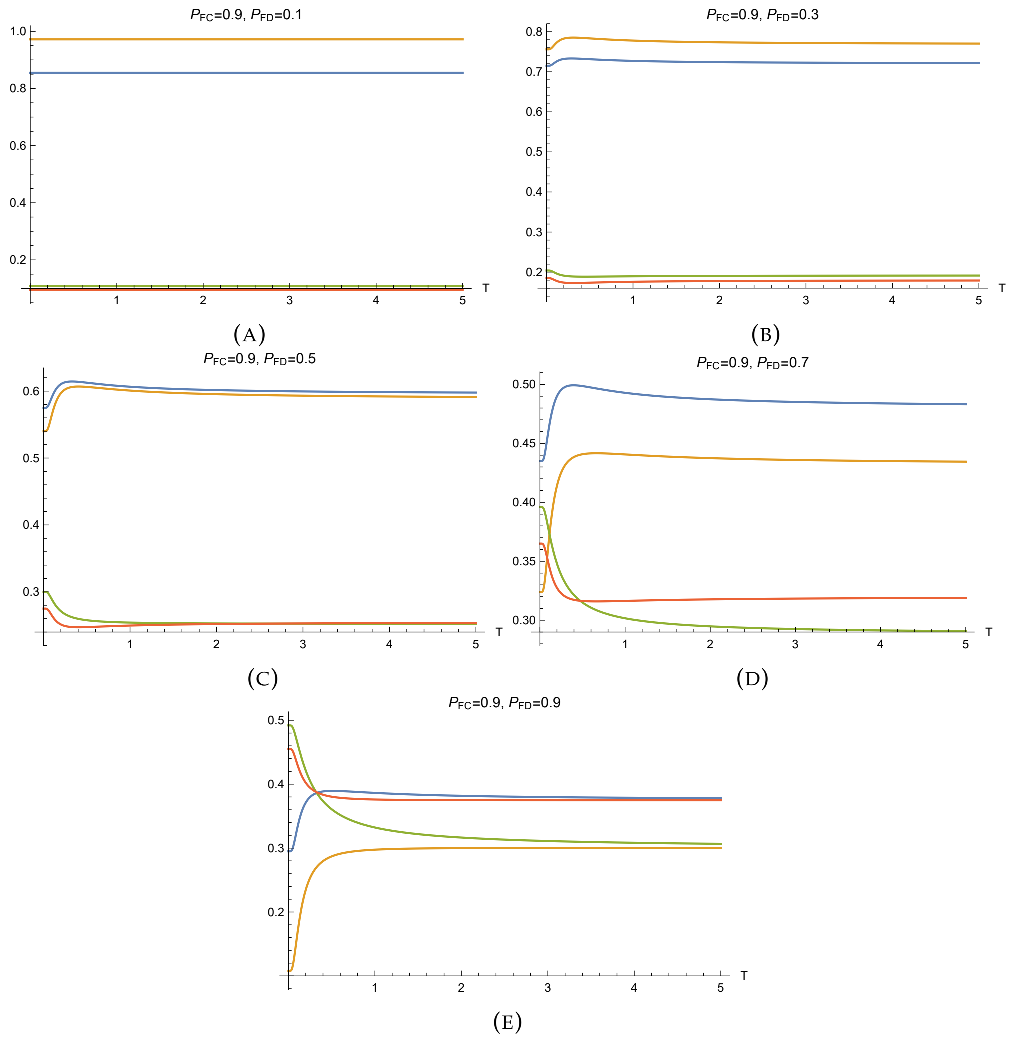


FIGURE 4.9: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.9$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

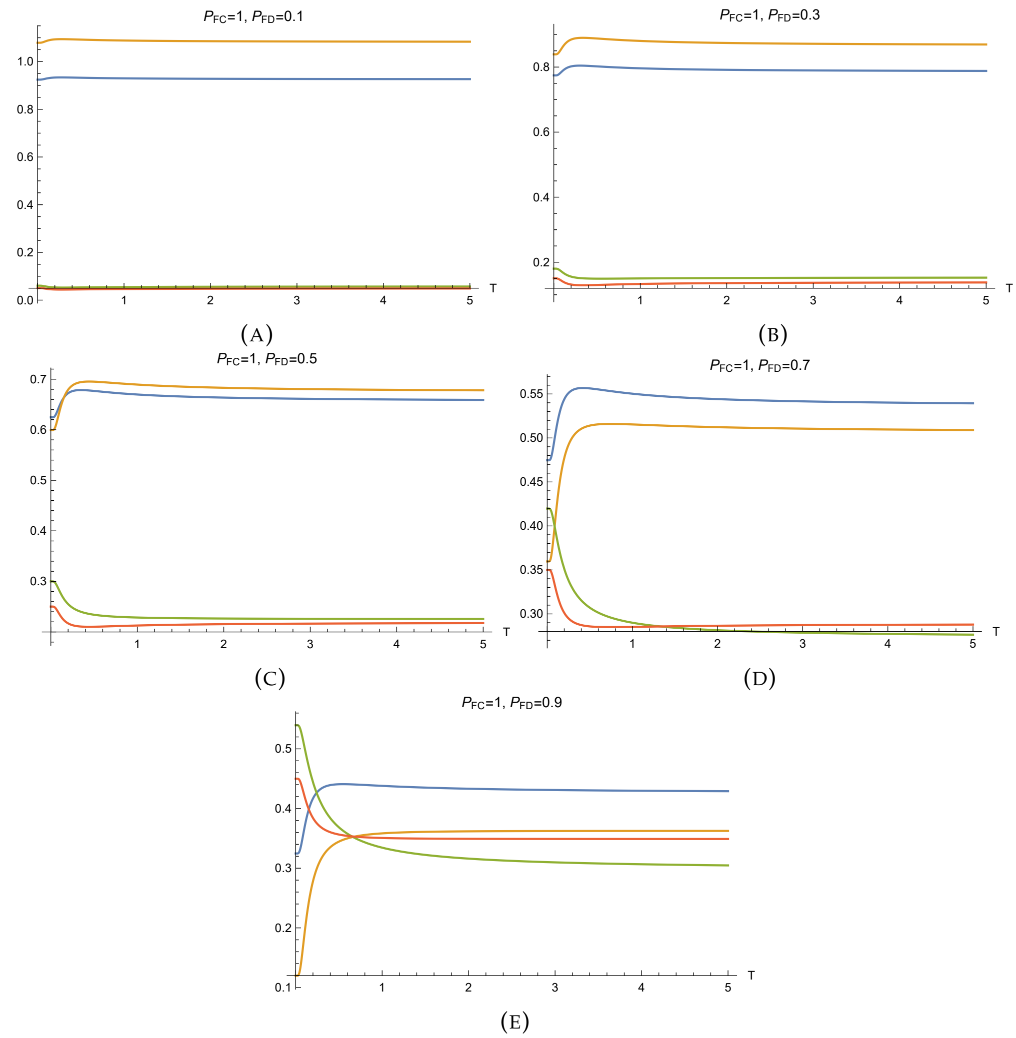


FIGURE 4.10: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 1$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.2$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red. Only in (a) the two conditions are met up to a certain temperature; after that the second condition is broken.

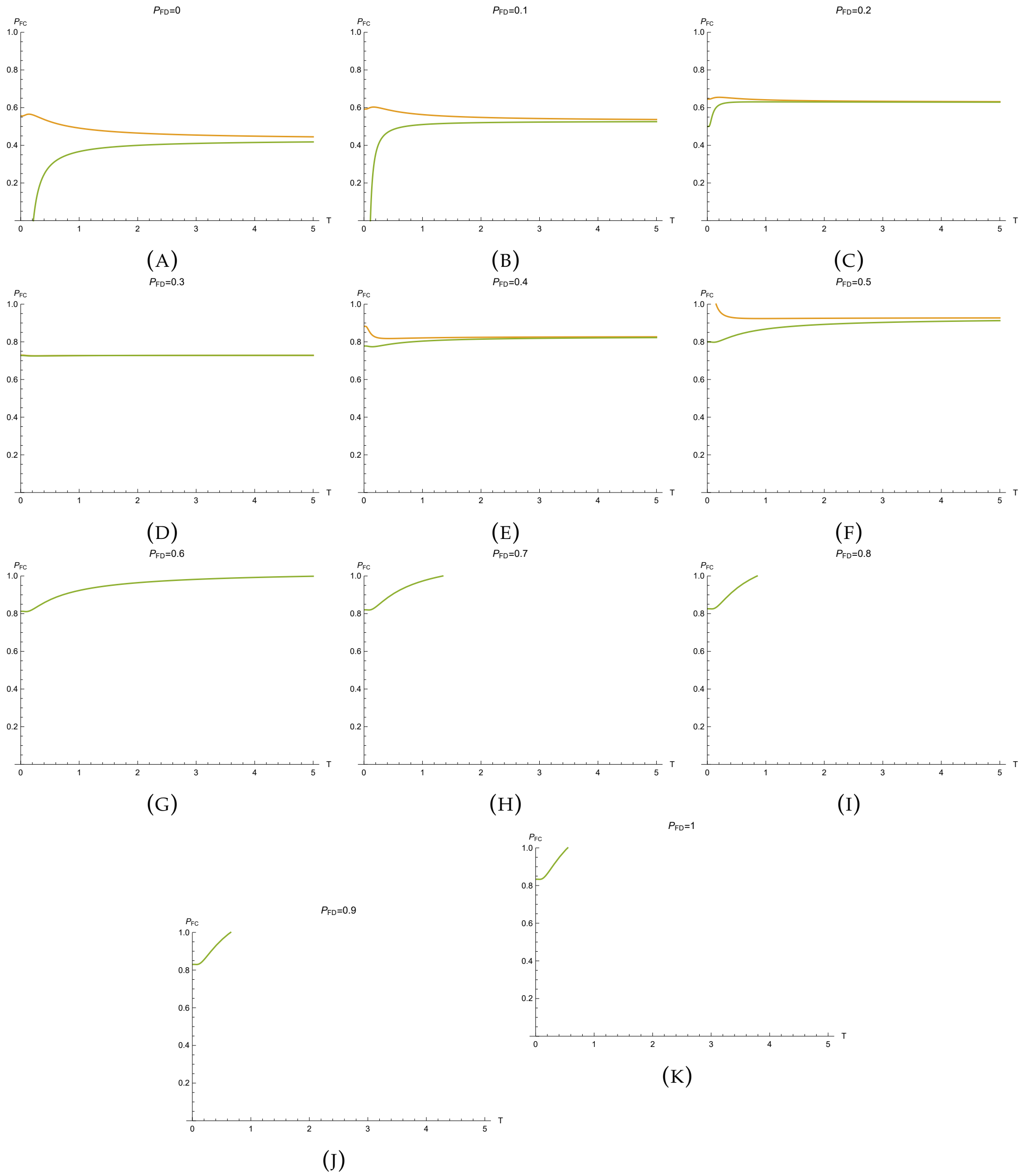


FIGURE 4.11: $P^*_{F_{1st}}$ (orange) and $P^*_{F_{2nd}}$ (green) as functions of temperature, for the SD game at $s = 0.5$ and $t = 1.2$. The solutions are plotted for increasing values of P_{F_D}. The correlation is in equilibrium above the green curve and bellow the yellow curve. The solutions of P_{F_C} that are outside this interval do not represent stability changes.

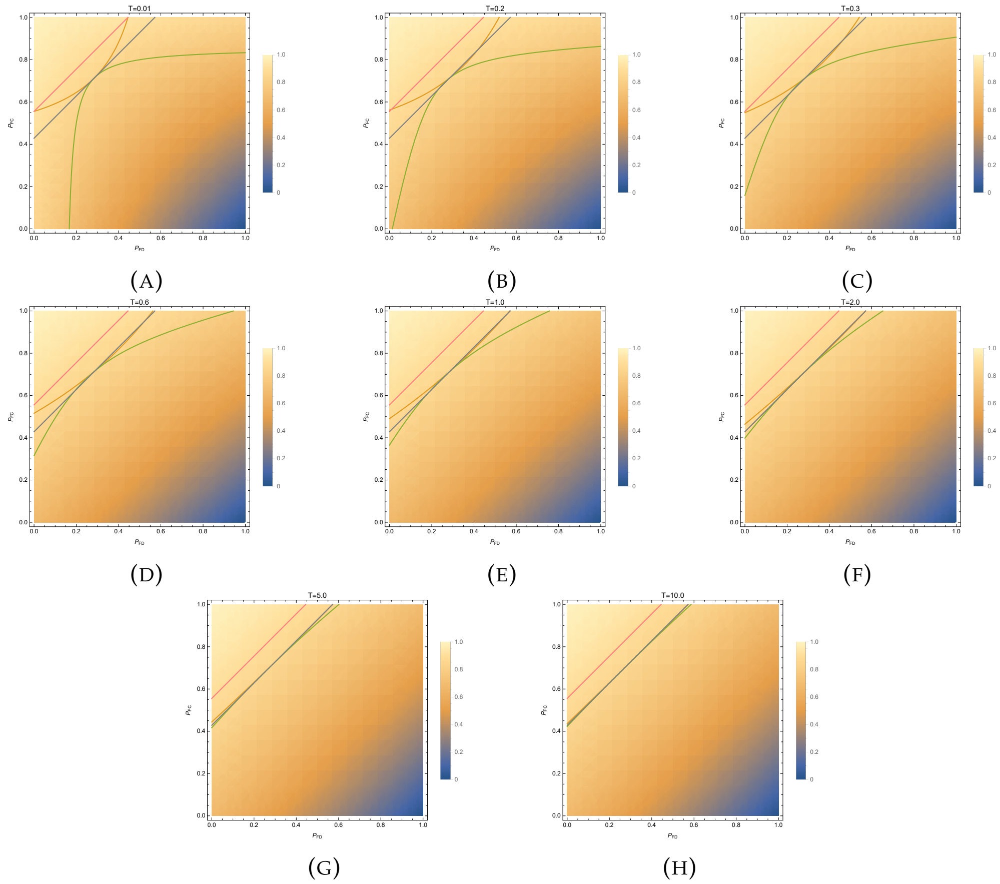


FIGURE 4.12: $P^*_{F_{1st}}$ (orange) and $P^*_{F_{2nd}}$, for $s = 0.5$ and $t = 1.2$, as functions of temperature, and payoff for the same parameters in the density plot. The pink line connects the low temperature limit point given by eq. 3.77 and the gray line marks the mixed strategy equilibrium from eq. 3.50.

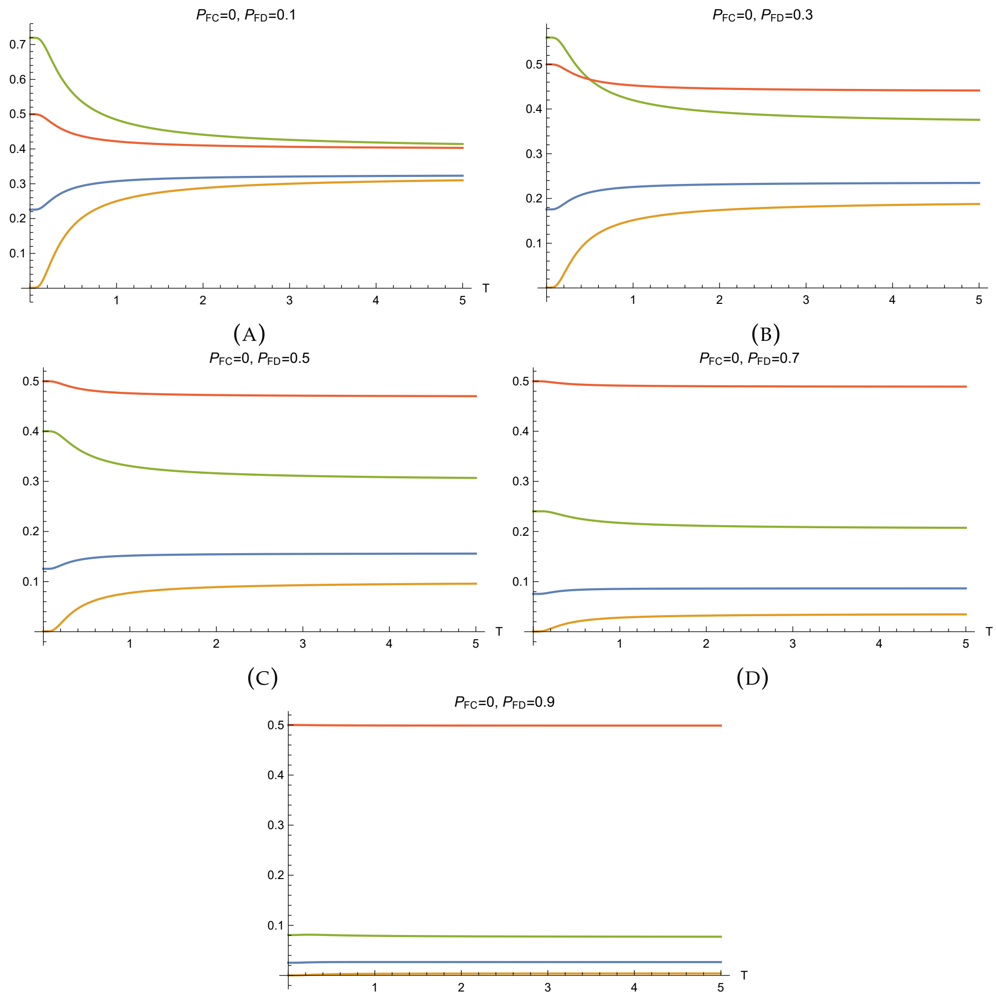


FIGURE 4.13: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

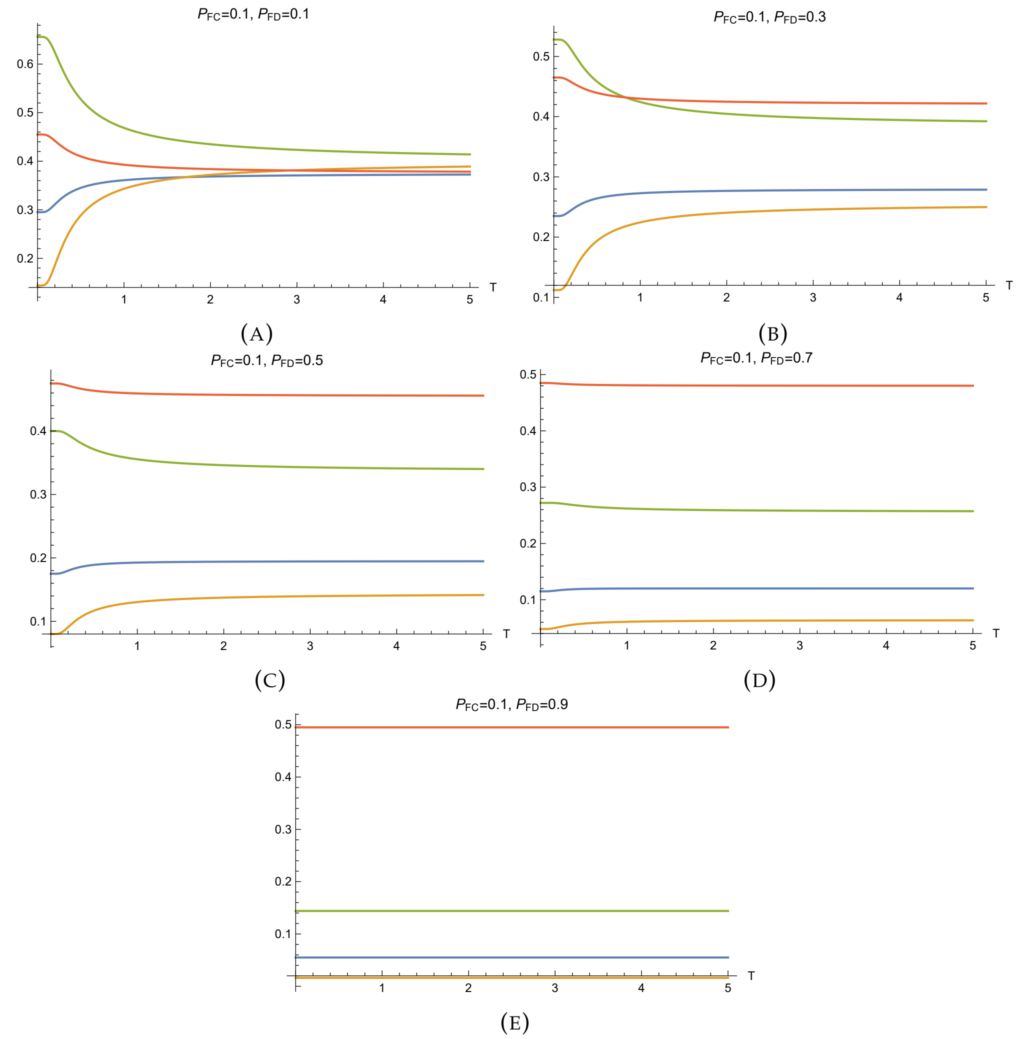


FIGURE 4.14: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.1$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

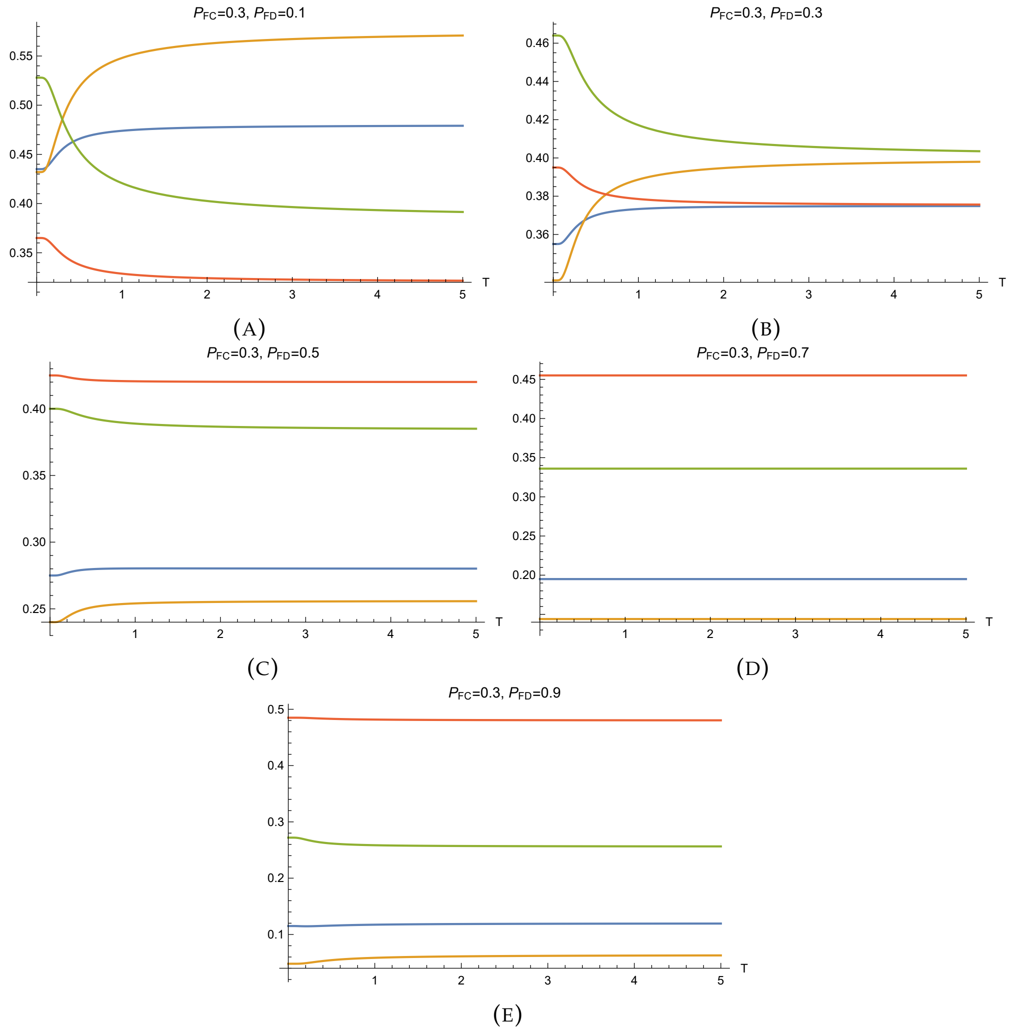


FIGURE 4.15: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.3$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

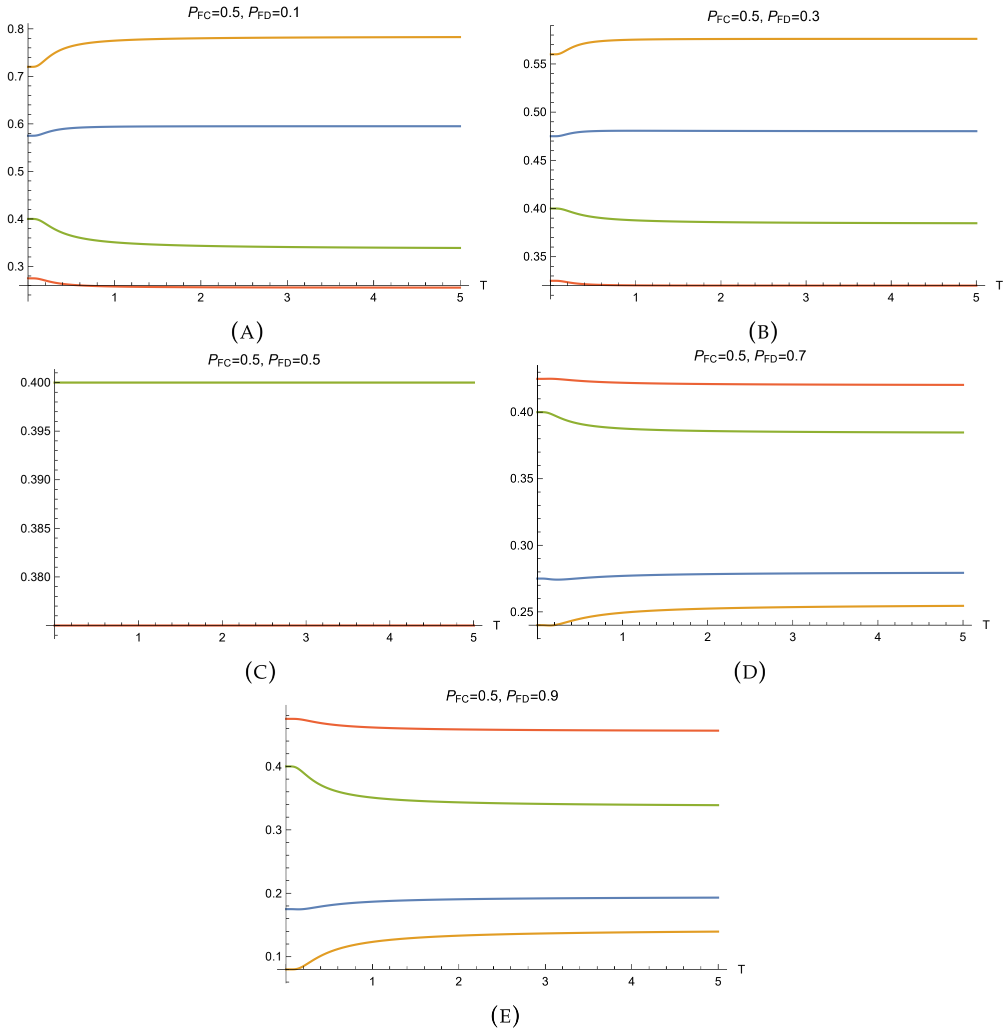


FIGURE 4.16: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 0.5$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

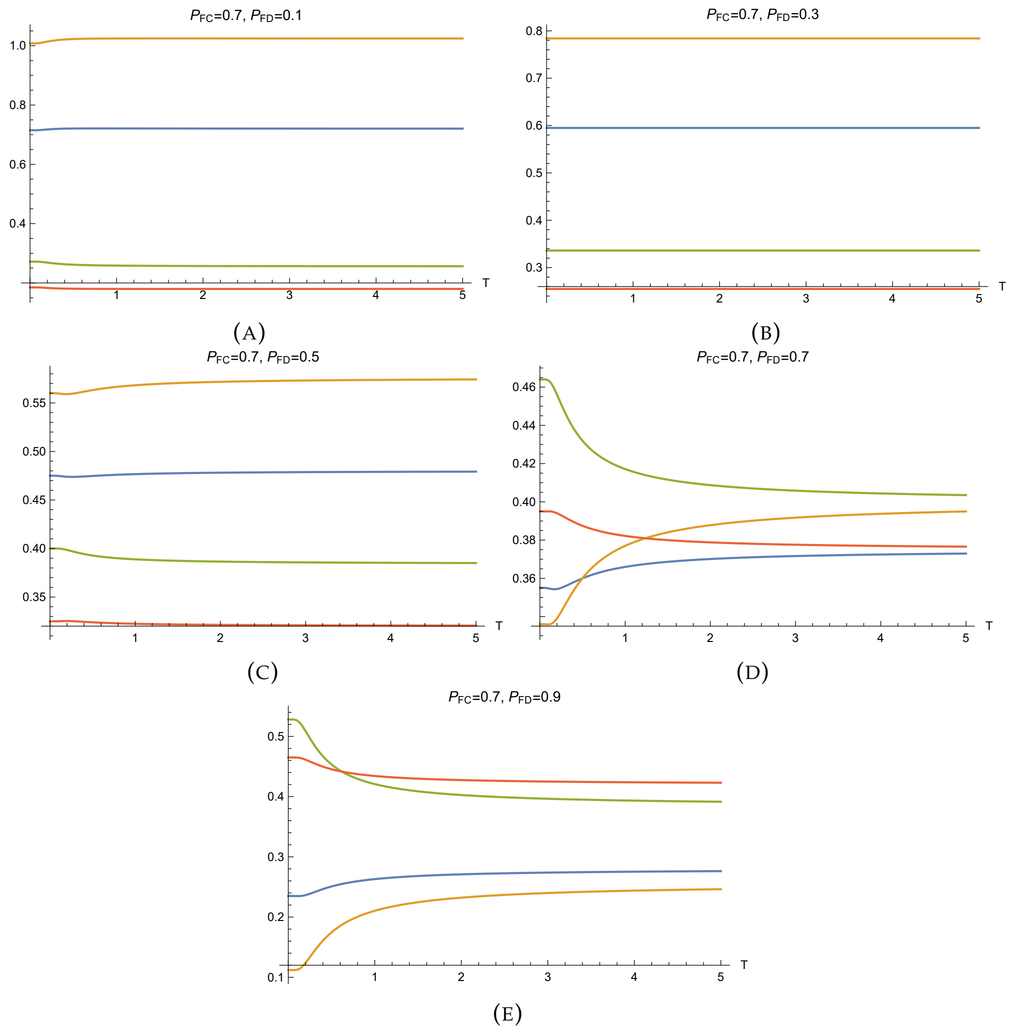


FIGURE 4.17: LHS and RHS of first and second equilibium conditions for $P_{F_C} = 0.7$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

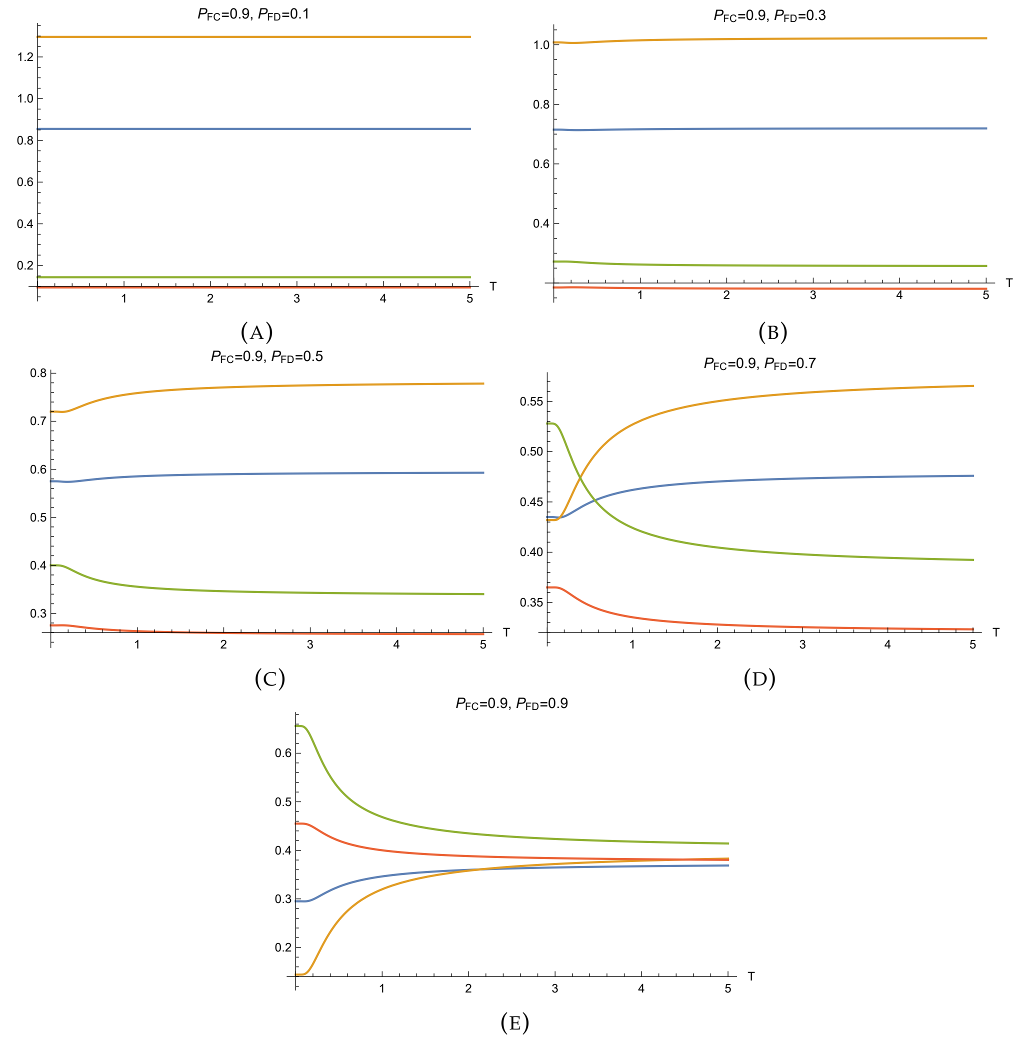


FIGURE 4.18: LHS and RHS of first and second equilibium conditions for $P_{F_C} = 0.9$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

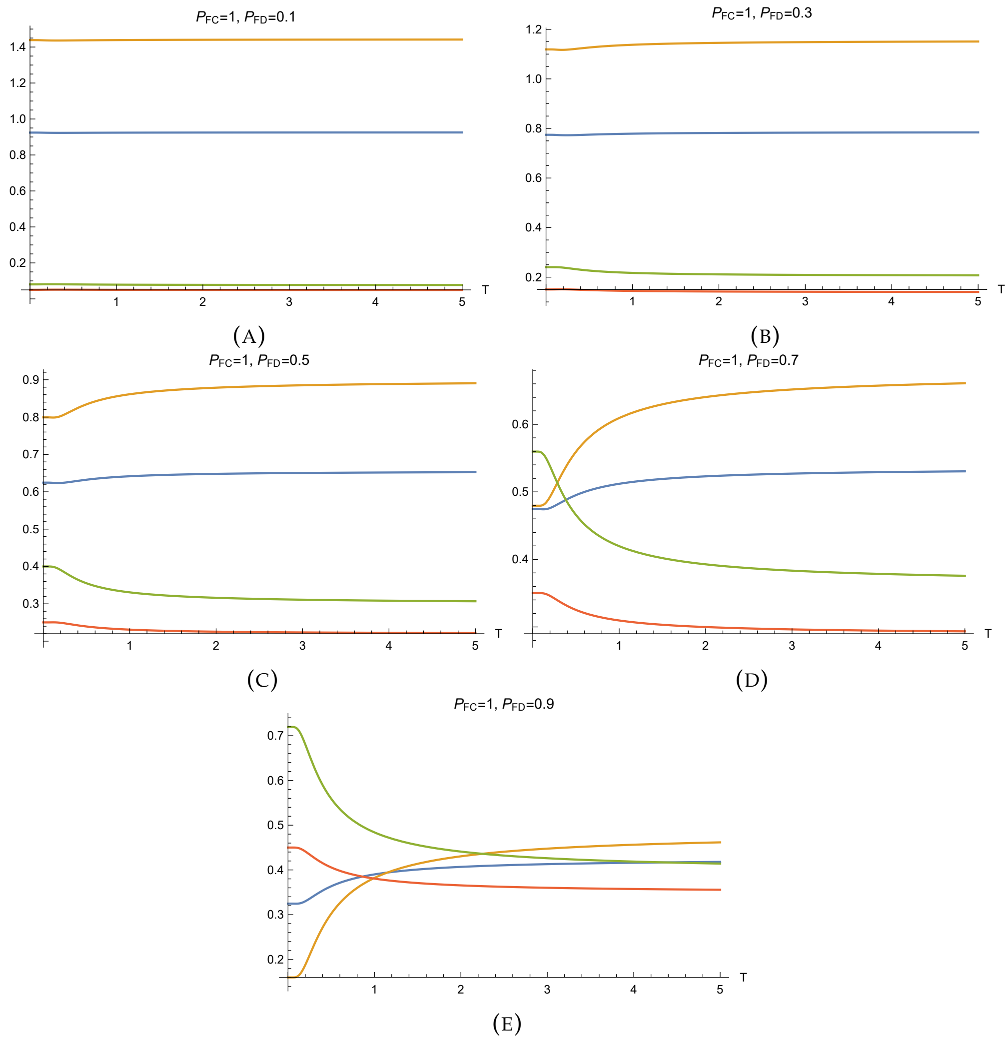


FIGURE 4.19: LHS and RHS of first and second equilibium conditions given by 3.42 for $P_{F_C} = 1$ and several values of P_{F_D}, for $s = 0.5$ and $t = 1.6$. In blue: LHS 1st condition; in orange: RHS 1st condition. In green: LHS 2nd condition; in red: RHS 2nd condition. The first condition holds when blue is above orange, while the second one holds when green is above red.

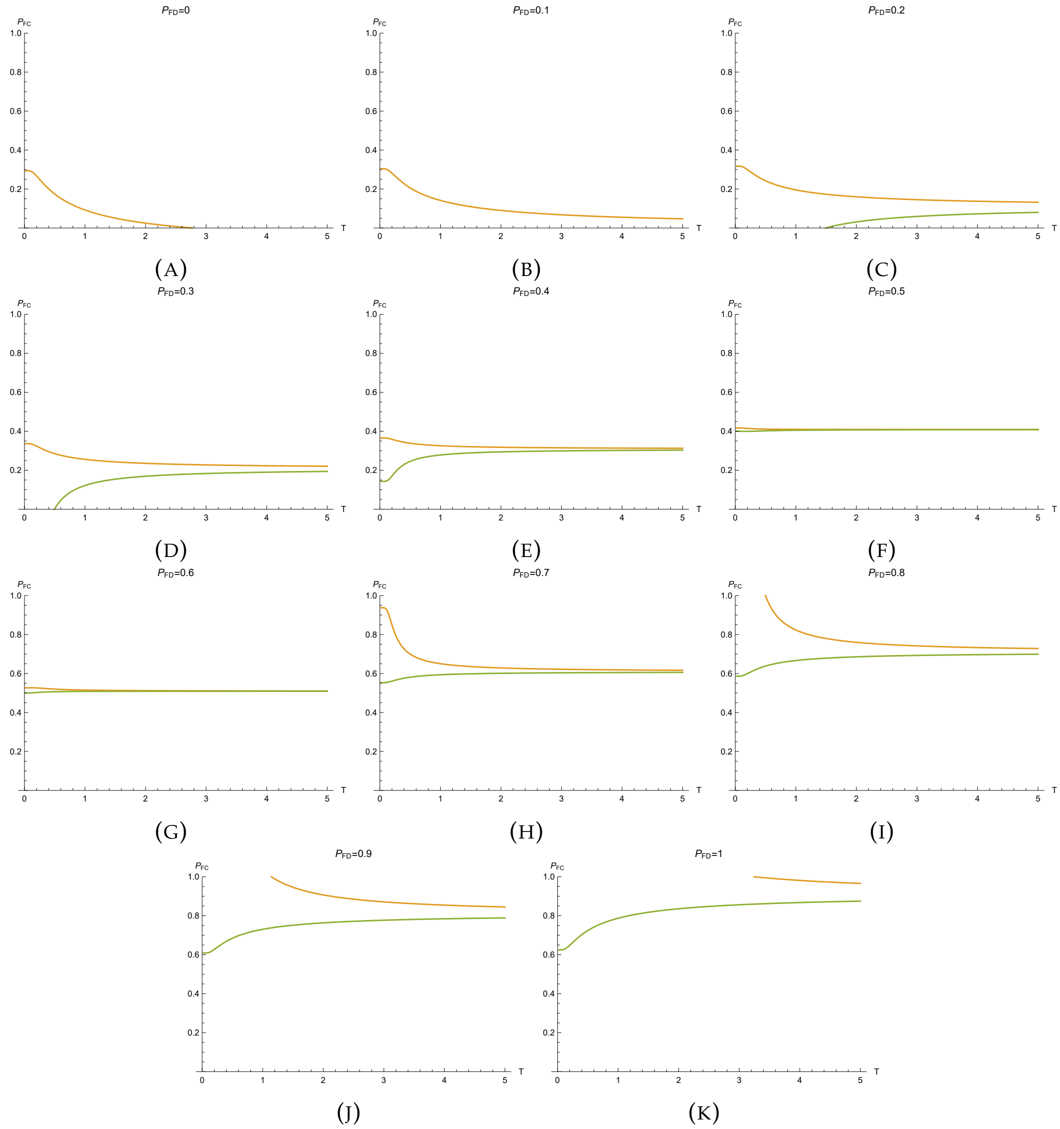
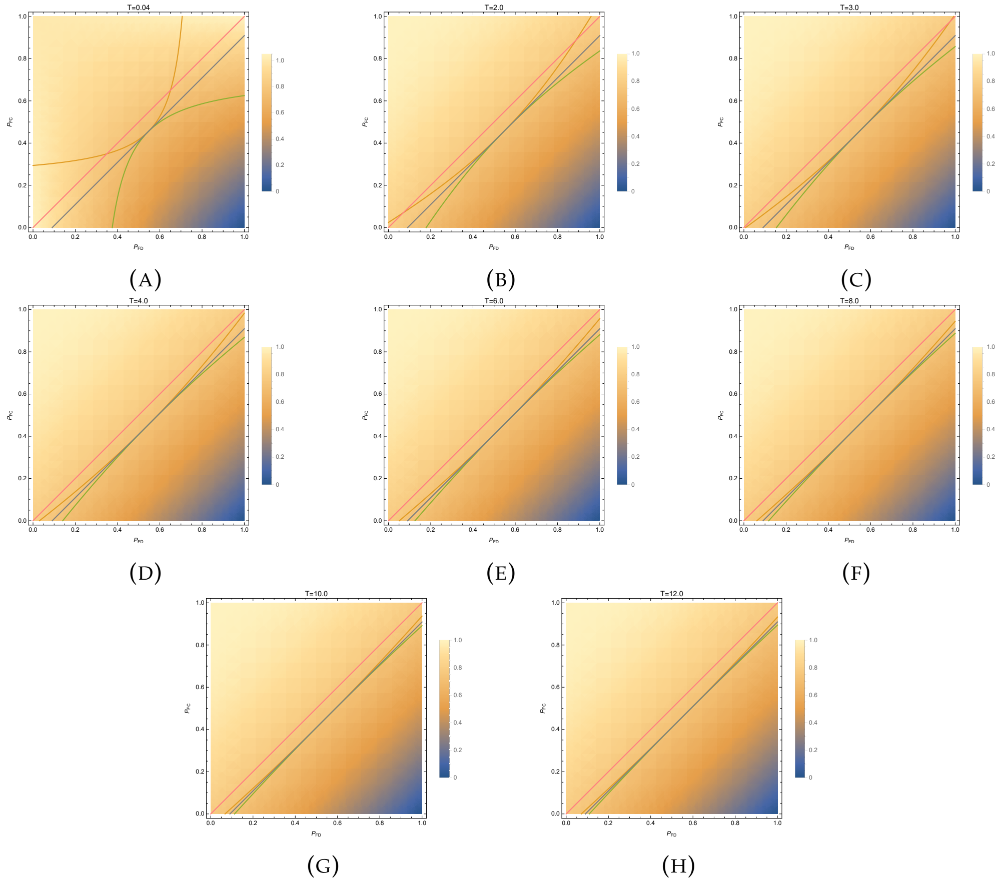


FIGURE 4.20: $P^*_{F_{1st}}$ (orange) and $P^*_{F_{2nd}}$ (green) as functions of temperature, for the SD game at $s = 0.5$ and $t = 1.6$. The solutions are plotted for increasing values of P_{F_D}. The correlation is in equilibrium above the green curve and bellow the yellow curve. The solutions of P_{F_C} that are outside this interval do not represent stability changes.



(A) (B) (C) (D) (E) (F) (G) (H)

FIGURE 4.21: $P^*_{F_{1st}}$ (orange) and $P^*_{F_{2nd}}$, for $s = 0.5$ and $t = 1.2$, as fucntions of temperature, and payoff for the same parameters in the density plot. The pink line connects the low temperature limit point, at $P_{F_C} = P_{F_D}$, and the gray line marks the mixed strategy equilibrium from eq. 3.50.

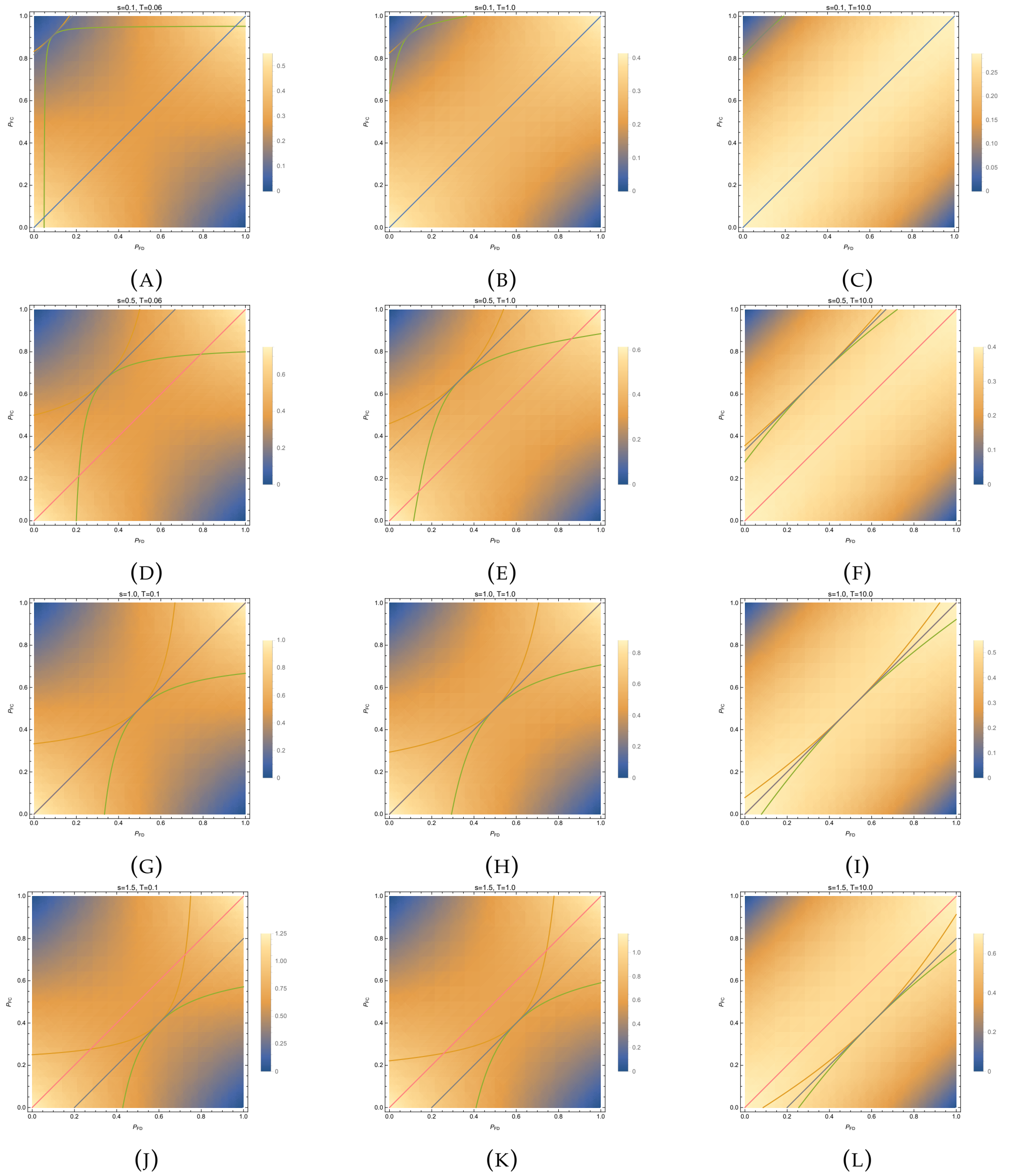


FIGURE 4.22: $P^{*BoS}_{FC_{1st}}$ (orange) and $P^{*BoS}_{FC_{2nd}}$ (green), for $s = 0.5$ and $t = 1.6$, as a fucntion of temperature, and payoff for the same parameters in the density plot. The pink line is given by $P_{F_C} = P_{F_D}$, connecting the low temperature limits, and the gray line marks the mixed strategy equilibrium from eq. 3.50.

Chapter 5

Optimal Payoffs

In this section we present the results obtained by maximizing the payoff as a function of P_{F_C} and P_{F_D}, between the domain of these variables and within the applicable equilibrium conditions, and for both the **SD** and **BoS** games.

5.1 Optimization Method

We are interested in studying how the maximum payoff behaves in the top-right corner of the $P_{F_C} - P_{F_C}$ parameter space, and on the bottom-left corner. These are regions of interest because this is where the equilibrium region is.

After setting what the criteria are for checking if a certain set of correlated probabilities is stable, we are now concerned with finding out what are the probabilities of following C and D that optimize the average payoffs of the players. We can use Lagrange multipliers to find out how the payoff surface behaves between the first and second condition equalities, or we could use numerical methods. For practical reasons, we chose the latter.

We want to find the values of P_{F_C} and P_{F_D}, as a function of temperature, and also to know what that payoff is. We will find that payoff bounded by the solutions $P^*_{FC_{1st}}$ and $P^*_{FC_{2nd}}$.

Looking at figs. 4.12 and 4.21, we expect the maximum to be either around the right-upper corner, or the left-lower one.

To study what happens in each of these regions, we used *Mathematica*'s function `FindMaximum`, which finds local maxima around inserted initial coordinates, in this case in P_{F_C} and P_{F_D}, and restricted it to the $P^*_{FC_{1st}}$ and $P^*_{FC_{2nd}}$ solutions, and the domain of the probability variables [1].

The color scheme we use to represent our findings is as follows: the doted curves represent the payoffs that the players would obtain if they would never follow, in blue, or if they would always follow, in red. The full curves represent the results found for the best payoff by the simulation: blue in the lower-left corner, associated with lower probabilities of following, and orange in the upper-right corner, representing the best payoffs when there is a high chance of following. The grey line represents the payoff for the mixed strategy (given by eq. 2.56 for **SD** game and eq. 2.67 for **BoS** game) and the pink line represents our calculated payoffs for low temperatures (given by eq. 3.68 if $s \geq 2 - t$ or eq. 3.76 if $s < 2 - t$ for the **SD** game, and by 3.87 for the **BoS** game [2]. We also present the values in the $P_{F_C} - P_{F_D}$ space where these values of the maxima were found; if they are of a color different than blue or orange, they correspond to values found at lower temperatures.

We present the results for both corners to be able to study how the system behaves. This should not be confused with finding two optimal payoffs: if the payoffs are different in both corners, *the optimal payoff overall will be found at the corner with the best payoff.*

[1]Starting at high temperature at either one of the corners, we updated the initial conditions to match the solutions found for the previous temperature for faster and more reliable convergence. Because sometimes the location of the maxima changed completely below a certain temperature, we often had to evaluate the maxima distinctively for higher and lower temperatures.

[2]The line at low temperatures is just connecting the two points in the intersection with the domain of P_{F_C} and P_{F_D}, where we are actually expected to find the best payoff at low temperature. This line in itself has no meaning. For the high temperature limits, however, we find the same payoff solution across the whole gray line.

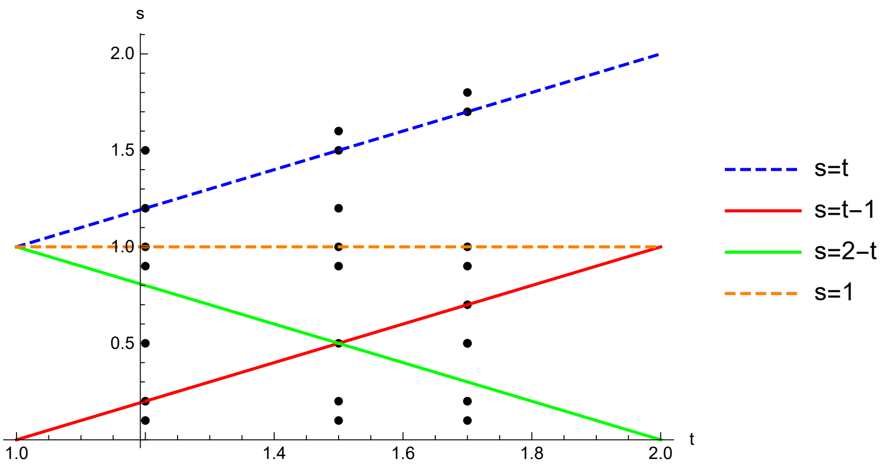


FIGURE 5.1: Parameter space for **SD** game. The full lines represent the regions where there is a change in the relation between the maximum payoffs and the payoffs for always follow/never follow. The dotted lines represent potential reagions of interest. The black bullet points represent the position in this parameter space of the cases that we are going to look at.

5.2 Snowdrift

The solutions for the best payoff in the **SD** game can be divided in the parameter space $s - t$ as in fig. 5.1. In this figure, the full lines represent the regions across which there is a change in the equilibrium behaviour, while nothing is expected to change across the doted lines, although we will look at these regions to check our predictions. Across the red line we expect a change in the behaviour regarding the location of the maxima (closer to always follow or to never follow), and across the green line we expect a change in the location of the low temperature maxima (coinciding with the always follow or coinciding with the limit of the equilibrium condition).

5.2.1 $s < t - 1$

We are in the situation analyzed in section 4.2.1, so it helps to have that topology in mind, but here we get to see the results for a wider range of parameters. The results are presented in figs. 5.2. If $s < 2 - t$, optimal payoff at lower temperature is always better than if the players always follow, or never follow. However, if $s > 2 - t$, the best payoff at low temperatures is found on always follow or never follow. In both cases, the payoff is independent from the corner we are looking at, but that changes immediately for higher temperatures.

The characteristic feature of $s < t - 1$ is that, after initially the best payoff being independent from the corner that we are looking at, immediately after that the best payoff is found at the lower corner (blue line). If we increase temperature, the best payoff crosses over to the upper corner (orange). The solution crosses from $P_{F_D} = 0$ to $P_{F_D} = 0$, and from $P_{F_C} = 1$ to $P_{F_D} = 1$. This is done in a continuous way, which means that at the point where this crossover happens, both corners have the same payoff. It is also interesting to notice that, because $P^*_{F_D} < P^*_{F_C}$, the location of the best payoff has the same behaviour as $P^*_{FC_{1st}}$ at the limits of the variables.

Because this condition crosses the the diagonal line, this justifies that after a certain temperature the best payoff starts being smaller than the always follow and never follow payoffs, but this is still the best payoff *inside the equilibrium conditions*.

This means that the best payoff is associated with the first equilibrium condition, which has a similar movement in the these parameters across the temperature. All of the payoff plots, at the high temperature values, seem to converge to the mixed strategy solution. So, using the correlations by carefully following or not following, the players can still have better payoffs than if they just played with an uncorrelated game, obtaining the mixed strategy.

5.2.2 $s = t - 1$

For $s = t - 1$, $P^*_{F_C} = P^*_{F_D}$, which means that the payoffs for always follow and never follow are always stable. There is an added symmetry for this set of parameters, such that the payoff surfaced is "leveled", and the payoffs are symmetric relative to the diagonal line, $P_{F_C} = 1 - P_{F_D}$.

The results are found in fig. 5.4, and because of this symmetry the results for both corners are overlapping. Despite the always follow and never follow being stable, we see that that the best payoff is always better than that for $s < 2 - t$ ($t = 1.2$), and the best payoffs are always equal or better compared with the corner payoffs for $s > 2 - t$ ($t = 1.5$). In $t = 1.5$, $s = 0.5$ we are in the special case of $s = 2 - t$, and the payoff is the same at both the equilibrium condition and the corners, which is why we see the value of P_{F_D} changing for low temperatures in fig. 5.4d. Note that, because the pairs of probabilities associated with the best payoffs are always found for $P_{F_C} = 1$ and $P_{F_D} = 0$, we can conclude that the best payoffs come associated with the first condition, because only $P^*_{FC_{1st}}$ intersects with these lines.

We also see that all of the maximum payoffs seem to approach the mixed strategy solution. However, while for $t = 1.2$ the value of P_{F_D} and a fucntion of temperature seems to decrease, for $t = 1.7$ the P_{F_D} is equal to 1 until a certain temperature, until it moves very fast towards $P^*_{FC_{1st}}$, and converges to the mixed equilibrium as this curve does.

5.2.3 $s > t - 1$

Now we are in the situation analyzed in section 4.2.1. If we look at figs. 5.5 to 5.10, we see that the best payoff always happens at the upper-right corner, and for $P_{F_C} = 1$. This is Due to the fact that now playing C is always a more advantageous, because the payoffs are better. Because $P^*_{F_C} > P^*_{F_D}$.

When $s < 2 - t$, as in fig. 5.5, the maximum payoff at low temperatures is above the always or never follow payoffs, as is characteristic of this combination of parameters. In this particular figure we see the optimal payoffs that correspond to the discussion in 4.2.1.

When $s > 2 - t$, the payoffs at low temperature are always coincident with always follow, and from a certain point on they move away from those values, although still for $P_{F_C} = 1$ and progressively converge to the mixed strategy. The convergence to the mixed strategy can be for increasing values of P_{F_D}, for example in fig. 5.6b, but it can also happen for decreasing values of P_{F_D}, as in fig. 5.6b. The difference resides in the fact that, depending on the parameters, the intersection of $P^*_{FC_{1st}}$ with $P_{F_C} = 1$ can cross the mixed strategy line, before it starts converging back to it.

5.2.4 Summary

After analyzing how the best payoff behaves for different s and t parameters and temperatures, and what values of P_{F_C} and P_{F_D} generate it, we can make a summary of this behaviour across the parameter regions:

- $s < t - 1$: the best payoff starts off corresponding to $P_{F_D} = 0$ and then moves to be at $P_{F_C} = 0$; after a certain temperature, it jumps to $P_{F_C} = 1$, although the change in the value of the payoff itself is continuous; finally, the best payoff is found at $P_{F_D} = 1$.

- $s = t - 1$: The best payoff corresponds to $P_{F_C} = 1$ or $P_{F_D} = 0$; the value of the payoff is the same in these two lines.

- $s > t - 1$: the best payoff is always found at $P_{F_C} = 1$.

- $s > 2 - t$: the best payoff coincides with always and never follow for low temperatures. The range of the temperatures for which this happens seems to increase with t.

- $s < 2 - t$: the best payoff never coincides with always follow and/or never follow; instead, it is always associated with the first equilibrium condition.

It is worth mentioning that the fact that the independent calculation of the best payoffs using numerical simulation agrees with our predictions for low and high temperature confirms the consistency of our approach.

It is also remarkable that the best payoffs are always found for an extreme value of either P_{F_C} or P_{F_D}, and associated with the first equilibrium condition. This allows us to study the evolution of the payoff as a function of only one parameter, for the whole temperature in $s \geq t-1$ and for big intervals of the temperature at a time for $< t-1$.

5.2.5 Battle of the Sexes

Now we study how the best payoff behaves for an asymmetric game.

Given that the probabilities $p^R(CC)$ and $p^R(DD)$ are the same for the **BoS** game, we expect that the payoffs of the two players are the same, and that they are also symmetric in relation with $P^1_{F_C}$ and $P^2_{P_D}$. This is what we see indeed when we look for the maximum payoff.

We see that the payoffs for $P^1_{F_C} = P^1_{F_D} = 0$ are the same as those for $P^1_{F_C} = P^1_{F_D} = 1$. We also see that the maximum payoff found closer to any of these regions is the same as the one found on the other one. Furthermore, we have that, if $s = 1$, the best payoff coincides with the payoff at the always follow or never follow corners. When $s \neq 1$, the best payoff is smaller than on the corners after a certain temperature. When $s < 1$, the touching point of the two solutions for the equality of the conditions is bellow the line of $P^1_{F_C} = P^1_{F_D}$, so eventually this line stops being inside of the equilibrium; the best payoff here is associated with the first condition. When $s > 1$, the touching point is above that line, and, although the behaviour of the maximum payoff is very similar to that of $s < 1$, the maximum are actually found associated with the second equilibrium condition; this indicated that the maximum payoff should be around $P^1_{F_C} = P^1_{F_D} = 1$. Similarly to the **SD** game, we always find the best values of the payoff at extreme values of $P^1_{F_C}$ or $P^1_{F_D}$. Also as in the symmetric game, the low temperature payoffs coincide with the theoretical predictions, namely that it is at $P^1_{F_C} = P^1_{F_D} = 1$ or $P^1_{F_C} = P^1_{F_D} = 0$, and the high temperature payoffs coincide with the mixed strategy calculated for the **BoS** game.

In 5.11 we present the numerical results for a number of selected representative values of s. We see that the low temperature limits are met, and that in high temperature the best payoffs converge to the mixed strategy. For $s < 1$ and $s > 1$, the best strategy is to follow the CD up to a temperature, and when this strats being unstable, we can still find a better payoff than to fall into the mixed strategy. For $s < 1$, the best payoffs are associated with P^{*BoS}_{2nd}, so that they are found for $P_{F_C} = 1$ and $P_{F_D} = 0$. For $s > 1$, the best payoffs are associated with P^{*BoS}_{1st} instead, so that they are found for $P_{F_D} = 1$ and $P_{F_C} = 0$. For $s = 1$, the best payoff is found when the players follow what the bank tells them to do.

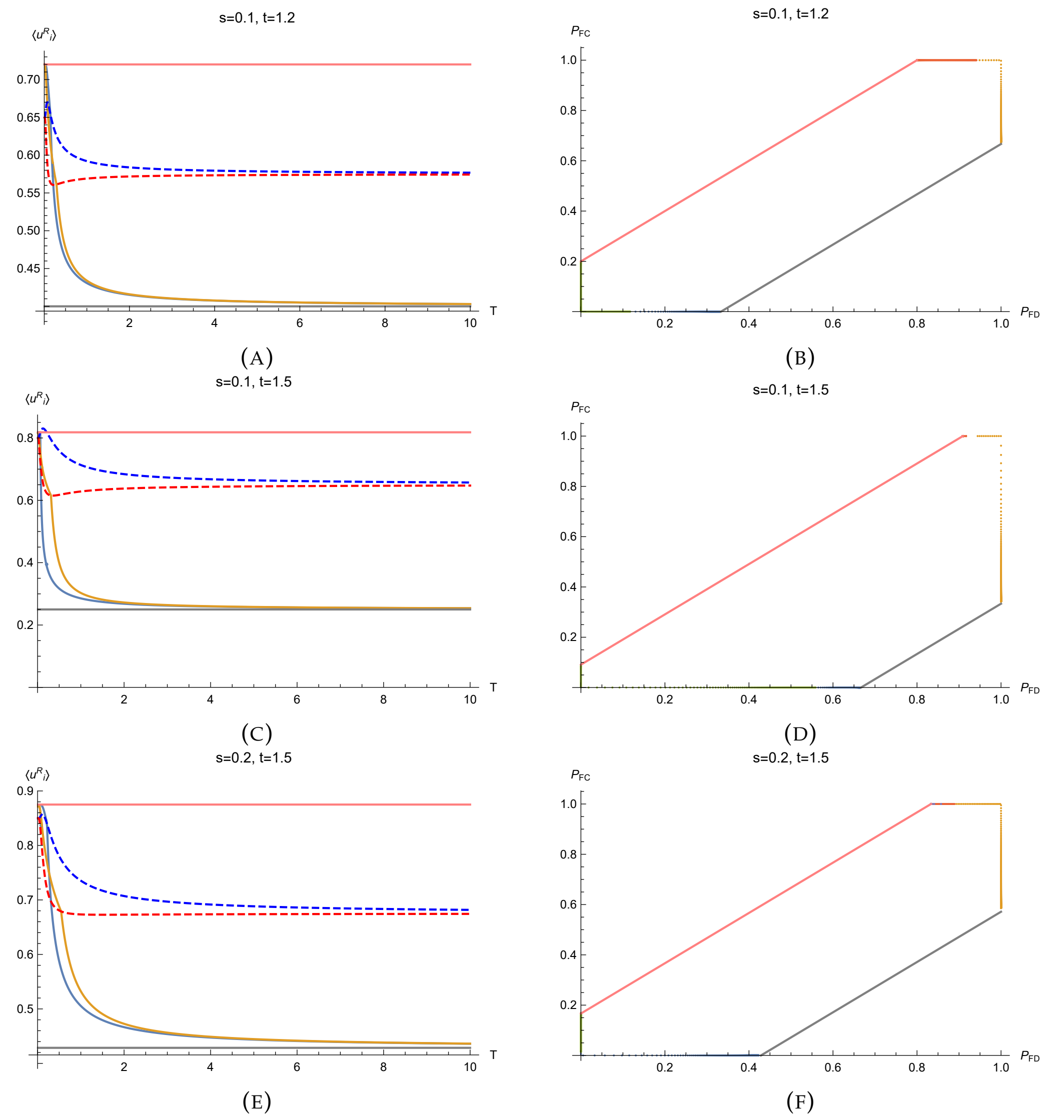
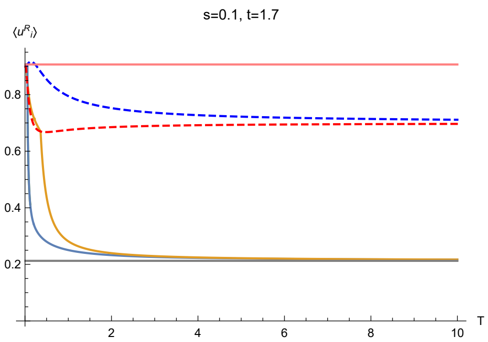
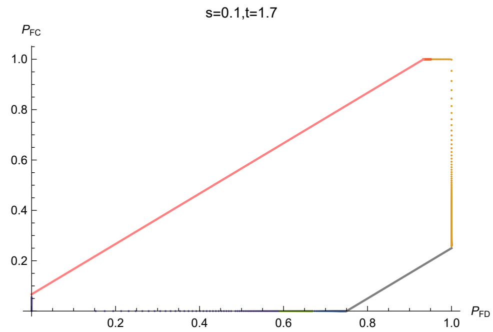


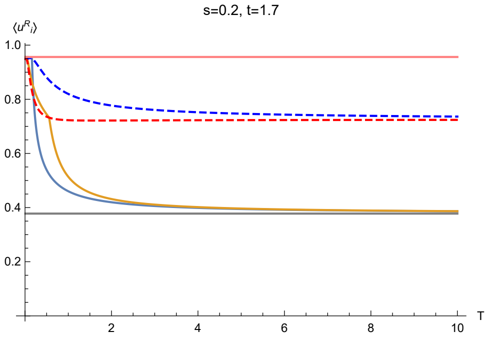
FIGURE 5.2: Average payoffs (first collumn) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right collumn), for $s < t - 1$, in the **SD** game. In the first collum, the pink line represents the limit in low temeprature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy; the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature; the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.



(A)



(B)



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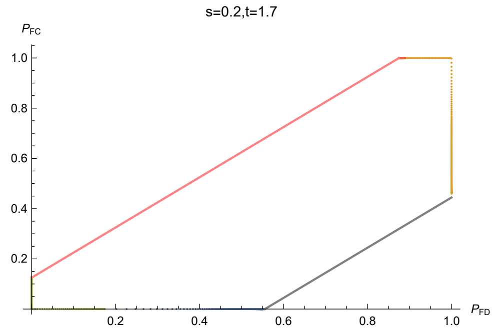

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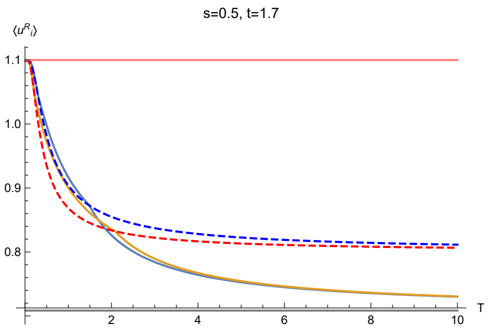

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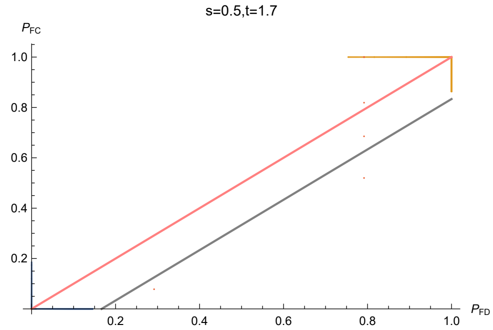


(F)

FIGURE 5.3: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s < t - 1$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

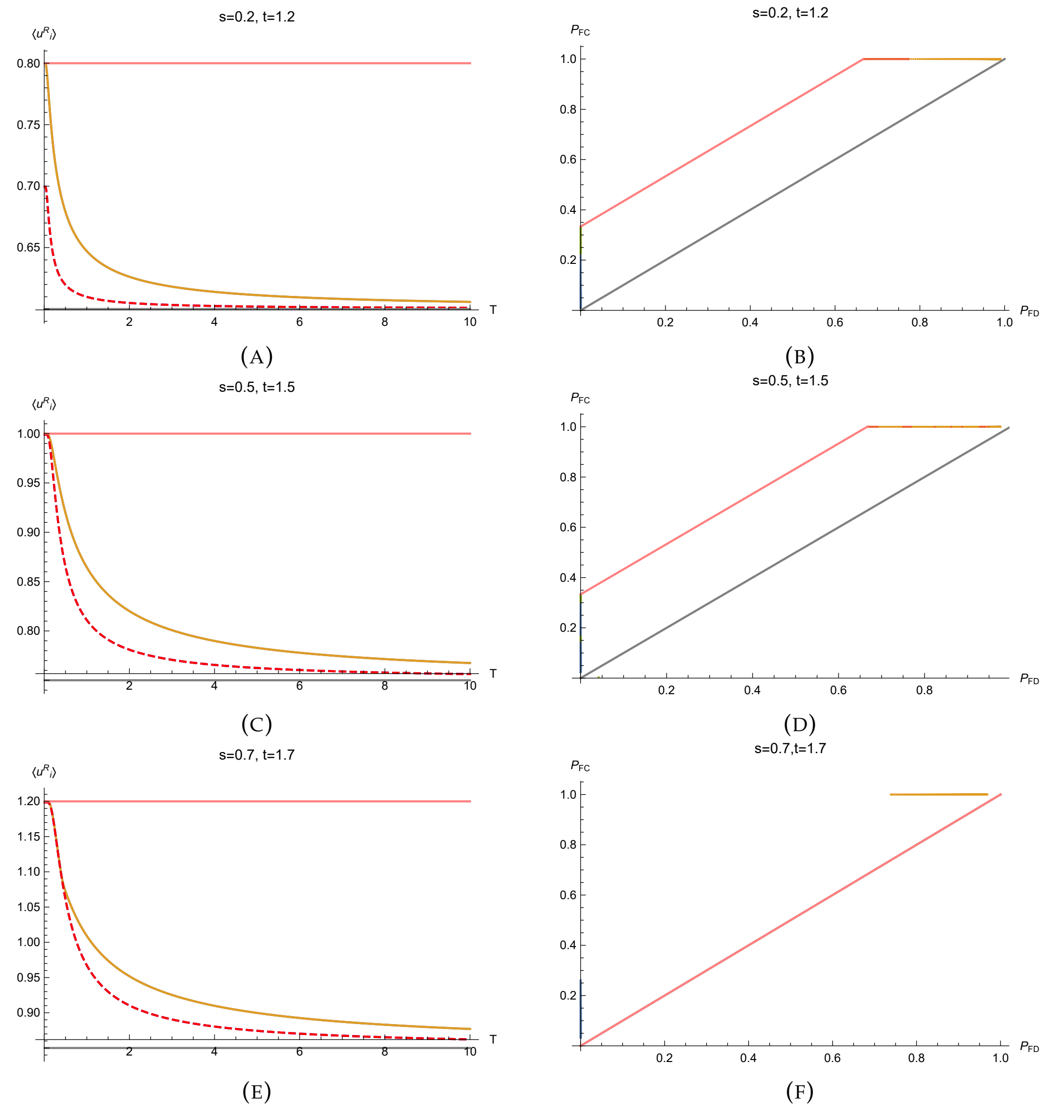


FIGURE 5.4: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s = t - 1$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner; these lines are overlapping because they have the same values. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

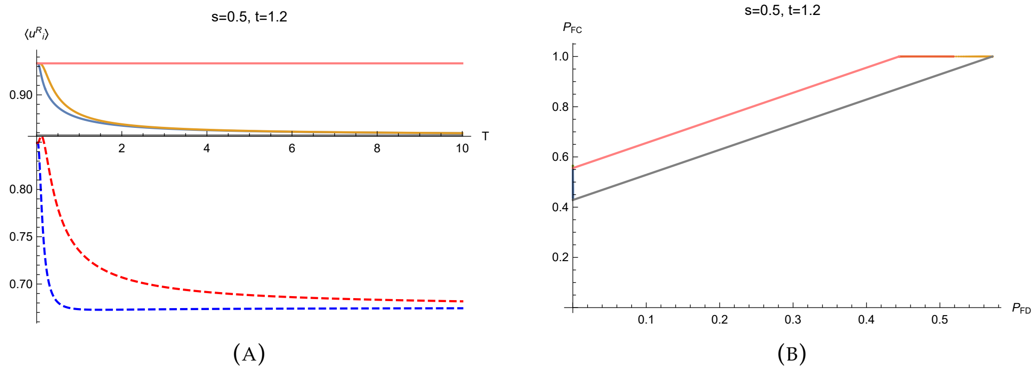


FIGURE 5.5: Average payoffs as a function of temperature in (A) and values of P_{F_C} and P_{F_D} where the best payoffs were found in (B), for $s > t - 1$ and $s < 2 - t$, in the case of $s = 0.5$ and $t = 1.2$, in the **SD** game. In the left figure, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the right figure, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

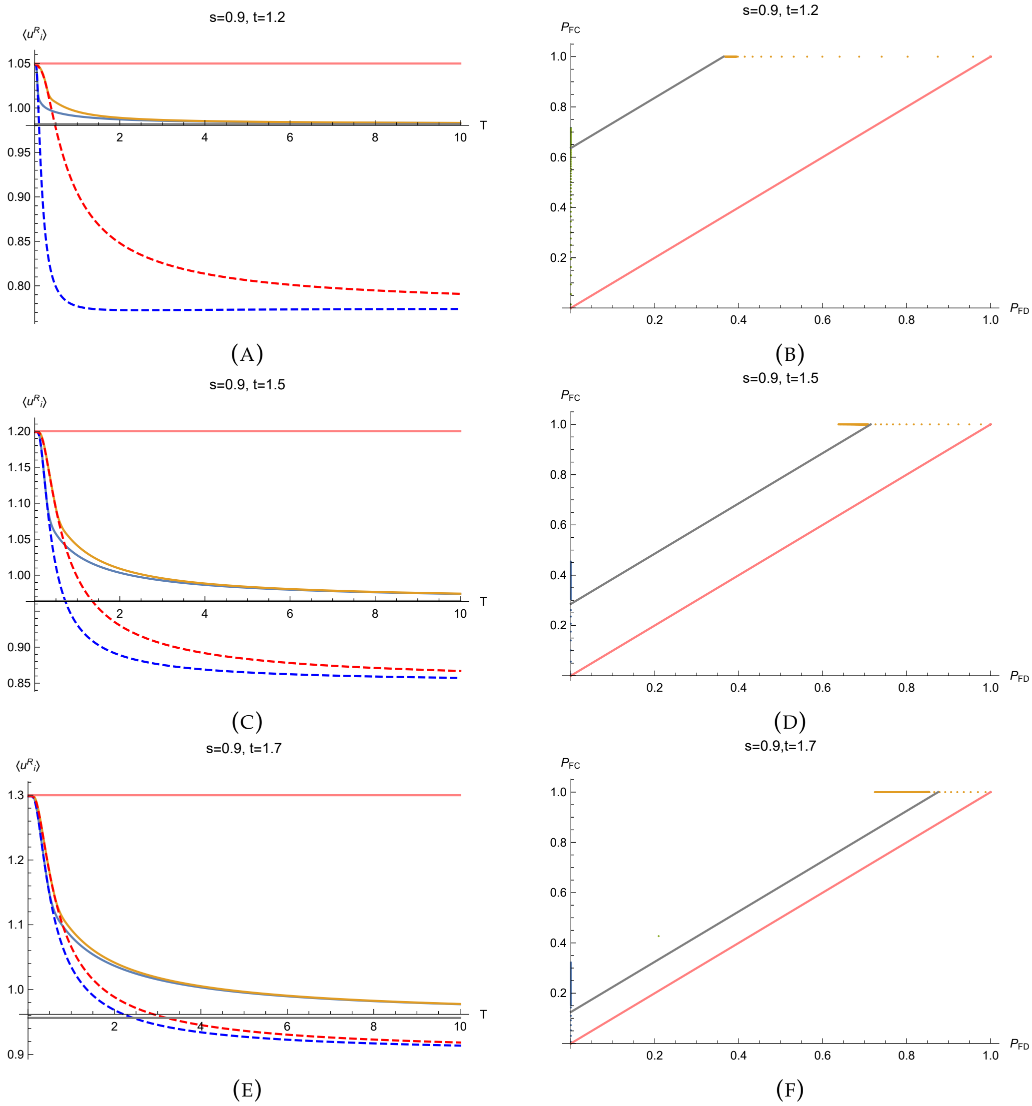


FIGURE 5.6: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s < t - 1$, in particular for $s = 0.9$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

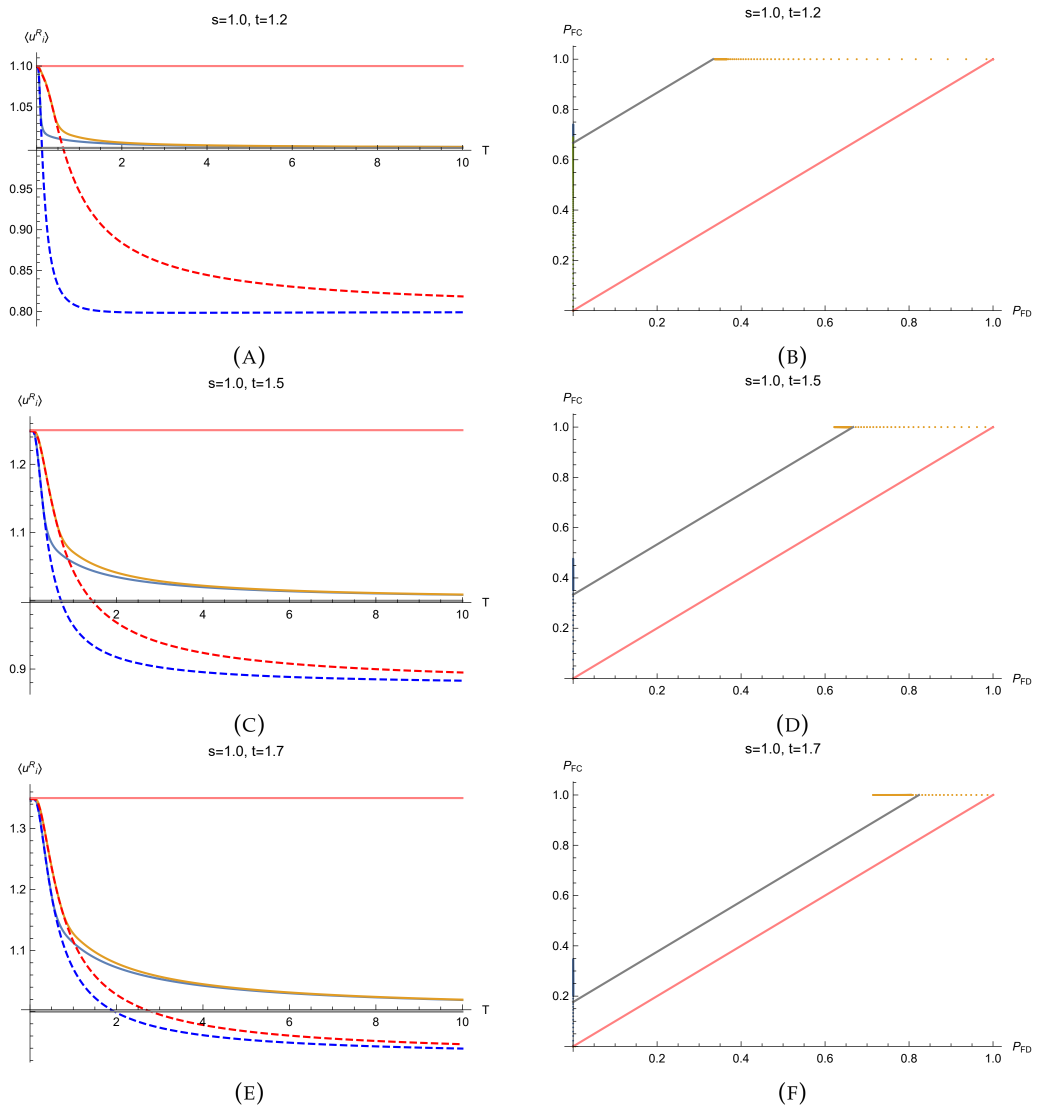
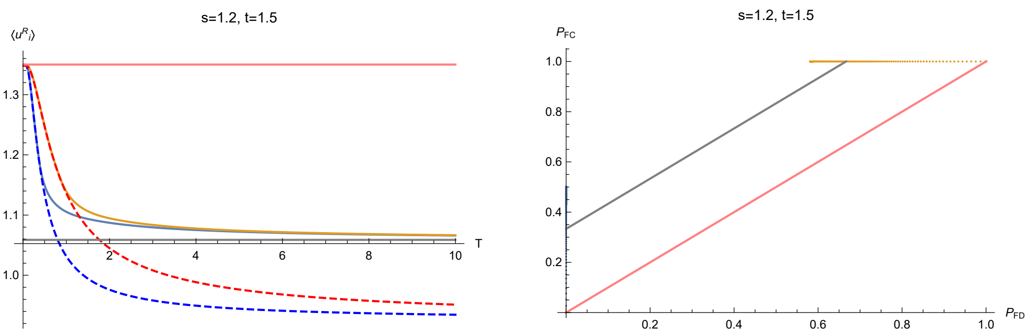


FIGURE 5.7: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s = 1$, in particular for $1 < s < t$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.



(A) (B)

FIGURE 5.8: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s < t - 1$, in particular for $1 < s < t$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

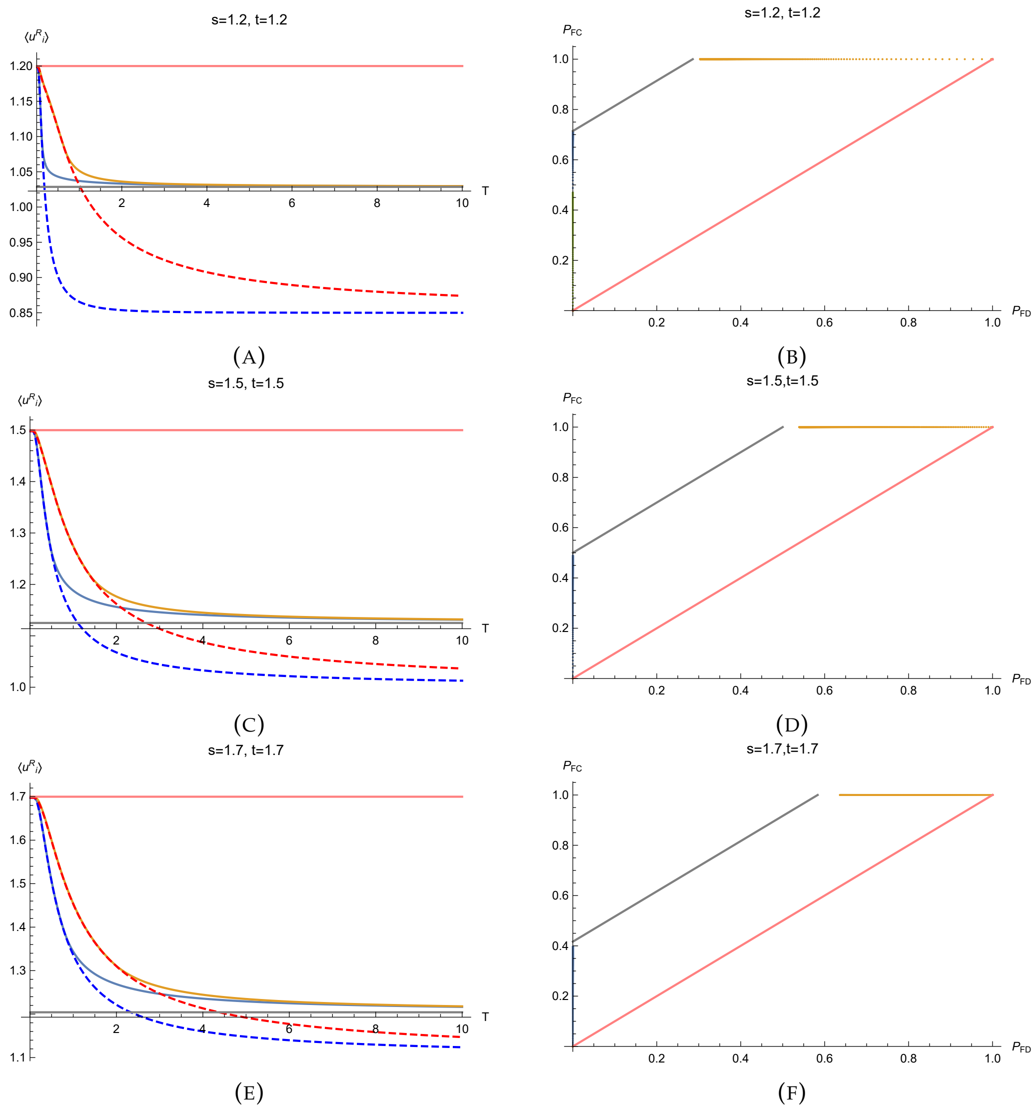


FIGURE 5.9: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s < t - 1$, in particular for $s = t$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

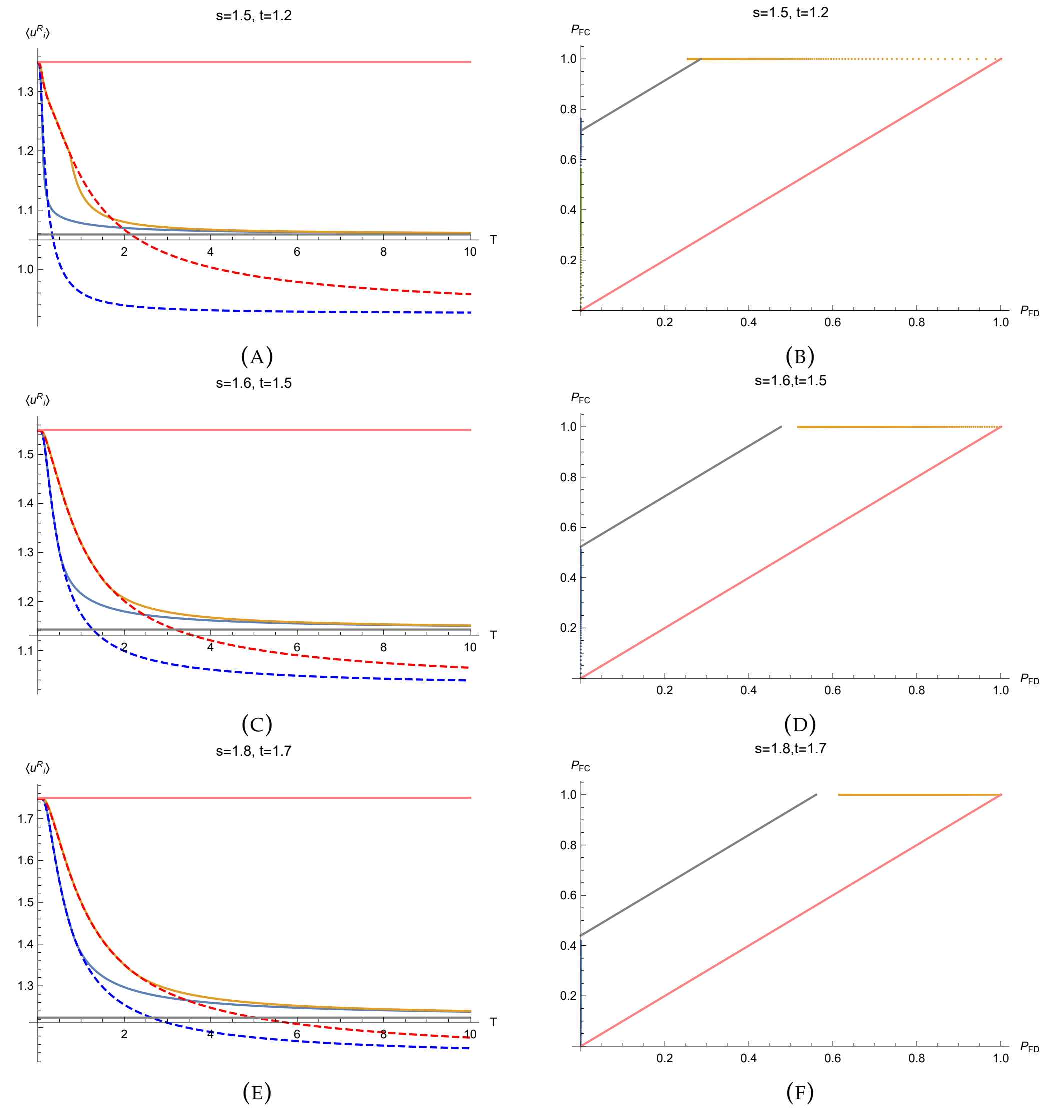


FIGURE 5.10: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), for $s < t - 1$, in particular for $s > t$, in the **SD** game. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the red and orange points represent, respectively, where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant for low and high temperature, the green and blue points represent, respectively, where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

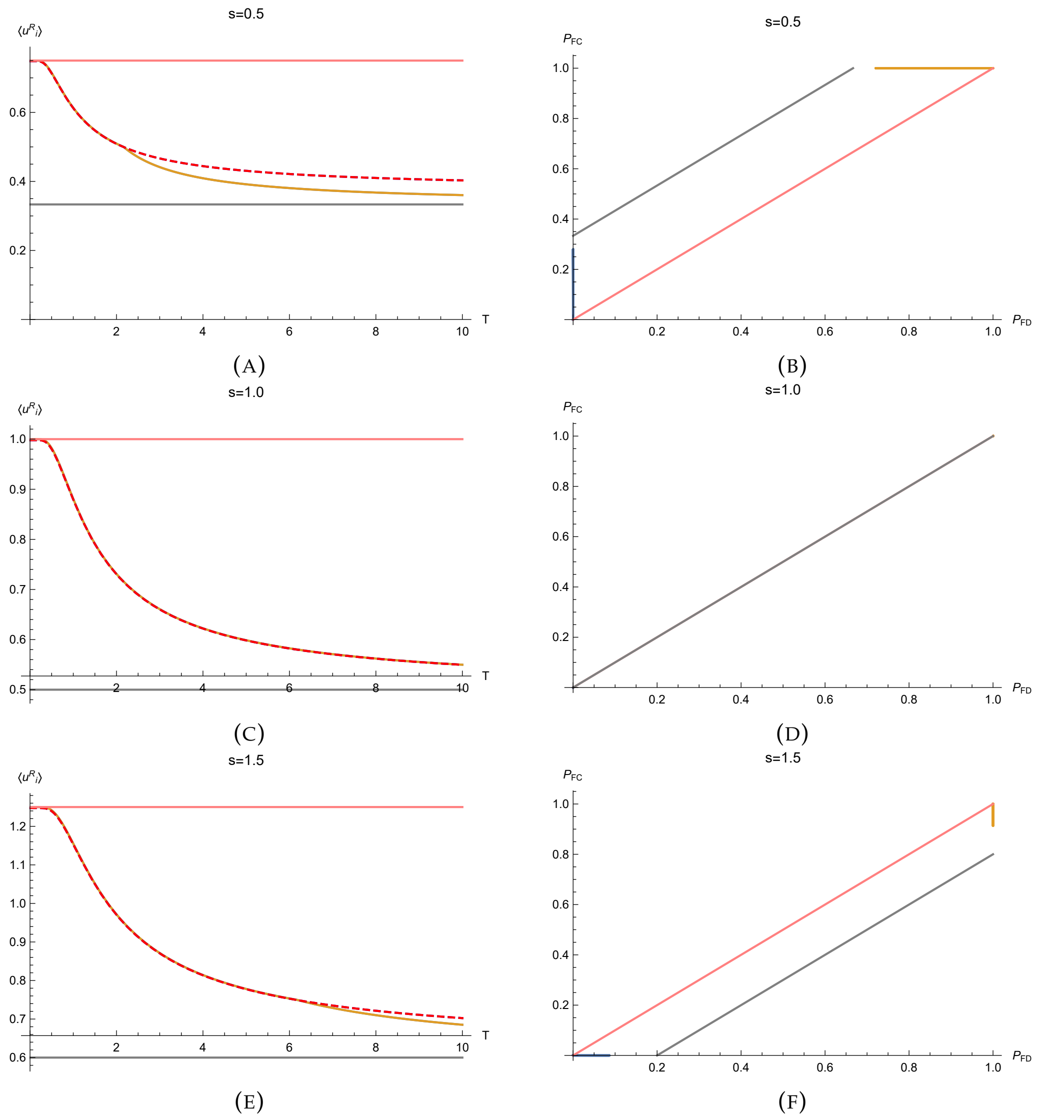


FIGURE 5.11: Average payoffs as a function of temperature (left column) and values of P_{F_C} and P_{F_D} where the best payoffs were found (right column), in the **BoS** game, for different values of s. In the first column, the pink line represents the limit in low temperature, the gray line is the mixed strategy solution, the dotted blue line is the payoff for $P_{F_C} = P_{F_D} = 0$, the dotted red line is the payoff for $P_{F_C} = P_{F_D} = 1$, the full blue line is maximum payoff in the lower-left corner, and the orange full line is the maximum payoff in the upper-right corner; the values corresponding to the two quadrants are overlapping, so we only see one line. In the second column, the pink line connects the values of P_{F_C} and P_{F_D} where the maximum payoff at low temperatures is obtained, the gray line represents P_{F_C} and P_{F_D} for the mixed strategy, the orange points represent where the maximum payoffs (represented in the left plot by the orange line) were found in that first quadrant, the blue points represent where the maximum payoffs (represented in the left plot by the blue line) were found in that third quadrant for low and high temperatures.

Chapter 6

Previous and Further Outlook

The beginning of this project had in view the use of the improvement of the players payoff through the use of the new set of probabilities 3.2, with the goal of extending these energies to be used in a network where several players would play against each other, and could do better because they had a modified set of energies.

Initially we procured to calculate the best payoff for two players, in an analogous way to the mixed strategy, by equating *the payoff if they were told to play C by a correlating device* with *the payoff if they were told to play D*, weighted by the probabilities generated from the upgraded energies. This had been done for the extreme limits of $T \to 0$ and $T \to \infty$ and we extended the previous results for the whole temperature range. The idea was to check that they had better payoffs under this condition for these new energies than they did with the initial energies, or without correlations (mixed strategy). This approach relied on the idea that it was the players themselves who changed these probabilities, and they therefore had no reason to play against them.

Expanding on the idea that this should be more connected with the concept of correlated strategies from game theory, we introduced the correlated equilibrium conditions. In this view, both the initial and the updated set of probabilities are seen as correlating devices, which tell the players externally what they should play. Then, we can analyze both sets of probabilities at the same level, and the players can still choose not to follow the result given by the instructions of the correlating device, similarly for the initial one or the updated. Like this, the new probabilities have exactly the same role as the initial ones, and we apply to them equilibrium conditions, that guarantee that the players always follow what it tells them to (in order to obtain average payoffs given by the probability distribution of the correlating device). While for the initial CD we can't have a stable solution at all temperatures, the inclusion of the B parameter in the updated energies allows for an equilibrium in which they always follow what they are told, for a range of values of B depending on the temperature, and within this interval they can choose the value of B that gives them a better payoff. We learn at this stage that it is not necessary to impose a mixed-strategy like condition to obtain the best payoff, and that the way that we were introducing it before required that they were at a correlated equilibrium, because it required that they always followed, which wasn't necessarily true.

At this point we knew how to make sure how to construct a set of probabilities that the players could always follow - but would they? What was their role in choosing those parameters for the new energies, if they were still just following? Was this just a different kind of CD that they were following, or was this new CD really representative of how they wanted to react to the information they received from the initial CD? What if we didn't classified the CDs simply as stable or unstable, and allowed the players to not follow sometimes, and follow other times? We find out that, indeed, these new energies could not be representative of their own choice to act upon their information, but that they could rather be seen as a better way to perhaps introduce an initial matrix, still liable to be followed or not. We go on to develop the formalism that we describe in this thesis, which has embeded from start to finish the idea that, while we will always have to describe the game as an external CD in relation to the players, we want to find the one CD, for a certain temperature and parameters, that the players *will always want to follow*. By allowing the players to not follow the initial information in a controlled way, we can create a new correlating device that is in equilibrium and for which the probability of following is 1. The implication from the point of view of the

Ising model is that the whole interaction energy will have to change in order to achieve this, not only the part associated with the external magnetic field.

We introduce the follow/not follow ideas as they were understood by the end of this thesis, and calculate the best correlating device possible under these conditions starting from the initial correlating device, for the whole range of temperatures and parameters, and create a taxonomy according to these.

Naturally, much is still left to do. Just for this formalism with two players, we can improve on the taxonomy by finding an analytic form for the maximum payoffs, for example with Lagrange multipliers constrained to the domain of P_F and the equilibrium conditions, to understand better how the payoffs change with the following probabilities. We can also calculate the probabilities that a player has of playing C or D in the renormalized probabilities, and try to work on a simplified formalism, where these might correspond to the mixed strategy probabilities given by some renormalized payoffs[1]. To extend it further we can apply the follow/not follow mechanism to a bigger range of initial CDs, be it by introducing the parameter B as initially, or by changing its anatomy completely.

On an exploratory level, we might want to see the theoretical connections between our formalism and stochastic games, which impose small perturbations on the payoff through a stochastic transformation, which can evolve to a Markov equilibrium (Fudenberg and Tirole, 1991). Since we also use stochastic transformations to change our payoffs, although not for small transformations, these concepts might shed some light on further development. If we want to work with small perturbations, we might also want to analyze how they scale using a renormalization-group flow (Stoof, Gubbels, and Dickerscheid, 2009), that, if properly defined, should converge to the follow/not follow probabilities that we found [2].

Next, we are still looking to find out how these ideas scale for more players, and if the maxima that we found are the same when there are more games being played at the same time. It should depend on if the game is symmetric or asymmetric, and in this case against what type of players they are playing. If we then want to study this in the context of a network, we have to check if all the games in the network are simultaneous (the network is already formed), or sequential (there are more players being added at any time step), and how their follow/not follow probabilities change with that. As a first toy model, we might want to study the statistical physics for the probabilities that we just found, without caring for the network structure; this will probably be a good approximation for symmetric games in a formed network, probably not so much for asymmetric.

Last but not least, it would be interesting to check how well our model would describe real games being played. We could see how well our model compares with simulations of games on several types of networks. Eventually, we could even do real life experiments in a sociology context to check how well people do when playing such games and are given the initial probabilities, by comparing how often they followed or not with the theoretical results that we found.

[1]Since trying to find a mixed strategy for the players was the initial approach, realizing this idea means that we have gone full circle. This realization also eliminates virtually the correlations, being possibly a better approach for the network implementation. One has for the **SD** that $P_C^R = p^R(CC) + p^R(CD) = \frac{s'}{t'+s'-1}$ and $\langle u_i^R \rangle = \frac{s't'}{s'+t'-1}$, while for the **BoS** that $P_C^{R^1} = \frac{1}{1+s'}$.

[2]The idea of using the RG-flow was first introduced when we didn't know if our follow/not follow probabilities would renormalize properly. However, this idea might still be of use since it seems that the payoff changes along a constant line of either P_{F_C} or P_{F_D} for large intervals of temperature, allowing us to track the flow in the variable that changes in temperature. If we want to look at a renormalization in "time", although we could achieve renormalization in one step, in principle we might be able to break it down in smaller steps to study it in this context.

Chapter 7

Conclusion

In this thesis we developed a new method to study correlated games inspired by the Ising model. Because we wanted to study how the players react to what a correlating device tells them to do, we developed a way to introduce the players choices in a statistical manner, developing a renormalized correlating device that represents the final statistics of the game. By defining a continuum of correlation matrices, we observed how the system behaves while the correlations start inside a correlated equilibrium and then break this equilibrium, and how the players react at a range of temperatures and parameters, for symmetric and asymmetric games. We found that if the initial correlation is stable the players can sometimes do better than to always follow, and when the initial correlation is not stable they can still use the initial correlations to create a new stable set of probabilities, that gives them a better payoff than the mixed strategy equilibrium.

By finding the probabilities of a new correlating device, we can effectively map these games onto an Ising model, which is related with the actual statistics of the game. This will hopefully be useful for the study of correlated games on networks, by comparing the results of numerics with the statical physics predictions, giving insight about the microscopic behaviour.

As for Bob and Alice? We hope they can find a suitable correlating device and use these methods to solve their arguments.

Appendix A

Generalization of Initial Correlating Device for Snowdrift

A.1 Snowdrift

A.1.1 Pure and Mixed equilibria

We want to study how the correlated equilibrium conditions given by 2.74 act on the correlations given by table 3.2.

Since the players are symmetric, we assume that $B_1 = B_2$. Our correlation matrix is then

TABLE A.1

	C	D
C	$e^{\beta(1+2B)}/Z_{new}$	$e^{\beta(t)}/Z_{new}$
D	$e^{\beta(t)}/Z_{new}$	$e^{\beta(t-s-2B)}/Z_{new}$

where $ZZ_{new} = e^{\beta(1+2B)} + 2e^{\beta(t)} + e^{\beta(t-s-2B)}$.

The conditions for this correlation game to be in correlated equilibrium are

$$\frac{e^{\beta(1+2B)} + e^{\beta t}s}{e^{\beta(1+2B)} + e^{\beta t}} \geq \frac{e^{\beta(1+2B)}t}{e^{\beta(1+2B)} + e^{\beta t}} \tag{A.1}$$

and

$$\frac{e^{\beta t}t}{e^{\beta t} + e^{\beta(t-s-2B)}} \geq \frac{e^{\beta t} + e^{\beta(t-s-2B)}s}{e^{\beta t} + e^{\beta(t-s-2B)}}. \tag{A.2}$$

It is reasonable to assume that the magnetic field has a temperature dependency, so we will describe the magnetic field as $\frac{B}{T}$.

In fig. A.1 we can see what happens as $\frac{B}{T}$ increases: in the first place the equilibrium is broken by the second condition, which means that the players will want to choose to play C if told to play D above a certain temperature, while they will always want to play C if told so; as it becomes big enough, the situation is reversed and the players will want to play D if told to play C and will want to play D if told accordingly.

What we verify is the following: if $\frac{B}{T}$ is bellow the coefficient of $\frac{1}{T}$ that corresponds to the mixed strategy solutions, namely $\frac{1}{2}\ln\left(\frac{s}{(t-1)}\right)$, then eq. A.1 is always true, while eq. A.2 is only met bellow a certain temperature; the reverse is true if $\frac{B}{T}$ is above that value. If we are at exactly that value then both conditions are met for all temperatures. We will now elaborate further on these conclusions .

The solution of equality for equation A.1 is

$$\frac{B}{T} = \frac{t-1}{2T} + \frac{1}{2}\ln\left(\frac{s}{(t-1)}\right). \tag{A.3}$$

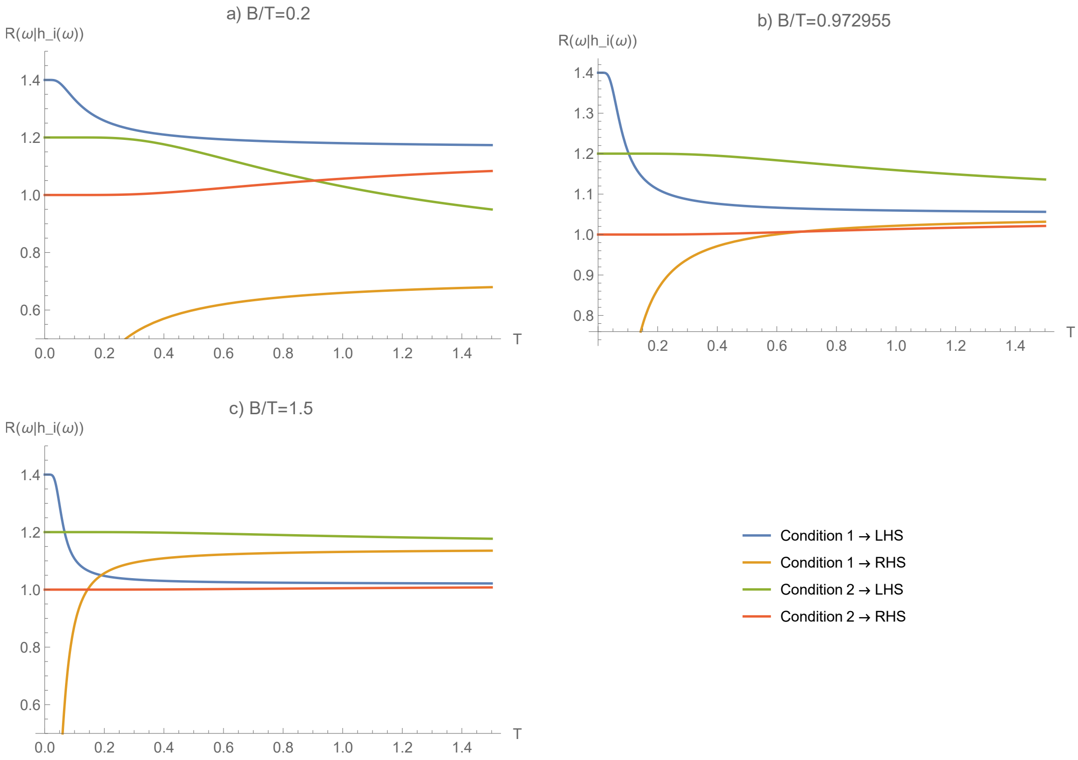


FIGURE A.1: Equilibrium conditions for $s = 1.4$ and $t = 1.2$. If the system is in correlated equilibrium the LHS is higher than the RHS for each condition. In figure a) we have a B/T that is smaller than than $\frac{1}{2}\frac{s}{(t-1)}$, the mixed strategy slope, which in this case is 0.972955, and we see that at a certain temperature the RHS of the second condition, which means that the stability is broken, but not on the first condition. In figure b) we are at exactly the mixed strategy slope, and at both sides of each equation only meet at infinity. In figure c) the slope is higher than that of the mixed strategy and it is the first equation that brings instability from a certain temperature.

and for equation A.2 is

$$\frac{B}{T} = -\frac{s}{2T} + \frac{1}{2}\ln\left(\frac{s}{(t-1)}\right) \tag{A.4}$$

If we plot both of these equations (fig. A.2) we see that these are the equations of two hyperbolas, with vertical asymptote corresponding to $T = 0$ and with horizontal asymptote $\frac{B}{T} = \frac{1}{2}\ln\left(\frac{s}{(t-1)}\right)$. One of the solutions approaches this asymptote from below (corresponding to the break of equilibrium in the second condition) and one from above (breaking the equilibrium in the first condition), both converging in the mixed strategy coefficient. Further we will refer to these two curves as, respectively, the *lower* and *upper* curves.

We can then talk of an equilibrium interval: for each value of the temperature there is a range in $\frac{B}{T}$ from which the players can choose any value and end up at a correlated equilibrium. It is interesting to note that the high temperature limit of the equilibrium lines is the mixed strategy value, since the mixed strategy solution is the solution that we found when we looked for the high temperature solution. This indicates that for high temperatures this uncorrelated solution is the only solution for the system, and in the absolute limit we recover that the probability for every strategy is $\frac{1}{2} \times \frac{1}{2}$.

Now we must answer the following question: given a certain temperature, what are the values of $\frac{B}{T}$ the yield the highest payoff inside the stability interval?

In fig. A.3 we see, first of all, that the maximum payoff is, in any of the cases, higher than the mixed strategy payoff, which justifies the use of the correlated equilibrium by the players (find a better way to argue for this). Secondly, we notice that the point where we find this

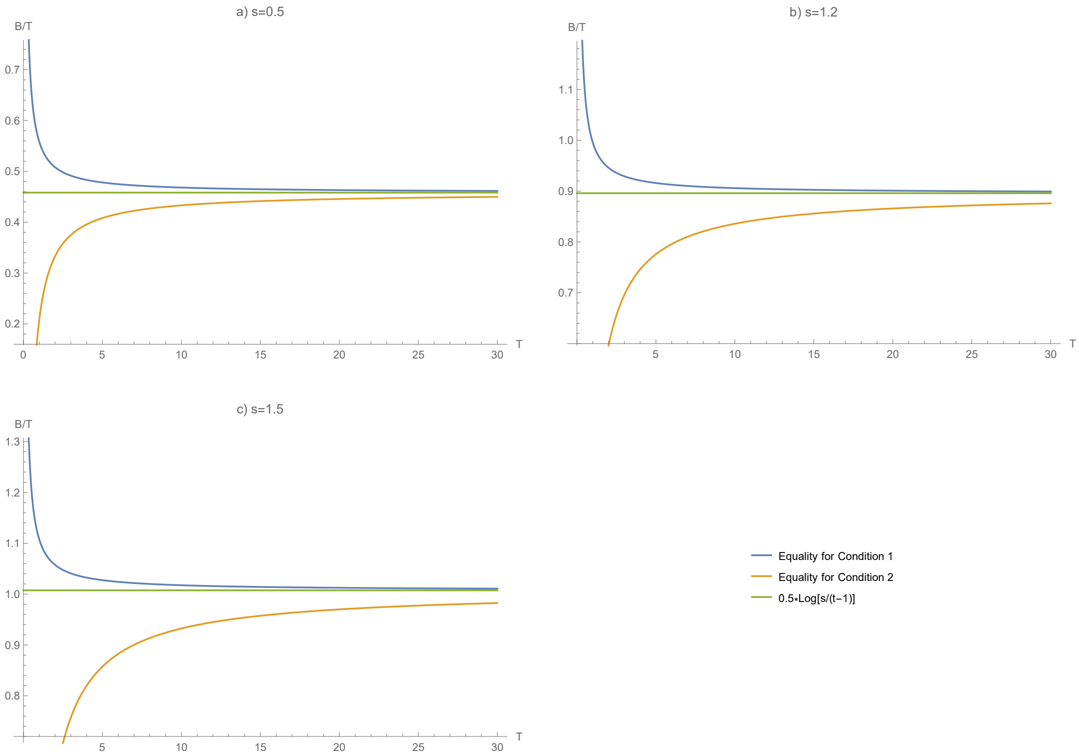


FIGURE A.2: Equilibrium range for $t = 1.2$, and several values of s. Below the blue line and above the orange line the players can choose any value of $\frac{B}{T}$ and be in a correlated equilibrium. The green line corresponds to the slope of the mixed strategy, for which borht equilibrium lines converge.

maximum can change depending on the parameters: for $a)$ it is on the higher equilibrium limit in $\frac{B}{T}$, by truncation; for $b)$ and $c)$ it is inside the interval; and for $d)$ it is, again by truncation, in the lower equilibrium limit in $\frac{B}{T}$.

We can inspect the maximum value of the function for varying values of s and T, keeping t constant as a control parameter, which we can do because the changes in the system only take place as the relative value between s and t changes, as we will find out later on.

In figure A.4 we see how $\frac{B}{T}$ changes along the temperature, for the same vales of s as in A.3 and some more. For $T = 3$ we find the maximum where we saw it was, but we also see that the place where we expect to find it is not constant long the temperature.

This is mirrored in what we can see in fig. A.5, from a different perspective. Most of the payoff curves have a region on the left side, then it changes abruptly to a different curve with a different slope, and then it changes again to a more linear form (find a better way of describing this). Comparing what happens when the vertical lines cross the payoff curves in this figure with the payoff curves along temperature in fig. A.4 allows us to see what these different regions are. Let's take for example the the vertical line corresponding to $s = 1.0$: for low temperatures it crosses on the middle part, and for higher temperatures it crosses in the left part, which means that the left part corresponds to having the limit by truncation in the upper $\frac{B}{T}$ curve, the middle part has the maximum inside the equilibrium interval, and the left part has the maximum by truncation in the lower curve (which we can see by seeing the vertical line of $s = 1.3$).

Since the payoff is an analytical function, we thus expect that the maxima inside the interval corresponds to a zero in the derivative. The expression of the derivative is

$$\frac{\partial\langle u_i\rangle}{\partial\frac{B}{T}} = \frac{2e^{\frac{2B+s+t}{T}}\left((s+t-2)\left(-e^{\frac{4B+s+1}{T}}\right)+2e^{\frac{2B+1}{T}}+(s+t)e^{t/T}\right)}{\left(2e^{\frac{2B+s+t}{T}}+e^{\frac{4B+s+1}{T}}+e^{t/T}\right)^2} \tag{A.5}$$

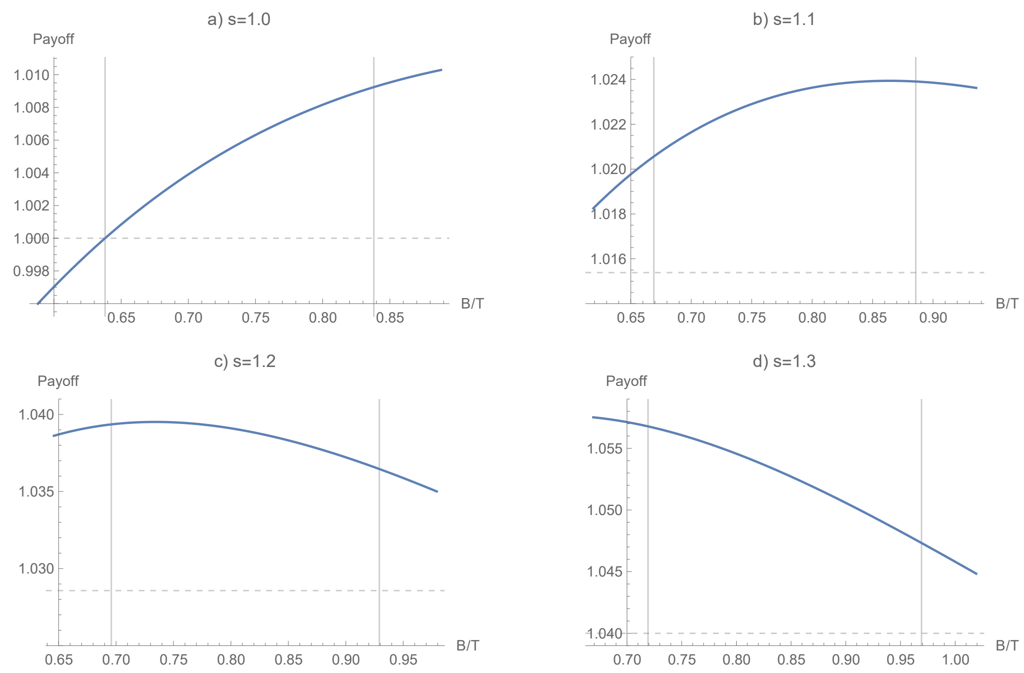


FIGURE A.3: Payoff as given by equation (insert equation) for $t = 1.2$ and $T = 3$ and several values of s. The veritical lines define are the equilibrium boundaries for the specific parameters, while the dashed line corresponds to the mixed strategy payoff.

The values of this derivative are plotted in fig. A.6 for the values of $\frac{B}{T}$ of fig. A.5. As expected, the middle region corresponds a zero derivative. For low temperatures, the function never leaves the middle region after a certain value of s, which means the maximum will always go asymptotically towards one of the curves, but never reach it.

There are four formal solutions in $\frac{B}{T}$ that correspond to eq. A.5 being zero, however, most of them have imaginary parts. The only solution that contains strictly real values is:

$$\frac{B}{T}_{sol} = \log\left(\sqrt{\frac{e^{-\frac{s+1}{T}}\left(\sqrt{e^{1/T}\left(\left(s^2+2s(t-1)+(t-2)t\right)e^{\frac{s+t}{T}}+e^{1/T}\right)}+e^{1/T}\right)}{s+t-2}}\right) \tag{A.6}$$

(show that this derivative=0 corresponds to a maximum for this solution)

When the above solution is real, it corresponds to a maximum in the payoff. If this maximum is outside the equilibrium interval, then the maximum we are interested in corresponds to one of the limit curves. When the solution has an imaginary part, then the solution is in one of the equilibrium curves as well.

$T \to \infty$

To see what happens to eq. A.6 on the high temperature limit, we can perform a series expansion in $\frac{1}{T}$ around zero, which gives

$$\lim_{T\to\infty} \frac{B}{T}_{sol} \approx \frac{t-2-s}{4T} + \frac{1}{2}\log\left(\frac{s+t}{s+t-2}\right) \tag{A.7}$$

This equation has a very similar form to eqs. A.1 and A.2. In fact, both are equations of hyperbolas and the horizontal asymptote is given by the logarithmic term. Since the coefficient is the same, the asymptote will be the same if the argument of the logarithms is the same, or $\frac{s}{t-1} = \frac{s+t}{s+t-2}$. The solution for this equality is, given the restriction in our parameters, $s = t$, so we conclude the following: if $s > t$, the asymptote of $\lim_{T\to\infty} \frac{B}{T}_{sol}$ is smaller than that of the equilibrium conditions, meaning that eventually the maximum payoff will cross the lower equilibrium condition, at which point it will become the solution; if $s = t$, we are guaranteed that the maximum payoff is always inside the equilibrium conditions; if $s < t$, the asymptote is bigger than that of the equilibrium curves, and the maximum will cross the higher equilibrium curve.

$T \to 0$

When the temperature gets very small, we can drop the exponential expressions with lower coefficients of $\frac{1}{T}$:

$$\begin{aligned}
\lim_{T\to 0} \frac{B}{T}_{sol} &\approx \log\left(\sqrt{\frac{e^{-\frac{s+1}{T}}\left(\sqrt{e^{1/T}\left((s^2+2s(t-1)+(t-2)t)\,e^{\frac{s+t}{T}}\right)}+e^{1/T}\right)}{s+t-2}}\right) \\
&\approx \log\left(\sqrt{\frac{e^{-\frac{s+1}{T}}\left(\sqrt{(s^2+2s(t-1)+(t-2)t)\,e^{\frac{s+t+1}{T}}}+e^{1/T}\right)}{s+t-2}}\right) \\
&= \log\left(\sqrt{\frac{\sqrt{(s^2+2s(t-1)+(t-2)t)\,e^{\frac{t-1-s}{T}}}+e^{-s/T}}{s+t-2}}\right) \\
&\approx \log\left(\sqrt{\frac{\sqrt{(s+t-2)(t+s)e^{\frac{t-1-s}{T}}}}{s+t-2}}\right) \\
&= \frac{1}{4}\log\left(\frac{(t+s)e^{\frac{t-1-s}{T}}}{s+t-2}\right) \\
&= \frac{t-1-s}{4T} + \frac{1}{4}\log\left(\frac{t+s}{s+t-2}\right)
\end{aligned} \tag{A.8}$$

where in the fourth line we used that $\lim_{T\to 0} e^{-s/T} = 0$ because $s > 0$.

Here we see that one of three things can happen:

- $s < t - 1$:the coefficient of $1/T$ is positive, and since the temperature is very small B/T diverges to $+\infty$, and B will have a constant positive value for absolute zero;
- $s = t - 1$: the coefficient of $1/T$ is zero and B/T has a constant value that corresponds to $\frac{1}{4}\log\left(\frac{2s+1}{2s-1}\right)$, while B will be zero for absolute zero; also, the horizontal asymptote in the equilibrium conditions is $B/T = 0$;
- $s > t - 1$: the coeficient of $1/T$ is negative, so that B/T diverges to $+\infty$, while B will be a negative constant for absolute zero.

Characterization of the maximum payoff B/T depending on the parameters

In the subsections above we saw that the behavior of the curve of $B/T(T)$ that corresponds to the maximum payoff inside the correlated equilibrium changes characteristics, either regarding the limits to very high or very low temperature, at $s = t - 1$ and at $s = t$. One relation has not yet been considered: eq. A.6 becomes imaginary if the denominator inside the square root is negative, or if $s < 2 - t$. At this point, there is not a maximum inside the equilibrium and, because it belongs to the region of $s < t$, the maximum always happens at the higher equilibrium curve. As the horizontal asymptote at $T \to \infty$ becomes increasingly higher, the max payoff curve crosses the upper equilibrium curve at increasingly lower temperatures, until the asymptote becomes infinite, which only meets the upper curve at $T = 0$, which, because it is its vertical asymptote, means in practice that the maximum is always at that equilibrium curve.

The fact that, at $s = 2 - t$, the maximum at zero temperature changes from being in the limit condition to being inside the interval is rather interesting. At this temperature, without a magnetic field, the correlation matrix becomes equiprobable in the (C,D) and (D,C) strategies, and the corresponding average payoff is $\frac{s+t}{2}$, as we saw when we defined our basis correlation matrix. If we add a magnetic field, this is only the payoff if the arguments of the exponential functions corresponding to (C,C) and (D,D) are smaller than those of the off-diagonal, or in the cases of $1 + 2B < t$ or $t - s - 2B < t$. If one of those cases doesn't happen, we are back in a pure strategy solution in the diagonal, which, as we saw in the beginning, are not Nash equilibria. If we sill have a small temperature, there will be values of B/T big and small enough that permit, respectively, one of the previous conditions not to be respected, but within these values the payoff is nearly independent from the magnetic field, and corresponds to $\frac{s+t}{2}$ [1]. Nonetheless, this is only profitable if $\frac{s+t}{2} > 1$, which is the next best payoff, corresponding to a pure strategy. Because of this, our stability conditions guarantee that, if $s < 2 - t$, the best payoff is not 1, but something slightly smaller, coinciding with the stability condition. In fig. A.8 we can see where the maximum payoffs can be found for several parameters at $T = 0.001$, illustrating what has just been described.

We are then able to define six different profiles for the maximum payoff curve, that are portrayed in fig. A.7. The configuration of its types according with the temperature limits are as described in table **??**.

[1] Despite the limits of B/T in eq. A.8 predicting a maximum for a very large or very small value of B at small temperatures, this almost only relates to the sign of the majority of the values that we can adopt, as the matrix will decay very quicly to the off-diagonal values

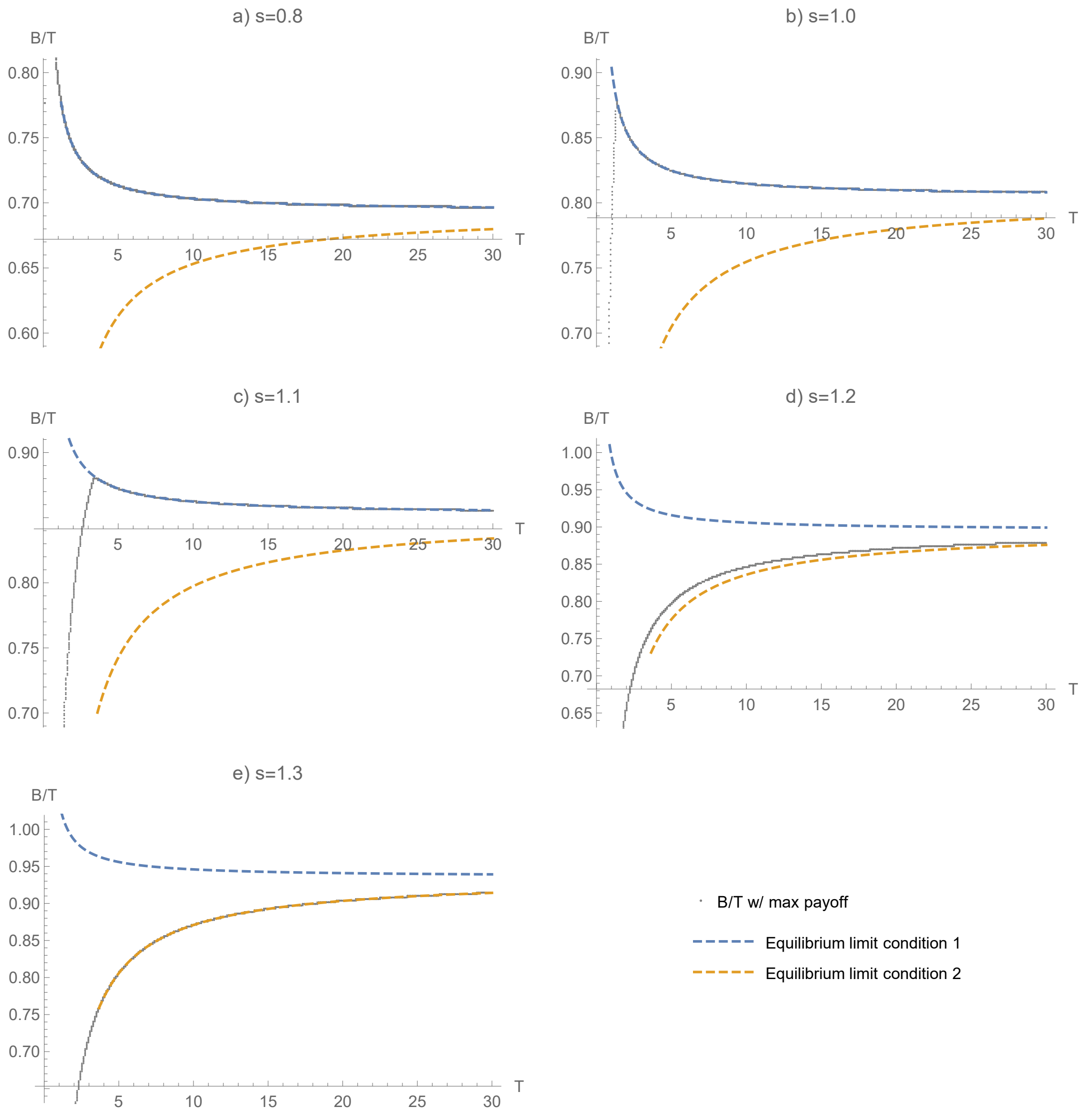


FIGURE A.4: Magnetic field corresponding to maximum payoff, for $t = 1.2$ and several values of s, depending on T. Depending on the value of s, the maximum payoff is either inside the interval of equilibrium, or corresponds to a truncation. For $a)$, $b)$ and $c)$ below a certain temperature the maximum happens inside the interval, tending asymptotically to the lower curve, and above it happens at the higher curve; for $d)$ the maximum is always inside the interval; finally for $e)$ the limit happens at the lower curve above a certain temperature, and below it happens inside, but tending asymptoticaly to the same curve.

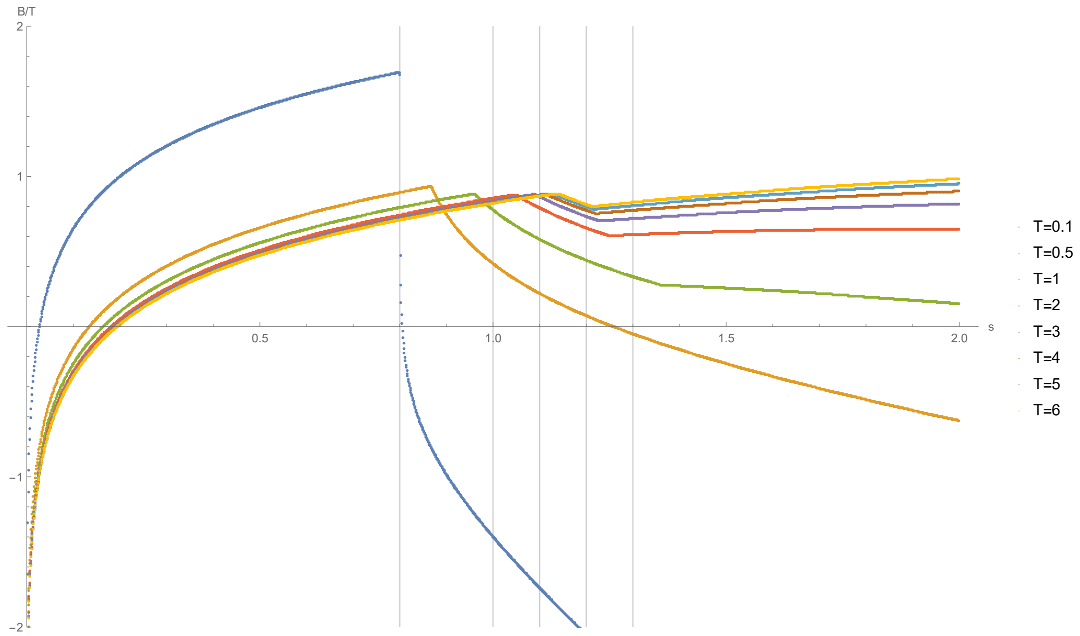


FIGURE A.5: Values of B/T corresponding to the maximum payoff for $t = 1.2$ at several temperatures, depending on s. The vertical lines correspond to the values of s that are represented in fig. A.4, namely to a) until e), from left to right.

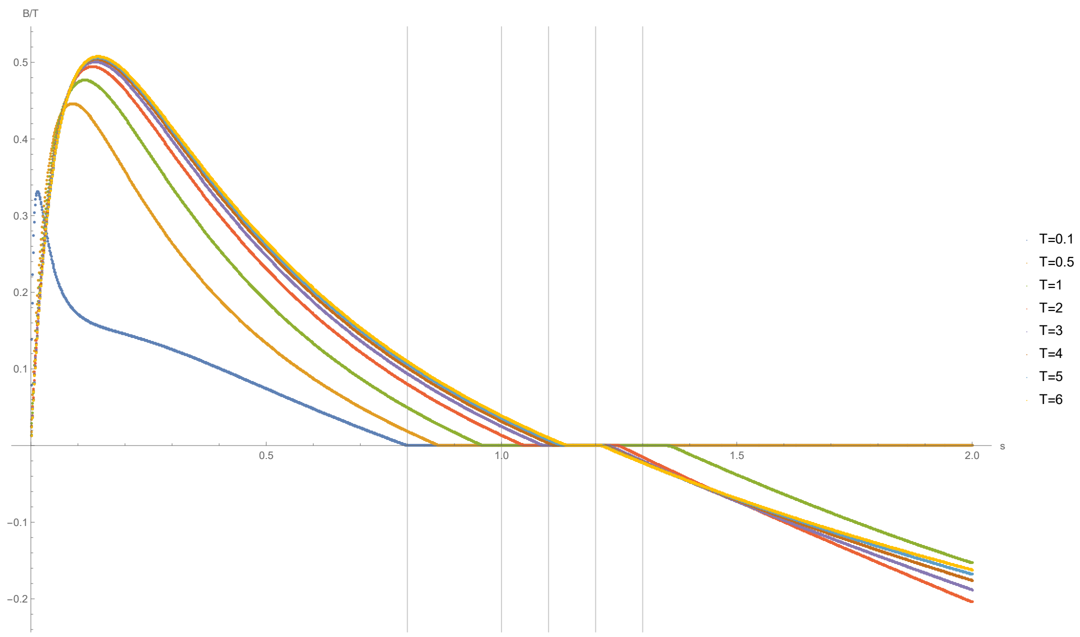


FIGURE A.6: Values of the derivative of the payoff at the value of $\frac{B}{T}$ with highest payoff, for a certain s, at $t = 1.2$ and several temperatures. Just as in A.5, the vertical lines correspond to the values of s that are represented in fig. A.4, namely to a) until e), from left to right. The curves for $T = 0.1$ and $T = 0.5$ are always zero from a certain value of s.

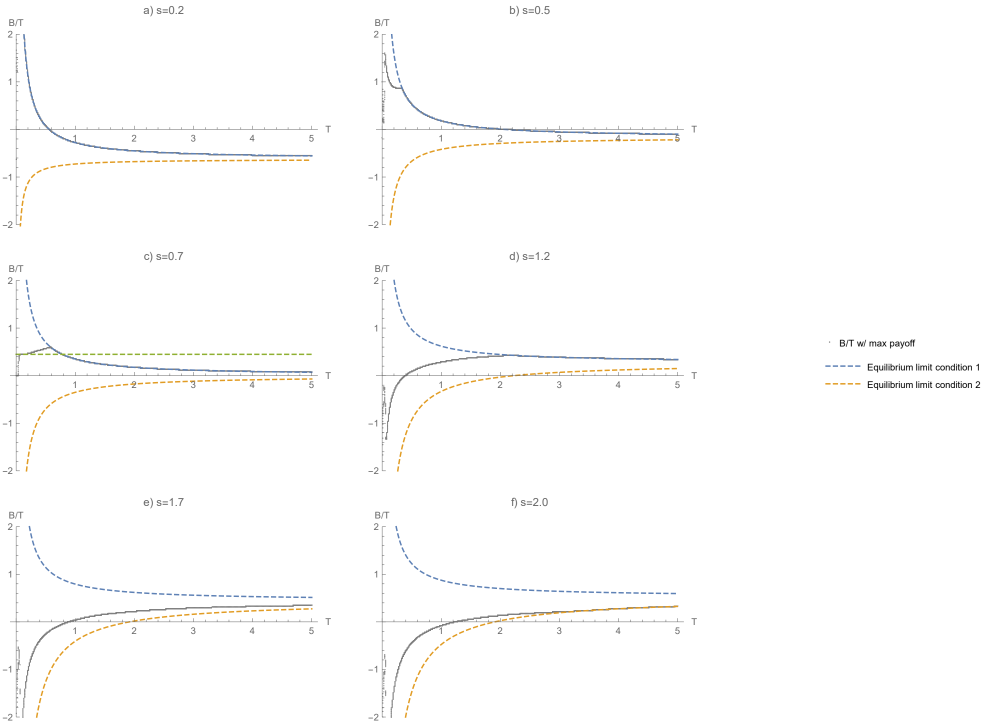


FIGURE A.7: Magnetic field corresponding to maximum payoff, for $t = 1.7$ and several values of s, depending on T. These plots represent the typical topology for: $a)$ $s < 2 - t$; $b)$ $2 - t < s < t - 1$; $c)$ $s = t - 1$ (green line corresponds to constant limit at low temperatures); $d)$ $t - 1 < s < t$; $e)$ $s = t$; $f)$ $s > t$. (Redo without temperatures below 1)

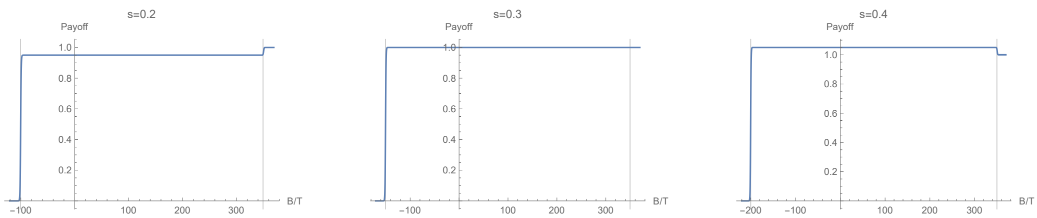


FIGURE A.8: Payoff at $T = 0.001$ for $t = 1.7$ and three different values of s. The vertical lines correspond to the stability limits. The flat line corresponds to the $\frac{s+t}{2}$, and throughout the a big range of B/T the payoff is independent from it. If it drops from this values, it goes either to 0 or 1, which are the payoffs of the pure strategies (D,D) and (C,C). As expected, for $s < 2 - t$ the best payoff allowed is at the limit, and not at 1, whereas in the other cases the best payoff is inside the interval.

Appendix B

Transformation matrix for the probabililities

In this appendix we explore the relation between the initial and final sets of probabilities for each state, and how it depends on that transforms one in other, and the need for the follow/not follow probabilities. We justify why we chose to move from the approach in appendix A into the model that is explored in the main body of this thesis.

B.1 Transformation matrix for simple exponential probabilities

The initial correlation matrix is given in table 2.8. We can decide that the probabilities of finding the final states are just imposed by a change in the magnetic field of the Ising system, such that we end with the probabilities described in table 3.2.

We will express these probabilities as vectors, each row representing the state to which we map the corresponding probability that that state is chosen (this probability is calculated by conserving the average of the energies that we attribute at each state, cf PME).

We want a matrix that transforms the probability vector

$$\begin{pmatrix} e^{\beta}/Z_B \\ e^{\beta t}/Z_B \\ e^{\beta t}/Z_B \\ e^{\beta(t-s)}/Z_B \end{pmatrix} \tag{B.1}$$

with $Z_B = e^{\beta} + 2e^{\beta t} + e^{\beta(t-s)}$ into the probability vector

$$\begin{pmatrix} e^{\beta(1+B_1+B_2)}/Z_{new} \\ e^{\beta(t+B_1-B_2)}/Z_{new} \\ e^{\beta(t-B_1+B_2)}/Z_{new} \\ e^{\beta(t-s-B_1-B_2)}/Z_{new} \end{pmatrix} \tag{B.2}$$

with $Z_{new} = e^{\beta(1+B_1+B_2)} + e^{\beta(t+B_1-B_2)} + e^{\beta(t-B_1+B_2)} + e^{\beta(t-s-B_1-B_2)}$.

This transformation should preserve that the elements of the vectors sum to 1.

Ideally, this transformation would come about by the action of two uncorrelated probabilities that depend each on B_1 and B_2, respectively. In the players interpretation, each of these probabilities would account for the chance with which a player would choose to play by the initial or the final set of probabilities, already having in mind that these sets of probabilities act in practice as correlating devices.

The probability that player i chooses to play by the old matrix, given that he was told to play μ by that same matrix, is $P_{NF\mu,i}$: *player i does Not Follow the new matrix if told to play* μ, implying that he plays ν. With probability $P_{F\mu,i}(B_i)$ the same player chooses to follow the new matrix, and for normalization we impose that $P_{F\mu,i}(B_i) = 1 - P_{NF\mu,i}(B_i)$, with $\mu, \nu \in \{C, D\}$ and $i, j \in \{1, 2\}$. Admittedly, this is a different interpretation of the follow/not follow probabilities than the one that appears in section **??**.

If the transformation from one probability to another should come about by the use of these uncorrelated probabilities, we should have

$$p^{new}(\mu\nu) = P^1_{F_\mu} P^2_{F_\nu} p^B(\mu\nu) + P^1_{F_\mu} P^2_{NF_\mu} p^B(\mu\mu) \tag{B.3}$$
$$+ P^1_{NF_\nu} P^2_{F_\nu} p^B(\nu\nu) + P^1_{NF_\nu} P^2_{NF_\mu} p^0(\nu\mu) \tag{B.4}$$

since there are only two values for both μ, ν and i, j, and, due to the symmetry of the players for the SD game, $p^{new}(\mu\nu) = p^{new}(\nu\mu)$, with $\nu \neq \mu$.

If this ansatz holds, we are allowed to write a transformation matrix as in eq. 3.9.

Since the probabilities that we want to transform depend exponentially on the variables that we want our transformation to depend on, namely B_1 and B_2, we can linearize the system, by Taylor expanding both the probabilities that we want to achieve, and the transition probabilities.

The linearization of the new probabilities is

$$\begin{aligned}
p^{new}(\mu\nu)(B_1, B_2) =& \\
=& p^{new}(\mu\nu)(0,0) + \left.\frac{\partial p^{new}(\mu\nu)(B_1,B_2)}{\partial B_1}\right|_{B_1,B_2=0} B_1 + \left.\frac{\partial p^{new}(\mu\nu)(B_1,B_2)}{\partial B_2}\right|_{B_1,B_2=0} B_2 + \mathcal{O}(B_1^2) + \mathcal{O}(B_2^2) + \mathcal{O}(B_1B_2) \\
=& p^0(\mu\nu) + \left.\frac{\partial p^{new}(\mu\nu)(B_1,B_2)}{\partial B_1}\right|_{B_1,B_2=0} B_1 + \left.\frac{\partial p^{new}(\mu\nu)(B_1,B_2)}{\partial B_2}\right|_{B_1,B_2=0} B_2 + \mathcal{O}(B_1^2) + \mathcal{O}(B_2^2) + \mathcal{O}(B_1B_2)
\end{aligned}$$

We can then expand our transformation matrix in terms of the first order expansions of the individual player's probabilities:

$$\begin{array}{llll}
P^1_{F_C} = 1 - C_1B_1 & P^1_{NF_C} = C'_1B_1 & P^2_{F_C} = 1 - C_2B_2 & P^2_{NF_C} = C'_2B_2 \\
P^1_{F_D} = 1 - D_1B_1 & P^1_{NF_D} = D'_1B_1 & P^2_{F_D} = 1 - D_2B_2 & P^2_{NF_D} = D'_2B_2
\end{array}$$

We choose that the zeroth order term in the *NF* probabilities is zero because the zeroth order expansion of the *new* probabilities should be exactly the old matrix (when the magnetic fields are zero we have our old matrix), which is allowed by this configuration. We choose to have the constants different in F and NF (for example C_1 and C'_1) for two reasons: the first is that, although the full probabilities of Follow and Not Follow should sum to one, there is no reason to assume that that has to happen at any arbitrary point in the truncation of the expansion, instead of the sum limit; the second is that, if this is not so, the generated matrices will have linearly dependent terms that yield the system unsolvable, as we will have a chance to see.

The transformation matrix with these probabilities becomes

$$\begin{aligned}
&\begin{pmatrix}
(1-B_1C_1)(1-B_2C_2) & B_2(1-B_1C_1)D'_2 & B_1(1-B_2C_2)D'_1 & B_1B_2D'_1D'_2 \\
B_2(1-B1C_1)C'_2 & (1-B_1C_1)(1-B_2D_2) & B_1B_2C'_2D'_1 & B_1(1-B_2D_2)D'_1 \\
B_1(1-B_2C_2)C'_1 & B_1B_2C'_1D'_2 & (1-B_2C-2)(1-B_1D_1) & B_2(1-B_1D_1)D'_2 \\
B_1B_2C'_1C'_2 & B_1C'_1(1-B_2D_2) & B_2C'_2(1-B_1D_1) & (1-B_1D_1)(1-B_2D_2)
\end{pmatrix} = \\
&= \mathbb{1} + \begin{pmatrix}
-C_1 & 0 & D'_1 & 0 \\
0 & -C_1 & 0 & D'_1 \\
C'_1 & 0 & -D_1 & 0 \\
0 & C'_1 & 0 & -D_1
\end{pmatrix} B_1 + \begin{pmatrix}
-C_2 & D'_2 & 0 & 0 \\
C'_2 & -D_2 & 0 & 0 \\
0 & 0 & -C_2 & D'_2 \\
0 & 0 & C'_2 & -D_2
\end{pmatrix} B_2 \\
&+ \begin{pmatrix}
C_1C_2 & -C_1D'_2 & -D'_1C_2 & D'_1D'_2 \\
C_1C'_2 & C_1D_2 & D'_1C'_2 & -D'_1D_2 \\
-C'_1C_2 & C'_1D'_2 & D_1C_2 & D_1D'_2 \\
C'_1C'_2 & -C'_1D_2 & -D_1C'_2 & D_1D_2
\end{pmatrix} B_1B_2
\end{aligned}$$

We see in the matrices associated with the linear coefficients that, if the prime constants were the same as those without prime, there would be linearly dependent rows (and columns) and for that we need to assume they are different.

The expansion of the New probabilities vector is as follows:

$$\begin{pmatrix} e^{\beta(1+B_1+B_2)}/Z_{new} \\ e^{\beta(t+B_1-B_2)}/Z_{new} \\ e^{\beta(t-B_1+B_2)}/Z_{new} \\ e^{\beta(t-s-B_1-B_2)}/Z_{new} \end{pmatrix} \approx \begin{pmatrix} e^{\beta}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta(t-s)}/Z_B \end{pmatrix} + \begin{pmatrix} \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta(s-t+1)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ \frac{2\beta e^{\beta s}\left(e^{\beta s}+1\right)}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta e^{\beta t}\left(e^{\beta t}+e^{\beta}\right)}{\left(e^{-\beta(s-t)}+2e^{\beta t}+e^{\beta}\right)^2} \\ -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)e^{\beta(s-t)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \end{pmatrix} B_1 + \begin{pmatrix} \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta(s-t+1)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta e^{\beta t}\left(e^{\beta t}+e^{\beta}\right)}{\left(e^{-\beta(s-t)}+2e^{\beta t}+e^{\beta}\right)^2} \\ \frac{2\beta e^{\beta s}\left(e^{\beta s}+1\right)}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)e^{\beta(s-t)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \end{pmatrix} B_2$$

We are then left to solve the following two systems, corresponding to the linear expansions in B_1 and B_2:

$$\begin{pmatrix} \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta(s-t+1)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ \frac{2\beta e^{\beta s}\left(e^{\beta s}+1\right)}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta e^{\beta t}\left(e^{\beta t}+e^{\beta}\right)}{\left(e^{-\beta(s-t)}+2e^{\beta t}+e^{\beta}\right)^2} \\ -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)e^{\beta(s-t)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \end{pmatrix} = \begin{pmatrix} -C_1 & 0 & D_1' & 0 \\ 0 & -C_1 & 0 & D_1' \\ C_1' & 0 & -D_1 & 0 \\ 0 & C_1' & 0 & -D_1 \end{pmatrix} \begin{pmatrix} e^{\beta}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta(t-s)}/Z_{old} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta(s-t+1)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta e^{\beta t}\left(e^{\beta t}+e^{\beta}\right)}{\left(e^{-\beta(s-t)}+2e^{\beta t}+e^{\beta}\right)^2} \\ \frac{2\beta e^{\beta s}\left(e^{\beta s}+1\right)}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \\ -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)e^{\beta(s-t)}}{\left(e^{\beta(s-t+1)}+2e^{\beta s}+1\right)^2} \end{pmatrix} = \begin{pmatrix} -C_2 & D_2' & 0 & 0 \\ C_2' & -D_2 & 0 & 0 \\ 0 & 0 & -C_2 & D_2' \\ 0 & 0 & C_2' & -D_2 \end{pmatrix} \begin{pmatrix} e^{\beta}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta t}/Z_{old} \\ e^{\beta(t-s)}/Z_{old} \end{pmatrix}$$

Solving these we get the following values:

$$C_1 = -\frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} \quad D_1 = \frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}} \quad C_2 = -\frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} \quad D_2 = \frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}}$$

$$C_1' = 0 \qquad D_1' = 0 \qquad C_2' = 0 \qquad D_2' = 0$$

As expected, because of the symmetry of the players, the coefficients for the same play and different player are the same. Interestingly enough, up to the linear order we have that the transformation matrices are diagonal, since the prime coefficients are zero, which means that the probability of Not Following the new matrix is zero. Could this be the case at all orders?

For that, we will start from the simplest transformation matrix that we know yields the desired *new* probabilities vector from the initial ones:

$$\begin{pmatrix} e^{\beta(B_1+B_2)}\frac{Z_B}{Z_{New}} & 0 & 0 & 0 \\ 0 & e^{\beta(B_1-B_2)}\frac{Z_B}{Z_{New}} & 0 & 0 \\ 0 & 0 & e^{\beta(-B_1+B_2)}\frac{Z_B}{Z_{New}} & 0 \\ 0 & 0 & 0 & e^{\beta(-B_1-B_2)}\frac{Z_B}{Z_{New}} \end{pmatrix} \tag{B.5}$$

To compare with the previous results, we Taylor-expand this matrix up to linear order, to obtain

$$
\mathbb{1}+\begin{pmatrix}
\frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} & 0 & 0 & 0 \\
0 & \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} & 0 & 0 \\
0 & 0 & -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}} & 0 \\
0 & 0 & 0 & -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}}
\end{pmatrix} B_1
$$
$$
+\begin{pmatrix}
\frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} & 0 & 0 & 0 \\
0 & -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}} & 0 & 0 \\
0 & 0 & \frac{2\beta\left(e^{\beta s}+1\right)e^{\beta t}}{2e^{\beta(s+t)}+e^{\beta s+\beta}+e^{\beta t}} & 0 \\
0 & 0 & 0 & -\frac{2\beta\left(e^{\beta t}+e^{\beta}\right)}{e^{\beta(t-s)}+2e^{\beta t}+e^{\beta}}
\end{pmatrix} B_2.
$$

This is exactly the same result as we obtained with the initial method. We can then infer that that transformation resumms to eq. B.5.

This has several implications. First of all, it seems that these are not probability distributions, as they don't add up to one in any form. Second, it seems that they cannot be separated in a multiplicative fashion (which would be required for uncorrelated probabilities) between terms that only depend on B_1 and B_2; this can be done with the numerator, but for the denominator we would have to have that

$$
\begin{aligned}
& e^{\beta(1+B_1+B_2)}+e^{\beta(t+B_1-B_2)}+e^{\beta(t-B_1+B_2)}+e^{\beta(t-s-B_1-B_2)} \\
& =\left(ae^{\beta}+be^{\beta t}+ce^{\beta t}+de^{\beta(t-s)}\right)\left(ee^{\beta B_1}+fe^{-\beta B_1}\right)\left(ge^{\beta B_2}+he^{-\beta B_2}\right),
\end{aligned}
$$

which would require that $aeg=bfg=beh=cfh=1$ and all other products are zero, a system that doesn't have a solution.

If we were working still in our first "mixed strategy" payoff, this would not be a problem because the partial payoffs, that cancel out the partition function (as do the payoffs at the equilibrium conditions), are equal to the full payoff, and we wouldn't need normalization. However, since this is not the case, any other full average payoffs need a normalized probability distribution. And the partition function carries the information of the other available states, so the probabilities without it are meaningless.

With this model we are basically saying that, if the initial correlating device makes a choice, then it would have made the same choice with the final set of probabilities. We can think of it in two ways:

1. If the CD draws from a correlation matrix in equilibrium, then if we change the magnetic field smoothly inside the correlated region, for the same temperature, there is no reason that the players would play differently than what the CD just told them to (because it is still a correlated equilibrium). However, in practice they would pretend that they were told to play what they were in fact told to play, but with the probability distribution that includes the magnetic field that maximizes their payoff, and they would calculate everything using that assumption. This is would be a possible working interpretation, but in reality we would just be hiding the fact that the drawing would have to happen directly from the final matrix if they want to claim their full prize. [1] [2]

[1]To obtain our new probabilities we had to maximize the entropy of the new energy distributions, which means that we tampered with the original energies and now we have a new average energy that is being conserved; hence they have a different payoff.

[2]Since the relation is one-to-one, it is indifferent whether the drawing is directly from the final matrix, or whether it was drawn from the first but they have the same playing information and use the final set of probabilities.

2. If the plays are seen as spin states, then when the bank tells the players to play something they are sent to one of the basis states, and our diagonal (unitary) transformation matrix only changes the "length" of the basis vector, not the assigned state.

B.2 General conservation of L^1 integral

When we talk about wanting to preserve the sum of the probabilities, we say more formally that we want to preserve the integral of an operator in L_1, which is the space where a length is defined by the sum of the coordinates in each dimension, instead of the square root of the sum of the squares (this would be measured in L_2.

One of the ways to preserve this sum is by using a generalized permutation matrix. One of these matrices has this form:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \frac{a}{a_0} & 0 & 0 & 0 \\ 0 & \frac{b}{b_0} & 0 & 0 \\ 0 & 0 & \frac{c}{c_0} & 0 \\ 0 & 0 & 0 & \frac{d}{d_0} \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}, \tag{B.6}$$

which is the form of eq. B.5.

Another way is by using Stochastic matrices, which are matrices for which the rows or columns sum to 1 and which are used to describe Markov chains, or, in other words, processes that only depend on the previous step. If we want to preserve a sum that is not necessarily 1, these transformation matrices still work:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ B_0 & B_1 & B_2 & B_3 \\ C_0 & C_1 & C_2 & C_3 \\ D_0 & D_1 & D_2 & D_3 \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{pmatrix}, \tag{B.7}$$

$$\begin{aligned} a + b + c + d = &(A_0 + B_0 + C_0 + D_0)a_0 \\ &+(A_1 + B_1 + C_1 + D_1)b_0 \\ &+(A_2 + B_2 + C_2 + D_2)c_0 \\ &+(A_3 + B_3 + C_3 + D_3)d_0. \end{aligned} \tag{B.8}$$

If the columns add up to one, we have a stochastic left matrix, and the previous condition amounts to having the sum of the vector elements preserved.

There are more general conditions that preserve the sum for a non-stochastic option (Krengel and Lin, 1987), but the stochastic transformation that we described above is precisely of the kind that we see in eq. 3.9 under the assumption of eq. 3.3. We see then why that is the right transformation that we want to use in order to arrive at the final correlation probability: it is completely independent of the values of the original correlation probabilities, and applying it again assumes the result of the first application, so we are confident that the players have converged at the best probabilities. This logic might also have important implications if we want to look at our system as a succession of decisions in time; in our case, we conclude that the game has converged in one time step. However, when we go to networks, we might want to change this in case we want to work with sequential games by adding links at already formed networks.

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