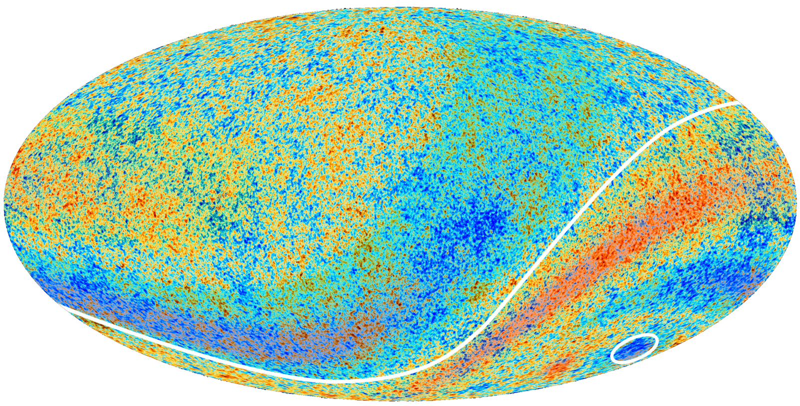
Utrecht University

Institute for Theoretical Physics

Master's Thesis

Vector Field in Cosmic Inflation: Construction and Analysis of Stable Models

CHÂU Lưu Đại-Thành



Supervisor: Assist. Prof. Dr. Enrico PAJER

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Abstract

We investigate the possibility of using a vector field to drive anisotropic cosmic inflation in contrast with the isotropic accelerated expansion driven by scalar field. The realization of such vector field models during inflation provides a promising link to the observed large scale Cosmic Microwave Background anomalies. Despite being a close upgraded step from the scalar field, the vector field models are troubled with instabilities due to the presence of ghost. In this work, we construct and analyze stable vector field models in simple form using ideas from chaotic inflation. We obtain stable evolution of slow-roll inflationary period driven by a vector field and present also different graceful exit scenarios.

To my family.

Acknowledgements

First and foremost, I would like to thank my supervisor, Dr. Enrico Pajer, for patiently giving me the chance to work on such an interesting topic. His guidance and dedication in addressing fundamental concepts throughout our weekly discussions have greatly influenced the accomplishment of this thesis. I must also acknowledge valuable contributions and friendship from my fellow students in Master's Programme at Utrecht University - Lǐ Chōngchuò, Enea Mauri, Leonardo Chataignier, Domingo Gallegos, Adriana Correia, Pedro Cal, Mohammad Waris, Andries Salm, Ward Vleeshouwers and Peter Cats. Finally, I would like to express my deep gratitude to my family for their unbounded love and uninterrupted support.

Units and Conventions

Unless stated otherwise, we use natural units:

$$c = \hbar = k_B \equiv 1\,,$$

and also set the reduced Planck mass $M_{\rm pl}$ to 1 for greater ease in computation:

$$M_{\rm pl} = (8\pi G)^{-1/2} \equiv 1\,.$$

We use the "mostly pluses" convention for metric signature $(-,+,+,+)$. Greek indices stand for the four spacetime coordinate labels $\mu, \nu = 0, 1, 2, 3$ while Latin indices stand for the three spatial coordinate labels $i, j = 1, 2, 3$. Partial derivatives are denoted by commas and covariant derivatives denoted by semi-colons.

We use following Fourier convention:

$$\mathcal{R}_{\mathbf{k}} = \int \mathrm{d}^3\mathbf{x}\,\mathcal{R}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}\,,$$

so that the power spectrum is:

$$\langle \mathcal{R}_{\mathbf{k}}\mathcal{R}_{\mathbf{k}'}\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_{\mathcal{R}}(k)\,, \qquad \Delta^2_{\mathcal{R}}(k) \equiv \frac{k^3}{2\pi^2}P_{\mathcal{R}}(k)\,.$$

Derivatives with respect to physical time are denoted by overdots, while derivatives with respect to conformal time are indicated by primes.

Contents

"The imagination of nature is far, far greater than the imagination of man."

Richard P. Feynman

1 Introduction

As early as the first emergence of mankind's civilization, people have unceasingly pondered, imagined and postulated ideas whenever they look for what is *"written in the stars."* Those mental adventures were in fact our cosmological theories in their utmost primitive forms. As fascinating as it is, the progress of knowledge advancement was extremely slow. Only after more than ten thousand years, we have recently come to realize that we Earthlings are *not* at the center of the Universe. Earth planet is just one of many planets orbiting the Sun. This milestone achievement is thanks to Nicolaus Copernicus and his seminal work *De revolutionibus orbium coelestium* (On the revolution of heavenly spheres), published in 1543. Over one hundred years passed by for Copernican heliocentrism to gain acceptance despite relentless suppression from ignorance.[1] Half a century later, in the mid-1680s, Sir Isaac Newton found that even our Sun is not at the center of Universe either because of its deviation from our solar system's center of gravity. Catching up with experimental evidences, on the theoretical side, Newton's law of universal gravitation and Cosmological Principle published in the *Principia* (1687) were established as the framework for cosmology by providing fantastic agreement with observational data.

The state of affair is maintained until the beginning of 20th century. This time, theoreti-

[1] This paradigm shift still cost us a hefty price with Giordano Bruno's *burned alive at the stake* incident in 1600 and Galileo Galilei's house arrest until his death for *vehemently suspect of heresy* in 1633.

cal side plays the pioneer role with the advent of General Relativity theory by Albert Einstein in 1915 [1]. On the observational side, the dominant cosmic view was that the Milky Way Galaxy represents the whole Universe. In 1920, we witnessed the *Great Debate* in astronomy concerned the nature of spiral nebulae and size of the Universe between astronomers Harlow Shapley and Heber Curtis. Shortly after, in 1924, Edwin Hubble successfully measured the distance to classical Cepheid variables in the Andromeda Galaxy and showed that these variables in fact do *not* belong to the Milky Way Galaxy. This settled the Great Debate with the more accurate side leaning toward Curtis. The Milky Way Galaxy is just one of many galaxies in the Universe! Furthermore, in 1929, Hubble demonstrated the approximately linear relationship between distances of the galaxies and their redshifts [2] based on measurements of galaxies' redshifts by Vesto Slipher and Milton Humason. Only after 5 years of our first realization that the Universe is actually much bigger than we previously thought, we received a significant paradigm shift since Copernicus: the Universe is also getting bigger over time! The theoretical side amazingly had a prediction beforehand. In 1927, by independently deriving Friedmann's equations, Georges Lemaître showed that an expanding Universe is a solution to Einstein's equations of General Relativity and the Hubble Law naturally follows. Modern physical cosmology era has unfolded.

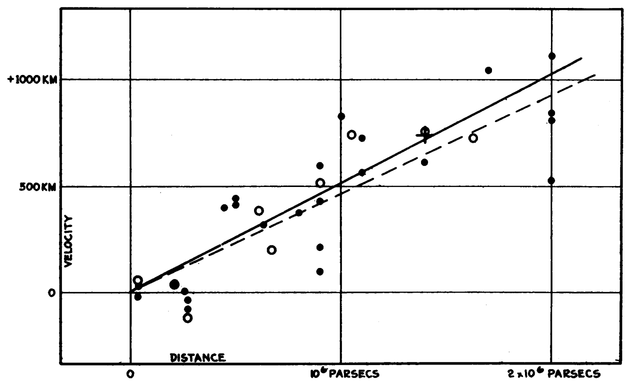

Figure 1.1: The original Hubble diagram [2]. Radial velocities (units should be km sec^{-1}), corrected for solar motion, are plotted against distances (units should be Mpc) estimated from involved stars and mean luminosities of nebulae in a cluster. The black discs and full line represent the solution for solar motion using the nebulae individually; the circles and broken line represent the solution combining the nebulae into groups; the cross represents mean velocity corresponding to the mean distance of 22 nebulae whose distances could not be estimated individually.

A question soon raised regarding an expanding Universe: "Does it have a beginning?"

There were two most popular theories that gained similar amount of supporters in the scientific community: 1) Fred Hoyle's Steady State theory asserts that an expanding Universe maintains its appearance over time by continually creating additional matter and therefore does not have a beginning; 2) Lemaître's Big Bang theory describes an expanding Universe began from a very high density and high temperature state. By a fortunate event in 1964, the Cosmic Microwave Background (CMB) was discovered by astronomers Arno Penzias and Robert Wilson [3, 4]. This event earned the discoverers the 1978 Nobel Prize in Physics and decisively tipped the balance in favor of the Big Bang model because the Steady State model is unable to generate a background spectrum *closer to a blackbody than any other source in nature* as measurements from the CMB shows [5].

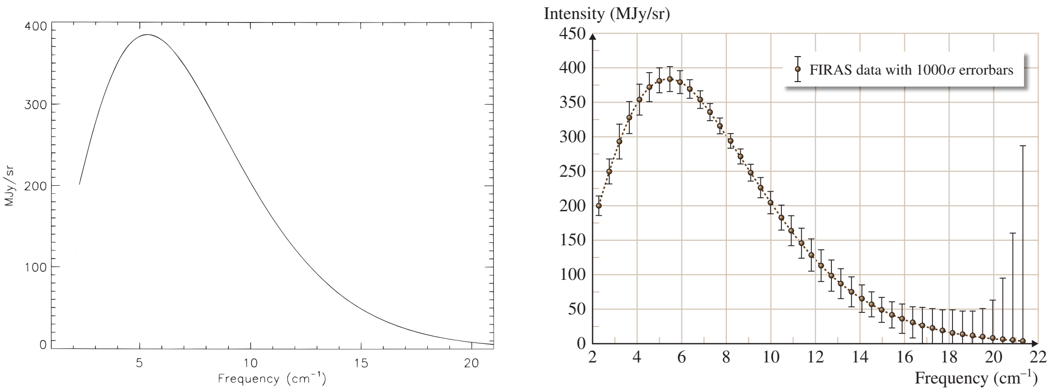

Figure 1.2: *Left:* Original plot of the CMB Spectrum from the full *COBE* FIRAS data set [6]. The FIRAS data match the curve so exactly, with *error uncertainties less than the width* of a blackbody curve. The solid curve shows the expected intensity from a single temperature blackbody spectrum as predicted by the hot Big Bang theory. The FIRAS data were taken at 43 positions equally spaced along this curve. *Right:* The error bars have been multiplied by a factor of thousand in the plot. The distribution is extremely well fit by a blackbody spectrum at a temperature of T_0 = 2.725 (±0.001) K – courtesy of Tuhin Ghosh (IUCAA).

Our currently accepted model for the Universe is the ΛCDM (Lambda Cold Dark Matter) Big Bang model, which claims that the Universe contains dark energy (associated with a cosmological constant, denoted by Λ) and cold dark matter. This model tells us the finite age of the currently *accelerated expanding* and *flat* Universe to be about 13.8 billion years [7]. Nowadays, any respectable theory which is capable of competing with the ΛCDM Big Bang model has to at least provide satisfactory explanations for *all* the following observations: i) Accelerating expansion, ii) Blackbody CMB and iii) Abundance of light elements. So far, there is *none*.

As successful as it is with vast amount of correct predictions, the ΛCDM Big Bang model still has thorny unresolved issues such as: i) Cosmological constant problem, ii) Nature of dark matter, iii) Validity of general relativity on large scales, iv) Existence of anomalies in the CMB and on small scales and v) Predictivity and testability of the inflationary paradigm [8]. The measurement in 2000 by BOOMERanG [9] and confirmed later in 2006 by three-year WMAP data [10] showed that our Universe is actually very close to be *flat* has revived significant interest in the Cosmic Inflationary paradigm, which was proposed earlier by Alan Guth in 1981 [11]. Although the primordial B-mode polarization – Cosmic Inflation's "smoking-gun" signature – has not been confirmed either by *Planck*-2015 experiment [7] or *BICEP2* and *Keck Array* collaborations [12], it is still *the best* shot among several other rival theories until today. This is largely because Cosmic Inflation in its generic form provides very good explanation for the following observations: i) Flat Universe, ii) Gaussian and adiabatic perturbation, iii) Nearly scale-invariant spectrum of density and iv) Specific peaks in the CMB's spectrum [13].

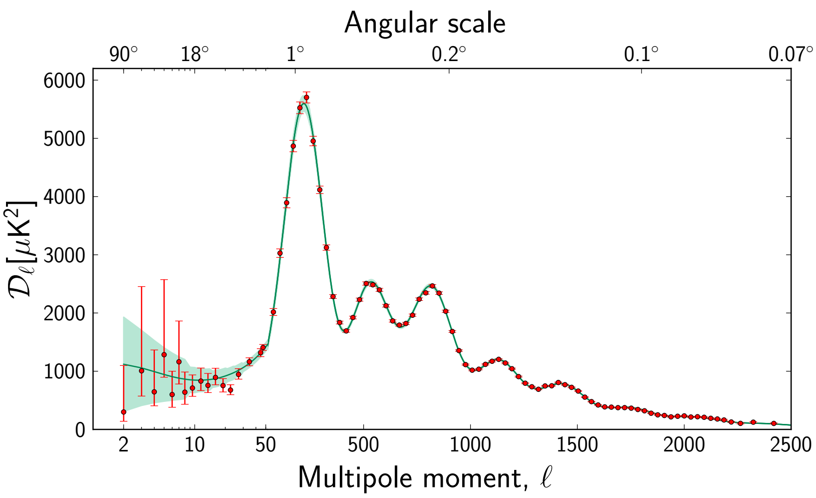

Figure 1.3: CMB data (*Planck* 2013) versus the predictions of one of the simplest inflationary models for flat Universe (full line) [14]. The error bars include cosmic variance, whose magnitude is indicated by the green shaded area around the best fit model. The low-ℓ values are plotted at 2, 3, 4, 5, 6, 7, 8, 9.5, 11.5, 13.5, 16, 19, 22.5, 27, 34.5 and 44.5.

Cosmic inflation in its simplest form involves only scalar field (the *inflaton*) to both driving inflation and generating primordial perturbations. Even at that level of simplicity, it has already given good predictions for above mentioned observations. Furthermore, there is one other noteworthy feature of scalar field inflation model: it produces both scalar and tensor perturbations but it does *not* produce vector perturbations. However, through high resolution data obtained from WMAP and confirmed by *Planck* experiment, we have recently noticed some unexplained features in the CMB. The most well-known among these features are: the low power in the quadrupole moment [15–19], the alignment of the lowest multipoles and claims of statistical anisotropy [19–25], an asymmetry in power between the northern and southern hemispheres [26–32] and a non-Gaussian deviation in the southern hemisphere, known as the "cold-spot" [33–38].

There are still disputes whether these anomalies are of any statistical significance. Nevertheless, the possibility that these unexplained features actually exist in the CMB has motivated theoretical cosmologists to extend Cosmic Inflation paradigm by including higher spin fields such as vector fields, spinors and p-forms. It seems that the most straightforward and simplest next step in making extension would be vector fields. However, this task is highly non-trivial. The appearance of *ghost* in many firstly proposed vector field models renders them unstable and therefore unviable [39–41]. Motivated by the flexibility of chaotic inflation, we explore in this thesis the possibility of using a vector field to drive anisotropic cosmic inflation. Our strategy in building a stable vector field model is to obtain firstly known behaviors of single scalar field in chaotic inflation and clear out subtleties such as initial conditions and positive Hubble parameter branch selection. The stability is ensured by constraining the mass term m^2 in the potential to be positive and real throughout the whole inflation period. Systematically, we obtain next a stable vector field model in simple form which provides the required amount of 60 e-folds slow-roll inflationary duration. We also observe that the inflation period does not end like the single scalar field case and instead goes on infinitely. Therefore, we further investigate different graceful exit scenarios by slightly modifying our original chaotic inflation potential. After we verify that inflation proceeds to exit as intended, we finish by performing stability analysis using different approaches.

This thesis is structured as follows. In chapter 2, we build up prerequisites by reviewing assumptions and basic equations starting from standard Big Bang cosmology until the formalism for calculation of spectrum perturbation in Cosmic Inflation. In chapter 3, we present the developments in vector field models for Cosmic Inflation and its constraints on stability by ghost fields. After that, we go into details about constructing simple ghost-free vector field models and performing stability analysis. Finally, in chapter 4, we conclude by discussing the implications in our findings and possible directions for further developments.[2]

[2] In the Appendices, we also provide the technicalities required for the conduct of our formalism.

"The Big Bang says nothing about what banged, why it banged, or what happened before it banged."

Alan H. Guth

2

Modern Precision Cosmology

More than twenty years ago, with data obtained by the Cosmic Background Explorer (COBE), we discovered *anisotropies* in the order of $\mathcal{O}(10^{-5})$ in the near-perfect blackbody spectrum CMB and therefore entered into the era of modern precision cosmology. In order to understand the origin of these anisotropies, we firstly review in this chapter the standard model of cosmology - the ΛCDM model - by presenting the equations which govern the evolution of the Universe. After that, we introduce the framework for the Cosmic Inflation paradigm which offers the *status quo* explanation to the formation of structures in the Universe via primordial fluctuations.[1]

2.1 ▪ THE STANDARD MODEL OF COSMOLOGY

The theoretical foundations of the ΛCDM model are: i) The Cosmological Principle and ii) Einstein's General Relativity theory. The energy content of an *expanding* Universe is modeled via cosmological fluids with *constant* equation of state: photons, baryons, neutrinos, cold dark matter (CDM) and dark energy.

In a powerful manner, the cosmological principle (CP) states that the Universe is homogeneous and isotropic on sufficiently large scales.[2] *Homogeneous* means that different patches

[1] For further reading, interested readers are recommended to follow excellent resources in [42–45].

[2] The CP is again strengthened by the recent *Planck* data to show that the Universe is highly isotropic [46].

in everywhere of the Universe have the same average physical properties such as energy density, pressure and temperature. *Isotropic* means that there are no preferred directions in the Universe. Any observer measuring a cosmological quantity – *e.g.* the photon flux or a galaxy count – in two different directions should find the same value.

Note that, homogeneity does not imply isotropy. A common example is an imagined Universe filled with a homogeneous magnetic field. That imagined Universe is homogeneous but not isotropic. Moreover, isotropy observed at one location does not guarantee homogeneity either. The simplest case is given by an observer at the center of an isotropic explosion. However, isotropy about two or more different locations is equivalent to homogeneity and isotropy about *all* locations.

On the one hand, physical theories can only be disproved, they can not be *proven* as in the case of mathematical theorems. On the other hand, we still can support the reliability of physical theories with more and more evidence. The CP is not true on small scales because galaxies, stars, planets and we ourselves should not exist in a perfectly homogeneous Universe. However, on scales roughly larger than $100h^{-1}$ Mpc (about the average distance between two galaxies), the Universe indeed becomes smooth.[3] Therefore, we can treat the dynamics of the cosmological fluids on the large scales as if the Universe were perfectly homogeneous and isotropic.[4]

2.1.1 ♦ The FLRW spacetime

Through observational data, Hubble told us the linear relationship between the radial speed with which a galaxy recedes from Earth and its distance to it:

$$v = H_0 r \, . \tag{2.1}$$

The proportionality H_0 is a positive number which is now called *Hubble constant*.[5] Using the CP, the Hubble's law then becomes universal. Isotropy enforces the radial motion and homogeneity ensures that the recession velocity is proportional to the distance; however, the CP

[3]By definition: $1\,\mathrm{Mpc} = 3.086 \times 10^{22}\,\mathrm{m} = 3.262 \times 10^{6}\,\mathrm{ly}$.

[4]It is impossible to observationally prove the homogeneity of the Universe without assuming first the Copernican principle. We can only probe the past light cone of here and now.

[5]By convention, we parametrize the Hubble constant by the pure number h where $H_0 \equiv 100h\,{}^{km/s}/_{\mathrm{Mpc}}$. The most accurate local measurements of h to date employ Cepheid variables and Type Ia supernovae in low-redshift galaxies give similar results as obtained by *Planck* [7] assuming ΛCDM model: $h = 0.6780 \pm 0.0077$ at 68% confidence level.

alone does not tell us the sign of H_0. The recession speed is not the speed of something moving through space, but of space itself. Because it is not a local phenomenon, it can exceed c – the speed of light. The implication of Hubble's law is groundbreaking: space itself is expanding and H_0 represents the homogeneous expansion rate.

Thanks to the CP, we can define a universal time variable – the *cosmic time* – which is defined as the time measured by observers at rest (or freely-falling) with respect to the matter in their neighborhood. By convention, to determine the age of the Universe, we choose the zero of the cosmic time to coincide with the Big Bang event. We also define *comoving observers* as observers who are at rest with the Hubble expansion. Comoving observers perceive the Universe as isotropic and see objects receding from them according to Hubble's law. The coordinate system where all comoving observers have constant spatial coordinates is called *comoving coordinates*. The metric that describes a homogeneous and isotropic expanding spacetime is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In comoving spherical coordinates (χ, θ, ϕ) and cosmic time t, it is given by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\chi^2 + \Phi_k(\chi^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right), \tag{2.2}$$

where

$$\Phi_k(\chi^2) \equiv \begin{cases} \sinh^2\chi & \text{Hyperbolic geometry} \quad k = -1 \\ \chi^2 & \text{Flat geometry} \quad k = 0 \\ \sin^2\chi & \text{Spherical geometry} \quad k = +1 \end{cases}. \tag{2.3}$$

The factor $a(t)$ parametrizes the uniform expansion of the Universe. We distinguish three different geometries for the Universe based on the value of the curvature constant k. Recent results from the WMAP [47] and *Planck* [7] CMB satellites strongly suggest that we live in a Universe with *flat* geometry. Therefore, from now on, we shall assume $k = 0$ and also $a(t_0) = 1$ where t_0 is current time. The Hubble constant H_0 is then just the present-day value of the *Hubble parameter* defined as:[6]

$$H \equiv \frac{1}{a}\frac{da}{dt}. \tag{2.4}$$

The FLRW metric can also be conveniently expressed using the *conformal time* defined as

[6] The expression for the radial velocity is $v = dr/dt \cdot \hat{\boldsymbol{r}} = \frac{1}{a}\frac{da}{dt} r + a\frac{dx}{dt} \cdot \hat{\boldsymbol{r}}$. Therefore, we obtain for comoving observers: $v = \frac{1}{a}\frac{da}{dt} r$.

$d\tau = dt/a$. Physically, conformal time may be interpreted as a "clock" which slows down with the expansion of the Universe.

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \left(d\chi^2 + \Phi_k(\chi^2)(d\theta^2 + \sin^2\theta d\phi^2)\right)\right]. \tag{2.5}$$

In an isotropic Universe, we may consider radial propagation of light as determined by the two-dimensional line element:

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + d\chi^2\right]. \tag{2.6}$$

which resembles a flat Minkowski metric multiplied by a time-dependent conformal factor $a(\tau)$. Light cone is therefore conveniently straight lines at angles ±45° in the $\tau-\chi$ plane. If we had used instead physical time t to study light propagation, the light cone for curved spacetime would be curved.

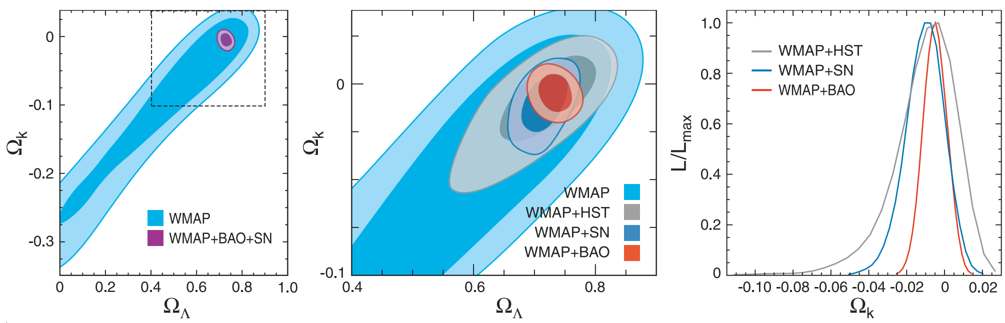

Figure 2.1: Joint two-dimensional marginalized constraint on the vacuum energy density, Ω_Λ, and the spatial curvature parameter, Ω_k. The contours show the 68% and 95% CL. *Left:* The WMAP-only constraint (light blue) compared with WMAP+BAO+SN (purple). Note that we have a prior on $\Omega_\Lambda > 0$. This figure shows how powerful the extra distance information is for constraining Ω_k. *Middle:* A blow-up of the region within the dashed lines in the left panel, showing WMAP-only (light blue), WMAP+HST (gray), WMAP+SN (dark blue), and WMAP+BAO (red). The BAO provides the most stringent constraint on Ω_k. *Right:* One-dimensional marginalized constraint on Ω_k from WMAP+HST, WMAP+SN, and WMAP+BAO. We find the best limit, $-0.0178 < \Omega_k < 0.0066$ (95% CL), from WMAP+BAO+SN, which is essentially the same as WMAP+BAO. Note that neither BAO nor SN alone is able to constrain Ω_k: they need the *WMAP* data for lifting the degeneracy. Note also that BAO+SN is unable to lift the degeneracy either, as BAO needs the sound horizon size measured by the *WMAP* data [48].

2.1.2 ♦ Hubble horizon vs. Particle horizon

The *Hubble time* t_H is defined as inverse of the Hubble parameter. Current value of the Hubble time is:

$$t_{H_0} \equiv \frac{1}{H_0} = 9.77h^{-1}\,\mathrm{Gyr}\ . \tag{2.7}$$

If we assume a constant expansion, *i.e.* $d^2a/dt^2 = 0$, the Hubble time is the time needed by the Universe to double in size. Therefore, if the expansion had been constant after the Big Bang, the Hubble time would be the *age of the Universe*. In the real case which accounts for different constituents of the Universe, Hubble time only serves as hint to the time-scale for the expansion of the Universe.[7]

The *Hubble radius* L_H is defined as the *physical distance* traveled by light in a Hubble time, its current value is given by:

$$L_{H_0} \equiv \frac{c}{H_0} = 2998h^{-1}\,\mathrm{Mpc}\ . \tag{2.8}$$

From Hubble's law, objects which are farther than a Hubble radius recede faster than light. Therefore, the Hubble radius is often referred to as *Hubble horizon*. The *comoving* Hubble horizon is then defined as:

$$r_{CHH}(t) = \frac{c}{a(t)H(t)}\ . \tag{2.9}$$

The comoving Hubble horizon is not to be confused with the *particle horizon*, which is defined as the maximum distance a particle could have traveled *since the Big Bang event* until a certain time t. With this definition, Since the speed of light is the limit velocity, the distance traveled by a photon[8] from the Big Bang event up to a certain time t coincides with the particle horizon. In comoving coordinates, it is obtained by:[9]

$$r_{CPH}(t) = \chi(t) \equiv \int_0^t c\frac{dt}{a(t)}\ . \tag{2.10}$$

[7] In the more realistic case of a decelerated expansion, the age of the Universe needs to be larger than $1/H$ so that the expansion parameter would have had enough time to reach its present value. The current best estimate of the age of the Universe for ΛCDM model is by *Planck* experiment: $t_0 = 13.813 \pm 0.038$ Gyr [7].

[8] Photon follow null geodesic ($ds^2 = 0$). Along a radial path, we have $d\chi = \frac{c}{a}dt$.

[9] The particle horizon is proportional to the conformal time τ where $\chi(t) = c\tau(t)$.

For an observer on Earth, the present-day particle horizon sets the size of the *observable Universe* which is about 28.5 Gpc. There is a subtle difference between the comoving particle horizon $\chi(t)$ and the comoving Hubble horizon $c/(aH)$: the former is a measure of the *past* light cone of an event given the previous expansion history, while the latter sets the extent of its *future* light cone based on the instantaneous value of H. An observer sitting at the center of a sphere with radius $\chi(t)$ did not have the possibility to establish contact with anything outside that sphere. The same observer, however, will be able to interact with all that is currently inside the Hubble sphere (provided that the expansion is not accelerated).[10]

2.1.3 ♦ The Dynamical Equations of Expanding Universe

In order to determine the dynamic of the Universe on large scale, we need to solve for the equation of the scale factor $a(t)$ in flat FLRW metric. These relations are given by the Einstein's equations. In our choice of units, we simply have:

$$G_{\mu\nu} = T_{\mu\nu} \,. \tag{2.11}$$

where the *Einstein tensor* $G_{\mu\nu}$, *Ricci tensor* $R_{\mu\nu}$ and the *Ricci scalar* R are defined as:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{2.12}$$

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \,, \tag{2.13}$$

$$R \equiv g^{\mu\nu} R_{\mu\nu} \,. \tag{2.14}$$

The *Christoffel symbols* (or *affine connection*) in metric space without torsion are given by:

$$\Gamma^{\mu}_{\alpha\beta} \equiv \frac{g^{\mu\nu}}{2} \left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right] \,. \tag{2.15}$$

[10] The *event horizon* is another concept of horizon in cosmology. It is the largest comoving distance from which light emitted now can ever reach the observer in the *future*. Another interesting concept is *future horizon* as discussed in [49] which leads to the speculation that we are living in a very privileged period in the evolution of the Universe. Observers when the Universe was an order of magnitude younger would not have been able to discern any effects of dark energy on the expansion, and observers when the Universe is more than an order of magnitude older will be hard pressed to know that they live in an expanding Universe at all, or that the expansion is dominated by dark energy.

We substitute the flat FLRW metric from (Eqn. 2.2) into the above equations to retrieve the Christoffel symbols and obtain the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar:[11]

$$\Gamma^0{}_{ij} = \delta_{ij}\dot{a}a \qquad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_{ij}\frac{\dot{a}}{a} \, , \tag{2.16}$$

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{ij} = \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \, , \tag{2.17}$$

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \, . \tag{2.18}$$

From above equations, all the components of the Einstein tensor $G_{\mu\nu}$ are determined as:

$$\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}}{a}\right)^2 \, , \\ G_{ij} &= -\delta_{ij}\left(\dot{a}^2 + 2a\dot{a}\right) \, , \\ G_{i0} &= G_{0i} = 0 \, . \end{aligned} \tag{2.19}$$

We observe from (Eqn. 2.19) that the Einstein tensor $G_{\mu\nu}$ is diagonal which means the energy-momentum tensor $T_{\mu\nu}$ is diagonal too. Therefore, in the simple FLRW model, at large scale, cosmological fluid is characterised only by its energy density $\rho(t)$ and its pressure $p(t)$. In a frame that is comoving with the fluid, we have:

$$T^\mu_\nu = g^{\mu\alpha}T_{\alpha\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \, . \tag{2.20}$$

By equating both left and right hand side of Einstein's equations (Eqn. 2.11), we finally obtain the desired equations for the evolution of the scale factor $a(t)$. These non-linear coupled

[11] An overdot denotes derivative with respect to physical time.

ordinary differential equations are also called the *Friedmann equations*:[12]

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho\,, \tag{2.21}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p)\,. \tag{2.22}$$

The total energy-momentum tensor is given by the *sum* of the energy-momentum tensors of the species in the Universe such as: photons, baryons, neutrinos, cold dark matter and dark energy.[13] We make an additional assumption that the fluids compose the Universe are *barotropic*. The relation between p and ρ is called the *equation of state* of the fluid; we parametrize it via the barotropic parameter w as:

$$p = w(\rho)\rho\,. \tag{2.23}$$

Relativistic species such as the photons, the neutrinos have $w_{\mathrm{R}} = \frac{1}{3}$. Non-relativistic species such as the baryons and cold dark matter after decoupling have no pressure and $w_{\mathrm{M}} = 0$. The dark energy can be treated as a cosmological constant, which is equivalent to a negative pressure fluid with $w_\Lambda = -1$. Eqns. (2.21) and (2.22) may be combined into *the continuity equation*. Although this equation does not contain any additional information beyond the above equations, it represents the energy conservation in more transparent way:

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0\,. \tag{2.24}$$

By rewriting above equation as $\frac{d\ln\rho}{d\ln a} = -3(1+w)$ and integrating it, we have:

$$\rho \propto a^{-3(1+w)}\,. \tag{2.25}$$

Plugging this back into the Friedmann equation (2.21), we obtain the time evolution of the scale factor as:

$$a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1\,, \\ e^{Ht} & w = -1\,. \end{cases} \tag{2.26}$$

---

[12] They were first theoretically derived by Alexander Friedmann in 1922 [50], before the obervational discovery by Hubble in 1929. The second Friedmann equation is also known as Raychaudhuri evolution equation.

[13] The cosmological constant is an example of a wider class of dark energy in which $-1 \leqslant w \leqslant -1/3$.

The table below summarizes the results for flat FLRW Universe:

	w	$\rho(a)$	$a(t)$	$a(\tau)$	$H(t)$	$H(a)$
MD	0	a^{-3}	$t^{2/3}$	τ^2	t^{-1}	$a^{-3/2}$
RD	1/3	a^{-4}	$t^{1/2}$	τ	t^{-1}	a^{-2}
Λ	-1	a^0	e^{Ht}	$-\tau^{-1}$	const	const

We have considered the contributions of different type of matter in the energy density separately. In the case that there are more than one matter species, we only need to sum their contributions in the energy density and the pressure of the Universe as:

$$\rho \equiv \sum_i \rho_i \, , \qquad p \equiv \sum_i p_i \, . \tag{2.27}$$

In order to emphasize the flatness of the Universe ($k = 0$). We also define the critical energy density as:[14]

$$\rho_{\rm crit}(t) \equiv 3H^2 \, . \tag{2.28}$$

The density of the species normalised to the critical density of the Universe is called the *density parameter*:

$$\Omega_i(t) \equiv \frac{\rho_i(t)}{\rho_{\rm crit}(t)} \, . \tag{2.29}$$

This allows us to write the Friedmann equation (Eqn. 2.21) as

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_{\rm k} a^{-2} \, . \tag{2.30}$$

We evaluate above equation at present-day value for a more explicit version. The subscript '0' denotes the value of a quantity at the present time, t_0:

$$H^2 = H_0^2 \left[\frac{\Omega_{\rm M0}}{a^3} + \frac{\Omega_{\rm R0}}{a^4} + \frac{\Omega_{\rm k0}}{a^2} + \Omega_{\Lambda 0} \right] , \tag{2.31}$$

with $\Omega_{\rm k0} \equiv -k/a_0^2 H_0^2$ parametrizing curvature.[15] As mentioned in (Sec 2.1.1) and shown in

[14] The present-day value of the critical energy density is $\rho_{\rm crit}(t_0) = 1.878 h^2 \times 10^{-26} \, \frac{\rm kg}{\rm m^3}$. Air is around 10^{26} times denser than the critical energy density.

[15] Ω_Λ, $\Omega_{\rm M}$, $\Omega_{\rm R}$ and $\Omega_{\rm k}$ each represents respectively the relative energy density for the dark energy, matter

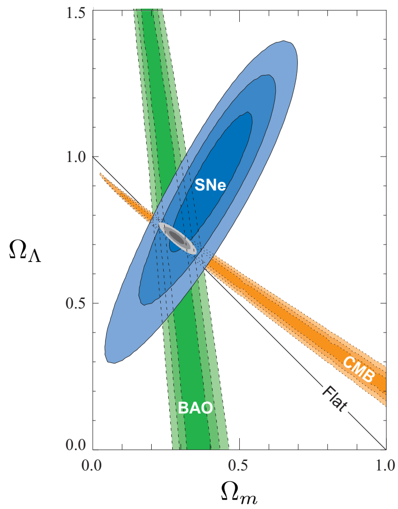

Figure 2.2: 68.3%, 95.4% and 99.7% confidence level contours on Ω_Λ and Ω_M obtained from CMB, BAO and supernovae (SNe) set as well as their combination (assuming $w_\Lambda \approx -1$) [51].

(Fig. 2.2), observations of the CMB and the large-scale structure told us that the Universe is flat:

$$\Omega_k \approx 0\,, \tag{2.32}$$

and composed of approximately 4.8% atoms (or baryons on astronomical scale), 26.7% (cold) dark matter and 68.5% dark energy ($w_\Lambda \approx -1$) [7]:

$$\Omega_b \approx 0.048\,,\quad \Omega_{DM} \approx 0.267\,,\quad \Omega_\Lambda \approx 0.685\,. \tag{2.33}$$

2.2 ▪ THE COSMIC INFLATION PARADIGM

2.2.1 ♦ Prelude

We refer to *physical theory* as notions and rules that describe how a physical system behaves from given conditions. When we restrict these conditions to fit the observed world, we obtain *models* such as the standard model in particle physics or the ΛCDM model in cosmology. Physical theory's merit is valued based on their explanation power for known facts and their prediction power for unknown phenomena.

In this sense, a physical theory is *not* required to justify or give answers to its own ini-

component (baryons plus the cold dark matter), the radiation and the curvature of the Universe.

tial conditions. We can describe a rocket's motion by theory of classical mechanics but it is inappropriate to do the same for combustion process which gives the rocket's initial conditions. Therefore, regarding i) "horizon or homogeneity problem" (the observed homogeneous CMB over $\approx 10^6$ causally-disconnected patches[16]) and ii) "flatness problem" (observed flatness of the Universe despite density curvature $|\Omega(a) - 1|$'s supposedly diverged behavior[17]), the answer from ΛCDM model are just: given correct initial conditions, let the equations evolve them and we will end up in the observed Universe today. There are theoretical physicists who consider the offered initial conditions explanation from ΛCDM model are just too "fine-tuned", "improbable", "unnatural", "unattractive" or even "disappointing".[18] They argue that the Universe should instead be able to come to existence from a much broader range of initial conditions by suggesting the Cosmic Inflation paradigm – a period of accelerated expansion from $\approx 10^{-36}$ s to $\approx 10^{-32}$ s after the Big Bang event. It is much debatable that the Cosmic Inflation paradigm really explains the initial conditions for ΛCDM model or only modifies the required initial conditions.

Historically, in 1981, Alan Guth [11] proposed the Cosmic Inflation paradigm as exponential expansion of the Universe in a supercooled false vacuum state but he also recognized that this model (now called as "old inflation") has a problem of properly reheating in order to generate radiation, known as *graceful exit problem*. One year later, in 1982, Andrei Linde [52], Andreas Albrecht and Paul Steinhardt [53] refined it by proposing the slow-roll version of inflation. In 1983, Linde [54] took next step further by removing the restriction of thermal equilibrium in the early Universe and the form of potential to be almost arbitrary provided having sufficiently flat region (now called as "chaotic inflation").

Before the discovery of dark energy, astronomers had incorrectly assumed that $\Omega \approx 0.3$, which compelled inflationary theorists to tweak their theory without success. In 1998, the first *direct* observational evidence for dark energy came from accelerated expansion of supernovaby Saul Perlmutter, Brian Schmidt and Adam Riess [55, 56]. This groundbreaking discovery led to the 2011 Nobel Prize in Physics and revived significant interest in the Cosmic Inflation paradigm. Since then, it has gained increasing popularity because its generic form provides very good explanation for observations such as: i) Flat Universe, ii) Gaussian and adiabatic per-

[16] The volume ratio obtained from physical length vs. horizon scale at the last scattering surface is $\approx 10^6$.

[17] Unless we assume initial condition of a flat geometry where $\Omega(a)$ is exactly 1, the *deviation from flatness* is more and more constrained from both sides as we look further to the past: $|\Omega(a_{\text{Planck}}) - 1| \leq \mathcal{O}(10^{-61})$.

[18] This line of thought seems to align with what Einstein proclaimed, "What I'm really interested in is whether God could have made the world in a different way."

turbation, iii) Nearly scale-invariant spectrum of density and iv) Specific peaks in the CMB's spectrum [13].

At this moment, the primordial B-mode polarization – Cosmic Inflation's "smoking-gun" signature – has not been confirmed either by *Planck*-2015 experiment [7] or *BICEP2* and *Keck Array* collaborations [12]. On the one hand, there have been raised concerns toward the Cosmic Inflation paradigm from technical aspects such as unlikeliness problem, new initial conditions problem, new measure problem [57–59] to a range of unresolved concepts: the entropy problem [60], the Liouville problem [61], the multiverse unpredictability problem [62, 63]. On the other hand, Linde and Guth [13, 64, 65] argued that "cosmic inflation is on stronger footing than ever" while insisting that a class of chaotic inflation's potential such as

$$V(\phi) = \frac{1}{2}m^2\phi^2(1 - a\phi + a^2b\phi^2)^2 \ .$$

can easily match *Planck's* constraint on A_s,n_s and r by using free parameters m,a and b in above potential, even with non-discovery of primordial B-mode polarization [66–69].

Within our best knowledge, dust at this fierce battle[19] between two camps still has not settled down. Until now, the Cosmic Inflation paradigm remains as theorists' *best* shot among several other rival theories such as pre-Big-Bang scenario, string gas cosmology, matter bounce, Ekpyrotic scenario, cosmology in Hořava-Lifshitz gravity and Galilean Genesis.

2.2.2 ♦ Conditions for Cosmic Inflation

During decelerating expansion, the increasing comoving Hubble radius, $1/(aH)$, is associated with an increasing comoving particle horizon. That is why the terms "comoving Hubble radius" and "comoving particle horizon" were used interchangeably although they are not the same concepts. This also implies comoving scales entering the horizon today have been far outside the horizon at CMB decoupling that they can not be causally connected unless we assume they were already homogeneous as initial conditions or invoke super-luminal propagation.

Cosmic Inflation paradigm suggests another scenario. The situation is radical different during the inflationary period: the Universe undergoes accelerating expansion and so the comoving Hubble radius instead shrinks!

As shown in (Fig. 2.3), the proposed solution to both the horizon problem and flatness

[19] The renown mathematical physicist Roger Penrose went as far as to make a remark during day three of the CMB@50 conference as "inflation is falsifiable and in fact has been falsified! [...] *BICEP* did a wonderful service by bringing all the Inflation-ists out of their shell, and giving them a black eye."

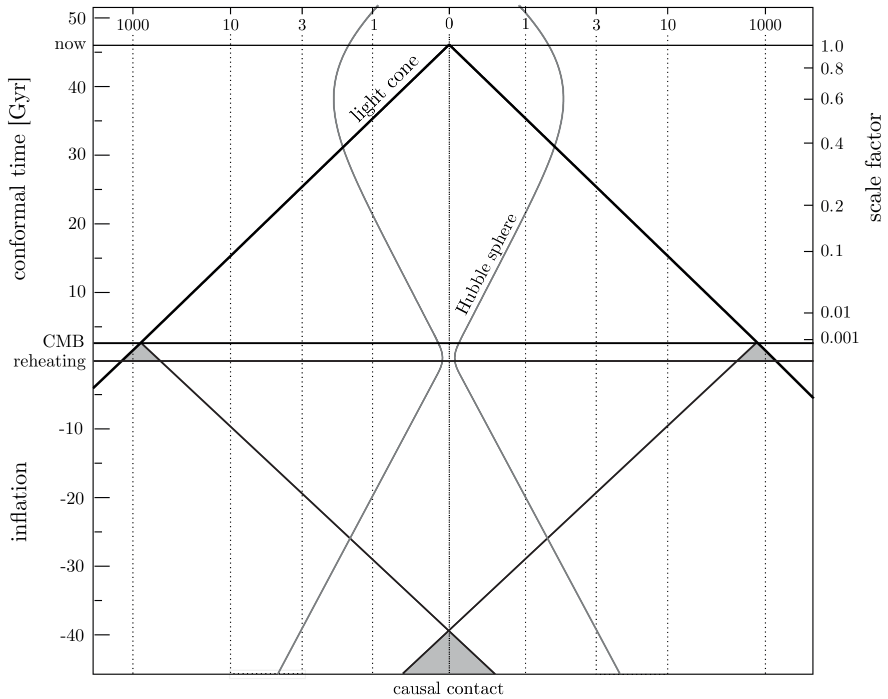

Figure 2.3: Evolution of the comoving Hubble radius, $1/(aH)$, in the inflationary Universe. The comoving Hubble sphere shrinks during inflation and expands during the conventional Big Bang evolution (at least until dark energy takes over). Conformal time during inflation is negative. The spacelike singularity of the standard Big Bang is replaced by the reheating surface: rather than marking the beginning of time, $\tau = 0$ now corresponds to the transition from inflation to the standard Big Bang evolution. All points in the CMB have overlapping past light cones and therefore originated from a causally connected region of space [70].

problem is the shrinking of Hubble sphere.[20] It gives "more conformal time" enough for previously mentioned $\approx 10^6$ patches to be causally connected in the past and also exponentially spreads out any primordial curvature. Assuming the Grand Unified Theory (GUT) energy scale of inflationary period, we can calculate backward and obtain the required duration in

[20] Originally, inflation idea came to Guth in 1980 while he was investigating why no magnetic monopoles are seen today. Cosmic Inflation solves *monopole problem* and flatness problem by the same mechanism. The monopole is conjectured to be abundantly produced using various *hypothetical* Grand Unified Theories at high temperature. This monopole problem however, arguably has a sense of "preventive medicine that is 100% effective against a disease that doesn't exist."

terms of number of e-foldings:[21]

$$N \equiv \ln\left(\frac{a_{\text{end}}}{a_{\text{start}}}\right) \gtrsim 60\,. \tag{2.34}$$

In flat FLRW metric, the following conditions for Cosmic Inflation period are equivalent: i) Shrinking Hubble sphere $\frac{d}{dt}\left(\frac{1}{aH}\right) < 0$, ii) Accelerated expansion $\frac{d^2a}{dt^2} > 0$, iii) Small fractional change of Hubble parameter $\varepsilon = -\frac{\dot{H}}{H^2} < 1$ and iv) Negative pressure $p < -\frac{1}{3}\rho$.

2.2.3 ♦ The Scalar Field Model

The ordinary matter does *not* satisfy the above negative pressure condition (violation of Strong Energy Condition) and therefore can not produce the accelerating expansion of the Universe. The simplest form of energy that can give rise to such a behavior is the cosmological constant. However, a pure cosmological constant is negligible in the early Universe and can not gracefully exit from inflation.

The simplest models of inflation involve a single *hypothetical* scalar field ϕ – the *inflaton*. Although the physical nature of the field ϕ is still unknown, we simply use it as an order parameter (or a clock) to parameterize the time evolution of the inflationary energy density. The dynamics of a scalar field *minimally-coupled* to gravity is governed by the following action :

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right] = S_{\text{EH}} + S_\phi\,. \tag{2.35}$$

The action (2.35) is the sum of the gravitational Einstein-Hilbert action, S_{EH}, and the action of a scalar field with canonical kinetic term, S_ϕ. The potential $V(\phi)$ describes the self-interactions of the scalar field.

We obtain the energy-momentum tensor for the scalar field using:

$$T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right). \tag{2.36}$$

The inflaton's equation of motion is:

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + V_{,\phi} = 0\,, \tag{2.37}$$

[21] We used the following relation during inflation: $\frac{|\Omega(a_{\text{end}})-1|}{|\Omega(a_{\text{start}})-1|} = \left(\frac{a_{\text{end}}}{a_{\text{start}}}\right)^{-2} = e^{-2N}$ and the constraint at Planck scale $|\Omega(a_{\text{end}}) - 1| \leq \mathcal{O}(10^{-61})$.

where $V_{,\phi} = \frac{dV}{d\phi}$. Assuming the flat FLRW metric (2.2) and restricting to the case of a homogeneous field $\phi(t, \mathbf{x}) \equiv \phi(t)$, the scalar energy-momentum tensor takes the form of a perfect fluid (2.20) with:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{2.38}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.39}$$

The resulting equation of state is:

$$w_\phi \equiv \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}\,, \tag{2.40}$$

shows that a scalar field can lead to negative pressure ($w_\phi < 0$) and accelerated expansion ($w_\phi < -1/3$) if the potential energy V dominates over the kinetic energy $\frac{1}{2}\dot{\phi}^2$.

The dynamics of the (homogeneous) scalar field and the FLRW geometry is determined by"

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \qquad \text{and} \qquad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\,. \tag{2.41}$$

For large values of the potential, the field experiences significant Hubble friction from the term $H\dot{\phi}$. The acceleration equation for a Universe dominated by a homogeneous scalar field

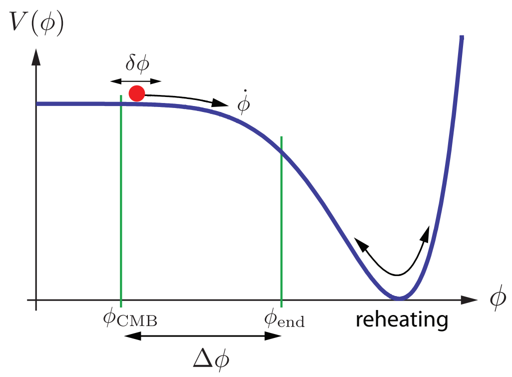

Figure 2.4: Example of a slow-roll inflationary potential. Acceleration occurs when the potential energy of the field, $V(\phi)$, dominates over its kinetic energy, $\frac{1}{2}\dot{\phi}^2$. Inflation ends at $\phi_{\rm end}$ when the kinetic energy has grown to become comparable to the potential energy, $\frac{1}{2}\dot{\phi}^2 \approx V$. CMB fluctuations are created by quantum fluctuations $\delta\phi$ about 60 e-folds before the end of inflation. During reheating period, the energy density of the inflaton is converted into converted into a thermal mix of elementary particles [45].

can be written as follows:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\phi + 3p_\phi) = H^2(1-\varepsilon)\,, \tag{2.42}$$

where

$$\varepsilon \equiv \frac{3}{2}(w_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}\,. \tag{2.43}$$

The so-called *slow-roll parameter* ε may be related to the evolution of the Hubble parameter as:

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}\,, \tag{2.44}$$

where we have used $\mathrm{d}N = H\mathrm{d}t$ to transform time variable into e-fold variable. Accelerated expansion occurs if $\varepsilon < 1$. In the de Sitter limit, $p_\phi \to -\rho_\phi$ corresponds to $\varepsilon \to 0$. In this case, the potential energy dominates over the kinetic energy: $\dot{\phi}^2 \ll V(\phi)$. Accelerated expansion will only be sustained for a sufficiently long period of time if the second time derivative of ϕ is small enough:

$$|\ddot{\phi}| \ll |3H\dot{\phi}|\,,\,|V_{,\phi}|\,. \tag{2.45}$$

This requires smallness of a *second slow-roll parameter*:

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{dN}\,, \tag{2.46}$$

where $|\eta| < 1$ ensures that the fractional change of ε per e-fold is small.

The slow-roll conditions, $\varepsilon, |\eta| < 1$, may also be expressed as conditions on the shape of the inflationary potential:

$$\epsilon_\mathrm{v}(\phi) \equiv \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2\,, \tag{2.47}$$

and

$$\eta_\mathrm{v}(\phi) \equiv \frac{V_{,\phi\phi}}{V}\,. \tag{2.48}$$

In the slow-roll regime:

$$\epsilon_\mathrm{v}, |\eta_\mathrm{v}| \ll 1\,, \tag{2.49}$$

the background evolution is:

$$H^2 \approx \frac{1}{3}V(\phi) \approx \mathrm{const} \qquad \text{and} \qquad \dot{\phi} \approx -\frac{V_{,\phi}}{3H}\,. \tag{2.50}$$

and the spacetime is approximately *de Sitter*. Inflation therefore realized as:

$$a(t) \sim e^{Ht} . \tag{2.51}$$

The parameters ϵ_{v} and η_{v} are called the *potential slow-roll parameters* to distinguish them from the *Hubble slow-roll parameters* ε and η. In the slow-roll approximation the Hubble and potential slow-roll parameters are related as follows:

$$\varepsilon \approx \epsilon_{\mathrm{v}} , \qquad \eta \approx \eta_{\mathrm{v}} - \epsilon_{\mathrm{v}} . \tag{2.52}$$

Inflation ends when the slow-roll conditions are violated:

$$\varepsilon(\phi_{\mathrm{end}}) \equiv 1 , \qquad \epsilon_{\mathrm{v}}(\phi_{\mathrm{end}}) \approx 1 . \tag{2.53}$$

The number of e-folds before inflation ends is:

$$\begin{aligned} N(\phi) &\equiv \ln \frac{a_{\mathrm{end}}}{a} \\ &= \int_t^{t_{\mathrm{end}}} H \mathrm{d}t = \int_{\phi}^{\phi_{\mathrm{end}}} \frac{H}{\dot{\phi}} \mathrm{d}\phi \approx \int_{\phi_{\mathrm{end}}}^{\phi} \frac{V}{V_{,\phi}} \mathrm{d}\phi , \end{aligned} \tag{2.54}$$

where we used the slow-roll results (2.50). The result (2.54) may also be written as

$$N(\phi) = \int_{\phi_{\mathrm{end}}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\varepsilon}} \approx \int_{\phi_{\mathrm{end}}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_{\mathrm{v}}}} . \tag{2.55}$$

As mentioned before, we need the total number of inflationary e-folds exceeds about 60 to solve the horizon and flatness problems. The precise value depends on the energy scale of inflation and on the details of reheating after inflation.[22] Therefore, above integral can be used to approximate the corresponding field value ϕ_{cmb}

$$\int_{\phi_{\mathrm{end}}}^{\phi_{\mathrm{cmb}}} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_{\mathrm{v}}}} \approx 60 . \tag{2.56}$$

[22] The unknown energy scale of inflation and unknown mechanism of reheating process have motivated variety of models of inflation in the literature.

2.3 ▪ PERTURBATIONS DURING INFLATION

The idea from Cosmic Inflation paradigm is that the structure we observe today, such as the CMB anisotropies and the galaxy distribution, formed starting from tiny quantum fluctuations set during inflationary period and later enhanced by gravitational instability. These primordial fluctuations are microscopic quantum vacuum fluctuations in the inflaton field. During inflation, they were stretched and imprinted on superhorizon scales by the accelerated expansion. Metric perturbations outside the horizon are preserved because there is no causal physics available.[23] These density fluctuations reentered the horizon after inflation ended and served as initial conditions for the anisotropy and the growth of structure in the Universe.

Because the anisotropies in the CMB is of order $\mathcal{O}(10^{-5})$, it is a very good approximation to split all quantities $X(t, \mathbf{x})$ (metric $g_{\mu\nu}$ and matter fields $T_{\mu\nu} \to \phi\ \rho,\ p$, etc.) into a homogeneous background $\bar{X}(t)$ that depends only on cosmic time and a spatially dependent perturbation:

$$\delta X(t, \mathbf{x}) \equiv X(t, \mathbf{x}) - \bar{X}(t)\,. \tag{2.57}$$

According to Einstein equations, inhomogeneities in the matter distribution induce the metric perturbations and vice versa. Expanding the Einstein equations at *linear order*, $\delta X \ll \bar{X}$, we have:

$$\delta G_{\mu\nu} = \delta T_{\mu\nu}\,. \tag{2.58}$$

During inflation we define perturbations around the homogeneous background solutions for the inflaton $\bar{\phi}(t)$ and the metric $\bar{g}_{\mu\nu}(t)$ as:

$$\delta\phi(t, \mathbf{x}) = \phi(t, \mathbf{x}) - \bar{\phi}(t)\,, \qquad h_{\mu\nu} \equiv \delta g_{\mu\nu}(t, \mathbf{x}) = g_{\mu\nu}(t, \mathbf{x}) - \bar{g}_{\mu\nu}(t)\,. \tag{2.59}$$

These perturbations can be decomposed into the Scalar, Vector and Tensor (SVT) parts according to Lifshitz's approach [71]. In the linear approximation, different types of perturbations evolve independently and can be *analyzed separately*.[24] The understanding of perturbations' behavior allow us to obtain the power spectra of both scalar and tensor fluctuations during inflation.

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[23] This is valid only when there are *no* anisotropic stress fluctuations.

[24] Steven Weinberg's "Cosmology" textbook is an excellent resource for this part [43].

Metric Perturbations

The metric of a flat homogeneous and isotropic FRW universe with tiny perturbations can be presented according to the *SVT decomposition* theorem as:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -(1+2\phi)dt^2 + 2aB_i dx^i dt + a^2\Big[(1-2\psi)\delta_{ij} + E_{ij}\Big]dx^i dx^j \ . \end{aligned} \tag{2.60}$$

According to the SVT decomposition, we can represent B_i and E_{ij} as:

$$B_i \equiv \partial_i B - S_i, \quad \partial^i S_i = 0 \ , \tag{2.61}$$

and

$$E_{ij} \equiv 2\partial_{ij}E + 2\partial_{(i}F_{j)} + h_{ij}, \quad \partial^i F_i = 0, \quad h^i_i = \partial^i h_{ij} = 0 \ . \tag{2.62}$$

We call ϕ, ψ, B_i and E_{ij} the *lapse* function, *spatial curvature*, *shift* vector and *shear* tensor respectively. The vector perturbations S_i and F_i are not created by inflation and decay as a^{-2} with the expansion of the Universe.[25]

Matter Perturbations

For a perfect fluid, we would have:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \,, \tag{2.63}$$

with the timelike velocity 4-vector $u^\mu \equiv \frac{dx^\mu}{d\tau}$ and the normalization condition

$$g^{\mu\nu}u_\mu u_\nu = -1 \,, \tag{2.64}$$

which gives $\delta u^0 = \delta u_0 = h_{00}/2$ while δu_i is an independent dynamical variable. Note that $\delta u^\mu \equiv \delta(g^{\mu\nu}u_\nu)$ is not given by $\bar{g}^{\mu\nu}\delta u_\nu$. The first order perturbation to the energy-momentum

[25] The case is different when there is vector source. For this reason we ignore vector perturbations in the case of flat FLRW metric.

tensor for a perfect fluid is:

$$\delta T_{ij} = \bar{p} h_{ij} + a^2 \delta_{ij} \delta p \ , \tag{2.65}$$

$$\delta T_{i0} = \bar{p} h_{i0} - (\bar{\rho} + \bar{p}) \delta u_i \ , \tag{2.66}$$

$$\delta T_{00} = -\bar{\rho} h_{00} + \delta \rho \ . \tag{2.67}$$

Using $\delta T^{\mu}_{\nu} = \bar{g}^{\mu\lambda}[\delta T_{\lambda\nu} - h_{\lambda\kappa} \bar{T}^{\kappa}_{\nu}]$, the perturbed mixed components are:

$$\delta T^0_0 = -\delta \rho \ , \tag{2.68}$$

$$\delta T^0_i = (\bar{\rho} + \bar{p}) \delta u_i \ , \tag{2.69}$$

$$\delta T^i_0 = a^{-2} (\bar{\rho} + \bar{p})(h_{i0} - \delta u_i) \ , \tag{2.70}$$

$$\delta T^i_j = \delta^i_j (\delta p) + \Sigma^i_j \ , \tag{2.71}$$

where Σ^i_j called *anisotropic stress perturbation* characterizes departures from the perfect fluid form of the energy-momentum tensor.

Gauge-Invariant Variables

The gauge transformation on the unperturbed FLRW metric and energy-momentum tensor will introduce *unphysical* perturbations.[26] Therefore, we need to either *fix the gauge*[27] to remove those extra degrees of freedom or choose to work only with special combinations of metric and matter perturbations called *gauge-invariant variable* [72].

Scalar Perturbations:

There are two different sources for the scalar perturbations: One comes from the scalar components of the metric, the other one comes from the scalar components of the matter field perturbations. By combining these transformations, we have the following gauge invariant quantities:

Curvature Perturbation on Uniform-Density Hypersurfaces: Geometrically, this measures the spatial curvature of the constant-density hypersurfaces:

$$-\zeta \equiv \psi + \frac{H}{\dot{\rho}} \delta \rho \ . \tag{2.72}$$

[26] This complication comes from the arbitrariness in choosing the way to map points on two different spacetimes: perturbed and unperturbed spacetime.

[27] There are various gauges like Newtonian gauge, synchronous gauge, co-moving gauge, constant density gauge, etc. from which we can pick and also do conversion to facilitate the computation in different cases.

Comoving Curvature Perturbations: Geometrically, this is the curvature perturbations that is transverse to the comoving world lines:

$$\mathcal{R} \equiv \psi - \frac{H}{\rho + p}\delta q \,. \tag{2.73}$$

Entropic Perturbation: This is the non-adiabatic part of the pressure:

$$\begin{aligned} \delta p_{nad} &= \delta p - \delta p_{ad} \\ &= \delta p - \frac{\dot{p}}{\dot{\rho}}\delta\rho \,. \end{aligned} \tag{2.74}$$

Note that there is a very important properties of ζ: for the adiabatic matters, where $\delta p_{nad} = 0$, it remains constant outside the horizon.

Comoving Density: This is the difference between the energy density and the scalar part of the 3-momentum, q:

$$\delta\rho_m = \delta\rho - 3H\delta q \,. \tag{2.75}$$

Mukhanov-Sasaki Variable: This is the field perturbation in the flat gauge, *i.e.* $\psi = 0$, and has the following definition:

$$\delta\varphi_\psi \equiv \delta\varphi + \frac{\dot{\varphi}}{H}\psi \,. \tag{2.76}$$

Note that, in the case of single field inflation, the 3-momentum is given by $\delta q = -\dot{\varphi}\delta\varphi$. So we obtain:

$$R \equiv \psi + \frac{H}{\dot{\varphi}}\delta\varphi \,. \tag{2.77}$$

We also have:

$$\delta\rho/\dot{\rho} \simeq \delta\varphi/\dot{\varphi} \,. \tag{2.78}$$

Therefore, in this case, ζ and R are equal to each other.

Vector Perturbations:

In general case, there are two different sources for the vector perturbations. For a FLRW background, we can safely assume that there is no pure vectors in the matter sector and thus we are only left with the vector perturbations in the metric sector. This assumption does not hold when we consider the anisotropic Universe.

The vectorial part of the metric perturbation is:

$$ds^2 = -dt^2 - 2a(t)S_i dx^i dt + a^2(t)[\delta_{ij} + 2F_{(i,j)}]dx^i dx^j \,, \tag{2.79}$$

where S_i and F_i are divergence free vectors. The gauge invariant combination is $\dot{F}_i + S_i/a$. This means among 4 degrees of freedom for these two vectors, there are only two independent components.

Tensor Perturbations:

The tensorial part of the metric is:

$$ds^2 = -dt^2 + a^2(t)[\delta_{ij} + h_{ij}]dx^i dx^j \,, \tag{2.80}$$

where h_{ij} are transverse, i.e. $h_{ij,i} = 0$, and traceless, i.e. $h^i_i = 0$. Tensor perturbations are gauge invariant.

2.3.1 ♦ Equations for Cosmological Perturbations

Scalar Perturbations

By using the Einstein field equations, we can correlate the metric perturbations with the matter field perturbations. We then have the following equations for the scalar perturbations:

$$3H(\dot{\psi} + H\phi) + \frac{k^2}{a^2}\left[\psi + H\left(a^2\dot{E} - aB\right]\right) = -4\pi G\delta\rho \,, \tag{2.81}$$

$$\dot{\psi} + H\phi = -4\pi G\delta q \,. \tag{2.82}$$

Combining above equations, we have:

$$\frac{k^2}{a^2}\Psi = -4\pi G\delta\rho_m \,, \tag{2.83}$$

which is a gauge invariant generalization of the Poisson equation.

In addition, the spatial parts of the Einstein Field equations gives the following equations:

$$\ddot{\psi} + 3H\dot{\psi} + H\dot{\phi} + \left(3H^2 + 2\dot{H}\right)\phi = 4\pi G\left(\delta p - \frac{2}{3}k^2\delta\Sigma\right) \,, \tag{2.84}$$

$$\left(\dot{E} - B/a\right)^{\cdot} + 3H\left(\dot{E} - B/a\right) + \frac{\psi - \phi}{a^2} = 8\pi G\delta\Sigma \,. \tag{2.85}$$

where the anisotropic stress is:

$$\delta\Sigma_{ij} = \left[\partial_i\partial_j + (k^2/3)\delta_{ij}\right]\Sigma \,. \tag{2.86}$$

If we rewrite Eqn. (2.85) in the *longitudinal gauge*, *i.e.* $E = B = 0$:

$$\Psi - \Phi = 8\pi G a^2 \delta\Sigma \,. \tag{2.87}$$

We can see that in the absence of the anisotropic stress, we have, $\Psi = \Phi$.

By using the covariant conservation of the energy-momentum tensor, *i.e.* $\nabla_\mu T^{\mu\nu} = 0$, we can obtain relation between different perturbations as:

$$\delta\dot{\rho} + 3H\left(\delta\rho + \delta p\right) = \frac{k^2}{a^2}\delta q + (\rho + p)\left[3\dot{\psi} + k^2\left(\dot{E} + B/a\right)\right] \,, \tag{2.88}$$

$$\delta\dot{q} + 3H\delta q = -\delta p + \frac{2}{3}k^2\delta\Sigma - (\rho + p)\,\phi \,. \tag{2.89}$$

This gives us the continuity equation and Euler equations respectively.

We can now rewrite Eqn. (2.88) in terms of the curvature perturbation, ζ, as:

$$\dot{\zeta} = -H\frac{\delta p_{nad}}{\rho + p} - \Pi \,, \tag{2.90}$$

where Π is the scalar shear along the comoving world-lines and can be measured with respect to the Hubble rate as:

$$\begin{aligned}\frac{\Pi}{H} &= -\frac{k^2}{3H}\left\{\dot{E} - (B/a) + \frac{\delta q}{a^2(\rho + p)}\right\} \\ &= -\frac{k^2}{3a^2H^2}\zeta - \frac{k^2\Psi}{3a^2H^2}\left[1 - \frac{2\rho}{9(\rho + p)}\frac{k^2}{a^2H^2}\right] \,.\end{aligned} \tag{2.91}$$

In the case of scalar field Lagrangian:

$$L = -\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi) \,, \tag{2.92}$$

We can find the energy density, pressure and the momentum density as:

$$\delta\rho = \left[\dot{\varphi}\left(\dot{\delta\varphi} - \dot{\varphi}\phi\right) + V_{,\varphi}\delta\varphi\right] , \tag{2.93}$$

$$\delta p = \left[\dot{\varphi}\left(\dot{\delta\varphi} - \dot{\varphi}\phi\right) - V_{,\varphi}\delta\varphi\right] , \tag{2.94}$$

$$\delta q_{,i} = -\sum_I \dot{\varphi}\delta\varphi_{,i} . \tag{2.95}$$

Using above equations, we can calculate the gauge invariant comoving energy density as:

$$\delta\rho_m = \left[\dot{\varphi}\left(\dot{\delta\varphi} - \dot{\varphi}\phi\right) - \ddot{\varphi}\delta\varphi\right] . \tag{2.96}$$

In addition, we can show that for the single field inflation the non-adiabatic perturbation is proportional to the comoving energy density

$$\delta p_{nad} = -\frac{2V_{,\varphi}}{3H\dot{\varphi}}\delta\rho_m . \tag{2.97}$$

On the other hand, according to Eqn. (2.83), since for the finite value of the Ψ, the comoving energy density is zero on the super-horizon scales, we can conclude that the perturbations are adiabatic in this limit. Using those relations, we can finally find the equation of motion for the scalar perturbations as:

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \frac{k^2}{a^2}\delta\varphi + V_{\varphi\varphi}\delta\varphi = -2V_{\varphi}\phi + \dot{\varphi}\left[\dot{\phi} + 3\dot{\psi} + \frac{k^2}{a^2}(a^2\dot{E} - aB)\right] . \tag{2.98}$$

Vector Perturbations

The vectorial part of the anisotropic stress is:

$$\delta\Sigma_{ij} = \partial_{(i}\Sigma_{j)} , \tag{2.99}$$

$$\Sigma_{i,i} = 0 . \tag{2.100}$$

In this case, there are only two Einstein equations:

$$\dot{\delta q_i} + 3H\delta q_i = k^2\delta\Sigma_i , \tag{2.101}$$

$$k^2(\dot{F}_i + S_i/a) = 16\pi G\,\delta q_i . \tag{2.102}$$

In the absence of anisotropic stress tensor, δq_i and the gauge invariant combination, $\dot{F}_i + S_i/a$, are diluted by the expansion of the Universe. The vector perturbations hence suppressed.

Tensor Perturbations

In this case, the anisotropic stress plays the role of the source. In the absence of anisotropic stress tensor, the equation of motion for the tensor part is:

$$\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2}h = 0 \ . \tag{2.103}$$

Eqn. (2.103) is the only equations which governs the evolution of the tensor components and describes the evolution of the tensor mode in an expanding Universe.

We have briefly built up prerequisites by reviewing assumptions and basic equations starting from standard Big Bang cosmology until the formalism for calculation of spectrum perturbation in Cosmic Inflation. Further details about primordial power spectrum of the scalar and tensor modes are given in Appendix A while the evolution of perturbations during inflation are given in Appendix B.

"My goal is simple. It is a complete understanding of the Universe, why it is as it is and why it exists at all."

Stephen W. Hawking

3

The vector models in cosmic inflation

The 2006 Nobel prize in Physics was awarded jointly to John C. Mather and George F. Smoot "for their discovery of the blackbody form and anisotropy of the cosmic microwave background radiation." From COBE (Cosmic Background Explorer) data, we discovered that the CMB is very smooth *but* it is not completely smooth. Assuming its statistical significance, this feature therefore demands explanation from the standard phase of early inflation.

3.1 ▪ CMB OBSERVATION AND INFLATIONARY PERIOD

We briefly describe here how CMB observations can be related to the primordial value of ζ after taking into account appropriate transfer functions to describe the sub-horizon evolution of the fluctuations. First of all, we need to figure out how ζ and h are related to the observables. Then, we also need to take into account the possible evolution of these parameters as soon as they have entered the horizon. Generically we can write:

$$Q_k(\tau) = T_Q(k, \tau, \tau_*)\zeta_k(\tau_*) \; , \tag{3.1}$$

where τ_* denotes the moment of the horizon crossing and T_Q refers to the transfer function between the time of horizon re-entry till the time of the observation, τ. The parameter Q may be the temperature of the CMB and the goal is to use the CMB data to put constrains on the primordial curvature power spectrum.

For this purpose, it is convenient to use the harmonic expansion of the CMB map as:

$$\Theta(\hat{n}) \equiv \frac{\Delta T(\hat{n})}{T_0} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}) \;, \tag{3.2}$$

where $\hat{n}$ denotes the direction in the sky and $a_{\ell m}$ is given by:

$$a_{\ell m} = \int d\Omega \, Y^*_{\ell m}(\hat{n})\Theta(\hat{n}) \;. \tag{3.3}$$

Here, $Y_{\ell m}(\hat{n})$ are the standard spherical harmonics and the magnetic quantum numbers satisfy $m = -\ell, \ldots, +\ell$. We can then combine the multipole moments $a_{\ell m}$ and calculate the rotationally-invariant angular power spectrum as:

$$C_\ell^{TT} = \frac{1}{2\ell+1}\sum_m \langle a^*_{\ell m} a_{\ell m}\rangle \;, \qquad \text{or} \qquad \langle a^*_{\ell m} a_{\ell' m'}\rangle = C_\ell^{TT}\delta_{\ell\ell'}\delta_{mm'} \;. \tag{3.4}$$

The angular power spectrum is a very important tool in the statistical analysis of CMB. Because CMB temperature fluctuations are dominated by the curvature perturbation, ζ, the transfer function, $\Delta_{T\ell}(k)$, correlates the curvature perturbation to that of the temperature fluctuation, ΔT through an integral over the k-space as:

$$a_{\ell m} = 4\pi(-i)^\ell \int \frac{d^3k}{(2\pi)^3}\, \Delta_{T\ell}(k)\, \zeta_{\mathbf{k}}\, Y_{\ell m}(\hat{\mathbf{k}}) \;. \tag{3.5}$$

Plugging Eq. (3.5) back into Eq. (3.4) and using the following identity:

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{k}}) Y_{\ell m}(\hat{\mathbf{k}}') = \frac{2\ell+1}{4\pi} P_\ell(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}') \;, \tag{3.6}$$

we obtain

$$C_\ell^{TT} = \frac{2}{\pi}\int k^2 dk P_\zeta(k)\Delta_{T\ell}(k)\Delta_{T\ell}(k) \;. \tag{3.7}$$

For a generic value of ℓ, the transfer functions are obtained numerically and the results depend on the background cosmology. So for a fixed background, the shape of the power-spectrum (according to *Planck* data, $k_p \simeq 0.05\,\mathrm{Mpc}^{-1}$) give us information about the initial spectrum, coming from the inflationary Universe:

$$P_\zeta(k_p) = 2.4 \times 10^{-9} \;. \tag{3.8}$$

3.2 ▪ CMB ANOMALIES

The CMB observations are in very good agreement with the suggestion from cosmic inflation paradigm. However, a number of studies of the data have obtained some features which seem to be anomalous within this picture. The most well-known among these features are: the low power in the quadrupole moment [15–19], the alignment of the lowest multipoles and claims of statistical anisotropy [19–25], an asymmetry in power between the northern and southern hemispheres [26–32] and a non-Gaussian deviation in the southern hemisphere, known as the "cold-spot" [33–38].

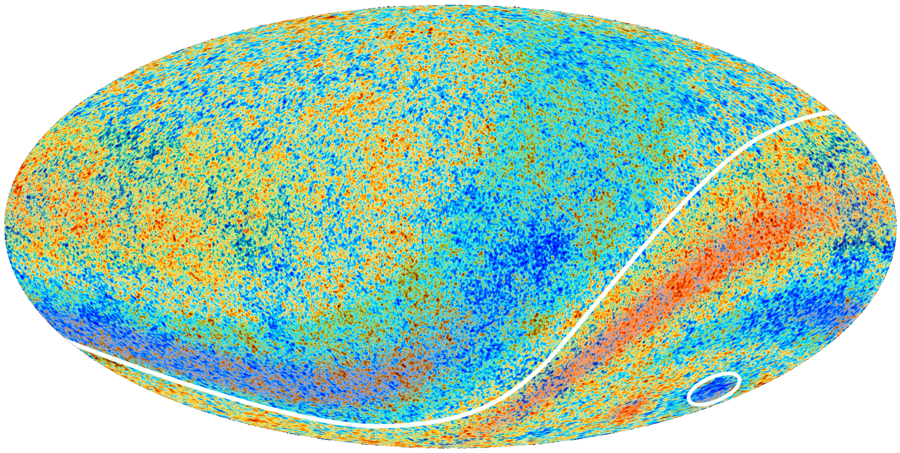

Figure 3.1: Two Cosmic Microwave Background anomalous features hinted at by Planck's predecessor, NASA's Wilkinson Microwave Anisotropy Probe (WMAP), are confirmed in the new high precision data from Planck. One is an asymmetry in the average temperatures on opposite hemispheres of the sky (indicated by the curved line), with slightly higher average temperatures in the southern ecliptic hemisphere and slightly lower average temperatures in the northern ecliptic hemisphere. This runs counter to the prediction made by the standard model that the Universe should be broadly similar in any direction we look. There is also a cold spot that extends over a patch of sky that is much larger than expected (circled). In this image the anomalous regions have been *enhanced* with red and blue shading to make them more clearly visible.

Although the statistical significance of these effects has been debated extensively in the literature, they has increased in the latest studies. For *anomalous alignment*: the probability for the obvious alignment to happen by chance is around 0.1%. In addition, the ecliptic plane seems to be correlated to this alignment, and separates a hot spot in the northern sky and a cold spot in the south. Up to today, there is no explanation for this correlation. For *power asymmetries* in the CMB: No known systematic effects or foregrounds are found to be able to explain this asymmetry. For *local features* such as "cold-spot": Systematics or foreground effects were ruled out to be responsible for this local feature, and its chance to happen accidentally is around or less then 1%, depending on the type of the analysis [26, 73, 74]. The possibility that these unexplained features actually exist in the CMB has motivated theoretical cosmologists to extend Cosmic Inflation paradigm by including higher spin fields such as vector fields, spinors and p-forms. It seems that the most straightforward and simplest next step in making extension would be vector fields. However, this task is highly non-trivial.

3.3 ▪ THE INSTABLE VECTOR FIELD MODELS

One can obtain prolonged anisotropic inflationary solutions by introducing some ingredients that violate the premises of Wald's theorem [75]. This has been realized through the addition of quadratic curvature invariants to the gravity action, with the use of the Kalb-Ramond axion, or of vector fields.

The first one among studies on vector fields during inflation is the Ackerman-Carroll-Wise (ACW) model, where the vector field is forced to have a fixed norm through a Lagrange multiplier field. This model has attracted considerable attention, since it has a simple background solution and a controllable anisotropy. However, it was shown in [40] that instabilities arise when the perturbations cross the horizon scale. This is due to the longitudinal vector polarization becomes a ghost at this moment. For the models where vector field non-minimally coupled to the curvature, the same problem of negative mass squared term and therefore ghost associated with its instability. This is because the requirements of slow-roll in the case of the vector curvaton leads again to negative mass squared term [39, 41].

The existence of ghost makes it hard to cure the model even if we assume that the model could be regularized above a certain scale such that the theory with the ghost is only an effective one.[1] There are however stringent experimental constraints on an effective theory with ghosts, coming from the amount of observed gamma ray background and the cut-off scale satisfies $\Lambda_{\rm UV} \lesssim$ MeV when the ghost has only gravitational couplings to matter [76]. The limit is even more stringent when there are stronger couplings. Inflationary physics takes place at much higher energy scales than MeV, therefore the inflationary predictions based on such models are still questionable.

[1] Another way out is to straight away provide the UV completion of the theory. However, this is a very hard task. For instance, in the Higgs mechanism, the field responsible for the mass would need to be itself a ghost.

3.4 ▪ STABLE MODEL BUILDING APPROACH

In previous sections, we have seen that the firstly proposed vector field models for cosmic inflation are unstable due to the presence of ghost. Motivated by the flexibility of simple chaotic inflation model using potential $V(\phi) = \frac{1}{2}m^2\phi^2 + V_0$, we attempt to build stable vector field models by making sure that the mass squared term m^2 in our potential remains real and positive during the whole inflation period.

In order to build stable models in simple form for vector field cosmic inflation, we first establish our base by obtaining known behaviors from single scalar field case in chaotic cosmic inflation model. The subtleties in initial conditions $(\dot{\phi})_0$ that satisfy slow-roll constraints and positive branch selection of the Hubble parameter are also addressed. Next, bringing along this knowledge, we move in similar way toward our goal of building up a stable vector field model for cosmic inflation. We obtain results aligned with our original goal: our stable vector field model can indeed provide required slow-roll inflation period longer than 60 e-folds.

Furthermore, we observe a new behavior, the inflation period does not end like the single scalar field case and instead goes on infinitely. This is because there is an important distinction from the single scalar field case: the "clock" parameter ξ in vector field case does not reach the chaotic potential's minimum in finite time as the "clock" parameter ϕ in single scalar field case. Therefore, we continue to investigate further by introducing minimal additional modifications at the end of inflation period to the original chaotic potential.[2] We obtain next results showing that our stable vector field models provide inflation period longer than 60 e-folds then proceed to exit as expected. Finally, we also briefly show the stability analysis of our models through different approaches.

3.4.1 ♦ The scalar model as our base

The action for single scalar field in cosmic inflation model is:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] . \tag{3.9}$$

For single scalar field in chaotic inflation model, the potential is:

$$V(\phi) = \frac{1}{2}m^2\phi^2 + V_0 , \tag{3.10}$$

[2] Although this move seems to be ad hoc/artificial and the result potentials are slightly more complicated than the original one, it is not a rare practice given the still unknown reheating era after inflation period.

which gives:

$$\frac{\partial V}{\partial \phi} = m^2 \phi \ . \tag{3.11}$$

In order to obtain evolution of *inflaton* ϕ and Hubble parameter H, we need to solve the following system of coupled non-linear differential equations:[3]

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \ , \tag{3.12}$$

$$\frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}m^2\phi^2 + V_0 - 3H^2 = 0 \ . \tag{3.13}$$

We work in e-fold variable instead of time variable in order to directly verify the required amount of more than 60 e-folds during inflation period. The relationship between e-fold variable and time variable is $\mathrm{d}N = H\mathrm{d}t$, which gives:

$$\dot{\phi} = H(\partial_N \phi) \ , \tag{3.14}$$

$$\ddot{\phi} = H^2(\partial_{NN}\phi) + (\partial_N \phi)H(\partial_N H) \ . \tag{3.15}$$

Using these relations for e-fold variable, our system of non-linear coupled partial differential equations is:

$$\partial_{NN}\phi + \left(3 + \frac{\partial_N H}{H}\right)\partial_N \phi + \frac{1}{H^2}\frac{\partial V}{\partial \phi} = 0 \ , \tag{3.16}$$

$$\frac{1}{2}H^2(\partial_N \phi)^2 + \frac{1}{2}m^2\phi^2 + V_0 - 3H^2 = 0 \ . \tag{3.17}$$

According to equation 2.55, the initial value of the *inflaton* $(\phi)_0$ can be determined from the required 60 e-folds duration of the inflation period.[4] Given the calculated value of $(\phi)_0$, we can specify the initial values of $(\partial_N \phi)_0$ and H_0. At the beginning of inflation, the slow-roll conditions are in effect, we have:

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{\partial_N H}{H} < 1 \ , \tag{3.18}$$

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\partial_{NN}\phi + (\partial_N \phi)\left(\frac{\partial_N H}{H}\right)}{\partial_N \phi} < 1 \ . \tag{3.19}$$

[3] They are the equation of motion for inflaton ϕ and the Friedmann equation.

[4] In this case, we have $(\phi)_0 = \sqrt{4 \times 60}$.

Using the above constraints to simplify equation 3.16, we can *selectively* obtain the initial *real* values of $(\partial_N\phi)_0$ and H_0 by solving the following algebraic system of equations:[5]

$$3(\partial_N\phi)_0 + m^2\frac{\phi_0}{{H_0}^2} = 0\ , \tag{3.20}$$

$$\frac{1}{2}{(\partial_N\phi)_0}^2{H_0}^2 + \frac{1}{2}m^2{\phi_0}^2 + V_0 - 3{H_0}^2 = 0\ . \tag{3.21}$$

The value of V_0 and m are free parameters in the single scalar field model. They can be determined when we know the energy scale of the inflation period. In the range of GUT scale, we picked [6] their values in reduced Planck mass framework to be $V_0 = 5\times10^{-15}$ and $m = 5\times10^{-8}$.

The calculated results below show expected behaviors from the single scalar field model. Inflation ends when the slow-roll conditions are violated. The graphs therefore show that the model provides stable evolution and graceful exit after required amount of 60 e-folds duration.

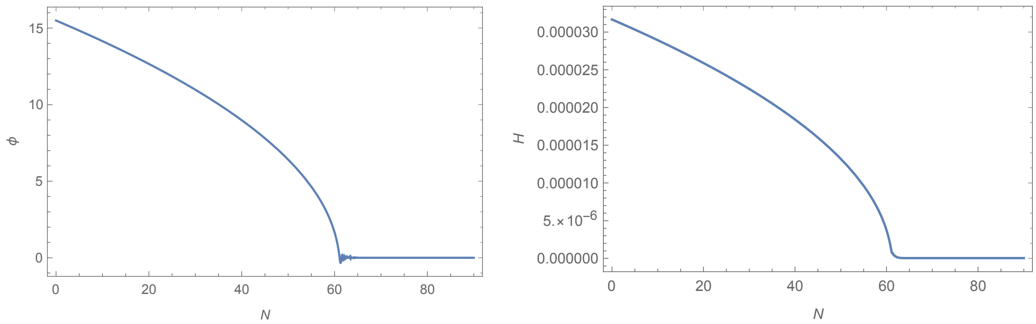

Figure 3.2: *Single scalar field in chaotic inflation model:* The evolution of the inflaton ϕ and Hubble parameter H during inflation period. The model provides stable evolution and graceful exit after required amount of 60 e-folds duration.

We also observed the discrepancy between two definitions of η. We adapted the definition $\eta \equiv \frac{-\ddot{\phi}}{H\dot{\phi}}$ because its straightforward extension to the case of vector field models in anisotropic background.

[5]Note that we have to select the *positive* branch of the Hubble parameter H due to the appearance of the H^2 term in our equations.

[6]The value of V_0 needs to be nonzero if we use e-fold variable due to the diverged behavior at zero value of the Hubble parameter H. We have also verified this behavior by re-solving this same system of equations in time variable when $V_0 = 0$.

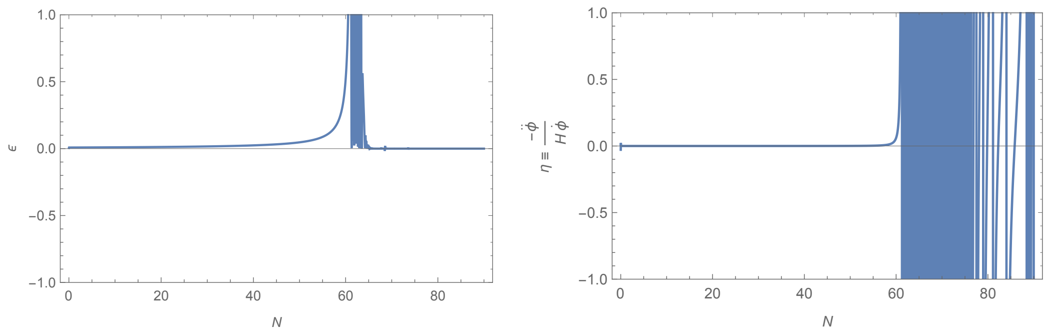

Figure 3.3: *Single scalar field in chaotic inflation model:* The evolution of the slow-roll parameters: ϵ and η during inflation period. Inflation ends when the slow-roll conditions are violated.

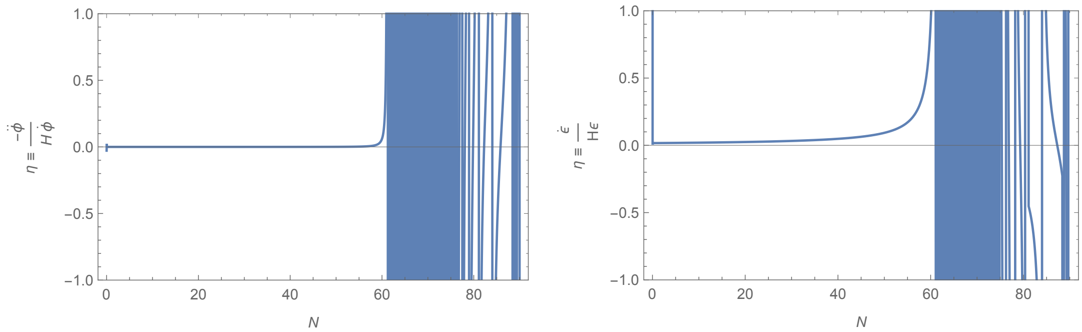

Figure 3.4: *Single scalar field in chaotic inflation model:* The evolution of different definitions of the second slow-roll parameter $\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\partial_{NN}\phi + (\partial_N\phi)\left(\frac{\partial_N\phi}{H}\right)}{\partial_N\phi}$ and $\tilde{\eta} \equiv \frac{1}{H}\frac{\dot{\epsilon}}{\epsilon} = \frac{-\partial_{NN}H + \frac{(\partial_N H)^2}{H}}{-\partial_N H}$ during inflation period.

3.4.2 ♦ The vector field model in simple form

Following the pioneer paper on building vector field model [77], we choose the Lagrangian for vector field in cosmic inflation model as:[7]

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(\xi)\ , \tag{3.22}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\xi = A_\alpha A^\alpha$ and V is a “potential”.

Motivated by the flexibility of chaotic inflation model, we started with the potential:

$$V(\xi) = \frac{1}{2}m^2\xi + V_0\ , \tag{3.23}$$

[7] The behavior of the system remains the same if we multiply its Lagrangian by a scalar. Nowadays, we usually see such Lagrangian instead written as: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\xi)$.

which gives:

$$\frac{\partial V}{\partial \xi} = \frac{1}{2} m^2 \, . \tag{3.24}$$

The energy momentum tensor is:

$$T_{\mu\nu} = F_{\mu\beta} F^{\beta}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - g_{\mu\nu} V + 2(\partial_\xi V) A_\mu A_\nu \, . \tag{3.25}$$

From this energy momentum tensor and assuming a perfect fluid case, we obtain the energy density and pressures for the vector field:

$$\begin{aligned} \rho = T^0_0 &= \frac{1}{2} \frac{\dot{A_z}^2}{b^2} + V \, , \\ p_z = T^3_3 &= -\rho + 2\xi(\partial_\xi V) \, , \\ p_x = p_y = T^1_1 = T^2_2 &= \frac{1}{2} \frac{\dot{A_z}^2}{b^2} - V \, , \end{aligned} \tag{3.26}$$

where

$$\xi = \frac{{A_z}^2}{b^2} \, . \tag{3.27}$$

Because the stress tensor need not be isotropic, the spacetime will not be the usual FLRW metric. For our purpose, we assume the Bianchi type-I metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2) + b^2(t)\mathrm{d}z^2 \, . \tag{3.28}$$

The Christoffel symbols, Ricci tensor $R_{\mu\nu}$ and Ricci scalar are routinely calculated from Bianchi type-I metric and give us following Einstein equations:

$$2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}^2}{a^2} = \rho \, , \tag{3.29}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -p_z \, . \tag{3.30}$$

The equation of motion for A_μ:

$$\Box A_\mu - \nabla_\nu(\nabla_\mu A^\nu) - 2(\partial_\xi V) A_\mu = 0 \, . \tag{3.31}$$

We are interested in homogeneous solutions, so $A_\mu = A_\mu(t)$. Therefore, equation for the only nonzero component A_z becomes:

$$\ddot{A}_z + \left[2\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right]\dot{A}_z + 2(\partial_\xi V)A_z = 0 \,. \tag{3.32}$$

In order to obtain the evolution of the vector field A_z and the Hubble parameters H_a and H_b, we need to solve the following system of coupled non-linear partial differential equations:[8]

$$2\frac{\dot{a}}{a}\frac{\dot{b}}{b} + \frac{\dot{a}^2}{a^2} - \left[\frac{1}{2}\frac{\dot{A}_z^{\,2}}{b^2} + V\right] = 0 \,, \tag{3.33}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \left[-\frac{1}{2}\frac{\dot{A}_z^{\,2}}{b^2} - V + 2\xi(\partial_\xi V)\right] = 0 \,, \tag{3.34}$$

$$\ddot{A}_z + \left[2\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right]\dot{A}_z + 2(\partial_\xi V)A_z = 0 \,. \tag{3.35}$$

Because $a(t) \neq b(t)$ in Bianchi-type I metric, we now have two different Hubble parameters $H_a \neq H_b$. Although the e-fold variables derived from H_a, H_b or their combinations such as $(2H_a + H_b)/3$ are equivalent, for facilitation in computation process, we pick the e-fold variable as N_b which corresponds to $\mathrm{d}N_b = H_b \mathrm{d}t$,. This gives us:

$$b = e^{N_b} \,, \tag{3.36}$$

$$\dot{A}_z = H_b(\partial_{N_b} A_z) \,, \tag{3.37}$$

$$\ddot{A}_z = H_b^{\,2}(\partial_{N_b N_b} A_z) + (\partial_{N_b} A_z) H_b (\partial_{N_b} H_b) \,. \tag{3.38}$$

Using these relations for e-fold variable, our system of non-linear coupled partial differential equations is:

$$2H_a H_b + H_a^{\,2} - \left[\frac{1}{2}\left(\frac{H_b}{b}\right)^2 (\partial_{N_b} A_z)^2 + V\right] = 0 \,, \tag{3.39}$$

$$2H_b(\partial_{N_b} H_a) + 3H_a^{\,2} + \left[-\frac{1}{2}\left(\frac{H_b}{b}\right)^2 (\partial_{N_b} A_z)^2 - V + 2\left(\frac{A_z}{b}\right)^2 (\partial_\xi V)\right] = 0 \,, \tag{3.40}$$

$$\partial_{N_b N_b} A_z + \frac{\partial_{N_b} H_b}{H_b}(\partial_{N_b} A_z) + \frac{2H_a - H_b}{H_b}(\partial_{N_b} A_z) + 2\frac{A_z}{H_b^{\,2}}(\partial_\xi V) = 0 \,. \tag{3.41}$$

[8] They are the equation of motion for A_z and the Einstein equations.

The initial value of the vector field $(A_z)_0$ can be adjusted to obtain the required 60 e-folds duration of the inflation period. Also, by observing the late time behavior of H_a and H_b that they converge to the same asymptotic value[9] of $\sqrt{V_0/3}$, we can attempt to specify initial value of H_b to be $(H_b)_0 = (1 - \frac{1}{\beta})\sqrt{V_0/3}$, where the value of $\beta = 10$ is chosen to influences the behavior of the slow-roll parameters at the beginning of inflation. Given the value of $(A_z)_0$ and $(H_b)_0$, we can specify the initial values of $(\partial_{N_b} A_z)_0$ and $(H_a)_0$. At the beginning of inflation, the slow-roll conditions are in effect, we have:

$$\varepsilon_b = -\frac{\dot{H}_b}{{H_b}^2} = -\frac{\partial_{N_b} H_b}{H_b} < 1\,, \tag{3.42}$$

$$\eta = -\frac{\ddot{A}_z}{H_b \dot{A}_z} = -\frac{\partial_{N_b N_b} A_z + (\partial_{N_b} A_z)\left(\frac{\partial_{N_b} H_b}{H_b}\right)}{\partial_{N_b} A_z} < 1\,. \tag{3.43}$$

Therefore, we can *selectively* obtain the initial *real* values of $(\partial_{N_b} A_z)_0$ and $(H_a)_0$ by solving the following algebraic system of equations:[10]

$$2(H_a)_0(H_b)_0 + {(H_a)_0}^2 - \left[\frac{1}{2}\left(\frac{(H_b)_0}{1}\right)^2 {(\partial_{N_b} A_z)_0}^2 + (V)_0\right] = 0\,, \tag{3.44}$$

$$\frac{2(H_a)_0 - (H_b)_0}{(H_b)_0}(\partial_{N_b} A_z)_0 + 2\frac{(A_z)_0}{{(H_b)_0}^2}(\partial_\xi V)_0 = 0\,. \tag{3.45}$$

Similar to the single scalar field model, the value of V_0 and m are free parameters. They can be determined when we know the energy scale of the inflation period. In the range of GUT scale, we picked [11] their values in reduced Planck mass framework to be $V_0 = 1 \times 10^{-15}$ and $m = 4.5 \times 10^{-9}$.

The calculated results below show behaviors of the vector field in chaotic inflation model. The graphs show that the model can provide stable evolution for the required amount of 60 e-folds duration. We observe that the vector component A_z and "clock" parameter ξ are suppressed quickly due to the exponential term e^{N_b} in the denominator that represents b. We also observe the confirmation of Wald's cosmic no hair theorem [75] as the model settles down to

[9] This is the consequence of Wald's cosmic no hair theorem [75] as the vector field $A_z \to 0$.

[10] Note that we have to select the *positive* branch of the Hubble parameter H_a due to the appearance of the ${H_a}^2$ term in our equations.

[11] The value of V_0 needs to be nonzero if we use e-fold variable due to the diverged behavior at zero value of the Hubble parameter H_b. We have also verified this behavior by re-solving this same system of equations in time variable when $V_0 = 0$.

isotropic expansion $H_a = H_b$ when $A_z \to 0$.

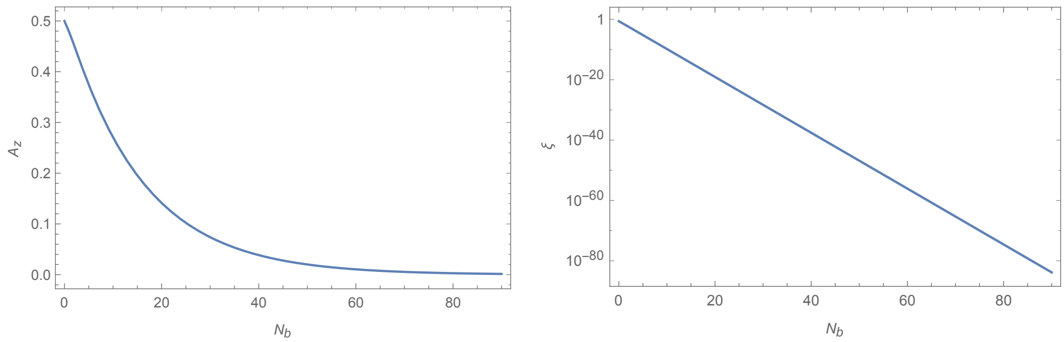

Figure 3.5: *Vector field in chaotic inflation model:* The evolution of the vector field A_z and "clock" parameter ξ during inflation period. The vector field A_z does not reach the minimum of the potential in finite time while ξ decreases exponentially but does not reach zero value.

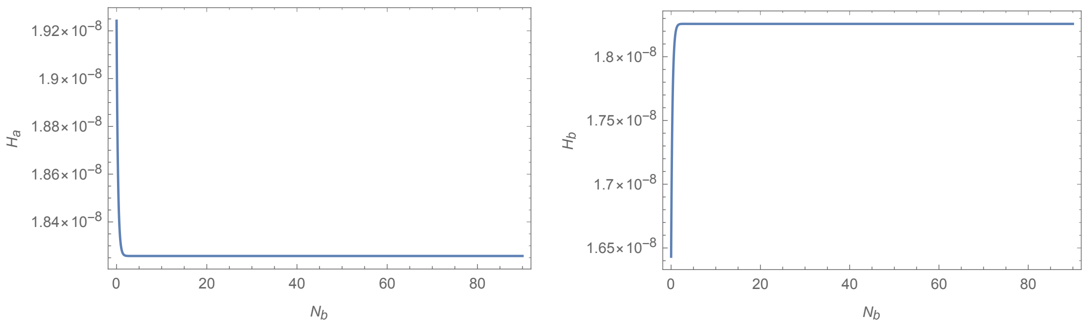

Figure 3.6: *Vector field in chaotic inflation model:* The evolution of the Hubble parameters H_a and H_b during inflation period. We also observe the confirmation of Wald's cosmic no hair theorem [75] as the model settles down to isotropic expansion $H_a = H_b$ when $A_z \to 0$.

The vector field A_z does not reach the minimum of the potential in finite time while ξ decreases exponentially but does not reach zero value. Therefore, inflation does not end and the slow-roll conditions are maintained as shown.

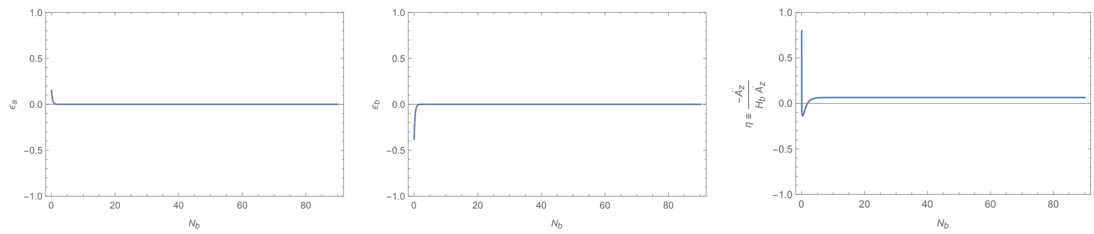

Figure 3.7: *Vector field in chaotic inflation model:* The evolution of the slow-roll parameters ε_a, ε_b and η during inflation period. The slow-roll conditions are maintained and show that inflation does not end.

3.4.3 ♦ The stable vector model for complete inflation period

We have successfully obtained slow-roll inflation for vector field in simple chaotic model which provide the required amount of 60 e-folds duration. We are going to investigate further by introducing minimal additional modifications at the end of inflation period to the original chaotic potential.[12] We look for the property of the potential that allows the same chaotic behavior as before when the vector field is large but provide a sharp decrease near 60 e-folds when the vector field becomes small to end the period of inflation. There are of course various kinds of potential that satisfy our requirements. We only provide below two candidates where the mass squared term is keep positive definite for the whole inflation period. As the discrepancy in behavior is negligible, the graphs below for the two different modified potentials are representative for *both* cases.

First candidate for vector field potential

$$V_1(\xi) = \frac{1}{2}m^2\xi + V_0 \exp\left(\frac{-\kappa}{\xi}\right) , \tag{3.46}$$

which gives:

$$\frac{\partial V_1}{\partial \xi} = \frac{1}{2}m^2 + V_0 \frac{\kappa}{\xi^2} \exp\left(\frac{-\kappa}{\xi}\right) . \tag{3.47}$$

[12] We also briefly investigated another possibility to obtain graceful exit scenario by using the combination of both vector field and the inflaton ϕ cascaded like a waterfall.

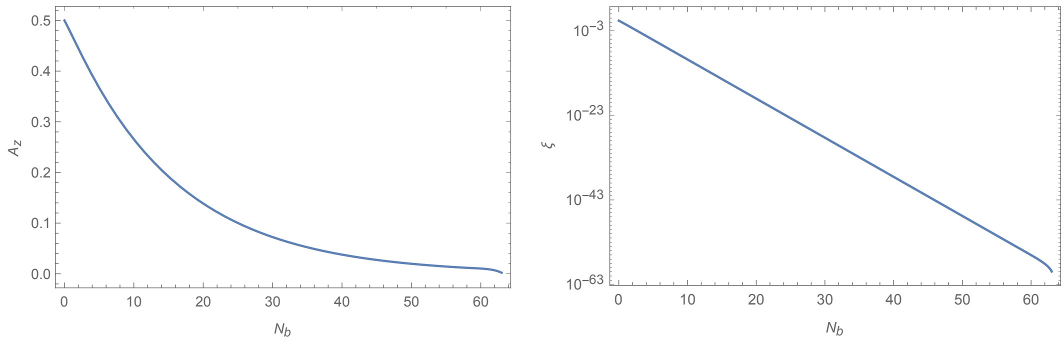

Figure 3.8: *Vector field in modified chaotic inflation model:* The evolution of the vector field A_z and "clock" parameter ξ during inflation period. We see now that ξ can reach zero value in finite time.

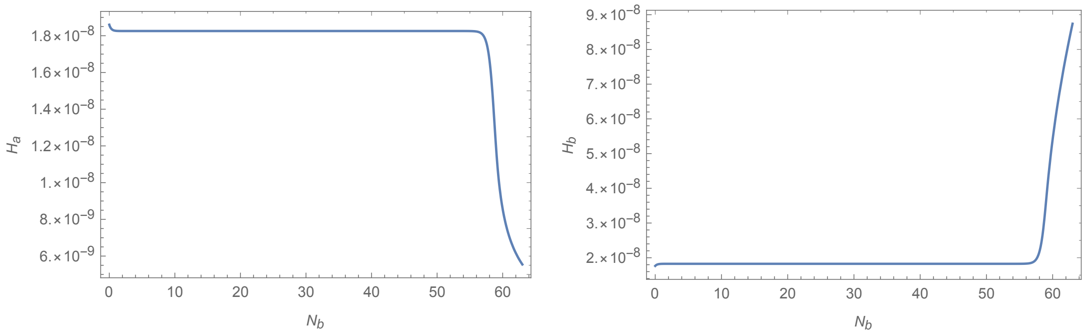

Figure 3.9: *Vector field in modified chaotic inflation model:* The evolution of the Hubble parameters H_a and H_b during inflation period. As in the case of the unmodified potential, the behavior of H_a and H_b after a few e-folds converged. However, after inflation exits, they depart into two different directions. This interesting phenomenon can be seen to have effect on the short scale power spectrum and links to the bound on the amounts of primordial black holes.

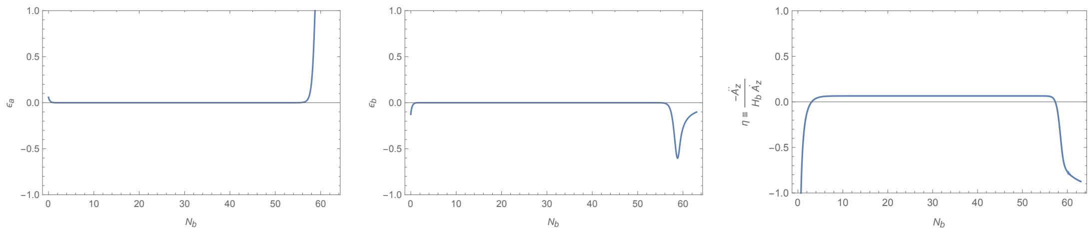

Figure 3.10: *Vector field in modified chaotic inflation model:* The evolution of slow-roll parameters ε_a, ε_b and η during inflation period. The inflationary duration proceeds to end as expected. This is clearly shown above by the violation of the slow-roll parameters

Second candidate for vector field potential

$$V_2(\xi) = \frac{1}{2}m^2\xi + V_0\left[1 - \frac{\kappa}{\kappa+\xi}\right], \tag{3.48}$$

which gives:

$$\frac{\partial V_2}{\partial \xi} = \frac{1}{2}m^2 + \frac{V_0 \kappa}{(\kappa + \xi)^2} \,. \tag{3.49}$$

We observe that the inflationary duration proceeds to end as expected. This is clearly shown above by the violation of the slow-roll parameters. Note also that, our choice of parameters and initial conditions are rather general because of the flexibility in chaotic inflation potential model. We observe again as in the case of the unmodified potential, the behavior of H_a and H_b after a few e-folds converged. However, after inflation exits, they depart into two different directions. This interesting phenomenon can be seen to have effect on the short scale power spectrum and links to the bound on the amounts of primordial black holes.

3.5 ▪ STABILITY ANALYSIS

This section will show that our system's stability is ensured by the constraints $\frac{\partial V}{\partial \xi} > 0$ and $V(\xi) > 0$. We show how this same conditions manifested using different approaches.

3.5.1 ♦ The Hamiltonian approach

First, we consider the following Lagrangian:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(A^2) \,. \tag{3.50}$$

The Euler-Lagrange equation is:

$$\frac{\partial L}{\partial A_\mu} - \partial_\lambda \left(\frac{\partial L}{\partial(\partial_\lambda A_\mu)} \right) = 0 \,, \tag{3.51}$$

which gives us the equation of motion:

$$-2\frac{\partial V}{\partial A^2}A^\mu + \partial_\lambda F^{\lambda\mu} = 0 \,. \tag{3.52}$$

We obtain the 0-component of the vector field A^0 from:

$$-2\frac{\partial V}{\partial A^2}A^0 + \partial_\lambda F^{\lambda 0} = 0 \,, \tag{3.53}$$

which gives us:

$$A^0 = -\frac{\nabla \cdot \boldsymbol{E}}{2\left(\frac{\partial V}{\partial A^2}\right)} \,. \tag{3.54}$$

Note that, A^0 is non-dynamical because $\Pi^{00} = 0$.
The momentum density is calculated as:

$$\Pi^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\nu)} = \partial^\nu A^\mu - \partial^\mu A^\nu = -F^{\mu\nu} \Rightarrow \Pi^{00} = 0, \Pi^{0i} = -E^i \ . \tag{3.55}$$

Finally, we obtain the Hamiltonian density:

$$H = \Pi^{0\mu}\partial_0 A_\mu - L = \Pi^{0i}\partial_0 A_i - \left(\frac{1}{2}(E^2 - B^2) - V(A^2)\right) \tag{3.56}$$

$$= -\boldsymbol{E}\cdot\dot{\boldsymbol{A}} + \frac{1}{2}(B^2 - E^2) + V(-(A^0)^2 + \boldsymbol{A}^2) \ . \tag{3.57}$$

We simplify further the above equation by using:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A^0 \Rightarrow \dot{\boldsymbol{A}} = -\boldsymbol{E} - \nabla A^0 \ , \tag{3.58}$$

and

$$\boldsymbol{E}\cdot\nabla A^0 = \nabla(\boldsymbol{E}A^0) - A^0(\nabla\cdot\boldsymbol{E}) = -A^0(\nabla\cdot\boldsymbol{E}) \ . \tag{3.59}$$

We have:

$$-\boldsymbol{E}\cdot\dot{\boldsymbol{A}} = -\boldsymbol{E}\cdot(-\boldsymbol{E} - \nabla A^0) = E^2 - A^0(\nabla\cdot\boldsymbol{E}) \tag{3.60}$$

$$= E^2 + \frac{(\nabla\cdot\boldsymbol{E})^2}{2\left(\frac{\partial V}{\partial A^2}\right)} \ . \tag{3.61}$$

In simple case when $A^0 = 0$, we obtain the Hamiltonian density as:

$$H = \frac{1}{2}(E^2 + B^2) + V(\boldsymbol{A}^2) \ , \tag{3.62}$$

which is bounded from below when $V(A^2) = V(\boldsymbol{A}^2) > 0$.
In general case, when $A^0 \neq 0$, we obtain the Hamiltonian density as:

$$H = \frac{1}{2}(E^2 + B^2) + V\left(-\left(-\frac{\nabla\cdot\boldsymbol{E}}{2\left(\frac{\partial V}{\partial A^2}\right)}\right)^2 + \boldsymbol{A}^2\right) + \frac{(\nabla\cdot\boldsymbol{E})^2}{2\left(\frac{\partial V}{\partial A^2}\right)} \ , \tag{3.63}$$

which is bounded from below when $\frac{\partial V}{\partial A^2} > 0$ and $V(A^2) > 0$.

Here we take an example, using the following potential:

$$V(A^2) = \frac{1}{2}m^2 A^2 \Rightarrow 2\left(\frac{\partial V}{\partial A^2}\right) = m^2 \,. \tag{3.64}$$

We have:

$$\begin{aligned} H &= \frac{1}{2}(E^2 + B^2) + \frac{1}{2}m^2\left(-\left(-\frac{\nabla\cdot\boldsymbol{E}}{m^2}\right)^2 + \boldsymbol{A}^2\right) + \frac{(\nabla\cdot\boldsymbol{E})^2}{m^2} \\ &= \frac{1}{2}(E^2 + B^2) + \frac{1}{2}m^2\boldsymbol{A}^2 + \frac{1}{2m^2}(\nabla\cdot\boldsymbol{E})^2 \,, \end{aligned} \tag{3.65}$$

which is bounded from below when $m^2 > 0$.

We apply this formalism to the following more general potential:

$$V(A^2) = \lambda A^{2n} \Rightarrow 2\left(\frac{\partial V}{\partial A^2}\right) = 2\lambda n A^{2(n-1)} \,, \tag{3.66}$$

We have:

$$\begin{aligned} H &= \frac{1}{2}(E^2 + B^2) + \lambda\left(-\left(-\frac{\nabla\cdot\boldsymbol{E}}{2\lambda n A^{2(n-1)}}\right)^2 + \boldsymbol{A}^2\right)^n + \frac{(\nabla\cdot\boldsymbol{E})^2}{2\lambda n A^{2(n-1)}} \\ &= \frac{1}{2}(E^2 + B^2) + \lambda\left(\frac{\nabla\cdot\boldsymbol{E}}{2\lambda n A^{2(n-1)}}\right)^{2n} - \lambda\begin{pmatrix} n \\ 1 \end{pmatrix}\left(A^2 + (A^o)^2\right)\left(\frac{\nabla\cdot\boldsymbol{E}}{2\lambda n A^{2(n-1)}}\right)^{2(n-1)} \\ &+ \ldots - \lambda\begin{pmatrix} n \\ n-1 \end{pmatrix}\left(A^2 + (A^o)^2\right)^{(n-1)}\left(\frac{\nabla\cdot\boldsymbol{E}}{2\lambda n A^{2(n-1)}}\right)^2 + \lambda\boldsymbol{A}^{2n} + \frac{(\nabla\cdot\boldsymbol{E})^2}{2\lambda n A^{2(n-1)}} \end{aligned} \tag{3.67}$$

Therefore, we can see that the conditions $\frac{\partial V}{\partial A^2} > 0$ and $V(A^2) > 0$ are used to ensure that the Hamiltonian is positive definite and free from ghost.

3.5.2 ♦ Other approaches

Although the following approaches to find ghost in a system are not given for the general case of $V(A^2)$ as above, they are still very illuminating. Hence we present them here closely to [40] in this section. We consider the action for the vector field in conformal time:

$$S = \int d\eta\, d^3x\left[-\frac{1}{4}F^2 - \frac{M^2}{2}A^2\right] . \tag{3.68}$$

The longitudinal vector (3.68) is a ghost in this model depends on the sign of the mass squared term. Following are three arguments to show that $M^2 < 0$ leads to a ghost. In all the discussion below, $M \equiv |M^2| > 0$.

The first approach is to using the propagator. The quadratic Lagrangian in (3.68) can be cast in the form $(1/2)A^\mu\, P^{-1}_{\mu\nu}\, A^\nu$, where

$$P_{\mu\nu} = -\frac{\eta_{\mu\nu} + \frac{k_\mu\, k_\nu}{M^2}}{k^2 + M^2} \ . \tag{3.69}$$

is the propagator. In general, the propagator needs to be diagonalized. However, we can choose a frame in which it is diagonal because the number and nature of physical modes does not depend on the frame. For a positive M^2, the pole is at $k^2 = -M^2 < 0$, and we can go in the rest frame, where $k^\mu = -k_\mu = (M,\, 0,\, 0,\, 0)$. In this case, $-\left(\eta_{\mu\nu} + k_\mu k_\nu / M^2\right) =$ diag $(0,\, -1,\, -1,\, -1)$, indicating that the theory has three well behaved physical particles (-1 indicates a well behaved mode due to the signature). For $M^2 < 0,$ we cannot go in the rest frame; however, we can choose a frame where the energy vanishes, $k^\mu = k_\mu = (0,\, 0,\, 0,\, M)$. In this case, $-\left(\eta_{\mu\nu} + k_\mu k_\nu / M^2\right) =$ diag $(1,\, -1,\, -1,\, 0)$, indicating that one mode (the longitudinal vector) is a ghost.

The second approach is to use the Stueckelberg formalism. For simplicity, here M is treated as a constant (the time dependence of M does not modify the quadratic kinetic term, but it complicates the diagonalization). If we redefine:

$$A_\mu = B_\mu + \frac{1}{M}\partial_\mu \phi \ . \tag{3.70}$$

We promote the action (3.68) to a gauge invariant action, with the symmetry:

$$\phi \to \phi + \xi \quad , \quad B_\mu \to B_\mu - \frac{1}{M}\partial_\mu \phi \ . \tag{3.71}$$

The action (3.68) is recovered in the unitary gauge, $\phi = 0$. But one can also choose a gauge in which B_μ is transverse, $\partial^\mu B_\mu = 0$. In this gauge, the action of the system is:

$$S = \int d\eta\, d^3x \left[-\frac{1}{4} F\,(B)^2 - \frac{M^2}{2}\, B^2 \mp \frac{1}{2}\,(\partial\phi)^2 \right] , \tag{3.72}$$

where the field strength $F_{\mu\nu}$ does not contain ϕ, and where the kinetic term of ϕ has opposite sign to M^2. We stress that the two actions (3.68) and (3.72) describe the same theory in two different gauges. We again see that the longitudinal component ϕ is a normal field for $M^2 > 0$, and a ghost for $M^2 < 0$.

The third approach to find the ghost is to decompose the vector as $A_\mu = \left(\alpha_0,\, \partial_i \alpha + \alpha_i\right)$, where $\partial_i \alpha_i = 0$. The action (3.68) then separates in two decoupled pieces:

$$S = \int d\eta d^3k \left\{ \frac{1}{2} \left[|\alpha_i'|^2 - \left(\boldsymbol{k}^2 + M^2 \right) |\alpha_i|^2 \right] \right.$$
$$\left. + \frac{1}{2} \left[\boldsymbol{k}^2 |\alpha'|^2 - \boldsymbol{k}^2 \left(\alpha'^* \alpha_0 + \text{h.c.} \right) - M^2 \boldsymbol{k}^2 |\alpha|^2 + \left(\boldsymbol{k}^2 + M^2 \right) |\alpha_0|^2 \right] \right\} \tag{3.73}$$

where prime denotes derivative with respect to conformal time, and in the second line we have Fourier transformed the modes. The action splits into two decoupled parts. The first part governs the two transverse polarizations, which are well behaved. The second part contains only one physical mode, since α_0 enters without time derivatives and must be integrated out. The equation of motion for this field obtained from (3.73) is $\alpha_0 = \left[\boldsymbol{k}^2 / \left(\boldsymbol{k}^2 + M^2 \right) \right] \alpha'$. Inserting this back into the second part of (3.73), we obtain:

$$S_{\text{longitudinal}} = \int d\eta\, d^3k \, \frac{\boldsymbol{k}^2 M^2}{2} \left[\frac{|\alpha'|^2}{\boldsymbol{k}^2 + M^2} - |\alpha|^2 \right] . \tag{3.74}$$

Again, we see that the longitudinal vector is well behaved if $M^2 > 0$, and a ghost if $M^2 < 0$.

Finally, as our choice of *positive* potentials satisfy the constraint $\frac{\partial V}{\partial \xi} > 0$ as:

$$\frac{\partial V_1}{\partial \xi} = \frac{1}{2} m^2 + V_0 \frac{\kappa}{\xi^2} \exp\left(\frac{-\kappa}{\xi} \right) > 0 \,, \tag{3.75}$$

and

$$\frac{\partial V_2}{\partial \xi} = \frac{1}{2} m^2 + \frac{V_0 \kappa}{(\kappa + \xi)^2} > 0 \,, \tag{3.76}$$

the systems are ensured to be free from ghost instability throughout more than 60-e folds slow-roll inflationary duration driven by a vector field.

"I have no special talents. I am only passionately curious."

Albert Einstein

4

Conclusion and Outlook

So far, the cosmic inflation paradigm remains the theorist's top choice to explain phenomena such as observed flat Universe, Gaussian and adiabatic perturbation, nearly scale-invariant spectrum of density and specific peaks in the CMB's spectrum. However, recent data obtained from COBE, WMAP and *Planck* have hinted at the possibility of anomalies in the CMB. These experimental findings have motivated the development of higher spin fields model such as vector fields, spinors and p-forms for cosmic inflation. Although being the close step upgraded from the scalar field, the vector field models for cosmic inflation are easily troubled with the instabilities associated with ghost in the system. We have shown the required constraint to keep the system ghost-free by keeping the mass squared term in the potential that keeps real and positive during the whole inflation period. Therefore, by carefully choosing appropriate potential, we have shown the realization of stable vector field model using chaotic inflation potential for slow-roll inflation throughout more than 60 e-folds. We also investigate and obtain different graceful exit scenarios. During this process, we also observe interesting phenomenon which can be seen to have effect on the short scale power spectrum and links to the bound on the amounts of primordial black holes. This suggests further development in the direction of this theoretical research as it suggests the revival of interest in the possibility of using simple vector field model for anisotropic cosmic inflation.

Furthermore, as another outlook, we also investigate the possibility of using a more complicated potential in order to directly influence behavior of the "clock" parameter ξ during

inflation period.
The idea is that we are going to use ξ directly as the "clock". Based on the following relation:

$$\log(\xi) = \log\left(A_\mu A^\mu\right) = \log\left(\frac{A_z}{e^{N_b}}\right)^2 , \tag{4.1}$$

We can construct the following potential:

$$V_3(\xi) = \lambda^2\left[\frac{1}{2}\log(\kappa+\xi) + N_{b_{\rm end}}\right]^2 + V_0 , \tag{4.2}$$

which gives:

$$\frac{\partial V_3}{\partial \xi} = \frac{\lambda^2\left[\frac{1}{2}\log(\kappa+\xi) + N_{b_{\rm end}}\right]}{\kappa+\xi} . \tag{4.3}$$

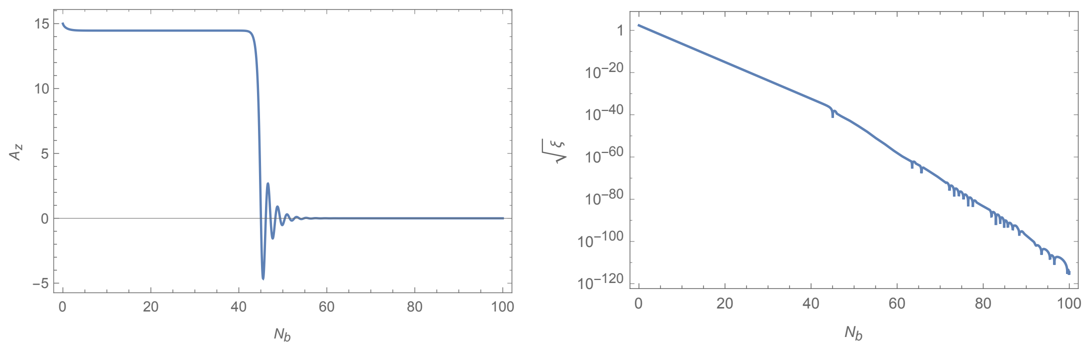

Figure 4.1: *Vector field in model where ξ is direct "clock":* The evolution of the vector field A_z and "clock" parameter ξ during inflation period.

Although that potential is rather complicated, the results appear to be very useful for further developments in vector field model building endeavor.

A

Primordial Spectra

The power spectrum of $\mathcal{R}$ (or ζ) is an important statistical measure of the primordial scalar fluctuations:[1]

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \, \delta(\mathbf{k} + \mathbf{k}') \, P_{\mathcal{R}}(k) \,, \qquad \Delta_s^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \,. \tag{A.1}$$

Here, $\langle \ldots \rangle$ defines the ensemble average of the fluctuations. The scale-dependence of the power spectrum is defined by the scalar spectral index (or tilt):

$$n_s - 1 \equiv \frac{d \ln \Delta_s^2}{d \ln k} \,, \tag{A.2}$$

where scale-invariance corresponds to the value $n_s = 1$. We may also define the running of the spectral index by:

$$\alpha_s \equiv \frac{d n_s}{d \ln k} \,. \tag{A.3}$$

The power spectrum is often approximated by a power law form:

$$\Delta_s^2(k) = A_s(k_\star) \left(\frac{k}{k_\star} \right)^{n_s(k_\star) - 1 + \frac{1}{2}\alpha_s(k_\star) \ln(k/k_\star)} \,, \tag{A.4}$$

[1]The normalization of the dimensionless power spectrum $\Delta_{\mathcal{R}}^2(k)$ is chosen such that the variance of $\mathcal{R}$ is $\langle \mathcal{R}\mathcal{R} \rangle = \int_0^\infty \Delta_{\mathcal{R}}^2(k) \, \mathrm{d} \ln k$.

where $k_\star$ is an arbitrary reference or pivot scale.

If $\mathcal{R}$ is Gaussian then the power spectrum contains all the statistical information. Primordial non-Gaussianity is encoded in higher-order correlation functions of $\mathcal{R}$. In single-field slow-roll inflation the non-Gaussianity is predicted to be small, but non-Gaussianity can be significant in multi-field models or in single-field models with non-trivial kinetic terms and/or violation of the slow-roll conditions. We restrict our computation to Gaussian fluctuations and the associated power spectra.

The power spectrum for the two polarization modes of h_{ij}, *i.e.* $h \equiv h^+, h^\times$, is defined as:

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3\, \delta(\mathbf{k} + \mathbf{k}')\, P_h(k)\,, \qquad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k)\,. \tag{A.5}$$

We define the power spectrum of tensor perturbations as the sum of the power spectra for the two polarizations:

$$\Delta_{\mathrm{t}}^2 \equiv 2\Delta_h^2\,. \tag{A.6}$$

Its scale-dependence is defined analogously to Eqn. (A.2) but for historical reasons without the -1:

$$n_{\mathrm{t}} \equiv \frac{d \ln \Delta_{\mathrm{t}}^2}{d \ln k}\,, \tag{A.7}$$

which gives

$$\Delta_{\mathrm{t}}^2(k) = A_{\mathrm{t}}(k_\star) \left(\frac{k}{k_\star} \right)^{n_{\mathrm{t}}(k_\star)}\,. \tag{A.8}$$

The results for power spectra of the scalar and tensor fluctuations created by inflation are:[2]

$$\Delta_{\mathrm{s}}^2(k) \equiv \Delta_{\mathcal{R}}^2(k) = \left. \frac{1}{8\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \frac{1}{\varepsilon} \right|_{k=aH}\,, \tag{A.9}$$

$$\Delta_{\mathrm{t}}^2(k) \equiv 2\Delta_h^2(k) = \left. \frac{2}{\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \right|_{k=aH}\,, \tag{A.10}$$

where

$$\varepsilon = -\frac{d \ln H}{dN}\,. \tag{A.11}$$

The horizon crossing condition $k = aH$ makes (A.9) and (A.10) functions of the comoving

[2] The additional factor of 2 in the tensor mode is due to the polarizations of the gravitational waves.

wavenumber k. The tensor-to-scalar ratio is:[3]

$$r \equiv \frac{\Delta_t^2}{\Delta_s^2} = 16\,\varepsilon_\star\,. \tag{A.12}$$

Scale-Dependence

The scale dependence of the spectra follows from the time-dependence of the Hubble parameter and is quantified by the spectral indices:

$$n_s - 1 \equiv \frac{d\ln\Delta_s^2}{d\ln k}\,, \qquad n_t \equiv \frac{d\ln\Delta_t^2}{d\ln k}\,. \tag{A.13}$$

We split this into two factors:

$$\frac{d\ln\Delta_s^2}{d\ln k} = \frac{d\ln\Delta_s^2}{dN} \times \frac{dN}{d\ln k}\,. \tag{A.14}$$

The derivative with respect to e-folds is:

$$\frac{d\ln\Delta_s^2}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN}\,. \tag{A.15}$$

The first term is just -2ε and the second term may be evaluated:

$$\frac{d\ln\varepsilon}{dN} = 2(\varepsilon - \eta)\,, \qquad \text{where} \quad \eta = -\frac{d\ln H_{,\phi}}{dN}\,. \tag{A.16}$$

The second factor in Eqn. (A.14) is evaluated by rewriting the horizon crossing condition:

$$\ln k = N + \ln H\,. \tag{A.17}$$

Hence,

$$\frac{dN}{d\ln k} = \left[\frac{d\ln k}{dN}\right]^{-1} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \approx 1 + \varepsilon\,. \tag{A.18}$$

We therefore find to first order in the Hubble slow-roll parameters:

$$n_s - 1 = 2\eta_\star - 4\varepsilon_\star\,. \tag{A.19}$$

[3] Here, $(\ldots)_\star$ indicates that a quantity is to be evaluated at horizon crossing.

Similarly, we find:

$$n_{\rm t} = -2\varepsilon_\star \,. \tag{A.20}$$

Any deviation from perfect scale-invariance ($n_{\rm s} = 1$ and $n_{\rm t} = 0$) is an indirect probe of the inflationary dynamics as quantified by the parameters ε and η.

Slow-Roll Results

In the slow-roll approximation the Hubble and potential slow-roll parameters are related as follows:

$$\varepsilon \approx \epsilon_{\rm v} \,, \qquad \eta \approx \eta_{\rm v} - \epsilon_{\rm v} \,. \tag{A.21}$$

The scalar and tensor spectra are then expressed purely in terms of $V(\phi)$ and $\epsilon_{\rm v}$ (or $V_{,\phi}$):

$$\Delta_{\rm s}^2(k) \approx \frac{1}{24\pi^2}\frac{V}{M_{\rm pl}^4}\frac{1}{\epsilon_{\rm v}}\bigg|_{k=aH} \,, \qquad \Delta_{\rm t}^2(k) \approx \frac{2}{3\pi^2}\frac{V}{M_{\rm pl}^4}\bigg|_{k=aH} \,. \tag{A.22}$$

The scalar spectral index is:

$$n_{\rm s} - 1 = 2\eta_{\rm v}^\star - 6\epsilon_{\rm v}^\star \,. \tag{A.23}$$

The tensor spectral index is:

$$n_{\rm t} = -2\epsilon_{\rm v}^\star \,, \tag{A.24}$$

and the tensor-to-scalar ratio is:

$$r = 16\epsilon_{\rm v}^\star \,. \tag{A.25}$$

We obtain an important consistency condition between the tensor-to-scalar ratio r and the tensor tilt $n_{\rm t}$ for single-field slow-roll models:

$$r = -8n_{\rm t} \,. \tag{A.26}$$

The Energy Scale of Inflation

Tensor fluctuations are often normalized relative to the (measured) amplitude of scalar fluctuations, $\Delta_{\rm s}^2 \equiv \Delta_{\mathcal{R}}^2 \sim 10^{-9}$. The *tensor-to-scalar ratio* is:

$$r \equiv \frac{\Delta_{\rm t}^2(k)}{\Delta_{\rm s}^2(k)} \,. \tag{A.27}$$

Since Δ_s^2 is fixed and $\Delta_t^2 \propto H^2 \approx V$, the tensor-to-scalar ratio is a direct measure of the energy scale of inflation

$$V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} 10^{16}\,\mathrm{GeV}\,. \tag{A.28}$$

Large values of the tensor-to-scalar ratio, $r \geq 0.01$, correspond to inflation occuring at GUT scale energies.

B

Evolution of Perturbations during Inflation

Scalar Perturbations

The equations of motion for scalar perturbations in flat gauge $\psi = 0$ is:

$$\ddot{\delta\varphi}_\psi + 3H\dot{\delta\varphi}_\psi + \left[\frac{k^2}{a^2} + V_{\varphi\varphi} - \frac{8\pi G}{a^3}\frac{d}{dt}\left(\frac{a^3\dot{\varphi}^2}{H}\right)\right]\delta\varphi_\psi = 0. \tag{B.1}$$

Now in order to further simplify the above equation, we can define the following variables:

$$v \equiv a\delta\varphi_\psi \qquad \text{and} \qquad z \equiv a\dot{\varphi}/H. \tag{B.2}$$

Using above variables, we can rewrite Eqn. (B.1) as:

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \tag{B.3}$$

where a prime denotes the derivative with respect to conformal time.

We rewrite the mass term in terms of the slow-roll variables as:

$$\frac{z''}{z} = (aH)^2\left[2 + 5\epsilon - 3\eta + 9\epsilon^2 - 7\epsilon\eta + \eta^2 + \xi^2\right], \tag{B.4}$$

where

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \quad \eta \equiv 2\epsilon - \frac{\dot{\epsilon}}{2H\epsilon}\,, \quad \xi^2 \equiv \left(2\epsilon - \frac{\dot{\eta}}{H\eta}\right)\eta. \tag{B.5}$$

We can neglect the time dependence of the slow-roll parameters:

$$\tau \simeq -\frac{1}{(1-\epsilon)aH}\,, \tag{B.6}$$

as well as:

$$\frac{z''}{z} = \frac{\nu_R^2 - (1/4)}{\tau^2}, \qquad \nu_R \simeq \frac{3}{2} + 3\epsilon - \eta. \tag{B.7}$$

Plugging these expressions back into Eqn. (B.3), we can easily find the solution of this differential equation as:

$$v \simeq \frac{\sqrt{\pi|\tau|}}{2} e^{i(1+2\nu_R)\pi/4}\left[c_1 H^{(1)}_{\nu_R}(k|\tau|) + c_2 H^{(2)}_{\nu_R}(k|\tau|)\right]. \tag{B.8}$$

There are still two free parameters that must be fixed in the above solution, *i.e.* c_1 and c_2. They can be fixed by assuming a preferred ansatz for the initial condition which we choose it to be the Baunch-Davis vacuum in the far past, $k\tau \to -\infty$:

$$v \to \frac{e^{-ik\tau}}{\sqrt{2k}}\,. \tag{B.9}$$

This leads to $c_1 = 1$ and $c_2 = 0$.

So, the power spectrum on the short scales, $k \gg aH$ is:

$$\begin{aligned}\mathcal{P}_{\delta\varphi} &\equiv \frac{4\pi k^3}{(2\pi)^3}\left|\frac{v}{a}\right|^2 \\ &= \left(\frac{k}{2\pi a}\right)^2.\end{aligned} \tag{B.10}$$

In addition, for the large scales $k \ll aH$, we have:

$$\mathcal{P}_{\delta\varphi} \simeq \left((1-\epsilon)\frac{\Gamma(\nu_R)}{\Gamma(3/2)}\frac{H}{2\pi}\right)^2\left(\frac{|k\tau|}{2}\right)^{3-2\nu_R}, \tag{B.11}$$

where we have used the following asymptotic behavior of the Hankel functions in the limit $k\tau \to 0$

$$H_\nu^{(1)}(k|\tau|) \to -(i/\pi)\Gamma(\nu)(k|\tau|/2)^{-\nu}. \tag{B.12}$$

For a massless scalar field, we would get the following power spectrum in the de Sitter Universe,

$$\mathcal{P}_{\delta\varphi} \to \left(\frac{H}{2\pi}\right)^2 \quad , \quad \frac{k}{aH} \to 0. \tag{B.13}$$

So far we were looking for the solution of Eqn. (B.3) at the slow-roll approximation. At later times, we need to use a large-scale limit which can be derived in terms of the comoving curvature perturbation, $\mathcal{R}$:

$$\frac{1}{a^3\epsilon}\frac{d}{dt}\left(a^3\epsilon\dot{\mathcal{R}}\right) + \frac{k^2}{a^2}\mathcal{R} = 0. \tag{B.14}$$

On large scale limit, we would have:

$$\mathcal{R} = C_1 + C_2 \int \frac{dt}{a^3\epsilon}. \tag{B.15}$$

where C_1 and C_2 are the constants of the integration. Since after the horizon exit the second term in Eqn. (B.15) would decay very fast, we are left with the constant term, C_1. By matching the power spectrum at the horizon crossing, we can find C_1 as:

$$C_1 = \frac{H^2}{\sqrt{2k^3}\dot{\varphi}}. \tag{B.16}$$

The power spectrum of the comoving curvature perturbation is:

$$\mathcal{P}_{\mathcal{R}} = \left(\frac{H}{\dot{\varphi}}\right)^2 \mathcal{P}_{\delta\varphi} \simeq \left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2_{k=aH}. \tag{B.17}$$

We can rewrite the above equation in terms of the potential as well as its first derivative as:

$$\mathcal{P}_{\mathcal{R}} = \left(\frac{128\pi}{3M_P^6}\frac{V^3}{V_\varphi^2}\right)_{k=aH}. \tag{B.18}$$

The amplitude of the comoving curvature perturbation is constant outside the horizon on very large scale. This means that its amplitude at the first and second horizon crossing which happen during the inflation and very late time, either during the radiation or matter epoch, respectively would be the same. The spectral indices for this model is:

$$\begin{aligned} n_s - 1 &= 3 - 2\nu_R \\ &= -6\epsilon + 2\eta. \end{aligned} \tag{B.19}$$

Since both of ϵ and η are quite small during inflation, we can conclude that the generated scalar perturbations during inflation are nearly scale invariant, $n_s \simeq 1$.

Tensor Perturbations

Similarly, we start from the main equation for the tensor mode, Eqn. (2.103), and rewrite it in terms of the new variable, $u \equiv ah/2\sqrt{8\pi G}$, as:

$$u'' + \left(k^2 - \frac{a''}{a}\right) u = 0. \tag{B.20}$$

Comparing this equation with that for the scalar field, we see that they are very similar except that z''/z has been replaced with a''/a:

$$\frac{a''}{a} = (aH)^2(2 - \epsilon). \tag{B.21}$$

Using the slow-roll approximation, we can simplify a''/a as:

$$\frac{a''}{a} \simeq \frac{\nu_{\mathrm{T}}^2 - (1/4)}{\tau^2} \quad , \quad \nu_T \simeq \frac{3}{2} + \epsilon. \tag{B.22}$$

We can now find the power-spectrum of the tensor mode on very large scale, $k \ll aH$, as:

$$\mathcal{P}_T \simeq \frac{64\pi}{M_P^2} \left((1-\epsilon) \frac{\Gamma(\nu_T)}{\Gamma(3/2)} \frac{H}{2\pi} \right)^2 \left(\frac{|k\tau|}{2} \right)^{3-2\nu_T}. \tag{B.23}$$

Quite similar to the scalar perturbations, we can also try to solve Eqn. (B.20) on large scales:

$$h = D_1 + D_2 \int \frac{dt}{a^3}. \tag{B.24}$$

So again one of the terms decays very quickly and we are left with the constant solution. Therefore, the power spectrum for the tensor mode is:

$$\mathcal{P}_T \simeq \frac{64\pi}{m_P^2}\left(\frac{H}{2\pi}\right)^2_{k=aH} \simeq \frac{128}{3}\left(\frac{V}{m_P^4}\right)_{k=aH}. \tag{B.25}$$

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