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Lie groups and spherical harmonics

Bachelor Thesis

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Abstract

In this thesis we introduce Lie groups and prove some important properties of them. Then we take a look at the general theory of representations of Lie groups. After that we take a look at irreducible representations and the decomposition of finite dimensional representations into irreducibles. This will allow us to prove an important theorem, the Peter-Weyl theorem, which states that the space $L^2(G)$ of square integrable functions on a compact Lie group G decomposes as a Hilbert direct sum of the linear span of matrix coefficients of irreducible representations. Then we will apply our knowledge of representations to find all irreducible representations of SO(3) using another important Lie group namely SU(2). Then, using Peter-Weyl, we show that any square integrable function on the two-sphere S^2 can be written in terms of spherical harmonics. Finally we apply this knowledge to solve the Schrödinger equation for an electron in the hydrogen atom.

Contents

1 Prerequisites

In this section we will discuss some important theorems and definitions about smooth manifolds. The proofs of these theorems can be found in [1].

1.1 Smooth manifolds

We'd like to generalize calculus to other mathematical spaces. However, we only know how to do calculus on $\mathbb{R}^n$. Since differentiation is a local property, we can solve this problem by limiting ourselves to spaces which locally look like $\mathbb{R}^n$, and for which we have a projection onto $\mathbb{R}^n$. Let's first forget about smoothness and look at what it means for a space to locally look like $\mathbb{R}^n$.

1.1.1 Topological manifolds

Definition 1.1.1 (Topological manifold)**.** A topological manifold M of dimension n is a Hausdorff, 2nd countable (ie. it admits a countable basis) space with the property that any point $x \in M$ has a neighborhood U which is homeomorphic to $\mathbb{R}^n$.

We note that any open set can be covered by open subsets homeomorphic to $\mathbb{R}^n$, so in the definition above we might as well require that any point has a neighborhood which is homeomorphic to some *subset* of $\mathbb{R}^n$.

We certainly hope that the dimension of the topological manifold is useful. The following theorem states this.

Theorem 1.1.2 (Topological Invariance of Dimension)**.** *The dimension of a topological manifold M is a topological invariant.*

Let's further look at the homeomorphism form the definition of a topological manifold.

Definition 1.1.3 (Coordinate chart)**.** A pair (U, ϕ), with $U \in M$ an open subset of a (topological) manifold M of dimension n and $\phi : U \to \widehat{U}$ a homeomorphism to an open $\widehat{U} = \phi(U) \subset \mathbb{R}^n$ is called a *coordinate chart* on M.
If $\phi(U)$ is a open ball, then $\phi(U)$ is called a *coordinate ball.* The components of ϕ are called *local coordinates.*
Given another (V, ψ) we have the composition $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$. This map is called *the change of coordinates form ϕ to ψ* or *the transition map from ϕ to ψ.*

Note that the transition map is a homeomorphism between two open subsets of $\mathbb{R}^n$. This is important for understanding Definition 1.1.9

One might wonder why we required the second-countability in the definition of a topological manifold. One reason for this is that every open cover of a second-countable space has a countable subcover, so M can be covered by countable many charts $\{(U_i, \phi_i)\}$. This is used to prove the following lemma.

Lemma 1.1.4. *Every topological manifold has a countable basis of precompact coordinate balls.*

Recall that a space X is called *locally path-connected* if it has a basis of path-connected open subsets. We then have the following corollary.

Corollary 1.1.5. *A topological manifold M is locally path-connected. Furthermore M is connected if and only if it is path-connected.*

The second statement follows from the fact that for a locally path-connected topological space, path-connectedness and connectedness are equivalent.

Recall that a space X is called *locally compact* if every point $x \in X$ has a compact neighborhood. Then Lemma 1.1.4 implies the following.

Corollary 1.1.6. *Every topological manifold is locally compact*

Local compactness and second-countability also imply *paracompactness*.

Definition 1.1.7 (Paracompactness)**.** Let M be a topological space and $\mathcal{X}$ a collection of subsets of M. $\mathcal{X}$ is called *locally finite* if for $x \in M$ there exists a neighborhood $U \subset M$ which intersects with at most finitely many elements of $\mathcal{X}$.
Let $\mathcal{U}, \mathcal{V}$ be two covers of M. Then $\mathcal{V}$ is called a *refinement of* $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $V \subset U$.
M is called *paracompact* if every open cover of M admits an open, locally finite refinement.

Theorem 1.1.8. *Every topological manifold is paracompact. Furthermore, for any open cover and for any basis of M one can create a locally finite refinement consisting of elements of the basis.*

1.1.2 Smooth atlas

Up to this point we have not discussed what it means for a manifold to be smooth. For this we have the following definition.

Definition 1.1.9 (Smoothly compatible charts)**.** Let M be a topological n-manifold. Given two charts (U, ϕ) and (V, ψ), we call these maps *smoothly compatible* if the transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism, ie. if it is a smooth bijective map, or if $U \cap V = \emptyset$.

Definition 1.1.10 (Atlas)**.** Let M be a topological n-manifold. An *atlas for M* is a collection of charts $\{(U_i, \phi_i)\}$, where $\{U_i\}$ is a cover of M.
This atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts are smoothly compatible.

We'd like to define a function $f : M \to \mathbb{R}$ to be smooth if and only if $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth for every chart (U, ϕ) in the atlas. Then we are interested in, for a given topological manifold M, which 'smooth structures' exists on M, ie. which functions f are smooth. For this we are not interested in a specific atlas. In general, however, there are more atlases which result in the same set of smooth functions. To resolve this issue, we limit ourselves to *maximal* atlases.

Definition 1.1.11 (Maximal atlas)**.** An smooth atlas $\mathcal{A}$ on a topological n-manifold M is called *maximal* if it is not properly contained in another atlas, ie. there exists no chart which both does not belong to $\mathcal{A}$ and is smoothly compatible with every chart in $\mathcal{A}$. Such an atlas is also called a *smooth structure on M*.

To see that this is a sensible definition we have the following proposition.

Proposition 1.1.12. *Let M be a topological manifold.*

- *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas.*

- *Two smooth atlases have the same maximal atlas if and only if their union is a smooth atlas*

1.1.3 Smooth manifold

We are now ready to define a smooth manifold

Definition 1.1.13 (Smooth manifold)**.** A *smooth manifold* is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

We call a chart (U, ϕ) from the maximal smooth atlas of a smooth manifold M a *smooth chart* or *smooth coordinate chart.* For all of the definitions given in Definition 1.1.3 we define a smooth version for when the chart (U, ϕ) is an element of the maximal smooth atlas. In fact, we also have a smooth version of Lemma 1.1.4.

Proposition 1.1.14. *Every smooth manifold has a countable basis of regular coordinate balls.*

Here we have used the following definition of a regular coordinate ball:

Definition 1.1.15. Let M be a manifold. A set $B \subset M$ is called a *regular coordinate ball* if it is a smooth coordinate ball and there exists a smooth coordinate ball $B' \supset \overline{B}$ together with a smooth coordinate map $\phi : B' \to \mathbb{R}^n$ such that $\phi(\overline{B}) = \overline{(}\phi B)$ and $\phi(B) \subsetneq \phi(B')$

1.2 Smooth maps

Definition 1.2.1 (Smooth maps)**.** Let M, N be smooth manifolds. We say that $F : M \to N$ a *smooth map* if for every $p \in M$ there exists smooth charts (U, ϕ) and (V, ψ) such that the coordinate representation $\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth in the ordinary sense.

When $N = \mathbb{R}^k$ for some $k \in \mathbb{N}$ and ψ is the identity map, we say F is a *smooth function.* Smooth maps satisfy the following properties.

Proposition 1.2.2. *Let M and N_i be smooth manifolds ,$1 \leq i \leq k$. Then*

- *The constant map $c : N_1 \to N_2$ is smooth.*
- *The identity map $I : M \to M$ is smooth.*
- *Suppose $F : N_1 \to N_2$ and $G : N_2 \to N_3$ are smooth, then $G \circ F : N_1 \to N_3$ is smooth*
- *Let $\pi_i : N_1 \times ... \times N_k \to N_i$ be the projection on M_i, then a map $F : M \to N_1 \times ... \times N_k$ is smooth if and only if for each i the map $F_i := \pi_i \circ F : M \to N_i$ is smooth.*

We say that two topological spaces are homeomorphic if there exists a homeomorphism between them, ie. a bijective continuous map with continuous inverse. This is also how we define a diffeomorphism.

Definition 1.2.3 (Diffeomorphisms)**.** Let M and N be smooth manifolds. A *diffeomorphism from M to N* is a smooth bijective map with a smooth inverse. We call M and N *diffeomorphic* if there exists a diffeomorphism between them.

1.3 Tangent vectors

Two important properties of a derivative are linearity and the product rule. Let's try to assume only those two properties and see what we can prove with it for manifolds.

Definition 1.3.1 (Derivation at a point and the tangent space)**.** Let M be a smooth manifold, $v : C^\infty(M) \to \mathbb{R}$ be a linear map and $p \in M$. Then v is called a *derivation at* p if

$$v(fg) = f(p)vg + g(p)vf \text{ for all } f, g \in C^\infty(M)$$

The set of all derivations at p is called the *tangent space to M at p* and denoted by T_pM. This is naturally a vector space, by linearity of v. An element $v \in T_pM$ is called a *tangent vector at p.*

Definition 1.3.2 (Differential)**.** Let M, N be smooth manifolds, $F : M \to N$ be a smooth map, $p \in M$. Then the map $dF_p : T_pM \to T_{F(p)}N$ defined by $dF_p(v)(f) = v(f \circ F)$ for $v \in T_p(M), f \in C^\infty(N)$ is called the *differential of F at p.*

We need to check that this is well defined, ie. that the image of the differential is indeed a derivation, so that it is linear and it satisfies the product rule. Use the definitions above and let $g \in C^\infty$. Then

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f) \end{aligned} \tag{1}$$

The differential satisfies some important properties, which are also satisfied by derivatives.

Proposition 1.3.3 (Properties of the differential)**.** *Let M, N, P be smooth manifolds, $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$. Then*

- *(linearity) $dF_p : T_pM \to T_{F(p)}N$ is linear*
- *(chain rule) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$*
- *(fixes the identity) $d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$*
- *(Inverse rule) Let F be a diffeomorphism, then this implies that $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism and the inverse satisfy $(dF_p)^{-1} = d(F^{-1})_{F(p)}$*

The following proposition is used to be able to identify T_pU with T_pM, where $U \subset M$.

Proposition 1.3.4. *Let M be a smooth manifold, $U \subset M$ an open subset, $\iota : U \hookrightarrow M$ the inclusion map and let $p \in U$. Then the differential $d\iota_p : T_pU \to T_pM$ is a linear isomorphism.*

When we take for example the gradient of a function, we see that this is a vector of the partial derivatives of the function. We have a similar result for the differential.

Proposition 1.3.5 (Tangent space to a product manifold)**.** *Let M_i be smooth manifolds, $1 \le i \le k$, $\pi_j : M_1 \times ... \times M_k \to M_j$ be the projection onto M_j and let $p = (p_1, ..., p_k) \in M_1 \times ... \times M_k$, then*

$$(d(\pi_1)_p(v), ..., d(\pi_k)_p(v)) : T_p(M_1 \times ... \times M_k) \to T_{p_1}M_1 \oplus ... \oplus T_{p_k}M_k$$

is an isomorphism.

1.3.1 Tangent bundle

So far we have only considered the tangent space at a specific point p. Especially for Lie groups it is useful to consider the set of all tangent spaces, where we keep track of the where the tangent space is calculated.

Definition 1.3.6 (Tangent bundle)**.** Let M be a smooth manifold. The *tangent bundle of M, TM*, is the disjoint union of tangent spaces at points of M:

$$TM := \coprod_{p \in M} T_pM$$

Elements of the tangent bundle are usually written as (p, v) where $p \in M$ and $v \in T_pM$.

This structure wouldn't be so useful if there was not an additional property of tangent bundles:

Proposition 1.3.7. *Let M be a manifold with tangent bundle TM, then TM has a natural topology and smooth structure, such that it is a 2n-dimensional smooth manifold, for which the projection $\pi : TM \to M$ is smooth.*

We can now construct the *global differential $dF : TM \to TN$* as a map whose restriction to each tangent space is $dF_p : T_pM \to T_pN$. If $F : M \to N$ is a smooth map, the global differential $dF : TM \to TN$ also turns out to be a smooth map.

Proposition 1.3.8. *If $F : M \to N$ is a smooth map, the global differential $dF : TM \to TN$ is also a smooth map.*

1.4 Immersions and submersions

First we generalise the idea of the rank of a linear map to smooth maps. We remark that the differential is a linear map. This results in the following definition.

Definition 1.4.1 (Rank)**.** Let $F : M \to N$ be a smooth map, M, N be smooth manifolds, $p \in M$. Then the *rank of F at p* is defined to be the rank of the differential $dF_p : T_pM \to T_{F(p)}N$. We denote the rank of F by rank F. We say that F has constant rank c if the rank at p equals c for all $p \in M$.

We use this definition to define submersions an immersions. We need these definitions to state a theorem about submanifolds. The following definition is also a proposition.

Definition 1.4.2 (Submersion and immersion)**.** Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. F is called a *smooth immersion* if rank $F = \dim M$, or equivalently if the differential is injective at every point of M.
F is called a *smooth submersion* if rank $F = \dim N$, or equivalently if the differential is surjective at every point of M.

We also need the definition of an embedding.

Definition 1.4.3 (Embedding)**.** Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. F is called a *smooth embedding of M into N* if it is a topological embedding and a smooth immersion.

It turns out that the rank also tells us something about the smoothness of maps.

Theorem 1.4.4 (Global rank theorem)**.** *Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. Suppose F has constant rank, that is its rank at every point $x \in M$ is equal to some constant $c \in \mathbb{N}$. Then*

- *If F is surjective, it is a smooth submersion.*
- *If F is injective, it is a smooth immersion.*

1.5 Submanifolds

A smooth manifold M is a set M which is a topological manifold together with a smooth structure. A submanifold should at least be a smooth manifold. Furthermore, there should be restrictions on the inclusion map.

Definition 1.5.1 (Embedded submanifold)**.** Let M be a smooth manifold, then a subset $S \subset M$ together with a smooth structure is called an *embedded submanifold* if it is a topological manifold in the subspace topology and the inclusion map $\iota : S \hookrightarrow M$ is a smooth embedding with respect to the smooth structure.

We now have two important characterizations of submanifolds. The first one says that, at least locally, a submanifold S is the level set of some submersion.

Proposition 1.5.2. *Let M be a smooth manifold of dimension n and $S \subset M$ a subset. Then S is an embedded submanifold of dimension k if and only if for each $p \in S$ there exists a neighborhood $U \subset M$ such that $U \cap S$ is a level set of a smooth submersion $\Phi : U \to \mathbb{R}^{n-k}$, ie. $U \cap S = \Phi^{-1}(c)$ where $c \in \mathbb{R}^{n-k}$*

The second has to do with immersions. It says that images of smooth immersions are submanifolds.

Proposition 1.5.3. *Let M, N be smooth manifolds, $F : N \to M$ an injective smooth embedding and $S = F(N)$. Then there exists a unique topology and smooth structure such that S is a submanifold of M and $F : N \to S$ is a diffeomorphism.*

1.6 Vector fields

When we study Lie groups, we want to define the exponential map and the Lie bracket. For this we need a notion of vector fields on manifolds. We think of vector fields as assigning a vector to each point in a space. This is exactly what we have done when we looked at the tangent bundle. What a vector field X should do is for each point p assign a tangent vector in that point, ie. an element $X(p) \in T_pM$.

Definition 1.6.1 (Vector field)**.** Let M be a smooth manifold and TM the corresponding vector bundle. A *vector field on M* is a continuous map $X : M \to TM$, $p \mapsto X_p := X(p)$ such that $X_p \in T_pM$ for each $p \in M$. A *smooth vector field* is a vector field X which as a map is smooth, with respect to the smooth structure on TM of Proposition 1.3.7

We denote by $\Gamma(TM)$ or by $\mathfrak{X}(M)$ the set of smooth vector fields on M. We can define addition and scalar multiplication on $\Gamma(TM)$ pointwise. Then $\Gamma(TM)$ becomes a linear space.

Proposition 1.6.2. *Let M be a smooth manifold, and $X : M \to TM$ be a vector field. Then X is smooth if and only if for every $f \in C^\infty(M)$ the function Xf is smooth.*

For a point $p \in M$, where M is a smooth manifold, we have constructed the tangent space to M at p as the set of all derivations at p. Derivations in this set are also called *tangent vectors at p*. Then we constructed the *tangent bundle* as the disjoint union of tangent spaces at points p in M. A *vector field* is then a map which for each point $p \in M$ assigns a tangent vector at p in the tangent bundle TM. To be able to compare the tangent bundle with derivations on the entire manifold we have extend Definition 1.3.1 of a *derivation at a point*.

Definition 1.6.3 (Derivation)**.** A *derivation* D on a smooth manifold M is a map $D : C^\infty \to C^\infty$ such that for each point $p \in M$ $D(\cdot)(p) : C^\infty \to \mathbb{R}$ is a derivation at p. The set of all derivations on M is denoted by $Der(C^\infty(M))$.

It would be nice, if similar as for points, the vector fields and the derivations would coincide. For this we have to create a derivation from a vector field.

Definition 1.6.4. Let $X \in \mathfrak{X}(M)$ for a smooth manifold M, then we define $d_X : C^\infty(M) \to C^\infty(M)$ as $(d_X f)(p) := X_p(f)$.

Note that this is well defined due to the smoothness of the vector field. Because $X_p(f)$ is by definition a derivation at p, d_X is a derivation. We will now show that every derivation is of this kind.

Proposition 1.6.5. *Let M be a smooth manifold. Then the map $\mathfrak{X}(M) \to Der(C^\infty(M), X \mapsto d_X$ is bijective.*

We are now able to identify $\mathfrak{X}(M)$ with $Der(C^\infty(M))$ and use this for the notation $Xf := d_X(f)$ for a smooth vector field X and a function $f \in C^\infty(M)$.

Suppose we have a smooth function $F : M \to N$, then given a vector field X on M we'd like to say something about vector fields on N. For this we have the following definition.

Definition 1.6.6 (F-related vector fields)**.** Let M, N be smooth manifolds and $F : M \to N$ be a smooth map. Let X be a vector field on M and Y a vector field on N. Then X and Y are F-related if for each $p \in M$ we have that $dF_p(X_p) = Y_{F(p)}$.

For diffeomorphisms $F : M \to N$ we expect to be able to relate vector fields on M with vector fields on N. The following proposition shows this.

Proposition 1.6.7. *Let M, N be smooth manifolds and $F : M \to N$ be a diffeomorphism. Then for every $X \in \Gamma(TM)$ there exists a unique smooth vector field $Y \in \Gamma(TN)$ such that Y is F-related to X. We call this vector field Y the* pushforward *of X* by *F and denote $Y = F_*X$.*

1.7 Integral curves

In the previous section we discussed vector fields. Now that we know how to define vector fields on manifolds and how to differentiate, we'd like to solve differential equations on manifolds. For this we need to have a notion of integral curves, ie. curves with velocities equal to a vector field.

Definition 1.7.1 (Curves and Integral curves)**.** A smooth *curve* is a smooth map $\gamma : I \to M$ from an interval $I \subset \mathbb{R}$ to a smooth manifold M.
An *integral curve of* V is a smooth curve γ for which at each point $\gamma(t)$ the velocity is equal to the vector field at that point, ie $\gamma'(t) = V_{\gamma(t)}$.
With the *velocity of* γ *at* t_0, $\gamma'(t_0)$ we mean the vector $\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}|_{t_0}\right) \in T_{\gamma(t_0)}M$, where $\frac{d}{dt}|_{t_0} \in T_{t_0}\mathbb{R}$.

In this thesis we will always mean a smooth curve when we say curve. We can apply the local existence and uniqueness theorem from the course on differential equations to local coordinates to conclude that for a given point $p \in M$ there exists a curve γ passing through p.

Proposition 1.7.2. *Let $V \in \mathfrak{X}(M)$ be a smooth vector field of a smooth manifold M and let $p \in M$. Then there exists an open neighborhood $I \subset \mathbb{R}$ of* 0 *and a smooth curve $\gamma : I \to M$ such that $\gamma(0) = p$.*

Suppose we are given a point $p \in M$ and an integral curve $\gamma : I \to M$ such that $\gamma(0) = p$. Then for each $t \in I$ we can get another point in M defined by following the integral curve for a time t, in other words 'going with the flow' for a time t. Now suppose every γ has domain $\mathbb{R}$. Then given a time $t \in \mathbb{R}$ and a point $p \in M$ we can get another point $\theta(t, p) \in M$.

Definition 1.7.3 (Smooth global flow)**.** Let M a smooth manifold. Then a *smooth global flow* θ is a smooth map $\theta : \mathbb{R} \times M \to M$ such that for all $t, s \in \mathbb{R}$ we have

$\theta(0, p) = p$ *(Flowing with time* 0 *results in the same point)*

$\theta(t, \theta(s, p)) = \theta(t + s, p)$ *(Adding times and applying the flow commutes)*

Furthermore we define the functions $\theta_t : M \to M$ and $\theta^{(p)} : \mathbb{R} \to M$ as $\theta_t(p) = \theta^{(p)}(t) = \theta(t, p)$.

It is clear that $\theta^{(p)}$ is a smooth curve, since it is the projection of the smooth map θ. One might wonder if it is an integral curve and if so, for which vector field.

Proposition 1.7.4. *Let M be a smooth manifold and let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow, then each curve $\theta^{(p)}$ is a integral curve with vector field defined (point wise) by $V_p := (\theta^{(p)})'(0)$. V is called the* infinitesimal generator *of θ.*

So given a global flow we can construct a smooth vector field V and a set of smooth integral curves of V. The converse is not always true, however it is true if the vector field is a left-invariant vector field on a Lie group. It is in general not true, because the flow can not always be defined for all of $\mathbb{R}$. To solve this issue, one could define a more general flow. We, however, will not do that here because we won't need it in our consideration of Lie groups.

2 Lie groups

In this section we introduce Lie groups, which are a both a group and a manifold. A closely related subject is that of the Lie Algebra and the Lie bracket. We will discuss these as well. Finally, there exists a map from the Lie Algebra of a Lie group to the Lie group itself, called the *exponential map*. This section is based on [1].

2.1 Lie brackets

An important method of creating new functions is by applying a derivation to it (see Proposition 1.6.5). If we composite two derivation in a special way, the result is again a derivation.

Lemma 2.1.1. *Let $X, Y \in \mathfrak{M}(M)$, and let d_X, d_Y be the corresponding derivations. Then $D := d_X d_Y - d_Y d_X \in Der(C^\infty)$.*

Proof. Firstly trivially $D : C^\infty \to C^\infty$. It is also clear that D is linear. We then only need to show Leibniz rule. Let $f, g \in C^\infty$.

$$\begin{aligned} D(fg) =& d_X d_Y - d_Y d_X(fg) = d_X d_Y(fg) - d_Y d_X(fg) \\ =& d_X(f d_Y(g) + g d_Y(f)) - d_Y(f d_X(g) + g d_X(f)) \\ =& f d_X(d_Y(g)) + d_Y(g) d_X(f) + g d_X(d_Y(f)) + d_Y(f) d_X(g) \\ &- f d_Y(d_X(g)) - d_X(g) d_Y(f) - g d_Y(d_X(f)) - d_X(f) d_Y(g) \\ =& f d_X(d_Y(g)) - f d_Y(d_X(g)) + g d_X(d_Y(f)) - g d_Y(d_X(f)) \\ =& f(D(g)) + g(D(f)) \end{aligned}$$

□

Then by Proposition 1.6.5) there exists a unique $V \in \mathfrak{X}(M)$ such that $D = d_V$.

Definition 2.1.2 (Lie bracket)**.** Let M be a smooth manifold, $X, Y \in \mathfrak{X}(M)$ vector fields on M. Then we define the *Lie bracket of X and Y* $[X, Y]$ by $[X, Y] = V$ for the unique vector field V such that $d_V = d_X d_Y - d_Y d_X$.

With the notation $Xf := d_X f$ introduced in Proposition 1.6.5) we can write down properties such as the bilinearity and the Jacobi-identity for Lie brackets.

Proposition 2.1.3 (Properties of the Lie bracket)**.** *The Lie bracket $[\cdot, \cdot]$ satisfies the following properties. For M a smooth manifold, $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$, $f, g \in C^\infty(M)$*

- (Bilinearity) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- (Antisymmetry) $[X, Y] = -[Y, X]$
- (Jacobi-identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

Proof. We note that because the tangent space is a vector space, the first property is well defined. Furthermore, since $[X, Y]$ is again a vector field for X, Y vector fields, the third property is also well defined. The rest follows from straightforward calculation. The first two are trivial. For the Jacobi-identity:

$$\begin{aligned} &[X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f = \\ &d_X d_Y d_Z f - d_X d_Z d_Y f - d_Y d_Z d_X f + d_Z d_Y d_X f + d_Y d_Z d_X f - d_Y d_X d_Z f \\ &- d_Z d_X d_Y + d_X d_Z d_Y f + d_Z d_X d_Y f - d_Z d_Y d_X f - d_X d_Y d_Z f + d_Y d_X d_Z f = 0 \end{aligned} \tag{2}$$

□

There is one more important property of Lie brackets. This will be important when we consider Lie algebra's.

Proposition 2.1.4 (Pushforwards of Lie brackets)**.** *Let M, N be a smooth map, let $F : M \to N$ be a diffeomorphism and let $X, Y \in \mathfrak{X}(M)$ then $F_*[X, Y] = [F_*X, F_*Y]$.*

Proof. By definition F_*X is F-related to X and F_*Y is F-related to Y. We need to show that $[F_*X, F_*Y]$ is F-related to $[X, Y]$, then by uniqueness and Proposition 1.6.7 we are done.

Claim 1. Let $X' \in \mathfrak{X}(M)$ and $Y' \in \mathfrak{X}(N)$ then X' and Y' are F-related if and only if for every $f \in C^\infty(U_f)$ where U_f is some open subset of N we have $X'(f \circ F) = (Y'f) \circ F$.

Proof. Let $p \in M$. Then for f such that U_f is a neighborhood of $F(p)$ we have

$$X'(f \circ F)(p) = X'_p(f \circ F) = dF_p(X'_p)f$$

by definition of a vector field and the differential. Furthermore, by a change of notation:

$$(Y'f) \circ F(p) = (Y'f)(F(p)) = Y'_{F(p)}f$$

The result follows. □

Let $X_* := F_*X$ and $Y_* := F_*Y$. Let f be as in the claim. Then, using the claim above twice.

$$XY(f \circ F) = X(Y(f \circ F)) = X((Y_*f) \circ F) = (X_*Y_*f) \circ F$$

And likewise

$$YX(f \circ F) = (Y_*X_*f) \circ F$$

So

$$\begin{aligned}
[X, Y](f \circ F) &= XY(f \circ F) - YX(f \circ F) \\
&= (X_*Y_*f) \circ F - (Y_*X_*f) \circ F \\
&= ([X_*, Y_*]f) \circ F
\end{aligned}$$

□

2.2 Lie algebra

Before we introduce the central topic of this section, we'd like to say something more about Lie brackets. We are going to define a *Lie algebra* as a map which satisfies the properties from Proposition 2.1.3. We have done this trick with derivations already. Here we know that a derivative is always linear and satisfies the Leibniz rule, and thus we defined a derivation as something linear which satisfies Leibniz rule.

Definition 2.2.1 (Lie algebra)**.** A *Lie algebra* (over $\mathbb{R}$) is a real vector space $\mathfrak{g}$ together with a *bracket* map $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$ which, for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{R}$, satisfies the properties:

- *(Bilinearity)* $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
- *(Antisymmetry)* $[X, Y] = -[Y, X]$
- *(Jacobi-identity)* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

A *Lie subalgebra* is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the bracket map β.

For Lie algebras we'd like to define something similar as homeomorphisms for groups.

Definition 2.2.2. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and let $A : \mathfrak{g} \to \mathfrak{h}$ be a linear map. A is called a *Lie algebra homeomorphism* if $A[X, Y] = [AX, AY]$ and a *Lie algebra isomorphism* if it is also invertible. If there exists a Lie algebra isomorphism between two Lie algebras $\mathfrak{g}, \mathfrak{h}$, then we say that $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic*.

The Lie algebra is in particular useful for a special kind of manifold, called the Lie group.

2.3 Lie groups

$\mathbb{R}^n$naturally has a group structure, namely addition and it is of course a smooth manifold. Also one could identify the space of invertible $n \times n$-matrices with $\mathbb{R}^{n^2}$, where addition, multiplication and taking the inverse is naturally defined. Lie groups are a generalisation of the idea of equiping $\mathbb{R}^n$with a group structure to smooth manifolds. To be able to prove useful theorems about these objects we require that the multiplication and the inverse map are smooth.

Definition 2.3.1 (Lie group)**.** A *Lie group G* is a smooth manifold G together with a group structure the multiplication map $\mu : G \times G \to G, \mu : (x, y) \mapsto xy$ and inverse map $\iota : G \to G, \iota : x \mapsto x^{-1}$ are smooth.

The two most important maps for Lie groups are *left translation* and *right translation.*

Definition 2.3.2 (Left and right translation)**.** Let G be a Lie group, then for any $g \in G$ we define *left translation* and *right translation* $L_g, R_g : G \to G$ by $L_g(h) := gh$ and $R_g(h) := hg$.

Both left and right translation are smooth and bijective, because the inverse is given by $L_g^{-1} = L_{g^{-1}}$ and $R_g^{-1} = R_{g^{-1}}$.
The most important aspect about Lie groups is that the inverse and multiplication maps are smooth. The following definition is then no surprise.

Definition 2.3.3 (Lie group homomorphism)**.** Let G, H be Lie groups, then $F : G \to H$ is called a *Lie group homomorphism from G to H* if it is smooth and a group homomorphism. If it is also a diffeomorphism, it is called a *Lie group isomorphism.* In that case we call G and H *Isomorphic Lie groups.*

Using this definition of a Lie group homomorphism, we can define a *Lie subgroup.*

Definition 2.3.4 (Lie subgroup)**.** Let G be a Lie group and $H \subset G$ be a subgroup. Then H is a *Lie subgroup* if it is equiped with a smooth structure such that the inclusion map $\iota : H \to G$ is a Lie group homomorphism.

It is important to note that the Lie subgroup does not need to have the subspace topology and smooth structure inherited from G. When we do require that, we call the subgroup an *embedded Lie subgroup.*

Definition 2.3.5 (Embedded Lie subgroup)**.** Let G be a Lie group and let $H \subset G$ be a subgroup and an embedded submanifold. Then H is called an *embedded Lie subgroup of G.*

The following theorem, which will be given without proof, gives many examples of Lie subgroups.

Theorem 2.3.6. *Let G be a Lie group. Then a subgroup $H \subset G$ is closed if and only if H is a embedded Lie subgroup.*

Corollary 2.3.7. *Let $F : G \to H$ be a Lie group homomorphism of G and H. Then* $\mathrm{Ker}(\phi)$ *is a Lie group of G.*

Proof. $\mathrm{Ker}(\phi) \subset G$ is a subgroup of G. In particular ϕ is continuous and $\mathrm{Ker} = \phi^{-1}(e_H)$ is the pre-image of a closed set. Therefore $\mathrm{Ker}(\phi)$ is a closed subgroup of G. □

Finally, we have that it is enough to require continuity for a bijective Lie group homomorphism.

Lemma 2.3.8. *Let $\phi : G \to H$ be a bijective Lie group homomorphism for Lie groups G and H. Then if ϕ has a continuous inverse, it is a diffeomorphism.*

Proof. We will show that ϕ has constant rank.Because ϕ is a homomorphism, for $g, g_0 \in G$

$$\phi(L_{g_0}(g)) = \phi(g_0 g) = \phi(g_0)\phi(g) = L_{\phi(g_0)}(\phi(g_0))$$

Now we take the differential and use the chain rule.

$$d\phi_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{\phi(g_0)})_{e_H} \circ dF_{e_G}$$

Since L_g is a diffeomorphism, we have that dF_{g_0} and dF_{e_G} have the same rank. This lemma now follows from the Global Rank Theorem (1.4.4). □

2.4 Lie algebra of a Lie group

Now that we know something about Lie groups, we can say a lot more about their Lie algebras. An important concept for this is a *left-invariant vector field*

Definition 2.4.1 (Left invariant)**.** Let G be a Lie group, $X \in \mathfrak{X}(G)$ is called *left-invariant* if it X is L_g-related to itself for every $g \in G$.

Since L_g is a diffeomorphism, X is the unique vector field L_g related to itself and $(L_g)_*(X) = X$ for every $g \in G$, by Proposition 1.6.7.

Proposition 2.4.2. *The set of smooth left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(M)$.*

Proof. We need to show that it is both a linear subspace and that it is closed under brackets.
Let X, Y be left-invariant vector fields and $a, b \in \mathbb{R}$. Then

$$(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y = aX + bY$$

We conclude that it is a linear subspace. With Proposition 2.1.4 we see that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$$

□

This allows us to do the following definition.

Definition 2.4.3 (Lie algebra of G)**.** Let G be a Lie group. The *Lie algebra of* G is the Lie algebra of smooth left-invariant vector fields on G, denoted by $\mathrm{Lie}(G)$.

Another example of a Lie algebra is the set of invertible matrices.

Example 2.4.4. We denote with $\mathrm{M}(n, \mathbb{R})$ the set of invertible $n \times n$ real matrices. Then this is a n^2-dimensional Lie algebra under the commutator bracket $[X, Y] = XY - YX$. We denote this Lie algebra of $\mathrm{M}(n, \mathbb{R})$ with $\mathfrak{gl}(n, \mathbb{R})$. △

Proof. The bilinearity and anti-symmetry are trivial. The Jacobi-identity follows from the same kind of calculations as in the proof of Proposition 2.1.3. □

Example 2.4.5. Let $End(V)$ be the set of linear endomophisms of a finite dimensional real vector space V. Then together with the commutator bracket $[X, Y] = X \circ Y - Y \circ X$ this is a Lie algebra. Here the addition and scalar multiplication of two linear endomophims X and Y is defined pointwise as $(aX + Y)(v) = aX(v) + Y(v)$ for $a \in \mathbb{R}$. The notation for this Lie algebra is $\mathfrak{gl}(V)$. △

Proof. We choose a basis for V and map these to the unit vectors in $\mathrm{M}(n, \mathbb{R})$. Then this map is a vector space isomorphism. The result follows. □

As it turns out, we can relate the entire Lie algebra to the tangent space at the identity element. Observe that for a left invariant vector field $X \in \mathrm{Lie}(G)$ we have for every $g, h \in G$, by definition of F-relatedness, that $d(L_g)_h(X_h) = X_{gh}$. First we introduce some notation. Let $v \in T_eG$ then we define a (not nessicarily smooth) vector field v^L on G as $v^L|_g = d(L_g)_e(v)$.

Theorem 2.4.6. *Let G be a Lie group, then the evaluation at the identity $\epsilon : \mathrm{Lie}(G) \to T_eG$ defined by $\epsilon(X) = X_e$ is a vector space isomorphism.*

Proof. By definition, ϵ is linear. To prove injectivity we note that the kernel is trivial: if $\epsilon(X) = X_e = 0$ for some vector field $X \in \mathrm{Lie}(G)$, then because of left-invariance we have, for any $g \in G$ that $X_g = d(L_g)_e(X_e) = 0$, so $X = 0$.
To see surjectivity, let $v \in T_eG$. Observe that for the value at the identity we have $v^L|_e = d(L_e)_e(v) = d(Id)_e(v) = v$. To prove surjectivity we need to show that this vector field is an element of $Lie(G)$, ie. that it is smooth and left-invariant.
First for smoothness, we note that $v^L|_g$ by definition is a derivation. We only need to check that D defined by $D(g) = v^L|_g$ is a map from C^∞ to C^∞ and we can apply Proposition 1.6.5 to construct a smooth vector field v^L. Now, pick a smooth curve $\gamma : (-\delta, \delta) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then for all $g \in G$

$$D(g) = v^L|_g f = d(L_g)_e(v)f = v(f \circ L_g) = \gamma'(0)(f \circ L_g) = \left.\frac{d}{dt}\right|_{t=0} (f \circ L_g \circ \gamma)(t)$$

Now, the composition of f, L_g and γ is a smooth function, therefore, $D(g) : C^\infty \to C^\infty$. We denote with v^L the corresponding smooth vector field.
We shall now prove left-invariance. First we remark that for $g, h \in G$ we have that $L_h \circ L_g = L_{hg}$. Therefore:

$$d(L_h)_g(v^L\big|_g) = d(L_h)_g \circ d(L_g)_e(v) = d(L_h \circ L_g)_e(v) = d(L_{hg})_e(v) = v^L\big|_{hg}$$

Then we see immediately that v^L is left-invariant and we have that ϵ is surjective. □

We use this theorem to identify $\mathrm{Lie}(G)$ with T_eG. Denote with $\mathfrak{gl}(n,\mathbb{R})$ the algebra of $n \times n$ dimensional real matrices.

Theorem 2.4.7 (Lie algebra of $\mathrm{GL}(V)$)**.** *Let V be a finite dimensional real vector space. Then there exists a Lie algebra isomorphism of* $\mathrm{Lie}(\mathrm{GL}(V))$ *to* $\mathfrak{gl}(V)$*.*

Proof. Since V is finite, we will prove the claim for $\mathbb{R}^n$ and the result follows using the isomorphism between $\mathrm{GL}(V)$ and $\mathrm{GL}(n,\mathbb{R})$ for suitable $n \in \mathfrak{n}$.
We are going to construct a Lie algebra isomorphism $\cdot^L$. Let $A \in \mathrm{GL}(n,\mathbb{R}) \subset \eth\lessdot(n,\mathbb{R})$. Denote with A^i_j the matrix coefficents of A. Then we have an ismorphism $\phi : \mathfrak{gl}(n,\mathbb{R}) \to T_{\mathrm{Id}}\mathrm{GL}(n,\mathbb{R})$ given by

$$\phi(A) = A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{\mathrm{Id}}$$

Now, let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$. Let A^L be the left-invariant vector field corresponding to A, determined by the previous theorem. Then using the properties of a global differential we see that

$$A^L\big|_X = d(L_X)_{\mathrm{Id}}(A) = d(L_X)_{\mathrm{Id}}\left(A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{\mathrm{Id}}\right)$$

Since L_x is just the linear map $A \mapsto XA$ restricted to $GL(n,\mathbb{R})$, then by linearity of L_X we have

$$A^L\big|_X = X^i_j A^j_k \frac{\partial}{\partial X^i_k}\big|_X$$

Let now also $B \in \mathfrak{gl}(n,\mathbb{R})$ then we have that

$$\begin{aligned}
[A^L, B^L] &= [X^i_j A^j_k \frac{\partial}{\partial X^i_k}, X^p_q B^q_r \frac{\partial}{\partial X^p_r}] = X^i_j A^j_k \frac{\partial}{\partial X^i_k}\left(X^p_q B^q_r\right)\frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial}{\partial X^p_r}\left(X^i_j A^j_k\right)\frac{\partial}{\partial X^i_k} \\
&= X^i_j A^j_k B^k_r \frac{\partial}{\partial X^i_r} - X^p_q B^q_r A^r_k \frac{\partial}{\partial X^p_k} = \left(X^i_j A^j_k B^k_r - X^i_j B^j_k A^k_r\right)\frac{\partial}{\partial X^i_r}
\end{aligned} \tag{3}$$

At Id this becomes

$$[A^L, B^L]_{\mathrm{Id}} = \left(A^i_k B^k_r - B^i_k A^k_r\right)\frac{\partial}{\partial X^i_r}\bigg|_{\mathrm{Id}}$$

Note that this is exactly the definition of left-invariant vector field of the commutator. Then the result follows, since the vector field is completely determined by its value at the identity. □

For complex finite dimensional vector spaces, we can the same theorem with the identification of $\mathbb{C}$ with $\mathbb{R}^2$. For finite dimensional vector spaces V, we can then use this theorem to identify $\mathrm{Lie}(\mathrm{GL}(V))$ with $\mathfrak{gl}(V)$.

Suppose we have a Lie group homomorphism between two Lie groups G and H. One might wonder what we can say about their Lie algebras.

Theorem 2.4.8. *Let G and H be Lie groups with Lie algebras* $\mathrm{Lie}(G)$ *and* $\mathrm{Lie}(H)$ *and let $F : G \to H$ be a Lie group homomorphism. Given a $X \in$* $\mathrm{Lie}(G)$ *there exists a unique vector field $Y \in$* $\mathrm{Lie}(G)$ *which is F-related to X. We denote $Y = F_*X$. The induced map $F_* :$* $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *is a Lie algebra homomorphism.*

Proof. First we note that if F is a diffeomorphism, there is nothing to prove.
We will try to construct this vector field Y. Define $Y' = (dF_e(X_e))^L$. F is a (group) homeomorphism so we have that $F(gg') = F(g)F(g')$ for $g, g' \in G$. By definition of left-translation we get that $F(L_g g') = L_{F(g)}F(g')$. This holds for all g', so we get $F \circ L_g = L_{F(g)} \circ F$. Then we apply the chain rule property of the global differential to conclude that

$$dF \circ d(L_g) = d(L_{F(g)}) \circ dF$$

Because of left-invariance of the vector field X we have that

$$dF(X_g) = dF(d(L_g)(X_e)) = d(L_{F(g)})(dF(X_e)) = d(L_{F(g)})(Y_e) = Y_{F(g)}$$

This is exactly the definition of F-relatedness. By definition $Y \in \mathrm{Lie}(H)$, see previous theorem. That F_* is a Lie algebra homomorphism follows from the fact that for F_* related fields it holds that $F_*[X,Y] = [F_*X, F_*Y]$. □

Proposition 2.4.9 (Properties of the induced homomorphism)**.** *Let G, H, K be Lie groups. Then the induced homomorphisms satisfy the following properties.*

- *(fixes the identity) The map $(\mathrm{Id}_G)_* : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ is the identity map on $\mathrm{Lie}(G)$.*
- *(Chain rule) Let $F_1 : G \to H$ and $F_2 : H \to K$ be Lie group homomorphisms. Then $(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_*$.*
- *Suppose $F : G \to H$ is a Lie group isomorphism, then $F_* : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie algebra isomorphism.*

Proof. The first two statements follow directly from the properties of the global differential. For the third property we show that F_* has an inverse. Using the second property:

$$F_* \circ (F^{-1})_* = (F \circ F^{-1})_* = Id = (F^{-1})_* \circ F_*$$

□

It so turns out that there is a deep relation between Lie groups and their algebras. This is also the case for the Lie algebra of Lie subgroups.

Theorem 2.4.10. *Let G be a Lie group, let $H \subset G$ be a Lie subgroup and let $\iota : H \hookrightarrow G$ be the inclusion map. Then the Lie subalgebra $\mathfrak{h} = \iota_*(\mathrm{Lie}(H)) \subset \mathrm{Lie}(G)$ is isomorphic to $\mathrm{Lie}(H)$.*

Proof. Because of Theorem 2.4.8, ι_* is a Lie group homomorphism. Therefore $\iota_*(\mathrm{Lie}(H))$ is closed under the bracket operator. It is also a linear subspace, so it is a Lie subalgebra of $\mathrm{Lie}(G)$. By definition of ι_* and because $d\iota_e : T_eH \to T_eG$ is trivially injective, ι_* is also injective. By definition, it is also surjective. We conclude that ι_* is a Lie algebra isomorphism between $\mathrm{Lie}(H)$ and $\mathfrak{h}$. □

2.5 Exponential map

Lets consider left-invariant maps in more detail. Theorem 2.4.6 asserts that a left-invariant vector field is uniquely defined by their value at the identity. We will also see that for a left-invariant vector field X there exists a unique flow θ such that X is the infinitesimal generator of θ (the converse of Proposition 1.7.4). Then given a tangent vector X, we can construct this flow and follow it for a time t. The result will be the exponential map $\exp(tX)$. This is actually a map from $\mathbb{R}$ to G. Let's consider that in the first place.

Definition 2.5.1 (One parameter subgroup)**.** Let G be a Lie group. A *one-parameter subgroup of* G is a smooth Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$.

We will denote the Lie group $(\mathbb{R}, +)$ from now on with just $\mathbb{R}$.

Proposition 2.5.2. *Let G be a Lie group. Then $\gamma : \mathbb{R} \to G$ is a one parameter subgroup if and only if γ is an integral curve of a left-invariant vector field with $\gamma(0) = e$.*

Proof. Suppose γ is a one parameter subgroup. Then because γ is a homomorphism, $\gamma(0) = e$. Let $X = \gamma_*(d/dt)$ where d/dt is the left-invariant vector field on $\mathbb{R}$.[1] Then by Theorem 2.4.8 $X \in Lie(G)$ and X is the unique vector field in $Lie(G)$ related to d/dt. Observe that the velocity $\gamma'(t_0)$ of a curve γ at an arbitrary $t_0 \in \mathbb{R}$ is defined as $\gamma'(t_0) = d\gamma(\frac{d}{dt}\big|_{t_0})$, which is $X_{\gamma(t_0)}$ by definition of γ-relatedness. We conclude that γ is an integral curve of $X \in \mathrm{Lie}(G)$.
Suppose now that γ is a maximal integral curve of a left-invariant vector field X with $\gamma(0) = e$. Since γ is already smooth we only need to show that γ is a homomorphism. Since X is a left-invariant vector field, the domain of γ is $\mathbb{R}$. We first show that $\sigma := L_g \circ \gamma : I \to G$ is an integral curve of X. For this we only need to show that $\sigma'(t) = X_{\sigma(t)}$, since it is a composition of two smooth maps, thus smooth. Using that X is L_g related to itself we have

$$\sigma'(t) = (L_g \circ \gamma)'(t) = d(L_g)_{\gamma(t)}(\gamma'(t)) = d(L_g)_{\gamma(t)}(X_{\gamma(t)}) = X_{L_g(\gamma(t)} = X_{\sigma(t)}$$

We apply this for $g = \gamma(s)$. Then we have an integral curve $t \mapsto L_{\gamma(s)}(\gamma(t)) = \gamma(s)\gamma(t)$ starting in $\gamma(s)$. We can get another integral curve $t \mapsto \gamma(s + t)$ starting at $\gamma(s)$ by doing a variable substitution. However, there is a unique maximal integral curve of X with the same start point, so it must hold that $\gamma(s)\gamma(t) = \gamma(s + t)$. □

This allows us to define the exponential map.

Definition 2.5.3 (Exponential map)**.** Let G be a Lie group with Lie algebra $\mathrm{Lie}(G)$. Let X and γ as in the previous proposition. Then we define the *exponential map* $\exp : \mathrm{Lie}(G) \to G$ by

$$\exp(X) = \gamma(1)$$

Proposition 2.5.4 (Properties of the exponential map)**.** *Let G be a Lie group. For $X \in \mathrm{Lie}(G)$ and $s, t \in \mathbb{R}$ the following holds.*

a) $\exp(sX) = \gamma(s)$

b) $\exp(s + t)X = \exp sX \exp tX$

c) $\exp : \mathrm{Lie}(G) \to G$ *is smooth*

d) *The differential $(d\exp)_0 : T_0Lie(G) \to T_eG$ is the identity map*

e) $\exp$ *restricts to a diffeomorphism of a neighborhood $V \subset \mathfrak{g}$ of 0 to a neighborhood $U \subset G$ of e*

f) *Let H be a Lie group with Lie algebra $\mathfrak{h}$ and let $\phi : G \to H$ be a Lie group homomorphism. Then $\exp(\phi_* X) = \phi(\exp X)$.*

[1] Here we used that d/dt is a derivation and the canonical identification of a derivation with a vector field

Proof. For a) define $\sigma(t) := \gamma(st)$ and let $\phi_s : \mathbb{R} \to \mathbb{R}, \phi_s(t) = st$. Then $\sigma(0) = \gamma(0) = e$ and $\sigma = \gamma \circ \phi$, so

$$\begin{aligned}\sigma'(t_0) &= d\sigma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) = d(\gamma \circ \phi_s)_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) = d\gamma_{\phi_s(t_0)} \circ d(\phi_s)_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right)\\ &= d\gamma_{st_0} \circ d(s\mathrm{Id})_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) = d\gamma_{st_0}\left(s\left.\frac{d}{dt}\right|_{t_0}\right) = sX_{\gamma(st_0)} = sX_{\sigma(t_0)} \in T_{\sigma(t_0)}G\end{aligned}$$

Therefore we see that σ is an integral curve of sX, so $\exp(sX) = \sigma(1) = \gamma(s)$
For b) we remark that $\gamma(s) = \exp(sX)$ is the one-parameter subgroup of G generated by X. Then we use that γ is a homomorphism of groups.
For c) let $X \in \mathrm{Lie}(G)$ and denote with θ_X the flow of X. Then we remark that $\exp(X) = \gamma(1) = \theta^e_X(1)$. Let $G \times \mathrm{Lie}(g)$ be the product manifold. Define the vector field Ξ on $G \times \mathrm{Lie}(g)$ as

$$\Xi_{(g,X)} = (X_g, 0) \in T_gG \oplus T_X\mathrm{Lie}(g) \simeq T_{(g,x)}(G \times \mathrm{Lie}(g))$$

Let X_i where $1 \leq i \leq n$ for some $n \in \mathbb{N}$ be a basis of $\mathrm{Lie}(g)$ and let x^i be local coordinates for $\mathrm{Lie}(g)$ and let w^i be smooth local coordinates for G. Then we define for $f \in C^\infty(G \times \mathrm{Lie}(g))$

$$\Xi f(w^i, x^i) = x^j X_j f(w^i, x^i)$$

Then from Proposition 1.6.2 it follows that Ξ is smooth. Then it follows that the flow $\Theta : \mathbb{R} \times G \times \mathrm{Lie}(G) \to G \times \mathrm{Lie}(G)$ given by

$$\Theta_t(g, X) = (\theta_X(t, g), X)$$

is smooth. It follows that exp is smooth. For d) let $X \in \mathrm{Lie}(G)$, and let $\sigma(t) = tX$ be a curve. Then trivially $\sigma'(0) = X$. Furthermore

$$(d\exp)_0(X) = (d\exp)_0(\sigma'(0)) = (\exp \circ \sigma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X$$

So indeed exp is the identity map. e) follows from d) and the inverse function theorem. For f) let $\sigma(t) = \exp(t\phi_* X)$ for $X \in \mathrm{Lie}(G)$. Then σ is a Lie group homomorphism, because it is the composition of two such homomorphisms. Furthermore

$$\sigma'(0) = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp(tX)) = d\phi_0\left(\left.\frac{d}{dt}\right|_{t=0} \exp tX\right) = d\phi_0(X_e) = (\phi_*)_e$$

Therefore the velocity at 0 satisfies the required condition for a one-parameter subgroup. The result follows. ◻

Then we ask what happens for subgroups.

Proposition 2.5.5 (Exponential map of a subgroup)**.** *Let G be a Lie group, $H \subset G$ be a Lie subgroup. Then using the identification of* $\mathrm{Lie}(H)$ *as a subalgebra of* $\mathrm{Lie}(G)$ *from Theorem 2.4.10, the exponential map* $\exp_H = \exp_G|_{\mathrm{Lie}(H)}$. *If furthermore H is an embedded subgroup we have* $\mathrm{Lie}(H) = \{X \in \mathrm{Lie}(G) | \exp(tX) \in H\ \forall t \in \mathbb{R}\}$

Proof. Let $\gamma : \mathbb{R} \to G$ be a one-parameter subgroup with $\gamma'(0) \in T_eH$. Then there exists a $\tilde{\gamma} : \mathbb{R} \to H$ such that $\tilde{\gamma}(0) = \gamma'(0)$. Then $\iota \circ \tilde{\gamma}$ is also a one-parameter subgroup in G for the same vector field. Here we use that the identification of $\mathrm{Lie}(G)$ with T_eG from Theorem 2.4.6. We conclude that $\iota \circ \tilde{\gamma}$ and γ must be equal.We conclude that γ can be restricted as a map to H. It then follows directly that $\exp_H = \exp_G |\mathrm{Lie}(H)$.
For the second part, suppose $X \in \mathrm{Lie}(H)$. Then using first part, $exp(tX) \in H$ for all $t \in \mathbb{R}$.
Suppose now that $X \in \mathrm{Lie}(G)$ and $\exp(tX) \in H\ \forall t \in \mathbb{R}$. Then because the inclusion map is smooth, $\gamma : \mathbb{R} \to H, \gamma(t) = \exp(tX)$ is smooth as a map to H. Therefore, $X_e \in T_eH$, so using the identification, $X \in \mathrm{Lie}(H)$. □

One might wonder why the map is called the *exponential* map.

Lemma 2.5.6. *For the Lie group $GL(V)$ the exponential map* $\exp$: *Lie*$(\mathrm{GL}(V)) \to \mathrm{GL}(V)$ *is given by* $e^{\cdot}$.

Proof. We will show that the one parameter subgroups of $\mathrm{GL}(n, \mathbb{R})$ are generated by $A \in \mathfrak{gl}(n, \mathbb{R})$ are $\gamma(t) = e^{tA}$. We use the correspondence of $\mathfrak{gl}(n, \mathbb{R})$ with $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{R}))$. Let $A \in \mathfrak{gl}(n, \mathbb{R})$. Let A^L be the left-invariant vector field for A. Then by definition of the velocity we have $\gamma'(t) = A^L\big|_{\gamma(t)}$ Furthermore, by definition of an integral curve we have $\gamma(0) = \mathrm{Id}$. Using Theorem 2.4.7 we have that $\gamma'(t) = \gamma(t)A$. The result follows. □

3 Representations

It would definately be useful if we could use our deep knowledge of vector spaces and apply it to the study of Lie groups. Lie group representations allow us to do this. Before we dive into representations, let's first consider group actions. This section is based on [2]

3.1 Group actions

There is one important aspect of group theory which we have not yet introduced in our consideration of Lie groups: group actions. Actually, in Section 1.7 we have already seen an example of a Lie group action on a manifold M, where the Lie group was $(\mathbb{R}, +)$.

Definition 3.1.1 (Smooth action)**.** Let M be a smooth manifold, G a Lie group. Then an *smooth left action of G on M* is a left action $\theta : G \times M \to M$ which is also smooth.
For $g \in G$ and $p \in M$ we define the maps $\theta_g : M \to M$ and $\theta^{(p)} : \mathbb{R} \to M$ as $\theta_g(p) = \theta^{(p)}(t) = \theta(t, p)$. Furthermore, we use the notation $g \cdot p := \theta_g(p)$.
Similary we define an *smooth right action* $\theta : M \times G \to M$. We use for the smooth right action the notation $p \cdot g := \theta_g(p)$.

One defines the *orbit* and *stabilizer* similarly as in group theory.

3.2 Representations

The most familiar representations of Lie groups are maps from the Lie group to the $n \times n$-dimensional matrices. However, we can define a representation a bit more general.

Definition 3.2.1 (Lie group representation)**.** Let V be a Banach space, G a Lie group and $\pi : G \times V \to V$ a continuous left action such that $\rho_g : V \to V, v \mapsto \pi(g, v)$ is a linear endomorphism of V for every $g \in G$, in other words a linear map from V to itself. We denote this with the pair (π, V). We call π a *finite-dimensional representation of* G if V is finite dimensional.

Proposition 3.2.2. *Let V be a finite dimensional vector space. Then we denote with* $\mathrm{GL}(V)$ *the group of invertible linear transformations of V. This group is isomorphic to* $\mathrm{GL}(n, \mathbb{R})$ *or* $\mathrm{GL}(n, \mathbb{C})$ *respectively by mapping the basis vectors of V to the unit vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. Here $n \in \mathbb{N}$ is the dimension of V.*
Let G be a Lie group. Let (π, V) be a representation of G. Then for every $g \in G$ the map $\pi(g) := \pi(g, \cdot) : G \to \mathrm{GL}(V)$ *is a Lie group homomorphism.*

Proof. Since π is a left-action, we have for every $v \in V$ that $\pi(g, \pi(h, v)) = \pi(gh, v)$, so $\rho_g \circ \rho_h = \rho_{gh}$. Therefore the map ρ is well defined, since it maps into the endomorphisms of V and each element $\rho(g)$ in the image has an inverse given by $\rho(g^{-1})$. The map ρ therefore is a homomorphism from G to $\mathrm{GL}(V)$. We only need to show that this map is smooth and then it will be a Lie homomorphism.
It is given that the map π is continuous. Therefore, because V is finite dimensional, ρ is continuous.

Claim 1. Let G, H be Lie groups, $\rho : G \to H$ be a continuous group-homomorphism. Then ρ is smooth.

Proof. Let $\Gamma = \{(g, \rho(g)) | g \in G\}$. Then Γ is the graph of ρ and is a subgroup of $G \times H$, since the map $x \mapsto (x, \rho(x))$ is a homomorphism to a subset of $G \times H$. Let $\gamma_n = (g_n, h_n)$ be a sequence in Γ such that $\lim_{n\to\infty} \gamma_n = (g, h)$, where $h_n = \rho(g_n)$ and $g_n \in G$. Then for each of the components we have $\lim_{n\to\infty} g_n = g$ and $\lim_{n\to\infty} h_n$. By continuity of ρ we have that $\lim_{n\to\infty} h_n = \lim_{n\to\infty} \rho(g_n) = \rho(g)$. Therefore $(g, h) \in \Gamma$ and we conclude that Γ is closed.
Now we apply the Closed Subgroup Theorem (2.3.6) to conclude that Γ is a smooth submanifold. Let $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ be the projection on the first and second component respectively. Then we define the restriction to Γ of π_1 as $\pi = \pi_1|\Gamma$. This is of course again a smooth map, with inverse $\pi^{-1} : g \mapsto (g, \rho(g))$. It is trivially a Lie group homomorphism and π^{-1} is continuous. Then we use Lemma 2.3.8 to conclude that π is a diffeomorphism. Note that $\rho(g) = \pi_2(\pi_1^{-1}(g))$. Therefore $\rho = \pi_2 \circ \pi^{-1}$ and we conclude that ρ is smooth. □

□

For a finite-dimensional representation (π, V) of G, we sometimes introduce the map $\pi : G \to \mathrm{GL}(V)$ and say that this is a finite dimensional representation of G in V. We then use the previous proposition to identify these statements with each other.

Given a representation, it is always possible to construct other representations. The following example illustrates this.

Example 3.2.3. Let $\pi : G \to \mathrm{GL}(W)$ be a finite-dimensional representation of the Lie group G in the vector space W. Then we define $V = W \times \mathbb{C}$ and $\tilde{\pi} : G \to \mathrm{GL}(V)$ as $\tilde{\pi}(x) = \pi(x) \times Id_{\mathbb{C}}$. Then the subset $\tilde{W} := \{(w, c) | w \in W, c \in \mathbb{C}\}$ is mapped onto itself by every element in the image of $\tilde{\pi}$. Intuitively a 'smaller' vector space than V would suffice. We call such a representation $\tilde{\pi}$ *reducible.* △

Definition 3.2.4 (Invariant subspace and irreducible representations)**.** Let π be a representation of a Lie group G in a linear Banach space V. Then $W \subset V$ is a *invariant subspace* if for every $g \in G$, $\pi(g)W \subset W$, in other words, every linear isomorphism of V in the image of π maps W onto itself.
If W is a closed invariant subspace implies that $W = \{0\}$ or $W = V$, then π is called *irreducible.*

From linear algebra we know that unitary matrices have some nice properties. We will need some of these properties for representations as well.

Definition 3.2.5 (Unitary representation)**.** Let $\mathcal{H}$ be a Hilbert space. Then a representation $\pi : G \to \mathcal{H}$ of a Lie group G in $\mathcal{H}$ is called *unitary* if for every $g \in G$, $\pi(g)$ is unitary, that is $\pi(g)^* = \pi(g)^{-1}$ or equivalently for the inner product $\langle \cdot, \cdot \rangle$ we have for $v, w \in \mathcal{H}$ that $\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle$.

An important fact is that finite dimensional representations on compact Lie groups can always be unitarized.

Lemma 3.2.6. *Let G be a compact Lie group, $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation on a Hilbert space V. Then π is unitarizable.*

Proof. Let dx be the normalized Haar measure on G and let $\langle \cdot, \cdot \rangle_1$ be an inner product on V. Then we define for $v, w \in V$

$$\langle v, w\rangle = \int_G \langle \pi(x)v, \pi(x)w\rangle_1 dx$$

We need to show that this is indeed an inner product. It is definitely complex linear and hermitian. Furthermore $\langle v, v\rangle = \int_G \langle \pi(x)v, \pi(x)v\rangle_1 dx \geq 0$. Suppose $v = 0$ then trivially $\langle v, v\rangle = 0$. Finally suppose $\langle v, v\rangle = 0$ then since $\langle \pi(x)v, \pi(x)v\rangle_1$ is continuous and non-negative, it follows from the properties of the integral that $\langle \pi(x)v, \pi(x)v\rangle_1 = 0$. We conclude that $v = 0$.
To see that π is unitary for this inner product, let $g \in G$ and $v, w \in V$. Then

$$\begin{aligned}\langle \pi(g)v, \pi(g)w\rangle &= \int_G \langle \pi(x)\pi(g)v, \pi(x)\pi(g)w\rangle_1 dx \\ &= \int_G \langle \pi(xg)v, \pi(xg)w\rangle_1 dx = \int_G \langle \pi(y)v, \pi(y)w\rangle dy = \langle v, w\rangle\end{aligned}$$

Here we used the right invariance of the normalized Haar integral. □

One of the properties is that unitary representations can be 'diagonalised', ie. written as a direct sum of invariant subspaces, such that the restriction of the representation to an invariant subspace is irreducible. For this we first need the following lemma.

Lemma 3.2.7. *Let $\pi : G \to \mathcal{H}$ be a unitary representation of the Lie group G in the Hilbert space $\mathcal{H}$. If $\mathcal{H}_1$ is an invariant subspace, then $\mathcal{H}_2 := \mathcal{H}_1^\perp = \{v \in \mathcal{H} | v \perp w \text{ for all } w \in \mathcal{H}_1\}$ is a closed invariant subspace. If furthermore $\mathcal{H}_1$ is also closed, $\mathcal{H}$ is given as a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of closed invariant subspaces.*

Proof. The last assertion is can easily be seen, since any vector is the sum of a vector in $\mathcal{H}_1$ and in the orthocomplement of $\mathcal{H}_1$.
So we only need to show that for any $v \in \mathcal{H}_2$ and $g \in G$ we have again that $\pi(g)v \in \mathcal{H}_2$. Well, since $\mathcal{H}_1$ is an invariant subspace, in particular we have for $w \in \mathcal{H}$ that $\pi(g^{-1})w \in \mathcal{H}_1$, so $v \perp \pi(g^{-1})w$. Then $\langle \pi(g)v, w\rangle = \langle v, \pi(g^{-1})w\rangle = 0$ and $\pi(g)v \perp w$, so $\pi(g)v \in \mathcal{H}_2$. □

Corollary 3.2.8. *Let $\pi : G \to \mathcal{H}$ be a unitarizable representation of the Lie group G a finite dimensional Hilbert space $\mathcal{H}$. Then there exists invariant subspaces $\mathcal{H}_i$ where $1 \leq i \leq n$ with $n \in \mathbb{N}_{\geq 1}$ such that $\mathcal{H} = \oplus_i \mathcal{H}_i$ and for every i the restriction $\pi_i : G \to \mathrm{GL}(\mathcal{H}_i)$ defined by $\pi_j(g) = \pi(g)|_{\mathcal{H}_i}$ is irreducible.*

Proof. Since π is unitarizable we may without loss of generality assume that π is unitary. Suppose $\mathcal{H} = \oplus_i \mathcal{H}_i$ for $1 \leq i \leq k$ is the direct sum of invariant subspaces $\mathcal{H}_i$. Now suppose that for each i, $\mathcal{H}_i$ is irreducible and we are done. Otherwise we apply the previous lemma for each $\mathcal{H}_i$. We see that $\mathcal{H} = \oplus_j \mathcal{H}_j$ where $1 \leq j \leq k$. Apply these step repeatedly. This process will terminate in finitely many times, because $\mathcal{H}$ has only finitely many distinct subspaces. □

We note that in linear algebra we can perform a basis transformation to get another matrix. The main use is that many matrices can be written as a diagonal matrix. The properties of the matrix with respect to the transformed basis are no different. The following states this for representations.

Definition 3.2.9 (Equivariant maps and equivalent representations)**.** Let π_i be a representation of G in a Banach space V_i for $i = 1, 2$. Then a continuous linear map $T : V_1 \to V_2$ is called *equivariant* if for every $g \in G$ we have that $\pi_2(g) \circ T = T \circ \pi_1(g) : V_1 \to V_2$. If a equivariant continuous linear map T with continuous inverse exists, π_1 and π_2 are called *equivalent*. We denote this with $\pi_1 \sim \pi_2$
We denote with $\mathrm{End}_G(V)$ the space of equivalent linear endomorphisms of V.

Definition 3.2.10 (G-module)**.** Let $\pi : G \to \mathrm{GL}(V)$ be a representation of a Lie group G in a complete locally convex space V. Then π is called a G-module.

The claim is that our discussion about basis transformations and equivalent representations makes sense. Lemma 3.2.12 states this. First we introduce some notation.

Definition 3.2.11. Let (π, V) be a finite dimensional representation of G. Then for a given basis of V, for any $g \in G$ we denote with $\mathrm{mat}\pi(g)$ the matrix associated with the element $\pi(g) \in \mathrm{GL}(V)$.

Lemma 3.2.12. *Let π_i be a representation of G in a finite dimensional Banach space V_i for $i = 1, 2$. Then π_1 and π_2 are equivalent if and only if there exist bases of V_1 and V_2 such that for each $g \in G$ we have $\mathrm{mat}\pi_1(g) = \mathrm{mat}\pi_2(g)$.*

Proof. Suppose first that π_1 and π_2 are equivalent. Then there exists a continuous linear isomorphism $T : V_1 \to V_2$ such that $T \circ \pi_1(g) = \pi_2(g) \circ T$. Let T^{-1} be the inverse. Then we have that $\pi_1 = T^{-1} \circ \pi_2 \circ T$. Now let v_i be a basis of V_1. Choose as a basis of V_2 the elements $w_i := T(v_i)$. Then one sees that the associated matrices are equal.
Conversely, suppose there exist bases of V_1 and V_2 such that for each $g \in G$ we have that $\mathrm{mat}\pi_1(g) = \mathrm{mat}\pi_2(g)$. Then in particular $mat\pi_1(g)$ and $\mathrm{mat}\pi_2(g)$ have the same dimensions and therefore V_1 and V_2 have the same dimensions. Let v_i be the basis of V_1 and w_i be the basis of V_2. Then we define a the linear map T such that $T(v_i) = w_i$, ie. really just the identity matrix. Then T is by definition linear, onto and injective. We conclude that π_1 and π_2 are equivalent. □

From linear algebra we remember for a finite dimensional Hilbert space V given an unitary operator $A \in \mathrm{GL}(V)$ we could find a basis transformation such that V decomposed into eigenspaces corresponding to eigenvalues of the operator A. A similar result holds for representations.

Lemma 3.2.13 (Schur's lemma)**.** *Let $\pi : G \to \mathrm{GL}(V)$ be a representation of the Lie group G in the finite dimensional Hilbert space V. Then*

a) *π is irreducible implies that* $\mathrm{End}_G(V) = \mathbb{C}\mathrm{Id}_V$

b) *If π is unitarizable and* $\mathrm{End}_G(V) = \mathbb{C}\mathrm{Id}_V$*, then π is irreducible.*

Proof. Given a representation π of G in V we define the representation $\tilde{\pi}$ of G in $\mathrm{End}(V)$ as $\tilde{\pi}(g)A = \pi(g)A\pi(g^{-1})$. Let $\mathrm{End}(V)^G = \{A \in \mathrm{End}(V) | \tilde{\pi}(g)A = A\}$ denote the set of G-invariants in V. Observe that this is exactly the set $\mathrm{End}_G(V)$ of G-equivalent endomorphisms of V. Let $A \in \mathrm{End}(V)^G$, with eigenvalue $\lambda \in \mathbb{C}$ and associated eigenspace $E_\lambda = ker(A - \lambda\mathrm{Id})$. By definition $\pi(g)A = A\pi(g)$, so for every $g \in G$ we have that $\pi(g)$ commutes with A and of course also with $\lambda\mathrm{Id}$).

Claim 1. Suppose $A, B \in \mathrm{End}V$, then $AB = BA$ implies that A leaves $\mathrm{Ker}(B)$ invariant.

Proof. Suppose $x \in \mathrm{Ker}(B)$ then $Bx = 0$, therefore $BAx = ABx = 0$, so also $Ax \in \mathrm{Ker}(B)$. □

We conclude that π leaves E_λ invariant. Since π is irreducible, we have that $E_\lambda = V$. Therefore $V = \mathrm{Ker}(A - \lambda\mathrm{Id})$ and $A = \lambda\mathrm{Id}$, which proves a).
For b) observe that since π is unitarisable, there exists an inner product $\langle \cdot, \cdot \rangle$ such that π is unitary. We will fix this inner product. Note that for π to be irreducible, the only invariant subsets of V should be V and $\{0\}$. Suppose W is an invariant subset and $W \neq \{0\}$ and define $P : V \to W$ as the orthogonal projection on W. We will show that P is the identity map and that therefore $W = V$. Since W is by assumption G-invariant, also $W^\perp$ is G-invariant. Since furthermore P is the identity on W and 0 on $W^\perp$ we have that for $w \in W$ $\pi(g)Pw = \pi(g)w = P\pi(g)w$ and for $w' \in W^\perp$ that $\pi(g)Pw' = 0 = P\pi(g)w'$. So for all $v \in V$ we have that $P\pi(g)v = \pi(g)Pv$. We conclude that $P \in \mathrm{End}_G(V)$. Now we apply a). and conclude that $P = \lambda\mathrm{Id}$ for some $\lambda \in \mathbb{C}$. Now, $W \neq \{0\}$, so $P \neq 0$. Furthermore, since P is a projection we have that $P^2 = P$. We conclude that $\lambda^2 = \lambda$ and $\lambda \neq 0$. Therefore $\lambda = 1$. We conclude that P is the identity map. □

When two representations (π, V) and (π', V') are equivalent we know, by definition, that there exists an equivariant linear isomorphism $T : V \to V'$. The following lemma stays something about equivariant maps $T : V \to V'$ if π and π' are not equivalent.

Lemma 3.2.14. *Let $\pi : G \to \mathrm{GL}(V)$ and $\pi' : G \to \mathrm{GL}(V')$ be two irreducible inequivalent finite dimensional representations of G. Then every equivariant map $T : V \to V'$ is trivial.*

Proof. The proof goes by looking at two important sets, $\ker T$ and $\mathrm{Im}\ T$ and show that they are G-invariant. Suppose that T is non-trivial. Then $\ker T \subsetneq V$. Let $v \in \ker T$ then $T(v) = 0$. By the equivariance $T(\pi(g)v) = \pi'(g)T(v) = 0$ for all $g \in G$. Therefore $\ker T$ is G-invariant, so since π is irreducible we have that $\ker T = 0$, so T is injective.
Since we may assume that $V \neq \{0\}$, then because T is injective we have that $\mathrm{Im}\ T \subset V'$ is non-trivial. Let $v' \in \mathrm{Im}\ T$ then there exist a v such that $T(v) = v'$. But since $T(\pi(g)v) = \pi'(g)T(v)$ we conclude that also $\pi'(g)v' \in \mathrm{Im}\ T$, so $\mathrm{Im}\ T$ is G-invariant. By irreducibility and non-triviality we conclude that $\mathrm{Im}\ T = V'$, so T is also surjective, hence bijective. But then π and π' are equivalent, proving a contradiction. □

3.3 Lie algebra representations

Similary as a Lie group representation, we can define a Lie algebra representation. The difference is that the representation doesn't map into $\mathrm{GL}(V)$, since the Lie algebra is not a group. It is however a vector space, so a Lie algebra representation should map into the set of linear maps from V to V, ie. into $\mathrm{End}(V)$.

Definition 3.3.1 (Representation of a Lie algebra)**.** Let $\mathfrak{g}$ be a Lie algebra, V a linear space. Then a *representation* of $\mathfrak{g}$ in V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

We also have a similar definition as a G-module for Lie algebra representations.

Definition 3.3.2. let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a Lie algebra representation in V. Then we call ρ a $\mathfrak{g}$-module.

Given a finite dimensional Lie group representation, we can construct a finite dimensional Lie algebra representation out of it. The converse is in general not true, since two distinct Lie groups can have the same Lie algebra, even if the Lie groups are not isomorphic. An example is $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$. These both have isomorphic Lie algebras, however the groups are not diffeomorphic. To see this note that $\mathrm{SU}(2)$ is simply connected, whereas $\mathrm{SO}(3)$ is not. Therefore the groups are not homeomorphic as topological manifolds, so they cannot be diffeomorphic as Lie groups.

Proposition 3.3.3. *Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional Lie group representation. Then $\pi_* : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, with $\mathfrak{g}$ the Lie algebra of G. We call the representation π_* the* induced Lie algebra representation*.*

Proof. This is a direct consequence of Theorem 2.4.7 and Theorem 2.4.8. □

One might expect that there is some connection with the exponential map. This is indeed the case.

Proposition 3.3.4 (Properties of the induced Lie algebra representation)**.** *Let (π, V) be a representation of the connected Lie group G and let $\pi_* : \mathrm{Lie}(G) \to \mathfrak{gl}(V)$ be the induced Lie algebra representation. Then for $v \in V$ and $X \in \mathrm{Lie}(G)$, π_* satisfies the following properties:*

a) $\pi_*(X)v = \frac{d}{dt}\big|_{t=0}\pi(\exp(tX))v$.

b) $\pi(\exp(X)) = e^{\pi_*(X)}$

Proof. For a). we use the chain rule for the smooth function π and exp. We have that

$$\frac{d}{dt}\bigg|_{t=0}\pi(\exp(tX))v = d\pi_e \circ d(\exp)_0(X)v$$

Now we note that from Proposition 2.5.4 it follows that $d(\exp)_0$ is the identity map. Also $d\pi_e : T_eG \to \mathfrak{gl}(V)$ is exactly the map π_*. We conclude a).
For b). we use also use Proposition 2.5.4 and then use Lemma 2.5.6. The equality follows. □

Also, there are some properties which do transfer between the Lie group and the Lie algebra. We have to expect that we need that the Lie group is connected, since the Lie algebras only depend on the connected component of the identity. This can be seen, because the Lie group is defined using smooth curves starting at the identity element. These of course stay in the connected component.

Lemma 3.3.5. *Let G be a connected Lie group, let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of G in V and let $\pi' : G \to \mathrm{GL}(V)$ be a finite dimensional representation of G in V'. Then*

π is irreducible if and only if π_* is irreducible.

Proof. We need to show, for a linear subspace $W \subset V$, that W is G-invariant if and only if W is $\mathfrak{g}$-invariant. Suppose W is $\mathfrak{g}$-invariant, then using the second property from Proposition 3.3.4 we see that W is invariant under the set generated by $\{\exp X | X \in \mathfrak{g}\}$. Now we use that this is equal to G_e, where G_e is the connected component of the identity element e. Using that G is connected we have that $G = G_e$. Conversely suppose W is G-invariant. Then $\pi(g)W \subset W$ for all $g \in G$. Then using the first property of Proposition 3.3.4 we see that $\pi_* W \subset W$. □

Also the unitary property 'transfers' between the Lie group and Lie algebra. We remark that for complex numbers $z \in \mathbb{C}$ we have $\overline{e^z} = e^{-z}$.

Lemma 3.3.6. *Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of the compact connected Lie group G in the Hilbert space V. Then π is unitary if and only if for all $X \in \mathfrak{g}$ we have $\pi_*(X)^* = -\pi_*(X)$.*

Proof. Suppose π is unitary. Let $X \in \mathfrak{g}$. Observe that π is a group homomorphism. Then using Proposition 3.3.4 b) we see that

$$\pi(\exp(tX)) = e^{t\pi_*(X)} \tag{4}$$

Then by definition of unitarity

$$\pi(\exp tX)^* = \pi(\exp tX)^{-1} = \pi((\exp tX)^{-1}) = \pi(\exp(-tX))$$

In the last step we used the addition property of exponentials, ie. $\exp(s+t)X = \exp sX \exp tX$. Therefore, using the (4)

$$e^{t\pi_*(X)^*} = e^{t\pi_*(X)} \tag{5}$$

Then the result follows using Proposition 3.3.4 a). Conversely suppose for all $X \in \mathfrak{g}$ we have $\pi_*(X)^* = -\pi_*(X)$. Then also (5) holds and trivially $\pi(x))$ is unitary for all $x \in \exp \mathfrak{g}$. Since G is connected we have that the same follows for $x \in G_e = G$. □

4 Peter-Weyl

In this section we will introduce a theorem about the decomposition of representations in irreducible representations and about the decomposition of the space of square integrable functions as a Hilbert direct sum, called the *Peter-Weyl Theorem.* This section is based on [2].

4.1 Measures

In this section we give an overview of how to define integration on a smooth manifold. It is definitely not our goal to be rigorous, it is more to give the reader a feeling of how one goes about doing this and to state some theorems without proof which we need in the proofs in later sections.

Recall that in real multivariable calculus we have that for a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(u) = v$, we have the following substitution rule

$$dv = |det(DT)|du = |det(T)|du$$

Let's apply our generalisation trick once again.

Definition 4.1.1 (Density)**.** Let V be an n-dimensional real vector space. A *density on* V is a map $\omega : V^n \to \mathbb{C}$ such that for $T \in \mathrm{End}(V)$ we have

$$T^*\omega := \omega \circ T^n = |\det T|\omega$$

The space of all densities on V is denoted by $\mathcal{D}V$.

Definition 4.1.2. Let $\phi : V \to W$ be a linear isomorphism from a real finite-dimensional vector space V onto a real finite-dimensional vector space W. Then we define the *pull-back map* ϕ^* as $\phi^* : \mathcal{D}W \to \mathcal{D}V,\ \omega \mapsto \omega \circ \phi^n$.

We now like to generalise this to smooth manifolds M. The first problem we encounter is that smooth manifolds are in general not finite dimensional vector spaces. However, the tangent space at a point is a vector space. Similary as for the tangent bundle, we define the *bundle* $\mathcal{D}TM$ *of densities on* M. Then following a similar approach as for the definition of a vector field, we can define the *space of continous densities on* M denoted by $\Gamma(\mathcal{D}TM)$. On this space we can define a *pull-back* map.

Definition 4.1.3. Let $\phi : M \to N$ be a diffeomorphism from the smooth manifold M to the smooth manifold N. Then we define the *pull-back* map $\phi^* : \Gamma(\mathcal{D}TN) \to \Gamma(\mathcal{D}TM)$ as

$$(\phi^*\omega)(x) = (d_x\phi)^*\omega(\phi(x))$$

for $\omega \in \Gamma(\mathcal{D}(TN))$ and $x \in M$.

This allows us to define an integral corresponding to a density ω, by relating it back to integrals on $\mathbb{R}^n$.

Definition 4.1.4. Let (U, ϕ) be a coordinate chart of a smooth manifold M. Let $\omega \in \Gamma(\mathcal{D}TM)$, with compact support $\mathrm{supp}\ \omega \subset U$. Then we define

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^*\omega$$

The right hand side is an integral over a subset of $\mathbb{R}^n$.

We use this to extend the integral to the entire manifold. This should be a sum over coordinate charts.

Definition 4.1.5. Let M be a smooth manifold and let $\mathcal{U} = \{U_\alpha | \alpha \in \mathcal{A}\}$ be an open cover of M. Let $\{\psi_\alpha | \alpha \in \mathcal{A}\}$ be a partition of unity subordinate to $\mathcal{U}$, ie a set of functions $\psi_\alpha : M \to \mathbb{R}$ such that

- For all $x \in M$ and $\alpha \in \mathcal{A}$ we have $0 \leq \psi_\alpha(x) \leq 1$.

- supp $\psi_\alpha = \overline{\{x \in M |\ \psi_\alpha(x) \neq 0\}} \subset U_\alpha$
- Every point $x \in M$ has a neighborhood V_x such that $V_x \cap U_\alpha \neq \emptyset$ for only finitely many $\alpha \in \mathcal{A}$
- For all $x \in M$ we have $\sum_{\alpha \in \mathcal{A}} \psi_\alpha(x) = 1$

Then for a compactly supported continous density ω we define the integral

$$\int_X \omega = \sum_{\alpha \in \mathcal{A}} \int_{V_\alpha} \psi_\alpha \omega$$

Note that for vector fields we had the concept of left-invariant vector fields. We have a similar definition for densities.

Definition 4.1.6. Let $\omega \in \Gamma(\mathcal{D}TM)$ be a density. Then ω is called *left-invariant* if $L_g^* \omega = \omega$ for all $g \in G$.

For compact Lie groups G there is a special left invariant density dx.

Lemma 4.1.7. *Let G be a compact Lie group. Then there exists a unique left-invariant density dx such that*

$$\int_G dx = 1$$

The Haar measure associated with this density is called the normalised Haar measure.

Now that we have defined an integral on a compact Lie group G, we define the space $L^2(G)$ as the space of (equivalence classes of) measurable functions $f \in \mathcal{F}(G)$ such that $\int_G |f(x)|^2 dx < \infty$. We equip this space with the inner product

$$\langle f, g \rangle := \int_G f(x)\overline{g(x)} dx, \quad f, g \in L^2(G)$$

Then the associated norm becomes $\|f\| = \sqrt{\int_G |f(x)|^2 dx}$.

4.2 Schur orthogonality

Later on we will try to find the basis elements for functions from a Lie group to the complex numbers. For instance, we want to find a basis for square integrable functions on the 2-sphere, in order to solve the spherical part of the Schrödinger equation of the hydrogen atom. It will turn out that matrix coefficients are well behaving functions which will help us finding these basis elements. Note that the definition below is nothing more than a matrix coefficient for a particular basis.

Definition 4.2.1 (Matrix coefficients)**.** Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of the Lie group G in the Hilbert space V. Then for $v, w \in V$ we define *a matrix coefficent of* π as a map $m : G \to \mathbb{C}$ where $m(g) = m_{v,w}(g) := \langle \pi(g)v, w \rangle$.

The choice of v and w is however a bit arbitrary. To resolve this, we will only look at the span of all such matrix coefficients.

Definition 4.2.2. Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of the Lie group G in the Hilbert space V. Then we define $C(G)_\pi$ as the linear span of the space of matrix coefficients $m : G \to \mathbb{C}$.

Remark that the notation $C(G)_\pi$ is derived from the notation $C(G)$ of continuous functions from G to $\mathbb{C}$. It would be nice if this span says something about the representation. This is indeed the case. In order to show this we first introduce the *left and right regular representations.*

Definition 4.2.3 (Left and right regular representation)**.** Let G be a Lie group, and let $C(G)$ be the set of continuous functions $f : G \to \mathbb{C}$. Then we define the *left regular representation* as the representation $L : G \to \mathrm{GL}(C(G))$ as $L(g)\phi(x) = \phi(g^{-1}x)$ for $x, g \in G$ and $\phi \in C(G)$.
Similary the *right regular representation* is given by $R : G \to \mathrm{GL}(C(G))$, $R(g)\phi(x) = \phi(xg)$.

Definition 4.2.4. Let G be a Lie group and let L and R be the left and right regular representations of G. Then we define the representation $R \times L$ of $G \times G$ on $C(G)$ as

$$(R \times L)(x, y) := R(x) \circ L(y)$$

Observe that trivially $R(x) \circ L(y) = L(y) \circ R(x)$. Here $C(G)$ is just the set of continuous functions $f : G \to \mathbb{C}$.

Now we can show that equivalent finite irreducible representations have the same linear span of matrix coefficients.

Definition 4.2.5. For a finite dimensional Hilbert space V and $v, w \in V$ we define the linear map $L_{v,w} : V \to V$ as $L_{v,w}(u) = \langle u, w\rangle v$.

To understand the previous definition, let V, $v, w \in V$ and $L_{v,w}$ as the definition. Choose an orthonormal basis on V. We note that we can write w as a row vector with respect to the basis. Then the inner product of u with w is a matrix multiplication. Then we have the following $L_{v,w} = vw^T u$. But since matrix multiplication is associative, we can also write this as $L_{v,w} = (vw^T)u$. Then vw^T can be seen as a matrix. We also define a map which gives the trace of a transformed endomorphism.

Definition 4.2.6. For an irreducible finite dimensional representation $\pi : G \to \mathrm{GL}(V)$ of the Lie group G in V, we define the map $T_\pi : \mathrm{End}(V) \to C(G)$ as $T_\pi(A)(g) = \mathrm{Tr}(\pi(x)A)$, where $\mathrm{Tr}(A)$ is the trace of a linear map $A \in \mathrm{End}(V)$.

This map has the property that it maps into the subspace spanned by matrix coefficients of π.

Lemma 4.2.7. *Let T_π as in the previous definition. Then the image of T_π is $C(G)_\pi$.*

Proof. Let $v, w \in V$. Remark that $\mathrm{Tr}(L_{v,w}) = \langle v, w\rangle$ by the previous discussion. Recall that a matrix coefficient was defined by $m_{v,w}(g) = \langle \pi(g)v, w\rangle$. Then, since $\pi(g)L_{v,w}(u) = \langle u, w\rangle \pi(g)v = L_{\pi(g)v,w}$ one immediately sees that

$$m_{v,w}(g) = \mathrm{Tr}(\pi(x)L_{v,w})$$

Now let $\{e_i | 1 \leq i \leq n\}$ with n the dimension of V be an orthonormal basis for V. Such a basis exists, because V is finite dimensional. Let $A \in \mathrm{End}(V)$. Then

$$\begin{aligned} Av &= \sum_{1\leq i\leq n} \langle Av, e_i\rangle e_i = \sum_{1\leq i,j\leq n} \langle A\langle v, ej\rangle ej, e_i\rangle e_i \\ &= \sum_{1\leq i,j\leq n} \langle Ae_j, e_i\rangle\langle v, e_j\rangle e_i = \sum_{1\leq i,j\leq n} \langle Ae_j, e_i\rangle L_{e_i,e_j}(v) \end{aligned}$$

Therefore

$$A = \sum_{1\leq i,j\leq n} \langle Ae_j, e_i\rangle L_{e_i,e_j}$$

Combining these results, we see that any function of the form $x \mapsto \mathrm{Tr}(\pi(x)A)$ can be expressed as a sum of matrix coefficients. Therefore, T_π maps into $C(G)_\pi$. □

Furthermore, we'd like to look at how endomorphisms behave if we do a transformation with $\pi(g)$. Therefore we give the following definition.

Definition 4.2.8. Let $\pi : G \to \mathrm{GL}(V)$ be an finite dimensional irreducible representation of the Lie group G in V. For $A \in \mathrm{End}(V)$, $x, y \in G$ define the representation $\pi \otimes \pi^*$ of $G \times G$ on $\mathrm{End}(V)$ as

$$[\pi \otimes \pi^*](x, y)A = \pi(x)A\pi(y)^{-1}$$

It turns out that a left or right action of an element of the Lie group G on a function in $C(G)_\pi$ results in another function in the same $C(G)_\pi$, in other words $C(G)_\pi$ is $R \times L$-invariant.

Lemma 4.2.9. *The map $T_\pi : \mathrm{End}(V) \to C(G)_\pi$ is surjective and equivariant for representations $R \times L$ and $\pi \otimes \pi^*$ of $G \times G$. In particular $C(G)_\pi$ is $R \times L$-invariant.*

Proof. First we prove that $T_\pi : \mathrm{End}(V) \to C(G)$ is equivariant. Then the result follows because $(Im)(T_\pi) = C(G)_\pi$. Now, for all $g \in G$

$$T_\pi([\pi \otimes \pi^*](x, y)A)(g) = \mathrm{Tr}(\pi(g)\pi(x)A\pi(y^{-1})) = \mathrm{Tr}(\pi(y^{-1}gx)A) = R_x L_y(T_\pi(A))(g)$$

This is exactly the definition of equivariant maps. The $R \times L$-invariance of $C(G)_\pi$ then follows directly from the $\pi \otimes \pi^*$-invariance of $\mathrm{End}(V)$. □

Corollary 4.2.10. *Let π and π' be equivalent finite dimensional irreducible representation of a Lie group G. Then $C(G)_\pi = C(G)_{\pi'}$.*

Proof. Since π and π' are equivalent, there exist an equivariant linear map $T : V \to V'$. We will now use the previous lemma.Then, for $A \in \mathrm{End}(V)$ and $g \in G$

$$T_{\pi'}(TAT^{-1})(g) = \mathrm{Tr}(\pi'(g)TAT^{-1}) = \mathrm{Tr}(T^{-1}\pi'(x)TA) = \mathrm{Tr}(\pi(x)A) = T_\pi(A)(x)$$

Here we used the definition of an equivariant map and that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for $A, B \in \mathrm{End}(V)$. We conclude that the images of T_π and $T_{\pi'}$ are equal and apply the claim. □

One might ask what happens if two representations are not equivalent. Is it possible that they have the same span of matrix coefficients? Secondly, since $C(G)_\pi$ is a linear space, does there exist an inner product on this space?

Theorem 4.2.11 (Schur orthogonality)**.** *Let $\pi : G \to \mathrm{GL}(V)$ and $\pi' : G \to \mathrm{GL}(V')$ be two finite dimensional irreducible representations of G in resp. V and V'. Then*

a) *Suppose π and π' are not equivalent, then $C(G)_\pi \perp C(G)_{\pi'}$.*

b) *Suppose V has an inner product and suppose π is unitary with respect to this inner product. Let $v, v', w, w' \in V$. Then for matrix coefficents $m_{v,w}$ and $m_{v',w'}$ the L^2 inner product is given by*

$$\int_G m_{v,w}(x)\overline{m_{v',w'}(x)}dx = \dim(\pi)^{-1}\langle v, v'\rangle\overline{\langle w, w'\rangle}$$

Proof. First we slightly extend our previous definition of $L_{v,w}$. We define $L_{w',w'} : V \to V'$ as $L_{w',w}(u) = \langle u, w\rangle w'$. Furthermore we define the map $I_{w',w} : V \to V'$ as

$$I_{w',w} = \int_G \pi'(x)^{-1} \circ L_{w',w} \circ \pi(x)dx$$

Then

$$\begin{aligned}\langle I_{w',w}v, v'\rangle &= \langle \int_G \pi'(x)^{-1} \circ L_{w',w}\pi(x)vdx, v'\rangle = \langle \int_G \pi'(x)^{-1}\langle \pi(x)v, w\rangle w'dx, v'\rangle \\ &= \int_G \langle \pi'(x)^{-1}w', v'\rangle\langle \pi(x)v, w\rangle dx = \int_G \langle \pi(x)v, w\rangle\overline{\langle \pi'(x)v', w'\rangle}dx \\ &= \langle m_{v,w}, m_{v',w'}\rangle_{L^2}\end{aligned} \tag{6}$$

Here we used that $\pi'(x)$ is unitary. $I_{w',w}$ is linear. Furthermore for $g \in G$

$$\begin{aligned}I_{w,w'} \circ \pi(g) &= \int_G \pi'(x)^{-1} \circ L_{w',w} \circ \pi(x) \circ \pi(g)dx = \int_G \pi'(x^{-1}) \circ L_{w',w} \circ \pi(xg)dx \\ &= \int_G \pi'(gy^{-1}) \circ L_{w',w} \circ \pi(y)dy = \pi'(g) \circ \int_G \pi'(y)^{-1} \circ L_{w',w} \circ \pi(y)dy \\ &= \pi'(g) \circ I_{w,w'}\end{aligned} \tag{7}$$

We conclude that $I_{w',w}$ is equivariant. Then we apply Lemma 3.2.14 to conclude that $I_{w',w}$ is trivial and apply (6). To prove b) we have that $V = V'$. Then since $I_{w',w}$ is equivariant for all $w, w' \in V$ we have that $I_{w',w} \in \mathrm{End}_G(V)$. Then we apply the Schur's Lemma 3.2.13 to conclude that $I_{w',w} = \lambda\mathrm{Id}$. Therefore, we only need to compute the trace of $I_{w',w}$. The trace is linear and the integral is normalised.

$$\mathrm{Tr}(I_{w',w}) = \int_G \mathrm{Tr}(\pi(x)^{-1}L_{w',w}\pi(x))dx = \int_G \mathrm{Tr}(L_{w',w})dx = \mathrm{Tr}(L_{w',w}) = \langle w', w\rangle$$

With $\dim(\pi)$ we denote the dimension of V, so $\mathrm{Tr}(\mathrm{Id}_V) = \dim(\pi)$. We conclude that

$$I_{w',w} = \dim(\pi)^{-1}\langle w', w\rangle$$

Filling this in in (6) yields the desired result.

$$\langle m_{v,w}, m_{v',w'}\rangle_{L^2} = \langle I_{w',w}v, v'\rangle = \langle \dim(\pi)^{-1}\langle w', w\rangle\mathrm{Id}_V v, v'\rangle = \dim(\pi)^{-1}\langle v, v'\rangle\overline{\langle w, w'\rangle}$$

□

We will use this theorem to prove the Peter-Weyl theorem. But before we look at Peter-Weyl, we first look into characters of representations.

4.3 Characters

Characters are just another way to study irreducible representations. They are a generalisation of the trace of a matrix to representations.

First we remark that according to Lemma 3.2.12 for two equivalent finite dimensional representations of a group G there exist bases such that the associated matrices of the representations are equal. This allows us to give the following definition.

Definition 4.3.1 (Character)**.** Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of the Lie group G. Then we define the *character* of π as the function $\chi_\pi : G \to \mathbb{C}$, $\chi_\pi(g) := \mathrm{Tr}(\mathrm{mat}\pi(g))$. Here $\mathrm{Tr}(A)$ is the trace of the matrix A.

Note that this is well defined, because from linear algebra we know that the trace of a matrix is not dependend on the particular basis we chose for the linear space V. Then we apply Lemma 3.2.12. Furthermore, suppose π and π' are two equivalent finite dimensional representations, then their characters are equal. Let us state that.

Lemma 4.3.2. *Let π, π' be two finite dimensional equivalent representations of G, then $\chi_\pi = \chi_{\pi'}$.*

Proof. Let T be the equivariant linear isomorphism. Then for all $g \in G$ we have $\pi'(g) = T \circ \pi(g) \circ T^{-1}$. Then we remark that from linear algebra we know that $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$, for $A, B \in \mathrm{GL}(V)$. □

Characters respect the unitary structure.

Lemma 4.3.3. *Let $\pi : G \to GL(V)$ be a unitarizable finite dimensional representation of G. Then for every $g \in G$ we have $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.*

Proof. Choose a basis on V such that π is unitary. For $g \in G$

$$\chi_\pi(g^{-1}) = \mathrm{Tr}(\pi(g^{-1})) = \mathrm{Tr}(\pi(g)^{-1}) = \mathrm{Tr}(\pi(g)^*) = \overline{\mathrm{Tr}(\pi(g))} = \overline{\chi_\pi(g)}$$

□

So also equivalent irreducible representations have the same characters. Furthermore, we have observed that equivalent irreducible representations have the same span of matrix coefficients. It therefore makes sense to look at equivalence classes of these representations. We need two definitions for this.

Definition 4.3.4 (Direct sum representation)**.** Let $\pi_i : G \to \mathrm{GL}(V_i)$ for $i = 1, 2$ be two representations of G. Then the *direct sum representation* $\pi = \pi_1 \oplus \pi_2 : G \to \mathrm{GL}(V_1 \oplus V_2)$ is defined as

$$\pi(g)(v_1, v_2) = (\pi_1(g)v_1, \pi_2(g)v_2) \quad \text{for } g \in G, v_1 \in V_1 \text{ and } v_2 \in V_2$$

Lemma 4.3.5. *Let $\pi_i : G \to \mathrm{GL}(V_i)$ be two finite dimensional representations of the Lie group G in the Hilbert space V_i. Then*

$$\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$$

Proof. Let $\mathcal{V}_1 := \{v_i |\ 1 \leq i \leq \dim V_1\}$ be a basis for V_1 and let $\mathcal{V}_2 := \{v_i |\ \dim V_1 + 1 \leq i \leq \dim V_1 + \dim V_2\}$ be a basis for V_2. Then $\mathcal{W} := \{v_j | 1 \leq j \leq \dim V_1 + \dim V_2\}$ is a basis of $V_1 \times V_2$. With respect to the basis $\mathcal{W}$, $\mathrm{mat}\pi(g)$ can be written as the matrix

$$\begin{pmatrix} \mathrm{mat}\pi_1(g) & 0 \\ 0 & \mathrm{mat}\pi_2(g) \end{pmatrix}$$

The result follows directly from the definition of a character. □

Definition 4.3.6. Let G be a Lie group. Then with $\widehat{G}$ we denote the set of equivalence classes of finite dimensional irreducible representations of G. Elements are written as $[\delta] \in G$, where δ is a representative of the equivalence class. Furthermore, let $n \in \mathbb{N}$ then we define with $n\delta := \oplus_{i=1}^{n} \delta$.

Using these definitions, we can decompose any finite dimensional representation into irreducible representations.

Lemma 4.3.7. *Let $\pi : G \to \mathrm{GL}(V)$ be a finite dimensional representation of a compact Lie group G in a Hilbert space V. Let $\langle \cdot, \cdot \rangle$ be the L^2 inner product with respect to the normalised Haar measure on G. Then*

$$\pi \sim \bigoplus_{[\delta] \in \widehat{G}} \langle \chi_\pi, \chi_\delta \rangle \delta$$

Proof. First we claim that the character functions constitute an orthonormal basis.

Claim 1. Let (π, V) and (π', V) be two finite dimensional irreducible representations of G. Then we denote with $\pi \sim \pi'$ that π and π' are equivalent representations. If π and π' are not equivalent we denote this with $\pi \nsim \pi'$. Then the following holds

a) $\pi \sim \pi'$ implies $\langle \chi_\pi, \chi_{\pi'} \rangle_{L^2} = 1$

b) $\pi \nsim \pi'$ implies $\langle \chi_\pi, \chi_{\pi'} \rangle_{L^2} = 0$

Proof. Suppose $\pi \sim \pi'$. Let n be the dimension of V and let e_i with $1 \leq i \leq n$ be an orthonormal basis of V. Then we have that

$$\chi_\pi(x) = \mathrm{Tr}(\pi(x)) = \sum_{i=1}^{n} \langle \pi(x) e_i, e_i \rangle = \sum_{i=1}^{n} m_{e_i, e_i}(x) \tag{8}$$

Since $\pi' \sim \pi$ it follows that $\chi_{\pi'} = \chi_\pi$.Then using Theorem 4.2.11 b) we have

$$\begin{aligned} \langle \chi_\pi, \chi_{\pi'} \rangle_{L^2} &= \int_G \chi_\pi(x) \overline{\chi_{\pi'}(x)} dx = \sum_{1 \leq i,j \leq n} \int_G m_{e_i,e_i}(x) \overline{m_{e_j,e_j}(x)} dx \\ &= \sum_{1 \leq i,j \leq n} \dim(\pi)^{-1} \langle e_i, e_j \rangle \overline{\langle e_i, e_j \rangle} = \sum_{i=0}^{n} \dim(\pi)^{-1} \langle e_i, e_i \rangle \overline{\langle e_i, e_i \rangle} \\ &= \dim(\pi)^{-1} \sum_{i=0}^{n} 1 = 1 \end{aligned} \tag{9}$$

Suppose now that $\pi \nsim \pi'$. Then using Theorem 4.2.11 a) we have that the matrix coefficients are orthogonal with respect to the L^2 norm. Then we use equation (8) and conclude that $\langle \chi_\pi, \chi_{\pi'} \rangle = 0$. □

Now we apply Lemma 3.2.6 to conclude that π is unitarizable. We conclude that π is equivalent to a direct sum of irreducible representations π_i: $\pi \sim \oplus_{i=1}^n \pi_i$. Then using Lemma 4.3.5 we see that the character of this representation is given by $\chi_\pi = \sum_{i=1}^n \chi_{\pi_i}$. We use the claim and this proves the lemma. □

In particular we can use this lemma to completely characterize the representations using their character.

Corollary 4.3.8. *Let π, π' be two finite dimensional representations of a compact Lie group G. Then π and π' are equivalent if and only if their characters are equal.*

Proof. We already proved one part of this corollary. Suppose that $\chi_\pi = \chi_{\pi'}$. Then for each $[\delta] \in \widehat{G}$ we have that $\langle \chi_\pi, \chi_\delta \rangle = \langle \chi_{\pi'}, \chi_\delta \rangle$. Then we use Lemma 4.3.7. □

4.4 Peter-Weyl

We are now ready to prove an important theorem about the decomposition of square integrable functions on a Lie group G. First a definition of a space of 'nice' functions.

Definition 4.4.1 (Space of representative functions)**.** Let G be a Lie group, $\widehat{G}$ be the set of equivalence classes of irreducible representations. Then we define the space of *representative functions* $\mathcal{R}(G)$ as the subset of $\mathcal{F}(G)$ of functions $f : G \to \mathbb{C}$ that can be written as a finite sum of functions $f_\delta \in C(G)_\delta$ for $[\delta] \in \widehat{G}$.

Let $f_\delta \in \mathbb{C}(G)_\pi$. Any representation can be decomposed in irreducible representations. We therefore expect that we can write $\mathcal{R}(G)$ as a direct sum of $C(G)_\pi$.

Proposition 4.4.2. *$\mathcal{R}(G)$ is the linear span of the set of matrix coefficients $C(G)_\pi$ of finite dimensional representations π of G in V.*

Proof. Let $f : G \to \mathbb{C}$ be in the span of matrix coefficients of finite dimensional representations of G. Then we need to show that f can be written as a linear combination of matrix coefficients of irreducible finite dimensional representations. It is enough to prove that a matrix coefficient of a finite dimensional representation can be written as a linear combination of matrix coefficients of irreducible finite dimensional representations. Choose a basis on V. Let $m_{v,w} : G \to \mathbb{C}$ be a matrix coefficient of π for $v = (v_1, ..., v_n)^T \in V$ and $w = (w_1, ..., w_n)^T \in V$. Now we use lemma 4.3.7 to write π as

$$\pi \sim \bigoplus_{[\delta] \in \widehat{G}} \langle \chi_\pi, \chi_\delta \rangle_{L^2} \delta$$

Let us index the $[\delta] \in \widehat{G}$. Say we have irreducible representatives δ_i where $1 \leq i \leq m$ for some $m \in \mathbb{N}$ and each δ_i appears $\langle \chi_\pi, \chi_{\delta_i} \rangle_{L^2}$ times. Denote with $\dim(\delta_i)$ the dimension of the representation δ_i. Let v^i be the corresponding vector in the direct sum, so

$$v^i := (0, ..., 0, v_{f(i)}, v_{f(i)+1}, ..., v_{f(i)+\dim(\delta_i)-1}, 0, ..., 0)^T$$

Here we have introduced a function f defined by $f(1) = 1$ and $f(i) = \sum_{k=1}^{i-1} \dim(\delta_k)$ for $1 < i < m$. Define w^i similar as v^i. Furthermore the vector starts with $f(i) - 1$ zero's and ends with a number of zero's such that $v^i \in V$. Then we see that

$$m_{v,w}(x) = \langle \pi(x)v, w \rangle = \sum_{i=1}^m \langle \delta_i(x) v^i, w^i \rangle$$

□

In order to construct all square integrable functions, there must be some kind of completion. This is the Hilbert direct sum.

Definition 4.4.3 (Hilbert direct sum)**.** Let $\mathcal{A}$ be an index set, let $\mathcal{H}_\alpha$ be a collection of Hilbert spaces for $\alpha \in \mathcal{A}$. Then together with the direct sum inner product $\langle \sum_\alpha v_\alpha, \sum_\alpha w_\alpha \rangle := \sum_\alpha \langle v_\alpha, w_\alpha \rangle$ the direct sum space

$$\bigoplus_{\alpha \in \mathcal{A}} \mathcal{H}_\alpha$$

is a pre-Hilbert space with completion denoted as

$$\widehat{\bigoplus_{\alpha \in \mathcal{A}}} \mathcal{H}_\alpha$$

The completion is the space of sequences $v = (v_\alpha)_{\alpha \in \mathcal{A}}$, with $v_\alpha \in \mathcal{H}$ such that $\|v\|^2 = \sum_{\alpha \in \mathcal{A}} \|v_\alpha\|^2 < \infty$.

Then the Peter-Weyl Theorem states that the Hilbert direct sum of the space of representative functions is $L^2(G)$. The significance of this theorem is that the space of square integrable functions can be decomposed into subspaces which stay invariant under the (left and right) group action of the Lie group on the space. The proof can be found in [2].

Theorem 4.4.4 (Peter-Weyl)**.** *The space $L^2(G)$ of square integrable functions on a Lie group G can be written as the Hilbert direct sum of $C(G)_\delta$, where each $C(G)_\delta$ is an irreducible invariant subspace of the representation $R \times L$ of $G \times G$. In other words*

$$L^2(G) = \widehat{\bigoplus_{\delta \in \widehat{G}}} C(G)_\delta$$

4.5 Class functions

Observe that the characters are invariant under conjugation. Let's study all functions which are invariant under conjugation.

Definition 4.5.1 (Class function)**.** A *class function* $f : G \to \mathbb{C}$ for a compact Lie group G is a function such that $f(g^{-1}hg) = f(h)$ for all $g, h \in G$, i.e. a function which is invariant under conjugation. This allows us to see it as a function on the conjugation classes of G, ie the sets $[h] = \{g \in G | g^{-1}hg = h\}$.
We denote with $L^2(G, \text{class})$ the set of square integrable class functions on G.

We will use these class functions in Section 5.1 to prove that we have found all irreducible representations of SU(2).

5 Spherical harmonics

In this section we will apply the Peter-Weyl theorem twice. First we use it to find all irreducible representations of $SU(2)$. Then we will use this result and the connection between $SU(2)$ and $SO(3)$ to find all irreducible representations of $SO(3)$. Finally we apply Peter-Weyl again to calculate the spherical harmonics. The subsections about SU(2) and SO(3) are based on [2]. The subsection about spherical harmonics is based on [3].

5.1 SU(2)

SU(2) is the group of unitary 2×2-matrices with determinant 1, ie. matrices $A \in \mathrm{GL}(2,\mathbb{C})$ which satisfy $A^*A = \mathrm{Id}$, with $A^* := \overline{A^T}$, complex conjugate of the transpose. Then it can easily be seen that elements $A \in \mathrm{SU}(2)$ are of the form

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C} \text{ and } |\alpha|^2 + |\beta|^2 = 1$$

We will now show that this is indeed a Lie group. First we show that it is a group. Let $A, B \in \mathrm{SU}(2)$ then $(AB^{-1})^*AB^{-1} = (B^{-1})^*A^*AB^{-1} = BA^{-1}AB^{-1} = \mathrm{Id}$, so SU(2) is a subgroup of $\mathrm{GL}(2,\mathbb{C})$, therefore is a group. To show that it is a compact and simply connected manifold, we define the natural map ι from $\mathbb{C}^2$ to $\mathrm{M}(2,\mathbb{C})$ by

$$\iota : \mathbb{C}^2 \to \mathrm{M}(2,\mathbb{C}), (\alpha, \beta) \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

Since ι is injective and by definition of SU(2), the restriction of ι to S^3 gives a diffeomorphism between S^3 and SU(2) as manifolds. In particular S^3 is simply connected and compact. We conclude that SU(2) is both simply connected as compact.

We introduce two special kind of matrices of SU(2).

Definition 5.1.1 (t_ϕ and r_ϕ)**.** Define for $\phi \in \mathbb{R}$ the matrix $r_\phi \in \mathrm{SU}(2)$ as

$$r_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Then $r_\phi^{-1} = r_{-\phi}$, since $r_{-\phi} = r_\phi^T = r_\phi^* = r_\phi^{-1}$. The last equation follows because all matrices in SU(2) are unitary. Furthermore we define $t_\phi \in \mathrm{SU}(2)$ as

$$t_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Then it can easily be seen that $t_\phi^{-1} = t_{-\phi}$. Let furthermore $T = \{t_\phi | \phi \in \mathbb{R}\}$.

Let us introduce some more notation. We denote with $P(\mathbb{C}^2)$ the space of polynomials $p : \mathbb{C}^2 \to \mathbb{C}$. Futhermore, we denote with $P_n(\mathbb{C}^2)$ the subspace of $P(\mathbb{C}^2)$ consisting of homogeneous polynomials of degree n.
Let $\pi : \mathrm{SU}(2) \to \mathrm{GL}(P(\mathbb{C}^2))$ be a representation of SU(2) in the space of polynomials $P(\mathbb{C}^2)$ defined by

$$\pi(g)p(z) = p(g^{-1}z), \text{ for } g \in \mathrm{SU}(2), p \in P(\mathbb{C}^2) \text{ and } z \in \mathbb{C}^2$$

Here g^{-1} is the natural action of SU(2) on $\mathbb{C}^2$ defined by matrix multiplication. We denote with π_n the restriction of π to P_n. It turns out that these representations are irreducible.

Proposition 5.1.2. *For $n \in \mathbb{N}$ let $(\pi_n, P_n(\mathbb{C}^2))$ be representations of* SU(2) *as defined above. Then for each n the representation $(\pi_n, P_n(\mathbb{C}^2))$ is irreducible.*

Proof. By the discussion above SU(2) is compact, therefore using Lemma 3.2.6 π_n is unitarizable. We will show that $\mathrm{End}_{\mathrm{SU}(2)}(P_n(\mathbb{C}^2)) = \mathbb{C}\mathrm{Id}_{P_n(\mathbb{C}^2)}$. Then we can apply Schur's Lemma 3.2.13 to conclude that π_n is irreducible.

Let $P_n(\mathbb{C}^2) \ni p_m(z) := z_1^{n-m} z_2^m$ be a homogeneous polynomial of degree n for $0 \leq m \leq n$. Then $(p_m)_{0\leq m\leq n}$ is obviously a basis for $P_n(\mathbb{C}^2)$. Now, let $\phi \in \mathbb{R}$. Then for all $z = (z_1, z_2)^T \in \mathbb{C}^2$

$$\pi(t_\phi)p_m(z) = (z_1e^{-i\phi})^{n-m}(z_2e^{i\phi})^m = e^{i(2m-n)\phi}z_1^{n-m}z_2^m = e^{i(2m-n)\phi}p_m(z)$$

We conclude that $\pi(t_\phi)p_m = e^{i(2m-n)\phi}p_m$ and that p_m is an eigenvector of $\pi(t_\phi)$. Furthermore we see that there exists a ϕ' such that for each m this results in a distinct eigenvalue. Take for example $\phi' = \frac{\pi}{n}$. Therefore we have found $n+1$ distinct eigenspaces $\mathbb{C}p_m$.

Now let $A \in \mathrm{End}_{\mathrm{SU}(2)}(P_n(\mathbb{C}^2))$. Then by definition $\pi(g)\circ A = A\circ\pi(g)$ for all $g \in \mathrm{SU}(2)$, so in particular for $g = t_{\phi'}$. Thus A commutes with $\pi(t_{\phi'})$ and since the eigenspace is one dimensional, we conclude that also p_m are eigenvectors of A. We call the corresponding eigenvalues λ_m and denote the eigenspaces with E_0. We will show that $\lambda_m = \lambda_0$ for all $0 \leq m \leq n$. Then $A = \lambda_0\mathrm{Id}$ and then the result follows.

$$\begin{aligned}\pi(r_\phi)p_0(z) = (\cos\phi\ z_1 + \sin\phi\ z_2)^n &= \sum_{m=0}^{n}\binom{n}{m}\cos^{n-m}(\phi)\sin^m(\phi)z_1^{n-m}z_2^m \\ &= \sum_{m=0}^{n}\binom{n}{m}\cos^{n-m}(\phi)\sin^m(\phi)p_m(z)\end{aligned}$$

By definition A also commutes with $\pi(r_\phi)$. Therefore $\pi(r_\phi)Ap_0(z) = A\pi(r_\phi)p_0(z)$. Therefore

$$\sum_{m=0}^{n}\binom{n}{m}\cos^{n-m}(\phi)\sin^n(\phi)\lambda_m p_m(z) = \sum_{m=0}^{n}\binom{n}{m}\cos^{n-m}(\phi)\sin^n(\phi)\lambda_m p_0(z)$$

We conclude that

$$\sum_{m=1}^{n}\binom{n}{m}\cos^{n-m}(\phi)\sin^n(\phi)(\lambda_0 - \lambda_m)p_m(z) = 0$$

However, the p_m constitute a basis of $P_n(\mathbb{C}^2)$ and therefore are linear independent. Furthermore this results holds for all $\phi \in \mathbb{R}$. We conclude that $\lambda_0 = \lambda_m$ for all $0 \leq m \leq n$. This proves the proposition. □

In fact, we have found all irreducible representations of SU(2) (up to an equivalence).

Proposition 5.1.3. *Let $\pi : \mathrm{SU}(2) \to \mathrm{GL}(V)$ be a finite dimensional irreducible representation of* SU(2). *Then π is equivalent to π_n for some $n \in \mathbb{N}$.*

Proof.

Claim 1. First we claim that every $A \in \mathrm{SU}(2)$ is conjugate to an element in T. Furthermore t_ϕ is conjugate to $t_{-\phi}$ for every $\phi \in \mathbb{R}$.

Proof. A is of the following form for $\alpha, \beta \in \mathbb{C}$

$$A = \begin{pmatrix}\alpha & -\bar{\beta} \\ \beta & \bar{\alpha}\end{pmatrix}$$

Then the characteristic polynomial is $P(\lambda) = (\alpha - \lambda)(\bar{\alpha} - \lambda) + |\beta|^2 = |\alpha|^2 - (\alpha + \bar{\alpha})\lambda + \lambda^2 + |\beta|^2 = \lambda^2 - (\alpha + \bar{\alpha})\lambda + 1$ First we remark that if we have a solution λ_1 of $P(\lambda) = 0$ then also since $\overline{P(\lambda)} = P(\bar{\lambda})$, we have that $\lambda_2 = \bar{\lambda}_1$ is also a solution. Observe that the trace of a matrix is the sum of the eigenvalues. Therefore $\mathrm{Tr}(A) = \alpha + \bar{\alpha} = \lambda_1 + \bar{\lambda}_1$. For λ_1 then the following holds

$$\begin{aligned}\lambda_1\bar{\lambda}_1 &= \frac{1}{2}\left((\lambda_1 + \bar{\lambda}_1)^2 - \lambda_1^2 - \bar{\lambda}_1^2\right) = \frac{1}{2}\left((\lambda_1 + \bar{\lambda}_1)^2 - (\alpha + \bar{\alpha})(\lambda_1 + \bar{\lambda}_1)\right) + 1 \\ &= \frac{1}{2}\left((\lambda_1 + \bar{\lambda}_1)^2 - (\lambda_1 + \bar{\lambda}_1)(\lambda_1 + \bar{\lambda}_1)\right) + 1 = 1\end{aligned}$$

We conclude that λ has norm one. Therefore there exists a ϕ such that $\lambda_1 = e^{\phi}$. Then A can be diagonalized with a unitary matrix B such that the resulting matrix $\tilde{A}$ has the eigenvalues λ_1 and λ_2 on the diagonal, ie. is an element of T. Suppose B does not have a determinant 1. Then we use $\tilde{B} = BC$ where

$$C := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\tilde{B}$ has a determinant 1. The resulting diagonal matrix will have the eigenvalues interchanged, but this is still an element of T. Observe further that C is indeed an element of SU(2) and that it conjugates t_ϕ with $t_{-\phi}$. □

Using this claim we see that a class function f on SU(2) is completely determined by its value on T. Also, since $t_\phi^{-1} = t_{-\phi}$ it follows that a class function f is invariant under substitution $t_\phi \mapsto t_\phi^{-1}$. We now introduce the notation $C(T)_{ev}$ as the space of continuous functions $g : T \to \mathbb{C}$ such that $g(t^{-1}) = g(t)$ for all $t \in T$. Then we claim that this space is isomorphic to the space of class functions on $SU(2)$.

Claim 2. Let $r : C(\mathrm{SU}(2), \mathrm{class}) \to C(T)_{ev}$ be the restriction map of SU(2) to T. Then r is bijective and preserves sup-norms.

Proof. The preservation of the sup-norm can easily be seen, because since a class function is completely determined by its restriction on T it follows in particular that the maximum (since SU(2) is compact) of $f \in C(\mathrm{SU}(2), \mathrm{class})$ is equal to the maximum on $C(T)_{ev}$.

Remark that using the previous claim the value of a class function f is completely determined by its value on T. Also t_ϕ is conjugate to $t_{-\phi} = t_\phi^{-1}$. This proves injectivity.

We only need to prove surjectivity. Let $g \in C(T)_{ev}$. As seen in the proof of the previous claim, the eigenvalues of A come in pairs: suppose λ_1 is an eigenvalue of A, then also $\bar{\lambda}_1$ is an eigenvalue of A and $\lambda_1\bar{\lambda}_2 = 1$. Now define the map $f : \mathrm{SU}(2) \to \mathbb{C}$ as $f(x) = g(h(x))$ for $x \in \mathrm{SU}(2)$, where

$$h(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \bar{\lambda}(x) \end{pmatrix}$$

where $\lambda : \mathrm{SU}(2) \to \mathbb{C}$ is a function which maps $x \in \mathrm{SU}(2)$ to the eigenvalue of x. Note that this is well-defined because $h(x) \in T$ for all $x \in \mathrm{SU}(2)$ and $g(t) = g(t^{-1})$ for all $t \in T$. Furthermore, f is indeed a class function because if $A, B \in \mathrm{SU}(2)$ are conjugate, they have the same eigenvalues. To see this we remark that $\mathrm{Tr}(A) = \lambda(A) + \bar{\lambda}(A)$ and that $\lambda(A)\bar{\lambda}(A) = 1$. This results in

$$\lambda(A) + \lambda(A)^{-1} = \mathrm{Tr}(A)$$

We see that $\lambda(A)$ only depends on the trace. Therefore it is conjugate invariant. Furthermore, we see that the functions λ and $\bar{\lambda}$ are continuous. Now we see that h is continuous. g is continuous by definition. We conclude that f is continuous, since it is a composition of continuous maps. Furthermore we have $r(f) = g$. This proves the claim. □

Let $\chi_n = \chi_{\pi_n}$ be the character of π_n.

Claim 3. The linear span of all characters χ_n with $n \in \mathbb{N}$ is dense in $C(\mathrm{SU}(2), \text{class})$, .

Proof. In the proof of Proposition 5.1.2 we have seen that $p_m = z_1^{n-m} z_2^n$ with $0 \leq m \leq n$ constitutes a basis of $P_n(\mathbb{C}^2)$ and that π_m are eigenvectors with eigenvalue $e^{i(n-2m)\phi}$. Then the trace is the sum of the eigenvalues, so $\chi_n(t_\phi) = \sum_{m=0}^{n} e^{i(n-2m)\phi}$.
Using the previous claim we see that we only have to show that the linear span of functions $\chi_n|_T$ is dense in $C(T)_{ev}$. Then this linear span is also spanned by functions $\gamma_n := \chi_n - \chi_{n-1} : T \to C(T)_{ev}, t_\phi \mapsto e^{in\phi} + e^{-in\phi}$. Since we have that for $g \in C(T)_{ev}$ that $g(t_\phi) = g(t_\phi^{-1}) = g(t_{-\phi})$, from Fourier theory it follows that the span of γ_n is dense in $C(T)_{ev}$. □

Suppose now that there exists a $[\delta] \in \widehat{G}$ such that for every $n \in \mathbb{N}$, $\delta \nsim \pi_n$. Then the for the class functions it holds that $\langle \chi_\delta, \chi_n \rangle = 0$. Since for $f \in C(G)$ where G is a compact Lie group holds that

$$\|g\|_{L^2}^2 = \int_G |g(x)|^2 dx \leq \|g\|_\infty^2 \int_G dx = \|g\|_\infty^2$$

the character functions χ_n are also dense in $C(\mathrm{SU}(2), \text{class})$ with respect to the L^2 norm. It follows that $\chi_\delta = 0$. This, however proves a contradiction, because $\chi_\delta(e) = \dim(\delta) \neq 0$. □

We will now calculate the Lie algebra of SU(2).

Lemma 5.1.4. *The Lie algebra* $\mathfrak{su}(2)$ *of* SU(2) *is equals the algebra of matrices* $X \in \mathrm{M}(2, \mathbb{C})$ *where* $X^* = -X$ *and* $\mathrm{Tr}(X) = 0$.

Proof. Using Theorem 2.4.6 we can identify the Lie algebra of SU(2) with the tangent space at the identity $T_e\mathrm{SU}(2)$. Then using Theorem 2.4.7 we can identify Lie(SU(2)) with $\mathfrak{su}(2)$. Also, we will use that the exponential exp is a map $\mathrm{Lie}(\mathrm{SU}(2)) \to \mathrm{SU}(2)$ by Proposition 2.5.4. Furthermore, we use that for matrices $X \in \mathrm{M}(n, \mathbb{C})$ we have that $\exp(X) = e^X$ from Lemma 2.5.6.
Then the goal is to find a representation of the Lie algebra $\mathfrak{su}(2)$ in $\mathrm{End}(\mathbb{C}^2) \simeq \mathrm{M}(2, \mathbb{C})$. Let $\pi : \mathrm{SU}(2) \to \mathrm{GL}(\mathbb{C}^2) \simeq \mathrm{GL}(2, \mathbb{C})$ be the natural representation of SU(2) defined as the inclusion of SU(2) into $\mathrm{GL}(\mathbb{C}^2)$. Then according to Proposition 3.3.4 we have that for the induced representation $\pi_* : \mathfrak{su}(2) \to \mathrm{End}(\mathbb{C}^2)$ the following holds

$$\pi_*(X)z = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))z \quad \text{where } z \in \mathbb{C}^2 \text{ and } X \in \mathfrak{su}(2)$$

Then using what we have mentioned above, this simplifies to

$$Xz = \frac{d}{dt} e^{tX} z \tag{10}$$

Now let $X \in \mathfrak{su}(2)$. Then, since $\exp tX \in \mathrm{SU}(2)$ for all $t \in \mathbb{R}$.

$$\mathrm{Id} = (e^{sX})^* e^{sX} = e^{sX^*} e^{sX}$$

Then using (10) and the chain rule, we see that $0 = X^* + X$. To see that X is traceless we note that for matrices it holds that

$$\mathrm{Tr}(X) = \frac{d}{dt}\bigg|_{t=0} \det(e^{tX}) = \frac{d}{dt}\bigg|_{t=0} 1 = 0 \tag{11}$$

Conversely suppose $X \in \mathrm{M}(2, \mathbb{C})$ for which $X^* = -X$ and $Tr(X) = 0$. Then we use the identity $1 = \det(e^X) = e^{\mathrm{Tr}(X)}$. It follows that $\mathrm{Tr}(X) = 0$. Furthermore

$$(e^{tX})^* e^{tX} = e^{tX^*} e^{tX} = e^{-tX} e^{tX} = e^0 = \mathrm{Id}$$

We conclude that $e^{tX} \in \mathrm{SU}(2)$ for all $t \in \mathbb{R}$. Then we use Proposition 2.5.5 and conclude that $X \in \mathrm{Lie}(\mathrm{SU}(2)) = \mathfrak{su}(2)$. $\square$

Using this lemma we find basis elements τ_j, $j = 1, 2, 3$ for $\mathfrak{su}(2)$ with

$$\tau_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \tau_3 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Using Example 2.4.4 and Theorem 2.4.7 we see that the Lie bracket is just the commutator of matrices. Straightforward calculation then results in the following relations:

$$[\tau_1, \tau_2] = 2\tau_3, \quad [\tau_2, \tau_3] = 2\tau_1, \quad [\tau_3, \tau_1] = 2\tau_2$$

5.2 SO(3)

$SO(3)$ is the group of matrices $X \in \mathrm{GL}(3, \mathbb{R})$ where $X^T X = \mathrm{Id}$ and $\det(X) = 1$.

Lemma 5.2.1. *The Lie algebra* $\mathfrak{so}(3)$ *of* $\mathrm{SO}(3)$ *is given by the algebra of matrices* $X \in \mathrm{M}(3, \mathbb{R})$ *where* $X^* = -X$.

Proof. The proof goes similar as in Lemma 5.1.4. $\square$

Lemma 5.2.2. $SO(3)$ *is the group of rotations in* $\mathbb{R}^3$.

Proof. Let $A \in \mathrm{M}(3, \mathbb{R})$ be a rotation in $\mathbb{R}^3$, for $x \in \mathbb{R}^3$. Then, because rotations leave the norm of the vector invariant:

$$\langle Ax, Ax \rangle = \langle x, x \rangle \tag{12}$$

We conclude that $A^* A = \mathrm{Id}$. Furthermore let also $e_1, e_2, e_3 \in \mathbb{R}^3$ be the orthonormal basis in $\mathbb{R}^3$. Then because rotations leave the cross product invariant

$$1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = e_1 \cdot (e_2 \times e_3) = Ae_1 \cdot (Ae_2 \times Ae_3) = \det(A) \tag{13}$$

We conclude that A in SO(3).
Let $A \in \mathrm{SO}(3)$. Then using (12) and (13) the other way around we see that A is a rotation. $\square$

Using Lemma 5.2.1, we find that $R \in \mathfrak{so}(3)$ is traceless and antisymmetric and therefore is of the form

$$R = \begin{pmatrix} 0 & a_1 & -a_2 \\ -a_1 & 0 & a_3 \\ a_2 & -a_3 & 0 \end{pmatrix} \quad \text{where } a_i \in \mathbb{R} \ \ 1 \leq i \leq 3$$

We see that basis elements R_j, $j = 1, 2, 3$ for $\mathfrak{so}(3)$ are given by

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Then it can be seen by straightforward calculation that

$$[2R_1, 2R_2] = 4R_3, \quad [2R_2, 2R_3] = 4R_1, \quad [2R_3, 2R_1] = 4R_2$$

and that

$$R_1^2 = R_2^2 = R_3^2 = -1 \tag{15}$$

Definition 5.2.3. Let R_i as in (14). Then we define the Casimir operator $\mathcal{C} := -(R_1^2 + R_2^2 + R_3^2)$.

Lemma 5.2.4. *The Casimir operator commutes with R_i for $1 \leq i \leq 3$.*

Proof. We will show that R_1 commutes with $\mathcal{C}$. The other two are similar.

$$\begin{aligned} -[\mathcal{C}, R_1] &= \sum_{j=1}^{3} [R_j^2, R_1] = [R_2^2, R_1] + [R_3^2, R_1] \\ &= R_2[R_2, R_1] + [R_2, R_1]R_2 + R_3[R_3, R_1] + [R_3, R_1]R_3 \\ &= -(R_2R_3 + R_3R_2) + (R_3R_2 + R_2R_3) = 0 \end{aligned} \tag{16}$$

□

It turns out that the Lie algebra of SU(2) is isomorphic to the Lie algebra of SO(3).

Lemma 5.2.5. *Let $\phi_* : \mathfrak{su}(2) \to \mathfrak{so}(3)$ be the linear map defined by $\phi_*(\tau_i) = 2R_i$, $i = 1, 2, 3$. Then this is a Lie algebra isomorphism.*

Proof. The map is by definition linear and bijective. Furthermore, for the basis elements the following holds:

$$\phi_*([\tau_i, \tau_j]) = \phi_*(2\epsilon_{ijk}\tau_k) = 2\epsilon_{ijk}R_k = [R_i, R_j] = [\phi_*(\tau_i), \phi_*(\tau_j)]$$

Here we used the calculations above. Since ϕ is linear and the Lie bracket is bilinear the result follows. □

One might wonder if the Lie groups themselves are also equal. This is not the case. There is however a natural homomorphism from SU(2) to SO(3).

Proposition 5.2.6. *Let $\Phi : \mathrm{SU}(2) \to \mathrm{End}(\mathfrak{su}(2)), \Phi(A) := \Phi_A$ where $\Phi_A \in \mathrm{End}(\mathfrak{su}(2))$ is defined by*

$$\Phi_A(X) = AXA^{-1}$$

Then the map $\phi : \mathrm{SU}(2) \to \mathrm{GL}(3, \mathbb{R})$ defined by $\phi(A) := \mathrm{mat}\ \Phi(A)$ is a group homomorphism. Here mat *is taken with respect to the basis elements τ_j of* SU(2)*. In particular it induces an isomorphism $\mathrm{SU}(2)/\{\pm \mathrm{Id}\} \simeq \mathrm{SO}(3)$.*

Proof. First we prove that the map Φ is well defined. For $A \in \mathrm{SU}(2)$ we have that $A^*A = \mathrm{Id}$, by definition. Then for $X \in \mathfrak{su}(2)$

$$AXA^{-1} = AXA^* = AX^*A^* = -AXA^*$$

Furthermore, $\mathrm{Tr}(AXA^{-1}) = \mathrm{Tr}(X) = 0$. We conclude that $\Phi_A(X) \in \mathfrak{su}(2)$.
Notice further that $\Phi_A^{-1} = \Phi_{A^{-1}}$, so we conclude that indeed ϕ maps into $\mathrm{GL}(n, \mathbb{R})$. To see that $n = 3$ note that mat is taken with respect to the basis elements of $\mathrm{SU}(2)$.
The map Φ is a homomorphism, because for $A, B \in \mathrm{SU}(2)$ and for all $U \in \mathfrak{su}(2)$

$$\Phi(A)\Phi(B)(U) = \Phi_A\Phi_B(U) = BAUA^{-1}B^{-1} = BAU(BA)^{-1} = \Phi(AB)(U)$$

We now also have proved that ϕ is a homomorphism.

We now have the following claim about the induced Lie algebra homomorphism.

Claim 1. The induced Lie algebra homomorphism of ϕ is ϕ_* from Lemma 5.2.5.

Proof. We note that by the chain rule for the induced Lie algebra homomorphism $\bar{\phi}_*$ we have

$$\bar{\phi}_*(X) = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp tX)$$

We will check the equality of ϕ_* and $\bar{\phi}_*$ on basis elements τ_i. Then using that $\exp = e^{\cdot}$ we see that this is just the tangent map of the corresponding element in $\mathrm{SO}(3)$. A simple calculation proves the claim. $\square$

From this claim it follows directly that ϕ maps into $\mathrm{SO}(3) \subset \mathrm{GL}(3, \mathbb{R})$. We first make a claim about surjective maps.

Claim 2. Let $f : U \to W$, $g : V \to W$ and $h : U \to V$. Suppose $f = g \circ h$ and f is surjective. Then g is surjective.

Proof. Let $w \in W$. Then by surjectivity of f there exists a $u \in U$ such that $f(u) = w$. Then also $g(h(u)) = w$, so we conclude that g is surjective. $\square$

We only need to prove that ϕ is surjective. Using Lemma 5.2.5 we have that ϕ_* is an isomorphism. In particular it is surjective. Suppose $\exp_{\mathrm{SO}(3)} : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is surjective. Then the composition $\exp_{\mathrm{SO}(3)} \circ \phi_*$ is surjective. Furthermore using Proposition 2.5.4 we have that $\phi \circ \exp_{\mathrm{SU}(2)} = \exp_{\mathrm{SO}(3)} \circ \phi_*$. We apply the previous claim and conclude that ϕ is surjective. What remains to be shown is that $\exp_{\mathrm{SU}(2)}$ and $\exp_{\mathrm{SO}(3)}$ are surjective.

Claim 3. $\exp_{\mathrm{SO}(3)}$ is surjective.

Proof. Let $r_{1,\phi} \in \mathrm{SO}(3)$, $\phi \in \mathbb{R}$. Assume that $r_{1,\phi}$ is of the form

$$r_{1,\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Let R_i for $1 \leq i \leq 3$ as in (14). Then using (15) and writing $e^{\phi R_1}$ written as a power series, one can see that $\exp(\phi R_1) = r_{1,\phi}$.

Similar, one also has that $\exp(\phi R_2) = r_{2,\phi}$ and $\exp(\phi R_3) = r_{3,\phi}$ where

$$r_{2,\phi} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \quad r_{3,\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can see that $r_{i,\phi}$ is a rotation around the e_i axis, for $1 \leq i \leq 3$. Then we use that any rotation can be decomposed in multiple rotations around the different e_i axes. We conclude that $\exp_{\mathrm{SO}(3)}$ is surjective. □

We conclude that ϕ is a surjective group homomorphism, so $SO(3) \simeq \mathrm{SU}(2)/\mathrm{Ker}(\phi)$. Let $x \in \mathrm{Ker}(\phi)$. Then $\tau_i x = x\tau_i$ for $i = 1, 2, 3$. Therefore $x = \mathrm{Id}$ or $x = -\mathrm{Id}$. We conclude that $\mathrm{Ker}(\phi) = \{\pm\mathrm{Id}\}$. □

Using this isomorphism, from the irreducible representations π_{2l} of $\mathrm{SU}(2)$ we can obtain representations of $\mathrm{SO}(3)$.

Proposition 5.2.7. *The representations π_{2l} of $\mathrm{SU}(2)$ for $l \in \mathbb{N}$ factorizes to a representations $\bar{\pi}_{2l}$ of $\mathrm{SO}(3)$, which are mutually inequivalent and exhaust $\widehat{\mathrm{SO}(3)}$.*

Proof. Let $l \in \mathbb{N}$ and let $x \in \{\pm\mathrm{Id}_{\mathrm{SU}(2)}\}$. Then $\pi_{2l}(x)p_m(z) = p_m(xz) = z_1^{2l-m}z_2^m = \pi_m(z)$, therefore $\pi_{2l}(x) = \mathrm{Id}_{P(\mathbb{C}^2)}$. Therefore, using the homomorphism $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ we can construct a representation $\bar{\pi}_{2l} : \mathrm{SO}(3) \to \mathrm{GL}(P(\mathbb{C}^2))$ of $\mathrm{SO}(3)$. We do this as follows. Let $x \in \mathrm{SO}(3)$ then choose some $y \in \phi^{-1}(x)$ and define $\bar{\pi}_{2l}(x) = \pi_{2l}(y)$. This is well defined because $\ker(\phi) = \{\pm\mathrm{Id}]\}$ and $\pi_{2l}(-x) = \pi_{2l}(x)$. The representations are mutually inequivalent. Suppose not, then there exist a l and n such that π_{2l} and π_{2n} are equivalent. But using Proposition 5.1.3 then we must have that $l = n$. This results in a contradiction.

To see that $\bar{\pi}_{2l}$ exhaust $\widehat{\mathrm{SO}(3)}$ let $\pi : \mathrm{SO}(3) \to \mathrm{GL}(V)$ be an irreducible representation of $\mathrm{SO}(3)$ in V. We need to show that π is equivalent to some $\bar{\pi}_{2l}$. Now $\pi \circ \phi$ is an irreducible representation on $\mathrm{SU}(2)$, so $\pi \circ \phi \sim \pi_n$ for some $n \in \mathbb{N}$ by Proposition 5.1.3. Then because $\pi \circ \phi = \mathrm{Id}$ on $\{\pm\mathrm{Id}\}$ we see that n must be even. Then $\bar{\pi}_n$ is the required representation. □

5.3 Spherical harmonics

We will now apply what we have learned about Lie group representations of $SO(3)$ and about Peter-Weyl to the study of spherical harmonics.

Definition 5.3.1 (Harmonic functions)**.** We call a function $f : \mathbb{R}^n \to \mathbb{C}$ *harmonic* if $f \in C^\infty(\mathbb{R}^n)$ and $\Delta f = 0$, where Δ is the Laplacian.
In the special case $n = 3$ we denote by

$$\mathfrak{H}_l = \{p \in P_l(\mathbb{R}^3) | \Delta p = 0\}$$

the space of homogeneous harmonic polynomials in $P_l(\mathbb{R}^3)$.

Definition 5.3.2. We define the $\mathrm{SO}(3)$-module $\mathfrak{H}_l$ as the representation $\rho_l : \mathrm{SO}(3) \to \mathrm{GL}(\mathfrak{H}_l)$ given by $\rho_l(g)p(x) = p(g^{-1}x)$, for $g \in \mathrm{SO}(3)$ and $x \in \mathbb{R}^3$.

Definition 5.3.3. Let G be a group, H a closed subgroup and $p : G \to G/H$ the projection map. Then we define the *pull-back* map $p^* : C(G/H) \to C(G), f \mapsto f \circ p$. Then this map is injective and $\mathrm{Im}(p^*) = C(G)^H$. We will use this to identify $C(G/H)$ with $C(G)^H$.
Suppose G is a Lie group and H is a Lie subgroup. Using the identification one defines $C(G/H)_\pi$ as $C(G)^H_\pi \subset C(G)_\pi$ as matrix coefficients which leave H invariant.

Proposition 5.3.4.

$$L^2(S^2) = \widehat{\bigoplus_{l \in \mathbb{N}} C(S^2)_l}$$

Here $C(S^2)_l \simeq P_{2l}(\mathbb{C}^2)$.

Proof. Let $\iota : \mathrm{SO}(2) \to \mathrm{SO}(3)$ be the injective group homomorphism defined by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

Then the map $a : \mathrm{SO}(3) \to S^2$ defined by rotation of the first unit basis $x \mapsto xe_1$ is a homomorphism of groups and induces an isomorphism $\hat{a} : \mathrm{SO}(3)/\mathrm{SO}(2) \to S^2$. Also on S^2 we can define a normalized Haar measure. Using Peter-Weyl for SO(3), the identification from previous definition and the identification of $\mathrm{SO}(3)/SO(2)$ with S^2 we have

$$L^2(S^2) = \widehat{\bigoplus_{l \in \mathbb{N}} C(S^2)_l}$$

This can be seen by observing that linear combinations of maps which a subgroup invariant, also leave the subgroup invariant.

Now $C(\mathrm{SO}(3)/\mathrm{SO}(2))_l$ is equivariantly isomorphic to $\mathrm{Hom}(P_{2l}(\mathbb{C}^2)^{\mathrm{SO}(2)}), P_{2l}(\mathbb{C}^2)$, the space of linear homomorphisms from $P_{2l}(\mathbb{C}^2)^{\mathrm{SO}(2)}$ to $P_{2l}(\mathbb{C}^2)$. Here $P_{2l}(\mathbb{C})^{\mathrm{SO}(2)}$ is the space that stays invariant under all group actions of SO(2) seen as a closed subgroup of SO(3). This follows from group theory.
Observe that $\mathrm{SO}(2) = \exp tR_1$ where R_1 is as defined above, $t \in \mathbb{R}$. This can be seen using calculations from the proof of Proposition 5.2.6. Then the pre-image of SO(2) under the map ϕ a defined in the same proposition is T. Therefore $P_{2l}(\mathbb{C}^2)^{SO(2)} = P_{2l}(\mathbb{C}^2)^T$. Now, we had that $\pi_{2l}(t_\phi)p_m(z) = e^{i(2m-2l)\phi}p_m(z)$. Therefore, we see that $C^(S^2)_l$ is isomorphic to $P_{2l}(\mathbb{C}^2)$. □

The action of $SO(3)$ induces a representation L of $\mathfrak{so}(3)$ in $C^\infty(\mathbb{R}^3)$ given by

$$L(R)f(x) = \frac{d}{dt} f(e^{tR}x)|_{t=0}$$

Here we used Proposition 3.3.4. Explicit calculation results in

$$L(R_i) = \epsilon_{ijk} x_m \frac{\partial}{\partial x_j} \tag{17}$$

We will use this in the proof of Proposition 5.3.5 and in Lemma 5.3.6.
Now we are ready to calculate all spherical harmonics.

Proposition 5.3.5. *For $l \in \mathbb{N}$, the polynomials*

$$Y_l^m = (-1)^m \left(\frac{(2l+1)(l+m)!}{4\pi(l-m)!} \right)^{\frac{1}{2}} e^{im\phi} P_l^m(cos\theta)$$

where

$$P_l^m(s) = \frac{(1-s^2)^{-m/2}}{2^l l!} \frac{d^{l-m}}{ds^{l-m}}(s^2-1)^l, \quad |m| \leq l$$

constitute a basis for $C(S^2)_l$ with respect to the L^2-norm. Here θ and ϕ are polar coordinates on S^2.

Proof. First we note that the polynomial $\mathrm{Y}_l(x) := (x_2 + ix_3)^l \in \mathfrak{H}_l$. We now use spherical coordinates $x = (\cos\theta, \sin\theta e^{i\phi})$, where we make the canonical identification of $\mathbb{R} \times \mathbb{C}$ with $\mathbb{R}^3$. Then we define the restriction $\mathrm{Y}_l := \mathrm{Y}_l|S^2$. Trivially $Y_l = (\sin\theta)^l e^{il\phi}$. Let $T : C(S^2)_l \to P_{2l}(\mathbb{C}^2)$ be the equivariant isomorphism from $C(S^2)_l$ to $P_{2l}(\mathbb{C}^2)$ of SU(2)-modules. We have that $T(Y_l)$ is a scalar multiple of $p_{2l} = z_2^{2l}$.

Define the matrix $X^- \in \mathfrak{su}(2)$ as

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then it can easily be seen that $\pi_{2l}(iX^-)^m p_{2l}$ constitute a basis for $P_{2l}(\mathbb{C}^2)$. Therefore $Y_l^{l-m} := (iX^-)^m Y_l$ constitute a basis for $C(S^2)_l$.

Since we had that $\pi_{2l}(t_\phi)p_m(z) = e^{i(2m-2l)\phi}p_m(z)$ we conclude that for $-l \leq m \leq l$ $Y_l^m = e^{ik\phi} f_m(\theta)$ for some $f_m(\theta)$. We remark that $X^- = \frac{1}{2}(\tau_2 - i\tau_3)$. Under the image of ϕ this becomes $R_2 - iR_3$. Therefore we get that $Y_l^{l-m} = (iR_2 + R_3)^m Y_l$.

Rewriting (17) in spherical coordinates results in

$$iR_2 + R_3 = e^{-i\phi} \left(\frac{\partial}{\partial\theta} - i\cot\theta \frac{\partial}{\partial\phi} \right)$$

Comparing $\mathrm{Y}_l = (\sin\theta)^l e^{il\phi}$, $Y_l^m = e^{im\phi} f_m(\theta)$ and $Y_l^{l-m} = (iR_2 + R_3)^m Y_l$ we see that

$$f_l = (sin\theta)^l$$

Therefore

$$-\frac{df_m}{d\theta} - k\cot\theta f_m = f_{m-1} \quad \text{where } |m| \leq l$$

Then for $0 \leq \theta \leq \pi$ we do the substitution $s = \cos\theta$ and define q_m as the function $q_m(s) = f_m(\arccos(s))$ for $-1 \leq s \leq 1$ We see that $q_m(\cos\theta) = f_m(s)$. Then we get that

$$(1-s^2)^{\frac{1}{2}} \left(\frac{dq_m}{ds} - \frac{ks}{1-s^2} q_m \right) = q_{m-1}$$

Now we define $u_m(s) := (1-s^2)^{\frac{m}{2}} q_m(s)$. Then the equation simplifies to

$$\frac{du_m}{ds} = u_{m-1}$$

For $m = l$ it holds in particular that

$$u_l(s) = (1-s^2)^{\frac{l}{2}} q_l(s) = (1-s^2)^l$$

Then by induction we see that

$$u_m(s) = \frac{d^{l-m}}{ds^{l-m}}(1-s^2)^l \quad \text{where } |m| \leq l \tag{18}$$

Then using that $Y_l^m = e^{im\phi} f_l(\theta)$, that $q_m(\cos\theta) = f_m(\theta)$, that $u_m(s) = (1-s^2)^{\frac{m}{2}} q_m(s)$ and equation (18) we see that for $0 \leq \theta \leq \pi$

$$Y_l^m = C_l^m e^{im\phi} q_m(\cos\theta)$$

where C_l^m is any constant form a basis for $C(S^2)$. Here

$$q_m = (1-s^2)^{-\frac{m}{2}} \frac{d^{l-m}}{ds^{l-m}}(1-s^2)^l$$

Then we note that these functions are also well defined for $\pi < \theta < 2\pi$. The result follows by choosing appropriate constants C_l^m. □

Lemma 5.3.6. *The action $L(\mathcal{C}) : C^2(\mathbb{R}^3) \to C(\mathbb{R}^3)$ of the Casimir operator acts as the scalar $l(l+1)$ on $\mathfrak{h}_l$.*

Proof. Using that $\mathcal{C} = -(R_1^2 + R_2^2 + R_3^2)$ and (17) straightforward calculation shows that

$$L(\mathcal{C}) = -x^2\Delta + E(E+1) \tag{19}$$

where $x^2 = \sum_j x_j^2$ and E is the Euler operator defined by

$$E := \sum_{j=1}^{3} x_j \frac{\partial}{\partial x_j}$$

Since $\Delta p = 0$ for every $p \in \mathfrak{h}_l$ we immediately see that $L(\mathcal{C})$ acts as the scalar $l(l+1)$. □

Now we have shown that any square integrable function on S^2 can be written in terms of spherical harmonics. This reminds us of Fourier theory, where we can also write any square integrable function on S^1 in terms of the harmonic function e^{x+iy}. In fact, the theory of Fourier series is a special case of the Peter-Weyl theorem. We will not prove our point here. An interested reader could look at the group $G = \mathbb{R}^n/2\pi\mathbb{Z}^n$ together with the character $\chi_l : x \mapsto e^{i(m\cdot x)}$.

6 The Schrödinger equation for the hydrogen atom

The time independent Schrödinger equation for the hydrogen atom is given by

$$E\psi = \frac{-\hbar^2}{2m}\Delta\psi + V(r)\psi$$

Here $\psi \in L^2(\mathbb{R}^3)$ is the wave function of the electron, E the energy, m the mass and $\hbar$ the reduced Planck constant. $V(r) = -\frac{e^2}{r}$ is the electric potential, which only depends on the radial distance $r := \|x\| \in \mathbb{R}^+$ with $x \in \mathbb{R}^3$. We will also sometimes simply write V instead of $V(r)$. Furthermore, e is a constant defined by $e = \frac{q}{2\sqrt{\pi\epsilon_0}}$, where q is the charge of the electron and ϵ_0 is the dielectric constant.

The goal of this section is to find all solutions of the Schrödinger equation for which the energy E is negative. This corresponds to states for which the electron is bound to the hydrogen atom. We won't look into the unbounded states, ie. the scattering states, since the required theorems needed for scattering states are beyond the scope of this thesis.

We will only consider the time-independent Schödinger equation. In that case the wave function ψ is a square integrable function on $\mathbb{R}^3$, so $\psi \in L^2(\mathbb{R}^3)$ and furthermore the wave function is at least twice continuously differentiable.

In the previous section we applied the Peter-Weyl theorem to calculate spherical harmonics. It is a well known fact that there is a deep relation between spherical harmonics and bounded eigenstates of the Schrödinger equation for the Hydrogen atom. It is, however, not at all clear why this is the case. In this section we will show why there exists such a relation. We first do a recap of the Hilbert tensor product, then we will show that $L^2(\mathbb{R}^3)$ can be decomposed in a Hilbert direct sum of SO(3)-invariant subspaces. Thirdly, we will show that in order to solve the Schrödinger equation on $L^2(\mathbb{R}^3)$ we only need to solve it on the invariant subspaces. Finally we will do the explicit calculations.

This section is based on [4].

6.1 Hilbert tensor product

For the separation of variables we need the Hilbert tensor product. Let us therefore recall some important aspects of the Hilbert tensor product.

Let $\mathcal{H}$ be a Hilbert space and let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Then every $x \in \mathcal{H}$ can be written as the (in general infinite) sum

$$x = \sum_{j\in\mathbb{N}} c_j e_j \qquad \text{where } c_j = \langle x, e_j\rangle \in \mathbb{C}$$

This sum converges with respect to the norm on $\mathbb{H}$ which is induced by the inner product.

Definition 6.1.1 (Algebraic tensor product)**.** Let V, W be two complex linear spaces. Then the algebraic tensor product of V and W is a complex linear space $V \otimes^{\text{alg}} W$ together with a bilinear map $\beta : V \times W \to V \otimes^{\text{alg}} W, (v, w) \mapsto v \otimes w$ such that for any complex linear space U and bilinear map $b : V \times W \to U$ there exists a unique linear map $\bar{b} : V \otimes^{\text{alg}} W \to U$ such that $b = \bar{b} \circ \beta$.

We assume that the reader has some knowledge about tensor product. In particular we assume that the reader knows that for complex linear spaces V, W the algebraic tensor product $V \otimes^{\text{alg}} W$ exists.

It can easily be seen that for finite sums $\sum_{j=1}^{k} v_j \otimes w_j$ we have

$$\bar{b}(\sum_{j=1}^{k} v_j \otimes w_j) = \sum_{j=1}^{k} b(v_j, w_j) \qquad \text{where } k \in \mathbb{N}$$

Using this relation it can be shown that if $(e_i)_{i\in I}$ is a linear basis of V and $(f_i)_{j\in J}$ is a linear basis of W, then $(e_i \otimes f_j)_{(i,j)\in I\times J}$ is a linear basis of $V \otimes^{\text{alg}} W$.

The following lemma is result from the theory of the Hilbert tensor product and we assume that the reader is already familiar with it.

Lemma 6.1.2. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and let $\mathcal{H}\otimes^{alg}\mathcal{K}$ be the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$. Then there exists a unique Hermitian inner product $\langle\cdot,\cdot\rangle$ such that*

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle = \langle v_1, v_2\rangle\langle w_1, w_2\rangle \qquad \textit{for all } v_1, v_2 \in \mathcal{H} \textit{ and } w_1, w_2 \in \mathcal{K}$$

Using this lemma we can define the Hilbert tensor product.

Definition 6.1.3 (Hilbert space tensor product)**.** Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $\mathcal{H} \otimes^{\text{alg}} \mathcal{K}$ be the Algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$. Then we define the *Hilbert space tensor product* $\mathcal{H} \otimes \mathcal{K}$ to be the Hilbert completion of $\mathcal{H} \otimes^{\text{alg}} \mathcal{K}$ with respect to the norm induced by the inner product from Lemma 6.1.2.

It can easily be seen that if $(e_i)_{i \in I}$ is an orthonormal basis of $\mathcal{H}$ and $(f_j)_{j \in J}$ is an orthonormal basis of $\mathcal{K}$ that $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is an orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$. Indeed, it is a basis of the dense subset $\mathcal{H} \otimes^{\text{alg}} \mathcal{K}$ and the following relation holds

$$\langle e_i \otimes f_j, e_k \otimes f_l \rangle = \langle e_i, e_k \rangle \langle f_j, f_l \rangle = \delta_{ik} \delta_{jl}$$

6.2 Decomposition of $L^2(\mathbb{R}^3)$

In order to solve the Schrödinger equation for the Hydrogen atom, physicists apply an important trick. Given a wave function ψ they assume it can be separated into a radial part $R(r) \in L^2(\mathbb{R}^+, r^2 dr)$ and a spherical part $Y(\theta, \phi) \in L^2(S^2)$, so that $\psi(r, \theta, \phi) = R(r) Y(\theta, \phi)$. Here $\mathbb{R}^+ := \{x \in \mathbb{R} | x > 0\}$. In this section we will show why the wave function can be separated into a radial and spherical part. This will be the main result of this section.

In order for us to be able to make sense of such a separation, we need to define a measure on S^2. We will do this for a general submanifold $M \subset \mathbb{R}^n$ by defining a density $d\sigma$ on M and look at the measure associated with this density. Recall from Section 4.1 that a density on a manifold is an element of the bundle $\mathcal{D}TM$ of densities on M. Furthermore for each point $m \in M$ we have that the restriction $(d\sigma)_m$ is a density on $T_m M$, in other words it is a map $(d\sigma)_m : (T_m M)^k \to \mathbb{C}$ for which the following relation holds, with $k = \dim M$:

$$(d\sigma)_m(Tv_1, ..., Tv_k) = |\det T| (d\sigma)_m(v_1, ..., v_k) \quad \text{for all } T \in \text{End}(T_m M) \text{ and } v_i \in T_m M \tag{20}$$

Definition 6.2.1. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $\dim M = k$. Let $m \in M$ be a point on M and let $T_m M$ be the tangent space at this point. Then, the tangent space $T_m M$ can be identified with a linear subspace of $\mathbb{R}^n$, so there exists a natural inner product $\langle \cdot, \cdot \rangle_m$ on $T_m M$. Now choose an orthonormal basis $(v_i)_{1 \le i \le k}$ on $T_m M$ with respect to this inner product. Then we define $(d\sigma)_m$ as the density on $T_m M$ satisfying:

$$(d\sigma)_m(v_1, ..., v_k) = 1 \tag{21}$$

Remark that we can always do this, by normalising a given density. Note furthermore that the density is uniquely defined by Equation 21 and Equation 20.

Lemma 6.2.2. *Let $m, M, (d\sigma)_m$ and $T_m M$ as in Definition 6.2.1. Then $(d\sigma)$ is a density on M and defines a measure $d\sigma$ on M.*

Proof. We only need to show that $(d\sigma)_m$ depends smoothly on m. This would require a notion of smoothness on the bundle of densities on M, which we will not do here. For the proof we refer to [1]. □

Now we are ready to define a measure on S^2. Look at the natural embedding $S^2 = \{x \in \mathbb{R}^3 | \|x\| = 1\} \hookrightarrow \mathbb{R}^3$ and apply Definition 6.2.1.

Let $a \in \mathrm{SO}(3)$, then the natural action of a on $\mathbb{R}^3$ induces a diffeomorphism $a|_{S^2} : S^2 \to S^2$ of the smooth manifold S^2 onto itself. We can look at the pull-back map from Definition 4.1.3. It turns out that for any a, this pull-back map leaves the density $d\sigma$ invariant. We say that $d\sigma$ is SO(3)-invariant.

Definition 6.2.3. Let G be a Lie group, M a smooth manifold and let $d\sigma$ be a density on M. Suppose for any $a \in G$, the pull-back map of the action of a on M maps $d\sigma$ onto itself, then we call $d\sigma$ G-invariant.

For our discussion it is important that the measure on S^2 is SO(3)-invariant.

Lemma 6.2.4. *The measure $d\sigma$ on S^2 is* SO(3)*-invariant.*

Proof. Let $a \in \mathrm{SO}(3)$. Then we need to show that for every $x \in S^2$ we have

$$(d_x a)^* d\sigma(a(x)) = d\sigma(x)$$

However we can identify $T_m S^2$ as a linear subspace of $\mathbb{R}^3$, and then we have that $(d_m a)\big|_{T_m S^2} = a\big|_{T_m S^2} : T_m S^2 \to T_m S^2$. Since one of the defining properties of SO(3) is that for $a \in \mathrm{SO}(3)$ we have that $\det a = 1$ we see that $(d_m a)\big|_{T_m S^2}$ is an isometry. Then we use Definition 4.1.1 to see that $a^*(d\sigma) = d\sigma$. We conclude that $d\sigma$ is SO(3) invariant. □

We use the following lemma to show that a given representation is unitary. It is especially useful for infinite dimensional representations.

Lemma 6.2.5. *Let $(\pi, \mathcal{H})$ be a representation of the Lie group G in the Hilbert space $\mathcal{H}$. Suppose the following two conditions hold:*

a) *For all $g \in G$, $\pi(g) : \mathcal{H} \to \mathcal{H}$ is unitary*

b) *There exist a dense subset $D \subset \mathcal{H}$ such that for every $v \in D$, $\lim_{g \to e} \pi(g)v = v$*

Then π is a continuous unitary representation of G in $\mathcal{H}$.

Proof. We only need to show that it is a continuous representation, that is we need to show the following:

$$\lim_{v \to v_0, g \to g_0} \pi(g)v = \pi(g_0)v_0 \qquad \forall g_0 \in G, v_0 \in V$$

We will first show that we can assume that $g_0 = e$. Using that π is a homomorphism

$$\|\pi(g)v - \pi(g_0)v_0\| = \|\pi(g_0)(\pi(g_0^{-1}g)v - v_0)\| = \|\pi(g_0^{-1}g)v - v_0\|$$

The last step follows from property a). Now $g_0^{-1}g \to e$, so without loss of generality we may assume that $g_0 = e$. Finally we will reduce the problem to $v_0 \in D$. First choose v_0 such that $\|v - v_0\| < \epsilon/4$. Then let $v_D \in D$ such that $\|v_0 - v_D\| < \epsilon/4$. Fix a neighborhood U of e such that for all $g \in U$ we have that $\|\pi(g)v_D - v_D\| < \epsilon/4$. Then we have the following estimate for $g \in U$.

$$\begin{aligned}
\|\pi(g)v - v_0\| &= \|\pi(g)v - \pi(g)v_D + \pi(g)v_D - v_D + v_D - v_0\| \\
&\leq \|\pi(g)(v - v_D)\| + \|\pi(g)v_D - v_D\| + \|v_D - v_0\| \\
&\leq \|v - v_0\| + \|v_0 - v_D\| + \|\pi(g)v_D - v_D\| + \|v_D - v_0\| \\
&< 4 \cdot \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

□

Let is now define some useful representations.

Definition 6.2.6. We define a representation L of SO(3) in $L^2(\mathbb{R}^+ \times S^2)$ as $L(g) = L_g$ where

$$L_g\psi(r,\sigma) = \psi(r, g^{-1}\sigma) \quad \text{where } r \in \mathbb{R}^+, \sigma \in S^2, g \in \mathrm{SO}(3), \psi \in L^2(\mathbb{R}^+ \times S^2)$$

Furthermore we define a representation $L^{\mathbb{R}^3}$ of SO(3) in $L^2(\mathbb{R}^3)$ as $L^{\mathbb{R}^3}(g) = L_g^{\mathbb{R}^3}$ where

$$L_g^{\mathbb{R}^3}\psi(x) = \psi(g^{-1}x) \quad \text{where } x \in \mathbb{R}^3, g \in \mathrm{SO}(3), \psi \in L^2(\mathbb{R}^3)$$

Finally we can define a representation L^{S^2} of SO(3) in $L^2(S^2)$ as $L^{S^2}(g) = L_g^{S^2}$ where

$$L_g^{S^2}\psi(\sigma) = \psi(g^{-1}\sigma) \quad \text{where } \sigma \in S^2, g \in \mathrm{SO}(3), \psi \in L^2(S^2)$$

We will often also denote $L_g^{\mathbb{R}^3}$ and $L_g^{S^2}$ with L_g when it is clear which one of the three is meant.

Lemma 6.2.7. *The representations L, $L^{\mathbb{R}^3}$ and L^{S^2} are unitary continuous representations of* SO(3)*.*

Proof. We will do the proof for $L^{\mathbb{R}^3}$. The proofs for L and L^{S^2} are similar.

It can easily be seen that $L_g^{\mathbb{R}^3}$ is unitary. Let $D = C_c(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$. Then D is a dense subset of the Hilbert space $L^2(\mathbb{R}^3)$. Let $\phi \in C_c(\mathbb{R}^3)$. Then we only need to show that

$$\lim_{g \to e} L_g^{\mathbb{R}^3}\phi = \phi$$

If this is the case we can apply Lemma 6.2.5 and we are done.

Now since ϕ has a compact support, we have that supp$\phi \subset \overline{B}(0,R)$ for some finite $R \in \mathbb{R}$. Then we have that

$$\begin{aligned}\|L_g\phi - \phi\|_{L^2}^2 &= \int_{\mathbb{R}^3} |\phi(g^{-1}x) - \phi(x)|^2 dx = \int_{\overline{B}(0,R)} |\phi(g^{-1}x - \phi(x))|^2 dx \\ &\leq \left(\sup_{g \in \mathrm{SO}(3), x \in \overline{B}(0,R)} |\phi(g^{-1}x) - \phi(x)| \right)^2 \cdot \mathrm{vol}\overline{B}(0,R)\end{aligned}$$

We have that ϕ is uniformly continuous, since it is continuous and has compact support. Therefore, we can estimate the supremum by

$$\sup_{g \in \mathrm{SO}(3), x \in \overline{B}(0,R)} |\phi(g^{-1}x) - \phi(x)| \leq C\|g^{-1}x - x\| \leq C\|g^{-1} - e\| \cdot \|x\| \leq CR\|g^{-1} - e\|$$

for some constant $C \in \mathbb{R}$. Since furthermore $g \to e$ we see that we can get $\|L_g^{\mathbb{R}^3}\phi - \phi\|^2 < \epsilon$, which was to be proven. $\square$

Then first step in the process of showing that we can separate the radial and spherical part is to do a substitution of variables into spherical coordinates.

Lemma 6.2.8. *The substitution of variables $\phi : \mathbb{R}^+ \times S^2 \to \mathbb{R}^3 \setminus \{0\}, \phi(r,\sigma) \mapsto r\sigma$ is a diffeomorphism. Furthermore, the pull back map ϕ^* defined by*

$$\phi^* : L^2(\mathbb{R}^3 \setminus \{0\}, dx) \to L^2(\mathbb{R}^+ \times S^2, r^2 dr d\sigma), f \mapsto f \circ \phi$$

is an SO(3)*-equivariant isometric isomorphism of $L^2(\mathbb{R}^+ \times S^2, r^2 dr d\sigma)$ with $L^2(\mathbb{R}^3 \setminus \{0\})$. Here $d\sigma$ is the measure on S^2.*

Proof. For the first claim we refer to a course on multidimensional analysis. The second statement follows directly from the first statement and the fact that the Jacobian is given by r^2. For an explicit calculation of the Jacobian we refer to a course on multidimensional analysis. To show that it is SO(3)-equivariant we note that both the measures dx and $d\sigma$ are SO(3)-invariant. □

Since the point 0 has measure 0, we can identify $L^2(\mathbb{R}^3 \setminus \{0\}, dx)$ with $L^2(\mathbb{R}^3, dx)$.

Definition 6.2.9. Let $\psi \in L^2(\mathbb{R}^+ \times S^2, r^2 dr d\sigma)$. Then one can see that almost every $r_0 \in \mathbb{R}^+$ we can define a function $\psi_{r_0} : S^2 \to \mathbb{C}$ as $\psi_{r_0}(\sigma) := \psi(r_0, \sigma)$. Let $\mathcal{F}(S^2)$ denote the set of functions from S^2 to $\mathbb{C}$. Let $\rho_\psi : \mathbb{R}^+ \to \mathcal{F}(S^2)$ be the almost everywhere defined map given by $\rho_\psi(r_0) := \psi_{r_0}$.

Let $\mathcal{F}(\mathbb{R}^+, r^2dr, L^2(S^2))$ be the set of functions from $\mathbb{R}^+$ to $L^2(S^2)$. Then we also define the map $\rho : L^2(\mathbb{R}^+ \times S^2) \to \mathcal{F}(\mathbb{R}^+, r^2dr, L^2(S^2)),\ \rho(\psi) = \rho_\psi$.

Lemma 6.2.10. *$\mathrm{Im}(\rho_\psi) = L^2(S^2)$ and $\mathrm{Im}(\rho) = L^2(\mathbb{R}^+, r^2dr, L^2(S^2))$. In other words $\psi_r \in L^2(S^2)$ for every $r \in \mathbb{R}^+$ and $\rho_\psi \in L^2(\mathbb{R}^+, r^2dr, L^2(S^2))$ for every $\psi \in L^2(\mathbb{R}^+ \times S^2, r^2drd\sigma)$.*

Proof. Let $\psi \in L^2(\mathbb{R}^+ \times S^2)$. Then

$$\int_{\mathbb{R}^+ \times S^2} |\psi(r,\sigma)|^2 r^2 dr d\sigma < \infty$$

Now we apply Fubini's Theorem:

$$\int_{\mathbb{R}^+ \times S^2} |\psi(r,\sigma)|^2 r^2 dr d\sigma = \int_{\mathbb{R}^+} \int_{S^2} |\psi_r(\sigma)|^2 d\sigma r^2 dr < \infty$$

In particular we see that for almost every $r \in \mathbb{R}^+$, $\int_{S^2} |\psi_r(\sigma)|^2 d\sigma < \infty$, so $\psi_r \in L^2(S^2)$. We conclude that $\rho_\psi \in L^2(\mathbb{R}^+, r^2dr, L^2(S^2))$. □

It can also easily be seen that the following relations hold for the previously defined representations.

$$(L_g\psi)(r_0, \sigma) = (L_g\psi_{r_0})(\sigma) = \psi_{r_0}(g^{-1}\sigma)$$

Lemma 6.2.11. *The map $\rho : L^2(\mathbb{R}^+ \times S^2, r^2drd\sigma) \to L^2(\mathbb{R}^+, r^2dr, L^2(S^2)),\ \rho(\psi) = \rho_\psi$ is an isometric* SO(3)*-equivariant isomorphism.*

Proof. The isometry follows from Fubini as in the proof of Lemma 6.2.10.

The SO(3)-equivariance follows directly from the fact that SO(3) leaves the radial coordinate r invariant.

To see that it is an isomorphism we note that $\rho(\psi)(r)(\sigma) = \rho_\psi(r)(\sigma) = \psi_r(\sigma) = \psi(r,\sigma)$. From this we immediately see that ρ is bijective. Furthermore convergence goes with respect to the L^2 norm, which is preserved by ρ. We conclude that ρ is an isomorphism. □

The next lemma is the last step of the isomorphism between the spaces $L^2(\mathbb{R}^+, r^2dr) \otimes L^2(S^2)$ and $L^2(\mathbb{R}^3)$.

Lemma 6.2.12. *Let $(X, d\mu)$ be a measurable space and let $\mathcal{H}$ be a Hilbert space. Then there exists a natural isomorphism $b : L^2(X, d\mu) \otimes \mathcal{H} \to L^2(X, \mathcal{H}, d\mu)$, which is unitary.*

Proof. Define the map $b : L^2(X, d\mu) \times \mathcal{H} \to L^2(X, \mathcal{H}, d\mu)$ by

$$b(f, v)(x) = f(x)v \qquad \text{for } x \in X$$

Then this map is bilinear. Using the results from section 6.1 we see that b induces a linear map $\bar{b}_0 : L^2(X, d\mu) \otimes^{\text{alg}} \mathcal{H} \to L^2(X, \mathcal{H}, d\mu)$. Let $(f_j)_{j\in J}$ be an orthonormal basis of $L^2(X, d\mu)$ and let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}$. Since $\bar{b}_0$ is linear, it is sufficient to show that $\bar{b}_0(f_j \otimes e_i)$ is a orthonormal basis of $L^2(X, \mathcal{H}, d\mu)$.

First we remark that $\bar{b}_0(f_j \otimes e_i) = b(f_j, e_i)$. Now orthonormality follows easily

$$\begin{aligned} \langle b(f_j, e_i), b(f_l, e_k)\rangle_{L^2} &= \int_X \langle f_j(x)e_i, f_l(x), e_k\rangle d\mu \\ &= \int_X \overline{f_j(x)} f_l(x) d\mu \delta_{ik} \\ &= \delta_{jl}\delta_{ik} \end{aligned} \tag{22}$$

To show that the basis is complete, let $\phi \in L^2(X, \mathcal{H}, d\mu)$ be a function in the orthocomplement of the space generated by $b(f_i, e_i)$. Then

$$0 = \langle \phi, b(f_i, e_i\rangle = \int_X \langle \phi(x), f_j(x)e_i\rangle d\mu = \int_X \langle \phi(x), e_i\rangle \overline{f_j(X)} d\mu$$

Since the inner product $\langle \cdot, e_i\rangle$ is continuous, the function $\phi_i : x \mapsto \langle \phi(x), e_i\rangle$ is an element of $L^2(X, d\mu)$ and it is perpendicular to f_j for all $j \in J$. It is given that $(f_j)_{j\in J}$ is an orthonormal basis, therefore we have that $\phi_i = 0$ for all $i \in I$. Since $(e_i)_{i\in I}$ is also an orthonormal basis, it follows that $\phi = 0$.

To prove that $\bar{b}_0$ is unitary we only need to show it for elements of the form $\sum_{i,j} c_{ij} f_j \otimes e_i$ for some $c_{ij} \in \mathbb{C}$, since the elements $f_j \otimes e_i$ form an orthonormal basis of $L^2(X, d\mu) \otimes^{\text{alg}} \mathcal{H}$. We have that

$$\begin{aligned} \|\bar{b}_0(\sum_{i,j} c_{ij} f_i \otimes e_j)\|^2 &= \int_X \| \sum_{i,j} c_{ij} f_i(x) e_j\|^2 d\mu(X) = \int_X \sum_{i,j} \|c_{ij} f_i(x) e_j\|^2 d\mu(X) \\ &= \int_X \sum_{i,j} |c_{ij}|^2 |f_i(x)|^2 \|e_j\|^2 d\mu(X) = \sum_{i,j} |c_{ij}|^2 \int_X |f_i(x)|^2 d\mu(X) \\ &= \sum_{i,j} |c_{ij}|^2 \|f_i\|^2 = \sum_{i,j} \|c_{ij} f_i\|^2 \|e_j\|^2 \\ &= \sum_{i,j} \|c_{ij} f_i \otimes e_j\|^2 = \| \sum_{i,j} c_{ij} f_i \otimes e_j\|^2 \end{aligned} \tag{23}$$

Here we used twice that $\{f_j\}$ and $\{e_j\}$ are orthonormal and therefore we could apply the Pythagorean theorem. We conclude that $\bar{b}_0$ is an isometry. In particular $\bar{b}_0$ is unitary. Furthermore $\bar{b}_0$ has a isometric continuation to the completion $L^2(X, d\mu) \otimes \mathcal{H}$ of the algebraic Pre-Hilbert tensor product $L^2(X, d\mu) \otimes^{\text{alg}} \mathcal{H}$. □

Corollary 6.2.13. *The map $\Phi = \phi \circ \rho^{-1} \circ b$ is a isometric* SO(3)*-equivariant isomorphism between the spaces $L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(S^2)$ and $L^2(\mathbb{R}^3)$. Here ϕ is the map from Lemma 6.2.8, ρ is the map from Lemma 6.2.11 and b is the map from Lemma 6.2.12.*

Proof. We use that we can identify $L^2(\mathbb{R}^3 \setminus 0)$ with $L^2(\mathbb{R}^3)$ and that all three maps are unitary equivariant isomorphisms. □

Now that we were able to separate the wave function into a spherical part and a radial part, it would be useful if we could use our knowledge about the spherical harmonics to calculate the wave function. For this we need the following lemma.

Lemma 6.2.14. *Let $\mathcal{H}_i$, $i = 1, 2$ be Hilbert spaces. Let $\mathcal{H}_2 = \widehat{\bigoplus}_{\delta\in\mathcal{D}}\mathcal{H}_{2,\delta}$ be the Hilbert direct sum of mutually orthogonal components $\mathcal{H}_{2,\delta}$. Let ι be the inclusion of the algebraic direct sum into the tensor product*

$$\iota : \bigoplus_{\delta\in\mathcal{D}} \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta} \hookrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

Then ι induces an isometric isomorphism

$$\widehat{\bigoplus}_{\delta\in\mathcal{D}} \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta} \xrightarrow{\simeq} \mathcal{H}_1 \otimes \mathcal{H}_2$$

Proof. First we note that the spaces $\mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$ are mutually orthogonal. Indeed, let $\delta_1, \delta_2 \in \mathcal{D}$ such that $\delta_1 \neq \delta_2$. Then $\mathcal{H}_{2,\delta_1} \perp \mathcal{H}_{2,\delta_2}$. Furthermore we have for the inner product that

$$\langle (v_1 \otimes w_1), (v_2 \otimes w_2) \rangle = \langle v_1, v_2\rangle\langle w_1, w_2\rangle = \langle v_1, v_2\rangle \cdot 0 = 0 \quad \text{ for } v_1, v_2 \in \mathcal{H}_1, w_i \in \mathcal{H}_{2,\delta_i}$$

Therefore, $\mathcal{H}_1 \otimes \mathcal{H}_{2,\delta_1} \perp \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta_2}$.

The inclusion map ι maps $\mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$ isometrically into $\mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$, since it is an embedding. Now using the Pythagorean Theorem we conclude that ι is isometric.

Let us now construct an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $(e_i)_{i\in I}$ be an orthonormal basis of $\mathcal{H}_1$. Let for each $\delta \in \mathcal{D}$, $(f_j)_{j\in J(\delta)}$ be an orthonormal basis of $\mathcal{H}_{2,\delta}$. Then using our consideration from Section 6.1 we see that $(e_i \otimes f_j)_{(i,j)\in I\times J(\delta)}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$. Let J be the disjoint union of $J(\delta)$, $\delta \in \mathcal{D}$. Then using the same reasoning as above, $(e_i \otimes f_j)_{(i,j)\in I\times J}$ is an orthonormal basis of the algebraic direct sum $\bigoplus_{\delta\in\mathcal{D}} H_1 \otimes H_{2,\delta}$, therefore also for the completion $\widehat{\bigoplus}_{\delta\in\mathcal{D}} H_1 \otimes H_{2,\delta}$.

Furthermore $(f_j)_{j\in J}$ is an orthonormal basis of $\mathcal{H}_2$. Since ι is isometric, the images $\iota(e_i \otimes f_j)$ form an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since the set generated by finite sums of these basis elements is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$, it follows that ι has a unique extension to an isometric isomorphism. ☐

Now we can prove that the inclusion map is indeed SO(3)-equivariant.

Proposition 6.2.15. *Let $\mathcal{H}_1 = L^2(\mathbb{R}^+, r^2 dr)$, $\mathcal{H}_2 = L^2(S^2)$ and let $\mathcal{D} = \mathrm{SO}(3)^\wedge$. Then the isomorphism of Lemma 6.2.14 is* SO(3)*-equivariant.*

Proof. Let ι be the inclusion map:

$$\iota : \bigoplus_{\delta\in\mathcal{D}} \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta} \hookrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

Let $a \in \mathrm{SO}(3)$, then we have that

$$\iota \circ \left(\bigoplus_{\delta\in\mathcal{D}} I \otimes L_a \right) = (I \otimes L_a) \circ \iota \tag{24}$$

So for algebraic direct sums it can easily be seen that the inclusion map is SO(3)-equivariant. The identity 24 holds on a dense subspace of the Hilbert space $\widehat{\bigoplus}_{\delta\in\mathcal{D}} \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$, so by continuity, we conclude that ι is also SO(3)-equivariant on $\widehat{\bigoplus}_{\delta\in\mathcal{D}} \mathcal{H}_1 \otimes \mathcal{H}_{2,\delta}$. ☐

Finally we have the main result of this section, which puts together a Corollary of Peter-Weyl, Corollary 6.2.13 and Proposition 6.2.15.

Proposition 6.2.16. *$L^2(\mathbb{R}^3)$ decomposes as the Hilbert direct sum*

$$L^2(\mathbb{R}^3) \simeq \widehat{\bigoplus}_{[\delta]\in \mathrm{SO}(3)^\wedge} L^2(R^+, r^2dr) \otimes L^2(S^2)_\delta$$

Here $\mathrm{SO}(3)^\wedge$ denotes $\widehat{\mathrm{SO}(3)}$, the set of equivalence classes of irreducible representations.

Proof. From Proposition 5.3.4 it follows that

$$L^2(S^2) = \widehat{\bigoplus}_{[\delta]\in \mathrm{SO}(3)^\wedge} L^2(S^2)_\delta$$

With Corollary 6.2.13 we now see that

$$L^2(\mathbb{R}^3) \simeq L^2(R^+, r^2dr) \otimes \widehat{\bigoplus}_{[\delta]\in \mathrm{SO}(3)^\wedge} L^2(S^2)_\delta$$

Then we apply Proposition 6.2.15. □

6.3 Reducing the problem to irreducibles

The first step in solving the Schrödinger is to show that any solution to the Schrödinger equation is the direct sum of irreducibles which are also solutions to the Schrödinger equation. Once we have shown that, we only need to solve the Schrödinger equation in each of the SO(3)-invariant subspaces. In order to show this, denote that the Hilbert direct sum from 5.3.4

$$L^2(S^2) \simeq \widehat{\bigoplus}_{[\delta]\in \mathrm{SO}(3)^\wedge} L^2(S^2)_\delta$$

gives rise to a set of orthogonal projection maps $P_\delta^{S^2} : L^2(S^2) \to L^2(S^2)_\delta$.

Lemma 6.3.1. *Let $P_\delta^{S^2} : L^2(S^2) \to L^2(S^2)_\delta$ be the orthogonal projection map of $L^2(S^2)$ onto $L^2(S^2)_\delta$, let $\chi_\delta \in C^\infty(\mathrm{SO}(3))$ be the character of the irreducible represenation δ and let dg be the normalized Haar measure on $\mathrm{SO}(3)$. Then we have that $P_\delta^{S^2}$ is given by*

$$P_\delta^{S^2}\phi(x) = \int_{\mathrm{SO}(3)} \dim(\delta)\chi_\delta(g)\phi(gx)dg \qquad \text{where } \phi \in L^2(S^2) \tag{25}$$

Furthermore, the projection operator $P_\delta^{\mathbb{R}^3} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)_\delta$ is also given by the same formula.

Proof. Firstly, we have from Definition 5.3.3 that $L^2(\mathrm{SO}(3))^{\mathrm{SO}(2)}$. Furthermore, from Proposition 5.3.4 it follows that there exists an equivariant isomorphism T such that

$$T : (\delta, V_\delta) \xrightarrow{\simeq} \left(L, L^2(\mathrm{SO}(3))_\delta^{\mathrm{SO}(3)}\right)$$

Here V_δ is the representation space of the representation δ. Now define the map $S : V_\delta \to V_\delta$ as

$$S(v) = \int_{\mathrm{SO}(3)} \chi_\delta(g^{-1})\delta(g)v dg$$

Using the $SO(3)$-invariance of the measure dg we see that

$$S(v) = \int_{\mathrm{SO}(3)} \delta(x)\chi_\delta(g^{-1})\delta(g)\delta(x)^{-1}dg$$

Now it can easily be seen that S is an equivariant linear map from V_δ to V_δ. Furthermore, by definition δ is irreducible. Then Schur's Lemma 3.2.13 implies that $\mathrm{End}_{\mathrm{SO}(3)}(V_\delta) = \mathbb{C}\mathrm{Id}_{V_\delta}$. Since $S \in \mathrm{End}_{\mathrm{SO}(3)}(V_\delta)$ we conclude that $S = c \cdot \mathrm{Id}_{V_\delta}$ for some $c \in \mathbb{C}$. Then we seee that $\mathrm{Tr}(S) = c\dim(\delta)$. Then, since the trace commutes with the integral:

$$\begin{aligned}\mathrm{Tr}(S) = \mathrm{Tr}\int_{\mathrm{SO}(3)} \chi_\delta(g^{-1})\delta(g)dg &= \int_{\mathrm{SO}(3)} \chi_\delta(g^{-1})\mathrm{Tr}(\delta(g))dg \\ = \int_{\mathrm{SO}(3)} \overline{\chi_\delta(g)}\chi_\delta(g)dg &= \langle \chi_\delta, \chi_\delta\rangle_{L^2} = 1\end{aligned}$$

We conclude that $S = \dim(\delta)^{-1}\mathrm{Id}_{V_\delta}$. Now see that $P_\delta^{S^2}|_{L^2(S^2)_\delta} = \mathrm{Id}_{L^2(S^2)_\delta}$. The first result follows.

The second statement follows from the equivariance of the map Ψ from 6.2.13. □

When it is clear which one of $P_\delta^{S^2}$ and $P_\delta^{\mathbb{R}^3}$ is meant, we also just use the notation P_δ.

For the Schrödinger equation we apply a second order differential operator on $\psi \in L^2(\mathbb{R}^3)$. In order to make sense of this, we must require that the restriction

$$\psi_* := \psi|_{\mathbb{R}^3\setminus\{0\}}$$

of ψ to $\mathbb{R}^3 \setminus \{0\}$ belongs to $C^2(\mathbb{R}^3 \setminus \{0\})$. Furthermore, the function we get by applying the operator applied to ψ_* should also be square integrable. Let us therefore define the following set of functions:

$$L^2_*(\mathbb{R}^3) := \{\psi \in L^2(\mathbb{R}^3) \mid \psi_* \in C^2(\mathbb{R}^3 \setminus \{0\}) \text{ and } c_0\Delta\psi_* + V\psi \in L^2(\mathbb{R}^3)$$

The point 0 is special because the potential is undefined there and not continuous. Here $c_0 := -\frac{\hbar^2}{2m}$.

Lemma 6.3.2. *The projection operator $P_\delta^{S^2}$ maps $L^2_*(S^2)$ onto itself. Also the projection operator $P_\delta^{\mathbb{R}^3}$ maps $L^2_*(\mathbb{R}^3)$ onto itself.*

Proof. We will do the proof for $P_\delta^{\mathbb{R}^3}$. The proof for $P_\delta^{S^2}$ goes similar.

We only need to show that elements in the image of $L^2_*(\mathbb{R}^3)$ under $P_\delta^{\mathbb{R}^3}$ also satisfy the extra conditions, since we already know that $P_\delta^{\mathbb{R}^3}$ maps $L^2(\mathbb{R}^3$ into $L^2(\mathbb{R}^3$.

Let $\psi \in C(\mathbb{R}^3 \setminus \{0\})$. Let $\tilde{P}_\delta : L^2(\mathbb{R}^3 \setminus \{0\}) \to L^2(\mathbb{R}^3 \setminus \{0\})$ defined by the integral from (25). Then the terms under the integral sign are continuous. Then by continuïty of the integral, we see that $\tilde{P}_\delta$ maps $C(\mathbb{R}^3 \setminus \{0\})$ onto itself. Differentiating both sides and interchanging the differentiation and integration we see that $\tilde{P}_\delta$ maps $C^2(\mathbb{R}^3 \setminus \{0\})$ onto itself. Furthermore Δ is rotation invariant, and therefore commutes with the SO(3) action. Again by differentiation under the integral sign we get that

$$\Delta\tilde{P}_\delta(\psi) = \tilde{P}_\delta(\Delta\psi) \quad \text{for } \psi \in C^2(\mathbb{R}^3 \setminus \{0\}) \tag{26}$$

Finally let $v(x) = \|x\|^{-1}$, then

$$v\tilde{P}_\delta(\psi) = \tilde{P}_\delta(v\psi) \quad \text{for } \psi \in C^2(\mathbb{R}^3 \setminus \{0\}) \tag{27}$$

Now let $\psi \in L^2_*(\mathbb{R}^3)$. By comparing the definitions of P_δ and $\tilde{P}_\delta$ and the previous results (26) and (27) we see that

$$P_\delta(\psi)|_{\mathbb{R}^3\setminus\{0\}} = \tilde{P}_\delta(\psi|_{\mathbb{R}^3\setminus\{0\}}) \in C^2(\mathbb{R}^3 \setminus \{0\})$$

Furthermore, applying the previous results and the linearity of $\tilde{P}_\delta$

$$\begin{aligned} c_0\Delta(P_\delta(\psi)|_{\mathbb{R}^3\setminus\{0\}}) + VP_\delta(\psi) &= \tilde{P}_\delta(c_0\Delta(\psi|_{\mathbb{R}^3\setminus\{0\}})) + \tilde{P}_\delta(V\psi) \\ &= \tilde{P}_\delta(c_0\Delta(\psi|_{\mathbb{R}^3\setminus\{0\}}) + V\psi) \\ &= P_\delta(c_0\Delta(\psi|_{\mathbb{R}^3\setminus\{0\}}) + V\psi) \in L^2(\mathbb{R}^3) \end{aligned} \tag{28}$$

We conclude that $P^{\mathbb{R}^3}_\delta(\psi) \in L^2_*(\mathbb{R}^3)$ □

Then we have the main result, which allows us to use separation of variables.

Theorem 6.3.3. *Let $\psi \in L^2_*(\mathbb{R}^3)$ be a solution to the time independent Schödinger equation and let $\delta \in \mathrm{SO}(3)^\wedge$ be an irreducible representation. Let E be the energy of ψ. Then the projection $P_\delta\psi$ is also a solution to the time independent Schrödinger equation with energy E.*

Proof. From Lemma 6.3.2 it follows that $P_\delta\psi \in L^2_*(\mathbb{R}^3)$, so that we can apply the Schrödinger equation on it. On $\mathbb{R}^3 \setminus \{0\}$ the following holds:

$$EP_\delta(\psi) = P_\delta(E\psi) = P_\delta(c_0\Delta(\psi|_{\mathbb{R}^3\setminus\{0\}}) + V\psi) = (c_0\Delta + V)P_\delta(\psi) \tag{29}$$

Here we used the commutation relations (26) and (27). □

Since the projections into the spaces $L^2_*(\mathbb{R}^3)_\delta$ are also solutions to the Schrödinger equation, it suffices to solve the Schrödinger equation in the spaces $L^2_*(\mathbb{R}^3)_\delta$ and then use the decomposition of $L^2(\mathbb{R}^3)$ from Proposition 6.2.16.

Recall that we had the isomorphism $\Phi : L^2(\mathbb{R}^+, r^2dr) \otimes L^2(S^2) \to L^2(\mathbb{R}^3)$ from Corollary 6.2.13. We will describe the image of $L^2_*(\mathbb{R}^3)_\delta$ under Φ^{-1}. Firstly, recall that $L^2(S^2)_\delta$ is finite dimensional and contained in $C^2(S^2)$.[2]. Therefore we denote this space by $L^2_*(S^2)_\delta$. Furthermore we define

$$L^2_*(\mathbb{R}^+, r^2dr) := L^2(\mathbb{R}^+, r^2dr) \cap C^2(\mathbb{R}^+)$$

Then $L^2_*(\mathbb{R}^+, r^2dr)$ and $L^2_*(S^2)_\delta$ resemble the definition of $L^2_*(\mathbb{R}^3)$. Then the space $\Phi^{-1}(L^2_*(\mathbb{R}^3)_\delta)$ is contained in the algebraic tensor product $L^2_*(\mathbb{R}^+, r^2dr) \otimes L^2_*(S^2)_\delta$

6.4 Explicit calculation

In this section we will explicitely try to find solutions to the Schrödinger equation in the space $L^2_*(\mathbb{R}^+, r^2dr) \otimes L^2_*(S^2)_\delta$.

First we introduce some notation.

Definition 6.4.1 (Spherical and radial Laplacian)**.** We define $\Delta_{rad} : C^2(\mathbb{R}^3 \setminus \{0\}) \to C(\mathbb{R}^3 \setminus \{0\})$ and $\Delta_{S^2} : C^2(\mathbb{R}^3 \setminus \{0\}) \to C(\mathbb{R}^3 \setminus \{0\})$ as

$$\Delta_{rad} := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$$

$$\Delta_{S^2} := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

[2]The space is even contained in $C^\infty(S^2)$

Then we see that $\Delta = \Delta_{rad} + \frac{1}{r^2}\Delta_{S^2}$. Furthermore we see that we can restrict Δ_{rad} to $C^2(\mathbb{R}^+)$ and Δ_{S^2} to $C^2(S^2)$ since they only depend on r and θ and ϕ respectively. We will also denote the restrictions $\Delta_{rad}\big|_{\mathbb{R}^+}$ with Δ_{rad} and $\Delta_{S^2}\big|_{S^2}$ with Δ_{S^2}.

Lemma 6.4.2. *The sperical Laplacian is given by* $\Delta_{S^2} = R_1^2 + R_2^2 + R_3^2 : C^\infty(S^2)_\delta \to C^\infty(S^2)_\delta$. *Furthermore it acts as a scalar* $-l(l+1)$ *on* $C^\infty(S^2)_\delta \simeq V_\delta$ *where* (δ, V_δ) *are the irreducible representations of* $\mathrm{SO}(3)$ *on* V_δ. *Here* $2l+1$ *is the dimension of* V_δ.

Proof. The proof of this lemma is based on [3].

Let $f \in C^\infty(S^2)_\delta$, then we define the extension $\tilde{f}$ to $\mathbb{R}^3$ as $\tilde{f}(tx) = f(x)$ for $x \in S^2$ and $t > 0$. Then the spherical Laplacian Δ_{S^2} is given by

$$\Delta_{S^2} f = (\Delta \tilde{f})|_{S^2}$$

From (19) we see that $L(\mathcal{C}) = -\Delta_{S^2}$. The last statements follows from the discusson about spherical harmonics in Section 5. □

Now we apply Lemma 6.4.2 to see that solutions $R \otimes Y$ where $R \in L^2_*(\mathbb{R}^+, r^2 dr)$ and $Y \in L^2_*(S^2)_\delta$ of the time-independent Schrödinger equation satisfy

$$\begin{aligned} 0 &= \left(\frac{\hbar^2}{2m}\Delta - V(r) + E\right)\Phi(R \otimes Y) \\ &= \Phi\left(\left(\frac{\hbar^2}{2m}\Delta_{rad} - V(r) + E - \frac{\hbar^2}{2mr^2}l(l+1)\right) \otimes \mathrm{Id}\ (R \otimes Y)\right) \end{aligned} \tag{30}$$

For this we need the additional requirement that

$$\left(\frac{\hbar^2}{2m}\Delta_{rad} - V(r) + E - \frac{\hbar^2}{2mr^2}l(l+1)\right) R \in L^2(\mathbb{R}^2, r^2 dr) \tag{31}$$

We conclude from (30) that R must satisfy

$$\left(\frac{\hbar^2}{2m}\Delta_{rad} - V(r) + E - \frac{\hbar^2}{2mr^2}l(l+1)\right) R = 0$$

$$\begin{aligned} \left(\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - V(r) + E - \frac{\hbar}{2mr^2}l(l+1)\right) R = 0 \\ \frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + (E + \frac{e^2}{r} - \frac{\hbar^2}{2mr^2}l(l+1))R = 0 \\ \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(\frac{2m}{\hbar^2}\left(E + \frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}R\right) = 0 \end{aligned} \tag{32}$$

Let $n \in \mathbb{R}^+$ be defined by the equation

$$E = -\frac{me^4}{2n^2\hbar^2}$$

and futhermore let $x \in \mathbb{R}^+$ be defined by the equation

$$r = \frac{n\hbar^2}{2me^2}x$$

Then (32) simplifies to

$$\frac{d^2R}{dx^2} + \frac{2}{x}\frac{dR}{dx} + \left(-\frac{1}{4} + \frac{n}{x} - \frac{l(l+1)}{x^2}\right) R = 0 \tag{33}$$

Now we make a guess and suppose that R is of the form $R = u(x)x^l \exp\left(-\frac{x}{2}\right)$ for some $u \in C^2(\mathbb{R}^+)$ such that $R \in L^2(\mathbb{R}^+, r^2dr)$ and R satisfies (31). This can be done without loss of generality. We fill this in in (33) to get a differential equation for u. Straightforward calculation results in

$$x\frac{d^2u}{dx^2} + (2l+2-x)\frac{du}{dx} + (n-l-1)u = 0 \tag{34}$$

Lemma 6.4.3. *n is an integer and $n \geq l+1$.*

Proof. Let $\beta = 2l+1$ and $\alpha = n+l$, then (34) becomes

$$x\frac{d^2u}{dx^2} + (\beta+1-x)\frac{du}{dx} + (\alpha-\beta)u = 0 \tag{35}$$

From a theory about differential equations it follows that u will be given as a series

$$u = \sum_{i=0}^{\infty} a_i x^{L+i}$$

We fill this series in in (35) and looking at the terms of power $L-1$ we get that $L(L+\beta) = 0$, so $L = 0$ or $L = -\beta$. The solution $L = -\beta$ results in a R which is not square integrable around $x = 0$, since $\beta \leq 1$. Then R would not satisfy the requirement (31). We will look at the solution $L = 0$.

Comparing terms with the same order we in (35) with the series expansion of u filled in we get the following recursion formula:

$$(i+1)(i+1+\beta)a_{i+1} + (\alpha-\beta-i)a_i = 0$$

Therefore

$$\lim_{i\to\infty}\frac{a_{i+1}}{a_i} = \lim_{i\to\infty}\frac{\beta-\alpha+i}{(i+1)(i+1+\beta)} = \lim_{i\to\infty}\frac{i+\beta-\alpha}{i^2+2i+i\beta+1+\beta} = \frac{1}{i}$$

This is the same ratio of convergence as the series expansion of e^x. In order for R to satisfy (31) we must have that $i+\beta-\alpha = 0$. Then it follows directly that $a_j = 0$ for $j > i$ and the series expansion of U is a finite sum. Furthermore $\alpha - \beta = n-l-1$ is an integer. Since we already know that l is an integer we conclude that n is an integer. It also follows that $n \leq l+1$. $\square$

Proposition 6.4.4. *$R(r)$ is given by*

$$R(r) = -\left(\left(\frac{2}{na_0}\right)^3 \frac{(n-l-1)!}{2n((n+l)!)^3}\right)^{\frac{1}{2}} \left(\frac{2r}{na_0}\right)^l \exp\left(\frac{-r}{na_0}\right) L_{n+l}^{2l+1}\left(\frac{2r}{na_0}\right)$$

with a_0 the 'radius of the first Bohr orbit' defined by

$$a_0 = \frac{\hbar^2}{me^2}$$

Proof. First we look at another differential equation:

$$x\frac{dy}{dx} + (x-\alpha)y = 0 \tag{36}$$

It can be shown that the Laguerre polynomials L_α defined by

$$L_\alpha(x) = e^x \frac{d^\alpha}{dx^\alpha}(x^\alpha e^{-x})$$

are solutions to this differential equation. We now differentiate (36) $\alpha + 1$ times to x. Then we have

$$\begin{aligned} x\frac{d^{\alpha+2}y}{dx^{\alpha+2}} + (x+1)\frac{d^{\alpha+1}}{dx^{\alpha+1}} + (\alpha+1)\frac{d^\alpha}{dx^\alpha} = 0 \\ x\frac{d^2L_\alpha}{dx^2} + (1-x)\frac{dL_\alpha}{dx} + \alpha L_\alpha = 0 \end{aligned} \tag{37}$$

Now define the Laguerre polynomials L_α^β by

$$L_\alpha^\beta = \frac{d^\beta}{dx^\beta}L_\alpha$$

Then we apply a similar trick and differentiate (37) β times. This will result in precisely (34). The normalisation is determined by the condition that

$$\int_0^\infty |R(r)|^2 r^2 dr = 1$$

□

Finally we apply the results from Section 5.3 to get the solutions Y_l^m of the spherical part and we have completely solved the bound states of the Schrödinger equation of the hydrogen atom. We remark that the solutions are completely determined by the quantum numbers n, l and m. Here n is related to the energy of the electron, l to the total angular momentum and m to the angular momentum in the z-direction. Any solution to the Schrödinger equation of the hydrogen atom is then a linear combination of the solutions we found here.

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