The Kakeya conjecture in two dimensions

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Abstract

We investigate the proof of the Kakeya conjecture in two dimensions, which states that every set which contains a unit line segment in every direction of the plane must have Hausdorff dimension of 2. In this case, we look at a theorem which states that every subset of the plane which contains a line in every direction must have Hausdorff dimension of at least 2.

We also construct a set with a line segment in every direction which has 2-dimensional Hausdorff measure 0.

Acknowledgments

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Contents

1 Introduction

1.1 Motivation and main result

In this bachelor thesis, we study properties of sets which contain a unit line segment in every direction. These sets are also called Besicovitch sets.

Definition 1.1. A Besicovitch set in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which contains a unit line segment in every direction.

This thesis is concerned with two questions, one of them being the following:

Question 1.2 (Dimensions of Besicovitch sets)**.** *Does there exist a Besicovitch set in $\mathbb{R}^n$ with Hausdorff dimension less than n?*

I will answer this question for the special case $n = 2$. For the other cases, $n > 2$ the proof has not yet been given. There have been found lower bounds of plane sets for the case $n = 3$, but these are still a lot lower than the Kakeya conjecture states. The former question is closely connected to another one, asked by Besicovitch in 1917.

Question 1.3 (Lebesgue measures of Besicovitch sets)**.** *Do there exist compact Besicovitch sets with Lebesgue measure zero?*

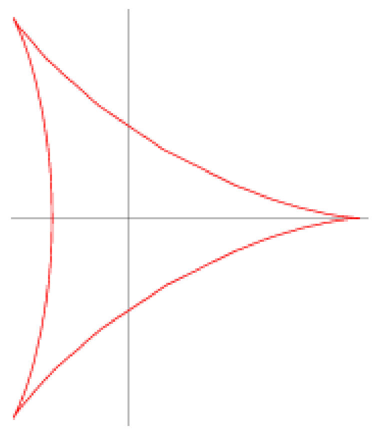
These questions concern two new terms, which will be explained in section 2 of the thesis. Intuitively, Hausdorff dimension can be thought of as a real number given to a subset of $\mathbb{R}^n$ which agrees with our concept of dimension for 'standard' shapes, such as cubes, but gives rise to a better understanding of the size of other subsets. The concept arises after the introduction of the s-dimensional Hausdorff measure; a measure which was defined such that it agrees with the volume of $\mathbb{R}^s$ if $s \in \mathbb{N}$. The Lebesgue measure is a special case of the Hausdorff measure, and it agrees with the n-dimensional Hausdorff measure when it is defined, for $n \in \mathbb{N}$.

Besicovitch was led to question 1.3 while considering the following question: "If f is a Riemann integrable function defined on the plane, is it always possible to find a pair of orthogonal coordinate axes with respect to which $\int f(x, y)dx$ exists as a Riemann integral for all y, and with the resulting function of y also Riemann integrable?"[Fal02, p. 95] Constructing a compact set F of the plane of Lebesgue measure zero containing a unit line segment in every direction would lead to a counterexample.

Theorem 1.4 (Besicovitch conjecture)**.** *There exists a compact Besicovitch set in $\mathbb{R}^2$ with Lebesgue measure zero.*

This theorem would give Besicovitch the necessary counterexample. By translating the compact Besicovitch set with Lebesgue measure zero F, F would contain no line segment which is both parallel and at rational distance to either axis, of a fixes pair of axes. Let F_0 be the subset of F of which consists of the points with at least one rational

Figure 1: This is an example of a non-convex set in which a unit line segment can continuously rotate and point in every direction. It is a smaller set than an equilateral triangle, since it has smaller area.



coordinate. f is the function on $\mathbb{R}^2$ gives 1 for each point in F_0 and 0 for each point not in F_0. Since F contains a line segment in every direction on which both F_0 and its complement are dense, there is a segment in every direction on which f is not Riemann integrable. However, the set of points of discontinuity of f is of Lebesgue measure 0, so f is Riemann integrable.

We want to prove the theorem by transforming a certain subset of the plane into a set of lines. For certain types of subsets we know that the sets of lines they form have 2-dimensional Hausdorff measure zero. By taking an appropriate set to start with, we can prove the theorem.

The other theorem we discuss in this thesis was first stated with an extra requirement by Kakeya. This extra requirement was that the unit line segment had to be continuously rotated in a convex set [Fal02, p. 95]. Kakeya conjectured that the equilaterial triangle was the smallest such set. He noted that dropping the convexity condition would lead to a smaller such set, the set portrayed in figure 1. Eventually, Pál reconjectured the theorem without the requirement of continuous rotation, and the following theorem is the result of that.

Theorem 1.5 (Kakeya Conjecture)**.** *A set with a line in every direction in* $\mathbb{R}^2$ *has a Hausdorff dimension of 2*

The theorem has applications in Fourier transforms, as described in a paper by Terrence Tao, [Tao00].

We will prove this theorem by looking at the set E which forms a set F with lines in every direction which is a subset of every set with lines in every direction. Since F contains a line in every direction, the projection of E onto the y-axis must contain the entire y-axis which gives that E has Hausdorff dimension larger than 1. We can apply theorems which show us that therefore the set F must have Hausdorff dimension 2, and so every set which contains a line in every direction in $\mathbb{R}^2$ has Hausdorff dimension 2.

1.2 Organization of this thesis

Section 2 contains background material on the proof and will explore the main concepts used in the proof. In it, I will define the most important concepts and prove some important theorems, characterizing the concepts I defined. We look at the concepts of Hausdorff measure and dimension and develop definitions and tools to be able to use these concepts. Furthermore, we define concepts necessary to prove useful lemmas.

In Section 3 we prove the lemmas and theorems we need to prove 1.4 and 1.5. With the necessary machinery developed and proven, in section 4 we will prove the main two theorems.

The proof is based on the book "The geometry of Fractal Sets" by Kenneth Falconer.

2 Background on Besicovitch sets

2.1 Definitions and characterization of concepts used in the theorems

The main theorems covered in this thesis introduce the concepts of Lesbesgue measure and Hausdorff dimension. In this section, I will explain and define these notions.

Both of these are related to the notion of an outer measure.

Definition 2.1. An outer measure on a set S is a function $\mu^* : \mathrm{P}(S) \to [0, \infty]$, with the following properties:

- $\mu^*(\varnothing) = 0$
- $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$
- $\mu^*(\bigcup_{i\in\mathbb{N}} U_i) \leq \sum_{i\in\mathbb{N}} \mu^*(U_i)$ for all $\{U_i\}_{i\in\mathbb{N}}, U_i \in X : \forall i \in \mathbb{N}$

Outer measures have an important property, they can be restricted to a collection of subsets of the set such that they become a measure on that set.

Theorem 2.2. *An outer measure μ^* gives rise to a measure μ when restricted to the collection of subsets $\mathscr{E}$, with $E \in \mathscr{E}$ when $\mu^*(A) = \mu^*(A\backslash E) + \mu^*(A \cap E) : \forall A \subset X.\mathscr{E}$ forms a σ-field.*

The proof of Theorem 2.2 can be found in the book of Falconer [Fal02, Theorem 1.2]

The outer Lebesgue measure is such an outer measure, and captures the intuitive idea of volume.

Definition 2.3. The volume of a box, $B = \prod_{i=1}^{n}[a_i, b_i)$, is given by $\mathrm{Vol}(B) = \prod_{i=1}^{n}(b_i - a_i)$.

Definition 2.4. The Lebesgue outer measure, a function $\mathscr{L}^* : \mathrm{P}(\mathbb{R}^n) \to [0, \infty]$, is given by:

$$\mathscr{L}^*(E) = \inf\{\sum_{B\in\mathscr{C}} \mathrm{Vol}(B) : \mathscr{C} \text{is a countable collection of boxes whose union covers E}\}.$$

Theorem 2.5. *The Lebesgue outer measure given is an outer measure.*

Proof. We have to check the 3 requirements given in the definition of an outer measure. For the first requirement, we have that since $\varnothing$ is contained in every box, the infinum of volumes of coverings is 0.

To proof the second requirement, we look at $A \subset B$. If $A \subset B$, $\mathscr{C}$ which covers B also covers A, so $\varphi(A)$ is upper bounded by $\varphi(B)$, $\varphi(A) \leq \varphi(B)$.

We can assume that $\mathscr{L}^*(U_i)$ is finite for all i, since the third requirement would be satisfied automatically if that was not the case. For a given ϵ, we know there exists a cover of rectangles R_{ij} of U_i such that $\sum_j \text{Vol}(R_{ij}) \leq \mathscr{L}^*(U_i) + \frac{\epsilon}{2^i}$. $\sum_{i,j} R_{ij}$ covers $\cup_{i=1}^n U_i$, so

$$\mathscr{L}^*(\cup_{i=1}^n U_i) \leq \sum_{i,j} \text{Vol}(R_{ij}) \leq \sum_i \left\{ \mathscr{L}^*(U_i) + \frac{\epsilon}{2^i} \right\} \leq \sum_i \mathscr{L}^*(U_i) + \epsilon.$$

Since ϵ was chosen arbitrarily small, the third requirement follows. □

The other term we encountered in the central question of the thesis was the term of Hausdorff dimension. Hausdorff dimension is defined using Hausdorff measure, so we need to define Hausdorff measure first. The Hausdorff measure agrees with the Lebesgue measure on integer n, only it is also defined on rational or irrational n.

Definition 2.6 (Hausdorff measure and content)**.** Let E be a subset of $\mathbb{R}^n$. Then the Hausdorff s-dimensional δ-content is defined by

$$\mathscr{H}_\delta^s = \inf \sum_{i=1}^{\infty} |2r_i|^s,$$

where the infinum is taken over all collections of spheres which cover E with radii smaller dan δ, $\mathscr{B}_\delta$.

This gives rise to a Hausdorff outer measure, defined by

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}_\delta^s.$$

The Hausdorff dimension concretizes the intuitive concept of dimension. The Hausdorff dimension of a set can be explained as the power with which its content (e.g. the volume in 3 dimensions) grows when you scale the set. For example, the volume of a cube grows with the power of three compared to the length of an axis of the cube, and is therefore 3 dimensional.

The following theorem gives us a definition of the Hausdorff dimension which gives us a unique dimension for every set, using Hausdorff measures.

Theorem 2.7. *For each subset E of $\mathbb{R}^n$ there is a unique value dim(E) such that*

$$\mathscr{H}^s(E) = \infty \text{ if } 0 \leq s < dimE, \mathscr{H}^s(E) = 0 \text{ if } dimE < s < \infty$$

Definition 2.8. We call dim(E) the Hausdorff dimension of E.

Proof. It is clear that $\mathscr{H}^s(E)$ decreases when s increases. Furthermore, if $s < t$, $\mathscr{H}_\delta^s(E) \geq \delta^{s-t}\mathscr{H}_\delta^t(E)$. When taking the limit of δ to ∞, we see that if $\mathscr{H}^t(E)$ is positive, $\mathscr{H}^s(E)$ is infinite. □

This definition agrees with our instinctive concept of dimension, but is defined on a wider range of sets. We see that it agrees with the usual definition of dimension on boxes: a straight line has Hausdorff dimension 1, a square has Hausdorff dimension 2 and a cube has Hausdorff dimension 3.

Figure 2: This portrays the steps involved in constructing the Cantor Set. The Cantor Set is constructed by taking the intersection of all steps shown below, continued indefinitely

Important to note is that Hausdorff dimension, contrary to the intuitive concept of dimension, does not have to be a whole number. Contrary to the naïve concept of dimension, it is possible for the Hausdorff dimension to be an arbitrary nonnegative real number. An example is the Hausdorff dimension of the Cantor set, as seen in figure 2 ([Azh17]). Its Hausdorff dimension turns out to be $\frac{\log 2}{\log 3}$ ([Fal02, Theorem 1.14]).

We want an easy way to refer to sets which have positive s-dimensional Hausdorff measure for their dimension.

Definition 2.9. If $0 < \mathscr{H}^s < \infty$, E is called an s-set.

Because of Theorem 2.7 we know E cannot be an s-set for multiple s.

We can also check that the Hausdorff measure s-dimensional measure is, in fact, an outer measure.

Theorem 2.10. *The Hausdorff measure as defined in definition 2.6 is an outer measure.*

Proof. For the first requirement of an outer measure, we know that since $\varnothing$ is contained in every ball, the infinum of volumes of coverings is 0 for every δ. Therefore $\mathscr{H}^s(\varnothing) = 0$.

If A in B, $\mathscr{B}_\delta$ which covers B also covers A, so $\mathscr{H}^s_\delta(A)$ is upper bounded by $\mathscr{H}^s_\delta(B)$, $\mathscr{H}^s_\delta(A) \leq \mathscr{H}^s_\delta(B)$ for every δ. Therefore, $\mathscr{H}^s(A) \leq \mathscr{H}^s(B)$.

As we look at the last requirement, we can assume that $\mathscr{H}^s_\delta(U_i)$ is finite for all i, since the third requirement would be satisfied automatically if that was not the case. For a given ϵ, we know there exists a cover of balls B_{ij} of U_i such that $\sum_j |r_{ij}|^s \leq \mathscr{H}^s_\delta(U_i) + \frac{\epsilon}{2^i}$. $\bigcup_{i,j} B_{ij}$ covers $\bigcup_{i=1}^n U_i$, so

$$\mathscr{H}^s_\delta(\bigcup_{i=1}^n U_i) \leq \sum_{i,j} |r_{ij}|^s \leq \sum_i \left\{\mathscr{H}^s_\delta(U_i) + \frac{\epsilon}{2^i}\right\} \leq \sum_i \mathscr{H}^s_\delta(U_i) + \epsilon$$

Since ϵ was chosen arbitrarily small, the third requirement follows for each δ. Taking the limit of δ to 0 for both sides of the inequality, it follows for the Hausdorff outer measure. □

Important to note is that for natural numbers n, the n-dimensional Hausdorff measure and the n-dimensional Lebesgue measure agree up to a constant. This theorem helps us to exchange these measures in a lot of proofs and use the one which is most convenient.

Theorem 2.11. $\mathscr{H}^n(E) = c(n) * \mathscr{L}^n(E)$ *for each* $E \subset \mathbb{R}^k$ *for all* k, n.

Proof. Theorem 8 in [Bel14] □

The Lebesgue measure is often also defined on the Borel sets. The Borel sets are a collection of subsets of $\mathbb{R}^n$ which is a σ-field (defined in the appendix, definition A.1).

Definition 2.12. The Borel sets of $\mathbb{R}^n$ is the σ-field generated by the open sets in $\mathbb{R}^n$, and so consists of the open sets, the countable union and intersection of open and closed sets, the countable intersection and union of those sets, etc.

Contained in the Borel sets are two types of subsets of $\mathbb{R}^n$ which we will use in the proof later, so I define them here.

Definition 2.13. The G_δ-sets are countable intersections of open sets. The F_σ-sets are countable unions of closed sets.

For the same reason, we also want to define the notion of Souslin sets.

Definition 2.14. The Souslin sets are sets of the form

$$E = \bigcup_{\{i_1,i_2,i_3,\ldots\}} \bigcap_{k=1}^{\infty} E_{\{i_1,i_2,i_3,\ldots\}},$$

where for each finite sequence $\{i_1, i_2, i_3, \ldots\}$ we have a closed set $E_{\{i_1,i_2,i_3,\ldots\}}$

Each Borel set is a Souslin set.

A property of measures which is important in this thesis is their continuity. This means we can for increasing or decreasing families of sets take the limit of the measure of the sets in the family to get the measure of the union of all sets for increasing families, or the intersection of all members of the family for decreasing families of sets.

Theorem 2.15 (Continuity of measures)**.** *For any increasing family of sets* $\{A_j\}, A_j \subset A_k$ *if* $j < k$ *with* $A = \bigcup_j A_j$, *we have*

$$\mu(A) = \lim_{j\to\infty} \mu(A_j) \tag{2.1}$$

For any decreasing family of sets $\{A_j\}, A_j \subset A_k$ *if* $j > k$ *with* $A = \bigcap_j A_j$, *we have*

$$\mu(A) = \lim_{j\to\infty} \mu(A_j) \tag{2.2}$$

Proof. Theorem 4.4 of [Sch11] □

We now have covered the new terms introduced in the questions asked in the introduction. For proving the main theorems of the thesis, some other concepts are also of vital importance.

2.2 Definitions used in the proof

2.2.1 Density

The concept of density is reoccuring in the proof of the main theorems and is therefore one we need to define properly. Intuitively, the density of a set at a point is how much of the available space around the point is occupied by the set.

Definition 2.16. We define upper and lower densities as

$$\overline{D}^s(E, x) = \limsup_{r\to 0} \frac{\mathscr{H}^s(E \cap B_r)}{(2r)^s}$$

and

$$\underline{D}^s(E, x) = \liminf_{r\to 0} \frac{\mathscr{H}^s(E \cap B_r)}{(2r)^s}$$

respectively. When they coincide, we write $D^s(E, x)$ for the common value.

Now that we have defined the concept of density, we note that it has a maximum of 1. A point having a density of 1 with respect to a set E has sets around it which differ by a set of s-Hausdorff measure 0 from a ball around the point for some small enough r and which are contained in E. We want to distill this in a property to which we can refer.

Definition 2.17. A point $x \in E$ at which $D^s(E, x) = 1$ is called a regular point of E, otherwise x is irregular.

2.2.2 Lines on the plane

For our final proof, we use an operation of subsets which transforms a subset of the plane into a collection of lines on the plane. By proving theorems about our original set and transforming it into the collection of lines on the plane, we can prove useful characteristics about the set of lines.

For example, an irregular set E in $\mathbb{R}^2$ turns out to have a line set L_E with $\mathscr{H}^2(L(E)) = 0$, which proves to be very useful.

Definition 2.18. For $(a, b) \in \mathbb{R}^2$, $L(a, b)$ is the set of the points on the line $y = a + bx$

Definition 2.19. For a set $E \subset \mathbb{R}^2, L(E) := \bigcup_{(a,b)\in E} L(a, b)$

Furthermore, we have to define a line with fixed x-constant, since there is no (a, b) such that $L(a, b)$ is a straight line with fixed x-constant. These lines will also come in handy when we intersect a given set with them, allowing us to apply Fubini's theorem.

Definition 2.20. For a constant $C \in \mathbb{R}, L_c$ is the set of points in $\mathbb{R}^2$ with $x = c$.

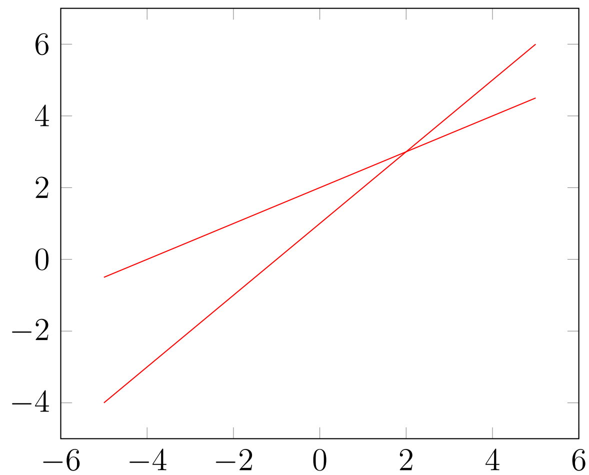


Figure 3: The line set $L(E)$ given by the set $\{\{1, 1\}, \{2, 0.5\}\}$

For convenience, we also want to define a line trough the origin with a certain angle with respect to the x-angle. Such a line is also denotable as a $L(a, b)$, but denoting it differently is

Definition 2.21. L_θ is the line through the origin which makes an angle of θ with the x-axis.

We also want to be able to refer to a line through a point x in a certain direction.

Definition 2.22. By $L_\theta(x)$ we define the line through x in direction θ.

2.2.3 Projection

Another useful concept we will consider is the concept of projection. In the proof of the main result, it is particularly useful to project irregular sets in the plane onto subsets of an axis. These subsets are often easier to work with. Furthermore, in the case of the line sets we defined the projection of E onto the y-axis captures the properties of the gradients of the lines in the lineset $L(E)$.

Definition 2.23. For a subspace of $\mathbb{R}^n$ denoted V, the orthogonal projection onto V is denoted as proj_V. The orthogonal projection is taken for a subspace V by writing $\mathbb{R}^n$ as $V \oplus V^\perp$, and an element $x \in \mathbb{R}^n = V \otimes G_\perp$ denoted as (x_1, x_2) is taken to X_1.

Definition 2.24. By proj_θ we denote the projection of $\mathbb{R}^2$ onto the line L_θ.

2.2.4 Support of Measures

For proving some essential lemma's which help us prove that the $L(E)$ of an irregular set E has 0 $\mathscr{H}^2$-measure, we need the concept of support of a measure defined on the Borel sets.

Definition 2.25. The support of a Borel measure μ is the smallest closed set S such that $\int f d\mu = 0$ for all continuous functions f that vanish on S.

Definition 2.26. A measure on the Borel sets μ on $\mathbb{R}^n$ with a compact support and with $0 < \mu(\mathbb{R}^n) < \infty$ is called a *mass distribution.*

The next definitions are necessary to prove important lemma's about the Hausdorff dimension of projections of 1-sets in $\mathbb{R}^2$.

Definition 2.27. The *t-potential* of a mass distribution μ is given by

$$\varphi_t(x) = \int \frac{d\mu(y)}{|x-y|^t}$$

Definition 2.28. The *t-energy of μ* is given by

$$I_t(\mu) = \int \varphi_t(x) d\mu(x) = \int\int \frac{d\mu(y) d\mu(x)}{|x-y|^t}$$

Definition 2.29. For E a compact subset of $\mathbb{R}^n$, the *t-capacity* is defined as

$$C_t(E) = \sup_\mu \left\{ \frac{1}{I_t(\mu)} : \text{E supports } \mu \text{ and } \mu(E) = 1 \right\}$$

Since $I_t(\mu)$ can be infinite, for infinity we use $\frac{1}{\infty} = 0$
If E is an arbitrary subset of $\mathbb{R}^n$, we take

$$C_t(E) = \sup\{C_t(F) : F \text{ is a compact subset, } F \subset E\} \tag{2.3}$$

2.2.5 Characterizing sets

In proving the theorems required for the final proof of the main theorems of the thesis, we use types of sets which reoccur. Therefore we want to define and characterize them here. We mainly use these definitions in the proof of theorem 3.13 but they can elsewhere.

The first type of set we want to define is a type of curve in $\mathbb{R}^n$. We need some definitions to be able to characterize the curve properly.

Definition 2.30. A curve Γ is the image of an injective map $\varphi : [a,b] \to \mathbb{R}^n$, with $[a,b] \subset \mathbb{R}$ a closed interval.

Since every curve is the image of a compact connected set, it is compact and connected itself and therefore any curve is a Borel set.

Definition 2.31. The length of a curve Γ is defined as

$$\mathscr{L}(\Gamma) := \sup \sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})|$$

with the supremum taken over all dissections of $[a, b] : a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$.

Definition 2.32. If $\mathscr{L}(\Gamma) < \infty$, Γ is called a rectifiable curve.

We also want to use sets which are combinations of rectifiable curve.

Definition 2.33. A Y-set is a 1-set contained in a countable union of rectifiable curves.

For the following definition of condensation directions we need to define a double sector.

Definition 2.34. We define as $C_r(x, I)$, called the double sector, the set consisting of the points in

$$\bigcup_{\theta:\theta \in I} B_r(x) \cap L_\theta(x)$$

2.2.6 Net measure

For the proof of theorem 3.10 we want to use net measures instead of the Hausdorff measure.

Definition 2.35. A class of sets $\mathscr{N}$ is a net of sets when if $U_1, U_2 \in \mathscr{N}$, then either $U_1 \cap U_2 = \emptyset$ or $U_1 \subset U_2$ or $U_2 \subset U_1$ and each set is contained in finitely many others.

For the s-dimensional net measure we want to define we specify a specific net of sets.

Definition 2.36. As a net set for the definition of our measure we take $\mathscr{N}$ in $\mathbb{R}^n$ we take the collection of all sets of the form

$$[2^{-k}m_1, 2^{-k}(m_1 + 1)) \times [2^{-k}m_2, 2^{-k}(m_2 + 1)) \times \ldots \times [2^{-k}m_n, 2^{-k}(m_n + 1))$$

with $\{m_i\}_{i=1}^n$ a collection of integers and k a non-negative integer.

We define as the s-dimensional δ-net measure content as

$$\mathscr{M}_\delta^s = \inf \sum_{i=1}^{\infty} |S_i|^s$$

with the infinum taken over all countable δ-covers of E by sets $\{S_i\} \subset \mathscr{N}$.

We define the s-dimensional net measure as

$$\mathscr{M}^s(E) = \sup_{\delta>0} \mathscr{M}_\delta^s(E)$$

We want a relation to the s-dimensional Hausdorff measure so we can use these net measures to prove theorems about the Hausdorff measure. Therefore we need the following theorem.

Theorem 2.37. *There exist constants b_n dependent which are only dependent on the dimension n such that for all $E \subset \mathbb{R}^n$ we have*

$$\mathscr{H}^s(E) \leq \mathscr{M}_\delta^s \leq b_n \mathscr{H}^s(E)$$

Proof. Theorem 5.1 in [Fal02] □

2.2.7 Condensation directions

For theorem 3.4, we need the concept of condensation directions. We define a condensation direction of the first kind and a condensation direction of the second kind and use these definitions to then define a point of radiation.

Definition 2.38. A direction θ is a condensation direction of the first kind at a point $x \in E$ of an irregular set E if $L_\theta(x)$, the line in direction θ through the point x, intersects E infinitely often in every neighborhood of x

Definition 2.39. For positive numbers ρ, ϵ, m a subset $T(x, \rho, \epsilon, m)$ of $[0, \pi)$ is constructed by taking $\theta \in T(x, \rho, \epsilon, m)$ if and only if there exists some r, $0 < r < \rho$ and some open interval I with $\theta \in I \subset [0, \pi)$ and $|I| < \epsilon$ such that

$$\frac{\mathscr{H}^1(E \cap C_r(x, I))}{r} \geq m|I|$$

Furthermore, we define the set T as

$$T := \bigcap_{\rho > 0} \bigcap_{\epsilon > 0} \bigcap_{0 < m < \infty} T(x, \rho, \epsilon, m)$$

For x in an irregular set E, a θ is a condensation direction of the second kind if $\theta \in T$.

Intuitively, we can understand a condensation direction of the second kind as direction from a point in which there is always enough of the set close enough to the direction.

We want to use these two definitions to define the concept of a point of radiation.

Definition 2.40. A point x is called a point of radiation for an irregular 1-set E if almost all $\theta \in [0, \pi)$ are a radiation direction of the first or second kind.

Two other sets connected to the concept of radiation directions are also used in the proof, and we will define them here.

Definition 2.41. $G_i := \{(x, \theta) : \theta$ is a condensation direction of the i-th kind of E at $x\}$. G_i is defined for $i \in \{1, 2\}$.

2.2.8 Vitali classes

For the proof of theorem 3.4, we need the concept of a Vitali class and a result from the book by [Fal02].

Definition 2.42. A collection of sets $\mathscr{V}$ is a Vitali class for E if for each $x \in E$ and $\delta > 0$ there is some $U \in \mathscr{V}$ such that $x \in U, 0 < |U| \leq \delta$.

Theorem 2.43. *Let E be a $\mathscr{H}^s$-measurable subset of $\mathbb{R}^n$. Let $\mathscr{V}$ be a Vitali class of closed sets, for E. Then we can select a countable (or finite) disjoint sequence $\{U_i\}, U_i \in \mathscr{V}$ for all i, such that either $\sum_i |U_i|^s = \infty$ or $\mathscr{H}^s(E \setminus \bigcup_i U_i) = 0$. If $\mathscr{H}^s(E) < \infty$ we may also require that for a given $\epsilon > 0$, $\mathscr{H}^s(E) \leq \sum_i |U_i|^s + \epsilon$.*

3 Lemmas necessary in the proof

In this section I prove the lemmas and theorems necessary to prove our two final theorems. I sometimes reference theorems which are further back in the section, this is because of the structure of this chapter. We look at some general useful theorems first and then look at theorems specifically mentioned in the proof of our two main theorems. In the last subsection we prove lemmas for the theorems proven earlier in the section, so to these lemmas I sometimes refer in the proof of our earlier theorems.

3.1 General useful theorems for working with Hausdorff measure

To characterize the Besicovitch set about which we are going to prove Theorem 1.5, we need to know more about the set than just that it contains a line segment in every direction.

The following theorem gives us a tool to generalize a Besicovitch set, such that we can construct a Besicovitch set with certain properties if we know one exists.

Theorem 3.1. *If E is any subset of $\mathbb{R}^n$, there is a G_δ set containing E with $\mathscr{H}^s(G) = \mathscr{H}^s(E)$*

Proof. We have two options. $\mathscr{H}^s(E) = \infty$, in which case $\mathbb{R}^n$ is a valid G_δ set, or $\mathscr{H}^s(E) < \infty$.

In that case we can choose for all $i \in \mathbb{N}$ an open $\frac{2}{i}$-cover of E (Since a $\frac{1}{i}$- cover is also a $\frac{2}{i}$-cover), $\{U_{ij}\}_j$ such that

$$\sum_j^\infty |U_{ij}|^s < \mathscr{H}^s_{\frac{1}{i}}(E) + \frac{1}{i}$$

Then we see $E \subset G$, and $G = \bigcap_{i=1}^\infty \bigcup_{j=1}^\infty U_{ij}$ is a G_δ set, since it is a countable intersection of a union of open sets which is an open set itself. We also see that G is covered by $\{U_{ij}\}_j$, and therefore $\mathscr{H}^s_{2/i}(G) = \mathscr{H}^s_{1/i}(E) + 1/i$. Letting i go to ∞, we get $\mathscr{H}^s(G) = \mathscr{H}^s(E)$. $\square$

The next theorem is similar to the previous one and is used in the proof of theorem 3.6.

Theorem 3.2. *Any $\mathscr{H}^s$-measurable subset of finite $\mathscr{H}^s$-measure contains an F_σ set of equal measure, and therefore contains a closed set differing from it by arbitrarily small measure.*

Proof. Let E be a set which is $\mathscr{H}^s$-measurable and has $\mathscr{H}^s(E) < \infty$. Using theorem 3.1 and by the definition of a G_δ-set, we know that we have a collections of open sets $\{O_i\}$, each containing E, and with

$$\mathscr{H}^s(\bigcap_{i=1}^\infty O_i \backslash E) = \mathscr{H}^s(\bigcap_{i=1}^\infty O_i) - \mathscr{H}^s(E) = 0.$$

Since every open set is an F_σ-set, we know that we can write each one of these O_i's as $\cup_{j=1}^{\infty} F_{ij}$, where we take as $\{F_{ij}\}$ an increasing sequence of closed subset, such that $F_k \subset F_l$ if $k \leq l$. Since the Hausdorff measure is continuous, by theorem 2.15, $\mathscr{H}^s(E \cap F_{ij}) = \mathscr{H}^s(E \cap O_i) = \mathscr{H}^s(E)$

By the definition of a limit, we can find a j_l for a given ϵ such that $\mathscr{H}^s(E \cap F_{ij_l}) < 2^{-i}\epsilon$ for all $i \in \mathbb{N}$.

If we take as F the closed set $F = \bigcap_{i=1}^{\infty} F_{ij_l}$, then we get:

$$\mathscr{H}^s(F) \geq \mathscr{H}^s(F \cap E) \geq \mathscr{H}^s(E) - \sum_{i=1}^{\infty} \mathscr{H}^s(E \backslash F_{ij_l}) > \mathscr{H}^s(E) - \epsilon$$

We have $F \subset \bigcap_{i=1}^{\infty} O_i$, since each $F_{ij_l} \subset O_i$ and therefore $\bigcap_{i=1}^{\infty} F_{ij_l} \subset \bigcap_{i=1}^{\infty} O_i$.

So, $\mathscr{H}^s(F \backslash E) \leq \mathscr{H}^s(\bigcap_{i=1}^{\infty} O_i \backslash E) = 0$, so $\mathscr{H}^s(F \backslash E) = 0$. By theorem 3.1, we know that there is a G_δ-set G with $F \backslash E \subset G$ such that $\mathscr{H}^s(G) = 0$.

We see that $F \backslash G$ is contained in E and that $F \backslash G$ is still an F_σ-set, since it is a finite intersection of two F_σ-sets (the complement of a G_δ-set is an F_σ-set and F G is the intersection of F with the complement of G).

For that reason,

$$\mathscr{H}^s(F \backslash G) \geq \mathscr{H}^s(F) = \mathscr{H}^s(G) > \mathscr{H}^s(E) - \epsilon.$$

So we have with $F \backslash G$ a closed set contained in G which differs by arbitrarily small measure from E since we can take ϵ as small as we want.

Taking a union over these F_σ-sets for $\epsilon = \frac{1}{n} : \forall n \in \mathbb{N}$ gives an F_σ-set F^* contained in E with $\mathscr{H}^s(F^*) - \mathscr{H}^s(E)$. □

To have general information about the Hausdorff measure of the projection of a set onto a subspace, we need the following theorem.

Theorem 3.3. *Let E be a subset of $\mathbb{R}^n$ and let G be any subspace of E. Then $\mathscr{H}^s(proj_G(E)) \leq \mathscr{H}^s(E)$.*

Proof. Since $|\text{proj}_G(x) - \text{proj}_G(y)| < |x - y| : \forall x, y \in E$, the theorem follows from 3.12 □

3.2 Theorems necessary for the proof of theorem 1.4

The theorems in this sections are used directly in the proof of theorem 1.4.

The following theorem and the corollary which follows from it are used to prove that our constructed set is really irregular, which is essential for the proof of the theorem.

Theorem 3.4. *Let E be an irregular 1-set in $\mathbb{R}^2$. Then $\mathscr{H}^1(proj_\theta E) = 0$ for almost all $\theta \in [0, \pi)$*

Proof. First, assume that E is closed.

Claim: $G = G_1 \cup G_2$ is an $(\mathscr{H}^1 \times \mathscr{H}^1)$-measurable subset of $E \times [0, \pi)$ for E a closed irregular 1-set.

Proof of claim: For G_1 we first want to look at the subset $G_{r,\rho}$ with $0 < r < \rho$, where we define $G_{r,\rho} := \{(x, \theta) : x \in E \text{ and } r \leq |x - y| \leq \rho \text{ for some } y \in E \cap L_\theta(x)\}$. Since E is closed, so is $G_{r,\rho}$, and therefore $G_{r,\rho}$ is a Borel set. We see that $G_1 = \bigcap_{\rho>0}(\bigcup_{r\leq\rho} G_{r,\rho})$. By taking the union and intersections over rational r and ρ, we see G_1 is also a Borel set by the definition of σ-algebra's: A.1.

For G_2 we look at $\mathscr{H}^1(E \cap C_r(x, I))$, which is continuous in x, r and I for an irregular 1-set E ([Fal02, p. 85]). Therefore, if we define a $T'(x, \rho, \epsilon, m)$ which is defined just like $T(x, \rho, \epsilon, m)$ but with a strict inequality in the equation

$$\frac{\mathscr{H}^1(E \cap C_r(x, I))}{r} > m|I|,$$

we get that $\{(x, \theta) : \theta \in T'(x, \rho, \epsilon, m)\}$ is an open set in $E \times [0, \pi)$. But we still have

$$G_2 = \bigcap_{\rho>0} \bigcap_{\epsilon>0} \bigcap_{0<m<\infty} \{(x, \theta) : \theta \in T'(x, \rho, \epsilon, m)\}.$$

By taking an intersection over rational values of ρ, ϵ and m, we get that G_2 is a countable intersection over open sets, so it is a Borel set and therefore an $(\mathscr{H}^1 \times \mathscr{H}^1)$-measurable set. □

So G is an $\mathscr{H}^1 \times \mathscr{H}^1$-measurable subset of $E \times [0, \pi)$ and we also have $\mathscr{H}(\{\theta : (x, \theta \in G)\}) = \pi$ for almost all $x \in E$, by lemma 3.18. Therefore we can apply Fubini, which gives us that for almost all θ, $(x, \theta) \in G$ for almost all x.

Now we take as a θ such a direction such that it is a condensation direction for almost all x, which was true for almost all θ. We take $\varphi = \theta + \frac{1}{2}\pi$, so φ is perpendicular to θ.

We want the projection of E onto L_φ to have 1-dimensional Hausdorff measure 0.

We take E as a union of three sets: $E = E_0 \cup E_1 \cup E_2$. E_0 is the set of points in E for which θ is not a condensation direction, while E_1 and E_2 are the sets of points for which θ is a condensation direction of the respectively first or second kind.

Obviously, we have $\mathscr{H}^1(E) = 0$, since θ was a condensation direction for almost all $x \in E$. By theorem 3.3, we have $\mathscr{H}^1(\text{proj}_\theta E_0) = 0$.

With the projection of the set E_1, we want to use theorem 3.10, which states:

Theorem 3.10: E be a subset of the plane, and let A be any subset of the x-axis. Suppose that for all $x \in A$ we have $\mathscr{H}^t(E_x) > c$ for some constant c, where E_x is the subset of E with x-coördinate x. Then

$$\mathscr{H}^{s+t}(E) \geq bc\mathscr{H}^s(A)$$

where b is a constant dependent on only s and t.

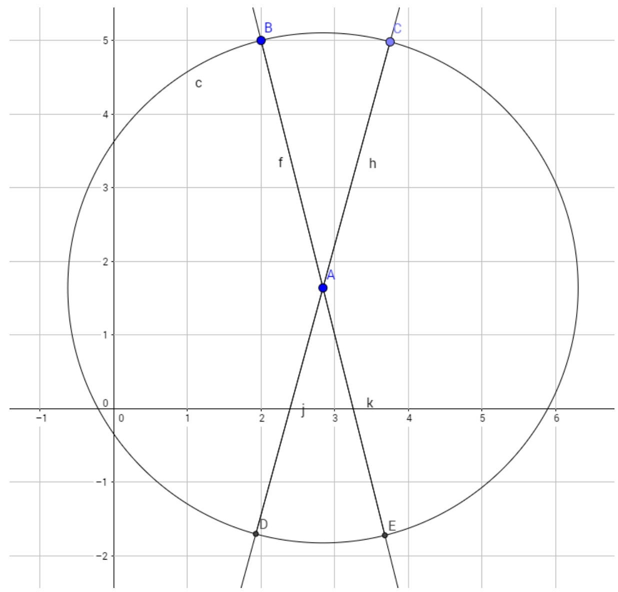


Figure 4: A double sector with radius r in direction θ around A

We view $\mathbb{R}^2$ as the Cartesian product of L_θ and L_φ. Since θ was a condensation direction of the first kind for almost all $x \in E$, that means that $L_\theta(x)$ intersects E on infinitely many points for almost all $x \in E$. Therefore, $\mathscr{H}^0(L_\theta(x) \cap E) = \infty$ for almost all x.

Therefore, if we take L_φ as the x-axis, we can apply theorem 3.10 with $t = 0, s = 1$, A the projection of E_1 onto L_φ, and we get $\infty > \mathscr{H}^{1+0}(E_1) \geq bc\mathscr{H}^1(\text{proj}_\varphi(E_1)$, which must hold for any c since $\mathscr{H}^0(L_\theta(x) \cap E) = \infty > c$ for all $c \in \mathbb{R}$. Therefore, $\mathscr{H}^1(\text{proj}_\varphi(E_1)) = 0$.

Last we take a look at E_2. We want to look at the projection of E_2 onto L_φ.

For a given positive number m, we let $\mathscr{V}$ be a class of intervals of L_φ, defined by

$$\mathscr{V} := \{J : \mathscr{H}^1(x \in E : \text{proj}_\varphi x \in J) > m|J|\} \tag{3.1}$$

We recall the definition of the second condensation direction which uses the following equation for $T(x, \rho, \epsilon, m)$: $\frac{\mathscr{H}^1(E \cap C_r(x,I))}{r} \geq m|I|$.

For all $x \in E_2$ this equation holds for all ρ, ϵ and m for some open interval I around θ and some $r < \rho$. T

he double sector $C_r(x, I)$ is contained in a rectangle of width less than $|Ir|$ and with the other sides in the direction of θ, because the projection of an interval on the circle onto L_φ has length less than its length on the circle, and the length of the outer boundary of one of the sectors (such as the segment of the circle BC in figure 4) has length $|Ir|$.

By projecting these rectangles onto L_φ for all x for the intervals I with which they are in $T(x, \rho, \epsilon, m)$, the projection is in the class of intervals $\mathscr{V}$. Since we can do this for all $x \in E_2$ with arbitrarily small intervals, we have that $\mathscr{V}$ is a Vitali class.

Now we can use theorem 2.43 to get for $\epsilon > 0$ some disjoint family of sets $\{J_i\}$, $J_i \in \mathscr{V}$ for all i, such that

$$\mathscr{H}^1(\text{proj}_\varphi E_2) \leq \sum_i |J_i| + \epsilon. \tag{3.2}$$

But because of our definition of the Vitali class, we also have the inequality

$$\sum_i |J_i| + \epsilon \leq \frac{1}{m}\mathscr{H}^1(E) + \epsilon, \tag{3.3}$$

since the $\{J_i\}$ are disjoint and so every $x \in E$ is in at most one of the sets $\{x \in E : \text{proj}_\varphi x \in J_i\}$.

Because we can choose ϵ and m arbitrarily small, we get $\mathscr{H}^1(\text{proj}_\varphi E_2) = 0$.

We already had $\mathscr{H}^1(\text{proj}_\varphi E_1) = 0$ and $\mathscr{H}^1(\text{proj}_\varphi E_0) = 0$, combining these facts gives us $\mathscr{H}^1(\text{proj}_\varphi E) = 0$ for every irregular compact 1-set E and for almost all φ.

Now we can take E as a union of a closed set F and a set G with $\mathscr{H}^1(G) < \epsilon$ for all ϵ by theorem 3.2.

Since F is a closed set, we know $\mathscr{H}^1(\text{proj}_\varphi F) = 0$ for almost all φ. So, by theorem 3.3, we know that for almost all φ we have $\mathscr{H}^1(\text{proj}_\varphi E)$ $\mathscr{H}^1(\text{proj}_\varphi G) \leq \mathscr{H}^1(G) < \epsilon$

Since we can take ϵ arbitrarily small, this gives us $\mathscr{H}^1(\text{proj}_\varphi E) = 0$ for almost all φ, which proves the theorem. □

Corollary 3.5. *A 1-set $E \subset \mathbb{R}^2$ is irregular if and only if it has projections of Hausdorff measure zero in two distinct directions*

Proof. From theorem 3.4 we know that for each irregular set there are certainly two distinct directions in which the projection have $\mathscr{H}^1$-measure zero.

If E has a regular part with $\mathscr{H}^1(E) > 0$, then theorem 3.13 gives that there is at most one direction for which $\mathscr{H}^1(\text{proj}_\theta(E)) = 0$, so if there are two distinct directions, E must be irregular. □

The next theorem combines well with the previous corollary to give us characteristics of a set we just proved to be irregular. Especially, we know that the line set of an irregular set has 0 2-dimensional Hausdorff measure, which turns out to be crucial for our proof.

Theorem 3.6. *If E is an irregular 1-set in $\mathbb{R}^2$, then $\mathscr{H}^2(L(E)) = 0$.*

Proof. Say $\mathscr{H}^1(E) = 0$. We can find a G_δ-set E_0 such that $E \subset E_0$ and $\mathscr{H}^1(E_0) = 0$, by theorem 3.1.

Claim: Since $\mathscr{H}^1(E_0) = 0, \mathscr{H}^1(\text{proj}_\theta(E_0)) = 0$ and therefore $\mathscr{H}^1(L(E_0) \cap L_c) = 0$ for all c.

Proof of claim: Theorem 3.3 gives $\mathscr{H}^1(\text{proj}_\theta(E_0)) = 0$. In combination with lemma 3.11 we get the result. □

Since $L(E_0)$ is a G_δ-set, because of theorem 3.7 it is a measurable subset of $\mathbb{R}^2$. We apply Fubini's theorem A.7 and integrate in the y-direction first, which gives us that $\mathscr{H}^2(L(E_0)) = 0$, so $L(E) \subset L(E_0)$ is measurable with $\mathscr{H}^s(L(E)) = 0$.

If E is any 1-set, theorem 3.2 gives us that we can write $E = E_0 \cup F$ with E_0 an F_σ-set and where $\mathscr{H}^s(F) = 0$. So, E is a union of two measurable sets and is therefore measurable itself.

Claim: If E is an irregular 1-set, then $\mathscr{H}^1(\text{proj}_\theta E) = 0$ for almost all θ.

Proof of claim: Theorem 3.4. □

When we apply lemma 3.11, we get $\mathscr{H}^1(L(E) \cap L_c) = 0$ for almost all c. Since we have proven that $L(E)$ is measurable, we can apply Fubini's theorem, A.7, in the same way we did before and we get $\mathscr{H}^2(L(E)) = 0$.

□

3.3 Theorem necessary for theorem 1.5

In the proof of theorem 1.5, we look at a set E which forms all of the lines contained in another set F. We therefore want to be able to know something about this set E by having information about the set F.

Theorem 3.7. *E is a G_δ-set $\iff$ $L(E)$ is a G_δ-set.*
E is an F_σ-set $\iff$ $L(E)$ is an F_σ-set.

Proof. If E is open in $\mathbb{R}^2$, so is $L(E)$, since if there exists an open ball around every point in E there also exists an open ball around every point in $L(E)$. For this reason, if $L(E)$ is open, so is E. If E is closed in $\mathbb{R}^2$, so is $L(E)$, since it contains all its limit points, which can also be turned around to show that is $L(E)$ is closed, so is E.

Therefore if E is a G_δ set, so is $L(E)$ and also if $L(E)$ is a G_δ-set, so is E, since they are both intersections of opens. The same logic can be applied to show the F_σ-equivalence. □

Then, we have the following two theorems which help us look at the projection onto L_θ for a set E for which we know the dimension is greater than or equal to 1. This set is used in the final proof. The cases of the dimension being 1 and the dimension being greater than 1 turn out to need to different theorems. We use the following theorem for the case that the dimension of the set E is one, while we use the theorem after that to look at the case of the dimension being greater than 1.

Theorem 3.8. *Let E be a Souslin subset of $\mathbb{R}^2$ with dim$(E) = s$ with $s \leq 1$. Then dim(proj$_\theta(E)) = s$ for almost all $\theta \in [0, \pi)$*

Proof. Theorem 3.3 gives $\dim(\text{proj}_\theta E) \leq s$ for all θ, so we only need to prove the other way; $\dim(\text{proj}_\theta E) \geq s$.

Corollary 3.15 gives us that we can choose a mass distribution μ with its support a compact subset S of E such that $I_t(\mu) < \infty$. Using this, following [Fal02, p. 81] we can define a positive linear functional $\varphi : f \to \int f(\vec{x} \cdot \vec{\theta}) d\mu(x)$, where $\vec{\theta}$ is the unit vector in the θ direction and $\vec{x}$ the direction of length x in the x-direction.

We want this functional since we can understand $f(x \cdot \theta)$ as the integral over the projection of E onto L_θ by identifying $u \in \mathbb{R}$ with $u\theta$, with θ the unit vector in the direction of θ

Using the Riesz representation theorem, theorem 6.2 in [Fal02] and following [Fal02, p. 81] we can now find a mass distribution μ_θ for each θ such that

$$\int f(u) d\mu_\theta(u) = \int f(x \cdot \theta) d\mu(x) \tag{3.4}$$

By Beppo Levi's theorem, theorem A.8 in the appendix, this holds for any nonnegative measurable function, since we can approach each measurable nonnegative function by a series of increasing continuous functions.

Now, by definition,

$$\begin{aligned} I_t(\mu_\theta) &= \int\int \frac{d\mu_\theta(x) d\mu_\theta(y)}{|x-y|^t} \\ &= \int\int \frac{d\mu_(u) d\mu(v)}{|\vec{u} \cdot \theta - \vec{v} \cdot \theta|^t} \\ &= \int\int \frac{d\mu(u) d\mu(v)}{|(\vec{u} - \vec{v}) \cdot \theta|^t} \end{aligned} \tag{3.5}$$

We now look at $\int_0^\pi I_t(\mu_\theta)$. As τ we take $\frac{(\vec{u}-\vec{v})}{|u-v|}$. We see we can write

$$\begin{aligned} \int_0^\pi I_t(\mu_\theta) &= \int\int\int \frac{d\mu(u) d\mu(v) d\theta}{|u-v|^t}) \frac{1}{|\tau \cdot \theta|^t} \\ &= \int_0^\pi \frac{d\theta}{|\tau \cdot \theta|^t} \int\int \frac{d\mu(u) d\mu(v)}{|u-v|^t}) \\ &= \int_0^\pi \frac{d\theta}{|\tau \cdot \theta|^t} I_t(\mu) \end{aligned} \tag{3.6}$$

since $\int_0^\pi \frac{d\theta}{|\tau \cdot \theta|^t}$ is independent on τ (because the inner product is $|\tau||\theta| \cos(\tau - \theta) = \cos(\tau - \theta)$, and integrating the absolute value of a cosinus over a length of π always gives the same value). Because $I_t(\mu) < \infty$, which we already knew, and $\int_0^\pi \frac{d\theta}{|\tau \cdot \theta|^t} < \infty$ since $t > 0$, we get $\int_0^\pi I_t(\mu_\theta) < \infty$.

Since $\int_0^\pi I_t(\mu_\theta) < \infty, I_t(\mu_\theta) < \infty$ for almost all θ. Corollary 3.15 gives us that $\dim(\text{proj}_\theta E) \geq t$, since the support of μ_θ was a subset of $\text{proj}_\theta E$.

This construction is true for all $t < s$, so $\text{proj}_\theta E = s$.

□

Theorem 3.9. *Let E be a Souslin subset of $\mathbb{R}^2$ with dim$(E) = s$ with $s > 1$. Then $\mathscr{H}^1(proj_\theta(E)) > 0$ for almost all $\theta \in [0, \pi)$*

Proof. Theorem 6.8 (b) in [Fal02]

□

The next theorem helps us in estimating the Hausdorff dimension and measure of a set E if we know that $E \cap L_c$ has a certain dimension for all c in a subset of the plane with a certain dimension. Specifically, it helps us determine the dimension of a set with a line in every direction and is therefore crucial to the proof of theorem 1.5.

Theorem 3.10. *Let E be a subset of the plane, and let A be any subset of the x-axis. Suppose that for all $x \in A$ we have $\mathscr{H}^t(E_x) > c$ for some constant c, where E_x is the subset of E with x-coördinate x. Then*

$$\mathscr{H}^{s+t}(E) \geq bc\mathscr{H}^s(A)$$

where b is a constant dependent on only s and t.

Proof. We can use net measure $\mathscr{M}^s$ on $\mathbb{R}^n$ because of theorem 2.37.

For $\delta > 0$, let $\{S_i\}_{i=1}^\infty$ be a collection of sets in $\mathscr{N}$ forming a $2^{\frac{1}{2}}\delta$-cover of E. We see that for each $x \in A$, $E_x \subset \bigcup_1^\infty (S_i)_x$. Therefore, $\mathscr{M}_\delta^t(E_x) \leq \sum_1^\infty |(S_i)_x|^t$.

Let $A_\delta := \{x \in A : \mathscr{M}_\delta^t(E_x) > c\}$, then if $x \in A_\delta$,

$$c < \sum_{i=1}^\infty |(S_i)_x|^t = 2^{-\frac{1}{2}t} \sum_{i: x \in \text{proj} S_i}^\infty |S_i|^t \tag{3.7}$$

Where projS_i is the projection of S_i onto the x-axis of at most length δ. Because of that

$$\sum_1^\infty |S_i|^{s+t} = \sum_1^\infty |S_i|^s |S_i|^t = 2^{\frac{1}{2}s} \sum_1^\infty |S_i|^t |\text{proj} S_i|^s \geq 2^{\frac{1}{2}(s+t)} c\mathscr{M}_\delta^s(A_\delta)$$

by applying lemma 3.17 with $y_i = |S_i|^t$ and with $I_i = \text{proj}S_i$, since its condition is satisfied by equation 3.7 with constant $2^{\frac{1}{2}t}c$.

Because this is true for every $2^{\frac{1}{2}}\delta$-cover of E with binary squares, so therefore

$$bc\mathscr{M}_\delta^s(A_\delta) \leq \mathscr{M}_{2^{\frac{1}{2}}}^{s+t}(E) \leq \mathscr{M}^{s+t}(E)$$

for $b = 2^{\frac{1}{2}(s+t)}$. Since $\bigcup_{\delta > 0} A_\delta = A$ and A_δ is an increasing family of sets (for δ tending to 0), we know $\mathscr{M}_\delta^s(A_\rho) \leq \mathscr{M}_\delta^s(A_\delta) \leq b^{-1}c^{-1}\mathscr{M}^{s+t}(E)$ for $\delta \leq \rho$.

By letting δ tend to zero, therefore we get for $\rho > 0$ that $\mathscr{M}^s(A_\rho) \leq b^{-1}c^{-1}\mathscr{M}^{s+t}(E)$.

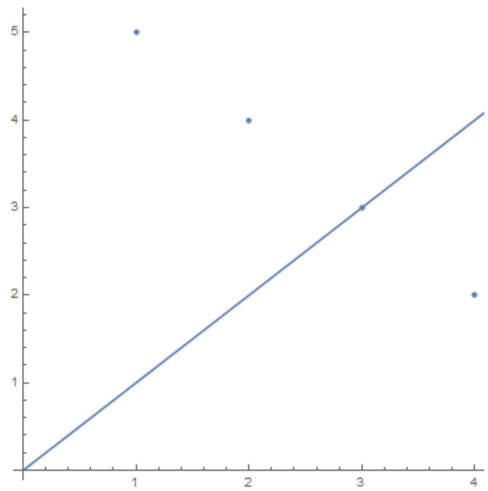


Figure 5: A set of points of which the line set intersects L_1 at the same point

Since measures are continuous by theorem 2.15, therefore

$$\mathscr{M}^s(A) = \mathscr{M}^s(\bigcap_{\rho>0} A_\rho) = \lim_{\rho\to 0} \mathscr{M}^s(A_\rho) \leq b^{-1}c^{-1}\mathscr{M}^{s+t}(E)$$

This proves the theorem for the net measure and therefore for the Hausdorff measure by choosing our constants in accordance with theorem 2.37.

□

3.4 Proof of lemma's necessary for the theorems

We use the following lemma often. It gives us a geometric similarity between two sets which we use often to translate characteristic from a set E to its line set $L(E)$.

Lemma 3.11. *$L(E) \cap L_c$ and $\mathscr{H}^1(proj_\theta(E))$ are geometrically similar, and therefore $\mathscr{H}^1(L(E) \cap L_c) = 0 \Leftrightarrow \mathscr{H}^1(proj_\theta(E)) = 0$ and $dim(L(E) \cap L_c) = dim(proj_\theta E)$*

Proof. We want $L(E) \cap L_C$ to be geometrically similar to $\text{proj}_\theta E$. We look at lines which cross L_c at the same point. The points in E forming these lines lie on a straight line as in figure 5. We see in figure 6 that if one of two lines crosses the y-axis k higher, to still intersect the vertical line at $x = c$ we get the equation

$$a + bc = (a + k) + b'c \tag{3.8}$$

with a for the points of intersection with the y-axis and b and b' the gradients of the line. By solving this equation, we get that the line of points which form lines which intersect at one point on the line L_c has gradient $-\frac{1}{c}$. Therefore its perpendicular line

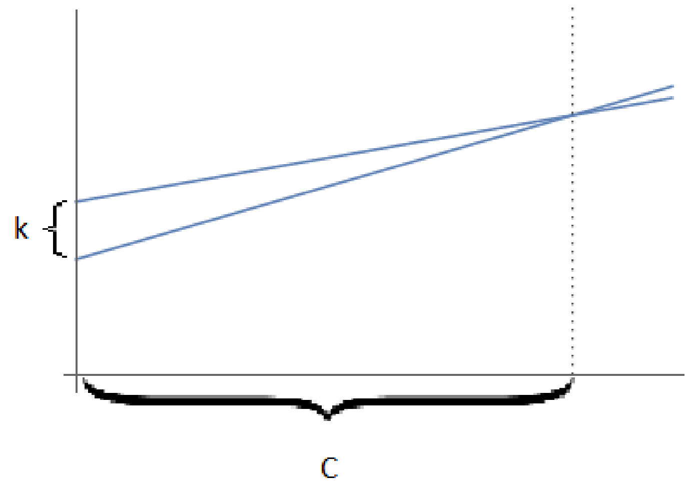


Figure 6: A sketch from which we can derive the equation of the angle of the line L_θ on which points whose line sets intersect the same point on L_c project onto a single point

onto which all these points project into a single point has gradient c and therefore makes an angle with the x-axis of arctan$c$. Therefore, $L(E) \cap L_c$ and $\text{proj}_\theta(E)$ are geometrically similar for $c = \tan\theta$.

This geometric similarity gives that if one has $\mathscr{H}^1$-measure zero, so does the other and that these sets will have the same Hausdorff dimension. □

A next lemma is useful in looking at the Hausdorff measures of projections of sets.

Theorem 3.12. *Let $\varphi : E \to F$ be a surjective Lipschitz map. Then $\mathscr{H}^s(F) \leq c^s \mathscr{H}^s(E)$.*

Proof. We know that for each i, $|\varphi(U_i \cap E)| \leq c\ |U_i \cap E|$. So if $\{U_i\}$ is a δ-cover of E, $\varphi(U_i \cap E)$ is a $c\delta$-cover of F. Therefore,

$$\sum_i |\varphi(U_i \cap E)|^s \leq c^s \sum_i |U_i|^s$$

and therefore $\mathscr{H}^s_{c\delta}(F) \leq c^s \mathscr{H}^s_\delta(E)$. Taking the limit of δ to 0 gives the theorem. □

The next theorem is used to prove that a set whose projection has 1-dimensional Hausdorff measure 0 in two directions must be irregular. This turns out to very useful for the proof of theorem 1.4, since constructing an irregular set is crucial in the proof.

Theorem 3.13. *Let E be a regular 1-set in $\mathbb{R}^2$. Then $\mathscr{H}^1(proj_\theta(E)) = 0$ for at most one value of θ*

Proof. We look at E as a Y-set, since a regular 1-set is the union of measurable subset of a Y-set and a set of $\mathscr{H}^1$-measure zero according to theorem 3.25 of [Fal02]. Since if we can prove the theorem for a 1-set contained in a rectifiable curve it follows for E (since E

necessarily contains a 1-set contained in a rectifiable curve), we try and prove that. We call this 1-set contained in a rectifiable curve F.

We take a point x which is a regular point both of F and the rectifiable curve Γ in which F is contained. There exists such a point, since every Y-set is a regular 1-set (corollary 3.9, [Fal02]) and every regular point of F is a regular point of Γ.

Therefore, because of our definition of density given in definition 2.16 and the definition of a limit, for any ϵ there is an r such that both

$$\mathscr{H}^1(F \cap B_r(x)) > (1-\epsilon^2)2r \text{ and } \mathscr{H}^1(\Gamma \cap B_r(x)) < (1+\epsilon)2r \tag{3.9}$$

Multiplying the right of equation 3.9 with $(1-\epsilon)$ and combining both inequalities gives $\mathscr{H}^1(F \cap B_r(x)) > (1-\epsilon)\mathscr{H}^1(\Gamma \cap B_r(x))$.

By splitting Γ up into a part which is a subset of F and a part which is disjoint of F, this equation gives us $\mathscr{H}^1((\Gamma \backslash E) \cap B_r(x)) < \epsilon \mathscr{H}^1(\Gamma \cap B_r(x))$.

Now, for any rectifiable curve we know that the intersection $\Gamma \cap B_r(x)$ consists of only countably many arcs, which are disjoint since the curve Γ is formed by an injective map from $[a,b] \to \mathbb{R}^2$, there must be some arc Γ_0 such that both $\Gamma_0 \subset \Gamma \cap B_r(x)$ and $\mathscr{H}^1(\Gamma_0 \backslash E) < \epsilon \mathscr{H}^1(x)$, since if there would not consist such a curve we would have

$$\mathscr{H}^1(\Gamma \backslash E) = \sum_{i=0}^{\infty} \mathscr{H}^1(\Gamma_i \backslash E) \geq \epsilon \sum_{i=0}^{\infty} \mathscr{H}^1(\Gamma_i) = \epsilon \mathscr{H}^1(\Gamma \cap B_r(x))$$

which is in contradiction with our earlier equation.

Because the curve is rectifiable, we can take by the definition of rectifiability take a subset of this curve $\Gamma_1 \subset \Gamma_0$ such that $\mathscr{H}^1(\Gamma_1 \backslash E) < \epsilon \mathscr{H}^2(\Gamma_1) < 2\epsilon|y-z|$, with y and z endpoints of Γ_1.

Now we want some line L_θ with θ making an angle of φ with the line from y to z for which we want $\cos\varphi > 2\epsilon$.

Now, by theorem 3.3 we have

$$\begin{aligned} \mathscr{H}^1(\text{proj}_\theta E) &> |y-z|\cos\varphi - \mathscr{L}^1(\text{proj}_\theta(\Gamma_0 \backslash E)) \\ &\geq |y-z|\cos\varphi - \mathscr{H}^1(\Gamma_0 \backslash E) > |y-z|(\cos\varphi - 2\epsilon) > 0 \end{aligned} \tag{3.10}$$

So, except for a set of directions for which $\cos\varphi \leq 2\epsilon$ for all ϵ, we have $\mathscr{H}^1(\text{proj}_\theta E) > 0$, which means that there is only one direction for which this is not the case.

This proves the theorem. □

We use the next the next theorem and the corollary which follows from it in the proof of theorem 3.8. That theorem helps us in the final proof by helping us prove that our dimension of a set which contains a line in every direction is big enough.

Theorem 3.14. *Let E be a subset of $\mathbb{R}^n$. If E is a Souslin set with $C_t(E) = 0$, then $\mathscr{H}^s(E) = 0$ for all $s > t$. If $\mathscr{H}^s(E) < \infty$, then $C_s(E) = 0$.*

Proof. Theorem 6.4 in [Fal02]. □

Corollary 3.15. *We have E a Souslin subset of $\mathbb{R}^n$.*

If $I_t(\mu) < \infty$ for some mass distribution μ supported by E, then $t \leq dimE$.

If $t < dimE$, there exists a mass distribution μ with support in E such that $I_t(\mu) < \infty$.

Proof. These statements both follow from another claim:

Claim: If E a Souslin subset of $\mathbb{R}^n$, then we know

$$\dim E = \inf(t : C_t(E) = 0) = \sup(t : C_t(E) > 0) \tag{3.11}$$

Proof of claim: For all $t > \dim E$, we have $\mathscr{H}^t(E) = 0 < \infty$ and therefore by theorem 3.14 we know $C_t(E) = 0$. However, for all $t < \dim E$ we have $\mathscr{H}^t(E) = \infty$. Therefore, by theorem 3.14, $C_t(E) > 0$ since $C_t(E) = 0$ would imply $\mathscr{H}^s(E) = 0$. So since $C_t(E) = 0$ for all $t < \dim E$ and $C_t(E) > 0$ for all $t > \dim E$, we get $\dim E = \inf(t : C_t(E) = 0) = \sup(t : C_t(E) > 0)$ which is the claim. □

So if $I_t(\mu) < \infty$ for some t, we know $C_t(E) > 0$ and therefore $t \leq \dim E$. If $t < \dim E$, the claim shows that $C_t > 0$ and therefore there must exist some mass distribution μ with support in E such that $I_t(\mu) < \infty$. This proves the corollary.

□

Theorem 3.16. *Let E be a Souslin subset of $\mathbb{R}^n$, $\mathscr{H}^s(E) = \infty$. Then there is a compact subset F of E such that $\mathscr{H}^s(F) > 0$ and $\mathscr{H}^s(B_r(x) \cap F) \leq br^s$ for $x \in \mathbb{R}^n$ and $r \leq 1$, for some constant $b \in \mathbb{R}$.*

Proof. A Souslin subset E of $\mathbb{R}^n$ with $\mathscr{H}^s(E) = \infty$ has a closed subset S with $\mathscr{H}^s(S) = \infty$, see [Rog98].

Claim: Let E be a closed subset of $\mathbb{R}^n$ with $\mathscr{H}^s(E) = \infty$. Then there is a compact subset F of E such that $\mathscr{H}^s(F) > 0$ and $\mathscr{H}^s(B_r(x) \cap F) \leq br^s$ for all $x \in \mathbb{R}^n, r \leq 1$.

Proof of claim: Theorem 5.4 (b) in [Fal02].

Combining these two facts proves the theorem.

□

Lemma 3.17. *Let A be any subset of $\mathbb{R}$, let $\{I_i\}$ be a countable δ-cover of A by binary intervals, and let $\{y_i\}$ be a set with $y_i \in \mathbb{R}_+ : \forall i \in \mathbb{N}$. Suppose that c is a constant and $\sum_{\{i:x\ \in I_i\}} y_i > c$ for all $x \in A$. Then*

$$\sum_i y_i |I_i|^s \geq c\mathscr{M}^s_\delta(A)$$

Proof. Lemma 5.7 in the book [Fal02] □

Our final lemma has a lengthy proof and is used in theorem 3.4 to help us prove that the set we use in the proof of theorem 1.4 is actually irregular.

Lemma 3.18. *Let E be a closed irregular 1-set in the plane. Then almost all points of E are points of radiation*

Proof. Following directly from theorem 3.30 in[Fal02], we have for almost all $x \in E$ and for any interval $I \subset [0, \pi)$ that

$$\limsup_{r \to 0} \frac{\mathscr{H}^1(E \cap C_r(x, I))}{2r} > \frac{1}{20}|I| \tag{3.12}$$

We want to show that for all x for which this is true, x is a point of radiation of E.

We show the set S of condensation directions of the first kind at x to be a Borel subset of $[0, \pi)$. As $S(r)$ we take the set of θ for which $L_\theta(x) \cap B_r(x)$ contains a point of E other than x. We see that $S(r)$ is the union of of all sets $S(r, \delta) : \{\theta : \exists y \in L_\theta$ such that $\delta \leq |x - y| \leq r\}$. $S(r, \delta)$ is a closed set. Therefore, $S(r)$ is a Borel set.

We note that $S = \bigcap_{j=1}^{\infty} S(\frac{1}{j})$, since if $\theta \in S(r)$ for all r, then E must have infinite points on the line L_θ, for if E had finitely many points on the line there would be a minimum distance of these points to x.

So S is the countable intersection of Borel sets and therefore is a Borel set itself.

We now want that for postive ρ, ϵ and m that $\theta \in T(x, \rho, \epsilon, m)$ for almost all $\theta \notin S$. We define Lebesgue density on $[0, \pi)$:

$$\frac{\lim_{r \to 0} \mathscr{L}^1(S \cap [\theta - r, \theta + r]}{2r})$$

We look at $\theta \in [0.\pi)$ which has Lebesgue density of S zero, which are almost all $\theta \notin S$ according to theorem 1.13 of [Fal02].

Therefore, there must exist ϵ such that $\mathscr{L}^1(S \cap I) < |I|/20m$ for all I with $|I| < \epsilon$ by the definition of a limit.

Let I be any such interval. Since $\bigcap_{r=0}^{\infty} S(r) = S$ and $S(r)$ is an increasing family of sets, there must be a $\rho_1 \leq \rho$ such that for $r < \rho_1$, $\mathscr{L}^1(S(r) \cap I) < \frac{|I|}{20m}$.

Using equation 3.12 there must exist an $r < \rho_1$ such that $\mathscr{H}^1(E \cap C_r(x, I)) > \frac{2r|I|}{20}$ by the definition of lim sup and multiplying both sides with $2r$.

Then we get 2 possible cases. Either we have $\mathscr{H}^1(E \cap C_r(x)) > mr|I|$, such that $\theta \in T(x, \rho, \epsilon, m)$ already, or that is not yet the case.

If that is not the case, we have that since $\mathscr{L}^1(S(r) \cap I) < \frac{|I|}{20m}$, there exists a countable covering of disjoint intervals $\{I_i\}$ such that $S(r) \cap I \in \cup + i = 1^\infty I_i$ and $\mathscr{L}^1(\cup_{i=1}^{\infty} I_i) < \frac{|I|}{20m}$, since if we cover with half-open squares we can always take the squares in a way such that they do not overlap, and since we know $\mathscr{L}^1(S(r) \cap I) < \frac{|I|}{20m}$, there must exist some covering of half open squares for which it is the case that the union of the Lebesgue measures of these squares is also smaller than $\frac{|I|}{20m}$

We construct a new set Q, which consists of the indexes for which $\mathscr{H}^1(E \cap C_r(x, I_i)) > mr|I_i|$. We get that

$$\sum_{i \notin Q} \mathscr{H}^1(E \cap C_r(x, I_i)) \leq mr\mathscr{L}^1(\bigcup_{i \notin Q} I_i) \leq \frac{r|I|}{20}, \tag{3.13}$$

because since the i are not in Q,$\sum_{i \notin Q} \mathscr{H}^1(E \cap C_r(x, I_i)) < \sum_{i \notin Q} \mathscr{L}^1(I_i)$ and because the I_i are disjoint $\sum_{i \notin Q} \mathscr{L}^1(I_i) = \mathscr{L}^1(\bigcup_{i \notin Q} I_i)$.

Since the points of $E \cap C_r(x, I)$ lie on lines $L_\theta(x)$, where $\theta \in S(r) \cap I$ and we have $S(r) \cap I \in \bigcup_{i=1}^{\infty} I_i$, we have that $E \cap C_r(x, I) \subset E \cap C_r(x, I \cap S(r)) \subset \bigcup_{i=1}^{\infty}(E \cap C_r(x, I_i))$.

Now we have

$$\sum_{i \in Q} \mathscr{H}^1(E \cap C_r(x, I_i) = \sum_{i=1}^{\infty} \mathscr{H}^1(E \cap C_r(x, I_i)) - \sum_{i \notin Q} \mathscr{H}^1(E \cap C_r(x, I_i) \tag{3.14}$$

We had already established that $\mathscr{H}^1(E \cap C_r(x, I) > \frac{2r|I|}{20}$ and $\sum_{i \notin Q} \mathscr{H}^1(E \cap C_r(x, I_i)) < \frac{r|I|}{20}$. Combining this with equation 3.14 we get:

$$\sum_{i \in Q} \mathscr{H}^1(E \cap C_r(x, I_i)) > \mathscr{H}^1(E \cap C_r(x, I)) - \frac{r|I|}{20} > \frac{r|I|}{20} \tag{3.15}$$

Now we can expand all intervals in Q slightly and combine them if they start to overlap, so that we get a disjoint collection of open intervals $\{J_j\}_{j \in \mathbb{N}}$, with $\bigcup_{i \in Q} I_i \subset \bigcup_{j=1}^{\infty} J_j = J$ and with $\mathscr{H}^1(E \cap C_r(x, J_j)) = mr|J_j|$.

This is possible because the intervals in Q are defined using $\mathscr{H}^1(E \cap C_r(x, I_i)) > mr|I_i|$ which gives us a little wiggle room to expand and combine the sets to turn the inequality into an equality.

Since we did not yet have $\mathscr{H}^1(E \cap C_r(x)) > mr|I|$, or it would be immediately clear that θ was already in $T(x, \epsilon, \rho, m)$, this is possible in such a way that $J \subset I$ (we expand J by taking bits from I_i for which $i \notin Q$, so this is not in contradiction with $\bigcup_{i \in Q} I_i \subset \bigcup_{j=1}^{\infty} J_j = J$).

By application of the equation $\mathscr{H}^1(E \cap C_r(x, J_j)) = mr|J_j|$, we have $\mathscr{H}^1(J) = \sum |J_j|$ (since the J_j are disjoint), and so $\mathscr{H}^1(J) = \frac{\sum_{j=1}^{\infty} \mathscr{H}^1(E \cap C_r(x, J_j)}{mr}$. Since $\bigcup_{i \in Q} I_i \subset \bigcup_{j=1}^{\infty} J_j$, we have $\sum_{j=1}^{\infty} \mathscr{H}^1(E \cap C_r(x, J_j)) > \sum_{i \in Q} \mathscr{H}^1(E \cap C_r(x, I_i)) > \frac{r|I|}{20}$.

So, by combining equations, $\mathscr{H}^1(J) > \frac{|I|}{20m}$

Now, for all $\theta' \in J$, we have θ' in some J_j and therefore we also know $\frac{\mathscr{H}^1(E \cap C_r(x,J_j))}{r} = m|J_j|$, so therefore $\theta' \in T(x, \rho, \epsilon, m)$.

Since this is true for all $\theta' \in J$, we must have $J \subset T(x, \rho, \epsilon, m)$ and since J was also a subset of I, we have $J \subset I \cap T(x, \rho, \epsilon, m)$.

Therefore, $\mathscr{H}^s(I \cap T(x, \rho, \epsilon, m) > \frac{I}{20}$, since this was true for $J \subset I \cap T(x, \rho, \epsilon, m)$.

This was true for all I with $\theta \in I$ and $I < \epsilon$ and true for all $\theta \notin S$, where S was the set of all condensation directions of the first kind at x. So for all $\theta \notin S$, we either have $\theta \in T(x, \rho, \epsilon, m)$, or the Lebesgue density of $T(x, \rho, \epsilon, m)$ at θ is at least $\frac{1}{20m}$.

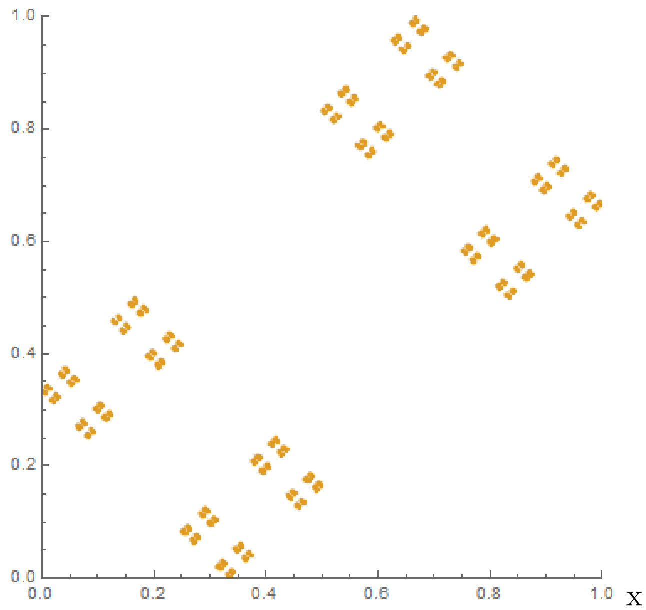
By theorem 1.13 in [Fal02], the set of $\theta \notin T(x, \rho, \epsilon, m)$ with density not 0 has Hausdorff measure zero. Therefore, almost all $\theta \notin S$ belong to $T(x, \rho, \epsilon, m)$.

This is all true for all $\rho > 0, \epsilon > 0, 0 < m < \infty$, we can take countable intersections over decreasing values of ρ, m and ϵ which gives us that almost all $\theta \notin S$ are condensation directions of the second kind.

This proves the theorem.

□

Figure 7: The plot of the irregular subset of $\mathbb{R}^2$ we defined in the proof of theorem 4.1



4 Proof of Theorems 1.4 and 1.5

Finally, we can prove our main two theorems. We will recall the two theorems here for clarity.

Theorem 4.1 (Besicovitch conjecture)**.** *There exists a compact Besicovitch set in* $\mathbb{R}^2$ *with Lebesgue measure zero*

Theorem 4.2 (Kakeya Conjecture)**.** *Every Besicovitch set in* $\mathbb{R}^2$ *has a Hausdorff dimension of 2*

Proof of theorem 4.1. Let E an irregular 1-set in $\mathbb{R}^2$ such that its projection onto the y-axis contains the segment $[-1, 1]$.

Claim: such a set exists.

Proof of claim: We will construct such a set by looking at the graph of a specific function. We look at the function $f : [0, 1) \to [0, 1)$ which takes the element $0.x_1x_2x_3x_4...$, denoted in base 4, to $0.y_1y_2y_3y_4....$ by the formula $y_i = \text{mod}_4(5 - x_i)$. If there are two possible decimal notations, we choose the finite one. This makes sure that there is a single decimal representation for every element in $[0, 1)$. The graph of this function, the set of points $(x, (f(x))$ is an irregular subset of $\mathbb{R}^2$.

The graph has projection of $\mathscr{H}^1$-measure 0 in two distinct directions: the lines making a line of angle $\pm\frac{\pi}{4}$ with the x-axis.

We know this since the set is upper bounded by the line $y = x + \frac{1}{3}$, since for points on the graph (x, y) we can only have at maximum $y = x + 0.1111111\ldots$ with $0.1111111\ldots$ in base 4. This is true since our transformation of decimals switches 0 for 1, 1 for 0, 2 for 3 and 3 for 2, so if every decimal switches up $0.1111111\ldots$ is the maximal gain. We see $0.1111111\ldots = \frac{1}{3}$, so the graph is upper bounded by the line $y = x + \frac{1}{3}$. By the same logic, the graph is lower bounded by the line $y = x - \frac{1}{3}$.

We see that the subsets $[0, 0.1)$ and $[0.2, 0.3)$ have already gained 0.1 by applying f to the first decimal. That means that the (x, y) on the graph for $x \in [0, 0.1) \cup [0.2, 0.3)$ $y \geq x + 0.1 - 0.01111\ldots = x + \frac{1}{6}$, so these points on the graph are lower bounded by the line $y = x + \frac{1}{6}$. By the same logic, the (x, y) on the graph for $x \in [0.1, 0.2) \cup [0.3, 1)$ are upper bounded by the line $y = x - \frac{1}{6}$

So all points on the graph are either between the lines $y = x + \frac{1}{3}$ and $y = x + \frac{1}{6}$ or between the lines $y = x - \frac{1}{6}$ and $y = x\frac{1}{6}$. When we project these lines onto $L_{-\frac{\pi}{4}}$, we see that the projection of the graph is contained in two intervals of equal length with an empty interval of the same length inbetween them. This can be interpreted as the first set used in the construction of the Cantor set.

But since the graph is self similar, since $f([0, 1))$ follows the same rules as $f([0, 0.1))$ just as a smaller resolution. So the graph is also a subset of the second set used in constructing the Cantor set, and by the same logic also a subste of the third set, etc. So the set is a subset of the Cantor set, which gives us that it has Hausdorff dimension $\frac{\log 2}{\log 3}$ by [Fal02, Theorem 1.14]. Since $1 > \frac{\log 2}{\log 3}$, we have $\mathscr{H}^1(\text{proj}_{-\frac{\pi}{4}} G) = 0$.

The same logic can be applied to the direction $\frac{\pi}{4}$ Therefore, corollary 3.5 gives us that the graph is irregular.

The projection of this graph onto the y-axis is clearly the line segment $[0, 1)$. By adding the point $(1, 1)$ and mirroring the graph in the x-axis, we obtain a subset of $\mathbb{R}^n$ whose projection onto the y-axis contains the segment $[-1, 1]$. This gives us the proof of the claim. □

We note that $L(E)$ contains a line with every gradient from -1 to 1.

Claim: $\mathscr{H}^2(L(E)) = 0$

Proof of claim: theorem 3.6

The union of E and a copy of E rotated by $\pi/2$ gives a set with Hausdorff measure 0 which contains a line segment in every direction.

□

Proof of theorem 4.2. Every set is contained in a G_δ set of the same s-dimensional Hausdorff measure and therefore of the same Hausdorff dimension, see theorem 3.1.

Therefore we can without loss of generality assume that our Besicovitch set is a G_δ-set. We take as a subset of F the set $E = \bigcap_{r=1}^{\infty} \{(a, b) : L(a, b) \cap B_r(0) \subset F \cap B_r(0)\}$, which is the set of points (a, b)in $\mathbb{R}^2$ for who $L(a, b) \subset F$.

Claim: If F a G_δ-set, so is $E = \bigcap_{r=1}^{\infty}\{(a, b) : L(a, b) \cap B_r(0) \subset F \cap B_r(0)\}$, and therefore E is Borel measurable.

Proof of claim: Theorem 3.7

Since F contains a line segment in every direction, E projected onto the y-axis contains the entire y-axis and therefore, $\mathscr{H}^1(E) = \infty$ by theorem 3.3. From the definition of Hausdorff dimension follows $\dim(E) \geq 1$.

Since every Borel set is a Souslin set therefore since E is a Borel set, it is also a Souslin set. We have to possibilities: $\dim(E) = 1$ or $\dim(E) > 1$. In the first case we can apply theorem 3.8 to get $\dim(\text{proj}_\theta E) = 1$ for almost all θ. If $\dim(E) > 1$, we can apply the projection theorem, theorem 3.9 such that $\mathscr{H}^1(\text{proj}_\theta E) > 0$ for almost all θ. Since the projection of a set in the plane always has Hausdorff dimension 1, this gives us that $\dim(\text{proj}_\theta E) = 1$ for almost all θ.

Theorem 3.8 gives us that since $\mathscr{H}^1(L(E) \cap L_c) = 0$ and $\mathscr{H}^1(\text{proj}_\theta E)$ are geometrically similar for $c = \tan\theta$, so $\dim(L(E) \cap L_c) = 1$ for almost all c.

Now we want to apply theorem 3.10. We see that since $\dim(L(E) \cap L_c) = 1$ for almost all c, we can take as A a subset of the x-axis the c for which $\dim(L(E) \cap L_c) = 1$. We see $\mathscr{H}^t(L(E) \cap L_x) > k$ for any $t < 1$, for every k, since $\dim(L(E) \cap L_x) = 1$ for all $x \in A$. Since $\mathscr{H}^1(A) = \infty$, we get $\mathscr{H}^l(L(E)) = \infty$ for all $l < 2$ if we apply theorem 3.10. So $\dim(L(E)) \geq 2$. Since $L(E)$ is a subset of $\mathbb{R}^2$, we also have $\dim(L(E)) \leq 2$. Therefore, $\dim(L(E)) = 2$. Since $L(E)$ is a subset of F, F has Hausdorff dimension 2.

□

A Appendix

We need the concept of sigma algebra to understand the definition of measure.

Definition A.1. A σ-field Σ on X is a set of subsets of X which has the following three properties:

- $X \in \Sigma$
- If $A \in \Sigma, A^c \in \Sigma$
- If $A_i \in \Sigma, i \in \mathbb{N}, \bigcup_{i\in\mathbb{N}} A_i \in \Sigma$

The following set of definitions is necessary to properly define the concept of an integral. We use the concept of the integral repeatedly in the thesis, so a proper definition is necessary.

Definition A.2. $\mathcal{E}^+$ is a set of functions g of the form $g := \sum_{i=1}^m y_i * \mathbb{1}_{A_i}$ for finitely many A_j in the σ-algebra.

Definition A.3. $\Sigma_\mu(g)$ is defined as $\sum_{i=1}^m y_i * \mu(A_i)$.

Definition A.4. The integral of a positive measurable function u which is positive and measurable is given by

$$\int ud\mu := \sup\Big\{\Sigma_\mu(g) : g \leq u, g \in \mathcal{E}^+\Big\} \in [0, \infty].$$

([Sch11])

Definition A.5. u_+ is defined as $u_+(x) = u(x)$ if $u(x) \geq 0$, $u_+(x) = 0$ if $u(x) < 0$ and u_- is defined as $u_-(x) = -u(x)$ if $u(x) \leq 0$, $u_-(x) = 0$ if $u(x) > 0$.

Definition A.6. If $\int u_+ d\mu < \infty$ and $\int u_- d\mu < \infty$, the integral of a measurable function u is given by

$$\int ud\mu = \int u_+ d\mu - \int u_- d\mu.$$

We use Fubini's theorem in the thesis, but it is not connected to the rest of the thesis.

Theorem A.7 (Fubini's Theorem). *For two spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we have that if*

$$\int_X \left(\int_Y |f(x,y)|\nu(dy) \right) \mu(dx)$$

$$\int_Y \left(\int_X |f(x,y)|\mu(dx) \right) \nu(dy)$$

$$\int_{X\times Y} |f(x,y)|d(\mu \times \nu)$$

is finite, then

$$\int_X \left(\int_Y f(x,y)dy \right) dx = \int_Y \left(\int_X f(x,y)dx \right) dy = \int_{X\times Y} f(x,y)d(x,y)$$

Proof. See corollary 13.9 of [Sch11] □

Theorem A.8 (Beppo Levi's theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. For an increasing sequence of positive numerical measurable functions $(u_j)_{j\in\mathbb{N}}$ with $0 \leq u_j \leq u_{j+1} \leq \ldots$, we have*

$$\int \sup_{j\in\mathbb{N}} u_j d\mu = \sup_{j\in\mathbb{N}} \int u_j d\mu \tag{A.1}$$

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