G-$\mathcal{I}$-SPACES

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Contents

Introduction

The non-equivariant part

The category of finite sets and injective maps has a relatively short history that goes back to M.Bockstedt famous preprint from 1985. But the formal development of the homotopy theory of $\mathcal{I}$-spaces, that is, diagrams from the category $\mathcal{I}$ to the category of unbased spaces $\mathcal{U}$, denoted by $\mathcal{U}^{\mathcal{I}}$, has only been recently started to develop and they have become parts in various research programs. In the homotopical study of the category $\mathcal{U}^{\mathcal{I}}$, the most important model structure is called the *projective $\mathcal{I}$-model structure* and has the property that it is Quillen equivalent to the usual model structure on spaces. Thus, the homotopy category of $\mathcal{I}$-spaces equipped with the projective $\mathcal{I}$-model structure is equivalent to the usual homotopy category of spaces.

More closely for the purposes of this thesis, is the relationship of the homotopy theory of $\mathcal{I}$-spaces and stable homotopy theory. The category of $\mathcal{I}$-spaces is closely related to one model of structured spectra, the category of symmetric spectra, Sp^{Σ}. In particular, there is a pair of adjoint functors that relates these two categories. The importance of the projective $\mathcal{I}$-model structure mentioned in the previous paragraph, is that this adjunction becomes a Quillen pair, when the category of symmetric spectra is equipped with the stable model structure. As a formal consequence we have an adjunction between the homotopy category of $\mathcal{I}$-spaces and the stable homotopy category $\mathcal{SH}$. So, it is not a far fetched analogy to say that the homotopy theory of $\mathcal{I}$-spaces serve as an "unstable"analogue for the homotopy theory of symmetric spectra.

The equivariant part

Stable equivariant homotopy theory (S.E.H.T.) has seen various applications both in equivariant and non-equivariant topology. One spectacular application is in the paper [14]. Roughly speaking, S.E.H.T. studies homology and cohomology theories of spaces with symmetries, that is, with group actions. The subject is very well developed and an overview of one its models can be be found in [11] and the state of the art is [20]. But we are following the point of view given in [32] and [31, Chapter III] which has the advantage that the objects considered are the most concrete. To be more precise the objects of study in the second approach are just G-objects in the category spectra (symmetric, orthogonal, etc.), that is, spectra $\{X_n : n \in \mathbb{N}\}$ which every pointed space, is a $G \times \Sigma_m$ or $G \times O(n)$ space respectively with appropriate G-equivariant structure maps. These categories support many model categories but the most interesting one is the G-stable model structure, which has the property that its homotopy category is the equivariant analog of the stable homotopy category, $\mathcal{SH}(G)$.

The main point of this thesis is to generalize the Quillen adjunction between the homotopy theory of $\mathcal{I}$-spaces and symmetric spectra in the equivariant context for a finite group G.

Structure and summary of the thesis

We now give a summary of the contents of this thesis. In Part 3, Section 4 we will define formally the category of $\mathcal{I}$-spaces with a finite group action, in Section 6 we define three "level"model structures and in Section **??** we their existence and various properties. In Section 8 we will construct left Bousfield localizations of these model structures and in Section 9 we compare these localizations. Finally, in Part 4 we show the relationship between the homotopy theory of $\mathcal{I}$-spaces with G-action and the homotopy theory of G-symmetric spectra.

For the following proposition we fix a finite group G. We will call $\mathcal{I}$-spaces with group action G-$\mathcal{I}$-spaces. We will properly define this category in Definition 4.1. The notation $G\mathcal{U}$ stands for the category of G-spaces and G-equivariant maps.

Proposition 0.1. *There is a model structure on the category of G-$\mathcal{I}$-spaces whose weak equivalences are detected by the homotopy colimit functor*

$$\operatorname*{hocolim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}.$$

Furthermore, this model structure is Quillen equivalent to the genuine model structure on $G\mathcal{U}$.

The construction of this model structure can be done in two ways. One is provided in Proposition 8.9 by appealing to the recognition principle of cofibrantly generated model structures, [16, Theorem 2.1.19], after identifying a set of generating cofibrations and generating acyclic cofibrations. The second way is by a tautological application of a theorem proved by D.Dugger in [5, Theorem 5.2]. Both approaches have their advantages as we now briefly explain. One the one hand, D.Dugger's method is a left Bousfield localization of the projective model structure, so the fibrations are a bit mysterious to describe explicitly. The advantage of this approach is that the formal properties follow tautologically from [5, Theorem 5.2]. One the other hand, by appealing to the recognition principle of cofibrantly generated model structrures, Proposition 8.9 provides an explicit form of the fibrations, since we know the set of generating acyclic cofibrations.

The next proposition is the main result of this thesis.

Proposition 0.2. *Fix a complete G-universe U.*

(1) There is a cofibrantly generated G-topological model structure on the category G-$\mathcal{I}$-spaces, called the level model structure, with weak equivalences (resp. fibrations) those maps $f : X \longrightarrow Y$ such that for every finite G-set M, the map after evaluation $f(M) : X(M) \longrightarrow Y(M)$ is a genuine G-weak equivalence (resp. G-fibration),

(2) There is a left Bousfield localization of the above model structure, at a set of maps, such that the adjunction

$$\mathbb{S}^{\mathcal{I}}[-] : G\mathcal{U}^{\mathcal{I}} \rightleftarrows G\mathrm{Sp}^{\Sigma} : \Omega^{\mathcal{I}}$$

is a Quillen adjunction, for the corresponding G-stable model structure on the category of G-symmetric spectra, and

(3) Moreover, the model structure on the category of G-$\mathcal{I}$-spaces of part (2) is Quillen equivalent to the model structure of Proposition 0.1

The existence of the level model structure of part (*1*) is a consequence of the existence of the strong level model structure, given in Definition 6.6 and proven in Subsection 6.6. The relationship between the level model structure and the strong level model structure is made clear in Proposition 6.7, Remark 6.11 and in Proposition 6.12. For claim (*2*), the set of maps which, with respect to, we form the left Bousfield localization is defined in Subsection 8.13. The Quillen adjunction with the G-stable model structure on G-symmetric spectra is proven in Subsection 9.12. The proof of claim (*3*) is a consequence of Proposition 9.6 and Proposition 9.4.

0.3. **Limitations and Restrictions.** We have to note the limitations and restrictions of the theory developed in this thesis. The first limitation is that the group action on an $\mathcal{I}$-space must be a *finite* group. This is due to the fact, that both $\mathcal{I}$-spaces and symmetric spectra have internal symmetries encoded by actions of the symmetric groups, and thus, we cannot have continuous group homomorphisms $G \longrightarrow \Sigma_m$ for any $m \geq 0$ if G is, say, a compact Lie group. For compact Lie groups we would have to consider the equivariant analogue of *orthogonal* spaces, that is, diagrams of G-spaces over the index category $\mathbf{L}$ of finite dimensional real inner-product vector spaces and linear isometries. The other limitation is that in this thesis, we work only at single group at a time. In the book in progress [31] and in the preprint [13] the authors study orthogonal spaces and symmetric spectra in which all compact Lie groups or finite groups, respectively, act at the same time in a compatible way. Both of these approaches lie outside the scope of this thesis.

0.4. **General Prerequisites.** We assume that the reader is acquainted with the theory of model categories and we do not provide an introduction to it. Standard references include [15] and [16]. We will review some main definitions of G-symmetric spectra but the main reference for the homotopy theory of G-symmetric spectra is [12].

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Part 1. Preliminaries, Notation and Conventions

0.5. **Notation and Conventions.** As is standard we denote by $\mathcal{I}$ the category of finite sets $\mathbf{n} = \{1, \dots, n\}$, inclusing the empty set $\mathbf{0}$ and morphisms the injections. We will work in the category $\mathcal{U}$ of unbased compactly generated weak Hausdorff spaces which for us is synonymous to "spaces" .

In what follows G will **always** denote a finite group and the notation $G\mathcal{U}$ will mean the category of G-spaces and G-equivariant maps. The space of G-equivariant maps is denoted as $\mathrm{map}_G(-,-)$. We will reserve the notation $\mathrm{map}(-,-)$ of the G-space of all continuous maps between two G-spaces where G acts by conjugation. The fixed points of the G-space $\mathrm{map}(-,-)$ is precisely the space of G-equivariant maps $\mathrm{map}_G(-,-)$ We fix a G-set universe U, *i.e.*, a countably infinite G-set, which we always assume to be complete which means that every action on a finite set embeds into U.

If $\mathcal{C}$ is a category we will write $\mathcal{C}(X,Y)$ or $\hom_{\mathcal{C}}(X,Y)$ for the set of $\mathcal{C}$-morphisms from X to Y. We reserve the symbol $\otimes$ for the tensor product in a enriched category $\underline{\mathcal{C}}$ over a symmetric monoidal category $(\mathcal{V}, \times, *)$ As a last note, normally function spaces in model categories are built from framings or function complexes and do not refer to topological mapping spaces. However, if the model category is simplicial, the source is cofibrant and the target is fibrant, the mapping space created by function complexes are weakly equivalent to the simplicial mapping spaces. For simplicial model categories, we can use then the simplicial mapping space instead of function complexes to form function spaces and consequentially to form the left Bousfield localization with respect a map f if f is a map of cofibrant objects. The simplicial mapping spaces in a topological model category are $\mathrm{Sing}\,\mathrm{Map}(X,Y)$, where Sing is the singular complex functor. But Sing preserves and reflects weak equivalences, thus for topological model categories we can use topological mapping spaces. All the model structure that we will be dealing with are topological and we will use this convention throughout.

0.6. **Subgroups of product groups and representations.** Let A and B be finite groups. We denote the projections as $\mathrm{pr}_1 : A \times B \longrightarrow A$ and $\mathrm{pr}_2 : A \times B \longrightarrow B$.

Definition 0.7. Let G be a finite group. A *family* of subgroups is a non-empty set of subgroups of G such that it is closed under conjugation and passage to subgroups.

Definition 0.8 ($\mathcal{F}$-equivalence)**.** A G-map $f : X \longrightarrow Y$ of G-spaces is called an $\mathcal{F}$-equivalence if for all subgroups H of H that lie in the family $\mathcal{F}$, the fixed point map $f^H : X^H \longrightarrow Y^H$ is a weak equivalence.

Definition 0.9. Let $\mathcal{F}^{A,B}$ denote the family of subgroups $L \leq A \times B$ such that $L \cap (e \times B)$ is trivial.

We have the following lemma

Lemma 0.10. *A subgroup $L \leq A \times B$ belongs to the family $\mathcal{F}^{A,B}$ if and only if the restriction of the first projection function to L, i.e., $\mathrm{pr}_1\big|_L : L \to A$ is injective.*

Proof. We have the projection function $\mathrm{pr}_1 : A \times B \to A$ and its kernel is the following

$$\begin{aligned} \ker(\mathrm{pr}_1) &= \{(a,b) \in A \times B |\ \mathrm{pr}_1(a,b) = e\} \\ &= \{(e,b) \in A \times B |\ b \in B\} \\ &= e \times B \end{aligned}$$

Since by definition a subgroup $L \leq A \times B$ belongs to the family if $L \cap (e \times B) = \{(e,e)\}$ the claim follows. □

For the following lemma we use the notation $\hom(A,B)$ for the set of all group homomorphisms, for finite groups A and B.

Lemma 0.11. *Let $\mathcal{F}^{G,\Sigma_m}$ be the family of subgroups of $G \times \Sigma_m$ as in Definition 0.9. Let $\mathcal{A}$ be the set of pairs $\{(H,\phi)\,|H \leq G, \phi \in \hom(H,\Sigma_m)\}$. The map*

$$\begin{aligned} F : \mathcal{F}^{G,\Sigma_m} &\longrightarrow \mathcal{A} \\ L &\longmapsto \left(\mathrm{pr}_1(L), \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1}\right) \end{aligned}$$

is a bijection.

Proof. Suppose we have a pair (H,ϕ) where H is a subgroup of G and $\phi : H \to \Sigma_n$ a group homomorphism, that is, an action of H to the finite set $\mathbf{n}$. We have the inclusion $i : H \hookrightarrow G$ and let the group homomorphism $j := (i,\phi) : H \to G \times \Sigma_n$. Its kernel are those elements $h \in H$ such that $(h, \phi(h)) = (e,e)$ and the first coordinate imposes $h = e$. So the homomorphism $j = (i,\phi)$ is injective and its image $j(H) \leq G \times \Sigma_n$ is a subgroup with the property that $\mathrm{pr}_1 : j(H) \longrightarrow G$ is injective. This defines a map $G : \mathcal{A} \longrightarrow \mathcal{F}^{G,\Sigma_m}$ which is inverse to the function F. □

Part 2. Introduction to $\mathcal{I}$-spaces and (G-)Symmetric spectra

1. Symmetric Spectra

In this section we briefly recall one of the models of highly structured spectra, the category of Symmetric spectra. By definition a symmetric spectrum X is a spectrum in which each of the spaces X_n is equipped with a base-point preserving left action of the symmetric group Σ_n, with *structure maps*

$$\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$$

that are Σ_n-equivariant and the iterated structure maps $\sigma_n^m : X_m \wedge S^n \longrightarrow X_{m+n}$ are $\Sigma_m \times \Sigma_n$-equivariant. The $\Sigma_m \times \Sigma_n$-action on the space right hand side is the restriction action given by the inclusion $\Sigma_m \times \Sigma_n \hookrightarrow \Sigma_{n+m}$. A map $f : X \longrightarrow Y$ of symmetric spectra is a sequence of Σ_n-equivariant based maps $X_n \longrightarrow Y_n$. The category of symmetric spectra is most concisely given as a category of diagrams which we describe shortly below. For details we refer the seminal paper [21]. The approach that we describe below is given in more detail in [27, Subsection 3.1].

Let $\alpha : \mathbf{m} \longrightarrow \mathbf{n}$ a morphism in $\mathcal{I}$, let $\mathbf{n} - \alpha$ denote the complement of $\alpha(\mathbf{m})$ in $\mathbf{n}$ and let $S^{\mathbf{n}-\alpha}$ be the one-point compactification of $\mathbb{R}^{\mathbf{n}-\alpha}$. Given a morphism $\alpha : \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ there is an induced structure map $\alpha_* : X_m \wedge S^{\mathbf{n}-\alpha} \longrightarrow X_n$ defined as follows: Choose a bijection $\beta : \mathbf{l} \longrightarrow \mathbf{n} - \alpha$ for an object $\mathbf{l}$ in $\mathcal{I}$ and let $\{\alpha, \beta\} : \mathbf{m} \sqcup \mathbf{l} \longrightarrow \mathbf{n}$ be the resulting injection. Then α_* is defined by

$$\alpha_* : X_m \wedge S^{\mathbf{n}-\alpha} \to X_m \wedge S^{\mathbf{l}} \longrightarrow X_{m+l} \xrightarrow{\{\alpha,\beta\}_*} X_n$$

which is independent of the choice of β. With this convention, the subset inclusion $\iota : \mathbf{m} \hookrightarrow \mathbf{m} + \mathbf{1}$ induces the structure map $X_m \wedge S^1 \longrightarrow X_{m+1}$.

Consider then, the category $\mathcal{I}_S$ that has the same objects as $\mathcal{I}$ but whose morphisms are defined by

$$\mathcal{I}_S(\mathbf{m}, \mathbf{n}) := \bigvee_{\alpha \in \mathcal{I}(\mathbf{m},\mathbf{n})} S^{\mathbf{n}-\alpha}$$

We consider the category $\mathcal{I}_S$ enriched over the category of based spaces $\mathcal{T}$. Writing the morphisms of the category $\mathcal{I}_S$ in the form $(\mathbf{x}, \alpha)$ for $\mathbf{x} \in S^{\mathbf{n}-\alpha}$, the composition is defined by

$$\mathcal{I}_S(\mathbf{m}, \mathbf{n}) \wedge \mathcal{I}_S(\mathbf{l}, \mathbf{m}) \longrightarrow \mathcal{I}_S(\mathbf{l}, \mathbf{m}) \quad (\mathbf{x}, \alpha) \wedge (\mathbf{y}, \beta) \longmapsto (\mathbf{x} \wedge \alpha_* \mathbf{y}, \alpha\beta)$$

where $\mathbf{x} \wedge \alpha_* \mathbf{y}$ is defined by the canonical homeomorphism

$$S^{\mathbf{n}-\alpha} \wedge S^{\mathbf{m}-\beta} \cong S^{\mathbf{n}-\alpha\beta}, \quad \mathbf{x} \wedge \mathbf{y} \longmapsto \mathbf{x} \wedge \alpha_* \mathbf{y}$$

obtained by reindexing the coordinates of $S^{\mathbf{m}-\beta}$ via α. Thus, if $X : \mathcal{I}_S \longrightarrow \mathcal{T}$ is a continuous functor, then we have for each morphism $\alpha : \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathcal{I}$ a based continuous maps $\alpha_* : X_m \wedge S^{\mathbf{n}-\alpha} \longrightarrow X_n$.

Of particular importance is the stable model structure on the category Sp^Σ. It can be defined as localization of the projective model structure on the category of continuous functors $\mathcal{I}_S \longrightarrow \mathcal{T}$. The importance of this model structure is that its homotopy category, is one of the models of the stable homotopy category $\mathcal{SH}$. For details we refer to [21].

2. $\mathcal{I}$-spaces

In this section we will sketch some basic facts about the category of $\mathcal{I}$-spaces and their homotopy theory. The main references for this overview are the papers [26], [18, Section 2], and a survey on infinite loop spaces [1, Section 2]

As we said in the introduction, the category $\mathcal{I}$ has objects the finite sets $\mathbf{n} = \{1, \dots n\}$, including the empty set $\mathbf{0}$ and morphisms are the injections. An $\mathcal{I}$-space is a functor $X : \mathcal{I} \longrightarrow \mathcal{U}$. Every morphism in $\mathcal{I}$ can be factored as a composition of the canonical inclusion $\iota : \mathbf{n} \longrightarrow \mathbf{m}$ and a permutation $\sigma : \mathbf{m} \longrightarrow \mathbf{m}$. Therefore an $\mathcal{I}$-space $X : \mathcal{I} \longrightarrow \mathcal{U}$ determines a sequence of spaces $X(\mathbf{n})$ together with an induced action of the symmetric group Σ_n for $n \geq 0$, and structural maps $j_n : X(\mathbf{n}) \longrightarrow X(\mathbf{n} + 1)$ that are equivariant in the sense that $j_n(\sigma \cdot x) = \sigma \cdot j_n(x)$ for every $\sigma \in \Sigma_n$ and $x \in X(\mathbf{n})$. On the right hand side we see σ as element in Σ_{n+1} via the canonical inclusion $\Sigma_n \longrightarrow \Sigma_{n+1}$. Vice versa, given such a sequence of Σ_n-spaces $X(\mathbf{n})$ and compatible structure maps j_n, they give rise to an $\mathcal{I}$-space if and only if for $m \geq n$ and any two elements $\sigma, \tau \in \Sigma_m$ with identical restrictions to $\mathbf{n}$ we have $\sigma(x) = \tau(x)$ for all $x \in j(X(\mathbf{n}))$.

The fundamental notion of weak equivalence between $\mathcal{I}$-spaces is that of an $\mathcal{I}$-equivalence. These weak equivalences take part in various model structures but we will introduce here the most relevant to us, the so-called projective $\mathcal{I}$-model structure. We recall from [26, Subsection 3.1], the projective $\mathcal{I}$-model structure. We say that a map $f : X \longrightarrow Y$ of $\mathcal{I}$-spaces is an

- $\mathcal{I}$-equivalence if the induced map of homotopy colimits $X_{h\mathcal{I}} \longrightarrow Y_{h\mathcal{I}}$ is a weak equivalence of spaces,

- $\mathcal{I}$-fibration if it is a projective fibration and the diagram

$$\begin{array}{ccc} X(\mathbf{m}) & \longrightarrow & X(\mathbf{n}) \\ \downarrow & & \downarrow \\ X(\mathbf{m}) & \longrightarrow & Y(\mathbf{n}) \end{array}$$

  is homotopy cartesian for all morphisms $\mathbf{m} \longrightarrow \mathbf{n}$ in $\mathcal{I}$,
- cofibration if it has the left lifting property with respect to maps of $\mathcal{I}$-spaces that are level acyclic fibrations.

These classes specify a model structure on $\mathcal{U}^{\mathcal{I}}$. A proof can be found in [26, Proposition 3.2] and [18, Theorem 2.3].

Proposition 2.1. *The adjunction* $\operatorname{colim}_{\mathcal{I}} : \mathcal{U}^{\mathcal{I}} \rightleftarrows \mathcal{U} : \operatorname{const}_{\mathcal{I}}$ *defines a Quillen equivalence between the projective* $\mathcal{I}$*-model structure on* $\mathcal{I}^{\mathcal{I}}$ *and the usual model structure on* $\mathcal{U}$.

Thus, the homotopy category of $\mathcal{U}^{\mathcal{I}}$ is equivalent to the usual homotopy category of spaces. One should think of $\operatorname{hocolim}_{\mathcal{I}} X$ as the underlying space of the $\mathcal{I}$-space X.

The following proposition is the imporant link between the homotopy theory of $\mathcal{I}$-spaces and stable homotopy theory.

Proposition 2.2. [18, Proposition 2.4], [26, Proposition 3.19] *The adjunction*

$$\mathbb{S}^{\mathcal{I}}[-] : \mathcal{U}^{\mathcal{I}} \rightleftarrows \mathrm{Sp}^{\Sigma} : \Omega^{\mathcal{I}}$$

is a Quillen adjunction for the projective $\mathcal{I}$*-model structure and stable model structure respectively.*

3. Introduction to G-symmetric spectra

As we said in the introduction we are following the point of view of [32] and [12]. The G-symmetric spectra in this thesis are the direct analog of G-orthogonal spectra. The category is always that of G-objects in the category of non-equivariant symmetric spectra introduced in Section 1. Parallel to the orthogonal case, these G-spectra secretly inherit evaluations at arbitrary finite G-sets. Which of these evaluations are declared homotopically meaningful in the model structure is based on a G-set universe U, *i.e.*, a countable infinite G-set which is isomorphic to the disjoint union of two copies of itself. For this thesis we will always take the universe U to be complete, that is, every G-action on finite sets is contained in U.

Definition 3.1. [12, Definition 2.2] A G*-symmetric spectrum* of spaces is a symmetric spectrum together with a G-action via automorphisms of symmetric spectra, or equivalently a functor $G \longrightarrow \mathrm{Sp}^{\Sigma}$ where we regard the finite group G as a category with a single object. A map of G-symmetric spectra is a map that commutes with the given G-actions. We denote the category of G-symmetric spectra by $G\mathrm{Sp}^{\Sigma}$.

Equivalently, a G-symmetric spectrum is a symmetric spectrum X together with a G-action on each pointed space $X(n)$ which commutes with the Σ_n-action and for which all structure maps $\sigma : S^1 \wedge X_n \longrightarrow X_{n+1}$ are G-equivariant, for the trivial G-action on S^1.

As we said above, every G-symmetric spectrum secretly inherits evaluations on arbitrary finite G-sets.

Definition 3.2. [12, Definition 2.4] The evaluation of a G-symmetric spectrum X on a finite G-set M of cardinality m is defined by

$$\begin{aligned} X(M) &:= X(m) \wedge_{\Sigma_m} \mathrm{Bij}(m, M)_+ \\ & X_m \wedge \mathrm{Bij}(m, M)_+ / \{(\sigma_* x, f) \sim (x, \sigma^* f),\ \sigma \in \Sigma_m\} \end{aligned}$$

with diagonal G-action $g[x, f] = [gx, gf]$

Example 3.3. The *equivariant sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$ with action by Σ_n and with *trivial* action of the group G. This does *not* mean however that G acts trivially on the evaluation $\mathbb{S}(V)$ of $\mathbb{S}$ on a finite G-set V. Indeed, the map

$$\mathbb{S}(V) = S^n \wedge_{\Sigma_n} \mathrm{Bij}(n, V) \longrightarrow S^V, \qquad [x, \phi] \longmapsto \phi(x)$$

is a G-equivariant homeomorphism to the representation sphere of V

Example 3.4. Every pointed G-space A gives rise to a suspension spectrum $\Sigma^\infty A$ via

$$(\Sigma^\infty A)_n = A \wedge S^n.$$

The symmetric group acts though S^n, the group G acts through the action on A and the structure maps are the canonical homeomorphisms $(A \wedge S^n) \wedge S^1 \xrightarrow{\cong} A \wedge S^{n+1}$. For example the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty S^0$ where G acts trivially on S^0. If we evaluate the suspension spectrum on a finite G-set V we obtain

$$(\Sigma^\infty A)(V) \cong A \wedge S^V.$$

This homeomorphism is G-equivariant with respect the diagonal G-action on the right hand side.

Given two finite G-sets M and N, we set

$$\mathbf{\Sigma}(M,N) = \bigvee_{\alpha: M \to N \text{ injective}} S^{N-\alpha(M)}$$

where $N - \alpha(M)$ is the complement of the image of α in N and it has a G-action by conjugation.

Definition 3.5 (Free G-symmetric spectra)**.** [12, Definition 2.18] Let A be a based G-space and let M be a finite G-set. The free G-symmetric spectrum on A in level M is denoted by $\mathcal{F}_M A$ and defined via

$$(\mathcal{F}_M A)_n = A \wedge \mathbf{\Sigma}(M,n)$$

with diagonal G-action and Σ_n-action through $\mathbf{\Sigma}(M,n)$. The structure map is the composition

$$A \wedge \mathbf{\Sigma}(M,n) \wedge S^1 \hookrightarrow A \wedge \mathbf{\Sigma}(M,n) \wedge \mathbf{\Sigma}(n,n+1) \xrightarrow{A\wedge\circ} A \wedge \mathbf{\Sigma}(M,n+1)$$

The free G-symmetric spectrum $\mathcal{F}_\emptyset A$ is naturally isomorphic to the suspension spectrum $\Sigma^\infty A$, since there is only one injective function from the empty set to any other finite set.

The equivariant analog of infinite loop space is much more complicated and has a lot more structure.

Definition 3.6. [12, Definition 2.40] A G-projective level fibrant G-symmetric spectrum X is called a $G\Omega$-spectrum if for all subgroups $H \leq G$ and all finite H-subsets M and N of U, the adjoint generalized structure map induces a weak equivalence

$$(\widetilde{\sigma}_M^N)^H : X(M)^H \longrightarrow \mathrm{map}_H\left(S^N, X(M \sqcup N)\right)$$

on H-fixed points.

The $G\Omega$-spectra are the fibrant objects of a model structure on the category $G\mathrm{Sp}^\Sigma$, the G-stable model structure. Its homotopy category is the equivariant stable homotopy category $\mathcal{SH}(G)$. For further details we refer to [12, Section 2.10].

Part 3. Model Structures on G-$\mathcal{I}$-spaces

4. The category $G\mathcal{U}^{\mathcal{I}}$

Definition 4.1. A *G-$\mathcal{I}$-space* is a diagram $X : \mathcal{I} \longrightarrow G\mathcal{U}$. We write $G\mathcal{U}^{\mathcal{I}} := \mathrm{Fun}(\mathcal{I}, G\mathcal{U})$ for the category of G-$\mathcal{I}$-spaces with morphisms the natural transformations.

By Cartesian closedness, a diagram $X : \mathcal{I} \longrightarrow G\mathcal{U}$ can be considered equivalently as a functor $G \longrightarrow \mathcal{U}^{\mathcal{I}}$, where we regard G as a category with a single object or as a functor $\mathcal{I} \times G \longrightarrow \mathcal{U}$.

The next lemma recalls the basic formal properties of the category of G-$\mathcal{I}$-spaces.

Lemma 4.2. *The category $G\mathcal{U}^{\mathcal{I}}$ is bicomplete with limits and colimits constructed levelwise. Furthermore, $G\mathcal{U}^{\mathcal{I}}$ is enriched, tensored and cotensored over $\mathcal{U}$. For a G-$\mathcal{I}$-space X and a space T the tensor $X \otimes T$ and cotensor X^T are the G-$\mathcal{I}$-spaces defined by*

$$(X \otimes T)(\mathbf{k}) := X(\mathbf{k}) \times T \tag{4.1}$$

$$(X^T)(\mathbf{k}) := \mathrm{map}(T, X(\mathbf{k})). \tag{4.2}$$

with G-action though $X(\mathbf{k})$. For X and Y two G-$\mathcal{I}$-spaces let

$$\mathrm{Map}_{G\mathcal{U}^{\mathcal{I}}}(X, Y) \subset \prod_{\mathbf{k} \in \mathrm{ob}(\mathcal{I})} \mathrm{map}_{G \times \Sigma_k}(X(\mathbf{k}), Y(\mathbf{k}))\,. \tag{4.3}$$

denote the subspace of all collections of maps $(f(\mathbf{m}))_{\mathbf{m} \in \mathcal{I}}$ that are $G \times \Sigma_m$-equivariant in the product of the mapping spaces $\mathrm{Map}(X(\mathbf{m}), Y(\mathbf{m}))$ such that each collection determines a map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces. It is topologized as a subspace of this product of mapping spaces.

Every $\mathcal{I}$-space, that is, a functor $\mathcal{I} \longrightarrow \mathcal{U}$ gives rise to a diagram $\mathcal{I} \longrightarrow G\mathcal{U}$ by letting G act trivially on each $X(\mathbf{n})$. So, we have a functor $\mathrm{triv} : \mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}^{\mathcal{I}}$. This functor has a left adjoint $(-)/G : G\mathcal{U}^{\mathcal{I}} \longrightarrow \mathcal{U}^{\mathcal{I}}$

$$X/G : \mathcal{I} \longrightarrow \mathcal{U}, \quad \mathbf{n} \longrightarrow X(\mathbf{n})/G \tag{4.4}$$

by taking orbit spaces objectwise and a right adjoint $(-)^G : G\mathcal{U}^{\mathcal{I}} \longrightarrow \mathcal{U}^{\mathcal{I}}$

$$X^G : \mathcal{I} \longrightarrow \mathcal{U}, \quad \mathbf{n} \longmapsto X(\mathbf{n})^G \tag{4.5}$$

by taking fixed points objectwise.

For a subgroup $H \leq G$ we have the restriction functor

$$\mathrm{res}^G_H : G\mathcal{U}^{\mathcal{I}} \longrightarrow H\mathcal{U}^{\mathcal{I}} \tag{4.6}$$

which has a left adjoint, the induction functor

$$G \ltimes_H - : H\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}^{\mathcal{I}} \quad X \longmapsto G \ltimes_H X \tag{4.7}$$

The category $G\mathcal{U}^{\mathcal{I}}$ is also enriched over the category $G\mathcal{U}$. For a pair X and Y of G-$\mathcal{I}$-spaces, the underlying mapping space of non-G-equivariant maps $\mathrm{Map}_{\mathcal{U}^{\mathcal{I}}}(X, Y)$ carries a G-action by conjugation and the fixed points of this action are precisely the G-equivariant maps, that is, the space $\mathrm{Map}_{G\mathcal{U}^{\mathcal{I}}}(X, Y)$, as in (4.3). Let L be a G-space and $X : \mathcal{I} \longrightarrow G\mathcal{U}$. Then we can define the tensor $X \otimes L$ as in (4.1) with diagonal G-action and we can define the cotensor X^L as in (4.2) with G-action by conjugation.

Lemma 4.3. *The category $G\mathcal{U}^{\mathcal{I}}$ is enriched, tensored and cotensored over the category $G\mathcal{U}$.*

5. Other descriptions of the category $G\mathcal{U}^{\mathcal{I}}$

In this section we will sketch the outline for two different descriptions of the category of G-$\mathcal{I}$-spaces. Both of them follow from general considerations of equivariant phenomena.

5.1. As diagrams on $\mathcal{O}_G^{\mathrm{op}}$.

Let $\mathcal{C}$ be a category and we denote by $G\mathcal{C}$ the category of functors $\mathrm{Fun}(G, \mathcal{C})$ in which we regard G as a category with a single object. Given a subgroup $H \leq G$, we have a functor $(-)^H : G\mathcal{C} \longrightarrow \mathcal{C}$, defined by

$$G\mathcal{C} \xrightarrow{\mathrm{res}^G_H} H\mathcal{C} \xrightarrow{\lim_H} \mathcal{C}.$$

Let $\mathcal{O}_G$ be the orbit category of G; its objects are G/H for $H \leq G$ and morphisms the equivariant maps, that is, $\hom_{\mathcal{O}_G}(G/H, G/K) = (G/K)^H$. Then, an object $X \in G\mathcal{C}$, *i.e.*, a functor $X : G \longrightarrow \mathcal{C}$ defines a functor Φ from $\mathcal{O}_G^{\mathrm{op}}$ to the category $\mathcal{C}$ as follows

$$\Phi_X : \mathcal{O}_G^{\mathrm{op}} \longrightarrow \mathcal{C}, \qquad \Phi_X(G/H) \longrightarrow X^H \tag{5.1}$$

The functor Φ has a left adjoint defined by

$$\begin{aligned} \Lambda : \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}) &\longrightarrow G\mathcal{C} \\ Y &\longmapsto Y(G/e) \end{aligned} \tag{5.2}$$

Applying the above for our case of G-$\mathcal{I}$-spaces, we have the adjunction

$$\Lambda : \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{U}^{\mathcal{I}}) \rightleftarrows \mathrm{Fun}(G, \mathcal{U}^{\mathcal{I}}) : \Phi \tag{5.3}$$

5.2. **As enriched functors.** For basic facts about enriched categories over a symmetric monoidal category $\mathcal{V}$ we refer to [25, Chapter 3], [2, Chapter 6]. Let $\mathcal{I}_G$ be the category whose objects are pairs $(\mathbf{m}, \phi : G \longrightarrow \Sigma_m) := \mathbf{m}_\phi$, in which $\mathbf{m}$ is an object of $\mathcal{I}$ and ϕ is a homomorphism from G to Σ_m and thus endows $\mathbf{m}$ with the structure of a G-set. Morphisms $(\mathbf{m}, \phi) \longrightarrow (\mathbf{n}, \psi)$ are the underlying injective functions $\mathbf{m} \longrightarrow \mathbf{n}$, that is, the morphisms in $\mathcal{I}$. The group of automorphisms of $(\mathbf{m}, \phi)$ is Σ_m. Since we do not require morphisms to commute with the homomorphisms $G \longrightarrow \Sigma$, we have a G-action by conjugation on the hom-sets of $\mathcal{I}_G$; Concretely given a morphism $(\mathbf{m}, \phi) \longrightarrow (\mathbf{n}, \psi)$, an element of $g \in G$ acts on f

$$\mathbf{m} \xrightarrow{\phi(g^{-1})} \mathbf{m} \xrightarrow{f} \mathbf{n} \xrightarrow{\psi(g)} \mathbf{n}$$

which by a slight abuse of notation we write as $g \cdot f = gfg^{-1}$. We will write $\mathcal{I}_G$ for this category enriched in the monoidal category of G-sets. Similarly we define $\mathcal{U}_G$ to be the category whose objects are G-spaces and whose morphisms are continuous maps. The category $\mathcal{U}_G$ is enriched in the monoidal category of G-spaces via conjugation and we write $\mathcal{U}_G$ for the enriched category.

Definition 5.3. A $\mathcal{I}_G$-space is a G-enriched functor $\underset{\sim}{X} : \mathcal{I}_G \longrightarrow \mathcal{U}_G$. Morphisms between $\mathcal{I}_G$-spaces are G-natural transformations. We write G-$\mathrm{Fun}(\mathcal{I}_G, \mathcal{U}_G)$ for the above category.

Unwinding the definitions, given a G-functor $\underset{\sim}{X} : \mathcal{I}_G \longrightarrow \mathcal{U}_G$ we have a map

$$\mathcal{I}_G \ni \mathbf{m}_\phi \longrightarrow X(\mathbf{m}_\phi)$$

together with morphisms

$$\underset{\sim}{X}_{\mathbf{m}_\phi, \mathbf{n}_\psi} : \mathcal{I}_G(\mathbf{m}_\phi, \mathbf{n}_\psi) \longrightarrow \mathcal{U}_G\left(X(\mathbf{m}_\phi), X(\mathbf{n}_\psi)\right).$$

By definition of a $\mathcal{V}$-functor, the maps $\mathcal{I}_G(\mathbf{m}_\phi, \mathbf{n}_\psi) \longrightarrow \mathcal{U}_G\left(X(\mathbf{m}_\phi), X(\mathbf{n}_\psi)\right)$ must be G-equivariant with respect the respective conjugation actions, that is, given an injective function $f : \mathbf{m}_\phi \longrightarrow \mathbf{n}_\psi$ we have $g\underset{\sim}{X}(f)g^{-1} = \underset{\sim}{X}(gfg^{-1})$

A useful observation is that there is an adjoint equivalence of categories

$$\mathbb{P} : \mathrm{Fun}(\mathcal{I}, G\mathcal{U}) \rightleftarrows G\text{-}\mathrm{Fun}(\mathcal{I}_G, \mathcal{U}_G) : \mathbb{U} \tag{5.4}$$

where $\mathbb{P}$ is a form of extension and $\mathbb{U}$ is a form of restriction. Such an equivalence of categories is not something new. It was first observed in [34] in the case of G-Γ-spaces, that is, the category of diagrams $\Gamma \longrightarrow G\mathcal{U}$. It is very much analogous to the adjoint equivalence of categories that we mentioned in the introduction between G-spectra indexed on a complete universe and spectra with G-action.

We explain in brief the the adjoint equivalence (5.4). There is a fully faithful functor $\iota : \mathcal{I} \longrightarrow \mathcal{I}_G$ given by sending an object $\mathbf{m} \in \mathrm{ob}(\mathcal{I})$ to the object $(\mathbf{m}, \iota) \in \mathcal{I}_G$, where (by abuse of notation) ι denotes the unique homomorphism $G \longrightarrow \Sigma_m$ that factors through the trivial group. The functor ι induces the restriction functor $R : G\text{-}\mathrm{Fun}(\mathcal{I}_G, \mathcal{U}_G) \longrightarrow \mathrm{Fun}(\mathcal{I}, G\mathcal{U})$, which sends a G-enriched functor $\underset{\sim}{X}$ to its restriction to the objects with trivial G-action. Summing up the above discussion, given a G-enriched functor $\underset{\sim}{X} : \mathcal{I}_G \longrightarrow \mathcal{U}_G$, we precompose it with the inclusion $\iota : \mathcal{I} \longrightarrow \mathcal{I}_G$

$$\mathcal{I} \xrightarrow{\iota} \mathcal{I}_G \xrightarrow{\underset{\sim}{X}} \mathcal{U}_G$$

and setting $X := \underset{\sim}{X} \circ \iota$. Any injective morphism $f : \mathbf{m} \longrightarrow \mathbf{n}$ between trivial actions is G-fixed, and thus f must be sent to a G-equivariant map $X(f) : X(\mathbf{m}) \longrightarrow X(\mathbf{n})$. This implies that the restriction of $\underset{\sim}{X}$ to the category $\mathcal{I}$ in fact lands in $G\mathcal{U} \subset \mathcal{U}_G$. For the other direction, given a diagram $X : \mathcal{I} \longrightarrow G\mathcal{U}$, as we will see in Definition 5.5 we can always recover arbitrary G-actions by the formula $X(M) = X(\mathbf{m}) \times_{\Sigma_m} \mathrm{Bij}(\mathbf{m}, M)$ with diagonal G-action where M now stands for a finite G-set in the universe U

5.4. **Evaluations at finite G-sets.** As we said in the introduction, the main objects of study in equivariant stable homotopy theory are symmetric spectra with G-action but allowing arbitrary evaluations on finite G-sets. In parallel way we chose to work with the category of diagrams $\mathcal{I} \longrightarrow G\mathcal{U}$, that is, $\mathcal{I}$-spaces with G-action instead of $\mathcal{I}_G$-spaces, as in Definition 5.3. So, for the homotopy theory of G-$\mathcal{I}$-spaces the evaluations on arbitrary finite G-sets $M \subseteq U$ will play a major role. The definition is exactly the analogue of the case of G-symmetric spectra 3.2.

Definition 5.5 (Evaluation)**.** Let X be a diagram $\mathcal{I} \longrightarrow G\mathcal{U}$ and let M be a finite G-set in U of cardinality $|M| = m$. The *evaluation* of X at the finite G-set M is defined by

$$\begin{aligned} \mathrm{Ev}^{\mathcal{I}}_M X = X(M) :=& X(\mathbf{m}) \times_{\Sigma_m} \mathrm{Bij}(\mathbf{m}, M) \\ =& X(\mathbf{m}) \times \mathrm{Bij}(\mathbf{m}, M)/\left\{(\sigma_* x, f) \sim (x, \sigma^* f),\ \sigma \in \Sigma_m\right\} \end{aligned} \tag{5.5}$$

with diagonal G-action $g[x, f] := [gx, gf]$.

For every finite G-set M, the functor $\mathrm{Ev}^{\mathcal{I}}_M : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}$ has a left adjoint which we now define. For a finite G-set $M \subseteq U$ and $\mathbf{n} \in \mathrm{ob}(\mathcal{I})$ we denote by

$$\mathrm{Inj}(M, \mathbf{n}) \tag{5.6}$$

the set of injective functions $M \longrightarrow \mathbf{n}$. It comes with a $G \times \Sigma_n$-action which for an injective map $f : M \longrightarrow \mathbf{n}$, the group $G \times \Sigma_n$ acts by the rule $f \longmapsto \sigma f(g^{-1} \cdot x)$, for $g \in G$, $\sigma \in \Sigma_n$ and $x \in M$.

Definition 5.6. Let M be a finite G-set and L be a G-space. The *free G-$\mathcal{I}$-space on L in level M*, denoted by $F^{\mathcal{I}}_M(L)$ and defined via

$$\begin{aligned} F^{\mathcal{I}}_M(L)(-) : \mathcal{I} &\longrightarrow G\mathcal{U} \\ \mathbf{n} &\longmapsto \mathrm{Inj}(M, \mathbf{n}) \times L \end{aligned} \tag{5.7}$$

with diagonal G-action and Σ_n-action through $\mathcal{I}(M, \mathbf{n})$.

Proposition 5.7. *Let M be a finite G-set, let L be a G-space and let X be a G-$\mathcal{I}$-space. Then, the natural map*

$$\mathrm{Map}(F^{\mathcal{I}}_M(L), X) \longrightarrow \mathrm{map}(L, X(M)) \tag{5.8}$$

that sends a (non-necessarily equivariant) morphism of G-$\mathcal{I}$-spaces $f : F^{\mathcal{I}}_M(A) \longrightarrow X$ to the composite

$$A \cong A \times \{\mathrm{id}_A\} \hookrightarrow A \times \mathrm{Inj}(M, M) \cong F^{\mathcal{I}}_M(A)(M) \xrightarrow{f(M)} X(M)$$

is a G-isomorphism with the respective G-actions by conjugation on the spaces on 5.8.

Proof. By our discussion , the category G-$\mathcal{I}$-spaces considered as category of G-$\mathrm{Fun}(\mathcal{I}_G, \mathcal{U}_G)$, the above Proposition is just a consequence of the enriched Yoneda Lemma. □

Of particular importance for the course of this thesis is the following. Consider two finite G-sets $M, N \subseteq U$ and let $\alpha : M \longrightarrow N$ an injection (not-necessarily equivariant.) By the strong Yoneda Lemma, we have the following G-isomorphism $\mathrm{Map}(F^{\mathcal{I}}_M(*), X) \cong X(M)$, where the G-action on the left hand side is by conjugation. The injection $M \longrightarrow N$ induces a natural transformation of representable functors $F^{\mathcal{I}}_N(*) \longrightarrow F^{\mathcal{I}}_M(*)$ by precomposition. Consider the following commutative diagram for a G-$\mathcal{I}$-spaces X

$$\begin{array}{ccc} \mathrm{Map}(F^{\mathcal{I}}_M(*), X) & \longrightarrow & X(M) \\ \downarrow & & \downarrow {\scriptstyle X(\alpha)} \\ \mathrm{Map}(F^{\mathcal{I}}_N(*), X) & \longrightarrow & X(N) \end{array} \tag{5.9}$$

By the strong Yoneda lemma the top and bottom horizontal arrows are G-isomorphisms and the left vertical arrow is G-equivariant with the respective G-actions by conjugation. This implies that also the right vertical arrow is G-equivariant. For the course of this thesis, whenever we have an injection $M \longrightarrow N$, we will use this map as the induced map $X(M) \longrightarrow X(N)$ without any further comment.

6. Level Model structures on $G\mathcal{U}^{\mathcal{I}}$

We shall be mainly interested in three level model structures on the category of $G\mathcal{U}^{\mathcal{I}}$, the *projective model structure*, the *level model structure* and the *strong level model structure*. In this section we will state the definitions and some properties that follow directly from them. We will prove their existence and other model structure properties in the following section.

6.1. **Projective Model structure.** Consider the category $G\mathcal{U}$ equipped with the genuine model structure. Consider then, the category of diagrams $\mathcal{I} \longrightarrow G\mathcal{U}$ equipped with the *projective model structure*. By definition, in this model structure, we call a map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces

- a *projective equivalence* if for all objects $\mathbf{m}$ of $\mathcal{I}$, the map $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a G-weak equivalence,
- a *projective fibration* if for all objects $\mathbf{m}$ of $\mathcal{I}$, the map $f(\mathbf{m}) : X(\mathbf{m}) \longrightarrow Y(\mathbf{m})$ is a G-fibration, and
- a *projective cofibration* if it has the left lifting property with respect all maps that are projective acyclic fibrations

Proposition 6.2. *The class of projective equivalences, projective fibrations and projective cofibrations specify a cofibrantly generated, proper, cellular, topological model structure on the category $G\mathcal{U}^{\mathcal{I}}$.*

We write the category $G\mathcal{U}^{\mathcal{I}}$ equipped the above model structure as $G\mathcal{U}^{\mathcal{I}}_{\text{proj}}$.

Proof. By construction, the model structure $G\mathcal{U}^{\mathcal{I}}_{\text{proj}}$ exists and is cofibrantly generated by [15, Theorem 11.6.1]. It is proper by [15, Theorem 13.1.14] and cellular by [15, Proposition 12.1.5].

The evaluation functor $\text{Ev}^{\mathcal{I}}_{\mathbf{m}} : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}, \quad X \longmapsto X(\mathbf{m})$ has a left adjoint, which we write as $F^{\mathcal{I}}_{\mathbf{m}}$ and is defined as follows

$$\begin{aligned} F^{\mathcal{I}}_{\mathbf{m}}(L)(-) : G\mathcal{U} &\longrightarrow G\mathcal{U}^{\mathcal{I}} \\ L &\longmapsto \mathcal{I}(\mathbf{m}, -) \times L \end{aligned} \tag{6.1}$$

The set of generating cofibrations is

$$\begin{aligned} I^{proj} &= \left\{\mathcal{I}(\mathbf{m}, -) \times G/H \times \partial D^k \longrightarrow \mathcal{I}(\mathbf{m}, -) \times G/H \times D^k \,|\, \mathbf{m} \in \mathcal{I}, H \leq G, k \in \mathbb{N}\right\} \\ &= \left\{F^{\mathcal{I}}_{\mathbf{m}}\left(G/H \times \partial D^k\right) \longrightarrow F^{\mathcal{I}}_{\mathbf{m}}\left(G/H \times D^k\right) |\, \mathbf{m} \in \mathcal{I}, H \leq G, k \in \mathbb{N}\right\} \end{aligned} \tag{6.2}$$

and the set of generating acyclic cofibrations is

$$\begin{aligned} J^{proj} &= \left\{\mathcal{I}(\mathbf{m}, -) \times G/H \times D^k \longrightarrow \mathcal{I}(\mathbf{m}, -) \times G/H \times D^k \times [0,1] \,|\, \mathbf{m} \in \mathcal{I}, H \leq G, k \in \mathbb{N}\right\} \\ &= \left\{F^{\mathcal{I}}_{\mathbf{m}}\left(G/H \times D^k\right) \longrightarrow F^{\mathcal{I}}_{\mathbf{m}}\left(G/H \times D^k \times [0,1]\right) |\, \mathbf{m} \in \mathcal{I}, H \leq G, k \in \mathbb{N}\right\} \end{aligned} \tag{6.3}$$

□

6.3. **Level Model structure.** In this subsection we introduce the *level model structure* on G-$\mathcal{I}$-spaces in which we allow evaluations on finite G-sets in a complete universe U.

Definition 6.4 (Level model structure)**.** We call a map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces

- a *level equivalence* if for all finite G-sets $M \subseteq U$, the map after evaluation, $f(M) : X(M) \longrightarrow Y(M)$ is a G-equivalence,
- a *level fibration* if for all finite G-sets $M \subseteq U$, the map after evaluation, $f(M) : X(M) \longrightarrow Y(M)$ is a Serre G-fibration, and
- *level cofibration* if it has the left lifting property with respect all maps that are level acyclic fibrations.

We will write $G\mathcal{U}^{\mathcal{I}}_{level}$ for the above model structure.

6.5. **Strong level Model structure.** In this subsection we introduce the *strong level* model structure.

For the product group $G \times \Sigma_m$, we recall Definition 0.9 of the family of subgroups $\mathcal{F}^{G,\Sigma_m}$.

Definition 6.6 (Strong level m.structure)**.** We call a map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces

- a *strong level equivalence* if for all $\mathbf{n} \in \text{ob}(\mathcal{I})$ the $(G \times \Sigma_n)$-map $f(\mathbf{n}) : X(\mathbf{n}) \longrightarrow Y(\mathbf{n})$ is a $\mathcal{F}^{G,\Sigma_n}$-equivalence,
- a *strong level fibration* if for all $\mathbf{n} \in \text{ob}(\mathcal{I})$ the $(G \times \Sigma_n)$-map $f(\mathbf{n}) : X(\mathbf{n}) \longrightarrow Y(\mathbf{n})$ is a $\mathcal{F}^{G,\Sigma_n}$-fibration, and
- a *strong level cofibration* if it has the left lifting property with respect all maps that are strong level acyclic fibrations.

We will write $G\mathcal{U}^{\mathcal{I}}_{strong}$ for the category $G\mathcal{U}^{\mathcal{I}}$ equipped with the above model structure.

Proposition 6.7. *Let $f : X \longrightarrow Y$ be a map of G-$\mathcal{I}$-spaces. Then the following are equivalent:*

(i) The map f is a strong level equivalence (resp. strong level fibration)
(ii) For all subgroups $H \leq G$ and all finite H-sets M, the map after evaluation, $f(M) : X(M) \longrightarrow Y(M)$ is a weak equivalence (resp. Serre fibration) on H-fixed points, that is, the map $f(M)^H : X(M)^H \longrightarrow Y(M)^H$ is a weak equivalence (resp. Serre fibration).

Remark 6.8. By applying the above proposition for all subgroups of H with restricted action on M, we can replace the second condition by requiring $f(M)$ to be a genuine H-equivalence (resp. H-fibration) for every subgroup $H \leq G$.

Proposition 6.7 is a consequence of the following two lemmas which together show that subgroups $L \leq G \times \Sigma_n$ which lie in the family $\mathcal{F}^{G,\Sigma_m}$ correspond to pairs of a subgroup H of G and a finite H-subset M of U of cardinality n and that for a G-$\mathcal{I}$-space the respective fixed points $X(\mathbf{n})^L$ and $X(M)^H$ are naturally isomorphic. They also have an important implication which we will see later on Proposition 9.4.

Lemma 6.9 (Untwisting)**.** *Let X be a diagram $\mathcal{I} \to G\mathcal{U}$, let n be a natural number and let L be a subgroup in the family $\mathcal{F}^{G,\Sigma_n}$. Then there exists a subgroup $H \leq G$, a group isomorphism $j : H \to L$ and an H-structure on $\mathbf{n}$ such that there is an H-isomorphism $j^*(X(\mathbf{n})) \cong X(\mathbf{n}_\phi)$, where $\mathbf{n}_\phi$ is the finite set $\mathbf{n}$ equipped with the given H-action, $X(\mathbf{n}_\phi)$ is the evaluation at the finite H-set $\mathbf{n}_\phi$ and j^* is the restriction of the action along j.*

Proof. Since $L \in \mathcal{F}^{G,\Sigma_n}$, by Lemma 0.11, there is a subgroup H of G and a group homomorphism $\phi : H \to \Sigma_n$ such that the group L can be written as $L = \{(h, \phi(h)) : h \in H\}$. Define a map $j : H \to L,\ h \mapsto (h, \phi(h)) := (h, \sigma_h)$, which is an isomorphism. Define a map

$$F : X(\mathbf{n}) \to X(\mathbf{n}_\phi) \quad x \mapsto [x, \mathrm{id}]$$

This map is homeomorphism and it remains to show that it is H-equivariant with respect the restricted action on $X(\mathbf{n})$ by the homomorphism $j : H \to L$ and the diagonal action of H on $X(\mathbf{n}_\phi)$. More precisely we need to show that $j^*(X(\mathbf{n})) \to X(\mathbf{n}_\phi)$ is H-equivariant. For $x \in X(\mathbf{n})$, the restricted action of L along the homomorphism $j : H \to L$ is given by

$$\begin{aligned}(h, \phi(h)) \cdot x &= h \cdot \sigma_{h*}(x) \\ &= \sigma_{h*}(h \cdot x) \\ &= X(\phi(h))(h \cdot x)\end{aligned} \tag{6.4}$$

So, the map F maps $\sigma_{h*}(h \cdot x) \mapsto [\sigma_{h*}(h \cdot x), \mathrm{id}]$ and by Definition 5.5, the equivalence relation we have imposed on $X(\mathbf{n}) \times \mathrm{Bij}(\mathbf{n}, \mathbf{n}_\phi)$ this is, equivalent to $[h \cdot x, \sigma_h]$. But this is the diagonal action of an element h of the group H on $[x, \mathrm{id}]$. So it is equivariant and we are done. □

Lemma 6.10 (Twisting)**.** *Let H be a subgroup of G and M a finite H-set of order m. Then there is a subgroup $L \le G \times \Sigma_m$ in the family $\mathcal{F}^{G,\Sigma_m}$ and an isomorphism $j : H \to L$ such that there is an H-isomorphism $X(M) \cong j^*(X(\mathbf{m}))$ for every $X : \mathcal{I} \to G\mathcal{U}$. This H-isomorphism is natural in X.*

Proof. Choose a bijection $\phi : M \to \mathbf{m}$. Since M is a finite H-set, we can define an H-action on $\mathbf{m}$ by

$$\rho : H \longrightarrow \Sigma_m, \quad \rho(h) = \phi \circ l_h \circ \phi^{-1} \tag{6.5}$$

$$\begin{array}{ccc} M & \xleftarrow{\ \phi^{-1}\ } & \mathbf{m} \\ {\scriptstyle l_h}\downarrow & & \downarrow \\ M & \xrightarrow[\ \phi\ ]{} & \mathbf{m} \end{array}$$

The bijection ϕ is H-equivariant with respect these actions and so the induced homeomorphism $X(M) \cong X(\mathbf{m}_\phi)$ is H-equivariant. From above we know that $X(\mathbf{m}) \cong X(\mathbf{m}_\phi)$ where the action on $X(\mathbf{m})$ is the restriction along the homomorphism $j = (i, \phi) : H \to G \times \Sigma_m$ and the image of j, that is, the subgroup $j(H)$ lies in the family $\mathcal{F}^{G,\Sigma_m}$ □

Remark 6.11. Following [22, Remark 4.9.], given a map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces, the property of f being a strong level equivalence, that is, for every $H \le G$, the map after evaluation $f(M) : X(M) \longrightarrow Y(M)$ is an H-weak equivalence is equivalent to being a level equivalence as in Definition 6.4, part (i). Given a subgroup $H \le G$, let M be a finite H-set. Suppose that $f(T) : X(T) \longrightarrow Y(T)$ is a G-weak equivalence for every finite G-set T, *i.e.*, f is a level equivalence. Then, M is an H-retract of some G-set Q, and since the H-weak equivalences are closed under retracts we get also that $f(M)$ is an H-weak equivalence. So, f is a level equivalence implies the seemingly stronger condition that f is a strong level equivalence. The converse holds trivially.

Proposition 6.12. *We have the following Quillen equivalence*

$$\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{level} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{strong} : \mathrm{Id}$$

Proof. From Definition 6.4 and Proposition 6.7 we can easily see that the functor $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{strong} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{level}$ is right Quillen. From Remark 6.11, we also get immediately that the Quillen adjunction is a Quillen equivalence. □

Proposition 6.13. *The adjunction*

$$\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{proj} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{level} : \mathrm{Id}$$

is a Quillen adjunction

Proof. By Remark 6.13, we get immediately that the identity functor $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{level} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{proj}$ is right Quillen. □

7. Existence of the Level Model Structures and other properties

This section is devoted to proving the existence of the model structures that we defined in the previous section. We will appeal to the general recipe for constructing level model structures on the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{V})$ [31, Proposition 3.26, pp 565], for a small index category $\mathcal{D}$ that comes with a degree function $\deg : \mathrm{ob}(\mathcal{D}) \longrightarrow \mathbb{N}$ satisfying two axioms, namely

(i) If two objects d, e of $\mathcal{D}$ satisfy $\deg(e) < \deg(d)$, then $\mathcal{D}(e, d)$ is the empty set, and
(ii) if two objects d, e of $\mathcal{D}$ satisfy $\deg(e) = \deg(d)$, then d and e are isomorphic.

see [31, Skeletal filtration, pp.560] and $\mathcal{V}$ a symmetric monoidal category. We discuss a bit general recipe for constructing model structures on the functor category $\mathrm{Fun}(\mathcal{D}, \mathcal{V})$. As input we need, for every $m \geq 0$, a model structure $\mathcal{C}(m)$ on the category of $\mathcal{D}(m)$-objects. We call a morphism $f : X \longrightarrow Y$ in $\mathrm{Fun}(\mathcal{D}, \mathcal{V})$

- a *level equivalence* if $f(m) : X(m) \longrightarrow Y(m)$ is a weak equivalence in the model structure $\mathcal{C}(m)$ for all $m \geq 0$,
- a *level fibration* if $f(m) : X(m) \longrightarrow Y(m)$ is a fibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$,
- a *cofibration* if it has the left lifting property with respect to all maps that are level acyclic fibrations.

In fact, using the skeletal filtration defined by the degree function $\deg : \mathrm{ob}(\mathcal{D}) \longrightarrow \mathbb{N}$ we could be more explicit about the cofibrations using a form of “latching maps” which is a generalization of latching maps on diagram categories where the index category is Reedy.

The crucial property such that the above definitions define a model structure is the following *consistency condition* which we define.

Definition 7.1 (Consistency condition)**.** For all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the model structure $\mathcal{C}(m)$ on $\mathcal{D}(m)$-objects, every cobase change, in the category of $\mathcal{D}(m+n)$-objects, of the morphism

$$\mathcal{D}(m, m+n) \times_{\mathcal{D}(m)} i : \mathcal{D}(m, m+n) \times_{\mathcal{D}(m)} A \longrightarrow \mathcal{D}(m, m+n) \times_{\mathcal{D}(m)} B$$

is a weak equivalence in the model structure $\mathcal{C}(m+n)$.

7.2. **Existence.** We define a degree function on the category $\mathcal{I}$, $\deg : \mathrm{ob}(\mathcal{I}) \longrightarrow \mathbb{N}$, $\mathbf{n} \longmapsto |\,\mathbf{n}\,|$. We move on to prove the consistency condition for the strong level model structure. The consistency condition of the level model structure will be implied by the consistency of the strong level model structure.

For the proof of the consistency condition we will need the following lemma [12, Lemma A.2]

Lemma 7.3. *For every $\mathcal{F}^{G,\Sigma_m}$-equivalence between cofibrant $G \times \Sigma_m$-spaces $f : X \to Y$ and every cofibrant $G \times \Sigma_k$-space A the map $\Sigma_{m+k} \ltimes_{\Sigma_m \times \Sigma_k} (f \times A)$ is a $\mathcal{F}^{G,\Sigma_{m+k}}$-equivalence.*

We will need also the following situation. Suppose we have a G-space X. For the proof of the consistency condition, we would like to be able to write X in the form $G \ltimes_H A$, where H a subgroup of G and A an H-subspace of X. Let X be a G-space and $f : X \to G/H$ a G-map. Write $A = f^{-1}(eH)$ which is an H-subspace of X and we have a G -map $F : G \ltimes_H A \to X \quad [g, a] \mapsto ga$ Since we *only* deal with finite groups, the first condition of the following proposition is trivially satisfied, [37, Proposition 4.4, pp.32-33]

Proposition 7.4. *The map F constructed above is a homeomorphism if one of the following conditions are satisfied:*

(1) G is compact Hausdorff and H is closed in G
(2) $q : G \to G/H$ has a local section

With the above Lemma and Proposition now we can move to the proof of the consistency condition. Denote $\mathcal{I}(m, m+k)$ the set of injective functions from $\mathbf{m}$ to $\mathbf{m} \sqcup \mathbf{k}$. The group Σ_{m+k} acts by post-composition and the action is transitive. The stabilizer of the canonical inclusion

$$i_m : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m, m+1, \ldots, m+k\} \qquad j \longmapsto j$$

permutes the remaining coordinates so it is isomorphic to the group of permutations Σ_k. So, $\mathcal{I}(m, m+k)$ as a homogeneous Σ_{m+k}-space is isomorphic to $\Sigma_{m+k}/\,\mathrm{Stab}(i_m) \cong \Sigma_{m+k}/\Sigma_k$. Since the action is transitive any injective map can be written as a composition τi_m for some (non-unique) $\tau \in \Sigma_{m+k}$. Let f be the function

$$f : \mathcal{I}(m, m+k) \to \Sigma_{m+k}/\Sigma_k \quad \tau i_m \mapsto [\tau] = \tau \Sigma_k.$$

The inverse image of the function f is the following $A = f^{-1}(e\Sigma_k) = \{i \in \mathcal{I}(m, m+k) \,|\, i = \tau i_m \text{ such that } \tau \in \Sigma_k\}$ which consists of a single element $A = *$, namely the canonical inclusion, and we have the G-isomorphisms $\Sigma_{m+k} \ltimes_{\Sigma_k} * \cong \Sigma_{m+k}/\Sigma_k \cong \mathcal{I}(m, m+k)$

So, consider a generating acyclic cofibration in $(G \times \Sigma_m)\mathcal{U}$ with respect the family $\mathcal{F}^{G,\Sigma_m}$, that is, a map of the form $g : (G \times \Sigma_m)/L \times D^k \to (G \times \Sigma_m)/L \times D^k \times [0, 1]$ for some $k \in \mathbb{N}$ and for $L \in \mathcal{F}^{G,\Sigma_m}$. Every such map is a $\mathcal{F}^{G,\Sigma_m}$-equivalence between cofibrant $G \times \Sigma_m$ spaces.

So, let X be a $G\times\Sigma_{m+n}$-space and suppose we have the following pushout square

$$\begin{array}{ccc}(G\times\Sigma_m)/L\times_{\Sigma_m}\mathcal{I}(m,m+k)\times D^k & \longrightarrow & X\\ \downarrow & & \downarrow\\ (G\times\Sigma_m)/L\times_{\Sigma_m}\mathcal{I}(m,m+k)\times D^k\times[0,1] & \longrightarrow & Y\end{array}$$

The functor $-\times_{\Sigma_m}\mathcal{I}(m,m+k)$ is naturally isomorphic to $\Sigma_{m+k}\ltimes_{\Sigma_m\times\Sigma_k}(-\times\Sigma_k)$ and so we have that the map $g\times_{\Sigma_m}\mathcal{I}(m,m+n)$ is isomorphic to the map $\Sigma_{m+k}\ltimes_{\Sigma_m\times\Sigma_k}(g\times\Sigma_k)$ and by Lemma 7.3 this map is a $\mathcal{F}^{G,\Sigma_{m+k}}$-equivalence. Moreover, it is an h-cofibration of $(G\times\Sigma_{m+n})$-spaces which means that $X\to Y$ is a $\mathcal{F}^{G,\Sigma_{m+n}}$ equivalence and we have shown the consistency condition. This proof contains also the consistency condition for the level model structure. Thus we have proved that the model structure defined in 6.4 and 6.6 exist.

Next, we describe the set of generating cofibrations and the set of generating acyclic cofibrations for the level and strong level model structure. The evaluation functor $\mathrm{Ev}^{\mathcal{I}}_{\mathbf{m}}: G\mathcal{U}^{\mathcal{I}}\longrightarrow(G\times\Sigma_m)\mathcal{U},\quad X\longmapsto X(\mathbf{m})$ has a left adjoint which we denote by $G^{\mathcal{I}}_{\mathbf{m}}$ and is defined as follows

$$G^{\mathcal{I}}_{\mathbf{m}}(L)(-):(G\times\Sigma_m)\mathcal{U}\longrightarrow G\mathcal{U}^{\mathcal{I}} \qquad L\longmapsto\mathcal{I}(\mathbf{m},-)\times_{\Sigma_m}L \tag{7.1}$$

By inspecting Definition 6.6 we have that the strong level acyclic fibrations are detected by the following set

$$I^{strong}=\left\{G^{\mathcal{I}}_{\mathbf{m}}(G\times\Sigma_m/L\times\partial D^k)\longrightarrow G^{\mathcal{I}}_{\mathbf{m}}(G\times\Sigma_m/L\times D^k)\,\middle|\,\mathbf{m}\in\mathcal{I},L\in\mathcal{F}^{G,\Sigma_m},k\in\mathbb{N}\right\} \tag{7.2}$$

and similarly the strong level fibrations are detected by the following set

$$J^{strong}=\left\{G^{\mathcal{I}}_{\mathbf{m}}(G\times\Sigma_m/L\times D^k)\longrightarrow G^{\mathcal{I}}_{\mathbf{m}}\left(G\times\Sigma_m/L\times D^k\times[0,1]\right)\,\middle|\,\mathbf{m}\in\mathcal{I},L\in\mathcal{F}^{G,\Sigma_m},k\in\mathbb{N}\right\}. \tag{7.3}$$

By Remark 6.8, we can also write the set that detects strong level acyclic fibrations as follows

$$\begin{aligned}I^{strong}&=\left\{G\ltimes_H\left(\mathcal{I}(M,-)\times H/K\times\partial D^k\right)\longrightarrow G\ltimes_H\left(\mathcal{I}(M,-)\times H/K\times D^k\right)\right\}\\&=\left\{G\ltimes_H\left(F^{\mathcal{I}}_M(H/K\times\partial D^k)\right)\longrightarrow G\ltimes_H\left(F^{\mathcal{I}}_M(H/K\times D^k)\right)\right\}\end{aligned} \tag{7.4}$$

for every finite $H\le G$, every H-set M and every generating cofibration of the genuine $H\mathcal{U}$ model structure. And similarly the set that detects strong level fibrations as follows

$$\begin{aligned}J^{strong}&=\left\{G\ltimes_H\left(\mathcal{I}(M,-)\times H/K\times D^k\right)\longrightarrow G\ltimes_H\left(\mathcal{I}(M,-)\times H/K\times D^k\times[0,1]\right)\right\}\\&=\left\{G\ltimes_H\left(F^{\mathcal{I}}_M(H/K\times D^k)\right)\longrightarrow G\ltimes_H\left(F^{\mathcal{I}}_M(H/K\times D^k\times[0,1])\right)\right\}\end{aligned} \tag{7.5}$$

For the case of the level model structure, by inspecting definition 6.4 the following set of maps

$$\begin{aligned}I^{level}&=\left\{\mathcal{I}(M,-)\times G/H\times\partial D^k\longrightarrow\mathcal{I}(M,-)\times G/H\times D^k\,|M\in U,H\le G,k\in\mathbb{N}\right\}\\&=\left\{F^{\mathcal{I}}_M(G/H\times\partial D^k)\longrightarrow F^{\mathcal{I}}_M(G/H\times D^k)\,|M\in U,H\le G,k\in\mathbb{N}\right\}\end{aligned} \tag{7.6}$$

and the set of maps

$$\begin{aligned}J^{level}&=\left\{\mathcal{I}(M,-)\times G/H\times D^k\longrightarrow\mathcal{I}(M,-)\times G/H\times D^k\times[0,1]\,|\,M\in U,H\le G,k\in\mathbb{N}\right\}\\&=\left\{F^{\mathcal{I}}_M(G/H\times D^k)\longrightarrow F^{\mathcal{I}}_M(G/H\times D^k\times[0,1])\,|\,M\in U,H\le G,k\in\mathbb{N}\right\}\end{aligned} \tag{7.7}$$

detect level acyclic fibrations and level fibrations respectively. Sources and targets of all the maps of G-$\mathcal{I}$-spaces in the sets that we listed above are small with respect their respective cells. This follows from the fact that all objects in question in $G\mathcal{U}$ $H\mathcal{U}$ are small with respect to sequential colimits and we apply left adjoint functors.

Remark 7.5. The projective model structure is related to the level and strong level model structure in the following sense. Let $H\le G$ be subgroup and consider the trivial homomorphism $\phi:H\longrightarrow\Sigma_m,\quad h\longmapsto e$. By Lemma 0.11, this group homomorphism identifies the subgroup H of G as a subgroup of $G\times\Sigma_m$. As cosets, we have an isomorphism $G\times\Sigma_m/H\cong G/H\times\Sigma_m$ and so $G^{\mathcal{I}}_{\mathbf{m}}(G/H\times\Sigma_m)=\mathcal{I}(\mathbf{m},-)\times_{\Sigma_m}(G/H\times\Sigma_m)\cong\mathcal{I}(\mathbf{m},-)\times G/H$. So, the projective model structure can be considered as restricting to only finite sets with *trivial* G-actions.

7.6. **Compatibility with Enrichments.** In this subsection we will prove that the model structures $G\mathcal{U}^{\mathcal{I}}_{\text{level}}$ and $G\mathcal{U}^{\mathcal{I}}_{\text{strong}}$ are $\mathcal{U}$-model structures and $G\mathcal{U}$-model structures. Recall from Lemma 4.2 that the category $G\mathcal{U}^{\mathcal{I}}$ is tensored, cotensored and eriched over the category $\mathcal{U}$ of spaces. Recall from [16, Corollary 4.2.5] that a $\mathcal{U}$-model category (or a topological model category) $\mathcal{M}$ is a category that is enriched, tensored and cotensored over $\mathcal{U}$ and equipped with a model structure such that if $f : X \longrightarrow Y$ is a cofibration in $\mathcal{M}$ and $g : S \longrightarrow T$ a cofibration on $\mathcal{U}$, then the pushout-product map

$$f\Box g : Y \otimes S \cup_{X\otimes S} X \otimes T \longrightarrow Y \otimes T$$

is a cofibration in $\mathcal{M}$ which is acyclic if either f or g is.

Proposition 7.7. *The level model structure and the strong level model structure on the category $G\mathcal{U}^{\mathcal{I}}$ are topological model structures.*

Proof. Let $I_{\{e\}}$ be the set of generating cofibrations in $\mathcal{U}$, *i.e.*, $i_k : \partial D^k \longrightarrow D^k$ for every $k \in \mathbb{N}$ and $J_{\{e\}}$ be the set of generating acyclic cofibrations in $\mathcal{U}$, *i.e.*, $j_k : D^k \longrightarrow D^k \times [0, 1]$ for every $k \in \mathbb{N}$. By [16, Corollary 4.2.5] it suffices to show that $I^{level}\Box I_{\{e\}}$ consists of cofibrations and both $I^{level}\Box J_{\{e\}}$ and $J^{level}\Box I_{\{e\}}$ consists of acyclic cofibrations. Similarly, it suffices to show $I^{strong}\Box I_{\{e\}}$ consists of cofibrations and both $I^{strong}\Box J_{\{e\}}$ and $J^{strong}\Box I_{\{e\}}$ consists of acyclic cofibrations.

For the case of the strong level model structure, let $f : G\times\Sigma_m/L\times\partial D^k \longrightarrow G\times\Sigma_m/L\times D^k$ be a generating cofibration in the model structure $\mathcal{F}^{G,\Sigma_m}$-$\mathcal{U}$ so, $G^{\mathcal{I}}_{\mathbf{m}}(f)$ is a generating cofibration in $G\mathcal{U}^{\mathcal{I}}_{strong}$ as in the defining equation (7.2). Since the functor $G^{\mathcal{I}}_{\mathbf{m}}$ preserves colimits and tensors, the map $G^{\mathcal{I}}_{\mathbf{m}}(f)\Box i_k$ can be identified with the map $G^{\mathcal{I}}_{\mathbf{m}}(f\Box i_k)$. Since the $\mathcal{F}^{G,\Sigma_m}$-$\mathcal{U}$ is a $\mathcal{U}$-model category, the pushout -product map $f\Box i_k$ is a cofibration, hence $G^{\mathcal{I}}_{\mathbf{m}}(f\Box i_k)$ also is a cofibration in $G\mathcal{U}^{\mathcal{I}}_{strong}$. The argument for $I^{strong}\Box J_{\{e\}}$ and $J^{strong}\Box I_{\{e\}}$ is similar to the above.

For the case of the level model structure, it suffices to notice that the functor $F^{\mathcal{I}}_{M}(L)(-)$ preserves colimits and tensors. The proof is similar to the one above.

□

Proposition 7.8. *The level model structure and the strong level model structure on the category $G\mathcal{U}^{\mathcal{I}}$ are G-topological model structures.*

Proof. We know from Lemma 4.2 that the category $G\mathcal{U}^{\mathcal{I}}$ is, enriched, tensored and cotensored over $G\mathcal{U}$. By [16, Corollary 4.2.5], it suffices to consider the generating (acyclic) cofibrations for the respective model structures. Let $i_G : G/H \times \partial D^k \longrightarrow G/H \times D^k$ be a generating cofibration of the genuine model structure on $G\mathcal{U}$ and let f a generating cofibration of the $\mathcal{F}^{G,\Sigma_m}$-$\mathcal{U}$, so $G^{\mathcal{I}}_{\mathbf{m}}(f)$ is a strong level generating cofibration. Since G preserves colimits and tensors, the map $G^{\mathcal{I}}_{\mathbf{m}}(f)\Box i_G$ is isomorphic to the map $G^{\mathcal{I}}_{\mathbf{m}}(f\Box i_G)$. Since the $\mathcal{F}^{G,\Sigma_m}$-$\mathcal{U}$ is G-topological, this implies that $f\Box i_G$ is a cofibration in $\mathcal{F}^{G,\Sigma_m}$-$\mathcal{U}$ and henceso is $G^{\mathcal{I}}_{\mathbf{m}}(f\Box i_G)$. By the same reasoning, if either i_G or f is a generating acyclic cofibration in the respective model structures, so is $f\Box i_G$, hence so is $G^{\mathcal{I}}_{\mathbf{m}}(f\Box i_G)$. This concludes the proof.

□

7.9. **The class of h-cofibrations.** In this subsection we prove some results concerning the class of h-cofibrations which will be essential in later sections.

Definition 7.10. Let $\mathcal{C}$ be a category tensored over the category $\mathcal{U}$. A morphism $f : A \longrightarrow B$ is an *h-cofibration* if it has the homotopy extension property, *i.e.*, given morphism $\phi : B \longrightarrow X$ and a homotopy $H : [0, 1]\otimes A \longrightarrow X$ such that $H_0 = \phi f$, there is a homotopy $\overline{H} : [0, 1] \otimes B \longrightarrow X$ such that $\overline{H} \circ ([0, 1] \otimes f) = H$ and $\overline{H}_0 = \phi$.

Lemma 7.11. *Let $\mathcal{C}$ be a complete category, tensored and cotensored over the category of spaces.*

- *(i) If $\mathcal{C}$ is a topological model category in which every object is fibrant, then every cofibration is an h-cofibration*
- *(ii) The h-cofibrations are preserved under retracts, cobase change, coproducts, sequential compositions and transfinite compositions*
- *(iii) If $\mathcal{C}'$ be another category that is tensored and cotensored over the category $\mathcal{U}$ of spaces and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a continuous functor that commutes with colimits and tensors with [0,1]. Then F takes h-cofibrations in $\mathcal{C}$ to h-cofibrations in $\mathcal{C}'$.*

Proof. [31, Corollary 1.20] □

Proposition 7.12. *(i) Every cofibration in the level model structure and strong level model structure is an h-cofibration.*
(ii) For every finite G-set $M \subseteq U$, the functor $\mathrm{Ev}^{\mathcal{I}}_{M} : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}$ preserves h-cofibrations.

Proof. For the claim (i) by Proposition 7.6, the model structures $G\mathcal{U}^{\mathcal{I}}_{level}$ and $G\mathcal{U}^{\mathcal{I}}_{strong}$ are topological model structures. Since all objects are fibrant, the claim follows from Lemma 7.10 (i)

For claim (ii), recall from Definition 5.5, for a G-$\mathcal{I}$-space X and given a finite G-set $M \subseteq U$ of cardinality m the evaluation of X at M is $X(M) = X(\mathbf{m}) \times_{\Sigma_m} \mathrm{Bij}(\mathbf{m}, M)$. Hence, by construction the evaluation commutes with colimits, and obviously commutes with tensoring with $[0,1]$, that is, $(X \otimes [0,1])(M) = X(M) \times [0,1]$. Hence by Lemma 7.10, part (iii), the functor $\mathrm{Ev}_M^{\mathcal{I}}$ must preserves h-cofibrations. □

Remark 7.13. We know that for a small category $\mathcal{K}$, the h-cofibrations in $\mathcal{U}^{\mathcal{K}}$ are object-wise h-cofibrations. If we consider the category of diagrams $G\mathcal{U}^{\mathcal{I}}$ equivalently as the category of functors $\mathcal{I} \times G \longrightarrow \mathcal{U}$, we get that if $f : X \longrightarrow Y$ is cofibration in the level or strong level model structure in the category $G\mathcal{U}^{\mathcal{I}}$, then for any $\mathbf{k} \in \mathrm{ob}(\mathcal{I})$, the map $f(\mathbf{k}) : X(\mathbf{k}) \longrightarrow Y(\mathbf{k})$ is an h-cofibration of $(G \times \Sigma_k)$-spaces. In particular, $f(\mathbf{k})$ is an h-cofibration of the underlying spaces hence a closed embedding.

7.14. **Properness of Level Model structures.** In this subsection we prove that the level model structure and strong level model structure are proper, that is, left proper and right proper [15, Definition 13.1.1]. We will prove first the version of the gluing lemma.

Lemma 7.15 (Gluing lemma)**.** *Consider the following diagram in the category* $G\mathcal{U}^{\mathcal{I}}$

$$\begin{array}{ccccc} Y & \longleftarrow & X & \xrightarrow{\;i\;} & Z \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ Y_1 & \longleftarrow & X_1 & \xrightarrow[i_1]{} & Z_1 \end{array} \tag{7.8}$$

in which i, i_1 *are* h*-cofibrations of* G-$\mathcal{I}$*-spaces. If the maps* α, β, γ *are strong level equivalences, then the induced map of pushouts* $Y \cup_X Z \longrightarrow Y_1 \cup_{X_1} Z_1$ *is a strong level equivalence*

Proof. Let H be a subgroup of G and let M be a finite H-set in U of cardinality m. Then, evaluating the diagram (7.8) at M we get the induced diagram

$$\begin{array}{ccccc} Y(M) & \longleftarrow & X(M) & \xrightarrow{\;i\;} & Z(M) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ Y_1(M) & \longleftarrow & X_1(M) & \xrightarrow[i_1]{} & Z_1(M). \end{array} \tag{7.9}$$

Since $i : X \longrightarrow Z$ and $i_1 : X_1 :\longrightarrow Z_1$ are h-cofibrations it follows from Lemma 7.11, part (ii) that $i(M) : X(M) \longrightarrow Z(M)$ and $i_1(M) : X_1(M) \longrightarrow Z_1(M)$ are h-cofibrations of H-spaces. By assumption, the maps α, β, γ are strong level equivalences so the induced map of evaluations $\alpha(M), \beta(M), \gamma(M)$ are H-weak equivalences. Now the gluing lemma implies that $Y(M) \cup_{X(M)} Z(M) \longrightarrow Y_1(M) \cup_{X_1(M)} Z_1(M)$ is a H-weak equivalence. $(Y \cup_X Z)(M) \longrightarrow (Y_1 \cup_{X_1} Z_1)(M)$ is an H-weak equivalence. Doing the above for every $H \leq G$ and every finite H-set $M \subseteq_H U$, proves the result. □

Using the gluing lemma we get immediately the following corollaries.

Corollary 7.16. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \xrightarrow[k]{} & D \end{array}$$

be a pushout square such that f *is a strong level equivalence. If in addition* g *is an* h*-cofibration, then the morphism* k *is a strong level equivalence*

Proof. Let $g : A \longrightarrow C$ be an h-cofibration and consider the following diagram

$$\begin{array}{ccccc} A & = & A & \xrightarrow{\;g\;} & C \\ {\scriptstyle f}\downarrow & & \| & & \| \\ B & \xleftarrow{\;f\;} & A & \xrightarrow[g]{} & C \end{array}$$

which from the Gluing lemma, 7.14 we get immediately that the map $C \longrightarrow B \coprod_A C$ is a strong level equivalence. □

Corollary 7.17. *The model structures* $G\mathcal{U}^{\mathcal{I}}_{level}$ *and* $G\mathcal{U}^{\mathcal{I}}_{strong}$ *are proper.*

Proof. Left properness follows directly from Corollary 7.15, part (i) since by Lemma 7.11 all cofibrations in both model structures are h-cofibrations. Right properness follows directly from the definition, since weak equivalences and fibrations are defined levelwise and the pullback is computed levelwise. □

7.18. **Cellularity of Level Model structures.** In this subsection we prove that our model structures are cellular in the sense [15, Chapter 12]. Recall from [15, Definition 12.1.1] that a cellular model category is a cofibrantly generated model category $\mathcal{M}$ with generating cofibrations I and generating acyclic cofibrations J such that the domains and codomains of the maps in I are compact relative to I [15, Definition 10.8.1], the domains of the maps in J are small relative to the subcategory of relative I-cell complexes [15, Definition 10.4.1] and the cofibrations are effective monomorphisms [15, Definition 10.9.1]

Proposition 7.19. *The level model structure and the strong level model structure on the category $G\mathcal{U}^{\mathcal{I}}$ are cellular.*

Proof. We will prove the claim for the strong level model structure. For the compactness and smallness assertions, we recall from (7.2) that the objects in question are obtained by applying left adjoint functors to compact objects in $H\mathcal{U}$ for every subgroup $H \leq G$. The assertions hold because sequential colimits in $G\mathcal{U}^{\mathcal{I}}$ are created in $G\mathcal{U}$ and all cofibrations in the strong level model structure are h-cofibrations.

By definition of an effective monomorphism, we have to show that if $A \longrightarrow B$ is a cofibration, then it is the equalizer of the canonical maps $B \rightrightarrows B \coprod_A B$. Since equalizers are calculated objectwise we have to show that the diagram $A(\mathbf{m}) \to B(\mathbf{m}) \rightrightarrows B(\mathbf{m}) \coprod_{A(\mathbf{m})} B(\mathbf{m})$ is an equalizer diagram. By Remark 7.12 every strong level cofibration is an h-cofibration hence an object-wise closed embedding, that is, injective and closed map. The result now follows. □

8. Localizations of the Level Model structures

In this section will construct localizations of the three Level model structures that we defined in Section 6. We will not provide the definitions and main properties of left Bousfield localizations, instead we refer to [15, Chapter 3]. For the existence of the left Bousfield localizations we appeal to [15, Theorem 4.1.1].

8.1. ***G*-hocolim model structure.** In this subsection we will construct a model structure on the category $G\mathcal{U}^{\mathcal{I}}$ which, in a sense, is the direct generalization of the projective $\mathcal{I}$-model structure on the category $\mathcal{I}$-spaces given in Section 2 to the equivariant context.

As is generally the case for diagrams with values in the category of G-spaces, given a functor $X : \mathcal{I} \longrightarrow G\mathcal{U}$, the homotopy colimit of X, $\operatorname{hocolim}_{\mathcal{I}} X$, comes equipped with a natural G-action. We explain a bit here the induced G-action on the homotopy colimit. If X is a diagram $\mathcal{I} \longrightarrow G\mathcal{U}$ the usual Bousfield-Kan formula for homotopy colimit is by geometric realization of the simplicial replacement of the diagram X as follows

$$\operatorname*{hocolim}_{\mathcal{I}} X : \left| [n] \longmapsto \coprod_{\mathbf{k}_0 \leftarrow \ldots \leftarrow \mathbf{k}_n} X(\mathbf{k}_n) \right| \tag{8.1}$$

Since for every $\mathbf{n}$ in $\mathcal{I}$ the space $X(\mathbf{n})$ has a G-action, the simplicial replacement $X_\bullet$ is a G-simplicial space, that is, a functor $\Delta^{\mathrm{op}} \longrightarrow G\mathcal{U}$ and then composing with the the geometric realization of G-simplicial spaces to G-spaces, $|-| : \Delta^{\mathrm{op}} G\mathcal{U} \longrightarrow G\mathcal{U}$ we get the homotopy colimit.

Considered as a functor, $\operatorname{hocolim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}$ has a right adjoint such that, to a G-space L associates the G-$\mathcal{I}$-space

$$\mathbf{n} \longmapsto \operatorname{map}(B(\mathbf{n} \downarrow \mathcal{I}), L)$$

with G-action through L.

A crucial property of the above homotopy colimit is that, for a subgroup $H \leq G$, the H-fixed points of the G-space $\operatorname{hocolim}_{\mathcal{I}} X$ is naturally homeomorphic to the homotopy colimit (non-equivariant) of the fixed-point functor $X^H : \mathcal{I} \longrightarrow \mathcal{U}$, $\quad \mathbf{m} \longmapsto X(\mathbf{m})^H$, that is,

$$\left(\operatorname*{hocolim}_{\mathcal{I}} X\right)^H \cong \operatorname*{hocolim}_{\mathcal{I}}(X^H) \tag{8.2}$$

An informal proof of the above homeomorphism can be found in [6, Remark 5.6].

Remark 8.2. A more formal proof can be found in [7, Proposition 1.8]. The proposition as stated in [7] requires a discrete group G acting on a index category $\mathcal{C}$, encoded by a functor $\alpha : G \longrightarrow \mathrm{Cat}$, where we regard G as a category with a single object. The fundamental notion considered in the paper is that of a G-functor $X_\alpha : \mathcal{C} \longrightarrow \mathcal{U}$, which loosely speaking is a functor $X : \mathcal{C} \longrightarrow \mathcal{U}$ that is compatible with the G-action on $\mathcal{C}$ in an appropriate sense. For the formal definition we refer to [7, Definition 1.1] and [10, Definition 2.2]. G-functors or G-diagrams generalize the notion of a functor with values in G-spaces. Indeed, if the G-action on the index category is trivial, then a G-functor is precisely a functor $\mathcal{C} \longrightarrow G\mathcal{U}$ which is the case we are interested in. For further details about G-diagrams we refer to the papers [7], [8] and [10].

We will need the following two lemmas which we reproduce here for the reader's convenience.

Lemma 8.3. [24, Proposition 4.4] *Let $\mathcal{C}$ be a small category, let $X \longrightarrow Y$ be a map of $\mathcal{C}$-diagrams in $\mathcal{U}$, let $\alpha : \mathbf{k} \longrightarrow \mathbf{l}$ be a morphism in $\mathcal{C}$. Consider the two squares*

$$\begin{array}{ccc} X(\mathbf{k}) & \longrightarrow & Y(\mathbf{k}) \\ {\scriptstyle X(\alpha)}\downarrow & & \downarrow{\scriptstyle Y(\alpha)} \\ X(\mathbf{l}) & \longrightarrow & Y(\mathbf{l}) \end{array} \qquad\qquad \begin{array}{ccc} X(\mathbf{k}) & \longrightarrow & Y(\mathbf{k}) \\ \downarrow & & \downarrow \\ X_{h\mathcal{C}} & \longrightarrow & Y_{h\mathcal{C}}. \end{array} \tag{8.3}$$

If the left hand is homotopy cartesian for every α, then the right hand square is homotopy cartesian for every object $\mathbf{k}$.

Remark 8.4. [26, Remark 6.13] The above lemma as stated in [24], is only stated for simplicial sets, but the analogous result for our category $\mathcal{U}$ is an immediate consequence. Indeed, recall that a square diagram of topological spaces is homotopy cartesian if and only if applying the singular complex functor Sing gives a homotopy cartesian square of simplicial sets. Conversely, a square diagram of simplicial is homotopy cartesian if and only if the geometric realization is homotopy cartesian. Thus, given a map $X \longrightarrow Y$ of $\mathcal{C}$-diagrams of topological spaces such that the left hand squares are homotopy cartesian, the lemma implies that the diagram

$$\begin{array}{ccc} \mathrm{Sing}X(\mathbf{k}) & \longrightarrow & \mathrm{Sing}Y(\mathbf{k}) \\ \downarrow & & \downarrow \\ (\mathrm{Sing}X)_{h\mathcal{C}} & \longrightarrow & (\mathrm{Sing}Y)_{h\mathcal{C}} \end{array}$$

is homotopy cartesian. This in turn implies that the geometric realization is homotopy cartesian and the natural transformation $|\mathrm{Sing}X| \longrightarrow X$ defines a natural weak equivalence between this realization and the right hand square in the lemma

We continue with a construction that has been widely used in the homotopy theory of spectra and diagram spaces. Instead of proving another version of it, we state the formal construction and we refer to [31, Proposition 3.16] for the proof.

Construction 8.5. Let $j : A \longrightarrow B$ be a morphism in a topological model category. We factor j through the mapping cylinder construction as the composite

$$A \xrightarrow{c(j)} Z(j) = ([0,1] \otimes A) \cup_j B \xrightarrow{r(j)} B$$

where $c(j)$ is the front mapping cylinder inclusion and $r(j)$ is the projection, which is homotopy equivalence. We will be interested in the case where A and B are cofibrant, and then the morphism $c(j)$ is a cofibration by the pushout-product property. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$i_k \Box c(j) : D^k \otimes A \cup_{\partial D^k \otimes A} \partial D^k \otimes Z(j) \longrightarrow D^k \otimes Z(j) \tag{8.4}$$

for $k \geq 0$ where $i_k : \partial D^k \longrightarrow D^k$ the inclusion

Lemma 8.6. [31, Proposition 3.16] *Let $\mathcal{C}$ be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ be a fibration. Then the following two conditions are equivalent*

(i) The square of spaces

$$\begin{array}{ccc} \mathrm{map}(B, X) & \longrightarrow & \mathrm{map}(A, X) \\ \downarrow & & \downarrow \\ \mathrm{map}(B, Y) & \longrightarrow & \mathrm{map}(A, Y) \end{array}$$

is homotopy cartesian.

(ii) The morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$

We make the following definition.

Definition 8.7. A map $f : X \longrightarrow Y$ of G-$\mathcal{I}$-spaces is

- *G-hocolim equivalence* if the induced map $\mathrm{hocolim}_{\mathcal{I}} X \longrightarrow \mathrm{hocolim}_{\mathcal{I}} Y$ is a G-weak equivalence, and
- a *G-hocolim fibration* if it is projective fibration with the additional property that every morphism $\mathbf{k} \longrightarrow \mathbf{l}$ in $\mathcal{I}$, induces a homotopy cartesian square in $G\mathcal{U}$

$$\begin{array}{ccc} X(\mathbf{k}) & \longrightarrow & X(\mathbf{l}) \\ \downarrow & & \downarrow \\ Y(\mathbf{k}) & \longrightarrow & Y(\mathbf{l}). \end{array}$$

  Unwinding a bit, this means that for every subgroup $H \leq G$ and every $\mathbf{k} \in \mathrm{ob}(\mathcal{I})$ the map $f(\mathbf{k})^H : X(\mathbf{k})^H \longrightarrow Y(\mathbf{k})^H$ is a fibration of spaces and for every morphism $\mathbf{k} \longrightarrow \mathbf{l}$ in $\mathcal{I}$ the following diagram

$$\begin{array}{ccc} X(\mathbf{k})^H & \longrightarrow & X(\mathbf{l})^H \\ \downarrow & & \downarrow \\ Y(\mathbf{k})^H & \longrightarrow & Y(\mathbf{l})^H \end{array}$$

  is homotopy cartesian square of spaces, that is, the map $X(\mathbf{k})^H \longrightarrow X(\mathbf{l})^H \times_{Y(\mathbf{l})^H} Y(\mathbf{k})^H$ is a weak equivalence.

Consider a morphism $\alpha : \mathbf{k} \longrightarrow \mathbf{l}$ and a subgroup $H \leq G$. This induces a map of G-$\mathcal{I}$-spaces $\alpha^* \times G/H : F^{\mathcal{I}}_{\mathbf{l}}(G/H) \longrightarrow F^{\mathcal{I}}_{\mathbf{k}}(G/H)$

Lemma 8.8. *The map $\alpha^* \times G/H : F^{\mathcal{I}}_{\mathbf{l}}(G/H) \longrightarrow F^{\mathcal{I}}_{\mathbf{k}}(G/H)$ is a G-hocolim equivalence.*

Proof. By definition of the homotopy colimits, $F_{\mathbf{l}}^{\mathcal{I}}(G/H)_{h\mathcal{I}}$ and $F_{\mathbf{k}}^{\mathcal{I}}(G/H)_{h\mathcal{I}}$ can be identified by the product $B(\mathbf{l} \downarrow \mathcal{I}) \times G/H$ and $B(\mathbf{k} \downarrow \mathcal{I}) \times G/H$ respectively. Both of the categories $(\mathbf{l} \downarrow \mathcal{I})$ and $(\mathbf{k} \downarrow \mathcal{I})$ have an initial object $(\mathbf{l}, \mathrm{id})$ and $(\mathbf{k}, \mathrm{id})$, respectively so their classifying spaces are contractible. Therefore the map induced by $\alpha^* \times G/H$ is a G-equivalence. □

We now use the tensor with an interval in $\mathcal{U}$, to factor the map $\alpha^* \times G/H$ through the mapping cylinder in the usual way

$$F_{\mathbf{l}}^{\mathcal{I}}(G/H) \xrightarrow{c(j)} M(j) \xrightarrow{r(j)} F_{\mathbf{k}}^{\mathcal{I}}(G/H).$$

The map $c(j)$ is a projective cofibration and $r(j)$ is a homotopy equivalence. Let K be the set of morphisms of the form $i_k \Box c(j)$ where $c(j)$ is as above, i_k is a generating cofibration in $\mathcal{U}$, and $\Box$ as usual the pushout-product map associated to the tensor with an object in $\mathcal{U}$. We define $I^{hocolim} = I^{proj}$ and $J^{hocolim} = J^{proj} \cup K$.

Proposition 8.9. *The G-hocolim equivalences together with the projective cofibrations and G-hocolim fibrations specify a cofibrantly generated model structure on the category $G\mathcal{U}^{\mathcal{I}}$ with generating cofibrations $I^{hocolim}$ and generating acyclic cofibrations $J^{hocolim}$.*

We shall refer to this as the G-hocolim model structure on G-$\mathcal{I}$-spaces.

Proof. We will use the criterion [16, Theorem 2.1.19] for the recognition principle for cofibrantly generated model structures. As in the projective model structure, the $I^{hocolim}$-injective maps are the maps $X \longrightarrow Y$ such that for every $\mathbf{k} \in \mathrm{ob}(\mathcal{I})$ and all subgroups $H \leq G$, the induced map $X(\mathbf{k})^H \longrightarrow Y(\mathbf{k})^H$ is an acyclic fibration. Moreover from Lemma 8.6 we can see immediately that a map $X \longrightarrow Y$ is $J^{hocolim}$-injective if and only if it is G-hocolim fibration. Thus, a map $X \longrightarrow Y$ which is $I^{hocolim}$-injective is clearly both a $J^{hocolim}$-injective and a G-hocolim equivalence. Suppose then that $f : X \longrightarrow Y$ is $J^{hocolim}$-injective and a G-hocolim equivalence. The first condition implies for every subgroup $H \leq G$ and every morphism $\mathbf{k} \longrightarrow \mathbf{l}$ the square

$$\begin{array}{ccc} X(\mathbf{k})^H & \longrightarrow & X(\mathbf{l})^H \\ \downarrow & & \downarrow \\ Y(\mathbf{k})^H & \longrightarrow & Y(\mathbf{l})^H \end{array}$$

is homotopy cartesian square. Consider the commutative square

$$\begin{array}{ccc} X(\mathbf{k})^H & \longrightarrow & Y(\mathbf{k})^H \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_{\mathcal{I}}(X^H) & \longrightarrow & \mathrm{hocolim}_{\mathcal{I}}(Y^H). \end{array} \tag{8.5}$$

Since $f : X \longrightarrow Y$ is a G-hocolim equivalence, by definition, the induced map of fixed points $(\mathrm{hocolim}_{\mathcal{I}} X)^H \longrightarrow (\mathrm{hocolim}_{\mathcal{I}} Y)^H$ is a weak equivalence for every subgroup $H \leq G$. But $(\mathrm{hocolim}_{\mathcal{I}} X)^H \cong \mathrm{hocolim}_{\mathcal{I}}(X^H)$ and similarly $(\mathrm{hocolim}_{\mathcal{I}} Y)^H \cong \mathrm{hocolim}_{\mathcal{I}}(Y^H)$, hence by Lemma 8.3 the square (8.5) is homotopy cartesian, so we get that $X(\mathbf{k})^H \longrightarrow Y(\mathbf{k})^H$ is a weak equivalence. Hence the map $f : X \longrightarrow Y$ is projective level equivalence, so $I^{hocolim}$-injective.

The last thing to check is that $J^{hocolim}$-cell also belong to the class $I^{hocolim}$-cof and G-hocolim equivalences. For the second part we first observe that the maps in $J^{hocolim}$ are G-hocolim equivalences by Lemma 8.8. We next observe that the functor $\mathrm{hocolim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}} \longrightarrow G\mathcal{U}$ takes the class I^{hocolim}-cof to cofibrations in $G\mathcal{U}$. Since the functor $\mathrm{hocolim}_{\mathcal{I}}$ is left adjoint, it preserves colimits so it suffices to check that it takes the elements in $I^{hocolim}$ to cofibrations in $G\mathcal{U}$. Indeed, this follows from the fact that for a map of G-$\mathcal{I}$-spaces of the form $\mathcal{I}(\mathbf{m}, -) \times G/H \times \partial D^k \longrightarrow \mathcal{I}(\mathbf{m}, -) \times G/H \times D^k$, the induced map on homotopy colimits may be identified with the map $B(\mathbf{m} \downarrow \mathcal{I}) \times G/H \times \partial D^k \longrightarrow B(\mathbf{m} \downarrow \mathcal{I}) \times G/H \times D^k$ and $B(\mathbf{m} \downarrow \mathcal{I})$ is a cell complex. By definition a map in $J^{hocolim}$-cell is the transfinite composition of a sequence of maps which is a pushout of a map in $J^{hocolim}$. The induced map $X_{h\mathcal{I}} \longrightarrow Y_{h\mathcal{I}}$ is therefore the transfinite composition of a sequence of maps each of which is a pushout of an acyclic cofibration. So, it is transfinite composition of G-equivalences which are also h-cofibrations hence the induced map is a G-weak equivalence.

□

8.10. ***G*-hocolim model structure as localization.** In this section we follow the paper [5] and we will discuss in the next subsection how the G-hocolim model structure can be constructed equivalently as a left Bousfield localization of the projective model structure.

Recall that given $\mathcal{M}$ a cofibrantly generated model category and $\mathcal{K}$ a small category, then the category of diagrams $\mathcal{M}^{\mathcal{K}}$ may be given the projective model structure. A map of diagrams $X \longrightarrow Y$ is called a *hocolim-equivalence* if the induced map of corrected homotopy colimits is a weak equivalence in $\mathcal{M}$. One of the author's goal in the paper [5] is to localize the projective model structure by inverting the hocolim-equivalences.

Unfortunately these form a proper class, so the author proceeds to identify a *set* of maps such that forming the left Bousfield localization with respect this set of maps is enough. We describe shortly this set of maps.

Assume that $\mathcal{M}$ is left proper and cellular. Then by [5, Proposition A.5], there exists a set W of cofibrant objects that detect level equivalences, in the sense that a map $f : X \longrightarrow Y$ is a weak equivalence if and only if it induces weak equivalences on function complexes $\underline{\mathcal{M}}(A, X) \longrightarrow \underline{\mathcal{M}}(A, Y)$, for every $A \in W$.

Let $\mathcal{K}$ be a small category. For $i \in \mathrm{ob}(\mathcal{K})$ and $X \in \mathcal{M}$ let $F_i(X)$ the diagram

$$F_i(X)(-) : \mathcal{K} \longrightarrow \mathcal{M}^{\mathcal{K}}$$
$$j \longmapsto \coprod_{\mathcal{K}(i,j)} X = \mathcal{K}(i, j) \times X$$

Let S be the set of diagrams

$$F_j(A) \longrightarrow F_i(A) \tag{8.6}$$

for every morphism $i \longrightarrow j$ in $\mathcal{K}$ and $A \in W$.

Then the author proceeds to prove [5, Theorem 5.2], which states that if $\mathcal{K}$ has contractible nerve, then there is a model structure on the category $\mathcal{M}^{\mathcal{K}}$ with weak equivalences the class of S-local equivalences, cofibrations the cofibrations of the model structure $\mathcal{M}^{\mathcal{K}}_{proj}$ and the fibrant objects are the S-local objects. More precisely, this model structure is defined as a left Bousfield localization of $\mathcal{M}^{\mathcal{K}}_{proj}$ at the set of maps S as defined in (8.6). A map $f : X \longrightarrow Y$ belongs to the class of S-local equivalences if and only if it induces a weak equivalence on the corrected homotopy colimits. A diagram $X : \mathcal{K} \longrightarrow \mathcal{M}$ is S-local if and only if for every arrow $i \longrightarrow j$ in $\mathcal{K}$ the induced map $X(i) \longrightarrow X(j)$ is a weak equivalence in $\mathcal{M}$.

8.11. **Application to G-$\mathcal{I}$-spaces.** Since the category $\mathcal{I}$ has contractible nerve (it has an initial object) and the category $G\mathcal{U}$ with the genuine model structure is left proper and cellular we can apply [5, Theorem 5.2] for the projective model structure $G\mathcal{U}^{\mathcal{I}}_{proj}$ and we get tautologically the following corollary

Corollary 8.12. *There is a left Bousfield localization of $G\mathcal{U}^{\mathcal{I}}_{proj}$, at a set of maps $\mathcal{A}$, denoted by $G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.}$ such that the following hold*

- *(i) the weak equivalences in $G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.}$ are the G-hocolim equivalences*
- *(ii) the adjunction $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.} \rightleftarrows G\mathcal{U} : \mathrm{const}_{\mathcal{I}}$ is a Quillen equivalence, and*
- *(iii) the fibrant objects of $G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.}$, that is, the $\mathcal{A}$-local objects are those diagrams $X : \mathcal{I} \longrightarrow G\mathcal{U}$ such that for any morphism $\mathbf{n} \longrightarrow \mathbf{m}$ in $\mathcal{I}$ the induced map $X(\mathbf{n}) \longrightarrow X(\mathbf{m})$ is a G-weak equivalence.*

Since a model structure is completely specified by the cofibrations and the fibrant objects, the above model structure is equal to the model structure that we constructed in Proposition 8.9. Let's follow the recipe as is laid out in Subsection 8.10 to identify the set of maps of G-$\mathcal{I}$-spaces that do the job.

The (topological) genuine model structure $G\mathcal{U}$ has the set $\{G/H\,|H \le G\}$ that detect weak equivalences. The free functor that is used, in our case is defined by equation (6.1. So the set of maps $\mathcal{A}$ is

$$\alpha^* \times G/H : F^{\mathcal{I}}_{\mathbf{n}}(G/H) \longrightarrow F^{\mathcal{I}}_{\mathbf{m}}(G/H) \tag{8.7}$$

for every morphism $\alpha : \mathbf{m} \longrightarrow \mathbf{n}$ in $\mathcal{I}$ and every subgroup $H \le G$. By the above corollary and the proof of [5, Theorem 5.2], a map f of G-$\mathcal{I}$-spaces belongs to the class of $\mathcal{A}$-local equivalences if and only if the induced map $\mathrm{hocolim}_{\mathcal{I}} X \longrightarrow \mathrm{hocolim}_{\mathcal{I}} Y$ is a G-weak equivalence and the set of $\mathcal{A}$-local objects are those diagrams $X : \mathcal{I} \longrightarrow G\mathcal{U}$ that satisfy the condition in (*iii*) which are also called "locally constant functors".

8.13. **Localization of the Level Model structure.** In this subsection we will construct a left Bousfield localizations of the (topological) level model structure, $G\mathcal{U}^{\mathcal{I}}_{level}$. Recall the Definition 5.6 of a free G-$\mathcal{I}$-space on L in level M. Let M and N be finite G-sets and let $\alpha : M \longrightarrow N$ an injective map. This induces a natural transformation

$$\alpha^* : F^{\mathcal{I}}_N(*) = \mathcal{I}(N, -) \longrightarrow \mathcal{I}(M, -) = F^{\mathcal{I}}_M(*)$$
$$f \longmapsto \alpha^* f = f \circ \alpha$$

For a subgroup $H \le G$ we have as above a natural transformation

$$\alpha^* \times G/H : F^{\mathcal{I}}_N(G/H) \longrightarrow F^{\mathcal{I}}_M(G/H).$$

Let $\mathcal{B}$ be the set of all maps of G-$\mathcal{I}$-spaces,

$$\alpha^* \times G/H : F^{\mathcal{I}}_N(G/H) \longrightarrow F^{\mathcal{I}}_M(G/H) \tag{8.8}$$

for every $M, N \subset U$, every injection $\alpha : M \longrightarrow N$ and every $H \le G$. Since is left proper and cellular by [15, Theorem 4.1.1] the left Bousfield localization with respect to $\mathcal{B}$ exists and we write the resulting model structure by $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}$

The fibrant objects of the model structure $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}$ are precisely the $\mathcal{B}$-local objects. We have the following proposition.

Proposition 8.14. *Let X be a diagram $\mathcal{I} \longrightarrow G\mathcal{U}$. Then X is fibrant in $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}$ if and only if*

(1) X is fibrant in the model structure $G\mathcal{U}^{\mathcal{I}}_{\text{level}}$ and,
(2) for every M and N finite G-sets and every injection $M \longrightarrow N$ the induced map $X(M) \longrightarrow X(N)$ is a G-weak equivalence.

Proof. By definition, if S is a set of maps in a model structure $\mathcal{M}$, an object X is S-local if it fibrant in $\mathcal{M}$ and every element $f : A \longrightarrow B$ of S the induced map $f^* : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is a weak equivalence of spaces. The first condition is automatically satisfied since in $G\mathcal{U}^{\mathcal{I}}_{\text{level}}$ all objects are fibrant. Suppose we are given M and N finite G-sets, an injection $\alpha : M \longrightarrow N$ and a subgroup $H \leq G$. Let f be a map in the set $\mathcal{B}$. We have the following

$$f^* : \mathrm{Map}_{G\mathcal{U}^{\mathcal{I}}}(F^{\mathcal{I}}_M(G/H), X) \longrightarrow \mathrm{Map}_{G\mathcal{U}^{\mathcal{I}}}(F^{\mathcal{I}}_N(G/H), X) \quad \text{is w.e. if and only if}$$
$$f^*(M) : \mathrm{map}_G(G/H, X(M)) \longrightarrow \mathrm{map}_G(G/H, X(N)) \quad \text{which is w.e. if and only if}$$
$$f^*(M)^H : X(M)^H \longrightarrow X(N)^H \quad \text{is a w.e. for every } H \leq G$$

So, $X(M)^H \longrightarrow X(N)^H$ is a weak equivalence of spaces, when restricting the G-actions on M and N, by the inclusion $H \hookrightarrow G$. By letting H run through all subgroups of G we get that $X(M) \longrightarrow X(N)$ is a G-equivalence. Inductively doing the above for every $M, N \subset U$ and every equivariant injection $\alpha : M \longrightarrow N$ we get the proposition. □

Remark 8.15. If we chose to work in the G-topological model structure $G\mathcal{U}^{\mathcal{I}}_{level}$, the mapping spaces $\mathrm{Map}(F^{\mathcal{I}}_M(*), X)$ and $\mathrm{Map}(F^{\mathcal{I}}_N(*), X)$ have a G-action by conjugation. The set of maps $\mathcal{B}$, would be defined equivalently as the set of maps $F^{\mathcal{I}}_N(*) \longrightarrow F^{\mathcal{I}}_M(*)$. Hence a diagram X is $\mathcal{B}$-local if and only if

$$f^* : \mathrm{Map}(F^{\mathcal{I}}_M(*), X) \longrightarrow \mathrm{Map}(F^{\mathcal{I}}_N(*), X) \quad \text{is a } G\text{-weak equivalence}$$
$$\mathrm{map}(*, X(M)) \longrightarrow \mathrm{map}(*, X(N)) \quad \text{iff is a } G\text{-weak equivalence}$$
$$X(M) \longrightarrow X(N) \quad \text{iff is a } G\text{-weak equivalence}$$

8.16. **Localization of the Strong level model structure.** In this subsection we introduce a localization of the (topological) strong level model structure, $G\mathcal{U}^{\mathcal{I}}_{strong}$.

Similarly to the previous subsection, for M and N finite H-sets, with $M, N \subseteq_H U$ and $\beta : M \longrightarrow N$ an injection we have a natural transformation of G-$\mathcal{I}$-spaces

$$G \ltimes_H \beta^* : G \ltimes_H F^{\mathcal{I}}_N(*) \longrightarrow G \ltimes_H F^{\mathcal{I}}_M(*)$$

and for a subgroup $K \leq H$ we have

$$G \ltimes_H (\beta^* \times H/K) : G \ltimes_H F^{\mathcal{I}}_N(H/K) \longrightarrow G \ltimes_H F^{\mathcal{I}}_M(H/K).$$

Let $\mathcal{C}$ be the set of all maps of G-$\mathcal{I}$-spaces

$$G \ltimes_H (\beta^* \times H/K) : G \ltimes_H F^{\mathcal{I}}_N(H/K) \longrightarrow G \ltimes_H F^{\mathcal{I}}_M(H/K) \tag{8.9}$$

for every $H \leq G$, every M, N finite H-sets in U and injection $\beta : M \longrightarrow N$ and every $K \leq H$. Since the model structure $G\mathcal{U}^{\mathcal{I}}_{strong}$ is proper and cellular, the left Bousfield localization with respect the set $\mathcal{C}$ exists and we write the resulting left Bousfield localization as $G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$

Proposition 8.17. *Let X be a diagram $\mathcal{I} \longrightarrow G\mathcal{U}$. Then X is fibrant in $G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ if and only if*

(1) X is fibrant in the model structure $G\mathcal{U}^{\mathcal{I}}_{\text{strong}}$ and,
(2) for every $H \leq G$ and every M and N finite H-sets in U and every injection $M \longrightarrow N$ the induced map $X(M) \longrightarrow X(N)$ is an H-weak equivalence.

Proof. The proof is identical the proof of Proposition 8.14 and we omit it. □

Remark 8.18. Similarly to Remark 8.15, if we chose to work in the G-topological model structure $G\mathcal{U}^{\mathcal{I}}_{strong}$, then we would define the set of maps $\mathcal{C}$, as the set $G \ltimes_H F^{\mathcal{I}}_N(*) \longrightarrow F^{\mathcal{I}}_M(*)$, for every subgroup $H \leq G$, every $M, N \subseteq_H U$ and every injection $M \longrightarrow N$.

9. Comparing the localizations

In this section we will compare the three localizations that we constructed, namely, $G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.}, G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}, G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ As we will see in Discussion 9.7 these model structures are all Quillen equivalent. The important ingredients for this observation is Lemma 9.3 and Proposition 9.4.

For the next lemma recall the set of maps $\mathcal{A}$ from (8.7), the set of maps $\mathcal{B}$ from (8.8), and the set of maps $\mathcal{C}$ from (8.9), with respect to which we localize the projective model structure, the level model structure and the strong level model structure, respectively.

Lemma 9.1. *Consider the sets of maps $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ as sets of maps in the underlying category $G\mathcal{U}^{\mathcal{I}}$. Then*
- *(i) The set $\mathcal{B}$ is strictly contained in the set of maps $\mathcal{C}$, and*
- *(ii) The set of maps in the $\mathcal{A}$ is strictly contained in the set of maps $\mathcal{B}$.*

Proof. We prove first Part (i). Let M and N be finite G-sets contained in U, let $H \leq G$ and let $j : F^{\mathcal{I}}_N(G/H) \longrightarrow F^{\mathcal{I}}_M(G/H)$ be a map in the set $\mathcal{B}$. Restrict the G-action on M, to an H-action by the inclusion $\iota : H \hookrightarrow G$. Let $\mathbf{n} \in \text{ob}(\mathcal{I})$ and consider the G-set $\text{Inj}(M, \mathbf{n})$ that we defined in (5.6). Then the set $\text{Inj}(M, \mathbf{n})$ has an H-action that is a restriction of a G-action, which implies $G \ltimes_H \text{Inj}(M, \mathbf{n}) \cong G/H \times \text{Inj}(M, \mathbf{n})$ in which now $\text{Inj}(M, \mathbf{n})$ on the right hand side has the G-action. Since the isomorphism is natural we have

$$\begin{aligned} G \ltimes_H F^{\mathcal{I}}_M(*) &= G \ltimes_H \text{Inj}(M, -) \\ &\cong G/H \times \text{Inj}(M, -) \\ &= F^{\mathcal{I}}_M(G/H). \end{aligned}$$

Similarly for the finite G-set N, we have $G \ltimes_H \text{Inj}(N, -) \cong G/H \times \text{Inj}(N, -)$. So, every map $F^{\mathcal{I}}_N(G/H) \longrightarrow F^{\mathcal{I}}_M(G/H)$ can be written as a map $G \ltimes_H F^{\mathcal{I}}_N(*) \longrightarrow G \ltimes_H F^{\mathcal{I}}_M(*)$. Since not every H-action on a finite set is given by a restriction of a G-action by the inclusion $H \hookrightarrow G$, it follows that the containment is strict. Hence we can consider the set of maps $\mathcal{B}$ as a strict subset of the set of $\mathcal{C}$ restricting only to H-actions on finite sets that are restrictions of G-actions, induced by the inclusion $H \hookrightarrow G$.

We move on to Part (ii), which follows the same reasoning. Let $\mathbf{n}$ and $\mathbf{m}$ be objects in $\mathcal{I}$, let $H \leq G$ and let $F^{\mathcal{I}}_{\mathbf{n}}(G/H) \longrightarrow F^{\mathcal{I}}_{\mathbf{m}}(G/H)$ be a map in the set $\mathcal{A}$. The G-$\mathcal{I}$-space $F^{\mathcal{I}}_{\mathbf{n}}(G/H)$ is isomorphic to the G-$\mathcal{I}$-space $F^{\mathcal{I}}_M(G/H)$, after choosing a bijection $M \to \mathbf{m}$ and with trivial G-action on M. Hence every map in the set $\mathcal{A}$ is contained in the set $\mathcal{B}$ by restricting to trivial G-actions on all finite sets. That the containment is strict is evident. □

We have the following immediate corollaries.

Corollary 9.2. *The identity functors $\text{Id} : G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ and $\text{Id} : G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}$ are left Quillen.*

Proof. We consider first the case $\text{Id} : G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$. By Proposition 6.14, the identity functor is left Quillen, hence so is the composition $G\mathcal{U}^{\mathcal{I}}_{level} \xrightarrow{\text{Id}} G\mathcal{U}^{\mathcal{I}}_{level} \xrightarrow{\text{Id}} G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ So, by [15, Proposition 3.3.18], it suffices to show that the identity functor sends the maps in the set $\mathcal{B}$ to $\mathcal{C}$-local equivalences. But this follows from Lemma 9.1, part (i).

Similarly to the first case, by Proposition 6.14, it suffices to show the identity functor sends every map in the set $\mathcal{A}$, to $\mathcal{B}$-local equivalences. But this follows from Lemma 9.1, part (ii). As a consequence, we get also that the identity functor $\text{Id} : G\mathcal{U}^{\mathcal{I}}_{proj\text{-}loc.} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ is also left Quillen as a composition of left Quillen functors. □

Lemma 9.3. *Let L be a G-space. Consider the constant functor $F^{\mathcal{I}}_{\mathbf{0}}(L)(-) \cong \text{const}_{\mathcal{I}} L : \mathcal{I} \longrightarrow G\mathcal{U}$.*
- *(i) The functor $\text{const}_{\mathcal{I}} L$ is fibrant in $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.}$, that is, it is a $\mathcal{B}$-local object and,*
- *(ii) The functor $\text{const}_{\mathcal{I}} L$ is fibrant in $G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$, that is, it is a $\mathcal{C}$-local object.*

Proof. We will prove claim (ii). Recall that given a G-$\mathcal{I}$-space X and a finite H-set M of cardinality m, then the evaluation $X(M)$ is homeomorphic (non-equivariantly) to the underlying space $X(\mathbf{m})$ but has a new H-action given by the diagonal H-action on $X(\mathbf{m}) \times_{\Sigma_m} \text{Bij}(\mathbf{m}, M)$. By Lemma 6.10, after choosing a bijection $M \xrightarrow{\cong} \mathbf{m}$, we can define an H-action on $\mathbf{m}$ which we denoted as $\mathbf{m}_\phi$, such that $X(M)$ and such that $X(\mathbf{m}_\phi)$ are H-isomorphic. By Lemma 6.9, the new action of $h \in H$ on the underlying space $X(\mathbf{m})$ is given equivalently as follows

$$X\left(\phi(h)\right)(h \cdot x) \quad \text{for } x \in X(\mathbf{m}). \tag{9.1}$$

Since we consider the constant diagram $\text{const}_{\mathcal{I}} L : \mathcal{I} \longrightarrow G\mathcal{U}$, by definition, any morphism in $\mathcal{I}$ is sent to the identity. Hence for the constant diagram and for any $h \in H$ the induced automorphism $X(\phi(h))$ must be the identity. Hence we have $(\text{const}_{\mathcal{I}} L)(M) \cong L$ for any finite H- set $M \subseteq_H U$, where the H-action on the right hand side on L is just the restriction of the group action by the inclusion $\iota : H \hookrightarrow G$. This implies that for every finite H-sets $M, N \subseteq_H U$, and any injective function $M \longrightarrow N$, the induced map $(\mathrm{const}_{\mathcal{I}} L)(M) \longrightarrow \mathrm{const}_{\mathcal{I}}(L)(N)$ is an H-weak equivalence. This shows that $F^{\mathcal{I}}_{\mathbf{0}}(L) \cong \mathrm{const}_{\mathcal{I}} L$ is a $\mathfrak{C}$-local object. The proof of claim (i), is entirely analogous and we therefore omit it. □

The next proposition is an immediate corollary of the previous Lemma.

Proposition 9.4. *The adjunctions*

$$F^{\mathcal{I}}_{\mathbf{0}}(-) : G\mathcal{U} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.} : \mathrm{Ev}^{\mathcal{I}}_{\mathbf{0}} \tag{9.2}$$

$$F^{\mathcal{I}}_{\mathbf{0}}(-) : G\mathcal{U} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} : \mathrm{Ev}^{\mathcal{I}}_{\mathbf{0}} \tag{9.3}$$

are Quillen equivalences.

Proof. We will prove that the adjunction (9.2) is a Quillen equivalence. By Definition 6.6 of the strong level model structure, it is clear that the adjunction $F^{\mathcal{I}}_{\mathbf{0}} : G\mathcal{U} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{strong} : \mathrm{Ev}^{\mathcal{I}}_{\mathbf{0}}$ is a Quillen adjunction. By definition of the left Bousfield localization the identity functor $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{strong} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$ is left Quillen, so we have the Quillen adjunction $F^{\mathcal{I}}_{\mathbf{0}} : G\mathcal{U} \rightleftarrows G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.} : \mathrm{Ev}^{\mathcal{I}}_{\mathbf{0}}$. So, to show that the Quillen adjunction is a Quillen equivalence, we have to show that for every cofibrant G-space L and every fibrant X in $G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.}$, *i.e.*, a $\mathfrak{C}$-local object, a map $F^{\mathcal{I}}_{\mathbf{0}}(L) \longrightarrow X$ is a $\mathfrak{C}$-local equivalence if and only if its adjoint $L \longrightarrow X(\mathbf{0})$ is a G-weak equivalence.

By Lemma 9.3 part (ii), the G-$\mathcal{I}$-space $F^{\mathcal{I}}_{\mathbf{0}}(L)$ is $\mathfrak{C}$-local. Since by assumption X is $\mathfrak{C}$-local, the $\mathfrak{C}$-local equivalences between them are precisely the strong level equivalences. So it suffices to show that $F^{\mathcal{I}}_{\mathbf{0}}(L) \longrightarrow X$ is a strong level equivalence if and only if $L \longrightarrow X(\mathbf{0})$ is a G-weak equivalence. If a map $F^{\mathcal{I}}_{\mathbf{0}}(L) \longrightarrow X$ is a strong level equivalence then obviously we will have that the $L \longrightarrow X(\mathbf{0})$ is a weak equivalence. Conversely suppose that $L \longrightarrow X(\mathbf{0})$ is a G-weak equivalence. Since X is $\mathfrak{C}$-local object for any finite H-set M, we have an H-weak equivalence $X(\mathbf{0}) \longrightarrow X(M)$, which implies immediately that $F^{\mathcal{I}}_{\mathbf{0}}(L) \longrightarrow X$ is a strong level equivalence.

The proof that the adjunction (9.3) is a Quillen equivalence, follows similarly from Lemma 9.3, part (i), and the above proof so we omit it. □

Before proving our next proposition, we make the following remark about computing the colimits of spaces with G-action.

Remark 9.5. Recall that given a small category $\mathcal{C}$ and a functor $X : \mathcal{C} \longrightarrow G\mathcal{U}$, then the colimit of X, $\mathrm{colim}_{\mathcal{C}} X$ can be defined by first forgetting the G-action, compute the underlying colimit of the diagram $X : \mathcal{C} \longrightarrow \mathcal{U}$, and then give the induced G-action. Formally, for a discrete group G, the forgetful functor $G\mathcal{U} \longrightarrow \mathcal{U}$ creates the colimits. In particular importance to us is the following. Recall from Definition 5.6, the free G-$\mathcal{I}$-space in level M, for some finite G-set $M \subseteq U$ of cardinality m, that is, the functor

$$\begin{aligned} F^{\mathcal{I}}_{M}(*) : \mathcal{I} &\longrightarrow G\mathcal{U} \\ \mathbf{n} &\longmapsto \mathrm{Inj}(M, \mathbf{n}). \end{aligned}$$

with G-action on the source. To compute the colimit $\mathrm{colim}_{\mathcal{I}}(F^{\mathcal{I}}_{M}(*))$ as we said above, we first forget the G-action on the values $F^{\mathcal{I}}_{M}(*)(\mathbf{n})$ for every $\mathbf{n} \in \mathrm{ob}(\mathcal{I})$, compute the colimit, and then give the G-action. By forgetting the G-action on the finite set M, the free functor $F^{\mathcal{I}}_{M}(*)$ is naturally isomorphic to the functor $F^{\mathcal{I}}_{\mathbf{m}}(*)$, after choosing a bijection $M \xrightarrow{\cong} \mathbf{m}$. Now, the diagram $F^{\mathcal{I}}_{\mathbf{m}}(*)$ is cofibrant in the projective model structure, hence the natural comparison map, $\mathrm{hocolim}_{\mathcal{I}} F^{\mathcal{I}}_{\mathbf{m}}(*) \longrightarrow \mathrm{colim}_{\mathcal{I}} F^{\mathcal{I}}_{\mathbf{m}}(*)$ is a weak equivalence. The homotopy colimit of the diagram $F^{\mathcal{I}}_{\mathbf{m}}(*)$ may be identified with the classifying space $B(\mathbf{m} \downarrow \mathcal{I})$ hence it is contractible. This implies immediately that the underlying space of $\mathrm{colim}_{\mathcal{I}} F^{\mathcal{I}}_{\mathbf{m}}(*)$ is also contractible, hence any action on it induced by the diagram will be trivial. Now, let L be a G-space and consider the free functor

$$\begin{aligned} F^{\mathcal{I}}_{M}(L)(-) : \mathcal{I} &\longrightarrow G\mathcal{U} \\ \mathbf{n} &\longmapsto \mathrm{Inj}(M, \mathbf{n}) \times L. \end{aligned} \tag{9.4}$$

Consider the following adjunctions (left adjoints on the top)

$$G\mathcal{U} \underset{\mathrm{Ev}^{\mathcal{I}}_{M}}{\overset{F^{\mathcal{I}}_{M}}{\rightleftarrows}} G\mathcal{U}^{\mathcal{I}} \underset{\mathrm{const}_{\mathcal{I}}}{\overset{\mathrm{colim}_{\mathcal{I}}}{\rightleftarrows}} G\mathcal{U} \tag{9.5}$$

It follows that $\mathrm{colim}_{\mathcal{I}} \circ F^{\mathcal{I}}_{M}$ is left adjoint to $\mathrm{Ev}^{\mathcal{I}}_{M} \circ \mathrm{const}_{\mathcal{I}}$. By the proof of Lemma 9.3, we have that the composite functor $\mathrm{Ev}^{\mathcal{I}} \circ \mathrm{const}_{\mathcal{I}} : G\mathcal{U} \longrightarrow G\mathcal{U}$ is the identity functor $\mathrm{Id} : G\mathcal{U} \longrightarrow G\mathcal{U}$. So, we have the functor $\mathrm{colim}_{\mathcal{I}} \circ F^{\mathcal{I}}_{M}$ is left adjoint to Id. But the identity functor is also a left adjoint to itself, hence from [19, Corollary 1, Chapter IV], we have that $\mathrm{colim}_{\mathcal{I}} \circ F^{\mathcal{I}}_{M}$ is naturally isomorphic to the identity. Hence for the G-$\mathcal{I}$-space (9.4) we have $\mathrm{colim}_{\mathcal{I}} \circ F^{\mathcal{I}}_{M}(L) \cong L$.

In similar way to the above given a subgroup $H \le G$, and a finite H-set $M \subseteq_H U$ of cardinality m and consider the functor

$$\begin{aligned} G \ltimes_H F_M^{\mathcal{I}}(*) : \mathcal{I} &\longrightarrow G\mathcal{U} \\ \mathbf{n} &\longmapsto G \ltimes_H \operatorname{Inj}(M, \mathbf{n}) \end{aligned}$$

Then we have the following isomorphism since colimits commute

$$\operatorname*{colim}_{\mathcal{I}} \left(G \ltimes_H F_M^{\mathcal{I}}(*)\right) \cong G \ltimes_H \left(F_M^{\mathcal{I}}(*)\right).$$

By the above discussion, $\operatorname{colim}_{\mathcal{I}}(F_M^{\mathcal{I}}(*))$ is a point with trivial H-action. Hence, the induction $G \ltimes_H * \cong G/H$ with G action on the set G/H by translations. Finally, like above if L is a space with an H-action, we have that $\operatorname{colim}_{\mathcal{I}} \left(G \ltimes_H F_M^{\mathcal{I}}(L)\right) \cong G \ltimes_H L$

After this remark we can continue to compare our localized model structure. For the next proposition, we consider the category $G\mathcal{U}$ equipped with the fine (or genuine) model structure.

Proposition 9.6. *The adjunctions*

$$\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc} \rightleftarrows G\mathcal{U} : \operatorname{const}_{\mathcal{I}} \tag{9.6}$$

and

$$\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc} \rightleftarrows G\mathcal{U} : \operatorname{const}_{\mathcal{I}} \tag{9.7}$$

are Quillen adjunctions.

Proof. We prove first the claim that the adjunction (9.6) is a Quillen adjunction. In order to prove this, we first prove that the adjunction $\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{level} \rightleftarrows G\mathcal{U} : \operatorname{const}_{\mathcal{I}}$ is a Quillen adjunction. Recall from defining equations (7.6) and (7.7) the set of generating cofibrations and the set of generating acyclic cofibrations of $G\mathcal{U}^{\mathcal{I}}_{level}$, namely, I^{level} and J^{level}, respectively. Since the functor $\operatorname{colim}_{\mathcal{I}}$ is left adjoint is suffices to show that $\operatorname{colim}_{\mathcal{I}}$ sends the generating cofibrations to cofibrations in $G\mathcal{U}$, and that it sends the generating acyclic cofibrations to acyclic cofibration in $G\mathcal{U}$.

Let $f : F_M^{\mathcal{I}}(G/H \times \partial D^k) \longrightarrow F_M^{\mathcal{I}}(G/H \times D^k)$ be a map in I^{level} for some finite G-set $M \subseteq U$ of cardinality m, for some $H \le G$ and some $k \ge 0$. Consider the induced map of colimits

$$\operatorname*{colim}_{\mathcal{I}} \left(F_M^{\mathcal{I}}(G/H \times \partial D^k)\right) \longrightarrow \operatorname*{colim}_{\mathcal{I}} \left(F_M^{\mathcal{I}}(G/H \times D^k)\right) \tag{9.8}$$

By the discussion in Remark 9.5, the colimit $\operatorname{colim}_{\mathcal{I}} F_M^{\mathcal{I}}(G/H \times \partial D^k) \cong G/H \times \partial D^k$ with G-acting on G/H by translations. Applying the same for $\operatorname{colim}_{\mathcal{I}} \left(F_M^{\mathcal{I}}(G/H \times D^k)\right)$ we can conlude that the induced map of colimits (9.8) is the inclusion $G/H \times \partial D^k \xrightarrow{\mathrm{id} \times i_k} G/H \times D^k$ which is a cofibration in $G\mathcal{U}$. Following the same reasoning we can conclude immediately that $\operatorname{colim}_{\mathcal{I}}$ sends a generating acyclic cofibration to an acyclic cofibration in $G\mathcal{U}$.

We next prove that $\operatorname{colim}_{\mathcal{I}}$ is also left Quillen, considered now as a functor from $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc}$ to $G\mathcal{U}$. Since the model structure $G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc}$ is defined as a left Bousfield localization by [15, Proposition 3.3.18], it suffices to show that $\operatorname{colim}_{\mathcal{I}}$ sends the maps in the set $\mathcal{B}$ to weak equivalences in $G\mathcal{U}$. Recall, the set of maps $\mathcal{B}$ (8.8), which we localized the (topological) level model structure. Consider a map in $\mathcal{B}$, that is $f : F_N^{\mathcal{I}}(G/H) \longrightarrow F_M^{\mathcal{I}}(G/H)$ for some finite G-sets $M, N \subseteq U$, of cardinality m and n, respectively and $H \le G$ and some injection $M \longrightarrow N$. By the discussion in Remark 9.5 we have the G-isomorphisms $\operatorname{colim}_{\mathcal{I}} F_N^{\mathcal{I}}(G/H) \cong G/H$ and $\operatorname{colim}_{\mathcal{I}} F_M^{\mathcal{I}}(G/H) \cong G/H$ and it follows immediately that $\operatorname{colim} f$ is a weak equivalence in $G\mathcal{U}$. Hence, indeed we have the Quillen adjunction $\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} \rightleftarrows G\mathcal{U} : \operatorname{const}_{\mathcal{I}}$.

We move on to show that the adjunction (9.7) is a Quillen adjunction which follows the same reasoning. We prove first that $\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{strong} \longrightarrow G\mathcal{U}$ is left Quillen functor. So, consider a map f in I^{strong}, that is, $f : G \ltimes_H F_M^{\mathcal{I}}(H/K \ltimes \partial D^k) \longrightarrow G \ltimes_H F_M^{\mathcal{I}}(H/K \times \partial D^k)$, for some $H \le G$, a finite H-set $M \subseteq U$ of cardinality m, a subgroup $K \le H$ and some $k \ge 0$. By the discussion of Remark 9.5 we have the following G-isomorphisms

$$\begin{aligned} \operatorname*{colim}_{\mathcal{I}} G \ltimes_H \left(F_M^{\mathcal{I}}(H/K \times \partial D^k)\right) &\cong G \ltimes_H \left(\operatorname*{colim}_{\mathcal{I}} F_M^{\mathcal{I}}(H/K \times \partial D^k)\right) \\ &\cong G \ltimes_H \left(H/K \times \partial D^k\right) \\ &\cong G/K \times \partial D^k. \end{aligned}$$

Similarly we have $\operatorname{colim}_{\mathcal{I}} \left(G \ltimes_H F_M^{\mathcal{I}}(H/K \times D^k\right) \cong G/K \times D^k$. Hence the induced map of colimits , $\operatorname{colim} f$ is indeed a cofibration in $G\mathcal{U}$. Following the same reasoning if f is a generating acyclic cofibration, we get that $\operatorname{colim} f$ is an acyclic cofibration in $G\mathcal{U}$.

Finally, we show that the functor $\operatorname{colim}_{\mathcal{I}}$ is left Quillen when considered as a functor $\operatorname{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{storg\text{-}loc.} \longrightarrow G\mathcal{U}$. Since $G\mathcal{U}^{\mathcal{I}}_{storg\text{-}loc.}$ is defined as a left Bousfield localization, it suffices to show that $\operatorname{colim}_{\mathcal{I}}$ sends every map in the set $\mathcal{C}$ (8.9), to a weak equivalence in $G\mathcal{U}$. Let f be a map in the set $\mathcal{C}$, that is a map $G \ltimes_H F_N^{\mathcal{I}}(H/K) \longrightarrow G \ltimes_H F_M^{\mathcal{I}}(H/K)$ for a subgroup $H \le G$, for M, N finite H-sets in U and an injection $M \longrightarrow N$. By Remark

9.5 we have the G-isomorphisms $\mathrm{colim}_{\mathcal{I}}\left(G \ltimes_H F_N^{\mathcal{I}}(H/K)\right) \cong G \ltimes_H \left(\mathrm{colim}_{\mathcal{I}} F_N^{\mathcal{I}}(H/K)\right) \cong G \ltimes_H H/K \cong G/K$ and similarly $\mathrm{colim}_{\mathcal{I}}\left(G \ltimes_H F_M^{\mathcal{I}}(H/K)\right) \cong G/K$. Hence the functor $\mathrm{colim}_{\mathcal{I}}$ sends the maps in the set $\mathcal{C}$ to weak equivalences in $G\mathcal{U}$. The proof is complete. □

Discussion 9.7. We summarize our results so far from Proposition 9.6 and Proposition 9.4 and we explain their implications. We consider the following composition

$$G\mathcal{U} \xrightarrow{F_{\mathbf{0}}^{\mathcal{I}}} G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}} \xrightarrow{\mathrm{colim}_{\mathcal{I}}} G\mathcal{U}$$

and we have $\mathrm{colim}_{\mathcal{I}} \circ F_{\mathbf{0}}^{\mathcal{I}} = \mathrm{Id}$. The identity functor Id is of course a Quillen equivalence and by the 2-out-of-3 property of Quillen equivalences [16, Corollary 1.3.15], we can conclude that also the functor $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}} \longrightarrow G\mathcal{U}$ is a left Quillen equivalence. Following the same reasoning, we can also conclude that $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}} \longrightarrow G\mathcal{U}$ is a left Quillen equivalence. Consider now the following two commutative diagram of model categories

(9.9)
$$\begin{array}{ccc} & G\mathcal{U} & \\ {\scriptstyle \mathrm{colim}_{\mathcal{I}}} \nearrow & & \nwarrow {\scriptstyle \mathrm{colim}_{\mathcal{I}}} \\ G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} & \xrightarrow[\mathrm{Id}]{} & G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}} \end{array}$$

(9.10)
$$\begin{array}{ccc} & G\mathcal{U} & \\ {\scriptstyle \mathrm{colim}_{\mathcal{I}}} \nearrow & & \nwarrow {\scriptstyle \mathrm{colim}_{\mathcal{I}}} \\ G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} & \xrightarrow[\mathrm{Id}]{} & G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}} \end{array}$$

We know that the functors $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}} \longrightarrow G\mathcal{U}$ and $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}} \longrightarrow G\mathcal{U}$ are left Quillen equivalences. The functor $\mathrm{colim}_{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc}} \longrightarrow G\mathcal{U}$ is a Quillen equivalence from Corollary 8.12, which was provided by applying D.Dugger's result on hocolim-model structures. By Corollary 9.2, we have that the identity functors $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}}$ and $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}}$ are left Quillen functors. By applying again the 2-out-of-3 property of Quillen equivalences, we can conclude that $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}}$ and $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{\textit{proj-loc.}} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}}$ are left Quillen equivalences. This implies that also $\mathrm{Id} : G\mathcal{U}^{\mathcal{I}}_{\textit{level-loc.}} \longrightarrow G\mathcal{U}^{\mathcal{I}}_{\textit{strong-loc.}}$ is a left Quillen equivalence.

Part 4. **G-$\mathcal{I}$-spaces and G-symmetric spectra**

In this part we begin to develop the relationship between the homotopy theory of G-$\mathcal{I}$-spaces and the homotopy theory of G-symmetric spectra.

9.8. **Preliminaries.** We briefly recall the adjunction

$$\mathbb{S}^{\mathcal{I}}[-]:\mathcal{U}^{\mathcal{I}} \rightleftarrows \mathrm{Sp}^{\Sigma}:\Omega^{\mathcal{I}} \tag{9.11}$$

which level-wise, the left adjoint functor takes an $\mathcal{I}$-space to the symmetric spectrum $\mathbb{S}^{\mathcal{I}}[X]_n := S^n \wedge X(\mathbf{n})_+$, and the right adjoint $\Omega^{\mathcal{I}}:\mathrm{Sp}^{\Sigma} \longrightarrow \mathcal{U}^{\mathcal{I}}$, $E \longmapsto \Omega^n E_n$. The categories of G-$\mathcal{I}$-spaces and G-symmetric spectra are defined as the category of functors $G \longrightarrow \mathcal{U}^{\mathcal{I}}$ and $G \longrightarrow \mathrm{Sp}^{\Sigma}$, respectively. So, this adjunction prolongates to the adjunction

$$\mathbb{S}^{\mathcal{I}}[-]:G\mathcal{U}^{\mathcal{I}} \rightleftarrows G\mathrm{Sp}^{\Sigma}:\Omega^{\mathcal{I}} \tag{9.12}$$

which for a G-$\mathcal{I}$-space X we have $\mathbb{S}^{\mathcal{I}}[X]_n = S^n \wedge X(\mathbf{n})_+$ with diagonal Σ_n-action and G-action through $X(\mathbf{n})_+$. Similarly for a G-symmetric spectrum Y, the G-$\mathcal{I}$-space $\Omega^{\mathcal{I}}(Y) = \Omega^n Y_n$ with Σ_n-action by conjugation and G-action though Y_n.

For the next lemma recall, 3.5, [12, Definition 2.18] of a *free G-symmetric spectrum on A in level M* denoted by $\mathcal{F}_M A$. We have the following analogue of [26, Lemma 14.3]. We will use this in Subsection 9.12.

Lemma 9.9. *Let M be a finite G-set and let L be a G-space. Furthermore, let H be a subgroup of G, let N be a finite H-set and K an H-space. We have the following natural isomorphisms*

$$\mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(L)] \cong \mathcal{F}_M S^M \wedge L \qquad \textit{and} \qquad \mathbb{S}^{\mathcal{I}}[G \ltimes_H F^{\mathcal{I}}_N(K)] \cong G \ltimes_H \left(\mathcal{F}^{(H)}_N S^N \wedge K\right)$$

Proof. Let Y be a G-symmetric spectrum. We have the following equivalences

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}^{\Sigma}}\left(\mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(L)], Y\right) &\cong \mathrm{map}\left(L, \mathrm{Ev}^{\mathcal{I}}_M\, \Omega^{\mathcal{I}}(Y)\right)\\ &\cong \mathrm{map}\left(L, \Omega^M Y(M)\right)\\ &\cong \mathrm{map}\left(L, \mathrm{Map}_{\mathrm{Sp}^{\Sigma}}(\mathcal{F}_M S^M, Y)\right)\\ &\cong \mathrm{Map}_{\mathrm{Sp}^{\Sigma}}\left(\mathcal{F}_M S^M \wedge L, Y\right)\end{aligned}$$

The second isomorphism is a consequence of the first isomorphism and the fact that $\mathbb{S}^{\mathcal{I}}[-]$ preserves colimits as a left adjoint. □

If $\alpha : M \longrightarrow N$ is an injection and L is a G-space we have a natural transformation $\alpha^* \times L : F^{\mathcal{I}}_N(L) \longrightarrow F^{\mathcal{I}}_M(L)$ and applying the functor $\mathbb{S}^{\mathcal{I}}[-]$ we have the map

$$\mathbb{S}^{\mathcal{I}}[\alpha^*] \wedge L : \mathcal{F}_N S^N \wedge L \longrightarrow \mathcal{F}_M S^M \wedge L$$

The map $\mathbb{S}^{\mathcal{I}}[\alpha^*] : \mathcal{F}_N S^N \longrightarrow \mathcal{F}_M S^M$ is the adjoint of the following map

$$S^N \xrightarrow{(\mathrm{id},\alpha)} S^N \wedge \mathrm{Inj}(M,N)_+ \cong S^M \wedge \mathbf{\Sigma}(M,N) = \mathcal{F}_M S^M(N)$$

where the G-isomorphism (with diagonal G-actions) $\mathrm{Inj}(M,N)_+ \wedge S^N \cong S^M \wedge \mathbf{\Sigma}(M,N)$ is given in [12, Example 3.35].

9.10. **Quillen adjunction of Level Model Structures.**

Proposition 9.11. *The adjunction*

$$\mathbb{S}^{\mathcal{I}}[-]:G\mathcal{U}^{\mathcal{I}} \rightleftarrows G\mathrm{Sp}^{\Sigma}:\Omega^{\mathcal{I}}$$

is Quillen adjunction for the respective level and strong level model structures.

Proof. We will prove firstly the proposition for the case of the respective level model structures and we will do that by showing that $\mathbb{S}^{\mathcal{I}}[-]$ is left Quillen. Since it is left adjoint, it suffices to show that it sends generating cofibrations to cofibrations and generating acyclic cofibrations to acyclic cofibrations. Recall from defining equation (7.6) that a generating cofibration in the level model structure is of the form

$$F^{\mathcal{I}}_M(i_G) : F^{\mathcal{I}}_M(G/H \times \partial D^k) \longrightarrow F^{\mathcal{I}}_M(G/H \times D^k)$$

for $i_G : G/H \times \partial D^k \longrightarrow G/H \times D^k$ a generating cofibration in $G\mathcal{U}$. So, we need to show that

$$\mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(i_G)] : \mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(G/H \times \partial D^k)] \longrightarrow \mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(G/H \times D^k)]$$

is a cofibration in $G\mathrm{Sp}^{\Sigma}_{\mathrm{level}}$. By Lemma 9.9 we have an isomorphism $\mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(i_G)] \cong \mathcal{F}_M S^M \wedge i_G$. The G-symmetric spectrum $\mathcal{F}_M S^M$ is cofibrant in $G\mathrm{Sp}^{\Sigma}_{level}$ and since it is G-topological model structure we have that $\mathcal{F}_M S^M \wedge i_G$ is a cofibration in $G\mathrm{Sp}^{\Sigma}_{level}$, by [16, Remark 4.2.3]. Similarly if j_G is an acyclic cofibration in $G\mathcal{U}$ and $F^{\mathcal{I}}_M(j_G)$ we have the isomorphism $\mathbb{S}^{\mathcal{I}}[F^{\mathcal{I}}_M(j_G)] \cong \mathcal{F}_M S^N \wedge j_G$. Following the same reasoning as in the case of cofibrations we get that $\mathcal{F}_M S^N \wedge j_G$ is acyclic cofibration in $G\mathrm{Sp}^{\Sigma}_{level}$.

We proceed now to prove the proposition for the respective strong level model structures. Again, we will do that by showing that the functor $\mathbb{S}^{\mathcal{I}}[-]$ is left Quillen. Consider a subgroup H of G and a generating cofibration of the $H\mathcal{U}$ genuine model structure, that is, a map of the form

$$i_H : H/K \times \partial D^k \longrightarrow H/K \times D^k$$

for a subgroup $K \leq H$ and some $k \in \mathbb{N}$ and let the map

$$G \ltimes_H \left(F_M^{\mathcal{I}}(i_H)\right) : G \ltimes_H F_M^{\mathcal{I}}(H/K \times \partial D^k) \longrightarrow G \ltimes_H F_M^{\mathcal{I}}(H/K \times D^k)$$

for a finite H-set M, which is a generating cofibration of the model structure $G\mathcal{U}^{\mathcal{I}}_{strong}$.

By Lemma 9.9 we have the following isomorphism

$$\mathbb{S}^{\mathcal{I}}[G \ltimes_H \left(F_M^{\mathcal{I}}(i_H)\right)] \cong G \ltimes_H \left(\mathcal{F}_M^{(H)} S^M \wedge i_H\right)$$

For a subgroup $H \leq G$, the level model structure, $H\mathrm{Sp}^{\Sigma}_{level}$ is an H-topological model structure, so $\mathcal{F}_M^{(H)} S^M \wedge i_H$ is a cofibration in $H\mathrm{Sp}^{\Sigma}_{\mathrm{level}}$, by [16, Remark 4.2.3]. For the last step, by construction of the strong level model structure of G-symmetric spectra and observing the set of generating cofibrations [12, Equation 2.3, 2.4], the induction functor

$$G \ltimes_H - : H\mathrm{Sp}^{\Sigma}_{level} \longrightarrow G\mathrm{Sp}^{\Sigma}_{strong}$$

is left Quillen for every $H \leq G$, hence the induction $G \ltimes_H \left(\mathcal{F}_M^{(H)} S^M \wedge i_H\right)$ is a cofibration in $G\mathrm{Sp}^{\Sigma}_{strong}$. The argument that the functor $\mathbb{S}^{\mathcal{I}}[-]$ sends generating acyclic cofibrations to acyclic cofibrations follows the same reasoning as above and therefore we do not write it. Thus the proof is concluded.

□

9.12. **Quillen Adjunction and Localization.** In this subsection we prove, that the functor $\mathbb{S}^{\mathcal{I}}$ can be regarded as left Quillen functor from the localizations the we previously constructed on Section 8. To prove this, we will appeal to [15, Proposition 3.3.18], which we state below.

Proposition 9.13. *Let $\mathcal{M}$ be a model category and let S be a set of maps in $\mathcal{M}$. If $\mathrm{L}_S\mathcal{M}$ is the left Bousfield localization of $\mathcal{M}$ with respect to S, $\mathcal{N}$ is a model category, and $F : \mathcal{M} \longrightarrow \mathcal{N}$ is a left Quillen functor that takes every cofibrant approximation to an element of S into a weak equivalence in $\mathcal{N}$, then F is a left Quillen functor when considered as a functor $\mathrm{L}_S\mathcal{M} \longrightarrow \mathcal{N}$.*

Lets denote by $G\mathrm{Sp}^{\Sigma}_{stable,level}$ the G-stable localization of $G\mathrm{Sp}^{\Sigma}_{level}$ Consider the following diagram :

$$\begin{array}{ccccc} G\mathcal{U}^{\mathcal{I}}_{level} & \underset{\Omega^{\mathcal{I}}}{\overset{\mathbb{S}^{\mathcal{I}}[-]}{\rightleftarrows}} & G\mathrm{Sp}^{\Sigma}_{level} & \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} & G\mathrm{Sp}^{\Sigma}_{stable,level} \\ \downarrow{\scriptstyle\gamma} & & & \nearrow & \\ G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} & & & & \end{array}$$

Since $\mathrm{Id} : G\mathrm{Sp}^{\Sigma}_{level} \longrightarrow G\mathrm{Sp}^{\Sigma}_{stable,level}$ is left Quillen, by Proposition 9.11 the composition $G\mathcal{U}^{\mathcal{I}}_{level} \xrightarrow{\mathrm{Id} \circ \mathbb{S}^{\mathcal{I}}[-]} G\mathrm{Sp}^{\Sigma}_{stable,level}$ is also left Quillen. If we show the functor $\mathbb{S}^{\mathcal{I}}[-]$ takes every cofibrant approximation to an element of $\mathcal{B}$ into a G-stable equivalence of G-symmetric spectra then by the above proposition, 9.13, we will be done.

Consider an element of $\mathcal{B}$, that is, a map of the form $\alpha^* \times G/H : F_N^{\mathcal{I}}(G/H) \longrightarrow F_M^{\mathcal{I}}(G/H)$ for some M and N finite G-sets, an injection $\alpha : M \longrightarrow N$ and a subgroup $H \leq G$. By inspection of the set of generating cofibrations for the level model structure from defining equation (7.6) , we note that all objects in the set $\mathcal{B}$ are cofibrant, so we do not need to take cofibrant approximation. So, we need to show that the following map is a G-stable equivalence

$$\mathbb{S}^{\mathcal{I}}[\alpha^* \times G/H] : \mathbb{S}^{\mathcal{I}}[F_N^{\mathcal{I}}(G/H)] \longrightarrow \mathbb{S}^{\mathcal{I}}[F_M^{\mathcal{I}}(G/H)].$$

By Lemma 9.9 and the discussion below it, this is the map

$$\mathbb{S}^{\mathcal{I}}[\alpha^*] \wedge G/H : \mathcal{F}_N S^N \wedge G/H \longrightarrow \mathcal{F}_M S^M \wedge G/H$$

By [12, Proposition 4.1, (ii)] smashing with a cofibrant G-space preserves G-stable equivalences and since G/H is cofibrant G-space, it suffices to show that $\mathbb{S}^{\mathcal{I}}[\alpha^*] : \mathcal{F}_N S^N \longrightarrow \mathcal{F}_M S^M$ is a G-stable equivalence. Consider the injection $M \longrightarrow N$ as an inclusion $M \subseteq N$ of finite G-sets. We have the following map

$$\lambda_{M,N-M} : \mathcal{F}_{M \sqcup (N-M)} S^{N-M} = \mathcal{F}_N S^{N-M} \longrightarrow \mathcal{F}_M S^0$$

which by [12, Example 2.46] is a G-stable equivalence. Since the representation sphere S^M is cofibrant G-space smashing it with the map $\lambda_{M,N-M}$ is also a G-stable equivalence. The resulting map after smashing is

$$\mathbb{S}^{\mathcal{I}}[\alpha^*] = \lambda_{M,N-M} \wedge S^M : \mathcal{F}_N S^N \longrightarrow \mathcal{F}_M S^M$$

So, $\mathbb{S}^{\mathcal{I}}[\alpha^*] : \mathcal{F}_N S^N \longrightarrow \mathcal{F}_M S^M$ is a G-stable equivalence and we have proven the claim.

Consider now an element of $\mathcal{C}$, that is, choose finite H-sets M and N, an injection $\beta : M \longrightarrow N$ and a subgroup $K \leq H$. So, we have the following map

$$G \ltimes_H (\beta^* \times H/K) : G \ltimes_H F^{\mathcal{I}}_N(H/K) \longrightarrow G \ltimes_H F^{\mathcal{I}}_M(H/K).$$

After applying the functor $\mathbb{S}^{\mathcal{I}}[-]$, and by Lemma 9.9 we have the following isomorphism

$$\begin{aligned}\mathbb{S}^{\mathcal{I}}[G \ltimes_H F^{\mathcal{I}}_M(H/K)] &\cong G \ltimes_H \left(\mathcal{F}^{(H)}_M S^M \wedge H/K\right) \\ &\cong G \ltimes_H \mathcal{F}^{(H)}_M S^M \wedge G/K\end{aligned}$$

and so we have the following isomorphism

$$\mathbb{S}^{\mathcal{I}}[G \ltimes_H (\beta^* \times H/K)] \cong G \ltimes_H \mathbb{S}^{\mathcal{I}}[\beta^*] \wedge G/K$$

Since G/K is a cofibrant G-space, it clearly suffices to show that $G \ltimes_H \mathbb{S}^{\mathcal{I}}[\beta^*]$ is a G-stable equivalence. Consider the injection $\beta : M \longrightarrow N$ as an inclusion of finite H-sets. We have the map

$$\lambda^{(H)}_{M,N-M} : \mathcal{F}^{(H)}_N S^{N-M} \longrightarrow \mathcal{F}^{(H)}_M S^0$$

and smashing this map with the representation sphere S^M we have the map

$$\mathbb{S}^{\mathcal{I}}[\beta^*] = \lambda^{(H)}_{M,N-M} \wedge S^M : \mathcal{F}^{(H)}_N S^N \longrightarrow \mathcal{F}^{(H)}_M S^M.$$

By [12, Example 2.46], it follows that the induction

$$G \ltimes_H \left(\lambda^{(H)}_{M,N-M} \wedge S^M\right) = G \ltimes_H \mathbb{S}^{\mathcal{I}}[\beta^*]$$

is a G-stable equivalence.

Thus, by the above discussion and applying Proposition 9.13 we have shown the following proposition

Proposition 9.14. *We have the following Quillen adjunctions*

$$\mathbb{S}^{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{level\text{-}loc.} \rightleftarrows G\mathrm{Sp}^{\Sigma}_{stable,level} : \Omega^{\mathcal{I}} \tag{9.13}$$

and

$$\mathbb{S}^{\mathcal{I}} : G\mathcal{U}^{\mathcal{I}}_{strong\text{-}loc.} \rightleftarrows G\mathrm{Sp}^{\Sigma}_{stable,strong} : \Omega^{\mathcal{I}} \tag{9.14}$$

Part 5. An example

Let R be a ring with unit and consider $\mathrm{GL}_n(R)$, the *general linear group of degree* n, that is, the group of invertible $n \times n$ matrices with entries from the ring R. Identifying each $n \times n$ matrix A

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R)$$

gives an embedding $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$. The union of the resulting sequence

$$\mathrm{GL}_1(R) \subset \mathrm{GL}_2(R) \subset \dots \mathrm{GL}_n(R) \subset \mathrm{GL}_{n+1}(R) \subset \dots$$

is called the *infinite general linear group* $\mathrm{GL}(R)$. For every morphism $\alpha : \mathbf{m} \longrightarrow \mathbf{m}$ in $\mathcal{I}$ there is an induced map $\alpha_* : \mathrm{GL}_m(R) \longrightarrow \mathrm{GL}_n(R)$ which can be made functorial. This defines a functor

$$\mathrm{GL}(R) : \mathcal{I} \longrightarrow \mathrm{Grp}\,.$$

By taking the classifying space for each $\mathrm{GL}_n(R)$, that is, the geometric realization of the nerve we have the $\mathcal{I}$-space

$$B\mathrm{GL}(R) : \mathcal{I} \longrightarrow G\mathcal{U}, \quad \mathbf{n} \longmapsto B\mathrm{GL}_n(R)$$

Consider now a ring R with unit and a finite group G that acts on R by ring automorphisms. The action of G on R induces a G-action on $\mathrm{GL}_n(R)$ for every $n \in \mathbb{N}$ and the embedding $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ is equivariant. So we have the G-$\mathcal{I}$-space defined by

$$B\mathrm{GL}(R) : \mathcal{I} \longrightarrow G\mathcal{U} \quad \mathbf{n} \longmapsto B\mathrm{GL}_n(R).$$

Recall that given a G-$\mathcal{I}$-space X, and given a subgroup $H \leq G$, we can define the $\mathcal{I}$-space

$$X^H : \mathcal{I} \longrightarrow \mathcal{U}, \quad \mathbf{n} \longmapsto X(\mathbf{n})^H$$

From Subsection 8.11 we know that given a diagram $X : \mathcal{I} \longrightarrow G\mathcal{U}$, then its homotopy colimit $\mathrm{hocolim}_{\mathcal{I}} X := X_{h\mathcal{I}}$ has an induced G-action.

An important property of the classifying space functor is that if category $\mathcal{C}$ comes with a G-action then taking fixed points commutes with the classifying space functor. This is a consequence that the classifying space functor commutes with finite limits.

As an example consider $\mathbb{C}$ and $\mathbb{R}$ as discrete rings(fields) and let $\mathbb{Z}/2$, the cyclic group of order 2 act by conjugation on $\mathbb{C}$, that is, $z \mapsto \bar{z}$. By the above we have $\mathbb{Z}/2$-$\mathcal{I}$-space

$$B\mathrm{GL}(\mathbb{C}) : \mathcal{I} \longrightarrow \mathbb{Z}/2\mathcal{U} \quad \mathbf{n} \longmapsto B\mathrm{GL}_n(\mathbb{C})$$

and by taking objectwise fixed points we have the $\mathcal{I}$-space defined by

$$B\mathrm{GL}(\mathbb{C})^{\mathbb{Z}/2} : \mathcal{I} \longrightarrow \mathcal{U}, \quad \mathbf{n} \longmapsto B\mathrm{GL}_n(\mathbb{C})^{\mathbb{Z}/2}.$$

The $\mathbb{Z}/2$-fixed points of $\mathrm{GL}_n(\mathbb{C})$ is isomorphic to $\mathrm{GL}_n(\mathbb{R})$ (as groups), hence by taking realization we have the homeomorphism $B\mathrm{GL}_n(\mathbb{C})^{\mathbb{Z}/2} \cong B\mathrm{GL}_n(\mathbb{R})$. So, by taking the homotopy colimits we have $\left(B\mathrm{GL}(\mathbb{C})_{h\mathcal{I}}\right)^{\mathbb{Z}/2} = \left(B\mathrm{GL}(\mathbb{C})^{\mathbb{Z}/2}\right)_{h\mathcal{I}} = B\mathrm{GL}(\mathbb{R})_{h\mathcal{I}}$.

Another example that exhibits the same phenomenon, consider $\mathbb{F}_q$ the finite field with q elements and the extension $\mathbb{F}_q \longrightarrow \mathbb{F}_{q^n}$. Then $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z}$. So we have a $\mathbb{Z}/n$-$\mathcal{I}$-space defined by

$$\mathcal{I} :\longrightarrow \mathbb{Z}/n\mathcal{U}, \quad \mathbf{n} \longmapsto B\mathrm{GL}(\mathbb{F}_{q^n})$$

Following the reasoning above we have the homeomorphisms

$$\left(B\mathrm{GL}(\mathbb{F}_{q^n})_{h\mathcal{I}}\right)^{\mathbb{Z}/n} \cong \left(B\mathrm{GL}(\mathbb{F}_{q^n})^{\mathbb{Z}/n}\right)_{h\mathcal{I}} \cong B\mathrm{GL}(\mathbb{F}_q)$$

It is generally the case, that given a Galois extension L/K, with Galois group $G = \mathrm{Gal}(L/K)$, the fixed points $\mathrm{GL}_n(L)^G = \mathrm{GL}_n(K)$. Care has to be taken though since in general Galois groups are far from being finite groups, that is, they are profinite. This limits severely the above examples. We do not know, however, if the above consideration provides any new information about algebraic K-theory of rings (fields).

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