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Riemannian Manifolds

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Contents

1 Introduction

Riemannian manifolds are a pair of a smooth manifold and an inner product g_p. The inner product g_p is a tensor field, and it enables the definition of notions from Euclidean space on manifolds. For example lengths of curves and distances between points can be unambiguously defined on Riemannian manifolds. These definitions naturally allow us to proof theorems, two of the major theorems are the Gauss-Bonnet and Cartan-Hadamard theorem. The main goal of this thesis will be to proof these two theorems.

The first part will focus on the Gauss-Bonnet theorem, which relates the Gaussian curvature to the Euler characteristic of a manifold. Curvature is in a way a generalization of the concept of straightness, more formally it is a tensor field that measures how much the metric tensor is not locally isometric to the metric tensor of Euclidean space. For surfaces for example, it is a measure of how much the surface deviates from being flat. Curvature will be a generalization of this concept to n-dimensional Riemannian manifolds. From curvature, Gaussian curvature can be defined, it is a local concept, which the Gauss-Bonnet theorem relates to the Euler characteristic, a global number and a topological invariant associated with a manifold. It describes the topological space's structure independent of the way it is embedded.

After the Gauss-Bonnet theorem we will focus on the Cartan-Hadamard theorem, it is another example of a theorem that provides a link between local and global properties. And this theorem says that the local condition of having non-positive sectional curvature together with being simply-connected has as result that the Riemannian manifold is diffeomorphic to $\mathbb{R}^n$.

We will start off with a formal introduction of Riemannian manifolds and the immediate definitions that come with it, such as angles between tangent vectors and length of tangent vectors.

Then in chapter 3 we will define various curvatures, all of them are functions of the curvature tensor. As we will see the curvature tensor arises when trying to take derivatives of tangent vectors on manifolds. To take derivatives of tangent vectors, connections are introduced, after which the curvature tensor and some other forms of curvature, e.g. the Gaussian and sectional curvature, will conclude chapter 3.

In chapter 4 we will consider submanifolds, and more importantly how the connection of an ambient manifold is related to that of the submanifold. This combined with the results of the subsequent chapter, in which we will generalize plane geometry to 2-dimensional Riemannian manifolds, enables us to state and proof the Gauss-Bonnet theorem.

And in the final chapter we will consider geodesics, curves with 'constant' tangent vector, comparable to straight lines in $\mathbb{R}^n$, the distance function on manifolds. We will also define a map known as the exponential map, which is used to define coordinates known as normal coordinates in which the geodesics are linear functions of time. Next we will discuss Jacobi fields along curves and use them to proof the Cartan-Hadamard theorem.

The reader is assumed to be familiar with basic differential geometry. In particular, the reader should be familiar with tangent spaces, differential forms and Stokes' theorem. The main references used in writing this thesis were [Lee97, Pet98].

2 Riemannian manifolds

As the aim of this text is to obtain the aforementioned theorems from Riemannian geometry, we will begin with an introduction to Riemannian geometry. The main objects in Riemannian geometry are Riemannian manifolds, which will be the subject of this chapter together with the core definitions that follow from the metric g on a manifold.

2.1 Preliminary Definitions

Definition 2.1. : A pair (M, g) is *a Riemannian manifold*, where M is a smooth manifold and g is the Riemannian metric. The Riemanian metric is a 2-tensor field $g : \mathcal{T}^2(M) \to \mathbb{R}$, that is

1. Symmetric, i.e. $(g(X, Y) = g(Y, X)$ for all $X, Y \in T_p(M))$
2. Positive definite, i.e. $(g(X, X) \geqslant 0)$

The common shorthand for $g(X, Y)$ is $\langle X, Y \rangle$ with $X, Y \in T_p(M)$.

With the inner product defined as such, the tangent space is an inner product space. So the Cauchy-Schwarz inequality is valid for tangent vectors $X, Y \in T_pM$ on a Riemannian manifold:

$$|\langle X, Y \rangle| \leq \langle X, X \rangle \langle Y, Y \rangle$$

This Riemannian metric allows for the notions of length of tangent vectors and angles between them, in a way that is similar to the definitions of these quantities in Euclidean space.

Definition 2.2. *The length of a vector* $X \in T_p(M)$, denoted by $|X| := \langle X, X \rangle^{1/2}$.

The angle between tangent vectors $X, Y \in T_p(M)$ is the unique $\theta \in [0, \pi]$, such that $\cos(\theta) = \frac{\langle X, Y \rangle}{|X||Y|}$.

Two vectors are said to be *orthogonal* if the angle between them is $\pi/2$, from the definition of angles it can be seen that this is equivalent with $X, Y \in T_pM$ are orthogonal if and only if $\langle X, Y \rangle = 0$. Multiple vectors $E_1, ..., E_n$ are said to be *orthonormal* if $\langle E_i, E_j \rangle = \delta_{ij}$.

A diffeomorphism $\varphi : M \to \widetilde{M}$ is said to be an *isometry* if $\varphi^*(\tilde{g}) = g$. And (M, g) and $(\widetilde{M}, \tilde{g})$ are *isometric* if there exists an isometry between them.

For a local frame $(E_1, ..., E_n)$ with dual coframe $(\varphi_1, ..., \varphi_n)$ we can write g in terms of this local coframe as $g = g_{ij}\varphi^i \otimes \varphi^j$ To get an idea what this looks like, we can take Euclidean space, $\mathbb{R}^n$, with the Euclidean metric, $\bar{g}$, as an example. This is an example of a Riemannian manifold, and the Euclidean metric is defined as the usual inner product of tangent vectors at each $T_p\mathbb{R}^n$. Because $T_p\mathbb{R}^n = \mathbb{R}^n$ we can look at the inner product of two vectors $X, Y \in T_p\mathbb{R}^n$, written as $X = x^i E_i$ and $Y = y^i E_i$ for the unit vectors $\{E_1, ..., E_n\}$ in Euclidean space. Then the inner product $\bar{g}(X, Y) = \langle X, Y \rangle = \sum_i x^i y^i$. From how $\bar{g}$ acts on unit elements E_i, we see that $\bar{g}_{ij} = \delta_{ij}$.

Two important operations on tensors are commonly used, namely raising and lowering indices. They change the type of tensor from an $\binom{a}{b}$-tensor to a $\binom{a-1}{b+1}$-tensor or a $\binom{a+1}{b-1}$-tensor for raising and lowering an index respectively. On a Riemannian manifold TM is naturally isomporphic to T^*M, where the isomorphism is given by sending $X \in TM$ to the linear map $(Y \to g(X, Y)) \in T^*M$. As this is an isomorphism we can interchange one of the arguments of the tensor with the help of this isomorphism and therefore changing the type of tensor. For $\binom{0}{1}$-tensors in coordinates this results in the following operation.

Definition 2.3. *Raising an index* is a map from TM^* to TM, by sending a covector $\omega \to g^{ij}\omega^i$. Where g^{ij} is the inverse, $(g_{ij})^{-1}$, of the matrix of g_{ij}, which exists because the metric tensor is positive definite so non-singular.

And for $\binom{1}{0}$-tensors this map is:

Definition 2.4. *Lowering an index* is a map $TM \to TM^*$, by sending a vector X to the covector $g_{ij}X^i dx^j = g(X^i \partial_i, \cdot)$, whenever there are coordinates (x_i) with tangent vectors (∂_i) and covectors (dx^i).

Because Riemannian manifolds have additional structure when compared to smooth manifolds, there also is a unique volume form.

Definition 2.5. On any oriented Riemannian n-manifold (M, g), there is a unique n-form known as *the volume form* dV, satisfying the property that $dV(E_1, ..., E_n) = 1$, whenever $(E_1, ..., E_n)$ is an oriented orthonormal basis for a tangent space T_pM.

By definition dV can be written as $dV = E_1 \wedge ... \wedge E_n$ for an oriented orthonormal basis $(E_1, ..., E_n)$. We can proof uniqueness by considering $(\widetilde{E}_1, ..., \widetilde{E}_n)$ another oriented orthonormal basis. Then $dV = E_1 \wedge ... \wedge E_n = detA \cdot \widetilde{E}_1 \wedge ... \wedge \widetilde{E}_n$, where $detA$ is the transition matrix from E to $\widetilde{E}$. Since both basis are oriented and orthonormal $detA = 1$, which implies the defined volume form is unique. The expression for dV with respect to any oriented local frame E_i with dual coframe (φ^i) is: $dV = \sqrt{det(g_{ij})}\varphi^1 \wedge ... \wedge \varphi^n$ Because $\varphi^1 \wedge ... \wedge \varphi^n(E_1, ..., E_n) = det(g_{ij})$. The volume form is the first term we encouter that will appear in the Gauss-Bonnet theorem.

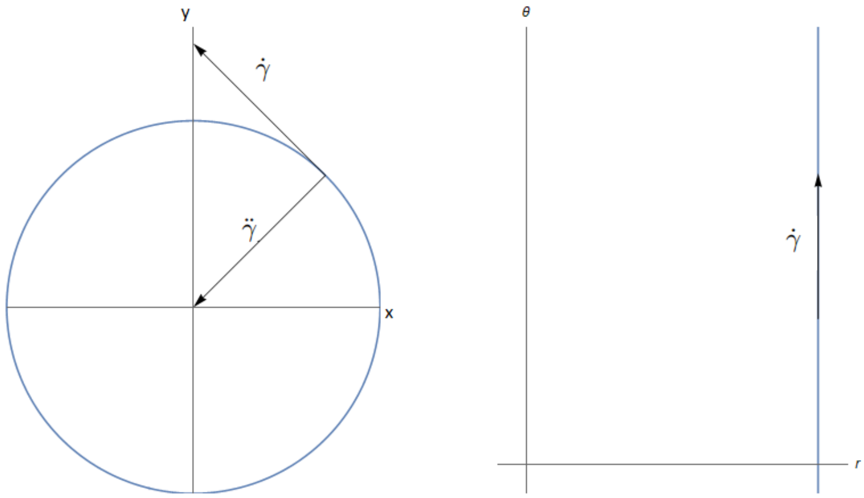
2.2 Curves and Connections

In this section we will introduce curves and connections, as said in the introduction connections are commonly used to define curvature. While curves are used to define geodesics, objects which are commonly studied in Riemannian geometry, and also play an important role in the Gauss-Bonnet theorem. We will start off with the general definition of curves and the definition of a connection will arise when we look at the unit cicle in $\mathbb{R}^2$, which is an example of a curve.

Definition 2.6. *A curve on a manifold* is a smooth map $\gamma : I \to M$, where I is some interval in $\mathbb{R}$.

A curve segment is a curve whose domain is a closed bounded interval.

The velocity of a curve is defined as follows, for $\gamma : I \to M$ at any time $t \in M$ the velocity is defined as the push-forward $\gamma_*(\frac{d}{dt})$, and is denoted by $\dot{\gamma}(t)$. In coordinates this corresponds to the usual definition of velocity, if the coordinate representation of γ is $\gamma(t) = (\gamma^1(t), ..., \gamma^n(t))$, then $\dot{\gamma}(t) = \gamma^i(t)\partial_i$.



(a) Cartasian coordinates (b) polar coordinates

Connections arise when we try to differentiate the velocity vectors. We can take a look at the unit circle in $\mathbb{R}^2$, then in Cartesian coordinates it is parametrized by $(x(t), y(t)) = (\cos(t), \sin(t))$. Differentiating twice gives $(x''(t), y''(t)) = (-\cos(t), -\sin(t))$. While in polar coordinates the circle is parametrized by $(r(t), \theta(t)) = (1, t)$, differentiating this twice gives $(r''(t), \theta''(t)) = (0, 0)$. This implies that acceleration is not invariant under a change of coordinates, therefore we introduce the notion of connections.

The aim of connections is to obtain a coordinate independent notion of acceleration of curves. Specifically they are defined as the directional derivative of sections of vector bundles.

Let $\pi : E \to M$ be a vector bundle over a manifold M, and let $\mathcal{E}(M)$ denote the space of smooth sections of E.

Definition 2.7. *A connection in E* is a map:
$$\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$$

written as $(X, Y) \to \nabla_X Y$, satisfying the following properties:

1. $\nabla_X Y$ is linear over $C^\infty(M)$ in X,

   i.e. $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$, for $f, g \in C^\infty(M)$

2. $\nabla_X Y$ is linear over $\mathbb{R}$ in Y,

   i.e. $\nabla_X aY_1 + bY_2 = a\nabla_X Y_1 + b\nabla_X Y_2$, for $a, b \in \mathbb{R}$)

3. ∇ satisfies the following product rule:

   $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$ for $f \in C^\infty(M)$

Then $\nabla_X Y$ is called *the covariant derivative* of Y in the direction of X

We mostly want to talk about the connection on the tangent bundle of M.

Definition 2.8. *A linear connection on M* is a connection in TM, so it is a map:

$$\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$$

Satisfying property 1. to 3. of a connection.

We can look at the expression for a linear connection in components of TM. Given a local frame (E_i) for TM on an open $U \subset M$, for any i, j, the expression for the linear connection is

$$\nabla_{E_i} E_j = \Gamma^k_{ij} E_k$$

This defines functions Γ^k_{ij} on U which are called the Christoffel symbols and leads to a formula for the linear connection in terms of the Christoffel symbols

$$\nabla_X Y = (XY^k + X^i Y^j \Gamma^k_{ij}) E_k$$

This follows immediately from the properties of connections:

$$\begin{aligned}
\nabla_X Y &= \nabla_X (Y^j E_j) \\
&= (XY^j)E_j + Y^j \nabla_{X^i E_i} E_j \\
&= (XY^j)E_j + Y^j X^i \nabla_{E_i} E_j = XY^k E_k + X^i Y^j \Gamma^k_{ij} E_k
\end{aligned}$$

Where in the last step the index name is just changed from j to k to get the desired result.

Theorem 2.9. *The value of a connection at p only depends on values of X and Y in an arbitrary neighborhood of p. Meaning that if two vector fields $\widetilde{X}$ and $\widetilde{Y}$ agree with X and Y on a neighborhood of p, then $\nabla_X Y|_p = \nabla_{\widetilde{X}} \widetilde{Y}|_p$.*

Proof. We can show that $\nabla_X(Y - \widetilde{Y})|_p = 0$ if $Y' := Y - \widetilde{Y}$ vanishes on a neighborhood $p \in U$, which implies that $\nabla_X(Y)|_p = \nabla_X(\widetilde{Y})|_p$.

By choosing a bump function $\varphi \in C^\infty(M)$ with support in U, such that $\varphi(p) = 1$. Then the hypothesis that Y' vanishes on U implies that $\varphi Y' \equiv 0$ everywhere on M, so $\nabla_X(\varphi Y') = 0\nabla_X(\varphi Y') = 0$. And therefore for any $X \in \mathcal{T}(M)$:

$$0 = \nabla_X(\varphi Y') = (X\varphi)Y' + \varphi(\nabla_X)Y') \tag{2.1}$$

Because $Y' \equiv 0$ on the support of φ the first term on the right is zero. And by evaluating at p this gives the desired result: $\nabla_X(Y - \widetilde{Y})|_p = 0$.

For X a similar argument holds: With the same bump function φ. For any $Y \in \mathcal{E}(M)$

$$0 = \nabla_{\varphi(X-\widetilde{X})}Y = \varphi(\nabla_X Y - \nabla_{\widetilde{X}} Y) \tag{2.2}$$

Which when evaluated at p gives $\nabla_X Y = \nabla_{\widetilde{X}} Y$. $\square$

Since $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$ is linear over $C^\infty(M)$ in its first argument, we can also think of ∇ as a map

$$\nabla : \mathcal{T}(M) \to \Gamma(T^*M \otimes TM)$$

Where $\Gamma(E)$ denotes the space of smooth sections of a vector bundle $\pi : E \to M$.

The covariant derivative can be extended to covector fields as follows. Let α be a covector field and X, Y vector fields on M, then its covariant derivative $(\nabla_X \alpha)$ is defined in a way that is compatible with the pairing between covectors and vectors and satisfies the product rule so.

$$(\nabla_X \alpha)(Y) = \nabla_X(\alpha Y) - \alpha(\nabla_X Y)$$

Then similar to the covariant derivatives of vector fields the covariant derivative is a map $\nabla : \Gamma(T^*M) \to \Gamma(T^*M \otimes TM)$.

Furthermore this can be extended to arbitrary tensor fields by imposing two conditions. For every pair of tensor fields φ and ψ

$$\nabla_X(\varphi \otimes \psi) = (\nabla_X \varphi) \otimes \psi + \varphi \otimes (\nabla_X \psi)$$

And for tensors of the same type

$$\nabla_X(\varphi + \psi) = \nabla_X \varphi + \nabla_X \psi$$

As a result of these two rules. For a tensor field S of type $\binom{n}{m}$ its covariant derivative is an $\binom{n}{m+1}$-tensor, and for $\alpha^1, ..., \alpha^n$ covector fields and $Y_1, ..., Y_n$ vector fields on M is given by

$$\begin{aligned}
\nabla S(X, \alpha^1, ..., \alpha^m, Y_1, ..., Y_n) =& (\nabla_X S)(\alpha^1, ..., \alpha^n, Y_1, ..., Y_n) \\
=& \nabla_X(S(\alpha^1, ..., \alpha^m, Y_1, ..., Y_n)) \\
&- \sum_{i=1}^{n} S(\alpha^1, ..., \alpha^m, Y_1, ..., \nabla_X Y_i, .., Y_n) \\
&- \sum_{i=1}^{m} S(\alpha^1, ..., \nabla_X \alpha^i, ..., \alpha^m, Y_1, .., Y_n)
\end{aligned}$$

Now we have the tools to define the acceleration of a curve, which we will do by defining the more general covariant derivative of vector fields along curves.

Given a curve $\gamma : I \to M$, *a vector field* V *along* γ is by definition a function $V : I \to TM$, with $V(t) \in T_{\gamma(t)}M$ for all $t \in I$.

Definition 2.10. A vector field V along a curve γ is said to be *extendible* if there exists a vector field $\widetilde{V}$ on a neighborhood of the image of γ, such that $V(t) = V_{\gamma(\tilde{t})}$.

Definition 2.11. Let ∇ be a linear connection on M, and $\gamma : I \to M$ a curve. Then define the operator

$$D_t : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$$

Satisfying the properties:

1. Linearity over $\mathbb{R}$,

   i.e. $D_t(aV + bW) = aD_tV + bD_tW$ for $a, b \in \mathbb{R}$
2. Product rule:

   $D_t(fV) = \dot{f}V + fD_tV$ for $f \in C^\infty(I)$
3. If V is extendible, then for any extension$\widetilde{V}$ of V,

   $D_tV(t) = \nabla_{\dot{\gamma}(t)}\widetilde{V}$

   For any $V \in \mathcal{T}(\gamma), D_tV$ is called the covariant derivative of V along γ.

This operator is uniquely determined by ∇, the values of D_tV at t_0 depend only on an arbitrary neighborhood of t_0. Let $V(t) = V^j(t)\partial_j$ for coordinates near $\gamma(t_0)$. Then by the properties of D_t:

$$D_tV(t_0) = \dot{V}^j(t_0)\partial_j + V^j(t_0)\nabla_{\dot{\gamma}(t_0)}\partial_j = (\dot{V}^k(t_0) + V^j(t_0)\dot{\gamma}^i(t_0)\Gamma^k_{ij}(\gamma(t_0))\partial_k$$

Because any covariant derivative satisfies this local expressions, this implies uniqueness.

So by introducing connections we were actually able to define the operator D_t as a coordinate independent way of taking derivatives of tangent fields.

2.3 Riemannian Connection

In this section we will look at the Riemannian connection, a connection that shares two properties with the Euclidean connection. The Euclidean connection is torsion free and compatibility with the Euclidean metric. And it turns out that there is a unique linear connection on all Riemannian manifolds that satisfies both properties. As this connection is more similar to the Euclidean connection than a general linear connection, and because compatibility with the metric is a nice property to work with the Riemannian connection can be thought of as the natural connection on a manifold.

Definition 2.12. A connection ∇ is said to be *compatible with a metric g* if the following product rule is satisfied, for all vector fields X, Y, Z

$$\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Proposition 2.13. *For a Riemannian manifold (M, g) the following conditions are equivalent*

1. *∇ is compatible with g*
2. *$\nabla g \equiv 0$*
3. *If V, W are vector fields along a curve γ, $\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$*

Proof. The equivalence of 1. and 2. is proven by the fact that

$$(\nabla g)(X, Y, Z) = (\nabla_X g)(Y, Z) = X(\langle Y, Z\rangle) - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle$$

And the equivalence of 1. and 3. by

$$\langle D_t V, W\rangle + \langle V, D_t W\rangle = \langle \nabla_{\dot\gamma} V, W\rangle + \langle V, \nabla_{\dot\gamma} W\rangle$$

Which can easily be seen to be equal to $\frac{d}{dt}\langle V, W\rangle$ if and only if g is compatible with the connection. □

The Euclidean connection $\overline{\nabla}$, defined as $\overline{\nabla}_X(Y^j\partial_j) = (XY^j)\partial_j$, is compatible with the Euclidean metric. This implies that it may be natural to look at connections that are compatible with the metric. However this property on its own is not enough to determine a unique linear connection on a manifold.

Definition 2.14. A linear connection is *torsion-free* if the torsion tensor

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

vanishes identically.

The requirements of torsion-freeness and compatibility with the metric are enough to determine a unique linear connection.

Theorem 2.15 (Fundamental theorem of Riemannian geometry)**.** *Let (M, g) be a Riemanian manifold. There exists a unique linear connection ∇ on M that is compatible with g and torsion-free.*

This connection is called the Riemannian connection or Levi-Civita connection

Proof. Compatibility with the metric implies that for $X, Y, Z \in \mathcal{T}(M)$

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Using the condition that the connection is torsion-free allows us to rewrite this to

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle + \langle Y, [X, Z]\rangle$$

By cyclically permutating X, Y, Z and adding the 3 resulting equations we obtain, with a minus sign for the $Z\langle X, Y\rangle$

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \\ 2\langle\nabla_X Y, Z\rangle + \langle Y,[X,Z]\rangle + \langle Z,[Y,X]\rangle - \langle X,[Z,Y]\rangle$$

Solving for $\langle\nabla_X Y, Z\rangle$ gives

$$\langle\nabla_X Y, Z\rangle = 1/2(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ - \langle Y,[X,Z]\rangle - \langle Z,[Y,X]\rangle + \langle X,[Z,Y]\rangle) \tag{2.3}$$

Since the right hand side of this equation does not depend on the connection and holds for all X, Y this implies that for 2 connections that are torsion-free and compatible with g: $\langle\widetilde{\nabla}_X Y - \nabla_X Y, Z\rangle = 0$. Therefore the conditions imply uniqueness. We proof this theorem by explicitly computing the Christoffel symbols in a coordinate chart, consider the standard coordinate vector fields ∂_i. Applying (2.3) to the coordinate vector fields gives

$$\langle\nabla_{\partial_i}\partial_j, \partial_l\rangle = \frac{1}{2}(\partial_i\langle\partial_j,\partial_l\rangle + \partial_j\langle\partial_l,\partial_i\rangle - \partial_l\langle\partial_i,\partial_j\rangle) \tag{2.4}$$

Recall from the definitions that

$$g_{ij} = \langle\partial_i,\partial_j\rangle$$

$$\nabla_{\partial_i}\partial_j = \Gamma^l_{ij}\partial_l$$

Combining this with (2.4) gives

$$\Gamma^m_{ij}g_{ml} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

Which implies

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

The connection obtained from this equation is the desired connection. □

Because of the fundamental theorem we can always use the Riemannian connection on any Riemannian manifold, so in the rest of the text the connection will always be assumed to be the Riemannian one. So connections on Riemannian manifolds will be assumed to be torsion-free and compatible with g. The Riemannian connection is also natural in the sense that any isometry takes Riemannian connections to Riemannian connections.

Proposition 2.16 (Naturality of the Riemannian Connection)**.** *Suppose $\varphi : (M,g) \to (\widetilde{M},\tilde{g})$ is an isometry. Then φ takes the Riemannian connection ∇ of (M,g) to $\widetilde{\nabla}$ of $(\widetilde{M},\tilde{g})$.*

$$\varphi_*(\nabla_X Y) = \widetilde{\nabla}_{\varphi_* X}(\varphi_* Y)$$

Proof. Define the map

$$(\varphi^*\widetilde{\nabla}) : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

as

$$(\varphi^*\widetilde{\nabla})_X Y = \varphi_*^{-1}(\widetilde{\nabla}_{\varphi_* X}(\varphi_* Y))$$

Then $\varphi^*\widetilde{\nabla}$ is a connection on M and it is symmetric and compatible therefore $\varphi^*\widetilde{\nabla} = \nabla$. $\square$

3 Curvature

Curvature in 2-dimensional Euclidean space is a concept used to measure the amount a smooth curve differs from a plane. The generalization to n dimensional Riemannian manifolds certainly is nontrivial, the motivation for the definition of curvature is the following.
Because for the Euclidean metric $\langle E_i, E_j \rangle = \delta_{ij}$ the Christoffel symbols are 0. Therefore in $\mathbb{R}^n$ with the Euclidean metric, for vector fields X, Y, Z

$$\begin{aligned}
\nabla_X \nabla_Y Z &= \nabla_X (Y Z^k \partial_k) = XYZ^k \partial_k \\
\nabla_Y \nabla_X Z &= \nabla_Y (X Z^k \partial_k) = YXZ^k \partial_k
\end{aligned}$$

With the difference between the two

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X = XYZ^k \partial_k - YXZ^k \partial_k = \nabla_{[X,Y]} Z$$

Then by the naturality of the Riemannian connection, this equation holds for any Riemannian manifold that is locally isometric to $\mathbb{R}^n$. For an arbitrary manifold however these equations do not have to hold. Therefore this motivates the definition of the curvature endomorphism, as some indication of the flatness of a manifold.

Definition 3.1. *The Curvature Endomorphism* is the map
$R : \mathcal{T}(M) \times \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.1}$$

Riemannian manifolds that are locally isometric to Euclidean space are called *flat*, it can be proven a manifold is flat if and only if its curvature tensor vanishes identically, see for example [Lee97, Theorem 7.3].

Lemma 3.2. *The Riemannian curvature endomorphism is a* $\binom{3}{1}$*-tensor field.*

Proof. By the tensor characterization lemma we need to show that R is linear over $C^\infty(M)$ in all of its arguments.

For $f \in C^\infty(M)$,

$$\begin{aligned}
R(X, fY)Z &= \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z \\
&= \nabla_X (f \nabla_Y Z) - f \nabla_Y \nabla_X Z - \nabla_{f[X,Y]+(Xf)Y} Z \\
&= (Xf) \nabla_Y Z + f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z - (Xf) \nabla_Y Z \\
&= f R(X, Y) Z
\end{aligned}$$

This shows that R is linear over $C^\infty(M)$ in Y. And from this it also follows that R is linear over $C^\infty(M)$ in X, as $R(X,Y)Z = -R(Y,X)Z$. For linearity over $C^\infty(M)$ in Z,

$$\begin{aligned}
R(X, Y)fZ &= \nabla_X \nabla_Y fZ - \nabla_Y \nabla_X fZ - \nabla_{[X,Y]} fZ \\
&= \nabla_X f \nabla_Y Z + \nabla_X (Yf) Z - \nabla_Y f \nabla_X Z - \nabla_Y (Xf) Z - f \nabla_{[X,Y]} Z - ([X,Y]f) Z \\
&= f \nabla_X \nabla_Y Z + (Xf) \nabla_Y Z + (Yf) \nabla_X Z + X(Yf) Z - f \nabla_Y \nabla_X Z - (Yf) \nabla_X Z \\
&- (Xf) \nabla_Y Z - Y(Xf) Z - X(Yf) Z + Y(Xf) Z - f \nabla_{[X,Y]} Z \\
&= f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z \\
&= f R(X, Y) Z
\end{aligned}$$

□

Because of the preceding lemma we can define the Riemannian curvature tensor Rm, a covariant 4-tensor field obtained from R by lowering an index.

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$$

Proposition 3.3. *The Riemannian curvature tensor $R(X, Y, Z, W)$ has the following properties:*

1. *R is antisymmetric in the first and last two entries:*

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z)$$

2. *R is symmetric between the first two and last two entries:*

$$R(X, Y, Z, W) = R(Z, W, X, Y)$$

3. *R satisfies a cyclic permutation property called Bianchi's first identity:*

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$$

4. *∇R satisfies a cyclic permutation property called Bianchi's second identity:*

$$(\nabla_Z R)(X, Y)W + (\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W = 0$$

Proof. The first part of 1. can be easily seen from the definition of the curvature endomorphism, for the second part we proof that $Rm(W, X, Y, Y) = 0$ then the desired result follows from $Rm(W, X, Y + Z, Y + Z) = 0$.

$$\begin{aligned}
WX|Y|^2 &= W(2\langle \nabla_X Y, Y\rangle) = 2\langle \nabla_W \nabla_X Y, Y\rangle + 2\langle \nabla_X Y, \nabla_W Y\rangle \\
WX|Y|^2 &= 2\langle \nabla_X \nabla_W Y, Y\rangle + 2\langle \nabla_W Y, \nabla_X Y\rangle \\
[W, X]|Y|^2 &= 2\langle \nabla_{[W,X]} Y, Y\rangle
\end{aligned}$$

Subtracting the second and third equation from the first we obtain

$$\begin{aligned}
0 &= 2\langle \nabla_W \nabla_X Y, Y\rangle - 2\langle \nabla_X \nabla_W Y, Y\rangle - 2\langle \nabla_{[W,X]} Y, Y\rangle \\
&= 2\langle R(W, X)Y, Y\rangle \\
&= 2Rm(W, X, Y, Y)
\end{aligned}$$

This finishes the proof for 1.
3. Extends to

$$\begin{aligned}
&(\nabla_W \nabla_X Y - \nabla_X \nabla_W Y - \nabla_{[W,X]} Y) \\
&\quad + (\nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W) \\
&\quad - (\nabla_Y \nabla_W X - \nabla_W \nabla_Y X - \nabla_{[Y,W]} X) \\
=&\nabla_W (\nabla_X Y - \nabla_Y X) + \nabla_X (\nabla_Y W - \nabla_W Y) + \nabla_Y (\nabla_W X - \nabla_X W) \\
&\quad - \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X \\
=&\nabla_W [X, Y] + \nabla_X [Y, W] + \nabla_Y [W, X] \\
&\quad - \nabla_{[W,X]} Y - \nabla_{[X,Y]} W - \nabla_{[Y,W]} X \\
=&[W, [X, Y]] + [X, [Y, W]] + [Y, [W, X]]
\end{aligned}$$

The Jacobi identity [Lee13, p. 188] for commutators says that this is zero.
2. Is obtained by combining the other identities.

$$\begin{aligned}
R(X, Y, Z, W) =& - R(Z, X, Y, W) - R(Y, Z, X, W) \\
=&R(Z, X, W, Y) + R(Y, Z, W, X) \\
=& - R(W, Z, X, Y) - R(X, W, Z, Y) \\
&\quad - R(W, Z, X, Y) - R(X, W, Z, Y) \\
=&2R(Z, W, X, Y) + R(R, X, Y, Z) + R(W, Y, X, Z) \\
=&2R(Z, W, X, Y) - R(Y, X, W, Z) \\
=&2R(Z, W, X, Y) - R(X, Y, Z, W)
\end{aligned}$$

Where the properties 3, 1, 3, 1, 3, 1 are applied in succession, this implies that $2R(X, Y, Z, W) = 2R(Z, W, X, Y)$.

The proof of 4. Because R is a tensor without loss of generality one can assume that X, Y, Z mutually commute, e.g. their Lie brackets are 0. Then

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_[X,Y]Z = [\nabla_X, \nabla_Y]Z$$

Because R is a tensor field

$$\begin{aligned}(\nabla_Z R)(X,Y)W =& \nabla_Z(R(X,Y)W) - R(\nabla_Z X, Y)W \\ &- R(X, \nabla_Z Y)W - R(X,Y)\nabla_Z W \\ =&[\nabla_Z, R(X,Y)]W - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W\end{aligned}$$

Therefore

$$\begin{aligned}&(\nabla_Z R)(X,Y)W + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W \\ &=[\nabla_Z, R(X,Y)]W + [\nabla_X, R(Y,Z)]W + [\nabla_Y, R(Z,X)]W \\ &\quad - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W \\ &\quad - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W \\ &\quad - R(\nabla_Z X, Y)W - R(Z, \nabla_Y X)W \\ &=[\nabla_Z, R(X,Y)]W + [\nabla_X, R(Y,Z)]W + [\nabla_Y, R(Z,X)]W \\ &\quad + R([X,Z],Y)W + R([Z,Y],X)W + R([Y,X],Z)W \\ &=[\nabla_Z, [\nabla_X, \nabla_Y]]W + [\nabla_X, [\nabla_Y, \nabla_Z]]W + [\nabla_Y, [\nabla_Z, \nabla_X]]W = 0\end{aligned}$$

Where the second equality follows because the connection is torsion-free, in the third equality the Lie brackets are used and the last step is again because of the Jacobi equality. □

Another important tensor is the Ricci Curvature, defined as follows:

Definition 3.4. *The Ricci curvature tensor Rc* is the tensor defined to be the trace of R. Meaning that if $e_1, ..., e_n \in T_pM$ is an orthonormal basis, then

$$Ric(v,w) = \sum_{i=1}^{n} g(R(e_i, v)w, e_i) = \sum_{i=1}^{n} g(R(v, e_i)e_i, w) \tag{3.2}$$

It can also be defined as the symmetric $\binom{1}{1}$-tensor:

$$Ric(v) = \sum_{i=1}^{n} R(v, e_i)e_i$$

Another type of curvature is the *sectional curvature sec*, defined by:

$$sec(v,w) = \frac{g(R(v,w)v,w)}{g(v,v)g(w,w) - g(v,w)^2} = \frac{Rm(v,w,w,v)}{g(v,v)g(w,w) - g(v,w)^2}$$

And the last one is *the scalar curvature S* defined as the trace of the Ricci curvature. For an orthonormal basis $e_1, ..., e_n \in T_pM$:

$$\begin{aligned} S &= \mathrm{tr}(Ric) \\ &= \sum_{i=1}^{n} g(Ric(e_j), e_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g(R(e_i, e_j)e_j, e_i) \end{aligned}$$

The sectional curvature will see much use in the last chapter. The most well-known example of an application of the Ricci curvature is in physics, as it appears in the Einstein field equations

$$Ric - \frac{1}{2}Sg = T \tag{3.3}$$

Which assumes that space-time is modeled by a $4-$manifold that carries a metric whose Ricci curvature satisfies (3.3). Where T is a symmetric 2-tensor called the stress-energy tensor that describes the physical properties of the matter. [Sus]

The geometric interpretation of the scalar curvature is the following, in chapter 6 we will define balls on manifolds. Then the scalar curvature tells you how the volume of a small ball differs from how much volume such a ball would have in Euclidean space.

4 Submanifolds

In this chapter we will look at Riemannian submanifolds and connections on these submanifolds. As these will allow us to get an interpretation of the curvature tensor. We will study the relation between the connections and curvatures of the ambient manifold and the submanifold. As well as expressing the different curvature tensors in terms of the Gaussian curvature, which is an intrinsic property of the Riemannian manifold. The Gaussian curvature is the main object of study in this chapter, as it is one of the terms in the Gauss-Bonnet theorem.

Definition 4.1 (Riemannian submanifold)**.** *A Riemannian submanifold M of a Riemannian manifold $(\widetilde{M}, \tilde{g})$ is an immersed submanifold of $\widetilde{M}$ together with the metric inherited from M. Meaning that for some immersion $\iota : M \to \widetilde{M}$, M has the induced metric $g = \iota^* \tilde{g}$ and in addition when ι is injective M is said to be a Riemannian submanifold of $\widetilde{M}$.*

All operations such as covariant derivatives and curvatures that are taken with respect to $\tilde{g}$ are denoted with a tilde, the inner product is unambiguously denoted as $\langle X, Y \rangle$, because g is just the restriction of $\tilde{g}$ to TM.

The restriction of a smooth vector bundle on $\widetilde{M}$ to M, is a smooth vector bundle on M.

$$T\widetilde{M}|_M = \amalg_{p \in M} T_p \widetilde{M}$$

As each local trivialization can be restricted to M and from the Vector bundle chart lemma (Lee lemma 10.6 Vector bundle chart lemma.) it follows that $T\widetilde{M}|_M$, called the ambient tangent bundle over M, is a smooth vector bundle over M. At each point $p \in M$ the ambient tangent bundle consists of a direct sum $T_p\widetilde{M} = T_pM \oplus N_pM$, by defining $N_pM = (T_pM)^\perp$ called the normal space at p. The union of which is called the normal bundle $NM = \amalg_{p \in M} N_pM$, which is also a vector bundle. For any point $p \in M$, there is a neighborhood $\widetilde{U}$ of p in $\widetilde{M}$ and a smooth orthonormal frame $(E_1, ..., E_m)$ on $\widetilde{U}$, called an adapted orthonormal frame, such that the restriction of this frame to M is a local orthonormal frame for TM: $(E_1, ..., E_n)$. Then the last $m - n$ vectors form a basis for N_pM at each $p \in M$, and the components of normal vectors in NM with respect to this basis can be used to construct a normal trivialization of NM. It can be checked that the transition functions are smooth, then by [Lee97, Lemma 2.2] it follows that NM is a vector bundle.

So we can project vectors at $p \in M$ orthogonally onto T_pM, by projecting onto $span(E_1, ..., E_n)$ and onto N_pM by projecting onto $span(E_{n+1}, ..., E_m)$. For a section X of $T\widetilde{M}|_M$, we use the notation $X^\perp$ and $X^\top$ to denote the orthogonal and tangential projections of X respectively.

Vector fields on M extend to vector fields on $\widetilde{M}$, by the extension lemma for vector fields on submanifolds [Lee13, p.201]. Therefore for looking at the shape of the manifold M inside of $\widetilde{M}$ with this decomposition

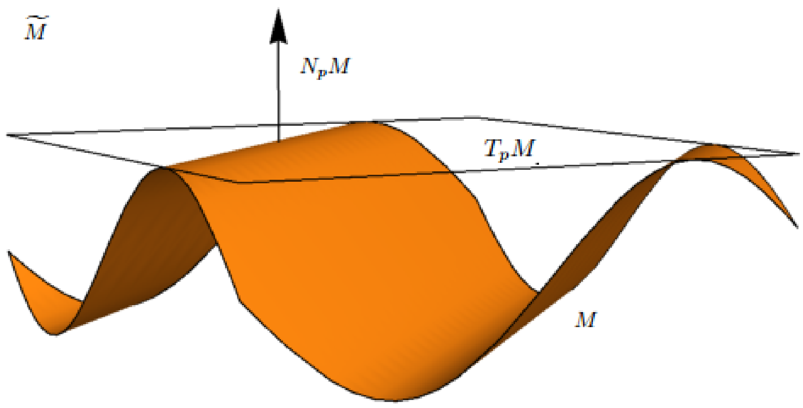


Figure 4.1: The normal space at p.

we can decompose the covariant derivative of $\widetilde{M}$ into:

$$\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^\top + (\widetilde{\nabla}_X Y)^\perp$$

And by extending vector fields on M to $\widetilde{M}$ we can define *the second fundamental* form $II : \mathcal{T}M \times \mathcal{T}M \to \mathcal{N}M$:

$$II(X,Y) = (\widetilde{\nabla}_X Y)^\perp$$

The second fundamental form is symmetric because $II(X,Y) - II(Y,X) = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X) = [X,Y]^\perp = 0$, where the last equality follows because both X, Y are tangent to M then so is their Lie bracket.

The tangential term in $\widetilde{\nabla}_X Y$ is just $\nabla_X Y$, as seen in the following theorem.

Theorem 4.2. *If $X, Y \in \mathcal{T}M$ are extended to arbitrary vector fields on $\widetilde{M}$, the following formula holds:*

$$\widetilde{\nabla}_X Y = \nabla_X Y + II(X,Y)$$

Proof. Because of the decomposition of $\widetilde{\nabla}_X Y$ we want to show that $\widetilde{\nabla}_X Y^\top = \nabla_X Y$ on M. One can easily check that $\widetilde{\nabla}_X Y$ satisfies the conditions to be a connection. And it is symmetric because:

$$\nabla^\top_X Y - \nabla^\top_Y X = (\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X)^\top = [X,Y]^\top = [X,Y]$$

Also it is compatible with g, the metric on M. For let $X, Y, Z \in \mathcal{T}M$ be extended to $\widetilde{M}$. Then because of he compatibility of $\widetilde{\nabla}$ and $\tilde{g}$.

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \widetilde{\nabla}_X Y, Z\rangle + \langle Y, \widetilde{\nabla}_X Z\rangle \\ &= \langle (\widetilde{\nabla}_X Y)^\top, Z\rangle + \langle Y, (\widetilde{\nabla}_X Z)^\top\rangle \end{aligned}$$

Therefore $\widetilde{\nabla}^\top$ is compatible with g, so the uniqueness property of Riemannian connections implies $\widetilde{\nabla}^\top = \nabla$ for vector fields on M. ☐

We can also look at covariant derivatives of vector fields normal to M, for which the following result holds.

Theorem 4.3 (The Weingarten equation). *Suppose $X, Y \in \mathcal{T}M$ and $N \in \mathcal{N}M$. When X, Y, N are extended to $\widetilde{M}$:*

$$\langle \widetilde{\nabla}_X N, Y \rangle = -\langle N, II(X, Y) \rangle$$

Proof.

$$\begin{aligned} 0 &= X\langle N, Y \rangle \\ &= \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, \widetilde{\nabla}_X Y \rangle \\ &= \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, \nabla_X Y + II(X, Y) \rangle \\ &= \langle \widetilde{\nabla}_X N, Y \rangle + \langle N, II(X, Y) \rangle \end{aligned}$$

□

Initially, we will be define Gaussian curvature for 2-dimensional manifolds embedded in $\mathbb{R}^3$. For such manifolds, we can pick a normal vector N, unique up to a minus sign. A generalization of this are hypersurfaces in Euclidean space, e.g. submanifolds of dimension n embedded in $\mathbb{R}^{n+1}$ and with the induced metric. By picking a sign for the normal vectors one gets a smooth section of NM, a vector field N. When N is normalized, we can replace the second fundamental form by a symmetric 2-tensor h:

$$h(X, Y) = \langle II(X, Y), N \rangle$$

From which a $\binom{1}{1}$-tensor field S can be constructed by raising an index, e.g. $\langle X, SY \rangle = h(X, Y)$. From the symmetry of h, it follows that s is a selfadjoint endomorphism of TM, for all $X, Y \in \mathcal{T}M$: $\langle SX, Y \rangle = \langle X, SY \rangle$. Therefore we can define *the Gaussian curvature*

$$K = detS$$

This Gaussian curvature seems to rely on the normal vector, so therefore one would expect it to be dependent on the way the manifold is embedded in Euclidean space. However as proven in the theorema Egregrium for 2 dimensions the Gaussian curvature does not depend on the embedding. For hypersurfaces in any dimension it is in fact an intrinsic invariant of the manifold (M, g). A reference for this is [Spi, Corollary 23].

Theorem 4.4 (Theorema Egregrium). *Let $M \subset \mathbb{R}^2$ be a 2 dimensional submanifold and g the induced metric on M. For any $p \in M$ and any basis (X, Y) for T_pM, the Gaussian curvature of M at p is given by*

$$K = \frac{RM(X, Y, Y, X)}{|X|^2|Y|^2 - \langle X, Y \rangle^2} \tag{4.1}$$

Before we proof this we first need another theorem by Gauss, the Gauss equation.

Theorem 4.5 (The Gauss equation)**.** *For any $X, Y, Z, W \in T_pM$, the following equation holds:*

$$\widetilde{Rm}(X,Y,Z,W) = Rm(X,Y,Z,W) - \langle II(X,W), II(Y,Z)\rangle + \langle II(X,Z), II(Y,W)\rangle$$

Proof of the Gauss equation: Let X, Y, Z, W be extended arbitrarily to vector fields on $\widetilde{M}$, such that they are tangent to M. Then from theorem 4.2 we know that:

$$\begin{aligned}\widetilde{Rm}(X,Y,Z,W) =&\langle \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W\rangle \\ =&\langle \widetilde{\nabla}_X(\nabla_Y Z + II(Y,Z)) - \widetilde{\nabla}_Y(\nabla_X Z + II(X,Z)) \\ &- \nabla_{[X,Y]} Z - II([X,Y],Z), W\rangle\end{aligned}$$

Since $W \in T_pM$, the inner product of W and $II([X,Y],Z)$ is zero, the rest of the terms involving the second fundamental form are given by the Weingarten equation.

$$\begin{aligned}\widetilde{Rm}(X,Y,Z,W) =&\langle \widetilde{\nabla}_X(\nabla_Y Z), W\rangle - \langle II(Y,Z), II(X,W)\rangle - \\ &\langle \widetilde{\nabla}_Y(\nabla_X Z), W\rangle + \langle II(X,Z), II(Y,W)\rangle \\ &- \langle \nabla_{[X,Y]} Z, W\rangle\end{aligned}$$

Again because W is tangential to M only the tangential part of $\widetilde{\nabla}$ survives which is exactly ∇ by theorem (4.2).

$$\begin{aligned}\widetilde{Rm}(X,Y,Z,W) =&\langle \nabla_X(\nabla_Y Z), W\rangle - \langle II(Y,Z), II(X,W)\rangle - \\ &\langle \nabla_Y(\nabla_X Z), W\rangle + \langle II(X,Z), II(Y,W)\rangle \\ &- \langle \nabla_{[X,Y]} Z, W\rangle \\ &= \langle R(X,Y)Z, W\rangle - \langle II(Y,Z), II(X,W)\rangle + \langle II(X,Z), II(Y,W)\rangle\end{aligned}$$

□

Proof of the Theorema Egregrium: For any basis X, Y of T_pM, the Gram-Schmidt procedure gives an orthonormal basis:

$$\begin{aligned}E_1 &= \frac{X}{|X|} \\ E_2 &= \frac{Y - \frac{\langle Y,X\rangle}{|X|^2} X}{|Y - \frac{\langle Y,X\rangle}{|X|^2} X|}\end{aligned}$$

These can be used to calculate K:

$$\begin{aligned}
K &= Rm(E_1, E_2, E_2, E_1) \\
&= \frac{Rm(X, Y - \frac{\langle Y,X\rangle}{|X|^2}X, Y - \frac{\langle Y,X\rangle}{|X|^2}, X)}{|X|^2|Y - \frac{\langle Y,X\rangle}{|X|^2}X|^2} \\
&= \frac{Rm(X,Y,Y,X)}{|X|^2\left(|Y|^2 - 2\frac{\langle Y,X\rangle}{|X|^2}^2 + \frac{\langle Y,X\rangle}{|X|^2}\right)} \\
&= \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}
\end{aligned}$$

Where the 2nd step follows because:

$$\begin{aligned}
&Rm(X, Y - \frac{\langle Y,X\rangle}{|X|^2}X, Y - \frac{\langle Y,X\rangle}{|X|^2}, X) = Rm(X, Y, Y - \frac{\langle Y,X\rangle}{|X|^2}X, X) \\
&- \frac{\langle Y,X\rangle}{|X|^2}Rm(X, X, Y - \frac{\langle Y,X\rangle}{|X|^2}X, X) = Rm(X, Y, Y - \frac{\langle Y,X\rangle}{|X|^2}X, X) \\
&= Rm(X,Y,Y,X)
\end{aligned}$$

And the fact that $Rm(X, X, \cdot, \cdot) = Rm(\cdot, \cdot, X, X) = 0$ is used, because of the symmetry property of the curvature tensor. □

By defining the Gaussian curvature K as (4.1), we gain a way to define the Gaussian curvature for abstract Riemannian 2-manifolds. Independent of how the manifold is embedded, not necessarily in Euclidean space, that agrees with our previous definition of K for manifolds that are embedded in $\mathbb{R}^3$.

5 Gauss-Bonnet Theorem

We now have all the differential geometry theory needed in order to prove the first big result, the Gauss-Bonnet theorem. For this proof we need two theorems, the rotation angle theorem which roughly says that the tangent vector to a piecewise smooth curve rotates an angle of 2π. We will use this result to proof the Gauss-Bonnet formula, a preliminary result of the Gauss-Bonnet formula. From this result the step to the Gauss-Bonnet theorem is quite small and we will conclude with the implications of the Gauss-Bonnet theorem.

5.1 Rotation angle theorem

We need a way to track the angular change of the tangent vector along a curve. So we will need to introduce a few notions concerning angles along a curve. We will also first prove the theorem in the plane and then extend all the definitions to manifolds, as that trivializes the proof on manifolds.

Definition 5.1. *An admissible curve* is a continuous map $\gamma : [a, b] \to M$, such that there exists a finite subdivision $a = a_0 < a_1 < ... < a_l = b$, $\gamma(t)$ is a smooth curve for all $t \in [a_{i-1}, a_i]$ and $\dot{\gamma}(t) \neq 0$ for all $t \in [a_{i-1}, a_i]$.

In this chapter $\gamma : [a, b] \to \mathbb{R}^2$ is a unit speed admissible curve in the plane. Such a curve γ is called *simple* if it is injective on the interval $[a, b)$, and *closed* if $\gamma(a) = \gamma(b)$.

For a smooth curve γ we define *the tangent angle* $\theta(t)$ as the unique continuous map $\theta : [a, b] \to \mathbb{R}$ by $\dot{\gamma} = (\cos(t), \sin(t))$ for all $t \in [a, b]$ and with the property that $\theta(a) \in (-\pi, \pi]$.

This tangent angle exists, because $\dot{\gamma}$ can be seen as a map from $[a, b]$ to S^1, since the tangent space to $\mathbb{R}^2$ can be naturally identified with itself and γ is unit speed. By the path lifting property [Hat01, p.60] the path $\dot{\gamma} : [a, b] \to S^1$ can be lifted for the universal covering $p : \mathbb{R} \to S^1$ given by $p(\theta) = (\cos\theta, \sin\theta)$. Then the tangent angle function θ is the unique continuous lift such that $\theta(a) \in (-\pi, \pi]$. So from the smoothness of $\dot{\gamma}$ and the fact that p is a local diffeomorphism, it follows that θ is a smooth map.

If γ is a unit speed smooth closed curve with $\dot{\gamma}(t) \neq 0$, for all $t \in [a, b]$, and $\dot{\gamma}(a) = \dot{\gamma}(b)$. Then the rotation angle of γ, denoted by $Rot(\gamma)$, is defined as $Rot(\gamma) = \theta(b) - \theta(a)$. So the rotation angle is the total amount of change in the tangent angle of the curve.

We can extend these definitions beyond smooth curves to admissible curves, at the points where the change in tangent vector is not continuous there is a 'jump' in tangent angle. So for admissible curves we need to define exterior angles describing how big this jump is.

The exterior angle measures the change in tangent angle. Let a_i be a point where the change in tangent angle is not continuous, then the oriented angle ϵ_i from $\dot{\gamma}(a_i^-)$ to $\dot{\gamma}(a_i^+)$ in the interval $[-\pi, \pi]$. This angle should have a positive sign if $(\dot{\gamma}(a_i^-), \dot{\gamma}(a_i^+))$ is an oriented basis for $\mathbb{R}^2$ and a negative sign otherwise. If $\dot{\gamma}(a_i^-) = -\dot{\gamma}(a_i^+)$ we have no unambiguous way to choose between $\pm\pi$ so we will not consider curves with angles of $\pm\pi$.

From now on we want to consider *curved polygons*, which are simple, closed, piecewise smooth, unit speed curve segments, whose exterior angles do not equal $\pm\pi$. If $a = a_0 < a_1 < ... < a_k = b$ is a subdivision of $[a, b]$ the restrictions of $\gamma|_{[a_{i-1},a_i]}$ are called the edges of the curved polygons and $\gamma(a_i)$ called its vertices.

Then by the Jordan curve theorem [Tve] the curve is the boundary of some open $\Omega \in \mathbb{R}^2$. On smooth pieces of γ, whenever $\dot{\gamma}$ is consistent with the orientation induced from Stoke's theorem, we call γ *positively oriented*. Sometimes curved polygons are defined as the Ω of which γ is the boundary, but for simplicity we will refer to γ as the curved polygon.

For a curved polygon γ we define *the tangent angle* $\theta : [a, b] \to \mathbb{R}$ to be the following: Starting with $\theta(a) \in (-\pi, \pi]$ we define $\theta(t)$ for $t \in [a, a_1)$ to be the tangent angle defined as above, as the angle between $\frac{\partial}{\partial x}$ and $\dot{\gamma}(t)$. At the point $\gamma(a_1)$ define:

$$\theta(a_1) = \lim_{t \to a_1} \theta(t) + \epsilon_1$$

Then define θ on $[a_1, a_2)$ as above and continue this inductively for all i, until $\gamma(b)$:

$$\theta(b) = \lim_{t \to b} \theta(t) + \epsilon_n$$

Taking the same definition of the rotation angle, $Rot(\gamma) = \theta(b) - \theta(a)$, we can obtain the following theorem on the rotation angle.

Theorem 5.2 (Rotation angle theorem). *If γ is a positively oriented curved polygon in the plane, the rotation angle of γ is 2π.*

Proof. In the simplest case, the curved polygon has no exterior angles. Therefore $\dot{\gamma}$ is continuous and $\dot{\gamma}(a) = \dot{\gamma}(b)$, so we can extend γ to a continuous map, with continuous first derivative, from $\mathbb{R} \to \mathbb{R}^2$ by requiring it to be periodic of period $b - a$.

By reparametrization we can consider γ as a function being defined on $[a', b']$ of length $b - a$, without changing $Rot(\gamma)$. Picking a' such that the y-coordinate of γ has a minimum for $t = a'$. For simplicity we relabel the new interval to $[a, b]$. Also by translation we may as well think of $\gamma(a)$ as the origin, without changing the rotation angle. Then the y-coordinate of γ is bigger than 0, and $\dot{\gamma}(a) = \dot{\gamma}(b) = \partial/\partial x$

The tangent angle function $\theta : [a, b] \to \mathbb{R}$ is continuous, because $\dot{\gamma}$ is continuous. We can define a continuous extension of the tangent angle function, the secant angle function $\varphi(t_1, t_2)$

Let T be the triangle $T = \{(t_1, t_2) : a \leqslant t_1 \leqslant t_2 \leqslant b\}$, and define the map $V : T \to \mathbb{S}^1$ as:

$$V(t_1,t_2)=\begin{cases}\frac{\gamma(t_2)-\gamma(t_1)}{|\gamma(t_2)-\gamma(t_1)|} & \text{if } t_1\leqslant t_2 \text{ and } (t_1,t_2)\neq(a,b)\\ \dot\gamma(t_1) & \text{if } t_1=t_2\\ -\dot\gamma(a) & \text{if } (t_1,t_2)=(a,b)\end{cases}$$

The map V is continuous along the line $t_1=t_2$, because

$$\begin{aligned}\lim_{(t_1,t_2)\to(t,t)} V(t_1,t_2) &= \lim_{(t_1,t_2)\to(t,t)} \frac{\gamma(t_2)-\gamma(t_1)}{|\gamma(t_2)-\gamma(t_1)|}\\ &= \lim_{(t_1,t_2)\to(t,t)} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\Big/\left|\frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\right|\\ &= \dot\gamma(t)/|\dot\gamma(t)|\\ &= V(t,t)\end{aligned}$$

Where the last step follows, because γ is unit speed.

V is also continuous at (a,b) as:

$$\begin{aligned}\lim_{(t_1,t_2)\to(a,b)} V(t_1,t_2) &= \lim_{(t_1,t_2)\to(a,b)} \frac{\gamma(t_2)-\gamma(t_1)}{|\gamma(t_2)-\gamma(t_1)|}\\ &= \lim_{(t_1,t_2)\to(a,b)} \frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\Big/\left|\frac{\gamma(t_2)-\gamma(t_1)}{t_2-t_1}\right|\\ &= \lim_{(a,t_2)\to(a,b)} \frac{\gamma(t_2)-\gamma(a)}{t_2-a}\Big/\left|\frac{\gamma(t_2)-\gamma(a)}{t_2-a}\right|\\ &= -\dot\gamma(b)/|\dot\gamma(b)|\\ &= -\dot\gamma(a)\\ &= V(a,b)\end{aligned}$$

Then from the fact that T is simply connected, the theory of covering spaces and the universal cover $p:\mathbb{R}\to\mathbb{S}^1$ implies that $V:T\to\mathbb{S}^1$ has a unique continuous lift we call $\varphi:T\to\mathbb{R}$, by requiring $\varphi(a,a)=0$. This functionφ is called the secant angle function and represents the angle between the x-axis and the vector from $\gamma(t_1)$ to $\gamma(t_2)$.

With this interpretation of the secant angle function we can express the rotation angle in terms of the secant angle function

$$Rot(\gamma)=\theta(b)-\theta(a)=\varphi(b,b)-\varphi(a,a)=\varphi(b,b)$$

As $\varphi(a,a)=0$. Looking at the part of T where $t_1=a$, the image of V lies in the upper half of $\mathbb{S}^1$, therefore $\varphi(a,t_2)\in[0,\pi]$. From this it follows that $\varphi(a,b)=\pi$, as φ is continuous and $\varphi(a,b)$ represents the angle between $-\dot\gamma(a)=-\partial/\partial x$ and $\partial/\partial x$.

In the part of T where $t_2=b$, the image of V lies in the lower part of $\mathbb{S}^1$. Which implies $\varphi(t_1,b)\in[\pi,2\pi]$, so since $\varphi(b,b)$ represents the angle between $\dot\gamma(a)=\partial/\partial x$ and $\partial/\partial x$ $\varphi(b,b)=k\cdot 2\pi$ for some $k\in\mathbb{Z}$. Then by continuity $\varphi(b,b)=2\pi$ which proofs the desired result for curved polygons without exterior angles.

For curved polygons γ with exterior angles we proof the theorem, by proving that there exists a curve with continuous tangent vector with the same rotation angle as γ. Intuitively this curve will be constructed rounding the corners of γ. In this part the interval $[a, b]$ will be chosen such that at $\gamma(a) = \gamma(b)$, $\dot{\gamma}(t)$ is continuous, and we call points at which $\dot{\gamma}(t)$ is not continuous vertices.

Let $\gamma(a_i)$ be a vertex, ϵ_i the corresponding exterior angle and let α be a small positive number depending on ϵ_i, we will define it later. $\theta(t)$ is continuous from the right by definition, and $\lim_{t \to a_i^-} \theta(t) = \theta(a_i) - \epsilon_i$, where the $-$ sign denotes a limit from the left. From the definition of limit it follows that there exists a δ small enough such that $|\theta(t) - (\theta(a_i) - \epsilon_i)| < \alpha$ when $t \in (a_i - \delta, a_i)$, and $|\theta(t) - \theta(a_i)| < \alpha$ when $t \in (a_i, a_i + \delta)$.

We can look at the set $\gamma([a, b] - (a_i - \delta, a_i + \delta))$ which is a compact set and $\gamma \notin \gamma([a, b] - (a_i - \delta, a_i + \delta))$, so a small enough disk $B_r(a_i)$ around $\gamma(a_i)$ with radius r can be chosen, such that $\gamma(t)$ is only in $\overline{B}_r(a_i)$ for $t \in (a_i - \delta, a_i + \delta)$. Denote the times when γ enters and leaves the ball by $t_1, t_2 \in (a_i - \delta, a_i + \delta)$ respectively. From our choice of δ, the total change in tangent angle $\theta(t)$ for $t \in [t_1, a_i)$ is smaller than α, and the same holds for $t \in (a_i, t_2]$. This means by choosing α small enough, the total change $\Delta\theta$ in tangent angle between t_1 and t_2 is between $\epsilon_i - 2\alpha$ and $\epsilon_i + 2\alpha$. So by choosing an $\alpha < 1/2(\pi - |\epsilon_i|)$, we get $-\pi < \Delta\theta < \pi$.

Now we can construct a smooth curve by following γ except for the interval $[t_1, t_2]$, where γ can be replaced by a smooth curve segment σ. Since for σ the tangent angle changes between $\dot{\gamma}(t_1)$ and $\dot{\gamma}(t_2)$, must be exactly equal to $\Delta\theta$. By repeating this process for every vertex, we obtain the a smooth curve, which is the edge of a curved polygon and which has the same rotation angle as γ, therefore completing the proof. □

Obviously we want to extend the definitions to oriented Riemannian 2-manifolds as the aim of this chapter is to proof the Gauss-Bonnet theorem on 2-dimensional manifolds. For a Riemannian 2-manifold (M, g) we begin with a curved polygon, which is a unit speed curve $\gamma : [a, b] \to M$ such that γ is the boundary of some open set Ω with compact closure, and such that there is a coordinate chart containing γ and Ω under whose image γ is a curved polygon in $\mathbb{R}^2$. Using the coordinates of the coordinate chart, we can transfer γ, Ω and g to the plane and assume g is a metric on some open $\mathcal{U} \in \mathbb{R}^2$.

Again we say that γ is positively oriented if the direction of the parametrization of γ is the same as the direction of Stokes theorem on the boundary of Ω. The exterior angle ϵ_i at a vertex a_i, a point where $\dot{\gamma}$ is not continuous, is defined as the angle from $\dot{\gamma}(a_i^-)$ to $\dot{\gamma}(a_i^+)$. Combining the definition of angles on a Riemannian manifold and the given the orientation on M, we get$\epsilon_i > 0$ if $(\dot{\gamma}(a_i^-), \dot{\gamma}(a_i^+))$ is a positively oriented frame for $\mathbb{R}^2$. The definition of tangent angle is in terms of angles, just as in $\mathbb{R}^2$ we can define the tangent angle on pieces of $[a, b]$ where $\dot{\gamma}$ is continuous. The tangent angle $\theta : [a, b] \to \mathbb{R}$ is defined as the continuous angle from $\partial/\partial x$ to $\dot{\gamma}$, with respect to the metric g. The rotation angle is $Rot(\gamma) = \theta(b) - \theta(a)$. And the same theorem can be proven as in the plane.

Theorem 5.3. *For a positively oriented curved polygon γ in M, the rotation angle is 2π.*

Proof. By using the coordinate chart we can consider γ as a curved polygon in the plane and compare the tangent angles on the Riemannian manifold and in the plane. So on the Riemannian manifold θ is a function of g, while in the plane it is a function of the Euclidean metric. In both cases, $Rot(\gamma) = 2\pi l$, for $l \in \mathbb{Z}$, as $\theta(a)$ and $\theta(b)$ represent the same angle, because $\dot{\gamma}(a) = \dot{\gamma}(b)$.

Now define $g_s = sg + (1-s)\overline{g}$, for $0 \leqslant s \leqslant 1$. From the same reasoning as for the other metrics $Rot_{g_s}(\gamma) = 2\pi l'$, for $l' \in \mathbb{Z}$. So the function $1/2\pi Rot_{g_s}(\gamma)$ is continuous and integer valued, therefore must be constant and equal to 2π. □

5.2 Gauss-Bonnet Formula and Theorem

We have done all the plane geometry necessary to prove the Gauss-Bonnet theorem. As mentioned before we will do this with the help of the Gauss-Bonnet formula as this relates the Gaussian curvature of a curved polygon to its signed curvature and exterior angles, a result that is already similar to the Gauss-Bonnet theorem.

In our 2-dimensional manifold, on segments where γ is smooth, each point of γ has a normal vector, which is unique up to a minus sign. So there exists a normal vector field $N(t)$. By adding the condition that $(\dot{\gamma}(t), N(t))$ is an oriented orthonormal basis for $T_{\gamma(t)}M$ we get a unique normal vector field to γ. The vector field $N(t)$ can then be thought of as the inward pointing normal vector.

We define *the signed curvature* $\kappa_N(t)$ at smooth points of γ by

$$\kappa_N(t) = \langle D_t\dot{\gamma}(t), N(t)\rangle$$

As the name suggests, there also exists a curvature κ. And the relation between the signed curvature and the curvature κ is $\kappa(t) = |\kappa_N(t)|$. It can be seen from differentiating $|\dot{\gamma}(t)|^2 \equiv 1$, that $\langle D_t\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0$. Therefore $D_t\dot{\gamma}(t) = \kappa_N(t)N(t)$ follows. Now we have everything we need for the statement and proof of the Gauss-Bonnet theorem.

Theorem 5.4 (Gauss-Bonnet Formula)**.** *Suppose γ is a curved polygon on an oriented Riemannian 2-manifold (M,g), and γ is positively oriented as the boundary of an open set Ω with compact closure. Then*

$$\int_\Omega K dA + \int_\gamma \kappa_N ds + \sum_i \epsilon_i = 2\pi \tag{5.1}$$

With K the Gaussian curvature of g and dA its Riemannian volume element.

Proof. Let $a = a_0 \leq ... \leq a_k = b$ be a subdivision into parts of $[a,b]$ such that γ is smooth on these parts. From the rotation angle theorem, and the fact that the rotation angle is the sum of all exterior angles plus the total change in tangent angle on the smooth parts of γ we get

$$2\pi = \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \dot{\theta}(t)dt \tag{5.2}$$

The Gauss-Bonnet formula will now be derived from expressing $\dot{\theta}$ in terms of K, κ_N.

First we will obtain an expression for κ_N, from the coordinates of the coordinate chart containing γ, we can obtain (x, y) oriented coordinates on the open U containing γ. Applying Gram-Schimdt algorithm to the frame $(\partial/\partial x, \partial/\partial y)$ results in an oriented orthonormal frame we call (E_1, E_2), where E_1 is in the direction of $\partial/\partial x$. Using the fact that $\theta(t)$ is defined as the angle between E_1 and $\dot{\gamma}(t)$ with respect to g, we get the following relations on points where γ is smooth

$$\begin{aligned} \dot{\gamma}(t) &= cos\theta(t)E_1 + sin\theta(t)E_2 \\ N(t) &= -sin\theta(t)E_1 + cos\theta(t)E_2 \end{aligned}$$

Differentiating the first relation yields

$$\begin{aligned} D_t\dot{\gamma} &= -\dot{\theta}(sin\theta)E_1 + (cos\theta)\nabla_{\dot{\gamma}}E_1 + \dot{\theta}(cos\theta)E_2 + (sin\theta)\nabla_{\dot{\gamma}}E_2 \\ &= \dot{\theta}N + (cos\theta)\nabla_{\dot{\gamma}}E_1 + (sin\theta)\nabla_{\dot{\gamma}}E_2 \end{aligned} \tag{5.3}$$

We also need the relations obtained from the covariant derivatives of E_1 and E_2.

$$\begin{aligned} 0 = \nabla_X|E_1|^2 &= 2\langle\nabla_X E_1, E_1\rangle \\ 0 = \nabla_X|E_2|^2 &= 2\langle\nabla_X E_2, E_2\rangle \\ 0 = \nabla_X\langle E_1, E_2\rangle &= 2\langle\nabla_X E_1, E_1\rangle \end{aligned}$$

Where the last one is true, because (E_1, E_2) is orthonormal.

Because the tangent space at each point is 2-dimensional, the first equation implies that $\nabla_X E_1$ is a multiple of E_2 and $\nabla_X E_2$ is a multiple of E_1. Therefore the 1-form ω defined by

$$\omega(X) = \langle E_1, \nabla_X E_2\rangle = -\langle E_1, \nabla_X E_2\rangle$$

Therefore by the definition of ω

$$\begin{aligned} \nabla_X E_1 &= -\omega(X)E_2 \\ \nabla_X E_2 &= -\omega(X)E_1 \end{aligned} \tag{5.4}$$

So ω completely determines the connection in U. Combining the results (5.3) and (5.4) we can compute κ_N

$$\begin{aligned} \kappa_N &= \langle D_t\dot{\gamma}, N\rangle \\ &= \langle\dot{\theta}N, N\rangle + cos\theta\langle\nabla_{\dot{\gamma}}E_1, N\rangle + sin\theta\langle\nabla_{\dot{\gamma}}E_2, N\rangle \\ &= \dot{\theta} - cos\theta\langle\omega(\dot{\gamma})E_2, N\rangle + sin\theta\langle\omega(\dot{\gamma})E_1, N\rangle \\ &= \dot{\theta} - cos^2\theta\omega(\dot{\gamma}) - sin^2\theta\omega(\dot{\gamma}) \\ &= \dot{\theta} - \omega(\dot{\gamma}) \end{aligned}$$

Thus (5.2) can be rewritten to

$$\begin{aligned}
2\pi &= \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \kappa_N(t)dt + \sum_{i=1}^{k} \int_{a_{i-1}}^{a_i} \omega_(\dot{\gamma}(t))dt \\
&= \sum_{i=1}^{k} \epsilon_i + \int_\gamma \kappa_N ds + \int_\gamma \omega
\end{aligned}$$

It remains to be shown that

$$\int_\gamma \omega = \int_\Omega K dA \tag{5.5}$$

We can use Stoke's theorem to smooth curves γ_j, obtained from a construction similar to the curve obtained in the proof of the rotation angle theorem, whose length approach the length of γ and which are the boundary of Ω_j, whose area approaches that of Ω. By taking the limit for $j \to \infty$, we get $\int_\gamma \omega = \int_\Omega d\omega$.

What is left is to show that $d\omega = KdA$. Which follows from a straightforward calculation.

$$\begin{aligned}
KdA(E_1, E_2) &= K = RM(E_1, E_2, E_2, E_1) \\
&= \langle \nabla_{E_1} \nabla_{E_2} E_2 - \nabla_{E_2} \nabla_{E_1} E_2 - \nabla_{[E_1, E_2]} E_2, E_1 \rangle \\
&= \langle \nabla_{E_1} (\omega(E_2)) E_1 - \nabla_{E_2} (\omega(E_1)) E_1 - \omega[E_1, E_2] E_1, E_1 \rangle \\
&= \langle E_1(\omega(E_2)) E_1 + \omega(E_2) \nabla_{E_1} E_1 - E_2(\omega(E_1)) E_1 \\
&\quad - \omega(E_1) \nabla_{E_2} E_1 - \omega[E_1, E_2] E_1, E_1 \rangle \\
&= E_1(\omega(E_2)) - E_2(\omega(E_1)) - \omega[E_1, E_2] \\
&= d\omega(E_1, E_2)
\end{aligned}$$

□

The Gauss-Bonnet formula can already be used for geometric results in the plane, such as the following:

1. The sum of the interior angles of a Euclidean triangle is π.
2. The circumference of a Euclidean circle of radius R is $2\pi R$.

1. Follows by writing the exterior angles in terms of the interior angles. While for 2. you need to know that the signed curvature of the sphere is $1/R$.

But we will use the Gauss-Bonnet formula to proof the Gauss-Bonnet theorem, which links the Euler characteristic with the curvature. Therefore we will first need to triangulate the surface.

A triangulation of M is a finite collection of 3-sided curved polygons, such that when (Ω_j) denotes the area bounded by a triangles, and such that the union of $\bar{\Omega}_j$ is M. Together with the condition that the intersection of 2 triangles is either empty, a single vertex, or a single edge. It was proven by Tibor Radò [Rad25] that every 2-dimensional topological manifold admits a triangulation, so every 2-dimensional Riemannian manifold admits a triangulation.

Definition 5.5. For a triangulated 2-dimensional manifold M, *the Euler characteristic* is defined as

$$\chi(M) = N_v - N_e + N_f$$

Where N_v,N_e and N_f are defined as the number of vertices, edges and faces of the triangulation of M. The Euler characteristic is a topological invariant and independent of the triangulation [Sie92]

Theorem 5.6 (The Gauss-Bonnet theorem)**.** *If M is a compact, oriented, Riemannian 2-manifold, then*

$$\int_M K dA = 2\pi\chi(M)$$

Proof. Denote each face the triangulation by $\Omega_i; i = 1, ..., N$. As each triangle has 3 edges, denote the edges of the i-th triangle by $\gamma_{ij} : j = 1, 2, 3$ and its interior angles by $\theta_{ij}; j = 1, 2, 3$. We can express the interior angles in terms of exterior angles, each interior angle is π minus the corresponding exterior angle. Now we can use the Gauss-Bonnet formula on the individual triangles and sum over them

$$\sum_{i=1}^{N_f} \int_{\Omega_i} K dA + \sum_{i=1}^{N_f}\sum_{j=1}^{3} \int_{\gamma_{ij}} \kappa_N ds + \sum_{i=1}^{N_f}\sum_{j=1}^{3}(\pi - \theta_{ij}) = \sum_{i=1}^{N_f} 2\pi \tag{5.6}$$

The 2nd term is zero, because each edge is the boundary of exactly 2 triangles. And the orientation of the edges is opposite therefore the integrals of κ_N cancels out. Also the faces add up to M, so (5.6) becomes:

$$\int_M K dA + 3\pi N_f - \sum_{i=1}^{N_f}\sum_{j=1}^{3}(\theta_{ij}) = 2\pi N_f$$

Furthermore, at each vertex the sum of all the interior angles is 2π. And as each interior angle appears only once we get

$$\int_M K dA = 2\pi N_v - \pi N_f$$

To rewrite this in terms of the Euler characteristic note that the total number of edges is three times the number of faces, but as each edge is part of 2 triangles, the total number of edges is $2N_e = 3N_f$. So $N_f = 2N_e - 2N_f$, this means that

$$\int_M K dA = 2\pi N_v - 2\pi N_e + 2\pi N_f = 2\pi\chi(M)$$

□

The Gauss-Bonnet theorem can be combined with the classification theorem for compact 2-manifolds, see for example [Mas67, Theorem I.5.1]. Which states that every compact orientable 2-manifold is homeomorphic to a sphere or the connected sum of tori, and every nonorientable compact 2-manifold is homeomorphic to the connected sum of projective planes. This yields the following corollary.

Corollary 1. *Let M be a compact Riemannian 2-manifold and K its Gaussian curvature.*

- *if M is homeomorphic to the sphere or the projective plane then $K > 0$ somewhere.*

- *If M is homeomorphic to the torus or the Klein bottle, then either $K \equiv 0$ or K takes on both positive and negative values.*
- *If M is any other compact surface, then $K < 0$ somewhere.*

Proof. First not that the Euler characteristic of a sphere is 2, for a connected sum of tori the Euler characteristic is $2 - 2g$, where g is the number of tori and for the connected sum of projective spaces the Euler characteristic is $2 - g$, with g the number of projective spaces. Then if M is orientable the corollary follows immediately by the Gauss-Bonnet theorem.

If M is nonorientable the result follows by applying the Gauss-Bonnet theorem to the orientable double cover $\pi : \widetilde{M} \to M$ with the lifted metric $\tilde{g} = \pi^* g$, using the fact that $\widetilde{M}$ is the sphere if $M = \mathbb{P}^2$, the torus if M is the Klein bottle, and otherwise has $\chi(\widetilde{M}) < 0$. $\square$

6 Comparison Theorems

The aim of this chapter will be the proof of the Cartan-Hadamard theorem. This theorem will be proven with the help of the theory of Jacobi fields, which requires three concepts to be defined first. The concepts of geodesics, which are the generalization of straight lines to manifolds, the definition of distances on Riemannian manifolds and also the exponential map, a map that maps tangent vectors to the manifold by following a geodesic in the direction of said tangent vector. With the definitions of distances and exponential map the Hopf-Rinow theorem will be stated, which says that for a connected Riemannian manifold geodesical completeness and completeness as a metric space are equivalent. This theorem allows us to define a single notion of completeness, which will be used in the Cartan-Hadamard theorem.

6.1 Geodesics

Now we start off with geodesics, this can be defined in terms of the operator D, but also in terms of parallel vector fields. The latter approach has the advantage that existence and uniqueness come quite naturally.

Given a curve γ a vector field V along γ is said to be parallel if $\dot{V} \equiv 0$.

Theorem 6.1 (Existence and uniqueness of parallel vector fields). *If $t_0 \in I$ and $V \in T_{\gamma(t_0)}M$, then there is a unique parallel vector field $V(t)$ defined on all of I with $V(t_0) = V$*

Proof. Begin by choosing a vector fields that form a basis for $T_{\gamma(t)}M$, denote the basis by $E_1(t), ..., E_n(t)$. Then any vector field $V(t)$ along γ can be written in the form, $V(t) = \sum V^i(t)E_i(t)$.

$$\begin{aligned}
\frac{d}{dt}V &= D_tV(t) = \dot{V^i}(t)E_i(t) + V^i(t)\nabla_{\dot{\gamma}}E_i(t) \\
&= \dot{V^i}(t)E_i(t) + V^i(t)\cdot\alpha_i^j(t)E_j(t), \text{ where } \nabla_{\dot{\gamma}}E_i = \sum \alpha_i^j E_j \\
&= \dot{V^j}(t) + V^i(t)\alpha_i^j(t))E_j(t)
\end{aligned}$$

So V is parallel whenever for all j

$$\dot{V^j}(t) = V^i(t)\alpha_i^j(t)$$

From the theory of ordinary differential equations it follows that these equations have a unique solution given initial values $(V^i(t_0))$. □

A curve $\gamma : I \to M$ is called a geodesic if $\dot{\gamma}(t)$ is parallel along γ, or equivalently if $D_t\dot{\gamma}(t) \equiv 0$. For a geodesic γ, $|\dot{\gamma}|$ is constant, and whenever $|\dot{\gamma}| \equiv 1$ we say that γ is parametrized by arc length. Geodesics are the equivalent to straight lines in Euclidean geometry. First of all we will prove that at every point there is a unique geodesic in each direction.

Theorem 6.2 (Existence and uniqueness of Geodesics)**.** *Let M be a manifold with a linear connection. For any $p \in M$, and any $V \in T_pM$, there exists an open interval $I \subset \mathbb{R}$ containing 0 and a unique geodesic $\gamma : I \to M$ satisfying $\gamma(0) = p$, $\dot{\gamma}(0) = V$.*

Proof. Given a coordinate chart $\varphi : U \to \mathbb{R}^n$ on an open U containing p with coordinate vector fields $\partial_1, ..., \partial_n$, $\partial_i = \partial/(\partial x^i)$. Recall that

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k$$

Then consider a curve $\gamma : I \to U$ for which we can write $(\varphi \circ \gamma)(t) = (\varphi^1(t), ..., \varphi^n(t))$. And $\dot{\gamma}(t)$ can be written as $\dot{\gamma}(t) = \dot{\varphi}^i(t)\partial_i|_{\gamma(t)}$. Then using the definition of the covariant derivative along curves we get

$$\begin{aligned}\ddot{\gamma} &= \ddot{\varphi}^i \partial_i|_{\gamma(t)} + \dot{\varphi}^i(t)\nabla_{\dot{\gamma}}\partial_i \\ &= \ddot{\varphi}^i \partial_i|_{\gamma(t)} + \dot{\varphi}^i(t)\dot{\varphi}^j(t)\Gamma_{ij}^k|_{\gamma(t)}\partial_k|_{\gamma(t)}\end{aligned}$$

So the curve is a geodesic if and only if its component functions satisfy the geodesic equations:

$$\ddot{\varphi}^k(t) + \dot{\varphi}^i(t)\dot{\varphi}^j(t)\Gamma_{ij}^k|_{\varphi^{-1}(\varphi_1(t),...,\varphi_n(t))}$$

As this is a system of second-order differential equations, existence and uniqueness is guaranteed for initial values of $\varphi^i(0)$ and $\dot{\varphi}^i(0)$, but because the system is not linear the solutions do not have to exist for all t. □

The uniqueness of geodesics ensures that there always exists a maximal geodesic for each combination of a point $p \in M$ and starting velocity $V \in T_pM$. This geodesic is usually denoted by γ_V, where the information of the starting point p is already embedded in V and can be recovered by the natural projection from $TM \to M$ which sends $V \to p$.

6.2 Distance on Riemannian manifolds

The metric g comes naturally with a notion of vector length. Similar to $\mathbb{R}^n$ we can integrate the length of the tangent vector of a curve to define the length of this curve. Because contrary to Euclidean space, there is no immediate distance function between points, so now we will construct this distance function.

Suppose γ is a piecewise smooth curve on (M, g), e.g. $\gamma : [a, b] \to M$ and there exists a finite subdivision of $[a, b]$ on which γ is smooth. Then define *the length* $l(\gamma)$ as follows

$$l(\gamma) = \int_a^b |\dot{\gamma}(t)| dt = \int_a^b (g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} dt$$

From the definition of piecewise smooth it is clear that the function $t \to |\dot{\gamma}(t)|$ is integrable, therefore $l(\gamma)$ is well-defined and nonnegative. It can also be shown that this definition is invariant under reparametrization of the curve. For let $\tilde{\gamma}$ be a reparametrization of γ, $\tilde{\gamma} = \gamma \circ \varphi$ for a smooth map $\varphi : [c, d] \to [a, b]$ with smooth inverse.

$$l(\tilde{\gamma}) = \int_c^d |\dot{\tilde{\gamma}}(s)|ds = \int_c^d \left| \dot{\gamma}(\varphi(s)) \frac{d\varphi(s)}{ds} \right| ds = \int_a^b |\dot{\gamma}(t)|dt = l(\gamma)$$

Where the integral was reparametrized by $\varphi(s) = t$. A piecewise smooth curve $\gamma : [a, b] \to M$ is said to be *parametrized by arc length* if $l(\gamma|_{[a,\lambda]}) = \lambda - a$ for all $\lambda \in [a, b]$ or equivalently if $|\dot{\gamma}| = 1$ at all smooth points $t \in [a, b]$. A piecewise smooth curve can always be reparametrized by arc length if at smooth points of the curve $|\dot{\gamma}(t)| > 0$. By defining the function $\varphi(s)$.

$$\varphi(s) = \int_a^s |\dot{\gamma}(t)|dt$$

one gets the reparametrization by arc length of γ to be $\gamma \circ \varphi^{-1} : [0, l(\gamma)] \to M$. This shows that one can consider only piecewise smooth curves parametrized by arc length, without any loss of generality.

For any pair $p, q \in M$ we can define the space of curves between these points , which we call *the path space* Ω.

$$\Omega(p, q) = \{\gamma : [0.1] \to M : \gamma \text{ is piecewise smooth and } \gamma(0) = p, \gamma(1) = q\}$$

Then we define *the distance function* d between points $p, q \in M$ as

$$d(p, q) = \inf\{l(\gamma) : \gamma \in \Omega(p, q)\}$$

It is obvious from the definition that $d(p, q) = d(q, p) \geq 0$, $d(p, p) = 0$ and $d(p, q) \leq d(p, r) + d(r, q)$ (curves from p to r and r to q can always be combined and reparametrized to form a curve from p to q).

We still need to check that $d(p, q) = 0$ only if $p = q$. Suppose we have $p \neq q$ with $d(p, q) = 0$. Choose a chart $\varphi : U \to V$ around p such that q is not in U. Then we can choose $\delta > 0$ and $C > 0$ such that on $B_\delta(\varphi(p)) \subset V$, $g(W, W) \geq C|D\varphi(W)|^2$, for $W \in T_pM$. Therefore for points r in $\varphi^{-1}(B_\delta(\varphi(p))$, we have $d(p, r) \geq C|\varphi(p) - \varphi(r)|$ But for q outside this set, any curve from p to q must first pass through $\varphi^{-1}(\partial B(\varphi(p))$ with ∂B the boundary of the sphere. So therefore this curve must have at least length $C\delta$.

Via the metric we can define various balls, the definitions are the same as for Euclidean balls

$$B(p, r) = \{x \in M : d(p, x) < r\}$$
$$\bar{B}(p, r) = D(p, r) = \{x \in M : d(p, x) \leq r\}$$

A curve $\gamma \in \Omega(p, q)$ is called a segment if $l(\gamma) = d(p, q)$ and γ is parametrized by arc length.

6.3 Exponential Map

This allows to define a map known as *the exponential map*, a map from the tangent bundle to M, by following γ_V for $t = 1$. More explicitly define the domain of the exponential map, known as $\mathcal{E}$.

$$\mathcal{E} = \{V \in TM : \gamma_V \text{ is defined on an interval containing } [0,1]\}$$

And the exponential map, $\exp : \mathcal{E} \to M$

$$\exp(V) = \gamma_V(1)$$

Notice that $\gamma_{\alpha V}(t) = \gamma_V(\alpha t)$, for $\alpha > 0$ whenever the geodesic is defined γ_V is defined for αt. For each $p \in M$ the domain can be restricted to $\mathcal{E}_p = \mathcal{E} \cap T_pM$ to define the map $\exp_p : \mathcal{E}_p \to M$. The theory of ordinary differential equations tells us that the exponential map is smooth and that $\mathcal{E}$ is open in TM, therefore each $\mathcal{E}_p$ is open and exp_p is smooth.

The next result is an important property of the exponential map.

Lemma 6.3. *If $p \in M$ and 0 denotes the zero vector in T_pM, then $exp_p : T_0(T_pM) \to T_pM$ is nonsingular. Consequently, there is an open neighborhood U of 0 in T_pM such that $exp_p : U_p \to M$ is a local diffeomorphism of U_p onto $exp_p(U_p) \subset M$. Or in other words, exp_p is a local diffeomorphism.*

Proof. There is a canonical isomorphism, between T_pM and $T_0(T_pM)$. We will call the isomorphism $I_0 : T_pM \to T_0T_pM$, e.g. $I_0(V) = \frac{d}{dt}(tV)|_{t=0}$. Recall if $V \in \mathcal{E}_p$ then $\gamma_V(t) = \gamma_{tV}(1)$ for all $t \in [0,1]$. Therefore

$$\begin{aligned}
(exp_p)_*(I_0(V)) &= \frac{d}{dt} exp_p(tV)|_{t=0} \\
&= \frac{d}{dt}\gamma_{tV}(1)|_{t=0} \\
&= \frac{d}{dt}\gamma_V(t)|_{t=0} \\
&= \dot{\gamma}_V(0)|_{t=0} \\
&= V
\end{aligned}$$

So $(exp_p)_* \circ I_0 = Id$ as a map from T_pM to T_pM, so in particular $(exp_p)_*$ is nonsingular. Therefore the Lemma follows by the inverse function theorem. □

This yields coordinates around p, by identifying T_pM and $\mathbb{R}^n$ via an isomorphism. For a neighborhood $p \in U$

$$exp_p^{-1} : exp_p(U) \to T_pM \simeq \mathbb{R}^n$$

This coordinate system is called normal coordinates at p, and they are unique up to the isomorphism. By requiring the identification to be a linear isometry the uniqueness is up to orthogonal transformations of $\mathbb{R}^n$. Furthermore if $\epsilon > 0$ is small enough such that exp_p is a diffeomorphism on the ball $B(0,\epsilon) \subset T_pM$, then the image of this ball under the exponential map $exp_p(B_\epsilon(0))$ is called a geodesic ball in M and its boundary is called a geodesic sphere. On the geodesic sphere we have the function $f(x) = |exp_p^{-1}(x)|$ is the distance from the origin in $B(0, \epsilon) \subset T_pM$, this Euclidean distance is usually denoted by $r(V) = |V| = (\sum_i (V^i)^2)^{1/2}$, and in a Cartesian coordinate chart on T_pM, $\partial_r = \frac{x^i}{r}\frac{\partial}{\partial x^i}$.

A straightforward result of lemma (6.3) is that in normal coordinates $\gamma_V(t) = (tV^1, ..., tV^n)$.

Next I want to mention completeness of Riemannian manifolds, a manifold on which all geodesics are defined for all time is equivalent to that manifold being a complete metric space. This result is fundamental in Riemannian geometry and is called the Hopf-Rinow theorem.

Theorem 6.4 (Hopf-Rinow). *Let (M, g) be a connected Riemannian manifold then the following statements are equivalent:*

1. *M is geodesically complete*
2. *M is complete as a metric space.*

I do not want to proof this theorem here as it is quite a long proof, so for that I refer to [Lee97]. Because of the Hopf-Rinow theorem we can simply refer to a Riemannian manifold that satisfies either property as a complete manifold, and we will need the notion of a complete manifold in the chapter on comparison theorems.

6.4 Jacobi Fields

In this section we will look at families of geodesics, and vector fields along them. And we will focus on special vector fields that satisfy the Jacobi equation, as the theory of Jacobi fields allows us to proof the Cartan-Hadamard theorem.

We start off by defining *an admissible family of curves* as a continuous map $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ that is smooth on each set of the form $(-\epsilon, \epsilon) \times [a_{i-1}, a_i]$ for a finite subdivision of $[a, b]$ and such that the main curves $\Gamma_s(t)$ defined as $\Gamma_s(t) := \Gamma(s, t)$, by setting s constant, is an admissible curve. An admissible curve is a piecewise smooth curve with nonzero first derivative at all t. The transverse curves $\Gamma^t(s) := \Gamma(s, t)$ can be defined in the same way by setting t constant. Then the partial derivatives

$$\partial_t \Gamma(s, t) := \frac{d}{dt}\Gamma_s(t) \qquad\qquad \partial_s \Gamma(s, t) := \frac{d}{ds}\Gamma^t(s)$$

are vector fields along the main and transverse curves of the admissible family, defined whenever Γ is smooth.

Lemma 6.5 (Symmetry Lemma). *Let $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ be an admissible family of curves in a Riemannian manifold. On any rectangle $(-\epsilon, \epsilon) \times [a_{i-1}, a_i]$ where Γ is smooth,*

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma$$

Proof. We proof this lemma by computing the derivatives of Γ at a point (s_0, t_0). Denote the components of Γ in coordinates (x^i) as $\Gamma(s,t) = (x^1(s,t), ..., x^n(s,t))$

$$\partial_t \Gamma = \frac{\partial x^k}{\partial t}\partial_k \qquad\qquad \partial_s \Gamma = \frac{\partial x^k}{\partial s}\partial_k$$

And the covariant derivatives of these terms is

$$\begin{aligned} D_s\partial_t\Gamma &= \left(\frac{\partial^2 x^k}{\partial s \partial t} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma^k_{ji}\right)\partial_k \\ D_t\partial_s\Gamma &= \left(\frac{\partial^2 x^k}{\partial t \partial s} + \frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial t}\Gamma^k_{ji}\right)\partial_k \end{aligned}$$

To see that these expressions are equal we can reverse the role of i, j, by symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ and therefore the desired result follows. $\square$

Theorem 6.6 (The Gauss Lemma)**.** *Let U, be a geodesic ball centered at p. The unit radial vector field $\partial/\partial r$ is orthogonal to the geodesic spheres in U.*

Proof. Let $q \in U$ and pick a vector $X \in T_qM$ which is tangent to the geodesic sphere through q. As the exponential map is a diffeomorphism, there are vectors $V \in T_pM$ such that $q = exp_pV$ and $W \in T_V(T_pM) = T_pM$ such that $X = (exp_p)_*W$. By how V and W are chosen it follows immediately that $V \in \partial B_R(0)$ and $W \in T_V\partial B_R(0)$ for $R = |V|$. Because of this diffeomorphism the image of the geodesic from p to q is the exponential map of V, therefore the geodesic γ from p to q $\gamma(t) = exp_p(tV)$, with tangent vector $\dot\gamma(t) = R\frac{\partial}{\partial r}$. We call this geodesic the radial geodesic from p to q. So proving Gauss lemma is equivalent to showing that X is orthogonal to $\dot\gamma$.

This will be done by choosing a curve $\delta : (-\epsilon, \epsilon) \to T_pM$ in $\partial B_R(0)$ satisfying $\delta(0) = V$ and $\dot\delta(0) = W$. Then consider the following variation Γ of γ

$$\Gamma(s,t) = exp_p(t\delta(s))$$

As $|\delta(s)| = R$ for each $s \in (-\epsilon, \epsilon)$, this implies Γ_s is a geodesic with constant speed R. Recall that $S = \partial_s\Gamma$ and $T = \partial_t\Gamma$, then the following equations hold

$$\begin{aligned} S(0,0) &= \frac{d}{ds}\Big|_{s=0} exp_p(0) = 0 \\ T(0,0) &= \frac{d}{dt}\Big|_{t=0} exp_p(t\delta(0)) = \frac{d}{dt}\Big|_{t=0} exp_p(tV) = V \\ S(0,1) &= \frac{d}{ds}\Big|_{s=0} exp_p(\delta(s)) = (exp_p)_*\dot\delta(0) = X \\ T(0,1) &= \frac{d}{dt}\Big|_{t=1} exp_p(t\delta(0)) = \frac{d}{dt}\Big|_{t=1} exp_p(tV) = \dot\gamma(1) \end{aligned}$$

Now we can look at the inner product $\langle S, T\rangle = 0$ at the point $(s,t) = (0,0)$ and $\langle S, T\rangle = \langle X, \dot\gamma(1)\rangle$ at $(s,t) = (0,1)$. Furthermore, the following computation shows that this inner product is independent of t

$$\begin{aligned} \frac{\partial}{\partial t}\langle S, T\rangle &= \langle D_tS, T\rangle + \langle S, D_tT\rangle \\ &= \langle D_sT, T\rangle + 0 \\ &= \frac{1}{2}\frac{\partial}{\partial s}|T|^2 = 0 \end{aligned}$$

The steps that were used in this equation are: the symmetry lemma, Γ_s is a geodesic which implies $D_t T \equiv 0$ and $|T| = |\dot{\Gamma}_s| \equiv R$. So because $\langle S, T\rangle = 0$ as it is independent of t, $0 = \langle S, T\rangle = \langle X, \dot{\gamma}(1)\rangle$. $\square$

Given a variation Γ of γ, such that $\Gamma_0(t) = \gamma(t)$, *a proper variation* is one such that $\Gamma_s(a) = \gamma(a)$ and $\Gamma_s(b) = \gamma(b)$ for all s. The variation field of Γ is the vector field $V(t) = \partial_s\Gamma(0, 1)$ along γ, a vector field along γ is proper if it vanishes at the endpoints of γ.

Suppose now that $\gamma : [a, b] \to M$ is a geodesic, and $\Gamma : (-\epsilon, \epsilon) \times [a, b] \to M$ is a variation of γ. We say that Γ is *a variation through geodesics* if each of the main curves $\Gamma_s(t)$ is also a geodesic. Then by the definition of geodesics we know that $D_t\partial_t\Gamma(s, t) \equiv 0$. And taking the covariant derivative with respect to s yields $D_sD_t\partial_t\Gamma(s, t) \equiv 0$.

For simplicity we adapt the notation $T(s, t) = \partial_t\Gamma(s, t)$ and $S(s, t) = \partial_s\Gamma(s, t)$ in this notation the last equation becomes $D_sD_tT \equiv 0$. To relate that equation to the variation field we need to commute the covariant derivatives.

Lemma 6.7. *If Γ is any smooth admissible family of curves, and V is a smooth vector field along Γ, then*

$$D_sD_tV - D_tD_sV = R(S, T)V$$

Proof. In suitable coordinates we can write $V(s, t) = V^i(s, t)\partial_i$, calculating its second covariant derivative gives

$$\begin{aligned}
D_tV &= \frac{\partial V^i}{\partial t}\partial_i + V^iD_t\partial_i \\
D_sD_tV &= \frac{\partial^2 V^i}{\partial s\partial t}\partial_i + \frac{\partial V^i}{\partial t}D_s\partial_i + \frac{\partial V^i}{\partial s}D_t\partial_i + V^iD_sD_t\partial_i \\
D_tD_sV &= \frac{\partial^2 V^i}{\partial t\partial s}\partial_i + \frac{\partial V^i}{\partial s}D_t\partial_i + \frac{\partial V^i}{\partial t}D_s\partial_i + V^iD_tD_s\partial_i
\end{aligned}$$

Because V is smooth, substracting these terms gives

$$D_sD_tV - D_tD_sV = V^i(D_sD_t\partial_i - D_tD_s\partial_i) \tag{6.1}$$

If we denote the coordinate functions of $\Gamma(s, t)$ by $x^j(s, t)$, then we can write S and T as

$$S = \frac{\partial x^j}{\partial s}\partial_j \qquad\qquad T = \frac{\partial x^k}{\partial t}\partial_k$$

Because D_t is the covariant derivative along the curve Γ^t and the fact that ∂_i is extendible we can write $D_t\partial_i$ according to the definition of covariant derivatives along curves as

$$D_t\partial_i = \nabla_T\partial_i = \frac{\partial x^k}{\partial t}\nabla_{\partial_i}\partial_j$$

Because $\nabla_{\partial_i}\partial_j$ is also extendible

$$\begin{aligned}
D_s D_t \partial_i &= D_s\left(\frac{\partial x^k}{\partial t}\nabla_{\partial_k}\partial_i\right) \\
&= \frac{\partial^2 x^k}{\partial s \partial t}\nabla_{\partial_k}\partial_i + \frac{x^k}{\partial t}\nabla_S\left(\nabla_{\partial_k}\partial_i\right) \\
&= \frac{\partial^2 x^k}{\partial s \partial t}\nabla_{\partial_k}\partial_i + \frac{x^k}{\partial t}\frac{x^j}{\partial s}\nabla_{\partial_j}\nabla_{\partial_k}\partial_i
\end{aligned}$$

The other term in the commutator has the j, k interchanged, so the commutator becomes

$$\begin{aligned}
D_s D_t \partial_i - D_t D_s \partial_i &= \frac{x^k}{\partial t}\frac{x^j}{\partial s}\left(\nabla_{\partial_j}\nabla_{\partial_k}\partial_i - \nabla_{\partial_j}\nabla_{\partial_k}\partial_i\right) \\
&= \frac{x^k}{\partial t}\frac{x^j}{\partial s}R(\partial_k, \partial_j)\partial_i \\
&= R(S, T)\partial_i
\end{aligned}$$

Where the last equality is true by linearity of the curvature. Combining this with (6.1) proofs the lemma. $\square$

Theorem 6.8 (The Jacobi Equation)**.** *Let γ be a geodesic and V a vector field along γ. If V is the variation field of a variation through geodesics, then V satisfies*

$$D_t^2 V + R(V, \dot{\gamma})\dot{\gamma} = 0 \tag{6.2}$$

Proof. By the symmetry lemma

$$\begin{aligned}
0 = D_s D_t T &= D_t D_s T + R(S, T)T \\
&= D_t D_t S + R(S, T)T
\end{aligned}$$

This expression yields the theorem when evaluated at $s = 0$, because there $S(0, t) = V(t), T(0, t) = \dot{\gamma}(t)$. $\square$

As the name suggests, equation (6.2) defines what is known as *a Jacobi field*, which is any vector field for which the Jacobi equation holds.

Lemma 6.9 (Existence and Uniqueness of Jacobi Fields)**.** *Let $\gamma : I \to M$ be a geodesic, $a \in I$, and $p - \gamma(a)$. For any pair of vectors $X, Y \in T_pM$, there is a unique Jacobi field J along γ satisfying the intial conditions*

$$J(a) = X \qquad\qquad D_t J(a) = Y$$

Proof. Begin by choosing an orthonormal basis $\{E_i\}$ for T_pM and extend it to a parallel orthonormal frame along γ. Substituting $J = J^i E_i$ into the Jacobi equation gives

$$\ddot{J}^i + R^i_{jkl} J^j \dot{\gamma}^k \dot{\gamma}^l = 0$$

This is just a system of linear second order ordinary differential equations and can be turned into a system of first order ODEs by substituting $V^i = J^i$. Therefore the theory of differential equations guarantees uniqueness and existance of Jacobi fields. □

Because of the preceding proposition, for some point p on γ, the map from the set of Jacobi fields along γ to $T_pM \oplus T_pM$ by sending $J \to (J(a), D_tJ(a))$ is bijective. So the dimension of the set of all Jacobi fields along a geodesic is $2n$, and it also is a linear subspace of $\mathcal{T}(\gamma)$.

Lemma 6.10. *Let $\gamma : I \to M$ be a geodesic, and $a \in I$. A Jacobi field J along γ is normal if and only if $J(a) \perp \dot{\gamma}(a)$ and $D_tJ(a) \perp \dot{\gamma}(a)$*

Proof.

$$
\begin{aligned}
\frac{d^2}{dt^2}\langle J, \dot{\gamma}\rangle &= \langle D_t^2 J, \dot{\gamma}\rangle \\
&= -\langle R(J,\dot{\gamma})\dot{\gamma}, \dot{\gamma}\rangle \\
&= -Rm(J,\dot{\gamma},\dot{\gamma},\dot{\gamma}) = 0
\end{aligned}
$$

This implies that $\langle J, \dot{\gamma}\rangle$ is linear in t. Denote $f(t) = \langle J(t), \dot{\gamma}(t)\rangle$, then $J(a)$ and $D_tJ(a)$ are both orthogonal to $\dot{\gamma}(a)$ if and only if f and its first derivative vanish at a. f and $\dot{f}$ vanish at a if and only if f vanishes at two points and is therefore 0 everywhere. □

From this lemma it follows that the dimension of the space of normal Jacobi fields is $2(n-1)$.

The converse of (6.2) is also true, therefore each Jacobi field tells us how some family of geodesics locally behaves.

Lemma 6.11. *Any Jacobi field J along a geodesic segment $\gamma(t)$ is the variation field of some variation of γ through geodesics.*

Proof. Let $\sigma : (-\epsilon, \epsilon) \to M$ be a curve with $\sigma(0) = \gamma(0)$, and $\dot{\sigma}(0) = J(0)$, also let W be a vector field along $\sigma(s)$ satisfying $W(0) = D_tJ(0)$. Then define a variation through geodesics of γ as

$$\Gamma(s,t) = exp_{\sigma(s)} tsW(s)$$

Then the variation field of Γ, $\partial_s\Gamma(s,t)|s = 0$. Then $\Gamma(s,0) = exp_{\sigma(s)}0 = \sigma(s)$ so its variation field here is

$$
\begin{aligned}
\partial_s\Gamma(s,0)|s = 0 = \dot{\sigma}(0) = J(0) \\
D_t\partial_s\Gamma(s,t)\big|_{s=0,t=0} = D_s\partial_t\Gamma(s,t)\big|_{s=0,t=0}
\end{aligned}
$$

Where

$$\partial_t\Gamma(s,t)\big|_{t=0} = (exp_{\sigma(s)})_*(sW(s)) = sW(s)$$

Because $(exp_{\sigma(s)})_*$ is the identity map at the origin of $T_{\sigma(s)}M$. Using these results $D_tJ(0) = D_t\partial_s\Gamma(s,t)\big|_{s=0,t=0} = W(s)$

But the variation field is also a Jacobi field as it is a variation field through geodesics. Therefore uniqueness of Jacobi fields proofs the lemma. □

The next lemma shows how to calculate Jacobi fields in normal coordinates on U, where $\gamma_V(t) = (tV^1, ..., tV^n)$.

Lemma 6.12. *Let $p \in M$, with normal coordinates (x^i) on a neighborhood U of p, and let γ be a radial geodesic, as in the proof of the Gauss Lemma, starting at p. For any $W \in T_pM$ the Jacobi field J along γ such that $J(0) = 0$ and $D_tJ(0) = W$ is given in normal coordinates by the formula*

$$J(t) = tW^i\partial_i$$

Proof. By defining $V = \dot{\gamma}(0) \in T_pM$, then we know that $\gamma(t) = (tV^1, ..., tV^n)$ and we can consider the variation of γ given by

$$\Gamma(s,t) = (t(V^1 + sW^1), ..., t(V^n + sW^n))$$

For each s this is a geodesic, so Γ is a variation through geodesics. Therefore its variation field $\partial_s\Gamma$ is a Jacobi field. $\frac{\partial}{\partial s}\Gamma(s,t) = J(t)$ □

Whenever a metric has constant sectional curvature, another explicit formula can be derived for Jacobi fields. Which gives Jacobi fields in terms of parallel vector fields.

Lemma 6.13. *Suppose (M,g) is a Riemannian manifold with constant sectional curvature C, and γ is a unit speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields*

$$J(t) = u(t)E(t)$$

Where E is a parallel normal vector field along γ, and $u(t)$ is

$$u(t) = \begin{cases} t & \text{if } C = 0 \\ R\sin\frac{t}{R} & \text{for } C = \frac{1}{R^2} > 0 \\ R\sin\frac{t}{R} & \text{for } C = -\frac{1}{R^2} < 0 \end{cases}$$

Proof. First consider a Riemannian manifold with constant sectional curvature. Then it can be shown that the following equality holds

$$R(X,Y)Z = C(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$$

By considering the $\binom{4}{0}$ - tensor.

$$Q(X,Y,Z,W) = \langle X,W\rangle\langle Y,Z\rangle - \langle Y,W\rangle\langle X,Z\rangle$$

Then for all $X, Y \in T_pM$, by definition of the sectional curvature

$$Rm(X,Y,Y,X) = C(|X|^2|Y|^2 - \langle X,Y\rangle^2) = CQ(X,Y,Y,X)$$

It can easily be checked that Q has the symmetries of the curvature tensor. Then by using [Lee97, Lemma 8.9] we can conclude that $Rm = Q$ or in other words

$$Rm(X, Y, Z, W) = CQ(X, Y, Z, W)$$

Therefore

$$R(X, Y)Z = C(\langle Y, Z\rangle X - \langle XZ\rangle Y)$$

Substituting this into the Jacobi equation gives for a normal Jacobi field J

$$\begin{aligned} 0 &= D_t^2 J + C(\langle\dot{\gamma}, \dot{\gamma}\rangle J - \langle J, \dot{\gamma}\rangle\dot{\gamma}) \\ &= D_t^2 J + CJ \end{aligned}$$

Let E be a normal parallel vector field along γ, then $J(t) = u(t)E(t)$ for some function u. Then J is a Jacobi field provided u satisfies

$$\ddot{u}(t) + Cu(t) = 0$$

With initial condition $u(0) = 0$, this yields constant multiples of the functions given in the theorem above. This construction actually gives all normal Jacobi fields vanishing at 0, since the there is a $(n-1)$ dimensional space of them, and the space of parallel normal vector fields has dimension $(n-1)$. And the dimension of the space of normal Jacobi fields in $2(n-1)$. $\square$

6.5 Comparison Theorems

In this section results will be proven regarding curvature, the most important result will be the Cartan-Hadamard theorem, which states that for a simply connected manifold with non-positive sectional curvature is diffeomorphic to Euclidean space via the exponential map.

We begin with a result known as the Sturm comparison theorem, a classical comparison theorem for ordinary differential equations.

Theorem 6.14 (Sturm Comparison Theorem)**.** *Suppose u and v are differentiable real-valued functions on $[0, T]$, and twice differentiable on $(0, T)$, and $u > 0$ on $(0, T)$. Suppose also that the following equations are satisfied*

$$\begin{gathered} \ddot{u}(t) + a(t)u(t) = 0 \\ \ddot{v}(t) + a(t)v(t) = 0 \\ u(0) = v(0) = 0, \qquad \dot{u}(0) = \dot{v}(0) > 0 \end{gathered}$$

for some $a : [0, T] \to \mathbb{R}$. Then $v(t) \le u(t)$ on $[0, T]$.

Proof. Consider $f(t) = v(t)/u(t)$ as a function on $(0, T)$. Using l'Hôpital's rule on f gives $\lim_{t\to 0} f(t) = \dot{v}(0)/\dot{u}(0) = 1$. Also $\dot{f} = \frac{d}{dt}\frac{v}{u} = \frac{\dot{v}u - v\dot{u}}{u^2}$, notice that $\dot{f} > 0$. As the numerator at $t = 0$ is $\dot{v}(0)u(0) -$

$v(0)\dot{u}(0) = 0$, and has nonnegative derivative

$$\frac{d}{dt}(\dot{v}u - v\dot{u}) = \ddot{v}u + \dot{v}\dot{u} - \dot{v}\dot{u} - v\ddot{u} = \ddot{v}u + avu \geq 0$$

Combining $\dot{f} > 0$ and the result from l'Hôpital's rule yields $f \geq 1$, therefore $v \geq u$ on $(0, T)$ and the continuity of both functions proves the theorem. □

Theorem 6.15 (Jacobi Field comparison theorem)**.** *Suppose (M, g) is a Riemannian manifold with all sectional curvatures bounded above by a constant C. If γ is a unit speed geodesic in M, and J is any normal Jacobi field along γ such that J(0) = 0, then*

$$|J(t)| \geq \begin{cases} t|D_tJ(0)| & \text{for } 0 \leq t & \text{if } C = 0 \\ R\sin\frac{t}{R}|D_tJ(0)| & \text{for } 0 \leq t \leq \pi R & \text{if } C = \frac{1}{R^2} > 0 \\ R\sin\frac{t}{R}|D_tJ(0)| & \text{for } 0 \leq t & \text{if } C = -\frac{1}{R^2} < 0 \end{cases}$$

Proof. The function $|J(t)|$ is smooth as long as $J(t) \neq 0$. By the Jacobi equation

$$\begin{aligned} \frac{d^2}{dt^2}|J| &= \frac{d}{dt}\frac{\langle D_tJ, J\rangle}{\langle J, J\rangle^{1/2}} \\ &= \frac{\langle D_t^2J, J\rangle}{\langle J, J\rangle^{1/2}} + \frac{\langle D_tJ, D_tJ\rangle}{\langle J, J\rangle^{1/2}} - \frac{\langle D_t^2J, J\rangle}{\langle J, J\rangle^{3/2}} \\ &= -\frac{\langle R(J, \dot{\gamma})\dot{\gamma}, J\rangle}{|J|} + \frac{|D_tJ|^2}{|J|} - \frac{\langle D_t^2J, J\rangle}{|J|^3} \end{aligned}$$

And using the Schwartz inequality yields $\langle D_tJ, J\rangle^2 \leq |D_tJ|^2|J|^2$, implying $\frac{|D_tJ|^2}{|J|} - \frac{\langle D_t^2J, J\rangle}{|J|^3} \geq 0$. Therefore

$$\frac{d^2}{dt^2}|J| \geq -\frac{Rm(J, \dot{\gamma}, \dot{\gamma}, J)}{|J|}$$

Because $\langle J, \dot{\gamma}\rangle = 0$, J and $\dot{\gamma}$ span a plane, of which the sectional curvature is $Rm(J, \dot{\gamma}, \dot{\gamma}, J/|J|^2 \leq C$ since $|\dot{\gamma}| = 1$. So $|J$ satisfies

$$\frac{d^2}{dt^2}|J| \geq -C|J|$$

We now wish to combine the results of the Sturm comparison theorem and lemma (6.13), for that we need to multiply J with a positive constant such that $|D_tJ(0)| = 1$ at $t = 0$.

From lemma (6.12), near $t = 0$ $J(t) = tW(t)$, for a smooth vector field W. Therefore

$$\begin{aligned} \frac{d}{dt}\Big|_0|J(t)| &= \lim_{t\to 0}\frac{|J(t)| - |J(0)||}{t} \\ &= \lim_{t\to 0}\frac{t|W(t)|}{t} = |W(0)| = |D_tJ(0)| = 1 \end{aligned}$$

We can now compare the function u from lemma (6.13) to $|J|$ with the Sturm comparison theorem, as they have the same value at $t = 0$. We only need to establish there is a neighborhood for which $|J|$ is nonzero, in order for it to be smooth. Which follows from $d|J|/dt = 1$ at $t = 0$, which implies there is an ϵ such that $|J| > 0$ on $(0, \epsilon)$. Together with $|J| \geq u$ implies that J cannot be zero before u. therefore

$|J| \geq u$ as long as $u \geq 0$. And substituting $u(t)$ yields the theorem. $\square$

A corollary of this theorem is the conjugate point comparison theorem. Which uses the notion of a conjugate point. Two points $p, q \in M$ are said to be conjugate along a geodesic segment, γ from p to q, if there is a nontrivial Jacobi field vanishing at both p and q.

Theorem 6.16 (Conjugate Point Comparison Theorem)**.** *Suppose all sectional curvatures of (M, g) are bounded above by a constant C. If $C \leq 0$, then no point of M has conjugate points along any geodesic. If $C = 1/R^2 > 0$, then the first conjugate point along any geodesic occurs at a distance of at least πR.*

Proof. If $C \leq 0$, the Jacobi field comparison theorem implies that any nonzero normal Jacobi field vanishing at $t = 0$ satisfies $|J(t)| > 0$ for all $t > 0$. Similarly, for $C > 0$, then $|J(t)| > 0$ for $0 < t < \pi R$. $\square$

Theorem 6.17 (The Cartan-Hadamard theorem)**.** *If M is a complete, connected manifold all of whose sectional curvatures are nonpositive, then for any point $p \in M$, $exp_p : T_pM \to M$ is a covering map. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$. If M is simply connected, then M itself is diffeomorphic to $\mathbb{R}^n$.*

Proof. By using the conjugate point comparison theorem, the nonpositive sectional curvature implies that p has no conjugate points along any geodesic. Therefore using [Lee97, p. 182] exp_p is a local diffeomorphism on all of T_pM.

So exp_p^* is nonsingular, and let $\tilde{g}$ be the 2-tensor field $exp_p^* g$, then $\tilde{g}$ is a Riemannian metric. And $exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry. Then by the following lemma it follows that exp_p is a covering map and the rest of the theorem follows by the uniqueness of the universal covering space. $\square$

Lemma 6.18. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, with $\widetilde{M}$ comeplete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a covering map*

The Cartan-Hadamard theorem is similar to the Gauss-Bonnet theorem in the sense that it provides a link between a global condition, simple-connectedness, and a local one, non-positive sectional curvature, which together imply a diffeomorphism to $\mathbb{R}^n$.

7 Notation

$\binom{k}{l}$ − tensor	The tensor that takes k vectors and l covectors as arguments.
T_pM	The tangent space at a point $p \in M$.
$\mathcal{T}^m_n(M)$	The $\binom{m}{n}$-tensor field on M.
$\mathcal{E}(M)$	The space of smooth sections of E, for a vector bundle E
TM	The tangent bundle over M

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