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Existence and regularity theorems for onedimensional variational problems with superlinear growth

Bachelor Thesis

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Contents

1 Introduction

In 1760, Euler first used the words "calculus of variations" [1], but this branch of mathematics is much older. In fact, the isoperimetric inequality, one of the oldest problems in mathematics, is a problem in the calculus of variations. Many, however, date the birth of the calculus of variations to seventeenth century Europe, where Galileo formulated the brachistochrone problem in 1638 [2]. It is the problem of finding a curve, connecting two points A and B, on which a point of mass moves under the influence of gravity and without friction from A to B.
But let us first take a step back, and look at what the calculus of variations entails. This branch has two main ingredients: a function space X and a functional $\mathcal{F} : X \to \mathbb{R}$. The goal then is to find a function $u \in X$ such that

$$\mathcal{F}(u) \leq \mathcal{F}(\nu),$$

either for all $\nu \in X$, or only for those $\nu \in X$ such that $||u - \nu||_X < \delta$ for some $\delta > 0$. In the first case, u is a *global minimizer*. In the second case, u is a *local minimizer*.
Let us now consider the brachistochrone problem again. We can take A as the origin and $B = (b, -\beta)$ with b, $\beta > 0$. Then the space of admissible functions is

$$\{u \in C^1 : u(0) = 0, u(b) = -\beta, \text{ and } u(x) > 0 \quad \forall x \in (0, b]\}$$

and the functional is

$$\mathcal{F}(u) = \frac{1}{\sqrt{2g}} \int_0^b \sqrt{\frac{1 + u'^2(x)}{u(x)}} dx.$$

[3], [2]
The functionals which we consider in this thesis are *variational integrals*, which means that they are of the form

$$\mathcal{F}(u) = \int_I F(x, u(x), u'(x)) dx.$$

Here $I = (a, b)$ is a bounded real interval, and the *Lagrangian* is a given function $F : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $F = F(x, z, p)$. Our goal will be to determine whether minimizers exist in the class $C^k(\bar{I})$, for $2 \leq k \leq \infty$.
We solve this using the *direct method of the calculus of variations*. This means that we split the problem of finding a minimizer in two parts. First we enlarge the space of functions so that we can apply a general existence theorem. Then we use regularity results to show that all minimizers have certain properties, which ensures that they lie in the original space. [4]
The direct method is a "modern" approach in the calculus of variations, and was mainly developed by Lorenzo Tonelli (1885-1946) [1]. Before that, the prevalent approach had been what we may call the *classical indirect approach*, which was based on the idea that every minimum problem has a solution (which we now know is not always the case). [3]
This thesis begins with Tonelli's Existence Theorem. This ensures that a minimizer exists in the class

$$\mathcal{C}(\alpha, \beta) := \{u \in H^{1,m}(I) : u(a) = \alpha, u(b) = \beta\}$$

(where α, $\beta \in \mathbb{R}$), if the Lagrangian F satisfies certain conditions on continuity and convexity and has polynomial (or superlinear) growth.
The space $H^{1,m}(I)$ is a *Sobolev space*, a vector space of functions which are differentiable in a weak sense. Moreover, u, $u' \in L^m(I)$ for all $u \in H^{1,m}(I)$. So especially, $C^k(\bar{I}) \subset H^{1,m}(I)$.
If we are lucky, full regularity holds. This means that any minimizer of $\mathcal{F}$ in $\mathcal{C}(\alpha, \beta)$ is a C^k-function, where k corresponds to the smoothness class of F. The minimizer will then also satisfy the Euler equation

$$\frac{d}{dx} F_p(x, u(x), u'(x)) - F_z(x, u(x), u'(x)) = 0. \tag{1}$$

Any weak extremal C^2-extremal of a variational integral $\mathcal{F}$ with C^2-Lagrangian F necessarily satisfies this equation. [3] In Chapter 4 we will look at regularity. In Section 4.1 we show that full regularity holds if the Lagrangian F has polynomial growth and satisfies certain conditions on boundedness and convexity. These conditions are stronger than the conditions in Tonelli's Existence Theorem, so full regularity may not hold

in every case.
Partial regularity may hold, however. A useful result for this is Tonelli's Partial Regularity Theorem, described in Section 4.2. This theorem tells us that if F is a smooth Lagrangian with strict convexity in p, then any absolutely continuous minimizer has a (possibly infinite) classical derivative everywhere on $\bar{I}$, and is smooth almost everywhere. Using this theorem we can prove another result which says that if a smooth Lagrangian satisfies certain conditions on boundedness and has superlinear growth, then any strong local minimizer is smooth and satisfies the Euler equation (1).
The last section of Chapter 4 is about the Lavrentiev phenomenon. If a Lagrangian F exhibits the phenomenon, it means that

$$\inf\{\mathcal{F}(u) : u \in AC(I), u(a) = \alpha, u(b) = \beta\} < \inf\{\mathcal{F}(u) : u \in Lip(I), u(a) = \alpha, u(b) = \beta\}.$$

We will discuss Manià's example of a polynomial Lagrangian exhibiting the phenomenon. Furthermore, we will give some remarks on the phenomenon. Especially, that we cannot carelessly substitute an admissible space of functions with a dense one, since this may gave different values for the infimum. And we will explain what is - and is not - the most reasonable extension of the problem

$$\inf\{\mathcal{F}(u) : u \in Lip(I), u(a) = \alpha, u(b) = \beta\}. \tag{2}$$

The first chapter after this introduction contains a number of definitions and preliminary results from the analysis of variations. For example, it contains a more detailed definition of a Sobolev space.
At the very end of this thesis, there is an appendix with theorems we used in the thesis. For example, it contains the Global Inverse Function Theorem used in Section 4.1.

2 Preliminaries

This chapter contains a number of definitions and lemmas from the calculus of variations which we will use in the rest of the thesis.
This chapter is based on [3].

Assume that $F \in C^1(\mathcal{U})$, where $\mathcal{U}$ is some open set in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ containing the 1-graph $\{(x, u(x), u'(x)) : x \in \bar{I}\}$. Then $\mathcal{F}(\nu)$ is defined for any $\nu \in C^1(\bar{I})$ satisfying $||\nu - u||_{C^1(I)} < \delta$ for some sufficiently small $\delta > 0$. It follows that the function

$$\Phi(\epsilon) := \mathcal{F}(u + \epsilon\phi)$$

is defined for any $\phi \in C^1(\bar{I})$ and $|\epsilon| < \epsilon_0$ where $\epsilon_0 := \delta/||\phi||_{C^(I)}$. Moreover, Φ is a C^1-function on $(-\epsilon_0, \epsilon_0)$. Furthermore,

$$\Phi'(0) = \int_I F_z(x, u, u') \cdot \phi + F_p(x, u, u') \cdot \phi' dx.$$

Now set

$$\delta\mathcal{F}(u, \phi) := \Phi'(0).$$

We call $\delta\mathcal{F}(u, \phi)$ the *first variation of $\mathcal{F}$ at u in the direction of ϕ*. Note that $\delta\mathcal{F}(u, \phi)$ is a linear functional of $\phi \in C^1(\bar{I})$.

Definition 2.1 (Weak extremal)**.** A function $u \in C^1(I)$ satisfying

$$\delta\mathcal{F}(u, \phi) = \int_I F_z(x, u, u') \cdot \phi(x) + F_p(x, u, u') \cdot \phi'(x) dx = 0 \tag{3}$$

for all $\phi \in C_c^\infty(I)$ is said to be a weak extremal of the functional $\mathcal{F}$.

Note that if $u \in C^1(\bar{I})$, then (3) is equivalent to

$$\delta\mathcal{F}(u, \phi) = 0 \text{ for all } \phi \in C_c^\infty(I).$$

Definition 2.2 (Weak minimizer)**.** The function $u \in C^1(\bar{I})$ is a weak minimizer of $\mathcal{F}$, if

$$\mathcal{F}(u) \leq \mathcal{F}(u + \phi)$$

for all $\phi \in C_c^\infty(I)$ with $||\phi||_{C^1(I)} < \delta$, where $0 < \delta \ll 1$.

Naturally, any weak minimizer of $\mathcal{F}$ is a weak extremal of $\mathcal{F}$.

Definition 2.3 (Field of extremals)**.** We say that u is embedded into a field of extremals of F, if there is a simply connected domain $\Gamma = \{(x, c) : c \in I_0, x \in I(c)\}$ in $\mathbb{R}^2$, where I_0 is a non-empty parameter set in $\mathbb{R}$ and $I(c)$ is an interval on the real axis, and a C^1-diffeomorphism $f : \Gamma \to G$ of Γ onto a simply connected domain G in $\mathbb{R}^2$ such that f is of the form

$$f(x, c) = (x, \phi(x, c))$$

with $\phi' = dy/dx \in C^1(\Gamma)$ and satisfies

$$u(x) = \phi(x, c_0) \text{ for all } x \in \bar{I}$$

and for some $c_0 \in I_0$, where $[a, b] \subset \text{int } I(c_0)$. Furthermore, it is assumed that, for any $c \in I_0$, the function $\phi(\cdot, c)$ is an extremal of $\mathcal{F}$, i.e.

$$F_z(\cdot, \phi(\cdot, c), \phi'(\cdot, c)) - \frac{d}{dx} F_p(\cdot, \phi(\cdot, c), \phi'(\cdot, c)) = 0.$$

Lemma 2.4 (The fundamental lemma of calculus of variations)**.** *Suppose that $f \in C^0(I)$ satisfies*

$$\int_I f(x)\eta(x)dx = 0 \textit{ for all } \eta \in C_c^\infty(I). \tag{4}$$

Then we have $f(x) = 0$ for all $x \in I$.

Proof. Let $x_0 \in I$ and $\delta > 0$ such that $I = (x_0 - \delta, x_0 + \delta) \subset I$, and let χ_0 be the characteristic function of I_0. Since C_c^∞ is dense in $L^2(I)$ with respect to the L^2-norm, we infer from (4) that

$$\int_{I_0} f(x)dx = \int_I f(x)\chi(x_0)dx = 0,$$

and therefore

$$\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)dx = 0.$$

Letting $\delta \to 0$ we obtain $f(x_0) = 0$ for any $x_0 \in I$. □

Lemma 2.5 (DuBois-Reymond's Lemma)**.** *Suppose that $f \in L^1(I)$ satisfies*

$$\int_I f(x)\eta'(x)dx = 0 \text{ for al } \eta \in C_c^\infty. \tag{5}$$

Then there is a constant $c \in \mathbb{R}$ such that $f(x) = c$ a.e. on I.

Proof. Fix two Lebesgue points of x_0, $\xi \in I$ of $f$and set $c := f(x_0)$. Suppose that $x_0 < \xi$ $(x_0 - \epsilon, \xi + \epsilon) \subset I$ for $\epsilon > 0$. Define the piecewise linear function $\zeta \in C_c^0(I)$, where

$$\zeta(x) := \begin{cases} 1 & \text{if } x \in [x_0, \xi] \\ \epsilon^{-1}(x - x_0 + \epsilon) & \text{if } x \in [x_0 - \epsilon, x_0] \\ \epsilon^{-1}(\xi - x + \epsilon) & \text{if } x \in [\xi, \xi + \epsilon] \\ 0 & \text{if } x \notin [x_0 - \epsilon, \xi + \epsilon] \end{cases}$$

It follows that

$$\begin{aligned} \left|\int_I f(x)\zeta'(x)dx\right| &= \left|\frac{1}{\epsilon}\int_{x_0-\epsilon}^{x_0} f(x)dx - \frac{1}{\epsilon}\int_\xi^{\xi+\epsilon} f(x)dx\right| \\ &\le \frac{1}{\epsilon}\left|\int_{x_0-\epsilon}^{x_0} f(x)dx\right| + \frac{1}{\epsilon}\left|\int_\xi^{\xi+\epsilon} f(x)dx\right| \\ &\le \frac{2}{\epsilon}\left|\int_I f(x)dx\right| = 0 \end{aligned}$$

because of (5). So

$$\int_I f(x)\zeta'(x)dx = 0$$

which is equivalent to

$$\frac{1}{\epsilon}\int_{x_0-\epsilon}^{x_0} f(x)dx - \frac{1}{\epsilon}\int_\xi^{\xi+\epsilon} f(x)dx = 0.$$

Letting $\epsilon \to 0$ we arrive at $f(x_0) = f(\xi)$, i.e. $f(\xi) = c$ for any Lebesgue point $\xi > x_0$. If $\xi < x_0$ we reverse the roles of x_0 and ξ, and we obtain the same result. Thus we have $f(\xi) = c$ for any Lebesgue point ξ of f. Hence, $f(x) = c$ a.e. on I. □

Definition 2.6 (Weak derivative)**.** We say that a function $u \in L^p(I)$, $1 \le p \le \infty$, has a function $u \in L^q(I)$, with $1 \le q \le \infty$, as a weak derivative if

$$\int_I u\phi' dx = -\int_I \nu\phi dx \quad \forall \phi \in C_c^\infty(I). \tag{6}$$

The weak derivative is denoted by u'.

Definition 2.7 (Sobolev space)**.** Let $1 \leq p < \infty$. Let X be the linear subspace of $L^p(I)$ for which

$$||u||_{H^{1,p}(I)} := \left(\int_I (|u|^p + |u'|^p)dx\right)^{1/p} < \infty, \tag{7}$$

i.e. all functions u in $L^p(I)$ whose weak derivative u' has a finite L^p-norm. Note that $||\cdot||_{H^{1,p}(I)}$ is a norm. The completion of X with respect to this norm is denoted by $H^{1,p}(I)$ and is referred to as a Sobolev space. The space $H^{1,1}$ is the space of absolutely continuous functions on I, and is also denoted by $AC(I)$.
The space of Lipschitz functions on I is also a Sobolev space, namely $H^{1,\infty}(I)$.

Theorem 2.8. *Let $u \in H^{1,1}(a,b)$. Then, by possibly changing u on a set of measure zero, we have that u is of class $C^0([a,b])$, we have that $u \in C^0([a,b])$, which is classically differentiable almost everywhere, and its classical derivative $[u']$ coincides almost everywhere with the weak L^1-derivative u'. Moreover, for all x, $y \in [a,b]$, the fundamental theorem of calculus holds:*

$$u(x) - u(y) = \int_y^x u'(t)dt.$$

3 Existence of minimizers

In this section, we will show that minimizers exist when the Lagrangian satisfies certain conditions on continuity and convexity, and which has certain growth properties.
We will prove the main result (Tonelli's Existence Theorem) using Tonelli's Semicontinuity Theorem.
The proofs in this section are based on [3].

Theorem 3.1 (Tonelli's Existence Theorem)**.** *Suppose that the Lagrangian $F(x,z,p)$ satisfies the following conditions:*

1. *$F(x,z,p)$ and $F_p(x,z,p)$ are continuous in (x,z,p);*
2. *$F(x,z,p)$ is convex in p, i.e. $F_{pp}(x,z,p) \geq 0$ for all $(x,z,p) \in I \times \mathbb{R} \times \mathbb{R}$;*
3. *$F(x,z,p)$ has polynomial growth of order m, i.e. there are positive constants c_0, c_1, c_2 and a constant $m > 1$ such that*
$$c_0|p|^m \leq F(x,z,p) \leq c_1|p|^m + c_2 \text{ for all } x,z,p.$$

Then there exists a minimizer of
$$\mathcal{F}(u) := \int_I F(x,u(x),u'(x))dx$$
in the class
$$\mathcal{C}(\alpha,\beta) := \{u \in H^{1,m}(I) : u(a) = \alpha, u(b) = \beta\}$$
where α, β are fixed constants in $\mathbb{R}$.

In order to prove this, we first need to prove the following lemma.

Lemma 3.2 (Tonelli's Semicontinuity Theorem)**.** *Let I be a bounded open interval in $\mathbb{R}$ and let $F(x,z,p)$ be a Lagrangian satisfying the following conditions:*

1. *F and F_p are continuous in (x,z,p);*
2. *F is non-negative or bounded below by an L^1-function;*
3. *F is convex in p.*

Then the functional
$$\mathcal{F}(u) = \int_I F(x,u(x),u'(x))dx$$
is sequentially lower weakly continuous in $H^{1,m}(I)$ for all $m \geq 1$, i.e. if $\{u_k\}$ converges weakly in $H^{1,m}(I)$ to u, then
$$\mathcal{F}(u) \leq \liminf_{k\to\infty} \mathcal{F}(u_k). \tag{8}$$
Equivalently we can say that (8) holds if $\{u_k\}$ converges uniformly to u and the L^1-norms of u_k' are equibounded.

First, we pass to a subsequence to ensure convergence almost everywhere. We use measurability results to find a compact subset of I on which uniform convergence and continuity hold. We then use Lebesgue's Absolute Continuity Theorem, convexity and compactness to obtain a lower bound, from which the result follows.

Proof of 3.2. It suffices to consider only the case $m = 1$, since if $\{u_k\}$ converges weakly to u in $H^{1,m}(I)$ for some $m \geq 1$, it also converges to u in $H^{1,1}(I)$.
Let $\{u_k\}$ be a sequence which converges weakly to u in $H^{1,1}(I)$. Passing to a subsequence we can assume that $\{u_k\}$ converges to u in L^q for every $q \geq 1$, hence almost everywhere. Assume now that $\mathcal{F}(u)$ is finite.
Because $\{u_k\}$ converges to u almost everywhere and u_k are measurable, we can apply Egorov's Theorem to find $K_1 \subset I$ such that $u_k \to u$ uniformly in K_1.
Because u and u' are measurable, we can apply Lusin's Theorem to find a compact set $K_2 \subset I$ such that u

and u' are continuous in K_2.
Now, let $K = K_1 \cap K_2$. Especially, K is compact. Then, by Lebesgue's Absolute Continuity Theorem

$$\int_K F(x,u,u')dx \geq \int_I F(x,u,u')dx - \epsilon$$

(if $\mathcal{F}(u) = +\infty$, we can assume that $\int_K F(x,u,u')dx > 1/\epsilon$).
Since F convex in p, we obtain

$$\begin{aligned}
\mathcal{F}(u_k) &\geq \int_K F(x,u_k,u_k')dx \\
&\geq \int_K F(x,u_x,u')dx + \int_K F_p(x,u_k,u')(u_k' - u')dx \\
&= \int_K F(x,u_k,u')dx + \int_K F_p(x,u,u')(u_k' - u')dx + \int_K [F_p(x,u_k,u') - F_p(x,u,u')](u_k' - u')dx.
\end{aligned}$$

Since u and u' are continuous on K, both functions are bounded on K. Because F_p is continuous as well, it follows that $F_p(x,u(x),u'(x))$ is bounded. Moreover, because $u_k \to u$ uniformly on K, it follows that $u_k' \to u'$ uniformly on K with respect to the L^1-norm. Hence,

$$\int_K F_p(x,u,u')(u_k' - u')dx \to 0 \text{ as } k \to \infty.$$

Since u, $u_k \in H^{1,1}(I)$, it follows that u' is bounded in L^1 and u_k' are equibounded in L^1. Therefore, $u_k' - u'$ are equibounded in L^1. Because $F_p(x,z,p)$ is continuous and $u_k \to u$ uniformly on K, it follows that $F_p(x,u_k,u') - F_p(x,u,u')$ converge uniformly to zero on K as $k \to \infty$. Therefore,

$$\int_K [F_p(x,u_k,u') - F_p(x,u,u')](u_k' - u')dx \to 0 \text{ as } k \to \infty$$

and

$$\int_K F(x,u_k,u')dx \to \int_K F(x,u,u')dx \text{ as } k \to \infty.$$

Thus,

$$\begin{aligned}
\liminf_{k\to\infty} \mathcal{F}(u_k) &\geq \liminf_{k\to\infty} \int_K F(x,u_k,u_k')dx \\
&\geq \liminf_{k\to\infty} \int_K F(x,u_k,u')dx \\
&= \lim_{k\to\infty} \int_K F(x,u_k,u')dx \\
&= \int_K F(x,u,u')dx \\
&\geq \int_I F(x,u,u')dx - \epsilon.
\end{aligned}$$

Since this holds for all ϵ and F satisfies the second condition, we have that

$$\liminf_{k\to\infty} \mathcal{F}(u_k) \geq \mathcal{F}(u).$$

□

Now we can prove Tonelli's Existence Theorem.

Proof of Theorem 3.1. Because F has polynomial growth, it follows that

$$\mathcal{F}(u) \geq \int_I c_0|u'(x)|^m dx > 0$$

for every $u \in H^{1,m}$. So the functional $\mathcal{F}$ is bounded from below. Let $\{u_k\}$ be a minimizing sequence in $\mathcal{C}(\alpha, \beta)$. Because of polynomial growth, we have

$$\mathcal{F}(u_k) \leq \int_I c_1 |u_k'(x)|^m + c_2 dx < \infty.$$

Because $H^{1,m}(I)$ is reflexive, and u_k is equibounded in $H^{1,m}(I)$, it follows that (a subsequence of) $\{u_k\}$ converges weakly in $H^{1,m}(I)$ to some function $u \in H^{1,m}(I)$. Lemma 3.2 then yields

$$\mathcal{F}(u) \leq \liminf_{k \to \infty} \mathcal{F}(u_k).$$

Because $\{u_k\}$ converges weakly to u in $H^{1,m}(I)$, it also converges to u in $H^{1,1}(I)$. Passing to a subsequence we can assume that $\{u_k\}$ converges to u in L^q for every $q \geq 1$, hence almost everywhere. Especially, $\alpha = \lim_{k\to\infty} u_k(a) = u(a)$ and $\beta = \lim_{k\to\infty} u_k(b) = u(b)$, so $u \in \mathcal{C}(\alpha, \beta)$. □

The result also holds if the Lagrangian satisfies the more general condition of superlinear growth. The minimizer will then be absolutely continuous. (For a proof of this, see for example [3]).

4 Regularity of minimizers

This chapter is divided into three sections, which all have to do with regularity of minimizers. First, we will look at the case that full regularity holds. Then, we look at a result from Tonelli for partial regularity, when a minimizer is regular almost everywhere. Lastly, we discuss the Lavrentiev phenomenon, which shows us that we cannot carelessly substitute an admissible space of functions with a dense one.

4.1 The regular case

In this section, we will show that full regularity holds for certain variational integrals of the form $\mathcal{F}(u) = \int_I F(x, u(x), u'(x))dx$. The Lagrangians of these integrals satisfy certain conditions on smoothness, boundedness and convexity. We will also show that the smoothness class of the minimizer is the same as the smoothness class of the Lagrangian. We end with three examples.

Theorem 4.1. *Let $I = (a, b)$ be a bounded interval in $\mathbb{R}$, and let $F(x, z, p)$ be a Lagrangian of class C^2 defined on $\bar{I} \times \mathbb{R} \times \mathbb{R}$, $N \geq 1$, satisfying the following conditions:*

1. *there are constants c_0, $c_1 > 0$ such that for all $(x, z, p) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$*
$$c_0|p|^m \leq F(x, z, p) \leq c_1(1 + |p|^m); \tag{9}$$
2. *there is a function $M(R) > 0$ such that*
$$|F_z(x, z, p)| + |F_p(x, z, p)| \leq M(R)(1 + |p|^m) \tag{10}$$
for all $(x, z, p) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$ with $x^2 + |z|^2 \leq R^2$;
3. *for all $(x, z, p) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$ we have*
$$F_{pp}(x, z, p) > 0. \tag{11}$$

Let
$$\mathcal{C}(\alpha, \beta) := \{u \in H^{1,m}(I) : u(a) = \alpha, u(b) = \beta\}$$
where α, β are fixed constants in $\mathbb{R}$. Suppose that u is a local minimizer of the variational integral
$$\mathcal{F}(u) := \int_I F(x, u(x), u'(x))dx$$
in $\mathcal{C}(\alpha, \beta)$. Then u belongs to $C^2(\bar{I})$ and satisfies the Euler equation
$$\frac{d}{dx}F_p(x, u(x), u'(x)) - F_z(x, u(x), u'(x)) = 0 \text{ on } I.$$

First, we prove that u is a weak $H^{1,m}$-extremal of $\mathcal{F}$. Next, we prove that u is of class C^1 using a global inversion argument. Then, we show that u is of class C^2 using the Implicit Function Theorem. Lastly, we prove that the Euler equation holds.
The proof is an adaptation of proofs in [3] and [5].

Proof of Theorem 4.1. Step 1: u is a weak $H^{1,m}$-extremal Because of (9)
$$-\,infty < c_0 \int_I |\nu'(x)|^m dx \mathcal{F}(\nu) \leq c_1 \int_I 1 + |\nu'(x)|^m dx <\leq \infty$$
for any $\nu \in \mathcal{C}(\alpha, \beta)$, so $\mathcal{F}(\nu)$ is well-defined. Let $\phi \in Lip(I)$ such that $|\phi(x)| \leq Q$ on I, and $|\phi'(x)| \leq Q$ a.e. on I for some $Q > 0$. Let $0 < \epsilon_0 \leq 1$, and let $\epsilon \in \mathbb{R}$ such that $|\epsilon| < \epsilon_0$. Using a generalized version of the fundamental theorem of calculus, we have
$$\begin{aligned} x^2 + (u(x) + \epsilon\phi(x))^2 &\leq b^2 + (u(x) + \epsilon\phi(x))^2 \\ &\leq b^2 + (u(x) + Q^2) \\ &\leq b^2 + (\int_I u'(x)dx + Q^2)^2 < \infty \end{aligned}$$

for all $x \in \bar{I}$. So there exists $R_0 \in \mathbb{R}$ such that

$$x^2 + (u(x) + \epsilon\phi(x))^2 \leq R_0^2 \text{ for all } x \in \bar{I}. \tag{12}$$

Moreover, we have that

$$|u'(x) + \epsilon\phi'(x)|^m \leq (|u'(x)| + |\epsilon\phi'(x)|)^m \leq (|u'(x)| + Q)^m \text{ a.e. on } I.$$

So, by (10) we have that

$$F_z(x, u + \epsilon\phi, u' + \epsilon\phi') + F_p(x, u + \epsilon\phi, u' + \epsilon\phi')$$

is a.e. dominated by the positive L^1-function

$$M(R_0)[1 + (|u'(x)| + Q)^m]. \tag{13}$$

We have that

$$\int_I |F_z(x, u + \epsilon\phi, u' + \epsilon\phi') + F_p(x, u + \epsilon\phi, u' + \epsilon\phi')|dx \leq \int_I M(R_0)[1 + (|u'(x)| + Q)^m]dx < \infty,$$

so $F_z(x, u+\epsilon\phi, u'+\epsilon\phi')+F_p(x, u+\epsilon\phi, u'+\epsilon\phi')$ is integrable, which means that the function $\Phi(\epsilon) := \mathcal{F}(u+\epsilon\phi)$ is of class C^1 on $(-\epsilon_0, \epsilon_0)$. Since u is a local minimizer of $\mathcal{F}(\nu)$ in $\mathcal{C}(\alpha, \beta)$, it follows that $\Phi(0) \leq \Phi(\epsilon)$ for $|\epsilon| < \epsilon_0$ if $\phi \in C_c^\infty(I)$, whence $\Phi'(0) = 0$ and therefore

$$\int_I F_z(x, u, u') \cdot \phi + F_p(x, u, u') \cdot \phi' dx = 0 \tag{14}$$

for all $\phi \in C_c^\infty(I)$. The formula above is the formula for the first variation of $\mathcal{F}$. Because it is equal to zero, it follows that $u \in AC(I)$ with $u' \in L^m(I)$ is a weak $H^{1,m}$-extremal of $\mathcal{F}$.

<u>Step 2: $u \in C^1(\bar{I})$</u> Because $u \in AC(I)$, we can find R_1 such that

$$x^2 + |u(x)|^2 \leq R_1^2 \text{ for all } x \in \bar{I}. \tag{15}$$

By virtue of (10), we have

$$\int_I |F_z(x, u, u')|dx \leq \int_I M(R_1)(1 + |u'(x)|^m)dx < \infty. \tag{16}$$

Therefore, $F_z(\cdot, u, u') \in L^1(I, \mathbb{R})$. An integration by parts leads to

$$\begin{aligned}\int_a^b F_z(x, u, u') \cdot \phi(x)dx &= \left[\phi(x) \cdot \int_a^x F_z(t, u, u')dt\right]_a^b - \int_a^b \left(\int_a^x F_z(t, u, u')dt\right) \cdot \phi'(x)dx \\ &= -\int_a^b \left(\int_a^x F_z(t, u, u')dt\right) \cdot \phi'(x)dx\end{aligned}$$

for all $\phi \in C_c^\infty(I)$. Combining this with (14), we get

$$\int_I \left[F_p(x, u, u') - \int_a^x F_z(t, u, u')dt\right] \cdot \phi'(x)dx = 0$$

for all $\phi \in C_c^\infty(I)$. Applying DuBois-Reymond's Lemma, we get that there is some constant $c \in \mathbb{R}$ such that

$$F_p(x, u(x), u'(x)) = c + \int_a^x F_z(t, u(t), u'(t))dt \text{ a.e. on } I. \tag{17}$$

Define

$$\pi(x) := c + \int_a^x F_z(t, u(t), u'(t))dt.$$

By (10) and (15), we have

$$\int_I |F_p(x,u,u')|dx \leq \int_I M(R_1)(1+|u'(x)|^m)dx < \infty,$$

Consider $\Psi : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \bar{I} \times \mathbb{R} \times \mathbb{R}$, defined by

$$\Psi(x,z,p) := (x,z,F_p(x,z,p)).$$

Because $F_{pp} > 0$, we have that $D\Psi(x,z,p) \neq 0$ for all $(x,z,p) \in \bar{I} \times \mathbb{R} \times \mathbb{R}$, so Ψ is injective and regular. By the Global Inverse Function Theorem (see for example [6]), we have that $\Psi : \bar{I} \times \mathbb{R} \times \mathbb{R} \to \Psi(\bar{I} \times \mathbb{R} \times \mathbb{R})$ is a C^1-diffeomorphism. Moreover, $F_{pp} > 0$ implies that $F_p(x,z,\cdot)$ is injective, and thus $\Psi(\bar{I}\times\mathbb{R}\times\mathbb{R}) = \bar{I}\times\mathbb{R}\times\mathbb{R}$. Define:

$$\begin{aligned}\sigma(x) &:= (x,u(x),u'(x))\\ e(x) &:= (x,u(x),\pi(x)).\end{aligned}$$

Then σ is defined almost everywhere on I, whereas $e(x)$ is defined for all $x \in \bar{I}$. Moreover,

$$\Psi(\sigma(x)) = e(x) \text{ a.e. on } I. \tag{18}$$

The image set $e(\bar{I})$ lies in the range of Ψ, and e is continuous on $\bar{I}$. Thus the function

$$(x,u(x),\nu(x)) := \Psi^{-1}(e(x)), \qquad x \in \bar{I},$$

is well-defined and continuous.
On the other hand, (18) implies that

$$(x,u(x),u'(x)) = \sigma(x) = \Psi^{-1}(e(x)) \text{ a.e. on } I$$

and therefore

$$u'(x) = \nu(x) \text{ a.e. on } I.$$

By a generalization of the fundamental theorem of calculus, we have that

$$u(x) = u(a) + \int_a^x u'(t)dt = u(a) + \int_a^x \nu(t)dt$$

and we obtain that $u \in C^1(\bar{I})$.

Step 3: $u \in C^2(\bar{I})$

Because $u \in C^1(\bar{I})$, we get that

$$F_p(x,u(x),u'(x)) = \pi(x) \text{ for all } x \in I.$$

Thus the mapping $G : \bar{I} \times \mathbb{R} \to \mathbb{R}$ defined by

$$G(x,p) := F_p(x,u(x),p) - \pi(x) \tag{19}$$

is of class $C^1(\bar{I} \times \mathbb{R})$, because F is of class C^2. Using (11), we have

$$G_p(x,u'(x)) = F_{pp}(x,u(x),u'(x)) > 0 \text{ for all } x \in \bar{I}. \tag{20}$$

Since

$$G(x,u'(x)) = 0 \text{ for } x \in \bar{I}, \tag{21}$$

it follows that $u' \in C^1(\bar{I})$ by the Implicit Function Theorem. Therefore, $u \in C^2(\bar{I})$.

Step 4: u satisfies the Euler equation
Since $u \in C^2(\bar{I})$, we can integrate (14) by parts. From this we get that

$$\int_I \left[F_z(x,u,u') - \frac{d}{dx}F_p(x,u,u') \right] \cdot \phi(x)dx = 0$$

for all $\phi \in C_c^\infty(I)$. Applying the fundamental lemma of calculus of variations, we get that

$$F_z(x,u,u') - \frac{d}{dx}F_p(x,u,u') = 0 \text{ on } I.$$

Hence, u satisfies the Euler equation.

□

A corollary of this is that regularity also holds for higher smoothness classes, i.e. if $F \in C^k$, then also $u \in C^k$.

Corollary 4.2. *If $F(x,z,p)$ is a Lagrangian of class $C^k(\bar{I} \times \mathbb{R}\mathbb{R})$, $2 \leq k \leq \infty$ which satisfies all conditions of Theorem 4.1, then any local minimizer $u \in \mathcal{C}(\alpha, \beta)$ is of class C^k.*

Proof. We only have to revise step 3 of the proof of Theorem 4.1.
Let $G : \bar{I} \times \mathbb{R} \to \mathbb{R}$ as in (19). Because $F \in C^k(\bar{I} \times \mathbb{R} \times \mathbb{R})$, it follows that $G \in C^{k-1}(\bar{I} \times \mathbb{R})$. Because (20) and (21) hold, it follows that $u' \in C^{k-1}(\bar{I})$ by the Implicit Function Theorem. Therefore, $u \in C^k(\bar{I})$. □

The following two examples will indicate that $F_{pp} > 0$ is necessary condition in 4.1. Both are taken from [3].

Example 4.3. Consider the variational integral

$$\mathcal{F}(u) := \int_0^1 (u'^2 - 1)^2 dx.$$

Every Lipschitz function u_0 in $(0,1)$ with the property that u_0' takes only the values 1 and -1 is a minimizer of $\mathcal{F}$ in the class $\{u \in H^{1,4}(0,1) : u(0) = u_0, u(1) = u_1\}$. So especially, the function

$$u_0(x) := \begin{cases} x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

is a minimizer of $\mathcal{F}$. Clearly, $u_0 \notin C^2([0,])$.
In this case, $F(x,z,p) = (p^2 - 1)^2$, and therefore $F_{pp}(x,z,p) = 12p^2 - 4$, so F does not satisfy $F_{pp} > 0$. △

The next example shows that the $F_{pp} \geq 0$ is not sufficient for regularity.

Example 4.4. Consider

$$\mathcal{F}(u) := \int_{-1}^1 u^2(2x - u')^2 dx.$$

A minimizer in $\{u \in H^{1,2}(0,1) : u(-1) = 0, u(1) = 1\}$ is the function

$$u(x) := \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$$

which is a $C^1([-1,1])$-function, but not of class C^2.
In this case, $F(x,z,p) = z^2(2x - p)^2$, so $F_{pp}(x,z,p) = 2z^2 \geq 0$.

△

The Lagrangian in the last example satisfies all conditions of Tonelli's Existence Theorem and Theorem 4.1, so a minimizer exists in $\{u \in C^\infty([1,2]) : u(1) = \alpha, u(2) = \beta\}$.
This example is by yours truly.

Example 4.5. Consider the variational integral

$$\mathcal{F}(u) := \int_1^2 (u'-1)^2(x^2+1)dx.$$

The Lagrangian $F(x,z,p) = (p-1)^2(x^2+1)$ is smooth on $[1,2]\times\mathbb{R}\times\mathbb{R}$, because it is a polynomial. We have for all $(x,z,p) \in (1,2)\times\mathbb{R}\times\mathbb{R}$ that

$$F_{pp}(x,z,p) = 2(x^2+1) > 0.$$

Moreover,

$$(p+1)^2(x^2+1) \leq 5(p+1)^2 \leq 10(p^2+1)$$

and

$$(p+1)^2(x^2+1) \geq 2(p+1)^2 = 2p^2+4p+1 \geq 2p^2+5 \geq 2p^2.$$

So F has polynomial growth of degree 2. We can thus apply Tonelli's Existence Theorem, which ensures that a minimizer u of $\mathcal{F}$ exists in the class $\mathcal{C}(\alpha,\beta) := \{u \in H^{1,2}((1,2)) : u(1)=\alpha, u(2)=\beta\}$.
Because F also satisfies

$$|F_z(x,z,p)| + |F_p(x,z,p)| = |F_p(x,z,p)| = |2(p+1)(x^2+1)| \leq 10|p+1| \leq 20(p^2+1)$$

for all $(x,z,p) \in [1,2]\times\mathbb{R}\times\mathbb{R}$. So F also satisfies (10) for $M(R)=20$. Hence, F satisfies all conditions of Theorem 4.1. Because $F \in C^\infty([1,2]\times\mathbb{R}\times\mathbb{R})$, we can apply Corollary 4.2, from which follows that any minimizer $u \in \mathcal{C}(\alpha,\beta)$ is smooth.

△

4.2 Partial regularity

Although full regularity may not hold, we may have partial regularity. This means that regularity holds *almost* everywhere. Tonelli's Partial Regularity Theorem tells us that partial regularity holds if the Lagrangian is smooth and satisfies certain conditions on growth and convexity. We will prove this theorem using two lemmas. Afterwards, we will prove a consequence of Tonelli's Partial Regularity Theorem.
The proofs in this section are based on [3] and [5].

Theorem 4.6 (Tonelli's Partial Regularity Theorem)**.** *Let $F(x,z,p)$ be a smooth Lagrangian of satisfying $F_{pp}(x,z,p) > 0$ for every $(x,z,p) \in \bar{I}\times\mathbb{R}\times\mathbb{R}$, and suppose that $u \in AC(I)$ is a strong local minimizer of the functional*

$$\mathcal{F}(u) = \int_a^b F(x,u(x),u'(x))dx \tag{22}$$

in the class

$$\mathcal{C}(\alpha,\beta) := \{u \in AC(I) : u(a)=\alpha, u(b)=\beta\}, \tag{23}$$

for some α, $\beta \in \mathbb{R}$. Then u has a (possibly infinite) classical derivative $[u'(x)]$ at each point of $[a,b]$, and $[u'] : [a,b] \to \mathbb{R}\cup\{+\infty,-\infty\}$ is continuous. Moreover, the singular set $E := \{x \in [a,b] : [u'(x)] = \pm\infty\}$ is closed and has measure zero. Finally, u is of class C^∞ outside of E.

To prove this theorem, we need the following two lemmas:

Lemma 4.7. *Let $F(x,z,p)$ be a smooth Lagrangian satisfying $F_{pp} > 0$. Moreover, let $A \subset \mathbb{R}^2$ be a bounded open set, and let $M > 0$ and $\delta > 0$. Then there exists $\epsilon > 0$ such that, if $(x_0,u_0) \in A$, $|\alpha| \leq M$, $|\beta| \leq M$, the solution $u(x;\alpha,\beta)$ of the Euler equation*

$$-\frac{d}{dx}F_p(x,u,u') + F_z(x,u,u') = 0 \tag{24}$$

satisfying the initial conditions

$$u(x_0;\alpha,\beta) = u_0+\alpha, \qquad u'(x_0;\alpha,\beta) = \beta \tag{25}$$

exists for $|x-x_0| < \epsilon$ and is unique. Moreover, we have

1. u and u' are C^1-functions of x, α, β in the set

$$S := \{(x, \alpha, \beta) : |x - x_0| < \epsilon, |\alpha| \leq M, |\beta| \leq M\} \tag{26}$$

2. on the set S we have

$$|u'(x : \alpha, \beta) - \beta| < \delta \tag{27}$$

$$\frac{\partial u}{\partial \alpha}(x; \alpha, \beta) > 0, \qquad sign \frac{\partial u}{\partial \beta}(x; \alpha, \beta) = sign(x - x_0). \tag{28}$$

Proof. Because $F_{pp} > 0$, solving (24) is equivalent to solving

$$u'' = f(x, u, u') \tag{29}$$

where $f(x, z, p) := (F_z - F_{px} - pF_{pz})/F_{pp}$.
Because F is smooth, f is continuous in $I \times \mathbb{R} \times \mathbb{R}$. Because u is a solution of the Euler equation, $u \in C^1$, so (29) is a continuous, explicit first-order differential equation. We can thus apply Peano's Existence Theorem, from which follows that there exists a solution $u(x)$ on $(x_0) - \epsilon, x_0 + \epsilon)$ for some $\epsilon > 0$. It also follows that u and u' are C^1-functions of x, α, β. Since

$$\frac{\partial u}{\partial \alpha}(x_0; \alpha, \beta) = 1, \qquad \frac{\partial u}{\partial \beta} = 0, \qquad \frac{\partial u'}{\partial \beta}(x_0; \alpha, \beta) = 1$$

it follows that we can choose ϵ sufficiently small so that the second condition also holds. □

Lemma 4.8. *Let m, ρ and M_1 be three positive constants. Then there exists $\epsilon > 0$ such that if $(x_0, x_1) \subset [a, b]$, $0 < x_1 - x_0 < \epsilon$, $|u_0| \leq m$, and $|(u_1 - u_0)/(x_1 - x_0)| < M_1$, then there exists a unique solution $\tilde{u} \in C^2([x_0, x_1])$ of the Euler equation (24) satisfying $\tilde{u}(x_0) = u_0$, $\tilde{u}(x_1) = u_1$, and $\max_{[x_0,x_1]} |\tilde{u}(x) - u_0| < \rho$. Moreover, $\tilde{u}$ is the unique minimizer of*

$$\mathcal{F}(u; (x_0, x_1)) := \int_{x_0}^{x_1} F(x, u, u')dx \tag{30}$$

over the set $\mathcal{A} := \{u \in H^{1,1}((x_0, x_1)) : u(x_0) = u_0, u(x_1) = u_1, \max_{[x_0,x_1]} |\tilde{u}(x) - u_0| < \rho\}$.

Proof. Let $\sigma := m + \rho$, $A = (a, b) \times (-\sigma, \sigma)$, $M > \max(M_1, 2\rho)$, and let $0 < \delta < M - M_1$. Let $\epsilon > 0$ be as in Lemma 4.7, and suppose in addition that $3M\epsilon < \rho$. Integrating (27) for $x \in [x_0, x_1]$, we get that

$$\delta(x - x_0) \geq \int_{x_0}^{x} |u'(t; \alpha, \beta) - \beta| dt = |u(x; \alpha, \beta) - u_0 - \alpha - \beta(x - x_0)|.$$

So,

$$|u(x; \alpha, \beta) - u_0 - \alpha - \beta(x - x_0)| \leq \delta(x - x_0). \tag{31}$$

Therefore, observing that by assumption

$$u_0 - M_1(x - x_0) \leq u_1 \leq u_0 + M_1(x - x_0),$$

we get that

$$u(x_1; 0, M) - u_0 - M(x_1 - x_0) \geq -\delta(x_1 - x_0)$$

So

$$\begin{aligned} u(x_1, 0, M) &\geq u_0 + M(x_1 - x_0) - \delta(x_1 - x_0) \\ &= u_0 + M_1(x_1 - x_0) + (M - M_1 - \delta)(x_1 - x_0) > u_1, \end{aligned}$$

and by a similar argument,

$$u(x_1; 0, -M) \leq u_0 - M_1(x_1 - x_0) - (M - M_1 - \delta)(x_1 - x_0) < u_1.$$

Because $x_1 - x_0 > 0$, we have by Lemma 4.7 that $\frac{\partial u}{\partial \beta}(x_1; 0, \beta) > 0$ for $\beta \in [-M, M]$. So there is a unique $\beta_0 \in [-M, M]$ such that $u(x_1; 0, \beta_0) = u_1$. Now define

$$\tilde{u}(x) := u(x; 0, \beta_0).$$

Setting $x = x_1$, $\alpha = 0$, $\beta = \beta_0$ in (31), we obtain

$$|u_1 - u_0 - \beta_0(x_1 - x_0)| \leq \delta(x_1 - x_0),$$

and thus

$$|\beta_0| - \left|\frac{u_1 - u_0}{x_1 - x_0}\right| \leq \left|\frac{u_1 - u_0}{x_1 - x_0}\right| \leq \delta.$$

So, we get,

$$|\beta_0| \leq \delta + \left|\frac{u_1 - u_0}{x_1 - x_0}\right| \leq \delta + M_1. \tag{32}$$

Therefore, again by (31), we have for $x \in [x_0, x_1]$ that

$$\begin{aligned}
|\tilde{u}(x) - u_0| &\leq (\delta + |\beta_0|)(x - x_0) \\
&\leq (2\delta + M_1)(x - x_0) \\
&< (2\delta + M_1)\epsilon \\
&< (2(M - M_1) + M_1)\epsilon \\
&< (2(M - M_1) + M)\epsilon \\
&< 3M\epsilon \\
&< \rho.
\end{aligned}$$

Now suppose that $\nu \in C^2([x_0, x_1])$ is also a solution of the Euler equation satisfying $\nu(x_0) = u_0$, $\nu(x_1) = u_1$, and $\max_{[x_0,x_1]} |\nu(x) - u_0| < \rho$. Then we have for some $\bar{x} \in (x_0, x_1)$

$$\nu'(\bar{x}) = \frac{u_1 - u_0}{x_1 - x_0}$$

and $(\bar{x}, \nu(\bar{x})) \in \mathcal{A}$. Applying Lemma 4.7 and in particular (27) with $(\bar{x}, \nu(\bar{x}))$ replacing (x_0, u_0), and $\nu'(\bar{x})$ replacing β, we deduce for $x \in [x_0, x_1]$ that

$$\left|\nu'(x) - \frac{u_1 - u_0}{x_1 - x_0}\right| \leq \delta.$$

In particular,

$$\begin{aligned}
|\nu'(x_0)| &\leq \delta + \left|\frac{u_1 - u_0}{x_1 - x_0}\right| \\
&< \delta + M_1 \\
&< M.
\end{aligned}$$

Since, as we have seen, there exists a unique $\beta_0 \in [-M, M]$ such that the solution of the Euler equation with initial values $u(x_0) = u_0$, $u'(x_0) = \beta_0$ has value u_1 at x_1, we deduce that $\nu'(x_0) = \beta_0$, and thus $\nu = \tilde{u}$.
To show that $\tilde{u}$ minimizes $\mathcal{F}(u; (x_0, x_1))$ in $\mathcal{A}$, we consider the one-parameter family of solutions $\{u(\cdot; \alpha, \beta_0) : |\alpha| \leq M\}$. By (31) and (32) we have for $x \in [x_0, x_1]$

$$u(x; M, \beta_0) - u_0 \geq M + (\beta_0 - \delta)(x - x_0) > M - (2\delta - M_1)\epsilon > 2\rho - \rho = \rho$$

and

$$u(x; -M, \beta_0) - u_0 \leq -M + (\beta_0 + \delta)(x - x_0) < -M + (2\delta + M)\epsilon < -2\rho + \rho = \rho.$$

Since $\frac{\partial u}{\partial \alpha}(x; \alpha, \beta_0) > 0$, it follows that $\tilde{u}$ is embedded in a field of extremals that simply covers the region $[x_0, x_1] \times [u_0 - \rho, u_0 + \rho]$. Since $F_{pp} > 0$, it follows from the Weierstrass formula that

$$\mathcal{F}(u; (x_0, x_1)) \geq \mathcal{F}(\tilde{u}; (x_0, x_1))$$

for all $u \in \mathcal{A}$, with equality if and only if $u = \tilde{u}$. $\square$

Now we can prove Tonelli's Partial Regularity Theorem.
The main ideas of the proof are the following. If at some point the difference quotients of u are equibounded, we can use lemma 4.8 to find a solution $\tilde{u}$ of the Euler equation in a small neighbourhood of such a point. Since $\tilde{u}$ can be embedded in a field of extremals, and $\alpha(x)$ (such that $u(x) = u(x;\alpha(x), M)$ in this neighbourhood) depends continuously on x, $\tilde{u}$ is a minimizer of $\mathcal{F}$ in such a small interval and coincides with u; in particular u is regular in a neighbourhood of our point. Since u is almost everywhere differentiable in the classical sense, we then infer that the difference quotients are almost everywhere bounded. Hence, u is regular in an open set Ω_0, and $E := [a, b] - \Omega_0$ has measure zero.

Proof of Theorem 4.6. Because u is a strong local minimizer in $\mathcal{C}(\alpha, \beta)$, there exists a constant $\delta_1 > 0$ such that $\mathcal{F}(u) \le \mathcal{F}(\nu)$ for all $\nu \in \mathcal{C}$ with $\max_{[a,b]} |u(x) - \nu(x)| \le \delta_1$. Let $\bar{x} \in [a, b]$ such that

$$M(\bar{x}) := \liminf_{x \to \bar{x}} \left| \frac{u(x) - u(\bar{x})}{x - \bar{x}} \right| < \infty. \tag{33}$$

Suppose that $\bar{x} \neq b$. Take $\bar{x_1} > \bar{x}$ with $\bar{x_1} - \bar{x}$ sufficiently small so that $\max_{[\bar{x},\bar{x_1}]} |u(x) - u(\bar{x})| < \delta_1/2$. Choose $M_1 > M(\bar{x})$. By (33), we can apply Lemma 4.8 with $x_0 = \bar{x}$, $u_0 = u(\bar{x})$, $\rho = \delta_1/2$, $u_1 = u(x_1)$, where $x_1 \in (\bar{x}, \bar{x_1})$ satisfies

$$x_1 - \bar{x} < \epsilon, \qquad \left| \frac{u(x_1) - u(\bar{x})}{x_1 - \bar{x}} \right| < M_1.$$

Let $\tilde{u}$ be the corresponding solution of the Euler equation, and let $\hat{u} \in AC(I)$ be defined as

$$\hat{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in [\bar{x}, x_1] \\ u(x) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \max_{[a,b]} |\hat{u}(x) - u(x)| &= \max_{[\bar{x},x_1]} |\tilde{u}(x) - u(x)| \\ &\le \max_{[\bar{x},x_1]} |\tilde{u}(x) - u(\bar{x})| + \max_{[\bar{x},x_1]} |u(\bar{x}) - u(x)| \\ &< \delta_1/2 + \delta_1/2 = \delta_1, \end{aligned}$$

and so

$$\mathcal{F}(\hat{u}) - \mathcal{F}(u) = \mathcal{F}(\tilde{u}; (\bar{x}, x_1)) - \mathcal{F}(u; (\bar{x}, x_1)) \ge 0.$$

Since $\tilde{u}$ is the unique minimizer of $\mathcal{F}(\cdot; (\bar{x_1}, x_1))$ with $\tilde{u}(\bar{x}) = u(\bar{x})$, $\tilde{u}(x_1) = u(x_1)$, and $\max_{[\bar{x},x_1]} |\tilde{u}(x) - \tilde{u}(\bar{x})| < \delta_1/2$, it follows that $\tilde{u} = u$ in $[\bar{x}, x_1]$, and hence that $u \in C^2([\bar{x}, x_1])$.
Now suppose $\bar{x} \neq a$. Take $\bar{x}_0 < \bar{x}$ with $\bar{x} - \bar{x}_0$ sufficiently small so that $\max_{[\bar{x}_0,\bar{x}]} |u(x) - u(\bar{x})| < \delta_1/2$. Choose $M_1 > M(\bar{x})$. By (33) we can apply Lemma 4.8 with $x_1 = \bar{x}$, $u_1 = u(\bar{x})$, $\rho = \delta_1/2$, $u_0 = u(x_0)$ where $x_0 \in (\bar{x}_0, \bar{x})$ satisfies

$$\bar{x} - x_0 < \epsilon, \qquad \left| \frac{u(\bar{x}) - u(x_0)}{\bar{x} - x_0} \right| < M_1.$$

Let $\tilde{u}$ be the corresponding solution of the Euler equation, and let $\hat{u} \in AC(I)$ be defined as

$$\hat{u}(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in [x_0, \bar{x}] \\ u(x) & \text{otherwise.} \end{cases}$$

Then $\max_{[a,b]} |\hat{u}(x) - u(x)| < \delta_1$, and so

$$\mathcal{F}(\hat{u}) - \mathcal{F}(u) = \mathcal{F}(\tilde{u}; (x_0, \bar{x})) - \mathcal{F}(u; (x_0, \bar{x})) \ge 0.$$

Since $\tilde{u}$ is the unique minimizer of $\mathcal{F}(\cdot, (x_0, \bar{x}))$ with $\tilde{u}(\bar{x}) = u(\bar{x})$, $\tilde{u}(x_0) = u(x_0)$, and $\max_{[x_0,\bar{x}]} < \delta_1/2$, it follows that $\tilde{u} = u$ in $[x_0, \bar{x}]$ and hence that $u \in C^2([x_0, \bar{x}])$.
Let U be a neighbourhood of the 1-graph of u on $[x_0, \bar{x}]$. Then there is a constant $c \in \mathbb{R}$ such that we can write

$$F_p(x, u(x), u'(x)) = \pi(x) \text{ for all } x \in I$$

where

$$\pi(x) := \int_a^x F_z(t, u(t), u'(t))dt + c.$$

(This follows from the equation of a weak extremal and DuBois-Reymond's Lemma. An explicit computation can be found in the proof in the previous section.) Because $u \in C^2[x_0, \bar{x}]$ and F is smooth, it follows that the mapping $G : [x_0, \bar{x}] \times \mathbb{R} \to \mathbb{R}$ defined by

$$G(x, p) := F_p(x, u(x), p) - \pi(x)$$

is of class $C^\infty([x_0, \bar{x}] \times \mathbb{R})$ and satisfies

$$\det G_p(x, u'(x)) \neq 0 \text{ for all } x \in [x_0, \bar{x}].$$

Since $G_p(x, u'(x)) = F_{pp}(x, u(x), u'(x))$ for all $x \in [x_0, \bar{x}]$. Since $p = u'(x)$, $x \in [x_0, \bar{x}]$, is a solution of the equation $G(x, p) = 0$ the Implicit Function Theorem yields $u' \in C^\infty([x_0, \bar{x}])$.
Since u is differentiable almost everywhere in $[a, b]$ it follows that

$$\Omega_0 := \{x \in [a, b] : M(x) \leq \infty\}$$

is an open subset of $[a, b]$ of full measure, and that $u \in C^\infty(\Omega_0)$.
It remains to show that the classical derivative $[u'(x)]$ of u exists everywhere, and is a continuous function with values in $\mathbb{R} \cup \{\pm\infty\}$. It suffices to consider points x_2 in $E := [a, b] \setminus \Omega_0$.
Let $x_2 \in E$, so that $M(x_2) = \infty$. Suppose that $(x_2) \in (a, b)$. By an appropriate reflection of the variables x and/or u we can suppose without loss of generality that there exist points $y_j \to x_2$, $y_j < x_2$ with

$$\lim_{j\to\infty} \frac{u(x_2) - u(y_j)}{x_2 - y_j} = +\infty.$$

Let $M > 0$, $\delta > 0$ be arbitrary and apply Lemma 4.7 with $u_0 = u(x_2)$. The solutions $\{u(\cdot; \alpha, M) : |\alpha| \leq M\}$ of the Euler equation form a field of extremals simply covering some neighbourhood of (x_2, u_0) in $\mathbb{R}^2$. Thus, for $|x - x_2|$ sufficiently small there exists a unique $\alpha(x)$ with $|\alpha(x)| \leq M$ such that $u(x) = u(x; \alpha(x), M)$, and by the Implicit Function Theorem and (28) α depends continuously on x. Clearly, $\alpha(x_2) = 0$.

Claim 1. $\alpha(x)$ is non-decreasing near x_2.

Proof. Suppose that there exist sequences $a_j \to x_2$, $b_j \to x_2$, $c_j \to x_2$ with $a_j < b_j < c_j$ and $\alpha(a_j) = \alpha(c_j) \neq \alpha(b_j)$. Then for large enough j the solution $\nu_j(x) := u(x; \alpha(a_j), M)$, $a_j \leq x \leq c_j$ satisfies $\nu_j(a_j) = u(a_j)$, $\nu_j(b_j) \neq u(b_j)$, $\nu_j(c_j) = u(c_j)$ and $\max_{[a_j,c_j]} |u(x) - \nu_j(x)| \leq \delta_1$. Since ν_j is embedded in a field of extremals and $F_{pp} > 0$, the Weierstrass formula gives

$$\int_{a_j}^{c_j} F(x, u, u')dx > \int_{a_j}^{c_j} F(x, \nu_j, \nu_j')dx$$

contradicting our hypothesis that u is a strong relative minimizer.
Thus α is either non-decreasing or non-increasing near x_2. The latter possibility is excluded, however, by noting that by integrating (27) we get

$$|u(y_j; \alpha(y_j), M) - u(x_2) - \alpha(y_j) - M(y_j - x_2)| \leq \delta(y_j - x_2)$$

and hence

$$\frac{\alpha(y_j)}{x_2 - y_j} \leq \delta + M - \frac{u(x_2) - u(y_j)}{x_2 - y_j}.$$

It follows that $\alpha(y_j) < 0$ for j sufficiently large. This contradicts $\lim_{j\to\infty} \alpha(y_j) = \alpha(x_2) = 0$, however. ∎

Now, let $x_j \to x_2$, $z_j \to x_2$ with $x_j > z_j$. Then, for large enough j,

$$\begin{aligned}\frac{u(x_j) - u(z_j)}{x_j - z_j} &= \frac{u(x_j;\alpha(x_j),M) - u(z_j;\alpha(z_j),M)}{x_j - z_j}\\ &\geq \frac{u(x_j;\alpha(z_j),M) - u(z_j;\alpha(z_j),M)}{x_j - z_j}\\ &= u'(w_j;\alpha(z_j),M)\\ &\geq M - \delta\end{aligned}$$

where $x_j \geq w_j \geq z_j$ and we have used (27). Since M and δ are arbitrary, we have

$$\lim_{j\to\infty} \frac{u(x_j) - u(z_j)}{x_j - z_j} = \infty. \tag{34}$$

In particular the classical derivative $[u'(x_2)]$ exists and equals $+\infty$. A similar argument applies if $x_2 = a$ or $x_2 = b$. We thus have shown that $[u'(x)]$ exists for all $x \in [a,b]$. The continuity of $[u'(x)]$ is obvious if $x \in \Omega_0$, and follows from (34) otherwise. □

As a consequence of Tonelli's Partial Regularity Theorem, we have the following theorem:

Theorem 4.9. *Suppose that $F(x,z,p)$ is a smooth Lagrangian with superlinear growth, i.e.*

$$\lim_{|p|\to\infty} \frac{F(x,z,p)}{|p|} = \infty$$

with $F_{pp} > 0$. Let $u \in AC(I)$ be a strong local minimizer of the functional

$$\mathcal{F}(u) = \int_I F(x,u,u')dx$$

with respect to its own boundary values, and suppose either that

$$F_z(\cdot,u,u') \in L^1(I) \tag{35}$$

or that

$$F_x(\cdot,u,u') \in L^1(I). \tag{36}$$

Then u is smooth and satisfies both the Euler equation

$$-\frac{d}{dx}F_p(x,u,u') + F_z(x,u,u') = 0 \tag{37}$$

and

$$\frac{d}{dx}[F(x,u,u') - u'(x)F_p(x,u,u')] = F_x(x,u,u'). \tag{38}$$

Proof. Let Ω_1 be a maximal interval in $\Omega_0 := [a,b] \setminus E$, where E is the singular set of u. By Theorem 4.6, u is smooth and satisfies (37), and thus DuBois-Reymond's equation, i.e. there is $c \in \mathbb{R}$ such that

$$F_p(x,u(x),u'(x)) = c + \int_a^x F_z(t,u(t),u'(t))dt. \tag{39}$$

Using (39), we get that

$$\begin{aligned}\frac{d}{dx}[F(x,u,u') - u'(x)F_p(x,u,u')] &= F_x(x,u,u') + u'(x)F_z(x,u,u') + u''(x)F_p(x,u,u') - u'(x)\cdot\frac{d}{dx}F_p(x,u,u') - u''(x)F_p(x,u\\ &= F_x(x,u,u') + u'(x)F_z(x,u,u') - u'(x)\cdot\frac{d}{dx}F_p(x,u,u')\\ &= F_x(x,u,u') + u'(x)F_z(x,u,u') - u'(x)F_z(x,u,u')\\ &= F_x(x,u,u').\end{aligned}$$

So, u satisfies (38).
The first case: $F_z(\cdot, u, u') \in L^1(a,b)$
Suppose that (35) holds. If we integrate (37) we get for $x \in \Omega_1$ that

$$-F_p(x, u(x), u'(x)) + \int_a^x F_z(t, u(t), u'(t))dt + c = 0$$

for some $c \in \mathbb{R}$ and $x \in \Omega_1$. Therefore,

$$\begin{aligned}|F_p(x, u(x), u'(x))| &= |\int_a^x F_z(t, u(t), u'(t))dt + c| \\ &\le \int_a^x |F_z(t, u(t), u'(t))|dt + |c| < \infty.\end{aligned}$$

Hence,

$$|F_p(x, u(x), u'(x))| \le c_1 \tag{40}$$

for some constant $c_1 \in \mathbb{R}$.
We claim that

$$|F_p(x, z, p)| \to \infty \tag{41}$$

as $|p| \to \infty$, uniformly in $x \in [a,b]$ and u in a compact subset of $\mathbb{R}$. Therefore, from (40) we see that u' is bounded in Ω_1, and thus $\Omega_1 = [a,b]$.
Let us now prove that (41) holds.
By the convexity of $F(x, z, p)$ in p we have that

$$F(x, z, 0) \ge F(x, z, p) - pF_p(x, z, p);$$

hence, for $p \neq 0$,

$$\frac{p}{|p|}F_p(x, z, p) \ge \frac{F(x, z, p)}{|p|} - \frac{F(x, z, 0)}{|p|}.$$

Therefore, for fixed x, z we deduce that

$$\lim_{p \to +\infty} F_p(x, z, p) = +\infty \qquad \lim_{p \to -\infty} F_p(x, z, p) = -\infty$$

because F has superlinear growth. Since F_p is increasing in p, we also have, for example, for $p \ge M$ that

$$F_p(x, z, p) \ge F_p(x, z, M).$$

From this we deduce that the limit in (41) is uniform in (x, z) in a compact set; otherwise there would exist a convergent sequence (x_j, z_j) and a sequence $p_j \to \infty$ such that $\liminf_{j\to\infty} F_p(x_j, z_j, p_j) < \infty$, in contradiction to

$$\liminf_{j\to\infty} F_p(x_j, z_j, p_j) \ge \liminf_{j\to\infty} F_p(x_j, z_j, M) \ge F_p(x, z, M)$$

for all M.
The case that $p \to -\infty$ follows from a similar argument.

The second case: $F_x(\cdot, u, u') \in L^1(a,b)$
Suppose that (36) holds. By integrating (38), we get for $x \in \Omega_1$ that

$$F(x, u, u') - u'(x)F_p(x, u, u') = \int_a^x F_x(t, u(t), u'(t))dt + c,$$

for some $c \in \mathbb{R}$. So especially,

$$\begin{aligned}|F(x, u, u') - u'(x)F_p(x, u, u')| &= |\int_a^x F_x(t, u(t), u'(t))dt + c| \\ &\le \int_a^x |F_x(t, u(t), u'(t))|dt + |c| < \infty.\end{aligned}$$

Hence,

$$|u'(x)F_p(x,u,u') - F(x,u,u')| \leq c_2 \tag{42}$$

for some constant $c_2 \in \mathbb{R}$.
We claim that

$$pF_p(x,z,p) - F(x,z,p) \to \infty \tag{43}$$

as $|p| \to \infty$, uniformly in $x \in [a,b]$ and u in a compact subset of $\mathbb{R}$. Therefore from (42) we see that u' is bounded in Ω_1, and thus $\Omega_1 = [a,b]$.
Let us now prove that 43 holds.
By convexity in p, we have that

$$F(x,u,1) \geq F(x,z,p) - (p-1)F_p(x,z,p),$$

and hence

$$pF_p(x,z,p) - F(x,z,p) \geq \frac{F(x,z,p)}{p}\frac{p}{p-1} - F(x,z,p)\frac{p}{p-1},$$

provided that $p > 1$. Therefore, for fixed x, u we have

$$\lim_{p\to\infty} pF_p(x,z,p) - F(x,z,p) = \infty$$

because F has superlinear growth. Since $pF_p - F$ is increasing in p, we also have, for example, for $p \geq M$ that

$$pF_p(x,z,p) - F(x,z,p) \geq MF_p(x,z,M) - F(x,z,M).$$

From this we deduce that the limit in (43) is uniform in (x,z) in a compact set; otherwise there would exist a convergent sequence (x_j,z_j) and a sequence $p_j \to \infty$ such that $\liminf_{j\to\infty} p_jF_p(x_j,z_j,p_j) - F(x_j,z_j,p_j) < \infty$, in contradiction to

$$\liminf_{j\to\infty} p_jF_p(x_j,z_j,p_j) - F(x_j,z_j,p_j) \geq \liminf_{j\to\infty} MF_p(x_j,z_j,M) - F(x_j,z_j,M) \geq MF_p(x,z,M) - F(x,z,M)$$

for all M. □

4.3 The Lavrentiev phenomenon

A Lagrangian F exhibits the Lavrentiev phenomenon if the infimum taken over the set of absolutely continuous functions is strictly lower than the the infimum over the Lipschitz functions. More specifically, if

$$\inf\{\mathcal{F}(u) : u \in AC(I), u(a) = \alpha, u(b) = \beta\} < \inf\{\mathcal{F}(u) : u \in Lip(I), u(a) = \alpha, u(b) = \beta\}.$$

It is clear that the phenomenon does not occur whenever a minimizer u is regular on $[a,b]$. On the other hand, if the Lavrentiev phenomenon does occur, then the singular set $E := \{x \in [a,b] : [u'(x)] = \pm\infty\}$ of u is nonempty (since otherwise, the minimizer would be regular).

An important example of a case where the Lavrentiev phenomenon occurs is Manià's Example.
The computations here are based on [1].

Example 4.10 (Manià's Example)**.** Consider the variational integral

$$\mathcal{F}(u) := \int_0^1 (u^3 - x)^2 u'^6 dx. \tag{44}$$

Obviously, $u(x) = x^{1/3}$ is a minimizer of $\mathcal{F}(u)$ in the class

$$\mathcal{C}(0,1) := \{u \in H^{1,1}(0,1) : u(0) = 0, u(1) = 1\}$$

as we have $\mathcal{F}(u) \geq 0$ for all $u \in \mathcal{C}(0,1)$ and $\mathcal{F}(x^{1/3}) = 0$. The function $x^{1/3}$ is also a minimizer of $\mathcal{F}$ in the class $\mathcal{C}(0,1) \cap C^1(0,1)$.

Let $\hat{u} \in \mathcal{C}(0,1) \cap Lip(0,1)$, and consider the function $\frac{1}{2}x^{1/3}$. By regularity of $\hat{u}$, there exists $a \in (0,1)$ such that $\hat{u}(x) \leq x^{1/3}/2$ for every $x \in [0,a]$ and $\hat{u}(a) = \frac{1}{2}a^{1/3}$.
Hence,

$$(\hat{u}^3(x) - x)^2 \xi^6 \geq ((x^{1/3}/2)^3 - x)^2 \xi^6 = \frac{7^2}{8^2} x^2 \xi^6 \tag{45}$$

for any $x \in [0,a]$ and $\xi \in \mathbb{R}$. Using the Hölder inequality, we see that

$$\frac{a^{1/3}}{2} = \hat{u}(a) = \int_0^a \frac{x^{1/3}}{x^{1/3}} \cdot \hat{u}'(x)dx \leq \left(\int_0^a x^{-2/5}dx\right)^{5/6} \cdot \left(\int_0^a x^2 \cdot \hat{u}'^6(x)dx\right)^{1/6} = \frac{5^{5/6}}{3^{5/6}} a^{1/2} \cdot \left(\int_0^a x^2 \cdot \hat{u}'^6 dx\right)^{1/6} \tag{46}$$

Using (45) and (46) we obtain that

$$\begin{aligned} \mathcal{F}(\hat{u}) &\geq \int_0^a (\hat{u}^3(x) - x)^2 \hat{u}'^6(x)dx \\ &\geq \frac{7^2}{8^2} \int_0^a x^2 \hat{u}'^6 dx \\ &\geq \frac{7^2}{8^2} \cdot \frac{3^5}{5^5} \cdot \frac{1}{2^6} \cdot \frac{1}{a} > 0. \end{aligned}$$

So for any $u \in \mathcal{C}(0,1) \cap C^1(0,1)$, we have $\mathcal{F}(u) > 0$. Hence,

$$0 = \inf_{\mathcal{C}(0,1)} \mathcal{F} < \inf_{\mathcal{C}(0,1) \cap Lip(0,1)} \mathcal{F}. \tag{47}$$

$\triangle$

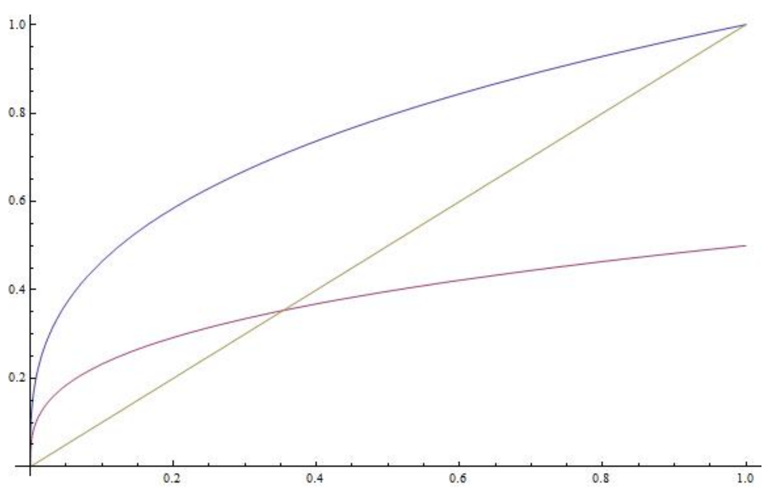

Figure 1: A plot of $x^{1/3}$, $\frac{1}{2}x^{1/3}$ and $\hat{u}$. Here, we use $\hat{u}(x) = x$.

Note that the strict inequality still holds if we replace $Lip(0,1)$ by $C^1(0,1)$, because a continuously differentiable function is especially a Lipschitz function.
Equation (47) shows that $x^{1/3}$ cannot be approximated in energy by functions in $\mathcal{C}(0,1) \cap Lip(0,1)$, i.e. there is no sequence $\{u_k\}$ in $\mathcal{C}(0,1) \cap Lip(0,1)$ such that

$$\lim_{k\to\infty} \mathcal{F}(u_k) \to \mathcal{F}(x^{1/3}).$$

It also shows us that by extending the functional in (44) from the class $\mathcal{C}(0,1) \cap Lip(0,1)$ to all of $\mathcal{C}(0,1)$, we have picked some semicontinuous extension of $\mathcal{F}$ which is not the *best extension*, i.e. the largest semicontinuous extension $\bar{\mathcal{F}}$ of $\mathcal{F}$. The largest extension is given by

$$\bar{\mathcal{F}}(u) = \inf\{\liminf_{k\to\infty} \mathcal{F}(u_k) : u_k \in \mathcal{C}(0,1) \cap Lip(0,1), u_k \rightrightarrows u \text{ on } [0,1]\},$$

where '$u_k \rightrightarrows u$ on $[0,1]$' means that $\{u_k\}$ converges uniformly to u on $[0,1]$.
The situation changes rather drastically if we do not require that the approximating sequence $\{u_k\}$ satisfies the boundary conditions $u_k(0) = 0$, $u_k(1) = 1$. In that case $u(x) = x^{1/3}$ can be approximated by functions $u_k \in Lip(0,1)$, defined by

$$u_k(x) := \begin{cases} x^{1/3} & \text{if } \frac{1}{n} \le x \le 1 \\ n^{-1/3} & \text{if } 0 \le x \le \frac{1}{n}. \end{cases}$$

Note that $\{u_k\}$ converges uniformly to u on $[0,1]$, and $\mathcal{F}(u_k) \to \mathcal{F}(u)$.
By taking $u_k = u$ we can approximate u by $u_k \in \mathcal{C}(0,1) \cap C^1(0,1)$ such that $u_k \rightrightarrows u$ on $[0,1]$ and $\mathcal{F}(u_k) \to \mathcal{F}(u)$.
From these considerations follows that the most reasonable generalization of the problem

$$\inf\{\mathcal{F}(u) : u \in \mathcal{C}(0,1) \cap Lip(0,1)\} \tag{48}$$

is in general not the problem

$$\inf\{\mathcal{F}(u) : u \in \mathcal{C}(0,1)\},$$

but the problem

$$\inf\{\bar{\mathcal{F}} : u \in \mathcal{C}(0,1)\}. \tag{49}$$

The problem (49) is the *relaxed minimum problem* associated with (48), and the functional $\bar{\mathcal{F}}$ is the *relaxed functional* associated with $\mathcal{F}$. In general, we have $\mathcal{F}(u) \le \bar{\mathcal{F}}(u)$.

A Appendix: theorems

This appendix is a list of results (without proofs) which we used in the thesis.

From [7]:

Theorem A.1 (Egorov's Theorem). *Assume $\mu(A) < \infty$. Let $\{u_n\}$ be a sequence of measurable functions on A that converges pointwise on A to the real-valued function u. Then for each $\epsilon > 0$, there is a closed set $B \subset E$ for which $u_n \to u$ uniformly on B and $\mu(A \setminus B) < \epsilon$.*

From [7]:

Theorem A.2 (Lusin's Theorem). *Let u be a real-valued measurable function on A. Then for each $\epsilon > 0$ there is a continuous function ν on $\mathbb{R}$ and a closed set $B \subset A$ for which $u = \nu$ on B and $\mu(A \setminus B) < \epsilon$.*

From [6]:

Theorem A.3 (Global Inverse Function Theorem). *Let U be open in $\mathbb{R}^n$ and let $\Psi \in C^1(U, \mathbb{R}^n)$. Then $V = \Psi(U)$ is open in $\mathbb{R}^n$ and $\Psi : U \to V$ is a C^1-diffeomorphism if and only if Ψ is injective and regular on U.*

From [6]:

Theorem A.4 (Implicit Function Theorem). *Let $k \in [1, \infty]$. Let W be open in $\mathbb{R} \times \mathbb{R}$ and $f \in C^k(W)$. Assume*

$$(x_0, y_0) \in W, \qquad f(x_0, y_0) = 0, \qquad \frac{d}{dx} f(x_0, y_0) \neq 0.$$

Then there exist open neighbourhoods U of x_0 and V of y_0 in $\mathbb{R}$ such that for every $y \in V$ there exists a unique $x \in U$ with $f(x, y) = 0$. In this way we obtain a C^k-mapping $\psi : V \to U$ with $\Psi(y) = x$ and $f(x, y) = 0$, which is uniquely determined by these properties.

From [8]:

Theorem A.5 (Peano's Existence Theorem). *Let A be an open subset of $\mathbb{R} \times \mathbb{R}$ with $f : A \to \mathbb{R}$ a continuous function and $y'(x) = f(x, y(x))$ a continuous, explicit first-order differential equation defined on A, then every initial value problem $y(x_0) = y_0$ for f with $(x_0, y_0) \in A$ has a local solution $z : J \to \mathbb{R}$ where J is a neighbourhood of x_0 in $\mathbb{R}$, such that $z'(x) = f(x, z(x))$ for all $x \in J$.*

From [9]:

Theorem A.6 (Hölder's inequality). *Assume that $u \in L^p(\mu)$ and $\nu \in L^q(\mu)$ where p, $q \in (1, \infty)$ are conjugate numbers: $\frac{1}{p} + \frac{1}{q} = 1$. Then $u\nu \in L^1(\mu)$, and the following inequality holds:*

$$\left| \int u\nu d\mu \right| \leq \int |u\nu| d\mu \leq ||u||_p \cdot ||\nu||_q.$$

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