Focusing on Brackets as Structuring Elements: The Effect on High School Students' Algebraic Expertise

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ABSTRACT. To understand the problems that high school students have with algebra and more specifically sense of structure in equations, we dive deeper in a specific part of their algebraic expertise: structure sense. This paper attempts to answer the question whether this structure sense can be developed by focusing on brackets as a structuring element. It elaborates on an effect study done on 108 Dutch 10th grade high school students in an attempt to develop structure sense by means of five interventions of ten minutes, given over two weeks during the regular mathematics lessons. These interventions were based on the idea of think-pair-share, with each intervention using problems in which structure sense played a role. Before the interventions, students had to do a pretest and after the interventions, a posttest. These tests were graded on mathematical correctness and effective use of structure. A statistically significant difference was found between the results of the pre- and posttest. These results would imply that structure sense was developed by means of the interventions. This is a promising result for mathematics education, although the long-term effects of the interventions remain yet to be seen.

Keywords: Mathematics education, structure, structure sense, algebra, brackets

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High school mathematics education very much stresses the importance of algebra, and any mathematics teacher knows that students tend to have problems with equations and the variety of symbols used in them. As Freudenthal (1973, p. 305) put it, symbols and the order of operations are essential aspects that distinguish "the algebraic language" from the spoken everyday language. This means that expertise in the algebraic language is of large importance to succeed in mathematics and related subjects, such as physics. This expertise helps to create a better understanding of relationships between variables and formulas, and also leads to more correct manipulations and calculations.

One of the recurring elements of high school algebra is the use of brackets or parentheses. In elementary school, students learn that they should prioritize calculating expressions between brackets first, abiding by the order of operations. This functionality is then expanded upon in high school: students get faced with expressions in between brackets they can not calculate due to variables being added. An example of this would be the expression $3(x+1)$. Thus, students have to learn a new strategy to remove brackets from equations: we can work out the multiplication to remove the brackets and end up with $3x+3$. This might be perceived as odd: if the expression would have been $3(5+1)$, we would have prioritized calculating $5+1$ and then multiplying by 3. If the expression would have had variables, as is the case with $3(x+1)$, we would have worked the other way around. This could possibly be a point of confusion for students.

In high school algebra, brackets mostly play the role of creating structure in, sometimes complicated, equations. They can thus be considered a structuring element of an equation; they tend to group up multiple terms and symbols to create a singular entity, which in turn creates structure. An example of this can be found in the following equation:

$(3x+1)(2x+5)=2x+5$. Here, brackets are used to first make entities out of both $3x+1$ and $2x+5$. They are then multiplied together. On the other side, the same term $2x+5$ can be found. This type of equation could be categorized as being of the structure $AB=B$, and can be solved by using the appropriate mathematical tools. To solve this particular equation, a basic understanding of the structure and the result of dividing by A would be enough. However, in order to create more meaningful and longer lasting knowledge, it is required to create a deeper understanding of the role of brackets and the structure of the equation.

A possible way to create this deeper understanding of the role of brackets in equations for students, would be to pay attention to this role of brackets in different situations. This study thus aims to evaluate whether explicitly focusing on brackets as structuring elements helps students' understanding of the structure of an equation and leads to more efficient and correct problem solving.

Literature

The understanding of the structure of an equation was first coined as "structure sense" by Linchevski and Livneh (1999). Hoch and Dreyfus (2004) further refined this definition as a collection of abilities:

> "[Structure sense is to] see an algebraic expression or sentence as an entity, recognize an algebraic expression or sentence as a previously met structure, divide an entity into sub-structures, recognize mutual connections between structures, recognize which manipulations it is possible to perform, and recognize which manipulations it is useful to perform." (p. 51)

An example is given for each of these abilities in table 1.

Table 1

Examples of structure sense

Ability	Example	Comments
See algebraic expression as an entity	Solve $(x+2)\cdot x^2 = x+2$	$x+2$ can be interpreted as single entity, as can x^2
Recognize expression as previously met	Solve $x^2+2x+1=0$	Students can recognize this as an equation of form $ax^2+bx+c=0$
Divide an entity into substructures	Calculate $999\cdot 1001$	Writing the sum like $(1000-1)(1000+1)$
Recognize mutual connections between structures	Solve $x^2+2x+1=0$	Can be written as $(x+1)^2=0$, which has a different structure, yet has the same solution.
Recognize which manipulations can be performed	Solve $x^2+2x+1=0$	One could use the quadratic formula or rewrite the equation.
Recognize which manipulations are useful to perform	Solve $(3x+1)(2x+5)=2x+5$.	It is most efficient to factor out $2x+5$.

In their study, a questionnaire was used to investigate high school students' structure sense, and whether usage of brackets increased this structure sense. To investigate this, they showed the students equations with brackets and without brackets. After evaluating the answers given by students, the four students who had used strategies implying a strong structure sense were interviewed. Hoch and Dreyfus (2004, p. 56) concluded from the interviews that "the presence of brackets also seemed to help students see structure, focussing their attention on like terms and breaking up the long string of symbols". Using multiple types of brackets could further help students with their structure sense when solving equations. However, it remains the question whether students with a weaker sense of structure were helped by the presence of brackets, as this was not covered by the interviews.

Development of structure sense is very rarely found in literature. Schüler-Meyer (2017) closely reported the development of structure sense for the distributive law in mathematics. Six groups of two grade 8 students were subjected to an intervention in which the distributive law was explored. An impotant takeaway from this research was that a “critical step in the development of structure sense” (p. 30) needs to be taken to gain a full understanding of the distributive law. This critical step entails to go from visual mapping similarities between expressions and guiding examples, to the structural properties of the distributive law. The teacher would be able to play an important role for students to take this critical step. The limitation to this research however, is that it only focuses on the distributive law and that it is aimed at smaller groups.

Hoch and Dreyfus (2010) were able to help ten students develop their structure sense by means of three one-on-one teaching interviews. The development of one student which showed remarkable retention was reported closely. The two elements of the interviews that were reported to be most important for this particular student were as follows. Firstly, naming and categorizing the structures used helped the student use them. Secondly, it was important that the teacher could intervene at any given moment. However, these teaching interviews were only designed for one-on-one interaction, so something similar for group work would have to be designed in order to develop structure sense in whole classrooms.

A teaching intervention that has been proven to be useful for a group-wide teaching is think-pair-share (TPS) (e.g. Fitzgerald, 2013; Kaddoura, 2013). It makes sense that it could also prove useful for group-wide teaching of structure sense, as TPS allows students to compare their ideas and re-evaluate in a safe setting: among their peers. The idea behind TPS is as follows. First, students are faced with a problem and asked to think for themselves in silence for a set period of time (e.g. a minute). Afterwards, the students may discuss among each other in small groups and voice their thoughts in a smaller setting, again for a set period of time. Finally, a classroom discussion is held to discuss what answers the students have found. This way of teaching allows for wait time: a concept that was extensively researched over twenty years and reported on by Rowe (1986). Wait time is simply the time in which a student is allowed to think about a problem before answering. While it sounds trivial, most teachers do not wait between asking a question and demanding an answer. Increasing this time allows for more meaningful and productive exchanges between teachers and students. Furthermore, TPS also allows for teacher intervention at several moments, in comparison to direct instruction. For example, while the groups are discussing, a teacher could ask critical questions in order to raise the levels of discussion to specific groups, or help those who have trouble with the problem at hand. Finally, cooperative learning has shown time and time again to have positive effects on learning and achievement, and can accelerate student learning considerably when each individual is held accountable (e.g. Slavin, 1990; Johnson and Johnson, 2008).

The research question now is: can focusing on brackets as structuring elements have an effect on students' structure sense? To answer this question, the following subquestions will need to be answered. Firstly, how can TPS be implemented most efficiently in the interventions? Secondly, does focusing on brackets as a structuring element increase students' structure sense? Finally, if there exists a difference in results between the pre- and posttest; to what extent can this difference be explained by an increase in students' structure sense versus simply practicing more efficiently?

Methods

Pilot study

To optimize the materials used in the interventions to raise structure sense, the interventions were first piloted on a small scale. During five separate lessons, two classes that the researcher teaches were subjected to five interventions, with one intervention per lesson. Each intervention was based on the idea of TPS. In this research, this idea was used as follows. First, students think for themselves for two minutes. No paper may be used during this time. Afterwards, students can discuss and revise their thoughts with their partner for three minutes. Paper may be used here to help them explain their thoughts more clearly. Finally, by means of a classroom discussion, they share their thoughts with their peers. Their peers and the teacher can then discuss their strategy and steer them in the right direction. Furthermore, the teacher explicitly refers to the type of structure that the problems feature, possibly even categorizing the problems, as was pointed out to be an important characteristic of learning structure sense by Hoch and Dreyfus (2010). The focus of these interventions was on the role of brackets in equations. While each intervention had the same time structure, the problems posed were slightly different each time, yet still retaining the focus on developing the individual aspects of structure sense. The problems were based on the work of Hoch and Dreyfus (2004), Kindt (2004) and on similar textbook problems, and can be found in appendix A. The pilots were recorded and reviewed by the researcher, in order to design the interventions as efficient as possible to raise structure sense.

An example of an intervention was as follows. Students were given a list of equations as can be found in figure 1, and were asked to add in up to two sets of brackets to make them right. The idea of this intervention was to raise awareness of how brackets can completely change an equation and to create the idea of two very different structures being mutually connected. After completing this task, they were asked to make a similar list to this one. If they managed to do so, it can be concluded that students have a good grasp of what the role of brackets was in this set of equations, and how students could manipulate equations by using this knowledge.

In the following equations, add up to two pairs of brackets to correct them.

$$a + 2 \cdot a + 7 = 3a + 7$$
$$a + 2 \cdot a + 7 = 3a + 14$$
$$a + 2 \cdot a + 7 = a^2 + 2a + 7$$
$$a + 2 \cdot a + 7 = a^2 + 9a + 14$$

Figure 1. Example of problem posed during TPS session (Kindt, 2004)

Effect study

In order to validate whether a student's structure sense is increased by means of these small interventions, a pre- and posttest method was adopted. Mathematics teachers from all over the Netherlands were contacted by means of an electronic newspaper for Dutch mathematics teachers. Four mathematics teachers from four different schools responded, resulting in five participating classes of students.

In total, 108 Dutch 10^{th} grade high school students participated, of which 49 were VWO-level (pre-university education) students and 59 were HAVO-level (higher general secondary education) students. These students were all enrolled in the "mathematics B" curriculum. The mathematics B curriculum mostly focuses on algebra, calculus and geometry, and is meant for students who wish to pursue a science-oriented career. This is in contrast with mathematics A, which is meant for students that who to pursue an economics-oriented career and mathematics C, which is meant for students who wish to pursue a cultural or societal-oriented career.

The pretest was then given to the students by their respective teachers. After the two weeks that followed this pretest, five interventions of ten minutes were done during the regular mathematics lessons by the teacher, in which attention was paid to brackets as a structuring element. These are the five interventions which were previously tested and improved. Teachers were instructed by means of mail contact by the researcher, and the interventions were discussed before the lessons in approximately fifteen minutes.

Furthermore, all teachers were familiar with TPS. After the interventions, a posttest was performed to see whether these interventions have had an effect on a student's structure sense.

The pre- and posttest had a variety of exercises: in total, seven exercises were on the test. The questions were all of open-form: no multiple-choice questions were used. However, they all had a unique answer (as is usually the case in mathematics). The focus of these exercises was on the six abilities that were defined by Hoch and Dreyfus (2004) as structure sense. As such, the exercises were carefully constructed and reviewed by peers and co-researchers to ensure that they are indeed testing for the intended abilities and not for something else. An example question would be the equation $(2x + 5)(x - 3) = x - 3$, where students were expected to recognize the structure AB = B and solve it by dividing by $(x - 3)$. The pre- and posttest can be found in appendix B and C respectively.

The answer sheets filled in by the students were reviewed and graded by the researcher. The reviewing was done according to a pre-made answer sheet with pre-determined point allocation. There are two different types of points: mathematical points and structure points. The mathematical points could be earned for answering the problem mathematically correctly. Structure points could be earned for using the structure of the equation in their solution of the problem as defined by Hoch and Dreyfus (2004). This was done in order to both find out whether students used structure sense more and whether they scored better. Note that it is possible to score the maximum amount of normal points without scoring any structure points. An example would be solving the equation $(x + 2)(x + 3) = x + 2$. For this question, one structure point and two mathematical points can be earned. A possible way of tackling this problem is by working out the brackets and using the quadratic formula. This is a mathematically correct way of solving the problem, thus acquiring two mathematical points. Yet a student with a high structure sense would immediately see that it is possible to divide by $(x + 2)$. Thus, the student who would work out the brackets and use the quadratic formula would miss out on the single structure point, whereas the student who would divide by $(x + 2)$ would receive the structure point. Both students would receive two mathematical points.

Furthermore, in order to check the reliability of the correction model, a second rater was asked to review a number of tests. First, three tests were reviewed together and discussed to work out any confusion. Fifteen tests were then reviewed seperately and compared afterwards. Cohen's κ was run in order to determine the interrater reliability between the researcher and a colleague. This resulted in $\kappa = 0.795$ ($p < 0.001$). This indicates substantial agreement between the researcher and the colleague according to Landis and Koch (1977), which in turn would mean that the correction model used is clear and usable for further research.

Finally, one of the contacted teachers offered to have a class be used as a control group. This control group (n = 20) was used to see whether or not only doing the pretest would also improve the results of the posttest. This control group only did the pre- and posttest, but were not subjected to the interventions.

Directly following the test, five randomly selected students from the researcher's school were invited to have a "stimulated recall interview" on how the different types of brackets affected their performance. This type of interview has been used several times to gain insight into how people think, with great effect (e.g., O'Brien, 1993; Lyle, 2003). In the interview, students were asked why they made certain decisions in their problem solving, while following their reasoning on their answer sheet. Interviewing students provided us with insight into how they experienced working with brackets as structuring elements and how it helped them or perhaps made them struggle with the questions. Also, it allowed the researcher to investigate to what extent students' reasoning with brackets, awareness of brackets and use of structure sense was increased. Each interview took about 20 minutes.

As is usual with a semi-structured interview, the interviewees were triggered to answer in more detail, until the researcher was provided with a clear picture. Furthermore, the given answers on the answer sheets by the students were discussed, to understand why choices were made. They were not shown whether their answer is right or wrong. The interviews were recorded and transcribed.

Results

The summarized results of the pre- and posttest can be found in table 2. On both pre- and posttest, students were able to obtain ten structure points and eleven mathematical points in total. As is clear from the table, in both structure points and normal points, the students scored a higher mean on the posttest compared to the pretest. On average, students scored 2.0 more structure points and 1.9 more mathematical points on the posttest than on the pretest. In total, 84 students were able to improve their previous amount of structure points, 18 students "tied" their amount of points and finally 10 students did worse on the posttest than on the pretest. For the mathematical points, the results were similar (respectively 79, 19 and 10).

A Wilcoxon Signed Ranks test further revealed that the differences in means are statistically significant. The test shows that there was a statistically significant difference between the results of the pretest and posttest concerning structure points ($Z = -7.675$, $p < 0.001$). Yet not only the structure points were higher in the posttest: the students also scored significantly higher on the posttest than on the pretest concerning their mathematical correctness ($Z = -6.656$, $p < 0.001$).

The results of the control group are as follows: the control group did not score significantly higher on the posttest compared to the pretest in terms of structure points ($Z = -1.628$, $p = 0.103$), nor in terms of mathematical points ($Z = -1.592$, $p = 0.111$). This would mean that the learning of structure sense did not necessarily result from doing the pretest only.

Table 2

Results of pre- and posttest for experiment and control group

		Pretest: SP [a]	Posttest: SP	Pretest: MP [a]	Posttest MP
	Maximum	10	10	11	11
Exp. group (n = 108)	Mean	2.0870	3.9815	3.7593	5.6759
	Std. dev.	1.7463	2.0370	2.3754	2.8046
Con. group (n = 20)	Mean	2.2667	2.8667	3.3333	4.1333
	Std. dev.	1.2907	1.3021	1.6762	1.9592

[a] SP stands for structure points, MP stands for mathematical points

The percentage of structure points that were scored per question by the entire group can be found in table 3. As is clear, each question was done better on the posttest than on the pretest when looking at the group as a whole. The most notable increases can be found in the results of questions 2a, 2b and 2d. Respectively, these were related to recognition of structure, seeing mutual connections between structures and seeing an expression as an entity. The increase in 2a is interesting in itself, because in the pretest only 9.5% of students were able to see through the structure of the equation. In the posttest, this amount was tripled. A similar story applies to the result of question 2b: the number of students here is nearly quadrupled (albeit still rather small). The increase in 2d is also interesting, because it is the biggest absolute increase. In the discussion, these results are further explored.

Table 3

Percentage of structure points earned per question

	1a	1b	2a	2b	2c	2d	3	Total
Pretest	43.1%	18.1%	9.5%	3.5%	50.0%	40.5%	8.6%	24.8%
Posttest	60.0%	30.9%	29.8%	13.3%	68.5%	74.3%	27.9%	45.2%
Difference	16.9%	12.8%	20.3%	9.8%	18.5%	33.8%	19.3%	21.4%

While the observed interventions were not recorded, field notes were made by the researcher on basis of the discussions that were held in the smaller groups during the pairing section of TPS and during the classroom discussion on all schools. The field notes show that TPS is indeed effective for student engagement and comparison of strategies, as it was reported that “none of the students were not trying to deal with the problems at hand”. It was also noted that a group managed to solve a problem in a roundabout manner, yet one of the students proposed to look “whether they could find a more efficient way to solve the problem”. In classroom discussions, the word “structure” was explicitly used by students several times to underpin why they chose a specific. It was also used in their explanation of how they knew how to handle the problems. In one classroom discussion, a strategy was proposed, but one of the students immediately intervened and proposed a different strategy. The teacher made use of this and let all students think for ten seconds before choosing a specific method. After the ten seconds, nearly every student chose the strategy which used the structure of the equation, realizing the usefulness.

The interviews further show an increase in structure sense. One of the students mentioned that he “had seen the *structure* of the sum”, thus providing him with what to do next and how to solve the problem. Another mentioned loosely that he “had seen *something like this* last week”. When asked to elaborate on what they meant by “structure” and “something like this”, they both provided a similar problem to the one at hand: indeed one with the exact same structure, namely that of $AB = A$ in these particular cases. This was not the case in all interviews however. One of the interviewed students scored only a single point for both categories in the posttest and called the problems “impossible”, claiming that he had never seen anything like it before. When asked whether he remembered what happened during the 10-minute breaks in the mathematics lessons, he responded with: “Those problems were far easier and clearer, I knew what to do immediately”. This suggests that the link between the interventions and tests were not always clear for every student.

Discussion of results

Showing students different types of structures and learning them to look at equations from a different angle turned out to be an efficient way of learning structure sense for most students. From the results of the Wilcoxon Signed Ranked test, it may be concluded that students performed better on the tests both in terms of their structure sense and in terms of their mathematical skill. This could imply that structure sense helps students with their general mathematical problem solving. At the very least, the results suggest that structure sense was developed thanks to the interventions, which may be concluded from the results of the control group. The results in pre- and posttest for the control group did not contain a significant difference, and the Z-score was many times lower compared to the experiment group. This answers the second question, namely that structure sense can in fact be developed in a short amount of time by focusing on brackets as a structuring element.

The third research question posed to what extent an increase in results could be explained away by "more efficient practicing" rather than an increase in structure sense. While this could be indeed the case, the steady increase in both structure points and mathematical points would indicate that students recognized the structure in the equations more often, which would indicate structure sense.

Despite seeing an increase in results, the results on both the pre- and posttest can still be considered subpar. The low mean numbers found in table 2 could indicate that the test was too difficult. For example, on the pretest, students only managed to score a mean of 2.0870 structure points, whereas 10 points could be acquired. It is the researcher's opinion, however, that the pre- and posttest exercises were of an appropriate level when compared to the normal curriculum the participants followed, partly due to the exercises on the tests being based on textbook exercises. The pilot of the pre- and posttest also did not indicate that the tests were too hard. Another factor that must be considered is that the group of participants contained both VWO- and HAVO-level participants, with the scores of the VWO-participants being significantly higher than the HAVO-level participants. As of such, it may be concluded that the pre- and posttest were more suited towards a VWO-level student than a HAVO-level student.

The increases in structure points on each individual question as found in table 3 can be explained to an extent. In intervention four, explicit attention was paid to looking at the structure featured in the equation on both sides of the equals sign and using it to their advantage. The question 2a then features an equation of structural form AB = B, which would explain why students scored better on the posttest. Furthermore, question 2d is very closely related to the fifth intervention, featuring a very similar expression. This could explain most of the increases in these questions. However, interestingly enough, the increase in 1a, 1b and 2c are not that large despite being rather similar to the items found in the first and fourth intervention. For 1a and 2c, this is perhaps due to these questions already being done fairly well on the pretest.

A possible limitation of this study is the generalizability of the results and the general usability of the interventions in mathematics education. This research was done on 10^{th} grade Dutch students ranging from 15 to 17 years old that followed the mathematics B curriculum on a HAVO- or VWO-level. It is unclear however whether the results of this study would also apply to different nationalities, as their mathematics programs and focus may be wildly different. Furthermore, for students who follow the mathematics A or C curriculum, there might be other aspects that need attention to raise structure sense.

A possible objection to the conclusion that structure sense was raised due to the interventions is that it is possible that structure sense was also raised because of the regular lessons. While none of the groups were actively developing structure sense during the regular lessons outside of the interventions according to teacher interviews, it could still have been raised passively simply by working out mathematical problems and thinking about mathematics. However, in the small timeframe in which the pretest and the posttest took place (three weeks), it seems unlikely that a change this big could be achieved simply by attending the mathematics lessons, while structure sense was not the focus of these lessons. It would also have been difficult to have a group of students who fulfilled the requirements of the study while also not following any regular mathematics classes.

One of the limitations of the study is that it did not investigate the long term effects of the interventions. It would be an interesting point of research to see whether the effects of these five interventions could be seen on a longer period of time, e.g. a year or multiple years. The idea of several small (e.g. 10 minute) interventions over the course of the year is an interesting one and could prove to be a very efficient teaching method. However, such a timeframe was beyond the scope of this study.

When grading the pre- and posttest, it was noted that a group of students did not manage to finish the pretest in time, whereas they were able to finish the posttest in time. It was hypothesized that students who did the pretest knew that they would be short of time. Thus, they would adopt a new strategy for the posttest: they would manage their time better and focus on the questions that they could solve quickly, rather than sticking with questions that take a long time. The most extreme example was a student that on the pretest only managed to get halfway through the fourth question. However, on the posttest, this student skipped this question altogether and managed to do all the questions that followed correctly. If this scenario happened multiple times, it would explain some of the results that were achieved as seen in table 3 as well, as it is possible that a larger group of participants suffered from a lack of time on the pretest and used a different type of strategy for the posttest. However, this effect can not be found in the results of the control group. This could mean that the interventions also helped students save time by using their structure sense, thus allowing them to spend more time on later exercises.

Conclusion

Structure sense is a relatively new subject in the world of mathematics education; very little literature exists around it. This study has shown that, according to the guidelines that Hoch and Dreyfus (2004) provided, structure sense can be developed using TPS alongside carefully designed problems in just a couple of interventions. This is good news for teachers, as structure sense is a vital part of students' algebraic expertise. Being able to teach it could then prove to be a valuable way to raise mathematical prowess, possibly even over a longer timespan.

An idea would be to instruct a class of students during the entirety of the year using interventions weekly, while keeping a similar class preoccupied with the regular materials. It could then be measured what the effect on structure sense is, while also seeing whether one class performed significantly better in general over the year. While it would be obvious that the class that was coached in structure sense would score better in terms of structure sense, it remains to be seen whether these results translate to a more general mathematical prowess. Still, the promising results ask for a follow-up study using a longer time frame, along with several retention tests. This could also lead to better results to the problems posed in the introduction: creating a better understanding of the role of brackets and more generally lead to finer algebraic expertise.

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Appendix A

Interventions Used and Additional Instructions for Teachers

Intervention 1. In the following expressions, you may insert a single pair of brackets: ( ). How many different outcomes are possible?

$$3 \cdot 2 - 4 \cdot 5$$

$$2 - 4 \div 5 - 4 \cdot 3$$

Intended learning result: students find that inserting a pair of brackets may or may not change an expression. Most importantly, they find that brackets can mean prioritization, but may also not have an effect at all. The teacher should focus on what the effect of brackets is, rather than giving the correct answer.

Intervention 2. Calculate:

$$(50 + 7)[50 - 7] =$$

$$5 + (2 + 12 \cdot (14 - (23 - 15)) + 3)$$

Intended learning result: students learn that brackets can be read in two ways: prioritization, or a grouping of terms. They also learn that different pairs of brackets may be used to structurize a longer expression. The teacher should aim to create discussion between what method is better for the first problem: using the structure (A+B)(A-B) or simply calculating 43 times 57.

Intervention 3. The following equations are wrong. You may add up to two pairs of brackets to make them correct.

$$a + 2 \cdot a + 7 = 3a + 7$$

$$a + 2 \cdot a + 7 = 3a + 14$$

$$a + 2 \cdot a + 7 = a^2 + 2a + 7$$

$$a + 2 \cdot a + 7 = a^2 + 9a + 14$$

Once you're finished: can you make a similar list of equations (with different numbers)?

Intended learning result: students learn, once again, that placing brackets may completely change the structure of an expression, thus changing the answer, but now with variables. Furthermore, rather than looking at each equation as a singular equation, they look at the list as a group of equations and have to create something "similar", which would indicate that they realized some sort of structure.

Intervention 4. Find x:

$$\frac{x^2 + 5x + 6}{(x + 2)} = 2x$$

$$\frac{(x - 4)}{x^2 - 3x - 4} = \frac{x}{x + 1}$$

Intended learning result: students find that expressions between brackets can be factored away in divisions, as they can be seen as a singular entity rather than multiple terms. The teacher should aim to create discussion about what type of equation the second problem is: it is of the form $\frac{A}{B} = \frac{C}{B}$ once (x - 4) is removed.

Intervention 5. Find x:

$$-\frac{1}{4} + \frac{x}{x^2 + 1} - x = 5 - \left(\frac{1}{4} - \frac{x}{x^2 + 1}\right)$$

$$(x^2 + 1) \cdot \left(\frac{1}{x} + \frac{1}{x^2 + 1}\right) = 2x \cdot \left(\frac{1}{x} + \frac{1}{x^2 + 1}\right)$$

Intended learning result: once again, students learn to see expressions between brackets as singular entities. The teacher should aim to make students notice that the first equation is of the form A = B + A, while the second one is of the form AB = AC. It is important that once these conclusions are reached, the problem is solved correctly.

Appendix B

Pretest

PRE | | | | |

- **Read the following instructions carefully.**
- Write in the empty boxes in the top right the first letter of your name, the first letter of your last name, and finally your month of birth. For example, **JS04** means John Smith, april. This is your **code.**
- No **calculator** may be used on this test.
- Write down all of your work. Make clear what you are doing at all times.
- You have **15 minutes** to finish the test.
- At the end, hand in the **test** (with the code) and your **answer sheet** (with the code).
- **Good luck!**

Exercise 1

Calculate:

a. $(1000 - 9)(1000 + 9) =$

b. $(1 + 12((4 + 3) - 14) - 2) - 5 =$

Exercise 2

Solve for x.

a. $(x - 6)(3x - 2) = x - 6$

b. $(2x + 3)^2 - 3(2x + 3) + 2 = 0$

c. $\frac{-5x-8}{x+2} = \frac{4x+1}{x+2}$

d. $\frac{1}{4} - \frac{x}{x-1} - x = 7 + \frac{1}{4} - \frac{x}{x-1}$

Exercise 3

Try to make an equation which looks complex, but one that, once you study it carefully, is easy to solve.

Appendix C

Posttest

POST | | | | |

- **Read the following instructions carefully.**
- Write in the empty boxes in the top right the first letter of your name, the first letter of your last name, and finally your month of birth. For example, **JS04** means John Smith, april. This is your **code.**
- No **calculator** may be used on this test.
- Write down all of your work. Make clear what you are doing at all times.
- You have **15 minutes** to finish the test.
- At the end, hand in the **test** (with the code) and your **answer sheet** (with the code).
- **Good luck!**

Exercise 1

Calculate:

a. $(500-7)(500+7)=$

b. $(12+9((3-4)+10)-3)-7=$

Exercise 2

Solve for x.

a. $(2x+5)(x-3)=x-3$

b. $(x-4)^2-5(x-4)+6=0$

c. $\frac{x-3}{x^2-x+1}=\frac{x-3}{x}$

d. $x+\frac{1}{3}-\frac{x^2}{x-1}=-3+\frac{1}{3}-\frac{x^2}{x-1}$

Exercise 3

Try to make an equation which looks complex, but one that, once you study it carefully, is easy to solve.