On the

Construction and Classification

of

Exotic Structures on the 7-Sphere

by

Jeroen van der Meer

Utrecht University

An honours thesis presented to Utrecht University
in fulfilment of the requirement for the degree of
Bachelor in Mathematics.

supervised by

dr. Stefan BEHRENS

and

prof. Marius CRAINIC

June 14, 2017

Contents

Chapter 1

Prologue

What mathematicians long for above all else is harmony, structure, and precision. One way in which this longing manifests is the countless classification theorems that exist throughout mathematics; indeed, one of the main objectives of mathematics is to classify patterns, where 'pattern' should be understood in its broadest sense.

In particular, when it comes to mathematical 'objects' and 'spaces', we need ways to formally describe what makes two objects or two spaces look alike. Not surprisingly, there are many ways to do this. When considering *topological spaces* for example, we have notions such as *homotopy equivalence*, *topological equivalence*, and many others.

We can apply these notions to, say, the *sphere*. While intuitively one of simplest objects in existence, the tendencies of mathematicians to complicate every tangible concept you can imagine, have resulted in *many* ways to describe the mysterious and elusive thing which we call a sphere. We have, for example, the *standard sphere*, which is literally a hyper-surface cut out of $\mathbb{R}^n$ by the equation $\sum_i x_i^2 = 1$. In addition, we have the *homotopy spheres*, closed smooth manifolds which is homotopy equivalent to the standard sphere, or the *topological spheres*, which are manifolds homeomorphic to the standard sphere, and so on.

We can also consider other properties of the sphere, and then look at spaces for which these properties coincide; for example, a *homology sphere* is any manifold having the homology groups of a standard sphere. We can even put some extra structures on spheres, like smooth structures or symplectic structures, then consider the objects for which not only the underlying set looks like the standard sphere, but which is also equivalent to the standard sphere with respect to this extra structure.

A natural question to ask is in what way these notions differ between each other, and whether some of these notions coincide; such questions lead to results which are vastly more complicated than the uninitiated may expect. This thesis is meant to provide a glimpse into some of these results; in particular, our main goal is to take a look at the difference between the notions of *homeomorphic* spaces and *diffeomorphic* spaces. Let's expand on this a bit.

A *homeomorphism* between two topological space is a continuous bijection with continuous inverse, while a *diffeomorphism* between them is a smooth bijection with smooth inverse. Let us consider the real line $\mathbb{R}$, and look at the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + |x|$. This is clearly a homeomorphism from $\mathbb{R}$ to itself, but it's not a diffeomorphism, because f has a 'kink' at $x = 0$. Intuitively, we can easily turn it into a diffeomorphism: we just 'iron out' the kink. This 'ironing' of non-differentiable things can always be done in the case of 'ordinary' (i.e., low-dimensional) spaces and functions.

This would easily lead us to believe that we can always do this. The problem is that this is spectacularly false. Given any object in $\mathbb{R}^2$ or $\mathbb{R}^3$ which admits a homeomorphism to the standard sphere S^1, S^2, we can iron out the kinks like we did in the example above so that the maps become diffeomorphisms. It turns out that this holds true for S^3, S^4, S^5, S^6, but that once we reach the seventh dimension we can no longer do this; more mathematically precise, there exist manifolds which are homeomorphic, but not diffeomorphic, to S^7.

The problem with this result is that it is *profoundly* counter-intuitive. You'd think (at least I did) that this process of removing the kinks can always be done. Well apparently you can't. The reason why the above methods don't quite work is that the nature of the non-differentiable kinks can be much more complicated than our pathetic little brains can visualize. The location of these kinks combine to form a subspace that you simply cannot just deform until they're gone.

This idea, on its own, is already hard to believe. Let me approach it in another way, by asking the reader a question: Why *would* we try to apply geometric intuition in higher dimensions? The answer is that to ordinary 3-dimensional human beings there *is* no other kind of intuition, so it is natural to assume (even if you have never explicitly given it a thought) that this intuition holds for higher dimensions. The reality is that our geometric intuition begins to fall apart in higher ones.

Roughly speaking, as you go up in dimension, the geometry of topological spaces start to 'reduce' to algebraic topology. For example, in higher dimensions, there is this thing called the *Whitney* *trick*, which allows us to 'remove' points of intersections between two submanifolds, basically by moving things around until they no longer intersect. It turns out that you need a few extra dimensions to carry out this process, and without these extra dimensions, there is no way to carry out such a trick. Because of that, you need a certain amount of dimensions to separate the concepts associated to smooth stuff, from the concepts associated to continuous stuff. At a sufficiently high dimension where this can be done, cool things start to happen; our thesis will focus on precisely these cool things.

Probably the first example of these cool things was the discovery of *exotic spheres*: manifolds which are homeomorphic, but not diffeomorphic, to the standard sphere. The first exotic spheres were constructed by John Milnor in 1956. He showed that there are *at least* three different exotic structures on S^7, by giving an explicit construction of spaces having this structure.

Milnor constructed the spaces (*Milnor manifolds*) as non-trivial S^3-bundles over S^4 (intuitively, this means we take S^4, and to each point we glue a copy of S^3). He showed that these *Milnor manifolds* are homeomorphic to S^7 by constructing a Morse function on these spaces which admits precisely two critical points; next, to show that they are not diffeomorphic to S^7, he used slightly more advanced machinery known as *characteristic classes* to develop a diffeomorphism invariant (the *Milnor invariant*). Construction of this invariant relies on, and may well have been inspired by, the *Hirzebruch Signature Theorem*, proved in 1954 by Friedrich Hirzebruch, which establishes a deep relation between two well-known diffeomorphism invariants. Next, Milnor computed the value of this Milnor invariant for the manifolds he constructed, and showed that these values often do not coincide with that of S^7.

It wasn't until 1963 that the same Milnor, together with Michel Kervaire, gave a full classification of exotic spheres, not only in dimension 7, but in all dimensions ≥ 5. To do this, they treat the set of smooth structures on S^n as a group under the connected sum operation, then used a result known as the H-*Cobordism Theorem*, proved by Stephen Smale in 1962, to view this group as the group of homotopy spheres modulo H-cobordism.

The classification theorem is non-constructive, in the sense that Milnor and Kervaire don't actually give an explicit construction of all exotic 7-spheres — twenty-eight of them, to be precise. In 1966, Egbert Brieskorn provided such an explicit construction. It relies on a completely different method: rather than considering fibre bundles, he constructed the spaces by considering zero-sets of non-trivial polynomials, then intersecting this zero-set with a sufficiently small sphere. The resulting sets can then be endowed with a natural smooth structure. Brieskorn gives twenty-eight such sets, each of homeomorphic to S^7, and each of them giving rise to a different smooth structure. By the results of Milnor and Kervaire, it follows that these must in fact be all of them.

This thesis gives an exposition of the three developments mentioned above. First, we give a basic overview of homology theory, characteristic classes, and the Hirzebruch Signature Theorem: topics which are necessary to understand Milnor's original construction. Next, we take a look at the h-Cobordism Theorem, and we outline the methods used by Milnor & Kervaire to produce their classification theorem. Then, we cover some basic ideas related to complex hyper-surfaces, and we use these ideas to give an exposition of the construction by Brieskorn. Finally, in the last chapter, we give a comparison of the two different constructions, and we sketch a rough overview of some more recent developments.

I am indebted to the guidance and support of several people. First and foremost, I'd like to thank dr. Stefan Behrens for spending his time and effort reviewing this thesis, and for helping me throughout the course of this project. Furthermore, I'd like to thank Arjen Baarsma (or is it *dr.* Arjen Baarsma by now?) for sparking my interest in differential geometry with his captivating personal lectures.

Before we move on, should you, dear reader, have something interesting to note — say, a question or comment related to the content, an error in the proofs, or neat results which are relevant to the topic of this thesis, feel free to send me an e-mail at `meer@vivaldi.net`.

Chapter 2

Homology

In this chapter, we recall some basis facts about homology. Homology will be used extensively throughout this thesis, and the concepts will be our basic tool upon which more advanced machinery will be built.

Roughly speaking, homology is a way of associating a sequence of algebraic objects (such as abelian groups) to other mathematical objects (such as topological spaces). They were originally defined in algebraic topology, but similar constructions are available in a wide variety of other context.

The contents of this chapter are largely based on [15] and [26], unless otherwise stated.

2.1 Axioms

Let G be an abelian group, and consider pairs of spaces (X, A). We will explain what it means when we say that G determines a homology theory on pairs (X, A).

Theorem 2.1: For integers k, there exist functors $H_k(X, A; G)$ from the category of pairs of spaces to the category of abelian groups, together with natural transformations $\partial : H_k(X, A; G) \to H_{k-1}(A, \varnothing; G)$ (the latter space usually being denoted by $H_{k-1}(A; G)$). These functors satisfy the following axioms:

- **Dimension axiom.** If X is a point then $H_0(X; G) = G$ and $H_k(X; G) = 0$ for all $k > 0$.
- **Exactness axiom.** The following sequence is exact, where the unlabelled arrows are induced by the natural inclusions:

$$\cdots \longrightarrow H_k(A; G) \longrightarrow H_k(X; G) \longrightarrow H_k(X, A; G) \xrightarrow{\partial} H_{k-1}(A; G) \cdots$$

- **Additivity axiom.** If $(X, A) = \bigsqcup_i (X_i, A_i)$, then the inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism $\bigoplus_i H_*(X_i, A_i; G) \xrightarrow{\sim} H_*(X, A; G)$.
- **Homotopy axiom.** If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then $f_* = g_*$.
- **Excision axiom.** If (X, A, B) is an **excisive triad** (i.e., a triple such that X is the union of the interiors of A and B), then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $H_*(A, A \cap B; G) \xrightarrow{\sim} H_*(X, B; G)$.

A particular example of a homology theory is given by singular homology. We will now review the basic definitions. Then in the next section we will present, without proof, some of its interesting properties.

The **standard topological n-simplex** is the subspace

$$\Delta_n = \big\{(t_0, \dots, t_n) : 0 \le t_i \le 1, \Sigma_i t_i = 1\big\}.$$

We define the natural **face maps** $\delta_i : \Delta_{n-1} \to \Delta_n$ by

$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$

Let G be an abelian group. For a space X, define $C_n(X; G)$ to be the free abelian group generated by all continuous maps $f : \Delta_n \to X$, with coefficients in G. Define the i-th face operator d_i, which sends a map $f : \Delta_n \to X$ to the map $d_i(f) : \Delta_{n-1} \to X$ defined by $d_i(f)(u) = f(\delta_i(u))$. We can linearly extend ∂_i to obtain a map $C_n(X; G) \to C_{n-1}(X; G)$. Defining $\partial = \sum_i (-1)^i \partial_i$, a straightforward verification shows that $d \circ d = 0$, from which it follows that $C_*(X; G)$ is a well-defined chain complex; its corresponding homology groups, called the **singular homology** groups, are denoted by $H_*(X; G)$. If $G = \mathbb{Z}$, we often denote the group by $H_*(X)$.

A slight generalization is as follows. Let A be a subspace of some topological space X. We note that if $C_k(X, A; G) = C_k(X; G)/C_k(A; G)$ then ∂ induces a well-defined map $C_k(X, A; G) \to C_{k-1}(X, A; G)$. This construction gives rise to a chain complex. We define the **relative homology** groups $H_k(X, A; G)$ with coefficients in G to be the homology groups associated to this chain complex. Finally, I'd like to mention the **reduced singular homology**, which is the modified homology group found by augmenting the chain complex $C_*(X; G)$ with $\varepsilon : C_0(X; G) \to G$ by sending $\sum_i n_i f_i$ to $\sum_i n_i$.

Parallel to homology, there exists a concept known as cohomology, which is basically a dualization of the concept of homology. The following result is nothing but the dualization of the previous theorem.

Theorem 2.2: For integers k, there exist *contravariant* functors $H^k(X, A; G)$ from the category of pairs of spaces to the category of abelian groups, together with natural transformations $\partial : H^k(X, A; G) \to H^{k+1}(A; G)$. These functors satisfy the following axioms:

- **Dimension axiom.** If X is a point then $H^0(X; G) = G$ and $H^k(X; G) = 0$ for all $k > 0$.
- **Exactness axiom.** The following sequence is exact, where the unlabelled arrows are induced by the natural inclusions:

$$\cdots \longrightarrow H^k(X, A; G) \longrightarrow H^k(X; G) \longrightarrow H^k(A; G) \longrightarrow H^{k+1}(X, A; G) \longrightarrow \cdots$$

- **Additivity axiom.** If $(X, A) = \bigsqcup_i (X_i, A_i)$, then the inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism $H_*(X, A; G) \xrightarrow{\sim} \bigoplus_i H_*(X_i, A_i; G)$.
- **Homotopy axiom.** If $f : (X, A) \to (Y, B)$ is homotopic to $g : (X, A) \to (Y, B)$, then $f^* = g^*$.
- **Excision axiom.** If (X, A, B) is an excisive triad, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism $H_*(X, B; G) \xrightarrow{\sim} H_*(A, A \cap B; G)$.

In particular, if we set $C^*(X, A; G) = \mathrm{Hom}(C_*(X, A); G)$, then the dual homomorphism of ∂ gives rise to a well-defined cochain complex $C^*(X; G)$, whose corresponding cohomology groups are the **singular cohomology** groups; as with homology, there exists a corresponding **reduced singular cohomology** theory as well. We denote the cohomology groups of X by $H^k(X; G)$, and in particular, if $G = \mathbb{Z}$, we denote the group by $H^k(X)$.

Unlike homology, cohomology has a natural product structure. From now on, we assume coefficients of homology and cohomology to lie in a commutative ring R.[1]

Given $\varphi_1 \in C^k(X; R)$ and $\varphi_2 \in C^l(X; R)$, we define their **cup product** $\varphi_1 \smile \varphi_2 \in C^{k+l}(X; R)$ to be the cochain complex such that

$$\varphi_1 \smile \varphi_2(f) = \varphi_1(f \circ i_{0,\ldots,k})\varphi_2(f \circ i_{k,\ldots,k+l}).^2$$

The cup product induces a product on the level of cohomology, which turns $H^*(X; R)$ into a graded-commutative R-algebra; in order words, we have $\varphi_1 \smile \varphi_2 = (-1)^{kl}\varphi_2 \smile \varphi_1$ for all $\varphi_1 \in H^k(X; R)$ and $\varphi_2 \in H^l(X; R)$.

[1] In this thesis, all rings are assumed to be unital.

[2] We will often omit the $\smile$, and denote the cup product simply by $\varphi_1\varphi_2$.

2.2 Some properties

In this section we present, without proof, some useful results related to singular homology and cohomology. All results mentioned here can be found, for example, in [15] and in [26].

Recall that excisive triads are a triple (X, A, B) such that X is the union of the interiors of A and B. Such triples are useful to consider, because they give rise to a natural long exact sequence known as the **Mayer-Vietoris sequences**. The result is as follows.

Theorem 2.3 (Mayer-Vietoris Theorem): Let (X, A, B) be an excisive triad, and set $C = A \cap B$. For each k, there is a linear map ∂ such that the following sequence is exact:

$$\cdots \longrightarrow H^{k-1}(C) \xrightarrow{\partial} H^k(A) \oplus H^k(B) \longrightarrow H^k(X) \longrightarrow H^k(C) \longrightarrow \cdots$$

Here, the unlabelled arrows are induced by inclusion.

Both homology and cohomology can be defined with arbitrary coefficients. It turns out that these groups are intimately related. This relation can be summarized with the following result. It is not as general as is possible, but it will be sufficient for our purposes.

Theorem 2.4 (Universal Coefficient Theorem): Let F be a field. For any space X, the homology group $H_k(X; F)$ is isomorphic to $H_k(X) \otimes_{\mathbb{Z}} F$. Furthermore, the cohomology group $H^k(X; F)$ is isomorphic to $\mathrm{Hom}(H_k(X; F); F)$.

Next, we look at a result which describes the homology or cohomology of a product space in terms of its factors. To obtain the result in its most general case it turns out that it is easier to work with homology than with cohomology. The result is as follows.

Theorem 2.5 (Künneth Theorem): Let X and Y be CW complexes and R a principal ideal domain. Then there are natural exact short sequences

$$0 \longrightarrow \bigoplus_{i=0}^{n} \big(H_i(X;R) \otimes_R H_{n-i}(Y;R)\big) \longrightarrow H_n(X \times Y; R) \longrightarrow \bigoplus_{i=0}^{n} \mathrm{Tor}_R\big(H_i(X;R), H_{n-i-1}(Y;R)\big) \longrightarrow 0$$

and these sequences split.

If the ring of coefficients is a field, the Künneth Theorem simplifies because the torsion terms are always zero. In this case, we find an isomorphism

$$\bigoplus_{i=0}^{n} \big(H_i(X;F) \otimes_F H_{n-i}(Y;F)\big) \cong H_n(X \times Y; F).$$

Before moving on, we recall an algebro-topological description of orientations. An R**-orientation** on an n-dimensional manifold[3] M is a function assigning to each $x \in M$ a generator $\mu_x = H_n(M, M \setminus \{x\}; R) \cong R$ in such a way that the function is locally compatible.

The above definition is a slight generalization of the 'intuitive' definition of orientability on smooth manifolds; that is, a locally compatible point-wise orientation on the tangent space. This intuitive orientation is equivalent to the above definition when we set $R = \mathbb{Z}$. However, the generalization is mostly for convenience, and it is not particularly far-reaching: any $\mathbb{Z}$-orientable manifold is R-orientable for all R, while a non-orientable manifold is R-orientable if and only if R contains a unit of order 2.

Theorem 2.6 (Poincaré-Lefschetz Duality): Let M be a compact R-orientable n-manifold whose boundary ∂M can be decomposed as the union of two compact $(n-1)$-manifolds A and B. Then $H^k(M, A; R) \cong H_{n-k}(M, B; R)$ for all k.

In particular, if the boundary is empty, we obtain the classical Poincaré Duality. There are other 'duality theorems' which occasionally turn up, although the Poincaré Duality is arguably the most important one. Another one which will need is as follows.

Theorem 2.7 (Alexander Duality): Let X be a compact, locally contractible subspace of S^n. Then $\widetilde{H}_k(X; R) \cong \widetilde{H}^{n-k-1}(S^n \setminus X; R)$ for all k.

Finally, we give two important results connecting homotopy theory with homology theory. The results are as follows.

Theorem 2.8 (Hurewicz Theorem): If a topological space X is $(n-1)$-connected for $n \geq 2$ then the homomorphism $\Phi : \pi_n(X, x) \to H_n(X)$, defined by $(f : S^n \to X) \mapsto f_*([S^n])$, is an isomorphism.

Theorem 2.9 (Whitehead Theorem): If $f : X \to Y$ is a map between CW complexes which induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.

2.3 Examples

As an example, we compute the homology and cohomology of several spaces. Some of the computations rely on results in the next chapters; I have decided to list them here for organizational reasons.

[3]Throughout this thesis, *all* manifolds are assumed to have a smooth structure, even though some definitions (including this one) work for arbitrary topological manifolds.

Example 2.10: The complex projective space $\mathbb{CP}^n$ can be realized as a CW complex. There is one cell in each dimension $2k$ ($k = 0, \ldots, n$). A concrete way of seeing this is as follows. Choose a basis for $\mathbb{C}^{n+1}$. Then the $2k$-skeleton is the subspace comprising those lines that lie inside the subspace spanned by the first $k+1$ basis vectors.

Any CW structure X gives rise to a cellular chain complex

$$\cdots \xrightarrow{d} H_{n+1}(X^{n+1}, X^n) \xrightarrow{d} H_n(X^n, X^{n-1}) \xrightarrow{d} \cdots$$

where d is the composition of the singular boundary map with inclusion. Denote its homology groups by $H_n^{\mathrm{CW}}(X)$. By excision, the k-th cellular chain group of X is $\mathbb{Z}^d$ where d is the number of k-cells. For the case of $\mathbb{CP}^n$, we see that there are no two adjacent non-zero cellular chain groups, so $H_{2k}^{\mathrm{CW}}(\mathbb{CP}^n) \cong \mathbb{Z}$. By [15, Thm. 2.35], cellular and singular homology coincide, so that we find

$$H_k(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \ldots, 2n; \\ 0 & \text{otherwise.} \end{cases}$$

We now look at the cohomology structure. Apply the Gysin sequence (3.16) to the canonical line bundle γ^1 over $\mathbb{CP}^n$. Since $c_1(\gamma^1) = e(\gamma^1_{\mathbb{R}})$, we have

$$\cdots \longrightarrow H^{k+1}(E_0) \longrightarrow H^k(\mathbb{CP}^n) \xrightarrow{\smile c_1} H^{k+2}(\mathbb{CP}^n) \xrightarrow{\pi_0^*} H^{k+2}(E_0) \longrightarrow \cdots$$

Geometrically, E_0 can be identified with $\mathbb{C}^{n+1} \backslash \{0\}$, so that it follows that $H^k(\mathbb{CP}^n) \cong H^{k+2}(\mathbb{CP}^n)$ for all k. Since $\mathbb{CP}^n$ is connected, each $H^{2k}(\mathbb{CP}^n)$ is infinite cyclic generated by $c_1(\gamma^1)^k$. Similarly, using the part

$$\cdots \longrightarrow H^{-1}(\mathbb{CP}^n) \longrightarrow H^1(\mathbb{CP}^n) \longrightarrow H^1(E_0) \longrightarrow \cdots$$

of the Gysin sequence, we find that the odd-dimensional cohomology groups are zero. Thus we find that

$$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/\langle \alpha^{n+1} \rangle, \quad \text{with} \quad \alpha = e(\gamma^1_{\mathbb{R}}) = c_1(\gamma^1) \in H^2(\mathbb{CP}^n).$$

Example 2.11: Like the complex projective space, the quaternionic projective space $\mathbb{HP}^n$ can be endowed with a cell structure, with one cell each in dimensions $0, 4, 8, \ldots$. This forces the cellular homology groups, and hence the singular homology groups, to be $\mathbb{Z}$ in exactly those dimensions.

As above, we apply the Gysin sequence to find the cohomology ring. Let γ^1 be the canonical line bundle over $\mathbb{HP}^n$. Using the inclusion mappings $\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$, it follows that there is an underlying complex rank-2 bundle $\gamma^1_{\mathbb{C}}$, and an underlying real rank-4 bundle $\gamma^1_{\mathbb{R}}$. From the Gysin sequence of $\gamma^1_{\mathbb{R}}$, we see that the cohomology ring $H^*(\mathbb{HP}^n)$ is again a truncated polynomial ring:

$$H^*(\mathbb{HP}^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\langle \alpha^{n+1}\rangle, \quad \text{with} \quad \alpha = e(\gamma^1_{\mathbb{R}}) = c_2(\gamma^1_{\mathbb{C}}) \in H^4(\mathbb{HP}^n).$$

Denoting this generator by u, it follows that the total Chern class of $\gamma^1_{\mathbb{C}}$ is $1+u$, and hence by Proposition 3.25 the total Pontryagin class is $p(\gamma^1_{\mathbb{R}}) = 1 - 2u + u^2$. In particular, we see that $p_1(\gamma^1_{\mathbb{R}}) = -2u$.

2.4 The signature

We give one more operation, which will turn out to be very important in the rest of this thesis. Let M be a connected closed oriented $4k$-manifold. Then the cup product on the middle cohomology group gives rise to a mapping $Q : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \to H^{4k}(M;\mathbb{R})$. By Poincaré Duality and connectedness, $H^{4k}(M;\mathbb{R})$ can be identified with $\mathbb{R}$. Furthermore, the cup product gives rise to a symmetric bilinear form on $H^{2k}(M;\mathbb{R})$. The form is non-degenerate due to Poincaré Duality, as it pairs non-degenerately with itself.

The quadratic form can be represented by a matrix with respect to any basis, which can then be diagonalized. By Sylvester's law of inertia we have that the difference in the number of positive and negative entries in the diagonalization is invariant under the choice of basis. This leads to the definition of the **signature** of a matrix to be the number of positive diagonal entries minus the number of negative ones after a choice of diagonalization.

Combining the above information, we find the following. Let M be a connected closed oriented $4k$-manifold. Then the **signature** of M, henceforth denoted by $\tau(M)$, is the signature of the matrix corresponding to the cup product on its middle cohomology group.

Example 2.12: We compute the signature of $\mathbb{CP}^{2k}$. By Example 2.10 it can be seen that $H^{2k}(\mathbb{CP}^{2k};\mathbb{Q})$ is generated by a single element α with $\langle \alpha \smile \alpha, \mu\rangle = 1$, so that the signature $\tau(\mathbb{CP}^{2k})$ is $+1$.

Chapter 3

Characteristic classes

This chapter is dedicated to the people out there who feel there is no absolute good in this world. For in this chapter, we study the absolute good that we call characteristic classes. Characteristic classes are good things. *Truly* good things. More specifically, they are global invariants associated to vector bundles[1] which measure, in some sense, the deviation of the local product structure from the global product structure.

Characteristic classes will be used to define diffeomorphism invariants, much like the signature invariant we just mentioned; these invariants provide us with a method of determining that two given manifolds are not diffeomorphic.

The results in this chapter are mostly based on the excellent source [34], unless otherwise stated.

3.1 Cobordism

Cobordism theories are an extremely important tool in modern algebraic topology. In this thesis, we will restrict our attention to the cobordism theory of smooth closed manifolds, but the reader should note that this is in fact a particularly elementary example. To motivate this section, we consider the problem of classifying smooth closed n-manifolds. The obvious first step would be to try an classify them up to diffeomorphism, but it turns out this is essentially impossible.

This 'essentially impossible' can, in fact, be made mathematically precise, but we need some background to explain this. The **word problem** for a finitely generated group G is the algorithm problem of deciding whether two words in the generators represent the same element. The related

[1] As with manifolds, we assume vector bundles are smooth, even though many concepts easily generalize.

triviality problem is the algorithmic problem of deciding, given as input a presentation P for some group G, whether this group is the trivial group or not.

In 1955, it was shown by Pyotr Novikov[37] that there exists a finitely presented group G such that the word problem for G is undecidable; in other words, it is impossible to construct a single algorithm that solves the word problem. In particular, it follows that the triviality problem is undecidable as well.

How does this relate to our discussion of classification of manifolds? In 1958, the following result was found.

Theorem 3.1: For any finite presentation P, one can construct a pair of smooth, closed, orientable 4-manifolds which are diffeomorphic if, and only if, P presents the trivial group.[24]

In particular, we know the following. Suppose there exists some classification up to diffeomorphism in dimension ≥ 4. Then in particular, given two 4-manifolds, there exists an algorithm that allows us to determine whether the manifolds are diffeomorphic or not. But by the above theorem, this implies the existence of an algorithm to determine whether P presents the trivial group.

Luckily for us, there exists a weaker equivalence relation, known as 'cobordism', up to which we *can* classify manifolds; this classification will come in handy throughout this thesis.

Two closed n-manifolds M_0, M_1 are said to be **cobordant** whenever there exists a compact manifold W such that ∂W is the disjoint union of M_0 and M_1. If we assume M_0 and M_1 are oriented manifolds, we say they are **oriented cobordant** if there exists a compact *oriented* manifold W such that the boundary (with the induced orientations) is $M_0 \sqcup -M_1$, where $-M_1$ denotes M_1 with the reversed orientation.

Cobordism is an equivalence relation. Denote by Ω_n and Ω_n^{SO} the set of cobordism classes and oriented cobordism classes, respectively. Both sets become an abelian group under disjoint union, the empty set being the zero element. The inverse element of a given manifold is itself in Ω_n, and its orientation-reversed self in Ω_n^{SO}. Furthermore, the Cartesian product turns $\bigoplus_{n\geq 0} \Omega_n$ into a graded commutative ring, and $\bigoplus_{n\geq 0} \Omega_n^{\mathrm{SO}}$ into a graded skew-commutative ring.

For the sake of completeness, let us list without proof the actual structures of the first few cobordism groups, both oriented and unoriented. At the end of Chapter 5, we will able to prove a small part of these results.

n	0	1	2	3	4	5
Ω_n	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$
Ω_n^{SO}	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Proposition 3.2: The signature gives rise to an algebra homomorphism from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.

Proof: We obviously have $\tau(M \sqcup M') = \tau(M) + \tau(M')$. Furthermore, $\tau(M \times M') = \tau(M) \times \tau(M')$ by the Künneth Theorem (2.5). Finally, suppose we have $M = \partial W$ and M, W are oriented. By the Poincaré-Lefschetz Duality (2.6) we have the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H^k(W;\mathbb{Q}) \arrow[r, "i^*"] \arrow[d] & H^k(M;\mathbb{Q}) \arrow[r] \arrow[d] & H^{k+1}(W,M;\mathbb{Q}) \arrow[r] \arrow[d] & H^{k+1}(W;\mathbb{Q}) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_{n+1-k}(W,M;\mathbb{Q}) \arrow[r] & H_{n-k}(M;\mathbb{Q}) \arrow[r, "i_*"] & H_{n-k}(W;\mathbb{Q}) \arrow[r] & H_{n-k}(W,M;\mathbb{Q}) \arrow[r] & \cdots \end{tikzcd}$$

Let $A^k = \mathrm{Im}(i^* : H^k(W) \to H^k(M))$. By the Universal Coefficient Theorem (2.4) we find that A^k is precisely the annihilator of A^{n-k}.

Since $\dim M = 4k$, this gives $H^{2k}(M) = A^{2k} \oplus B^{2k}$ with A, B dually paired and with dual bases a_i, b_j such that $a_i b_j = \delta_{ij}$ and $a_i a_j = b_i b_j = 0$. If we order the basis as $\{a_1, b_1, a_2, b_2, \ldots\}$ then the corresponding matrix consists of blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ along the diagonal, and thus $\tau(M) = 0$. ■

3.2 Stiefel-Whitney classes

In this section the coefficient group for cohomology will always be $\mathbb{Z}/2\mathbb{Z}$. We give an axiomatic characterization of the so-called **Stiefel-Whitney classes**. These are a set of invariants of real vector bundles that, in a sense, describe the obstructions to trivializing the vector bundle.

Note 3.3: We introduce some important notation; for convenience, we will use this notation several times in the rest of this chapter. We usually denote a vector bundle by ω. Its base space will be denoted by B, its total space by E, and the projection map by $\pi : E \to B$. By E_0 we denote the space of non-zero vectors of E. The associated restriction of the projection map is denoted by π_0. A fibre $\pi^{-1}(b)$ is usually denoted F_b, of F, if we don't care about b. By F_0 we denote $F \cap E_0$.

The axioms are as follows.

Axiom 1. To each rank-n vector bundle ω there is a sequence of cohomology classes $w_k(\omega) \in H^k(B; \mathbb{Z}/2\mathbb{Z})$ $(k = 0, 1, \ldots)$ called the **Stiefel-Whitney classes** of the vector bundle. The class $w_0(\omega)$ is equal to the unit element and $w_k(\omega)$ is zero for $k > n$.

Axiom 2. If $f : B' \to B$, then $w(f^*\omega) = f^*w(\omega)$, where $f^*\omega$ denotes the pull-back bundle over B'.

Axiom 3. If ω, ω' are two vector bundles over the same base space, then the class of the Whitney sum $\omega \oplus \omega'$ is given by

$$w_k(\omega \oplus \omega') = \sum_{i=0}^{k} w_i(\omega) \smile w_{k-i}(\omega').$$

Axiom 4. For the tautological line bundle over the circle, the first Stiefel-Whitney class is non-zero.

It is not obvious that such classes exist. We will assume their existence for now; a proof of this can be found in [34, §8]. Some quick corollaries of the axioms are as follows. If two vector bundles are isomorphic then their Stiefel-Whitney classes coincide. For $k > 0$, the k-th Stiefel-Whitney class of the trivial vector bundle is zero. Thus, taking the Whitney sum with a trivial vector bundle, the classes remain unaltered.

Given a rank-n vector bundle ω, we can define the **total Stiefel-Whitney class** to be the element $w(\omega) = 1 + w_1(\omega) + \cdots$ inside the Cartesian product of the groups $H^k(B; \mathbb{Z}/2\mathbb{Z})$.

Let us see, without proof, some of the power of this seemingly uninteresting invariant. For convenience, denote by T the tangent bundle of $\mathbb{RP}^n$.

Lemma 3.4: The total Stiefel-Whitney class of T is given by

$$w(T) = (1 + a)^{n+1},$$

where a is the generator of $H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$.

Corollary 3.5: The only projective spaces $\mathbb{RP}^n$ which can possibly be parallelizable are the ones for which $n + 1$ is a power of 2.

This result is intimately related to the question of the existence of real division algebras. Indeed, the existence of a bilinear product operation $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ without zero divisors implies that the projective space $\mathbb{RP}^{n-1}$ must be parallelizable. Hence we know that n must be a power of 2. It turns out the only parallelizable projective spaces are $\mathbb{RP}^1$, $\mathbb{RP}^3$ and $\mathbb{RP}^7$, and also the only known division algebras exist for $n = 1, 2, 4, 8$ (the real numbers, complex numbers, quaternions and the Cayley numbers).

Proof (of Corollary 3.5): First, we assume $n + 1 = 2^k$ for some integer k. Then we have

$$w(T) = (1 + a)^{2^k} = 1 + a^{2^k} = 1 + a^{n+1} = 1$$

where in the second step we used the fact that a is an order-2 element, and in the last step we used that the $(n+1)$-th cohomology of $\mathbb{RP}^n$ vanishes. Since triviality of the Stiefel-Whitney class is a necessary property of parallelizability, the result follows.

Conversely, suppose $n + 1 \neq 2^k$; say, $n + 1 = 2^k m$ for some odd m. Then a straightforward calculation shows that $(1 + a)^{n+1} \neq 1$, and thus, $\mathbb{RP}^n$ cannot be parallelizable. ■

Let M be a closed n-manifold. There exists a unique homology class $\mu_M \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ as mentioned in Section 2.2. Hence for any cohomology class $\nu \in H^n(M; \mathbb{Z}/2\mathbb{Z})$, we may associate the value $\langle \nu, \mu_M \rangle \in \mathbb{Z}/2\mathbb{Z}$. We can apply this idea to the discussion above. More precisely, let $r_1, \ldots, r_n$ be non-negative integers with $r_1 + 2r_2 + \cdots + nr_n = n$. A vector bundle ω then has the class $w_1(\omega)^{r_1} \cdots w_n(\omega)^{r_n} \in H^n(B; \mathbb{Z}/2\mathbb{Z})$. A particularly important case is when we look at the tangent bundle TM of a manifold M. The **Stiefel-Whitney number** of M associated with this n-tuple is then $\langle w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}, \mu_M \rangle \in \mathbb{Z}/2\mathbb{Z}$.

Example 3.6: We compute the Stiefel-Whitney numbers of $\mathbb{RP}^n$. Recall that by T we denote its tangent space. Suppose first that n is odd, say $n = 2k - 1$, then $w(T) = (1 + a)^{2k} = (1 + a^2)^k$ so $w_j(T) = 0$ when j is odd. Since every $(r_1, \ldots, r_n)$ must have a non-zero component r_j for some odd j, it follows that the Stiefel-Whitney numbers are all zero.

Next suppose that n is even. Then $w_n(T) = (n + 1)a^n$ is non-zero, and thus the Stiefel-Whitney number associated to $r_n = 1$ is non-zero. Similarly, the Stiefel-Whitney number associated to $r_1 = n$ is non-zero, since $w_1(T) = (n + 1)a \neq 0$. The remaining Stiefel-Whitney numbers can be computed using binomial coefficients; in any case, the outcomes are non-trivial.

The reason why we care about these numbers is because of the following powerful result, whose proof can be found in [48].

Theorem 3.7: Let M be a closed n-manifold. Then all of the Stiefel-Whitney numbers of M are zero if and only if M is the boundary of some smooth compact manifold.[2]

[2]In particular, $\mathbb{RP}^{2k-1}$ is the boundary of some compact $2k$-manifold. This can be seen more explicitly as follows. Let $S^1 \subseteq \mathbb{C}$ act on $S^{2k-1} \times D^2 \subseteq \mathbb{C}^k \times \mathbb{C}$ by $\lambda \cdot (x, y) = (\lambda x, \lambda^2 y)$; after quotienting out the group orbits, the resulting manifold has boundary $\mathbb{RP}^{2k-1}$.

Corollary 3.8: Two closed n-manifolds are cobordant if and only their Stiefel-Whitney numbers are equal.

3.3 The Euler class

Instead of working with coefficients in $\mathbb{Z}/2\mathbb{Z}$, we look at coefficients in $\mathbb{Z}$ again. It may not come as a surprise that this will reveal to us a lot of new structure. Unlike in the case $\mathbb{Z}/2\mathbb{Z}$, having a generator of the top cohomology is not a given; we need to impose the additional requirement that our vector bundles are oriented; in other words, for each fibre F there exists a preferred generator $u_F \in H^n(F, F_0) = H^n(F, F_0; \mathbb{Z})$ satisfying the local compatibility condition.

Theorem 3.9 (Thom Isomorphism Theorem): We use the notation of Note 3.3, with the additional orientability assumption as mentioned above. The cohomology group $H^k(E, E_0)$ contains a cohomology class u such that for each fibre F, the restriction of u to (F, F_0) is the class induced by the orientation of F. Moreover, the correspondence $x \mapsto x \smile u$ gives an isomorphism $H^k(E) \xrightarrow{\sim} H^{k+n}(E, E_0)$.

The proof can be found in [34, §10]. More generally, we can replace $\mathbb{Z}$ with any ring with unit; it turns out we won't need this. We define the **Thom isomorphism** $\phi : H^k(B) \to H^{k+n}(E, E_0)$ to be the map defined by $\phi(x) = (\pi^* x) \smile u$. By the above theorem, together with an application of homotopy invariance ($H^k(E) \cong H^k(B)$), this is indeed an isomorphism.

This result allows us to define a new characteristic class. Given an oriented rank-n vector bundle $\omega = (E, \pi, B)$, the inclusion $(E, \varnothing) \hookrightarrow (E, E_0)$ gives rise to a restriction homomorphism $H^*(E, E_0) \to H^*(E)$, which we can denote by $x \mapsto x|_E$. Applying this to the fundamental class $u \in H^n(E, E_0)$ we obtain a new cohomology class $u|_E \in H^n(E)$. As we just mentioned, $H^n(E)$ is isomorphic to $H^n(B)$ under the canonical isomorphism π^*. We define the **Euler class**[3] of the vector bundle ω to be the cohomology class $e(\omega) = (\pi^*)^{-1}(u|_E) \in H^n(B)$.

Proposition 3.10 (Properties of the Euler class): Let the notation be as in Note 3.3.

- Let ω, ω' are two oriented vector bundles, and let $f : B' \to B$ be covered by an orientation-preserving map $E' \to E$. Then $e(\omega') = f^* e(\omega)$. In particular, the Euler characteristic of the trivial bundle is zero, for we can take E' to be the bundle over a point.

[3] We call it the Euler class because, as we will soon see, it is intimately connected to the Euler characteristic.

- The Euler class of a vector bundle changes sign upon reversing the orientation. Thus, if the rank of the bundle is odd, by the orientation-reversing automorphism $(x, v) \mapsto (x, -v)$ we have $e(\omega) = -e(\omega)$.
- The Euler class of a Whitney sum is given by $e(\omega \oplus \omega') = e(\omega) \smile e(\omega')$.

Interestingly, the Stiefel-Whitney classes introduced above are strongly related to the Euler class; in fact, the natural homomorphism $H^n(B) \to H^n(B; \mathbb{Z}/2\mathbb{Z})$ carries the Euler class to the top Stiefel-Whitney class.

Let X be a finite-dimensional CW complex and F a field. Then we can define the **Euler characteristic** as the alternating sum

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim H^k(X; F).$$

In [15, Thm. 2.44], it is shown that this coincides with the alternating sum

$$\sum_{k=0}^{n} (-1)^k \cdot (\text{number of } k\text{-cells}),$$

and hence is independent of the coefficient field which is used.

Theorem 3.11: Let M be a compact oriented manifold. Then the Euler number $\langle e(TM), \mu \rangle$, using rational or integer coefficients, is equal to the Euler characteristic. Similarly, for a non-oriented manifold, the Stiefel-Whitney number $\langle w_n(TM), \mu \rangle$ is congruent to $\chi(M)$ modulo 2.

Proof: Only the first part will be relevant for us, although the second statement is proved analogously. We first need the following lemma, as found in [34, Cor. 11.2]:

Lemma 3.12: Let M be an n-manifold which is smoothly embedded in an $(n+k)$-manifold N. If M is closed in N as a subset of the topological space, then the cohomology ring $H^*(E, E_0; R)$ associated to the normal bundle E of M in N (with respect to some Riemannian metric on N) is canonically isomorphic to the cohomology ring $H^*(N, N \setminus M; R)$.

Proof: (Sketch) Consider the 'exponential map' which maps $E(\varepsilon) = \{(x, v) \in E : |v| < \varepsilon\}$ to N by assigning to $(x, v) \in E(\varepsilon)$ the endpoint of the arc $\gamma : [0, 1] \to N$ with initial point x and $d_t\gamma|_{t=0} = v$. It turns out if ε is sufficiently small, $E(\varepsilon)$ is mapped diffeomorphically onto some open set $N_\varepsilon \subseteq N$.

The sets N_ε and the complement $N \setminus M$ are open subsets with union N and intersection $N_\varepsilon \setminus M$; thus, there is an excision isomorphism $H^*(N, N \setminus M; R) \xrightarrow{\sim} H^*(N_\varepsilon, N_\varepsilon \setminus M; R)$.

The required map is found by composing this excision isomorphism with the cohomology isomorphism induced by the exponential map. ■

We apply this lemma as follows. The fundamental class $u \in H^k(E, E_0; \mathbb{Z}/2\mathbb{Z})$ naturally corresponds to a cohomology class $u' \in H^k(N, N \setminus M; \mathbb{Z}/2\mathbb{Z})$. If the normal bundle is orientable, we can do the same with coefficients in $\mathbb{Z}$.

Lemma 3.13: Using the notation as above, and assuming the normal bundle is orientable, the composition of the two restriction homomorphisms

$$H^k(N, N \setminus M) \longrightarrow H^k(N) \longrightarrow H^k(M)$$

maps the fundamental class u' to the Euler class of the normal bundle.

Proof: Let $s : M \to E$ denote the zero section of the normal bundle; it induces a canonical isomorphism $s^* : H^*(E) \xrightarrow{\sim} H^*(M)$. First we note that the composition

$$H^k(E, E_0) \longrightarrow H^k(E) \xrightarrow{s^*} H^k(M)$$

maps the fundamental class to the Euler class. The image of $s^*(u|_E)$ is mapped under the Thom isomorphism to $\phi(s^*(u|_E)) = (\pi^* s^*(u|_E)) \smile u = u \smile u$, hence $s^*(u|_E) = \phi^{-1}(u \smile u)$. But since we clearly have $\phi\big((\pi^{-1}(u|_E)\big) = u \smile u$ as well, and ϕ is an isomorphism, we conclude that $s^*(u|_E)$ is precisely the Euler class.

Next, replace (E, E_0) with the diffeomorphic (under the aforementioned exponentiation map) pair $(N_\varepsilon, N_\varepsilon \setminus M)$. The restriction homomorphisms

$$H^k(N_\varepsilon, N_\varepsilon \setminus M) \longrightarrow H^k(N_\varepsilon) \longrightarrow H^k(M)$$

thus map the class u to the Euler class. Using the commutative diagram

$$\begin{array}{ccc} H^k(N, N \setminus M) & \longrightarrow & H^k(N) \\ \downarrow & & \downarrow \\ H^k(N_\varepsilon, N_\varepsilon \setminus M) & \longrightarrow & H^k(M) \end{array}$$

we see that the result immediately follows. ■

In particular, we may apply this result to the following situation: Let M be a manifold endowed with some Riemannian metric. Then $M \times M$ naturally inherits a metric structure: if M has the inner product $(u, u') \mapsto u \cdot u'$ $(u, u' \in M)$, then $M \times M$ has the natural inner product $(u, v) \cdot (u', v') = u \cdot u' + v \cdot v'$.

Lemma 3.14: The diagonal mapping $x \mapsto \Delta(x) = (x, x)$ embeds M smoothly as a closed subset of $M \times M$. The normal bundle associated with the diagonal embedding of M in $M \times M$ is canonically isomorphic to the tangent bundle of M.

Proof: A tangent vector (u, v) of $T_xM \times T_xM \cong T_{(x,x)}(M \times M)$ is tangent to $\Delta(M)$ if and only if $u = v$ and normal to $\Delta(M)$ if and only if $u + v = 0$. Thus to $v \in T_xM$ there corresponds a normal vector $(-v, v) \in T_{(x,x)}(M \times M)$. This correspondence maps the tangent manifold TM diffeomorphically onto the total space of the normal bundle. ■

Using Lemmas 3.13 and 3.14, we find that the Euler class of the tangent bundle of M is precisely $\Delta^*(u')$. We give an alternative description of this fundamental class u'.

Lemma 3.15: Let $b_1, \ldots, b_r$ be a basis for $H^*(M)$. Then by Poincaré Duality, there exists a dual basis $b'_1, \ldots, b'_r$ for $H^*(M)$ such that $\langle b_i \smile b'_j, \mu\rangle = \delta_{ij}$ for all i, j. Assuming for now the existence of such basis elements, the diagonal cohomology class $u' \in H^n(M \times M)$ is equal to

$$\sum_{i=1}^{r} (-1)^{\dim b_i} b_i \times b'_i.$$

Proof: By the Künneth Theorem (2.5), the diagonal class can be expressed as an r-fold sum $u' = b_1 \times c_1 + \cdots + b_r \times c_r$, where $c_1, \ldots, c_r$ are certain well-defined cohomology class in $H^*(M)$ with $\dim b_i + \dim c_i = n$. Now for any $a \in H^*(M)$, the product $(a \times 1) \smile u'$ is equal to $(1 \times a) \smile u'$. This can be seen by noting that the two projection maps $p_1, p_2 : M \times M \to M$ are homotopic (since they coincide on $\Delta(M)$, and $\Delta(M)$ is a deformation retract of the tubular neighbourhood N_ε). Milnor & Stasheff apply the so-called *slant product* (see [34, p. 125] for details) to both sides of this identity, from which the result easily rolls out. ■

We thus find that $e(TM) = \sum_i (-1)^{\dim b_i} b_i \smile b'_i$. Applying the homomorphism $\langle \bullet, \mu\rangle$ to both sides, we obtain the required formula. ■

3.4 The Gysin sequence

In this section, we introduce the Gysin sequence. This is a long exact sequence which relates the cohomology classes of the base space and the fibre of our vector bundle. The relevance of this section is mostly the computations found in Section 2.3. The results of this section are based on [34, §12].

Theorem 3.16 (Gysin sequence): Take over the notation as in Note 3.3. Assume ω has rank n. Then there exists an exact sequence of the form

$$\cdots \longrightarrow H^k(B) \xrightarrow{\bullet\smile e(\omega)} H^{k+n}(B) \xrightarrow{\pi_0^*} H^{k+n}(E_0) \longrightarrow H^{k+1}(B) \xrightarrow{\bullet\smile e(\omega)} \cdots .$$

Proof: Start with the cohomology exact sequence of the pair (E, E_0):

$$\cdots \longrightarrow H^k(E, E_0) \longrightarrow H^k(E) \longrightarrow H^k(E_0) \xrightarrow{\partial} H^{k+1}(E, E_0) \longrightarrow \cdots$$

Use the Thom Isomorphism (3.9) to substitute $H^{k-n}(E)$ in place of $H^k(E, E_0)$. We then obtain the exact sequence of the form

$$\cdots \longrightarrow H^{k-n}(E) \xrightarrow{f} H^k(E) \longrightarrow H^k(E_0) \xrightarrow{\partial} H^{k-n+1}(E) \longrightarrow \cdots$$

where $f(x) = (x \smile u)|_E = x \smile (u|_E)$. Now substitute $H^k(B)$ in place of $H^k(E)$ and note that the cohomology class $u|_E$ of $H^n(E)$ corresponds to the Euler class $e(\omega)$. This yields the required sequence. ■

Similarly, for an unoriented bundle, there is a corresponding exact sequence with mod 2 coefficients, using the Stiefel-Whitney class w_n in place of the Euler class. Consider a 2-fold covering ω. Then we can construct a line bundle over B whose total space is obtained from $E \times \mathbb{R}$ by collapsing (x_1, t) with $(x_2, -t)$ for all t. This gives rise to an exact sequence of the form

$$\cdots \longrightarrow H^{k-1}(B) \xrightarrow{\bullet\smile w_1(\omega)} H^k(B) \longrightarrow H^k(E) \longrightarrow H^k(B) \longrightarrow \cdots$$

with mod 2 coefficients.

3.5 Chern classes

Note 3.17: We introduce notation that will be used in this section. By ω we denote a *complex* vector bundle of rank n, so that its underlying real vector bundle, $\omega_{\mathbb{R}}$, has rank $2n$. The base space of ω will be denoted by B, its total space by E, and the projection map by $\pi : E \to B$. By E_0 we denote the space of non-zero vectors of E. The associated restriction of the projection map is denoted by π_0. A fibre $\pi^{-1}(b)$ is usually denoted F_b, of F, if we don't care about b. By F_0 we denote $F \cap E_0$.

A point in E_0 is specified by a non-zero vector v in a fibre F. We can thus define a canonical rank-$(n-1)$ vector bundle, denoted ω_0, with base space E_0, by setting the fibre over v to be $F/\langle v \rangle$.

Consider a complex vector bundle ω. Then the underlying real vector bundle has a canonical preferred orientation. In particular, if we apply this to the tangent bundle of a complex manifold, we conclude that any complex manifold has a preferred orientation. As an application, for any rank-n complex vector bundle, the Euler class $e(\omega_{\mathbb{R}}) \in H^{2n}(B;\mathbb{Z})$ is well-defined.

The orientation can be found as follows. Let V be any finite-dimensional complex vector space, with basis $a_1, \dots, a_n$ over $\mathbb{C}$. The vectors $a_1, ia_1, \dots, a_n, ia_n$ form a basis for the underlying real vector space. This ordered basis determines the required orientation for $V_{\mathbb{R}}$. This orientation does not depend on the choice of complex basis. We can apply this idea fibre-wise to find the preferred orientation of our manifold.

We define the **Chern classes** $c_k(\omega) \in H^{2k}(B;\mathbb{Z})$ as follows, by induction on the rank n. The top Chern class $c_n(\omega)$ is set to be equal to the Euler class $e(\omega_{\mathbb{R}})$. For $k < n$, we set $c_k(\omega) = (\pi_0^*)^{-1} c_k(\omega_0)$. Notice that this works because the rank of ω_0 is one lower than that of ω, and because π_0^* is an isomorphism; indeed, by the Gysin sequence (3.16) we have the exact sequence

$$\cdots \longrightarrow H^k(B) \xrightarrow{\bullet\smile e(\omega)} H^{k+2n}(B) \xrightarrow{\pi_0^*} H^{k+2n}(E_0) \longrightarrow H^{k+1}(B) \xrightarrow{\bullet\smile e(\omega)} \cdots,$$

and if $k < 0$, we have $H^k(B) = H^{k+1}(B) = 0$. Thus π_0^* is an isomorphism. If $k > n$ we define $c_i(E)$ to be zero. The formal sum $c(\omega) = 1 + c_1(\omega) + \cdots$ in the Cartesian product of the cohomology groups is called the **total Chern class**.

Proposition 3.18: If $f : B' \to B$, then $c_k(f^*\omega) = f^* c_k(\omega)$, where $f^*\omega$ denotes the pull-back bundle over B'.

Proof (of Proposition 3.18): By naturality of the Euler class, the result follows for the top Chern class. For lower Chern classes, we apply induction. The bundle map f gives rise to another bundle map f_0 between ω_0 and ω_0'. Using the commutative diagram

$$\begin{tikzcd} E_0 \arrow[r, "f_0"] \arrow[d, "\pi_0"'] & E_0' \arrow[d, "\pi_0'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

together with the identities $c_i(\omega_0) = \pi_0^* c_i(\omega)$ and $c_i(\omega_0') = \pi_0'(\omega')$, where π_0^* is an isomorphism for $i < n$, it follows that $c_i(\omega) = f^* c_i(\omega')$, as was to be shown. ∎

Proposition 3.19 (Whitney Product Formula): If ω, ω' are two complex vector bundles over a common base space B, then $c(\omega \oplus \omega') = c(\omega) \smile c(\omega')$.

Proof: (Sketch) Milnor & Stasheff begin by showing that there exists a polynomial $f_{m,n}$ such that

$$c(\omega \oplus \omega') = f_{m,n}\big(c_1(\omega), \ldots, c_m(\omega), c_1(\omega'), \ldots, c_n(\omega')\big).$$

They consider a universal model of complex vector bundles over a common base space by taking the tautological vector bundles γ^m, γ^n over the Grassmannians $G_m = G_m(\mathbb{C}^\infty)$ resp. $G_n = G_n(\mathbb{C}^\infty)$, then pulling them back by the projection maps π_1, π_2 to produce vector bundles over $G_m(\mathbb{C}^\infty) \times G_n(\mathbb{C}^\infty)$. Using the fact that the external cohomology cross-product operation $(a, b) \mapsto \pi_1^* a \smile \pi_2^* b$ induces an isomorphism $H^*(G_m) \otimes H^*(G_n) \xrightarrow{\sim} H^*(G_m \times G_n)$, we see that $H^*(G_m \times G_n)$ is a polynomial ring over $\mathbb{Z}$ on the generators $\pi_1^* c_i(\gamma^m)$, $\pi_2^* c_j(\gamma^n)$. Hence the total Chern class of $\gamma_1^m \oplus \gamma_2^n$ can be uniquely expressed as a polynomial:

$$c(\gamma_1^m \oplus \gamma_1^n) = f_{m,n}\big(c_1(\gamma_1^m), \ldots, c_m(\gamma_1^m), c_1(\gamma_1^n), \ldots, c_n(\gamma_1^n)\big).$$

The same result holds true for *arbitrary* vector bundles ω, ω' over a common base space B, because we can *always* construct maps $f : B \to G_m$, $g : B \to G_n$ which pull the tautological bundles γ^m, γ^n back to ω, ω'.

Finally, to compute the polynomials, Milnor & Stasheff proceed by induction on $m + n$, and analyzing the vector bundles $\gamma_1^{m-1} \oplus \varepsilon^1$ and γ_2^n over $G_{m-1} \times G_n$ (and equivalently, γ_1^m and $\gamma_2^{n-1} \oplus \varepsilon^1$). ■

If ω is a complex vector bundle then the **conjugate bundle** $\overline{\omega}$ is defined to be the complex vector bundle with the same underlying real vector bundle but with the 'opposite' complex structure. Thus the identity map $f : E \to \overline{E}$ is conjugate linear.

Example 3.20: A complex bundle need not be isomorphic to its conjugate bundle. Consider the tangent bundle T of the complex manifold $\mathbb{CP}^1$. Suppose an isomorphism $T \to \overline{T}$ would exist. Then each tangent plane of the 2-sphere would be mapped onto itself so as to reverse the complex structure. Any such map must be obtained by reflection in some line. This would imply the existence of a continuous non-zero vector field on S^2 (since S^2 and $\mathbb{CP}^1$ are homeomorphic), which is known to be impossible.

The Chern class of a conjugate bundle is related to that of the original bundle.

Proposition 3.21: The Chern class $c_k(\overline{\omega})$ is precisely $(-1)^k c_k(\omega)$.

Proof: Any fibre F has a basis $v_1, \ldots, v_n$ over $\mathbb{C}$. Then the basis $v_1, iv_1, \ldots, v_n, iv_n$ determines the preferred orientation for $F_{\mathbb{R}}$. Similarly, the basis $v_1, -iv_1, \ldots, v_n, -iv_n$ prefers the orientation for the conjugate bundle. Using the properties of the Euler class, it follows that $c_n(E) = (-1)^n c_n(\overline{E})$. By an induction argument, the same holds true for $k < n$. ■

Example 3.22: We compute the total Chern class of $\mathbb{CP}^n$. Let γ^1 be its canonical line bundle. Let ω^n be its orthogonal complement with respect to the standard inner product. Note that we can identify the tangent bundle T of $\mathbb{CP}^n$ with the complex vector bundle $\mathrm{Hom}_{\mathbb{C}}(\gamma^1, \omega^n)$; intuitively, this can be seen as follows. Consider the tangent space of $\mathbb{CP}^n$ at a line L. Then $\mathrm{Hom}(L, L^\perp)$ can be identified with the neighbourhood of L in $\mathbb{CP}^n$ consisting of all lines L' which can be considered as graphs of linear maps from L to $L^\perp$.

Taking the Whitney sum with $\varepsilon^1 \cong \mathrm{Hom}(\gamma^1, \gamma^1)$ on both T and $\mathrm{Hom}(\gamma^1, \omega^n)$, we find that $T \oplus \varepsilon^1$ can be identified with the Whitney sum of $n+1$ copies of the dual bundle $\mathrm{Hom}(\gamma^1, \varepsilon^1) = \overline{\gamma^1}$. Using Propositions 3.19 and 3.21 we find that $c(\mathbb{CP}^n) = c(T) = (1+a)^{n+1}$, where $a = -c_1(\gamma^1)$, which, as mentioned in Example 2.10, is a generator of $H^2(\mathbb{CP}^n)$.

3.6 Pontryagin classes

Consider a *real* vector bundle ω. We construct a complexified bundle $\omega_{\mathbb{C}}$ by replacing the fibres F with $F \otimes \mathbb{C}$. The underlying real vector bundle of the complexification can be seen to be canonically isomorphic to $\omega \oplus \omega$.

Consider the total Chern class of the complexification $\omega_{\mathbb{C}}$. The complexification of a real vector bundle is isomorphic to its own conjugate bundle under the correspondence $x + iy \mapsto x - iy$. Thus, by Proposition 3.21, the odd Chern classes are elements of order 2.

Ignoring these elements of order 2, we can define the k-th **Pontryagin class** $p_k(\omega) \in H^{4k}(B; \mathbb{Z})$ to be the integral homology class $(-1)^k c_{2k}(\omega_{\mathbb{C}})$. The **total Pontryagin class** is again their formal sum. As before, we have some nice properties.

Proposition 3.23: If $f : B' \to B$, then $p_k(f^*\omega) = f^* p_k(\omega)$, where $f^*\omega$ denotes the pull-back bundle over B'.

Proof: This follows directly from the naturality of the Chern classes. ■

In analogy with the Chern classes, the Pontryagin classes satisfy a product formula. Since the odd Chern classes of $\omega_{\mathbb{C}}$ have been thrown away, the best we can do is as follows.

Proposition 3.24: If ω, ω' are two real vector bundles over a common base space B, then $p(\omega \oplus \omega')$ is congruent to $p(\omega) \smile p(\omega')$ modulo elements of order 2.

Proof: Modulo odd Chern classes (which are elements of order 2), we have, by Proposition 3.19, that

$$c_{2k}(\omega \oplus \omega') \otimes \mathbb{C}) = \sum_{i+j=k} c_{2i}(\omega \otimes \mathbb{C}) \smile c_{2j}(\omega' \otimes \mathbb{C}).$$

Multiplying both sides by $(-1)^k = (-1)^{i+j}$, the result follows. ■

Proposition 3.25: For a complex rank-n bundle E, the Chern classes $c_i(\omega)$ determine the Pontryagin classes $p_k(E_\mathbb{R})$ by the formula

$$1 - p_1 + p_2 - \cdots + (-1)^n p_n = (1 + c_1 + \cdots + c_n)(1 - c_1 + c_2 - \cdots + (-1)^n c_n).$$

Proof: This follows from the Whitney formula for the Chern classes, together with Proposition 3.21. ■

Example 3.26: We compute the Pontryagin classes of $\mathbb{CP}^n$. Since the total Chern class is $(1+a)^{n+1}$ by Example 3.22, a quick calculation shows that $p_k(\mathbb{CP}^n) = \binom{n+1}{k} a^{2k}$ for all $1 \leq k \leq n/2$.

3.7 Characteristic numbers

In this section, we introduce the Chern numbers and the Pontryagin numbers of a closed oriented manifold, both of them neat invariants, which will turn out to be important in the computation of cobordism groups.

Let M be a compact complex manifold of complex dimension n. Let $I = (i_1, \ldots, i_r)$ be a partition of n. Then the **Chern number** $c_I(M)$ is defined to be the integer

$$c_I(M) = \langle c_{i_1}(TM) \cdots c_{i_r}(TM), \mu_{2n} \rangle,$$

where μ_{2n} is, as usual, the fundamental homology class determined by the preferred orientation.

A complex n-dimensional manifold has precisely Part(n) different Chern numbers. These Chern numbers are unrelated, in the sense that there is no linear relation between them which is satisfied for all n-manifolds.

Now let M be a compact oriented $4n$-manifold. For each partition I of n, the **Pontryagin number** $p_I(M)$ is the integer defined by

$$p_I(M) = \langle p_{i_1}(TM) \cdots p_{i_r}(TM), \mu_{4n} \rangle.$$

Example 3.27: As an example, we consider again the complex projective plane $\mathbb{CP}^n$. If n is odd, then the dimension of $\mathbb{CP}^n$ is not divisible by 4. Usually, for such manifolds, the Pontryagin number is simply set to be zero. If n is even, then the Pontryagin numbers can be determined using Example 3.26; we find that

$$p_{i_1} \cdots p_{i_r}(\mathbb{CP}^n) = \binom{n+1}{i_1} \cdots \binom{n+1}{i_r}.$$

An application of Pontryagin numbers is as follows.

Proposition 3.28: Let M be a closed oriented $4n$-manifold. If some Pontryagin number $p_I(M)$ is non-zero, then M cannot be the boundary of a compact oriented $(4n+1)$-dimensional manifold with boundary.

This application, while already neat on its own, has the following immediate corollary.

Corollary 3.29: For any partition $I = (i_1, \ldots, i_r)$ of k, the I-th Pontryagin number induces a well-defined group homomorphism $\Omega_{4k}^{\mathrm{SO}} \to \mathbb{Q}$.

Chapter 4

Milnor's construction of exotic spheres

This chapter will be an exposition of the paper [28] by Milnor. In this paper, Milnor gives an explicit construction of exotic smooth structures on the 7-sphere, thus providing the first example of an exotic structure ever found.

4.1 The construction

The spaces Milnor constructs are S^3-bundles over S^4. In [47, Thm. 18.5], a general classification theorem of fibre bundles over S^n is given. As a special case, we have the following.

Consider S^3-bundles over S^4 with the rotation group SO(4) as structure group. The equivalence classes of such bundles are in bijective correspondence with elements of the group $\pi_3(\mathrm{SO}(4))$. This group is known to be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We construct an explicit isomorphism $\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\sim} \pi_3(\mathrm{SO}(4))$. For all $(h, j) \in \mathbb{Z} \oplus \mathbb{Z}$, define the map $f_{h,j} : S^3 \to \mathrm{SO}(4)$ as follows. We can endow $\mathbb{R}^4$ with the quaternion multiplication structure (and thus we can identify S^3 with the set of unit quaternions); using this multiplicaton structure, define $f_{h,j}(u) \cdot v = u^h v u^j$, for all $v \in \mathbb{R}^4$.

We denote by $\xi_{h,j}$ the sphere bundle corresponding to the map $f_{h,j}$. For each odd k we define the 7-manifold M_k to be the total space of $\xi_{h,j}$, where h, j are determined by $h - j = k$ and $h + j = 1$.

Applying the classification theorem to our situation allows us to give a more explicit construction of the total space M_k. The construction is as follows. We start with two copies of $\mathbb{R}^4 \times S^3$; then, we glue the two subsets $(\mathbb{R}^4 \setminus \{0\}) \times S^3$ together under the map

$$(u, v) \mapsto (u', v') = \left(\frac{u}{||u||^2}, \frac{u^h v u^j}{||u||} \right). \tag{4.1}$$

Note that this is a diffeomorphism because an explicit smooth inverse can be easily constructed. Since the gluing map is a diffeomorphism, the resulting space has a natural smooth structure (see [22]).

How do we see that this is an S^3-bundle over S^4? Recall that there exist natural stereographic projection maps from $S^4 \setminus \{\text{point}\}$ to $\mathbb{R}^4$. In particular, we obtain two such maps by removing the north pole and by removing the south pole. A point which corresponds to $u \in \mathbb{R}^4$ under one projection then corresponds to $u' = u/||u||^2$ under the other. Thus, when looking at how the copies of $(\mathbb{R}^4 \setminus \{0\})$ (the 'preliminary base spaces') get glued together, we see precisely that we end up with S^4.

Next, for a specific $u' = u/||u||^2$ on the 'glued base space', look at how the fibres of u and u' are glued together. We use, as before, the quaternion multiplication map, but now we also normalize by dividing by $||u||$. The result is indeed normalized, because we required that $h + j = 1$. Thus, we have indeed an S^3-bundle over S^4.

In his paper, Milnor constructs a diffeomorphism invariant λ for oriented, differentiable 7-manifolds satisfying the hypothesis that $H^3(M) = H^4(M) = 0$, and computes $\lambda(M_k)$ for all k; since $\lambda(M_k)$ does not coincide with $\lambda(S^7)$ for some k, we must conclude that not all M_k diffeomorphic to S^7.

Next, we give an elementary proof of the fact that the M_k are, in fact, mutually homeomorphic, by using some basic constructions from a field in mathematics called Morse theory.

4.2 Reeb's Theorem

In this section, we recall some basic facts about Morse theory, and then use a particular result known as Reeb's Theorem to prove that the M_k are mutually homeomorphic. The results in this section are based on [5] and [30].

Let $f : M \to \mathbb{R}$ be a smooth function on a compact manifold M. At a critical point of f, we can define the Hessian, or second-order derivative: Given a tangent vector v at the critical point z, let p be a path through $p(0) = z$ with $dp(0)/dt = v$. We then define $\ddot{f}(p(t)) = d^2 f(p(t))/dt^2$, which we will assume to be a well-defined definition, and we set the Hessian at z to be the quadratic function $\ddot{f}(p(0))$ of v.

Let M be a compact manifold. Then a **Morse function** on M is a smooth function $f : M \to \mathbb{R}$

such that all its critical points are **non-degenerate**: i.e., the Hessian at each critical point is non-degenerate. On the first sight, it is not immediately apparent that Morse functions exist at all. However, we have the following result, as found in [25, Thm. 2.20].

Theorem 4.2: Let M be a compact manifold. The set of Morse functions on M is a dense open subset of the topological space $C^\infty(M;\mathbb{R})$ endowed with the uniform topology.

A characterization of Morse functions becomes immediately apparent when we consider the following result.

Theorem 4.3 (Morse Lemma): Let p be a non-degenerate critical point for f. Then there exists a neighbourhood U of p together with a coordinate chart (known as a **Morse chart**) $\varphi : (U,p) \to (\mathbb{R}^n, 0)$, such that

$$f \circ \varphi^{-1}(x^1,\dots,x^n) = f(p) - \big((x^1)^2 + \cdots + (x^i)^2\big) + \big((x^{i+1})^2 + \cdots + (x^n)^2\big).$$

The value i in the above statement is known as the **index** of the critical point. As a corollary of this result, we see that the non-degenerate critical points of a function must be isolated. In particular, a Morse function on a compact manifold admits finitely many critical points. It thus makes sense to define the **Morse number** of a manifold (and in particular of a cobordism) to be the minimum over all Morse functions f of the number of critical points of f.

Let M be a compact manifold, and let $f : M \to \mathbb{R}$ be a Morse function. A **gradient-like vector field** adapted to f is a vector field X on M such that

- we have $(df)_x(X_x) \geq 0$ for all x, with equality if and only if x is a critical point, and
- in any Morse chart around a critical point, X coincides with the negative gradient of f: $f = f(p) - |x|^2 + |y|^2$, and $X = (-x_1,\dots,-x_\lambda, x_{\lambda+1},\dots,x_n)$.

A gradient-like vector field is a rather particular vector field, and it may not be obvious that such a field always exist. Fortunately, they do.

Proposition 4.4: All Morse functions have a gradient-like vector field adapted to it.

Proof: Let $c_1,\dots,c_r$ be the critical points on M. (There are only finitely many, since M is compact.) Let $h_1,\dots,h_r$ be Morse charts on $U_1,\dots,U_r$. Assume the images $\Omega_i = h_i(U_i)$ are disjoint. Given a function g on an open subset of $\mathbb{R}^n$, define a vector field X_i on Ω_i by pulling back $\operatorname{grad}(f \circ h_j)$ to M:

$$X_i(x) = \big(d_{h_i^{-1}(x)} h_i\big)\big(\operatorname{grad}_{h_i^{-1}(x)}(f \circ h_j)\big)$$

Assume that each Ω_i only covers c_i and no other critical point; in fact, we may assume Ω_i are disjoint. We can extend to a finite cover $(\Omega_i)_{1\leq i\leq N}$ of M by images of charts.

By definition, we know that $X_i \cdot f \geq 0$ on Ω_i, and X_i vanishes only at the critical points of f on U_i (so just on c_i, provided $i \leq r$). Hence the main result follows by a standard partition-of-unity argument. ■

Define $M^a = f^{-1}(]-\infty, a[)$. If a is a regular value of f, then M^a is a manifold with boundary, as follows from the Regular Value Theorem.

Theorem 4.5: Let a, b be two real numbers such that f does not have any critical value in the interval $[a, b]$. Suppose that $f^{-1}([a, b])$ is compact. Then M^b is diffeomorphic to M^a.

Proof: We use the flow of a gradient-like field X as map from M^a onto M^b. We fix a function $\rho : M \to \mathbb{R}$ with values

$$\begin{cases} \frac{1}{(df)_x(X)} & \text{on } f^{-1}([a,b]); \\ 0 & \text{outside of a compact neighbourhood of this subset.} \end{cases}$$

The vector field $Y = \rho X$ is zero outside of a compact set, hence it admits a well-defined global flow $\psi : \mathbb{R} \times M \to M$. For a fixed point $x \in M^a$, consider the function $s \mapsto f \circ \psi(s, x)$. If $\psi(s, x) \in f^{-1}([a, b])$ then we have

$$\begin{aligned} \frac{d}{ds} f \circ \psi(s,x) &= (df)_{\psi(s,x)}\left(\frac{d}{ds}\psi(s,x)\right) \\ &= (df)_{\psi(s,x)}\big(Y_{\psi(s,x)}\big) = 1. \end{aligned}$$

It follows that for $\psi(s, x) \in f^{-1}([a, b])$ we have $f \circ \psi(s, x) = s - f(x)$, hence the diffeomorphism $\psi(b - a, \bullet)$ sends M^a to M^b. ■

As a useful corollary, we have the result we were looking for.

Theorem 4.6 (Reeb's Theorem): Let M be a compact manifold, and suppose that there exists a Morse function on M that has only two critical points. Then M is homeomorphic to a sphere.

Proof: We may assume that f takes values on $[0, 1]$. Then for ε sufficiently small, the Morse Lemma asserts that $f^{-1}([0, \varepsilon])$ and $f^{-1}([1-\varepsilon, 1])$ are disks. By the previous theorem, the sublevel sets M^ε and $M^{1-\varepsilon}$ are diffeomorphic. Hence $M^{1-\varepsilon}$ is also a disk, and thus M is the union of two disks glued along their boundaries. This gluing is not strong enough to guarantee that M be diffeomorphic to a sphere; by an explicit mapping, however, we do know that two disks, glued along their boundaries, are *homeomorphic* to the sphere. ■

The above result gives us an easy method to ensure that a given manifold is homeomorphic to the standard sphere. It turns out that something much stronger holds true: the theorem remains true even if the two critical points are not assumed to be non-degenerate.[42] For us, however, the above result will be fine. We go on to explicitly construct such a Morse function on M_k.

Replacing the coordinates (u', v') by (u'', v'), where $u'' = u'(v')^{-1}$, we may consider the function $f : M_k \to \mathbb{R}$ defined by

$$f(u,v) = \frac{\operatorname{Re} v}{\sqrt{1+||u||^2}} = \frac{\operatorname{Re} u''}{\sqrt{1+||u''||^2}}.$$

Let us first see whether this is well-defined. We have $||v|| = 1$ and

$$||v'|| = \frac{||u^h v u^j||}{||u||} = ||u||^{h+j-1},$$

and this is 1 since we assumed $h + j = 1$. It follows, then, that

$$\sqrt{1+||u''||^2} = \sqrt{1+\frac{1}{||u||^2}} = \frac{\sqrt{1+||u||^2}}{||u||}.$$

Thus we get

$$f(u,v) = \frac{\operatorname{Re} v}{\sqrt{1+||u||^2}} = \frac{(\operatorname{Re} u'')||u||}{\sqrt{1+||u||^2}}.$$

We will see these definitions coincide on overlaps, since $(\operatorname{Re} u'')||u|| = \operatorname{Re} v$ (this suffices by Equation 4.1):

$$\begin{aligned}
2\operatorname{Re} u'' &= u'' + ||u''||^2 (u'')^{-1} \\
&= u'(v')^{-1} + ||u''||^2 \big(u'(v')^{-1}\big)^{-1} \\
&= \frac{1}{||u||}\big(u u^{-(1-h)} v^{-1} u^{-h} + u^h v u^{1-h} u^{-1}\big) \\
&= \frac{2}{||u||}\operatorname{Re}\big((||u||^{-1}u)^h v (||u||^{-1}u)^{-h}\big) \\
&= \frac{2}{||u||}\operatorname{Re} v.
\end{aligned}$$

Next, we show that these functions are as required. In other words, they have two critical points which are non-degenerate. Restrict your attention to the first chart. We denote $u = (x^1, x^2, x^3, x^4)$ and $v = (y^1, y^2, y^3, y^4)$. Since $||v||^2 = 1$, we may write

$$f(u,v) = \frac{\sqrt{1-(y^2)^2-(y^3)^2-(y^4)^2}}{\sqrt{1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2}}.$$

Let us take the positive root now — we will see we won't lose any generality. We find

$$\frac{\partial f}{\partial y^i} = \frac{-y^i}{\left(1-(y^2)^2-(y^3)^2-(y^4)^2\right)^2\sqrt{1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2}};$$
$$\frac{\partial f}{\partial x^i} = \frac{(-x^i)\sqrt{(1-(y^2)^2-(y^3)^2-(y^4)^2}}{\left(1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2\right)^{3/2}}.$$

We see there are only two solutions for $df = 0$, namely the points $(u, v) = ((0, 0, 0, 0), (\pm 1, 0, 0, 0))$. Next, we do the exact same thing in the other chart to conclude that there are indeed only two critical points.

We show they are non-degenerate, simply by explicitly taking the second-order derivatives. Let's show one.

$$\begin{aligned}\frac{\partial^2 f}{\partial (y^i)^2}(0,\pm 1) =& \left(\frac{1}{\sqrt{1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2}}\right)\times \\ & \left.\left(\frac{-1}{\sqrt{1-(y^2)^2-(y^3)^2-(y^4)^2}} + \frac{-(y^i)^2}{\left(1-(y^2)^2-(y^3)^2-(y^4)^2\right)^{3/2}}\right)\right|_{0,\pm 1} \\ =& -1.\end{aligned}$$

You will then probably believe me if I say that

$$\frac{\partial^2 f}{\partial x^i \partial y^j}(0,\pm 1) = 0; \quad \frac{\partial^2 f}{\partial (x^i)^2}(0,\pm 1) = -1; \quad \frac{\partial^2 f}{\partial x^i \partial x^j}(0,\pm 1) = 0; \quad \frac{\partial^2 f}{\partial y^i \partial y^j}(0,\pm 1) = 0.$$

The result is that the Hessian is a diagonal matrix, at least with respect to the coordinates we just considered. Since the choice of coordinates of irrelevant, we conclude that f is a Morse function. The conditions of Reeb's Theorem are satisfied, so that we may apply it to our situation.

4.3 The invariant λ

For every closed, oriented 7-manifold M with orientation generator $\mu \in H_n(M)$ satisfying $H^3(M) = H^4(M) = 0$, we define a residue class $\lambda(M)$ modulo 7. As we saw in Section 3.1, every closed oriented 7-manifold is the boundary of an 8-manifold B. The invariant λ will be defined as a function of the signature and the first Pontryagin class of B.

By the hypothesis $H^3(M) = H^4(M) = 0$, the inclusion homomorphism $i : H^4(B, M) \to H^4(B)$ is an isomorphism; thus, we can define a 'Pontryagin number'

$$q(B) = \left\langle \nu, \left(i^{-1}p_1(TB)\right)^2 \right\rangle.$$

Given a manifold M as above, we will be interested in the seemingly arbitrary number $2q(B) - \tau(B)$, where $\tau(B)$ is the signature of B (see Section 2.4). Of course, the question that should now come to the reader's mind is "Is this result independent of B?", and the answer is no, although we have something slightly weaker that we can work with.

Theorem 4.7: The residue class of $2q(B) - \tau(B)$ modulo 7 does not depend on the choice of the manifold B.

This theorem relies on a deep result known as the Hirzebruch Signature Theorem, whose proof we shall see in Chapter 5.

Proof (of Theorem 4.7): Let B_1, B_2 be two manifolds with boundary M. Then consider the closed 8-manifold C by taking the disjoint union of B_1 and B_2 and gluing them along their boundaries via the identity map. Choose an orientation ν for C that is consistent with the orientation ν_1 of B_1. Let $q(C)$ denote the Pontryagin number $\langle \nu, p_1^2(C) \rangle$.

The Hirzebruch Signature Theorem implies that

$$\tau(C) = \left\langle \nu, \frac{1}{45}\big(7p_2(C) - p_1^2(C)\big) \right\rangle, \tag{4.8}$$

and therefore

$$45\tau(C) + q(C) \equiv 2q(C) - \tau(C) \equiv 0 \mod 7.$$

Lemma 4.9: Using the notation as above, we have

$$\tau(C) = \tau(B_1) - \tau(B_2); \quad q(C) = q(B_1) - q(B_2).$$

Proof: (Sketch) The cohomology ring of C is the sum of the cohomology rings of $H^*(B_1, M)$ and $H^*(B_2, M)$, except in the top dimension. Furthermore, by the hypothesis $H^3(M) = H^4(M) = 0$, $H^4(B_1, M) \cong H^4(B_1)$ and $H^4(B_2, M) \cong H^4(B_2)$. This gives rise to the commutative diagram

$$\begin{tikzcd} H^4(C, M) \arrow[r, "h"] \arrow[d, "j"'] & H^4(B_1, M) \oplus H^4(B_2, M) \arrow[d, "i_1 \oplus i_2"] \\ H^4(C) \arrow[r, "k"'] & H^4(B_1) \oplus H^4(B_2) \end{tikzcd}$$

Given $\alpha = j \circ h^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C)$ we have

$$\langle \nu, \alpha^2 \rangle = \langle \nu, j \circ h^{-1}(\alpha_1^2 \oplus \alpha_2^2) \rangle = \langle \nu_1 \oplus -\nu_2, \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle, \tag{$\star$}$$

from which the formula for $q(C)$ becomes visible. Next, we set $\alpha_1 = i_1^{-1}p_1(B_1)$ and $\alpha_2 = i_2^{-1}p_1(B_2)$. Then by the relation $k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$ we find that

$$j \circ h^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C).$$

Equation $(\star)$ now shows us that

$$\langle \nu, p_1^2(C)\rangle = \langle \nu_1, \alpha_1^2\rangle - \langle \nu_2, \alpha_2^2\rangle,$$

which is precisely the formula for $\tau(C)$. ■

From the above lemma, the main result now follows. ■

Determining $\lambda(M_k)$

In this section, we explicitly compute the invariant λ associated to the manifolds M_k we constructed earlier. First, we consider the added value of finding these invariants.

Proposition 4.10: If $\lambda(M) \neq 0$, then M is not diffeomorphic to S^7.

Proof: If we reverse orientation on M we get $\lambda(-M) = -\lambda(M)$. Now suppose M admits an orientation-reversing diffeomorphism $f : M \to M$, and let $M = \partial B$. Then f induces an orientation-preserving diffeomorphism $M \to \partial(-B)$, and hence we can identify M with $\partial(-B)$. We conclude that $\lambda(M) = \lambda(-M) = 0$. The proof follows since the standard S^7 admits an orientation-reversing diffeomorphism. ■

We now go on to actually calculate the invariants. Of course, in order to actually calculate the invariants, we need a way construct 8-manifolds B_k which have boundary M_k. The obvious way would be to replace every fibre S^3 with D^4, simply by 'filling in' the fibres. Here one should be careful: not every smooth sphere bundle bounds a smooth disc bundle. In our case, things go fine, because M_k have structure group SO(4). More explicitly, the gluing map 4.1 can be linearly extended to obtain a rank-4 vector bundle, and we can consider its corresponding 'sphere subbundle' and 'disc subbundle'.

Let us first consider one specific case, from which we 'linearly' compute the invariants for the other cases. In the specific case that $k = 1$, Milnor states the following (without proof):

Lemma 4.11: The space B_1 is diffeomorphic to the quaternion projective plane $\mathbb{HP}^2$ with an 8-cell removed.

Proof: The space $\mathbb{HP}^2$ is the quotient of $\mathbb{H}^3 \setminus \{0\}$ under the identification $(u, v, w) \sim (xu, xv, xw)$ for all $x \in \mathbb{H} \setminus \{0\}$. Identifying S^4 with $\mathbb{HP}^1$, there exists a natural fibre bundle over S^4 with fibre $\mathbb{H}$ and total space $\mathbb{HP}^2 \setminus \{[0, 0, 1]\}$, which is obtained by setting $\pi[u, v, w] = [u, v]$. There is an easy way to compare this bundle with B_1: we consider two 'hemispheres' of $\mathbb{HP}^1$, namely $H_1 = \{[u, 1] : |u| \leq 1\}$ and $H_2 = \{[1, v] : |v| \leq 1\}$. Note that points on their overlap correspond via the identification $[u, 1] \sim [1, u^{-1}]$. We write down the local trivializations, and we look what happens on the 'equator' (the points $[u, 1]$ with $|u| = 1$). The trivializations are $\varphi_1 : H_1 \times \mathbb{H} \to \pi^{-1}(H_1)$, $\varphi_2 : H_2 \times \mathbb{H} \to \pi^{-1}(H_2)$, and they are given by

$$\varphi_1([u, 1], w) = [u, 1, w], \quad \varphi_2([1, v], w) = [1, v, w].$$

The transition function is then a map $(D_1 \cap D_2) \times \mathbb{H} \to (H_1 \cap H_2) \times \mathbb{H}$ given by

$$\varphi_2 \circ \varphi_1^{-1}([u, 1], w) = \varphi_2^{-1}([u, 1, w]) = \varphi_2^{-1}\big([1, u^{-1}, u^{-1}w]\big) = \big([1, u^{-1}], u^{-1}w\big).$$

We see that this is precisely the gluing map $f_{1,0}$.

Upon removing an 8-cell centred at the point $[0, 0, 1]$ the fibres at each point get 'restricted': More precisely, if we remove the points $\{[u, v, 1] : |u|^2 + |v|^2 < 1\}$, the fibre at a point $[u, v]$ becomes the set $\{[u, v, w] : |u|^2 + |v|^2 \geq |w|^2\}$. But we see that this is homeomorphic to D^4; furthermore, the transition maps don't change. We conclude that the total space $\mathbb{HP}^2 \setminus \{8\text{-cell}\}$ is a D^4 bundle over S^4 whose transition map coincides with that of B_1. Thus the spaces are in fact the same. ■

Lemma 4.12: The Pontryagin class $p_1(\xi_{h,j})$ equals $c(h - j)\mu$ for some constant c, where μ is the standard generator for $H^4(S^4)$.

Proof: (Sketch) First, recall from Section 4.1 that we constructed the maps $f_{h,j}$ as a homomorphism $\mathbb{Z} \oplus \mathbb{Z} \to \pi_3(\mathrm{SO}(4))$. Next, we have the following lemma, as found in [34, Thm. 5.6]:

Lemma 4.13: Every rank-n vector bundle over a paracompact base space admits a bundle map into the tautological bundle γ^n of the Grassmannian $G_n(\mathbb{R}^\infty)$.

We apply this to our situation to find that rank-4 vector bundles over S^4 are the pull-back of the tautological bundle on G_4. This induces a homomorphism $\pi_3(\mathrm{SO}(4)) \to \pi_4(G_4)$; let us denote by $F_{h,j}$ the bundle map corresponding to $\xi_{h,j}$. Finally, we consider the map $\pi_4(G_4) \to H^4(S^4)$ obtained by sending $[F_{h,j}]$ to $p_1\big(F_{h,j}^*(\gamma^4)\big)$, which is a homomorphism by naturality of the Pontryagin class. The resulting composition of homomorphisms $(h,j) \mapsto f_{h,j} \mapsto F_{h,j} \mapsto p_1\big(F^*_{h,j}(\gamma^4)\big)$ is precisely the map $(h,j) \mapsto p_1(\xi_{h,j})$, hence the latter is linear in h and j.

An alternative way to see this is by noting that $\xi_{h+h',j+j'}$ can be seen as a connected sum of $\xi_{h,j}$ and $\xi_{h',j'}$; one can then show directly that the first Pontryagin class is linear under such connected sums.

Next, consider the effect of reversing the fibre orientation. Interpreting the fibres as unit quaternions, this is equivalent to conjugation by the map $v \mapsto v^{-1}$. It is easily seen that under this conjugation, the sphere bundle $\xi_{h,j}$ gets sent to $\xi_{-j,-h}$. Thus $p_1(\xi_{h,j}) = p_1(\xi_{-j,-h})$, and it follows that $p_1(\xi_{h,j}) = c(h-j)\mu$ for some constant c. ■

Lemma 4.14: The invariant $\lambda(M_k)$ is precisely the residue class modulo 7 of $k^2 - 1$.

Proof: We can associate to each sphere bundle $M_k \to S^4$ a disk bundle B_k, with projection map ρ_k, whose total space is a smooth manifold with boundary M_k. The cohomology group $H^4(B_k)$ is generated by some $\alpha = \rho_k^*(\mu)$. Choose orientations ν_M, ν_B for M_k, B_k so that $\langle \nu, (i^{-1}\alpha)^2 \rangle = +1$; it immediately follows that the signature $\tau(B_k)$ becomes $+1$.

Next, we determine the first Pontryagin class of B_k. The tangent bundle splits into a 'vertical' piece (the pull-back of M_k) and a 'horizontal' piece (the pull-back of TS^4). Thus we have

$$TB_{h,j} = \rho_k^*(TS^4 \oplus M_k).$$

Therefore, by the Whitney product formula together with Lemma 4.12, we find that

$$p_1(B) = ck\alpha.$$

As we saw in Example 2.11, the Pontryagin class $p_1(\mathbb{HP}^2)$ is, up to a minus sign, twice the generator of $H^4(\mathbb{HP}^2)$, so the constant c must be ± 2.

This completes the proof of our lemma. Indeed, we have $q(B_k) = \langle \nu, i^{-1}(\pm 2k\alpha))^2 \rangle = 4k^2$, and thus $\lambda(M_k) \equiv 2q(B_k) - \tau(B_k) \equiv k^2 - 1 \mod 7$. ■

Theorem 4.15 (The Theorem to End All Theorems): If $k^2 - 1 \not\equiv 0 \mod 7$ then M_k is homeomorphic but not diffeomorphic to S^7. In particular, let us consider the cases where $k = 1, 3, 5, 7, 9, 11, 13$. For these k, respectively, we find that $k^2 - 1 \mod 7 = 0, 1, 3, 6, 3, 1, 0$. This gives four different results, hence there are *at least* four different structures on S^7.

Chapter 5

Hirzebruch Signature Theorem

The main result which led to the eventual discovery of exotic structures is the Hirzebruch Signature Theorem, a specific case of which we have used without proof in Theorem 4.7. In this chapter, we present the main result, and we give a sketch of the proof.

Coefficients of cohomology will henceforth be assumed to be in $\mathbb{Q}$.

5.1 Formulation

We first give the relevant definition necessary to formulate the Hirzebruch Signature Theorem. The contents of this section are based on [34, §19].

Let A^* be a commutative graded algebra over some commutative ring Λ with unit. To each A^* we associate the commutative ring A^Π consisting of all formal sums $a_0 + a_1 + \cdots$ with $a_i \in A^i$; by A_1^Π we denote the subring of elements of A^Π with leading term 1. The product of two such units is then given by the formula

$$(1 + a_1 + a_2 + \cdots)(1 + b_1 + b_2 + \cdots) = 1 + (a_1 + b_1) + (a_2 + a_1 b_1 + b_2) + \cdots .$$

Now consider a sequence of polynomials $K_1(x_1), K_2(x_1, x_2), \ldots$ with coefficients in Λ such that $x_i \in A^i$ is of degree i, and each K_n is of degree n. Given $a \in A_1^\Pi$, define a new element $K(a) \in A_1^\Pi$ by the formula

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots .$$

We say the K_n form a **multiplicative sequence** whenever we have $K(ab) = K(a)K(b)$ for all such Λ-algebras A^* and for all $a, b \in A_1^\Pi$.

Proposition 5.1: Given a formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots$ with coefficients in Λ, there exists one and only one multiplicative sequence (K_n) with coefficients in Λ such that $K(1+t) = f(t)$.

Example 5.2: We look at the second element K_2 of the multiplicative sequence belonging to an arbitrary formal power series $f(t) = 1 + \lambda_1 t + \cdots$. Let us expand the lower-order terms of the product $K(1+x_1)K(1+x_2)$:

$$\begin{aligned} K(1+x_1)K(1+x_2) &= K(1+x_1+x_2+x_1x_2) \\ &= 1 + K_1(x_1+x_2) + K_2(x_1+x_2, x_1x_2) + \cdots . \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} K(1+x_1)K(1+x_2) &= (1+\lambda_1 x_1 + \lambda_2 x_2^2 + \cdots)(1 + \lambda_1 x_2 + \lambda_2 x_2^2 + \cdots) \\ &= 1 + \lambda_1(x_1+x_2) + \lambda_1^2 x_1 x_2 + \lambda_2(x_1^2 + x_2^2) + \cdots . \end{aligned}$$

Comparing the terms of degree 2, then setting $a_1 = x_1 + x_2$, $a_2 = x_1x_2$, we obtain

$$K_2(a_1, a_2) = \lambda_2 a_1^2 + (\lambda_1^2 - 2\lambda_2)a_2.$$

In particular, we can verify that if we apply this example to the specific power series defined below, we obtain Equation 4.8 which we used without proof in the previous chapter. We also note that the proof of Proposition 5.1 follows by generalizing this construction to all K_n.

Now consider some multiplicative sequence of polynomials $\{K_n(x_1, \ldots, x_n)\}$ with rational coefficients. Let M be a compact oriented n-manifold. The K-**genus** $K[M]$ is zero if n is not divisible by 4, and is otherwise equal to the rational number

$$K_n[M] = \langle K_n(p_1, \ldots, p_n), \mu_n \rangle$$

where p_i denotes the i-th Pontryagin number of the tangent bundle.

We now formulate the main result we will be striving to prove.

Theorem 5.3 (Hirzebruch Signature Theorem): Let (L_k) be the multiplicative sequence of polynomials belonging to the power series

$$f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + \frac{(-1)^{k-1}2^{2k}B_k t^k}{(2k)!} + \cdots .$$

Then the signature $\tau(M)$ of any compact oriented manifold M coincides with the L-genus $L[M]$.

Recall from Section 2.4 that the signature defines a ring homomorphism from the cobordism ring to the real numbers (in fact, to the integers). A nice first step would be to prove the same for the L-genus, so that we need only check a set of generators for Ω_n^{SO}.

Proposition 5.4: For any multiplicative sequence (K_n) with rational coefficients, the correspondence $M \mapsto K[M]$ defines a ring homomorphism from the cobordism ring Ω_*^{SO} to $\mathbb{Q}$.

Proof: The mapping is evidently additive, and the K-genus of a boundary is zero. For a product manifold $M \times M'$ with total Pontryagin class congruent to $p \cdot p'$ modulo elements of order 2 (by the Whitney product formula), we have $K(p \cdot p') = K(p)K(p')$, and hence

$$\langle K(p \cdot p'), \mu \times \mu' \rangle = (-1)^{nn'} \langle K(p), \mu \rangle \langle K(p'), \mu' \rangle.$$

Since n, n' are divisible by 4, the sign is clearly $+1$ and so the result follows. ■

Combining this with Proposition 3.2, we see that it suffices to check this theorem on a set of generators for the algebra $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$. This gives rise to the following question: Is there a nice set of generators? The answer will be the main topic of the next section.

5.2 The cobordism ring Ω_n^{SO}

In Section 3.1 we explicitly showed the structures of the first few cobordism groups, both oriented and oriented. I promised the reader that I would present a proof of at least part of the structures. The author always keeps his promises, for he is a man of integrity. We spend the next section proving the following result.

Theorem 5.5 (Thom Cobordism Theorem): The oriented cobordism group Ω_n^{SO} is finite for $n \not\equiv 0 \mod 4$, and is a finitely generated group with rank equal to $\mathrm{Part}(k)$ when $n = 4k$.

The above result answers the question whether or not a finite set of generators can be found for the different cobordism groups. Of course, we have yet to find an explicit set of generators, but this will turn out not to be too hard, as we will see in the next section.

The results of this section are based on [34, §18], unless otherwise specified.

Let ξ be a rank-k vector bundle over B, endowed with a Euclidean metric, and let A be the subset of the total space consisting of all vectors v with $|v| \geq 1$. Then the space E/A will be called the **Thom space** $T(\xi)$ (or T). It has a preferred base point, denoted by ∞_T.

Proposition 5.6: If ξ is an oriented bundle of rank k, then $H_n(T, \infty_T)$ is canonically isomorphic to $H_{n-k}(B)$.

Proof: Note that T_0 is contractible, so that by the exact sequence belonging to (T, T_0, ∞_T) we find $H_n(T, \infty_T) \cong H_n(T, T_0)$. By excision, the latter is itself isomorphic to $H_n(E, E_0)$; composing this with the Thom Isomorphism $H_n(E, E_0) \xrightarrow{\sim} H_{n-k}(B)$, the result follows. ■

We use the above proposition to give a relation between homotopy and homology groups. The value of this relation will follow in a moment.

Theorem 5.7: If T is the Thom space of an oriented rank-k vector bundle over a finite CW complex B, then $\pi_{n+k}(T) \cong H_n(B)$ for all $n < k - 1$.

Proof: The proof relies on a variation of the Hurewicz Theorem (2.8).

Lemma 5.8: Let X be a finite $(k-1)$-connected CW complex $(k \geq 2)$. Then for all $r < 2k-1$, the Hurewicz homomorphism $\pi_r(X) \to H_r(X)$ has finite kernel and co-kernel.

The proof of the lemma, as found in [34, Thm. 18.3], relies on several results by Serre. Let us sketch why the Thom space of a rank-k vector bundle is $(k-1)$-connected, so that we can apply the lemma to obtain the desired result.

For an open n-cell e_α of B, the inverse image $\pi^{-1}(e_\alpha) \cap E_0$ is an open cell of dimension $n + k$; these open cells are mutually disjoint, and they cover the set $E \setminus A \cong T \setminus \infty_T$.

We construct the characteristic map of $\pi^{-1}(e_\alpha)$ in the Thom space. Let $f : D^n \to B$ be the characteristic map of e_α. Then $f^*(\xi)$ is trivial by the Covering Homotopy Theorem (see [47, §11.6]). It follows that the set of vectors of length ≤ 1 in the total space of $f^*(\xi)$ can be identified with $D^n \times D^k$. The composition $D^n \times D^k \subseteq E(f^*(\xi)) \to E(\xi) \to T(\xi)$ forms the required characteristic map. We conclude that T is a finite CW complex with no cells in dimensions 1 through $k - 1$. Thus T is $(k-1)$-connected and we can apply the lemma. ■

We show how this result can be applied to the computation of cobordism groups.

Once again let ξ be an oriented rank-k vector bundle, with base space B and total space E. Given a continuous map f from the sphere S^m to the Thom space T, we would like to approximate f by a 'smooth' map, in the sense that we construct a function g which coincides with f on $f^{-1}(\infty_T) = g^{-1}(\infty_T)$, and which is smooth on $T \setminus \infty_T$.

Theorem 5.9: Using the notation as above, every continuous $f : S^m \to T$ is homotopic to g. We may furthermore make sure that the pre-image $g^{-1}(B)$ is a smooth $(m-k)$-manifold. Its oriented cobordism class depends only on the homotopy class of g. Hence the correspondence $g \mapsto g^{-1}(B)$ gives rise to a homomorphism $\pi_m(T, \infty_T) \to \Omega^{\mathrm{SO}}_{m-k}$.

The necessary techniques to construct this approximation are described in [47, §6.7].

In his work on cobordisms, Thom applied the above result to a particular bundle, namely the tautological oriented rank-k vector bundle $\widetilde{\gamma}^k$ over the oriented Grassmannian $\widetilde{G}_k(\mathbb{R}^\infty)$. The result is as follows.

Theorem 5.10: For $k > n+1$, the homotopy group $\pi_{n+k}\big(T(\widetilde{\gamma}^k), \infty_T\big)$ is canonically isomorphic to the oriented cobordism group Ω^{SO}_n.

The uninitiated reader should not confuse $\widetilde{\gamma}^k$ over $\widetilde{G}_k$ with the vector bundle γ^k over G_k that we saw in Chapter 3. Interestingly, the latter vector bundle can be applied similarly to the above theorem to obtain a canonical isomorphism $\pi_{n+k}\big(T(\gamma^k), \infty_T\big) \xrightarrow{\sim} \Omega_n$.

Proof (of Theorem 5.10): (Sketch) We omit the general result, and only show that the homomorphism $\pi_{n+k}\big(T(\widetilde{\gamma}^k_p)\big) \to \Omega^{\mathrm{SO}}_n$ is surjective for $k \geq n$, $p \geq n$. We do this as follows. Let M be a compact oriented n-manifold. By the Whitney Embedding Theorem, we can embed M in $\mathbb{R}^{n+k}$. Let U be a neighbourhood of M in $\mathbb{R}^{n+k}$ which is diffeomorphic to the total space of the normal bundle N^k. (Such a neighbourhood exists by the Tubular Neighbourhood Theorem.) Recall from Lemma 4.13 that we can construct a map from the normal bundle into the tautological vector bundle of the Grassmannian $\widetilde{G}_{n+k}$; this gives us a map $E(N^k) \to E\big(\widetilde{\gamma}^k_n\big) \subseteq E\big(\widetilde{\gamma}^k_p\big)$. Composing with the canonical map $E\big(\widetilde{\gamma}^k_p\big) \to T\big(\widetilde{\gamma}^k_p\big)$, we obtain a map $g : U \to T\big(\widetilde{\gamma}^k_p\big)$. This map is transverse to the zero cross-section of E, hence its pre-image is a manifold; in fact, as one can verify, the pre-image is precisely M.

Next, simply extend g to the one-point compactification of $\mathbb{R}^{n+k}$ by mapping the point at ∞ to the point ∞_T of the Thom space. The resulting map $S^{n+k} \to T\big(\widetilde{\gamma}^k_p\big)$ gives rise to the cobordism class M, hence surjectivity follows. ■

Corollary 5.11: The oriented cobordism group Ω^{SO}_n is finite for $n \not\equiv 0 \mod 4$, and is finitely generated with rank $\mathrm{Part}(k)$ when $n = 4k$.

Proof: By the result above, the n-th cobordism group is the image of $\pi_{n+k}\big(T\big(\widetilde{\gamma}^k_p\big)\big)$ for some sufficiently large k, p. The latter group admits a homomorphism to $H_n\big(\widetilde{G}_k(\mathbb{R}^{k+p})\big)$ by Theorem 5.7.

Thus the result follows if we find the rank of the homology group $H_n\big(\widetilde{G}_k(\mathbb{R}^{k+p})\big)$. Computation of the cohomology of the Grassmannians is an essentially classical problem; the details are omitted, since the techniques are irrelevant for the rest of this thesis. The interested reader can find the computation in, for example, [15, Thm. 4D.4]. In any case, the ranks of the homology groups are precisely as in the statement of the corollary, hence the result follows. ■

5.3 Proof of the Signature Theorem

In the previous section, we explicitly the determined the rank of the oriented cobordism groups. We have not yet provided an explicit set of generators for these groups; it turns out that finding such a set is not too hard. The result is as follows.

Theorem 5.12: A basis for the vector space $\Omega_{4k}^{\mathrm{SO}} \otimes \mathbb{Q}$ is given by the products

$$\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r},$$

where $\{i_1, \ldots, i_r\}$ ranges over all partitions of k.

Proof: The naive approach to proving this result is as follows. Consider the $\mathrm{Part}(n) \times \mathrm{Part}(n)$ matrix of Pontryagin numbers given by $\big(p_{i_1} \cdots p_{i_r}(\mathbb{CP}^{2j_1} \times \cdots \times \mathbb{CP}^{2j_s})\big)$, whose explicit values have been calculated in Example 3.27. For a given n one can easily verify that this matrix is non-singular, so that the result follows by Corollary 3.29.

A more general approach is as follows. You may recall that we said in Section 3.7 that the Chern numbers (and Pontryagin numbers) are unrelated, in the sense that there is no linear relation between them. We can make this more precise to find the result as given in [34, Thm. 16.8], which considers, for given oriented manifolds $M^4, \ldots, M^{4n}$, the $\mathrm{Part}(n) \times \mathrm{Part}(n)$ matrix $\big(p_{i_1} \cdots p_{i_r}(\mathbb{M}^{4j_1} \times \cdots \times \mathbb{M}^{4j_s})\big)$. This result gives an easier to verify requirement for this matrix to be non-singular. Using this method, our result easily follows in the general case. ■

We thus need only verify the Hirzebruch Signature Theorem for the above set of generators. In Example 2.12 we calculated the signature of the space $\mathbb{CP}^{2k}$; by Proposition 3.2, the signature of the spaces $\mathbb{CP}^{2i_1} \times \cdots \times \mathbb{CP}^{2i_r}$ follows. Thus we need only show the following.

Lemma 5.13: $L[\mathbb{CP}^{2k}] = 1$.

Proof: Recall from Example 3.26 that $p(\mathbb{CP}^{2k}) = (1 + a^2)^{2k+1}$. By the definition of L we have

$$L(\mathbb{CP}^{2k}) = \big(L(1 + a^2)\big)^{2k+1} = \left(\frac{a}{\tanh a}\right)^{2k+1}.$$

To determine $L[\mathbb{CP}^{2k}]$, we need only know the coefficient of a^{2k} of the above series, since this will be the only non-zero contribution. If we consider a for now to be a complex-valued variable, then the a^{2k} term can be found by dividing by $2\pi i a^{2k+1}$ and then integrating around the origin:

$$\text{coefficient} = \frac{1}{2\pi i}\oint \frac{a^{2k+1}}{a^{2k+1}(\tanh a)^{2k+1}}\,da = \frac{1}{2\pi i}\oint \frac{1}{(\tanh a)^{2k+1}}\,da.$$

Introducing the substitution $u = \tanh a$, we find that $du/da = 1 - u^2$, hence $da = du/(1-u^2) = (1 + u^2 + u^4 + \cdots)\,du$, and thus we get

$$\text{coefficient} = \frac{1}{2\pi i}\oint \frac{1 + u^2 + u^4 + \cdots}{u^{2k+1}}\,du = \frac{1}{2\pi i}\oint \frac{u^{2k}}{u^{2k+1}}\,du = 1,$$

as follows by the Cauchy Integral Theorem. From this we conclude the result, hence the Signature Theorem is proven. ■

Chapter 6

The h-Cobordism Theorem

The full classification of exotic structures on the 7-sphere was established by Michel Kervaire and John Milnor, who showed that the exotic 7-spheres are the non-trivial elements of a cyclic group of order 28. Rather than explicitly looking at the group of smooth structures, Kervaire and Milnor shifted their focus to the group of homotopy spheres modulo an equivalence relation called h-cobordism.

It is the h-Cobordism Theorem which will tell us that these two perspectives are, in fact, equivalent. The theorem was first proven by Stephen Smale for which he received the Fields Medal. The result is as follows.

Theorem 6.1 (The h-Cobordism Theorem): Let (W, V, V') be a cobordism with the following properties.

- W, V and V' are simply-connected;
- $H_*(W, V) = 0$;
- $\dim W \geq 6$.

Then W is diffeomorphic to $V \times [0, 1]$.

We will provide a rough exposition of Milnor's proof, as found in [45]. While lengthy, the techniques that will be used are relatively intuitive. We begin by constructing a Morse function of W, and simplify the given Morse function until all critical points are eliminated. This elimination is motivated by the result that W admits a Morse function without critical points if and only if W is diffeomorphic to $V \times [0, 1]$. Without further ado, let us take a look at the proof.

6.1 Preliminary constructions

We start with the following result.

Theorem 6.2: If the Morse number of a cobordism is zero, then the cobordism is trivial.

Proof: Let $f : W \to [0,1]$ be a Morse function without critical points. By Proposition 4.4, there exists a gradient-like field ξ for f; this gradient-like field is strictly positive. Use the flow of this gradient-like field to construct a diffeomorphism $M_0 \times [0,1] \to W$. The details are similar to the proof of Theorem 4.6. ■

Corollary 6.3 (Collar Neighbourhood Theorem): Let W be any compact manifold with boundary. Then there exists a neighbourhood of ∂W which is diffeomorphic to $\partial W \times [0,1[$.

The corollary follows by constructing a smooth function on W without critical points in a neighbourhood of ∂W, then using Theorem 6.2 to construct the required diffeomorphism.

An **elementary cobordism** is a cobordism possessing a Morse function with one critical point. In this section, we will take a closer look at such cobordisms.

Let (W, V, V') be an n-dimensional cobordism with Morse function $f : W \to \mathbb{R}$ and gradient-like field ξ for f. Suppose $p \in W$ is a critical point, and $V_0 = f^{-1}(c_0)$, $V_1 = f^{-1}(c_1)$ are levels such that $c_0 < f(p) < c_1$ and that $f(p)$ is the only critical value in $[c_0, c_1]$. By $B^n(r)$ we denote the open ball of radius r in $\mathbb{R}^n$, and we set $B^n(1) = B^n$.

Let (U, φ) be a Morse chart of p. We may shrink U so that φ maps to $B^n(2\varepsilon)$. Define $V_\varepsilon = f^{-1}(f(p) + \varepsilon^2)$ and $V_{-\varepsilon} = f^{-1}(f(p) - \varepsilon^2)$, where ε is small enough so that $V_{-\varepsilon}$ lies between V_0 and $f^{-1}(c)$ and V_ε lies between $f^{-1}(c)$ and V_1.

We will go on the define what we call the **characteristic embedding** $\psi_L : S^{\lambda-1} \times B^{n-\lambda} \to V_0$. First, define an embedding $\psi : S^{\lambda-1} \times B^{n-\lambda} \to V_{-\varepsilon}$ by $\psi(u, cv) = \varphi^{-1}(\varepsilon u \cosh c, \varepsilon v \sinh c)$. Starting at the point $\psi(u, cv)$ in $V_{-\varepsilon}$, the integral curve of ξ is a non-singular curve leading from $\psi(u, cv)$ back to some point in V_0, which we denote $\psi_L(u, cv)$.

Define the **left-hand sphere** S_L of p in V_0 to be the image $\psi_L(S^{\lambda-1} \times \{0\})$. The **left-hand disc** D_L is the smoothly embedded disc with boundary S_L defined to be the union of the segments of those integral curves beginning in S_L, and ending at p.

The **right-hand sphere** is defined similarly. Define the embedding $\psi : B^\lambda \times S^{n-\lambda-1} \to V_\varepsilon$ by

$\psi(cu, v) = \varphi^{-1}(\varepsilon u \sinh c, \varepsilon v \cosh c)$, then follow the curve of ξ to V_1. The **right-hand disk** D_R is the union of segments on integral curves of ξ beginning at p and ending in S_R.

Given a manifold V of dimension $n-1$, and an embedding $\psi : S^{\lambda-1} \times B^{n-\lambda} \to V$, let $\chi(V, \psi)$ denote the quotient manifold obtained from

$$\big(V \setminus \psi(S^{\lambda-1} \times \{0\})\big) \sqcup (B^\lambda \times S^{n-\lambda-1}),$$

by identifying $\psi(u, cv)$ with (cu, v) for each $u \in S^{\lambda-1}$, $v \in S^{n-\lambda-1}$, $c \in]0, 1[$. We say $\chi(V, \psi)$ has been obtained from V by **surgery** of type $(\lambda, n-\lambda)$.

Of course, we have just seen such an embedding, namely the characteristic embedding. We want to see how applying surgery to V under the characteristic embedding influences our cobordism. To do this, we first have the following theorem, which Milnor proves by an explicit construction.

Theorem 6.4: If $\chi(V, \psi)$ is obtained from V by surgery of type $(\lambda, n-\lambda)$, then there exists an elementary cobordism $(W, V, \chi(V, \psi))$ and a Morse function $f : W \to \mathbb{R}$ with exactly one critical point of index λ.

The manifold W obtained from the construction will be denoted by $\omega(V, \psi)$ now. Their use follows from the following result.

Theorem 6.5: For any elementary cobordism (W, V, V') with characteristic embedding $\psi_L : S^{\lambda-1} \times B^{n-\lambda} \to V$, (W, V, V') is diffeomorphic to $\big(\omega(V, \psi_L), V, \chi(V, \psi_L)\big)$.

This can be used to prove the following result.

Theorem 6.6: Let (W, V, V') be an elementary cobordism with a Morse function f of index λ, possessing one critical point. Let D_L be the left-hand disk associated to a fixed gradient-like vector field ξ. Then $V \cup D_L$ is a deformation retract of W.

In particular, using this info, we can apply an excision argument to see that

$$H_k(W, V) = \begin{cases} \mathbb{Z} & k = \lambda; \\ 0 & \text{otherwise}; \end{cases}$$

a generator for $H_\lambda(W, V)$ is represented by D_L.

6.2 Cobordism rearrangement

The result we are interested in is the following.

Theorem 6.7 (Rearrangement Theorem): Any cobordism $c = (W, V_0, V_1)$ can be rearranged into a composition $c_0 \cdots c_n$ where $n = \dim W$ and each c_k admits a **self-indexing function**; i.e., Morse function possessing critical points of index k with the same critical value.

We know that a given cobordism $cc' = (W, V_0, V_1)$ admits a Morse function $f : cc' \to [0,1]$ with two critical points p, p' such that $\text{index}(p) = \text{index}(c)$, $\text{index}(p') = \text{index}(c')$ and $f(p) < \frac{1}{2} < f(p')$. Given a gradient-like field ξ for f, the trajectories from p meet $V = f^{-1}(\frac{1}{2})$ at the right-hand sphere S_R of p, and the trajectories going to p' meet V at the left-hand sphere S'_L. We will prove a preliminary version of the Rearrangement Theorem in the case that $S_R \cap S'_L = \varnothing$.

Theorem 6.8: Let (W, V_0, V_1) be a cobordism with Morse function f having two sets critical points $p = \{p_1, \dots, p_n\}, p' = \{p'_1, \dots, p'_m\}$, with all points of p and all points of p' at a single level. Suppose that for some choice of gradient-like field ξ the compact set K_p of points on trajectories going to or from points of p is disjoint from the compact set $K_{p'}$ of points on trajectories going to or from points of p'. If $f(W) = [0,1]$ and $a, a' \in]0,1[$, then there exists a new Morse function g such that

- ξ is a gradient-like field for g;
- the critical points of g are $p \cup p'$, and $g(p) = a$, $g(p') = a'$;
- g agrees with f near $V_0 \cup V_1$ and equals f plus a constant in some neighbourhood of every critical point.

The proof follows by an explicit construction of a suitable function g. The function is defined using a map assigning to a point $q \in W \setminus (K_p \cup K_{p'})$ the unique intersection of its trajectory with V_0.

Next, we give a way to move S_R out of the way of S'_L. Let us denote $\lambda = \text{index}(c)$, $\lambda' = \text{index}(c')$ and $n = \dim W$. We assume $\dim S_R + \dim S'_L < \dim V$ (or, equivalently, $\lambda \geq \lambda'$).

Theorem 6.9: If $\lambda \geq \lambda'$ then we may alter the gradient-like field ξ for f on a small neighbourhood of V so that the new spheres $\overline{S}_R$ and $\overline{S}'_L$ in V do not intersect.

The result relies on a construction, whose existence is asserted by a technical lemma. To state the lemma, we first need some definitions.

Let M be an m-submanifold of a v-manifold V. We say an open neighbourhood U of M which is diffeomorphic to $M \times \mathbb{R}^{v-m}$ in such a way that M corresponds to $M \times \{0\}$ is called a **product neighbourhood** of M in V. Next, consider two diffeomorphisms $h_0, h_1 : M \to M'$. They are said to be (**smoothly**) **isotopic** if there exists a map $f : M \times [0,1] \to M'$ such that f is smooth, each f_t is a diffeomorphism, $f_0 = h_0$ and $f_1 = h_1$.

Lemma 6.10: Let M, N be two submanifolds of dimension m resp. n in some v-manifold V. If M has a product neighbourhood in V and $m + n < v$ then there exists a diffeomorphism h of V onto itself, smoothly isotopic to the identity, such that $h(M)$ is disjoint from N.

The proof of the theorem relies mostly on the above lemma. Roughly speaking, it allows us to 'move' ξ in a product neighbourhood of S_R, so that the new spheres no longer intersect.

After applying Theorem 6.9, the requirements of Theorem 6.8 are satisfied; thus the main result (6.7) follows.

6.3 Cancellation theorems

Throughout this section, it is assumed that a cobordism (W, V_0, V_1) has exactly two critical points p, p' of indices λ and $\lambda + 1$, and that $f(p) < \frac{1}{2} < f(p')$. If ξ is a gradient-like vector field for f, then we have ourselves right-hand and left-hand spheres S_R, S'_L on $V = f^{-1}(\frac{1}{2})$.

Theorem 6.11: We may alter ξ to another gradient-like field so that S_R and S'_L intersect transversely in V.

The technique of the proof is very similar to that of Theorem 6.9: construct a suitable isotopy on a product neighbourhood of S_R, and then use this isotopy to alter ξ on this neighbourhood.

Notice that by our assumptions, $\dim S_R + \dim S'_L = \dim V$. For each $q \in S_R \cap S'_L$ there exists a local coordinate system $(x_1, \ldots, x_m)$ on an open neighbourhood U of q in V, q corresponds to $0 \in \mathbb{R}^{m-1}$, and $x_1 = \cdots = x_\lambda = 0$ on $U \cap S_R$, while $x_{\lambda+1} = \cdots = x_{m-1} = 0$ on $U \cap S'_L$. We may therefore assume that the intersection $S_R \cap S'_L$ only consists of finitely many points.

Theorem 6.12 (First Cancellation Theorem): Assume that S_R and S'_L intersect transversely in V, and $S_R \cap S'_L$ is just a single point. Then the cobordism is trivial.

Proof: (Sketch) Let f be a self-indexing function on W, and ξ a gradient-like vector field for f. Let T be the trajectory from p to p'. We alter ξ on an arbitrarily small neighbourhood of T to produce a nowhere-zero vector field ξ' that is a gradient-like field for a Morse function f' without critical points, and agreeing with f near $V_0 \cup V_1$.

More precisely, let $(U_1, g_1), (U_2, g_2)$ be Morse charts around the points p_1, p_2 such that

$$(f \circ g_1)(x) = x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2,$$
$$(f \circ g_2)(x) = -x_1^2 - x_2^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2,$$

and on which the map ξ has coordinates $(\pm x_1, -x_2, \ldots, -x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n)$. We want to find a chart g, defined on a neighbourhood U_T of T, that extends g_1, g_2 in such a way that p_1, p_2 correspond to the points $(0, \ldots, 0)$ and $(1, 0, \ldots, 0)$. Under this chart, ξ will have coordinates $(\nu_1(x_1), -x_2, \ldots, -x_{\lambda+1}, x_{\lambda+2}, \ldots, x_n)$ where $\nu_1 : [-2\delta, 1 + 2\delta] \to \mathbb{R}$ is a smooth function such that $\nu_1(x) \approx x$ if $x \approx 0$ and $\nu_1(x) \approx 1 - x$ if $x \approx 1$. The required construction takes some care. We will omit these details, and assume the above construction exists for now.

Lemma 6.13: Let U be a neighbourhood of T whose closure lies within U_T. Then there is a smaller neighbourhood U' of T contained in U such that no trajectory leads from U' to outside U and back into U'.

On U_T, integral curves of ξ satisfy the equations $dx_1/dt = \nu_1(x_1)$, $dx_i/dt = -x_i$ for $2 \leq i \leq \lambda+1$ and $dx_i/dt = x_i$ for $\lambda + 2 \leq i \leq n$. Using these equations, it can be shown that an integral curve $x(t)$ will leave $g(U)$ once t is both sufficiently large and once $-t$ is sufficiently large. If an integral curve is never in U' then it has to go from V_0 to V_1. If it ever gets into U' then it will eventually leave U and never re-enter U', so it will hit V_1. By the same reasoning it must have come from V_0. Therefore every trajectory goes from V_0 to V_1. We can use this trajectory to find an explicit diffeomorphism $V_0 \times [0, 1] \to W$ to conclude that W is trivial. ■

Let M, M' be submanifolds of dimensions r, s in a smooth manifold V of dimension $r + s$ that intersect transversely in points $p_1, \ldots, p_k$. Suppose M is oriented and that the normal bundle of M' in V is oriented as well. At p_i choose a positively oriented frame $\xi_1, \ldots, \xi_r$ spanning the tangent space $T_{p_i}M$. Since the intersection at p_i is transverse, $\xi_1, \ldots, \xi_r$ form a basis for the fibre of the normal bundle of M' at p_i. The **intersection number** of M and M' at p_i is then ± 1 depending on whether the $\xi_1, \ldots, \xi_r$ form a positively or negatively oriented basis for this fibre. By $M' \cdot M$ we denote the sum of intersection numbers of all i.

Lemma 6.14: With M' and V as above, there is a natural isomorphism $\psi : H_0(M') \xrightarrow{\sim} H_r(V, V \setminus M')$.

The proof is a corollary of the Thom Isomorphism Theorem (3.9). Next, let α be a generator for $H_0(M')$, and let μ be the orientation generator for $H_r(M)$. Then we have the following result, again based on the naturality of the Thom isomorphism.

Lemma 6.15: In the sequence

$$H_r(M) \xrightarrow{i_*} H_r(V) \xrightarrow{j_*} H_r(V, V \setminus M')$$

induced by the inclusion maps, we have $j_* i_* \mu = M' \cdot M\psi(\alpha)$.

Let (W, V_0, V_1) be a cobordism, again with Morse function f with two critical points of index λ and $\lambda + 1$. Suppose that S'_L is oriented, as is the normal bundle in V of S_R. Then we get the following strengthening of Theorem 6.12:

Theorem 6.16 (Second Cancellation Theorem): Suppose W, V_0, V_1 are simply-connected, $\lambda \geq 2$ and $\lambda + 1 \leq n - 3$. If $S_R \cdot S'_L = \pm 1$, then ξ can be altered near $V = f^{-1}(\frac{1}{2})$ so that S_R and S'_L intersect transversely at a single point.

Notice that the conclusions of the First Cancellation Theorem now immediately apply. Furthermore, observe that by Van Kampen's Theorem, V is simply-connected (note: this uses the assumptions $\lambda \geq 2$ and $\lambda + 1 \leq n - 3$), that $\dim S_R \geq 3$ and $\dim V = \lambda - 1$. If, furthermore, $\lambda \geq 3$, the conditions of the following hideous lemma will be satisfied.

Lemma 6.17: Let M, M' be transversely intersecting manifolds of dimensions r and s in the smooth $(r+s)$-manifold V. Suppose M and the normal bundle of M' are oriented in V. Furthermore, suppose $r+s \geq 5$ and $s \geq 3$, and, in case $r = 1$ or $r = 2$, suppose the inclusion-induced map $\pi_1(V \setminus M') \to \pi_1(V)$ is injective. Let $p, q \in M \cap M'$ be points with opposite intersection numbers such that there exists a loop L contractible in V that consists of a smooth arc from p to q in M followed by a smooth arc from q to p in M' where both arcs don't go through $M \cap M' \setminus \{p, q\}$. With these assumptions, there exists an isotopy h_t starting at the identity $i : V \to V$, such that the isotopy fixes i near $M \cap M' \setminus \{p, q\}$, and such that $h_1(M) \cap M' = M \cap M' \setminus \{p, q\}$.

Proof (of Theorem 6.16): By Theorem 6.11 we may assume S_R and S'_L intersect transversely. Since $S_R \cdot S'_L = \pm 1$, there exist two intersection points, say p_1 and q_1, with opposite intersection numbers. What we want to do is show that the assumptions of Lemma 6.17 hold for this situation. Then, using the techniques in the proof of Theorem 6.9, we can alter ξ to end up with two fewer intersection points. We may repeat this process until we have only one point, and the proof will be complete.

We already remarked that V is simply-connected, so if $\lambda \geq 3$ all the conditions are satisfied. If $\lambda = 2$, we also need the requirement that the inclusion-induced map $\pi_1(V \setminus S_R) \to \pi_1(V)$ is injective. Well, as I just said, V is simply-connected, so we want to show that $V \setminus S_R$ is simply-connected too. Here's how we can see this. Under the trajectories of ξ, we have a diffeomorphism from $V_0 \setminus S_L$ to $V \setminus S_R$, so we may as well show simply-connectedness of $V_0 \setminus S_L$. Now let U be a product neighbourhood of S_L in V_0. Since $\dim S_L \geq 3$, we have $\pi_1(U \setminus S_L) \cong \mathbb{Z}$. We may now apply Van Kampen's Theorem to the covering of V_0 by $V_0 \setminus S_L$ and U, from which immediately follows that $\pi_1(V_0 \setminus S_L)$ must be trivial. This completes the proof. ■

6.4 The main result

Suppose (W, V, V'), (W', V', V'') and $(W \cup W', V, V'')$ are some n-dimensional cobordisms. Let f be a Morse function on $W \cup W'$ with critical points $q_1, \ldots, q_l \in W$ on one level and of index λ, and critical points $q'_1, \ldots, q'_m \in W'$ on another and of index $\lambda + 1$, and let V' be a non-critical level in between them. Take a gradient-like vector field ξ and orient $D_L(q_1), \ldots, D_L(q_l)$ in W and $D'_L(q'_1), \ldots, D'_L(q'_m)$ in W'. The condition that $D_L(q_i)$ have intersection number $+1$ with $D_R(q_i)$ at the point q_i then determines an orientation for the normal bundle of the right-hand disks in W, which in turn determines an orientation for the normal bundle of $S_R(q_i)$. We conclude that once orientations have been chosen for the left-hand disks, the intersection number $S_R(q_i) \cdot S'_L(q'_j)$ of left-hand spheres with right-hand spheres in V' are well-defined.

Recall that by Theorem 6.7, a cobordism c can be factored into $c_0 \ldots c_n$ where c_λ admits a Morse function of index λ with one critical level. Let W_λ be $c_0 \ldots c_\lambda$ and set W_{-1} to be V, so that

$$V = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_n = W.$$

Define C_λ to be $H_\lambda(W_\lambda, W_{\lambda-1})$ and let $\partial : C_\lambda \to C_{\lambda-1}$ be the boundary homomorphism for the exact sequence belong to the triple $(W_{\lambda-2}, W_{\lambda-1}, W_\lambda)$.

Theorem 6.18: With the notation above, we have a chain complex; its corresponding homology H_λ is isomorphic to the singular homology group $H_\lambda(W, V)$.

The proof is found by considering an appropriate exact sequence. We will apply the result to the following theorem.

Theorem 6.19 (Basis Theorem): Suppose (W, V, V') is an n-dimensional cobordism with Morse function f such that all critical points of index λ are on the same level. Let ξ be its associated gradient-like vector field. Assume $2 \leq \lambda \leq n-2$ and that W is connected. Then, given any basis for $H_\lambda(W, V)$ there exist a Morse function f' and a gradient-like field field ξ' belonging to it, such that f' and ξ' agree with f and ξ in a neighbourhood of $V \cup V'$, such that f' has the same critical points as f, all on the same level, and such that the left-hand disks for ξ' determine the given basis.

Proof: Let $p_1, \dots, p_k$ be the critical points of f and denote $b_i = [D_L(p_i)]$ so that $\{b_1, \dots, b_k\}$ forms a basis of $H_\lambda(W, V)$. The matrix of intersection numbers $(D_R(p_i) \cdot D_L(p_j))_{ij}$ can be set to be the identity matrix if the choose the correct orientations for the normal bundles of the right-hand disks $D_R(p_1), \dots, D_R(p_k)$. Consider an oriented λ-dimensional disk D embedded in W with $\partial D \subseteq V_0$. Then $[D] \in H_\lambda(W, V_0)$ and therefore there are integers α_i such that $[D] = \alpha_1 b_1 + \cdots \alpha_k b_k$. Using a relative version of Lemma 6.15, we have

$$\begin{aligned} D_R(p_j) \cdot D &= D_R(p_j) \cdot (\alpha_1 b_1 + \cdots \alpha_k b_k) \\ &= \alpha_1 D_R(p_j) \cdot D_L(p_1) + \cdots + \alpha_k D_R(p_j) \cdot D_L(p_k) \\ &= \alpha_j . \end{aligned}$$

Thus, D represents the element $D_R(p_1) \cdot D b_1 + \cdots + D_R(p_k) \cdot D b_k$.

By Theorem 6.8 we have a Morse function f_1 which agrees on f outside a neighbourhood of p_1 such that $f_1(p_1) > f(p_1)$ and f_1 has the same critical points and gradient-like field as f. Choose t_0 between $f(p)$ and $f_1(p_1)$ and set $V_0 = f_1^{-1}(t_0)$. The left-hand sphere S_L of p_1 in V_0 and the right-hand spheres $S_R(p_i)$ $(i = 2, \dots, k)$ are mutually disjoint. Choose points $a \in S_L$ and $b \in S_R(p_2)$. $=$ We can construct an isotopy F_t of V_0 which moves S_L across $S_R(p_2)$. Now, use the

Collar Neighbourhood Theorem (Corollary 6.3), consider an embedding of the space $V_0 \times [0,1]$ into W on the 'right side' of V_0 such that $V_0 \times \{0\}$ corresponds with the copy $V_0 \subseteq W$, and such that the embedded $V_0 \times [0,1]$ contains no critical points. Use the isotopy F_t to alter the vector field ξ on this neighbourhood, using the techniques in the proof of Theorem 6.9 and set ξ' to be the new vector field on W. Since ξ, ξ' agree to the 'left' of V_0, it follows that the right-hand spheres are still $S_R(p_2), \dots, S_R(p_k)$. The left-hand sphere of p_1 is now $S_L' = F_0(S_L)$, which, by the given properties of H_t, is disjoint from $S_R(p_3), \dots, S_R(p_k)$.

We now apply Theorem 6.8 to find a Morse function f' with ξ' as its associated gradient-like field, and which has only one critical value.

It remains to be shown that the new left-hand disks give us the desired basis — we omit this result. ∎

Finally, let us recall the theorem we have been striving to prove.

Theorem 6.1 (The H-Cobordism Theorem): Let (W, V, V') be a cobordism with the following properties.

- W, V and V' are simply-connected;
- $H_*(W, V) = 0$;
- $\dim W \geq 6$.

Then W is diffeomorphic to $V \times [0, 1]$.

Proof: Let f be a self-indexing Morse function on (W, V, V'). Then the process is as follows.

Step 1. The critical points of index 0 can be cancelled against an equal number of critical points of index 1.

Step 2. If there are no critical points of index 0 one can insert for each index-1 critical point a pair of auxiliary index-2 and index-3 critical points and cancel the index-1 critical points against the auxiliary index-2 critical points.

Step 3. We can repeat the discussion by replacing f with $-f$ to eliminate the critical points of index n and $n-1$.

Step 4. Once there are no critical points of indices $0, 1, n-1, n$, the resulting cobordism is necessarily trivial.

Step 1: Let $W_k = f^{-1}[-\frac{1}{2}, k + \frac{1}{2}]$ $(k = 0, \ldots, n)$ and let $V_k^+ = f^{-1}(k + \frac{1}{2})$. Consider homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Since $H_0(W, V; \mathbb{Z}/2\mathbb{Z}) = 0$, by Theorem 6.18 the boundary homomorphism $\partial : H_1(W_1, W_0; \mathbb{Z}/2\mathbb{Z}) \to H_0(W_0, V; \mathbb{Z}/2\mathbb{Z})$ is surjective. But ∂ is given by the matrix of intersection numbers modulo 2 of the right-hand $(n-1)$-spheres and left-hand 0-spheres in V_0^+. Hence for any S_R there is at least one S_L with $S_R \cdot S_L \not\equiv 0 \mod 2$. So $S_R \cap S_L$ consists of an odd number of points, which can only be 1. Thus we can apply the First Cancellation Theorem (6.12) inductively.

Step 2: Given a critical point p of index 1, we want to construct an embedded 1-sphere S in V_1^+ that has one transverse intersection with a right-hand $(n-2)$-sphere and meets no other right-hand spheres. Certainly there exists a small embedded 1-disc $D \subseteq V_1^+$ which, at its midpoint q_0, intersects transversely with some S_R and does not intersect with other spheres. Translate the end-points of D along the trajectories of ξ to a pair of points in V. Since V is connected, and of dimension $n - 1 \geq 2$, these points may be joined by a smooth path in V which avoids the left-hand 0-spheres in V. This path may then be translated back to a smooth path that joins the end-points of D in V_1^+ and avoids all right-hand spheres. This gives us the required embedding.

Given a critical point of index 1, let S be as in the previous paragraph. After adjusting ξ if necessary to the right of V_2^+, we may assume S meets no left-hand 1-spheres in V_1^+. Then we can translate S to a 1-sphere S_1 in V_2^+.

In a collar neighbourhood extending to the right of V_2^+, we can choose coordinate functions $x_1, \ldots, x_n$ embedding an open set U into $\mathbb{R}^n$ such that $f|_U = x_n$. We can alter f on a compact subset of U inserting points q, r of index $2, 3$. Let S_2 be the left-hand sphere of q in V_2^+. Since V_2^+ is simply-connected, there is an isotopy of the identity $V_2^+ \to V_2^+$ that carries S_2 to S_1. After an adjustment of ξ to the right of V_2^+, the left-hand sphere of q in V_2^+ will be S_1. Then the left-hand sphere of q in V_1^+ is S, which we assumed to intersect the right-hand sphere of p transversely in a single point.

We can now alter f without changing ξ, using Theorem 6.8, so that we have

$$1 + \delta < f(p) < \frac{3}{2} < f(q) < 2 - \delta.$$

Next, apply the First Cancellation Theorem (6.12) to alter f and ξ so that we can eliminate the critical points p and q. Finally, using Theorem 6.8 again, move the critical points r to the right of V_3^+ again. Thus, we have replaced p with a point r of index 3. Repeat inductively until no critical points of index 1 remain.

Step 4: Let c denote the cobordism (W, V, V'). We can factor c as $c_2 \ldots c_{n-2}$. With the notation as in Theorem 6.18, we have a sequence of free abelian groups

$$C_{n-2} \xrightarrow{\partial} C_{n-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2. \qquad (\star)$$

For each $\lambda = 2, \ldots, n-2$ choose a basis $\left\{z_1^{\lambda+1}, \ldots, z_{k_{\lambda+1}}^{\lambda+1}\right\}$ for $\operatorname{Ker} \partial : C_{\lambda+1} \to C_\lambda$. Since $H_*(W, V) = 0$ it follows from Theorem 6.18 that this sequence is exact and hence that we may choose $b_1^{\lambda+1}, \ldots, b_{k_\lambda}^{\lambda+1} \in C_{\lambda+1}$ such that $\left\{z_1^{\lambda+1}, \ldots, z_{k_{\lambda+1}}^{\lambda+1}, b_1^{\lambda+1}, \ldots, b_{k_\lambda}^{\lambda+1}\right\}$ is a basis for $C_{\lambda+1}$.

Since $2 \leq \lambda \leq \lambda + 1 \leq n - 2$ we can use the Basis Theorem to find f' and its associated ξ' on W so that the left-hand disks of $c_\lambda, c_{\lambda+1}$ represent the chosen bases for $C_\lambda, C_{\lambda+1}$. Let p, q be the critical points in $c_\lambda, c_{\lambda+1}$ corresponding to z_1^λ and $b_1^{\lambda+1}$ respectively. By increasing f' in a neighbourhood of q we obtain $c_\lambda c_{\lambda+1} = c'_\lambda c_p c_q c'_{\lambda+1}$, where c_p, c_q only have the critical points p resp. q. Let $V_0 = (f')^{-1}(r)$ for some r between $f(p)$ and $f(q)$. Since $\partial b_1^{\lambda+1} = z_1^\lambda$ (by exactness of

($\star$), we may choose the $b_i^{\lambda+1}$ as such) the spheres $S_R(p)$ and $S_L(q)$ in V_0 have intersection number $+1$. This allows to us to apply the Second Cancellation Theorem[1] to see that $c_p c_q$ is a product cobordism, and that f' and ξ' can be altered in its interior so that f' has no more critical points there. We have now eliminated the critical points p, q. We may continue this process until no critical points are left. Then the result follows by Theorem 6.2. This concludes the main proof.■

6.5 Addendum

We end with some results which will be of great importance, especially in the next chapter. To start with, let us explain the name of the result. A cobordism (W, V, V') **H-cobordism** is said to be an **H-cobordism**, and V is said to be **H-cobordant** to V' if both V and V' are deformation retracts of W. Thus by the H-Cobordism Theorem we immediately have.

Theorem 6.19: Two simply-connected closed smooth manifolds of dimension ≥ 6 are H-cobordant if and only if they are diffeomorphic.

Theorem 6.20: Let W be a compact simply-connected n-manifold ($n \geq 6$) with a simply-connected boundary. Then W is diffeomorphic to the n-disc D^n if and only if W has the integral homology of a point.

Proof: We prove the non-trivial implication. Let D_0 is a smooth disc embedded into $\operatorname{Int} W$. Then $(W \setminus \operatorname{Int} D_0, \partial D_0, V)$ satisfies the condition of the H-Cobordism Theorem. Consequently $(W, \varnothing, V)$ is a composition of $(D_0, \varnothing, \partial D_0)$ with a product cobordism $(W \setminus \operatorname{Int} D_0, \partial D_0, V)$. It follows that W is diffeomorphic to D_0. ■

Theorem 6.21 (Generalized Poincaré Conjecture): If M is a closed simply-connected n-manifold ($n \geq 6$), then M is homeomorphic to S^n if and only if M has the homology of S^n.

Proof: Suppose D_0 is an embedded smooth n-disc again. We have

$$H_k(M \setminus \operatorname{Int} D_0) \cong H^{n-k}(M \setminus \operatorname{Int} D_0, \partial D_0) \cong H^{n-k}(M, D_0),$$

the first equality being Lefschetz Duality, and the second one excision. By an exact-sequence argument, the right-hand side is the homology of a point, hence is diffeomorphic to D^n. Consequently, M can be identified with two glued discs, such a glued space is homeomorphic (though not necessarily diffeomorphic) to S^n. This concludes the result. ■

[1] This is where we require that no index-1 critical points exist.

Chapter 7

Groups of homotopy spheres

This chapter will be mainly devoted to the paper [33], in which Milnor and Kervaire give an explicit classification of the exotic structures on higher-dimensional spheres. Although the details are beyond the scope of this thesis, we will at least attempt to give an overview of the general approach and results.

7.1 Some definitions

Recall that given two spaces X and Y, we say that they are of the **same homotopy type** whenever they are homotopy equivalent; i.e., there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity map.

Let n be a positive integer. By Θ_n we denote the set of equivalence classes of oriented manifolds with the homotopy type of S^n (i.e., homotopy n-spheres), up to H-cobordism. Throughout this chapter, we shall assume that $n \geq 5$. In these dimensions, we know that by Theorem 6.19 manifolds which are H-cobordant are diffeomorphic; furthermore, by the Generalized Poincaré Conjecture (6.21) we also know that in these dimensions, a homotopy n-sphere is homeomorphic to S^n. Thus Θ_n is the set of distinct differentiable structures on the topological n-sphere.

The set Θ_n turns out to have a natural group structure under the connected-sum operation. We first recall the definition of the connected sum $M_1 \# M_2$ of two connected n-manifolds M_1 and M_2. Choose embeddings $i_j : D^n \to M_j$ $(j = 1, 2)$ so that i_1 is orientation-preserving while i_2 is orientation-reversing. We obtain $M_1 \# M_2$ from the disjoint sum

$$\big(M_1 \setminus i_1(0)\big) \sqcup \big(M_2 \setminus i_2(0)\big),$$

by identifying $i_1(x)$ with $i_2\left(\frac{1-||x||}{||x||}x\right)$ for all $x \in D^n$ and $0 < ||x|| < 1$.

A lemma by Cerf[9] asserts that the connected-sum operation is well-defined up to orientation-preserving diffeomorphism.

Let us verify that this operation gives rise to a product structure on Θ_n.

Theorem 7.1: Under the connected-sum operation, Θ_n becomes an abelian group.

Proof: The connected sum of two homotopy n-spheres is again a homotopy n-sphere; this follows, for example, by the Generalized Poincaré Conjecture (6.21). Next, we show that the connected-sum operation is well-defined; we summarize this with a lemma.

Lemma 7.2: Let M_1, M_1', M_2 be closed and simply-connected manifolds. If M_1 is H-cobordant to M_1' then $M_1 \# M_2$ is H-cobordant to $M_1' \# M_2$.

Proof: Let $(W_1, M_1, -M_1')$ be an H-cobordism. Choose a differentiable arc A from a point $p \in M_1$ to a point $p' \in -M_1'$ within W_1 so that a tubular neighbourhood of this arc is diffeomorphic to $\mathbb{R}^n \times [0,1]$. This gives us an embedding i of $\mathbb{R}^n \times [0,1]$ into W_1, with $i(\mathbb{R}^n \times \{0,1\}) \subseteq M_1 \sqcup M_1'$ and $i(\{0\} \times [0,1]) = A$. Now construct the manifold W by starting with the disjoint sum

$$(W_1 \setminus A) \sqcup (M_2 \setminus i'(0)) \times [0,1],$$

and identifying $i(tu, s)$ with $(i'((1-t)u), s)$ for $0 < t < 1$, $0 \leq s \leq 1$ and $u \in S^{n-1}$.[1] Then W is a compact manifold bounded by the disjoint sum $(M_1 \# M_2) \sqcup -(M_1' \# M_2)$. We show that both boundaries are in fact deformation retracts of W.

Consider the inclusion map $j : M_1 \setminus \{p\} \to W_1 \setminus A$. We would like to show that this is a homotopy equivalence. Since $n \geq 5$, both of these manifolds are simply-connected. Consider now the following exact sequence associated with the pairs $(M_1, M_1 \setminus \{p\})$ and $(W_1, W_1 \setminus A)$:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(p) & \longrightarrow & H_n(M_1) & \longrightarrow & H_n(M_1, p) & \longrightarrow & H_{n-1}(p) & \longrightarrow \cdots \\
 & & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & & \\
\cdots \longrightarrow & H_n(A) & \longrightarrow & H_n(W_1) & \longrightarrow & H_n(W_1, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow \cdots
\end{array}$$

Since the homologies of $\{p\}$ and A are known, we see that j induces isomorphisms on the level of homology, hence by the Whitehead Theorem (2.9) it must be a homotopy equivalence. Using a Mayer-Vietoris sequence it then follows easily that the inclusion $M_1 \# M_2 \to W$ is also a

[1] Note the similarity of this construction with that of $\chi(V, \psi)$ in Section 6.1.

homotopy equivalence, hence that $M_1 \# M_2$ is a deformation retract of W. By symmetry, the same holds for $M_1' \# M_2$, which completes the proof. ■

We continue our proof of Theorem 7.1. Associativity and commutativity is intuitively clear; furthermore, $S^n \in \Theta_n$ acts as the identity. Given a homotopy n-sphere $E \in \Theta_n$, let $-E$ be the sphere with opposite orientation. This element will provide the inverse under the connected-sum operation. Notice that $E\#(-E)$ bounds the contractible manifold $(E \setminus D^n) \times [0,1]$. Then its boundary is simply-connected so that by Theorem 6.20 the manifold is diffeomorphic to D^{n+1} and thus $E\#(-E) = S^n$. ■

One of the main results in [33] is the following.

Theorem 7.3: If $n \neq 3$ then the group Θ_n is finite.

The methods that are used unfortunately break down for the case $n = 3$. Furthermore, it is already known that Θ_1, Θ_2 are both trivial. On the other hand not *all* groups will be trivial. For example, by the results in Chapter 4, we already know that $|\Theta_7| \geq 4$.

7.2 Parallelizability

Let M be a manifold with tangent bundle TM, and let ε^1 denote the trivial line bundle over M. We say a manifold is **parallelizable** if its tangent bundle is trivial. A manifold M will be called **s-parallelizable** if the Whitney sum of the tangent bundle with ε^1 is trivial.

Theorem 7.4: Homotopy spheres are s-parallelizable.

Proof: The general result relies on several technical results which we shall not present here. We will give an overview of the proof. Let Σ be a homotopy sphere. We can trivialize $T\Sigma$ on both hemispheres. The overlap map is thus a map $f : S^{n-1} \to \mathrm{SO}(n)$, therefore the bundle $TM \oplus \varepsilon^1$ will be trivial if the composition

$$S^{n-1} \xrightarrow{f} \mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$$

is homotopic to a constant map. By the homotopy exact sequence (see [26, §9.2]) we have $\pi_{n-1}(\mathrm{SO}(n+1)) \cong \pi_{n-1}(\mathrm{SO}(n+k))$ for all $k \geq 1$. A theorem of Bott on the stable homotopy of the classical groups[6] then ensures that no obstruction to trivialization can exist when $n \equiv 3, 5, 6, 7 \mod 8$.

If $n \equiv 0, 1, 2, 4 \mod 8$, Milnor and Kervaire give a more direct proof that no such obstruction can exist. The techniques rely on technical results due to Rohlin and Adams, which we will omit here. ■

By the Whitney Embedding Theorem, any closed n-manifold can be embedded into a sufficiently high-dimensional sphere S^{n+k}. It turns out that this embedding is unique up to smooth isotopy of S^{n+k} (recall from Section 6.2 that two diffeomorphisms $h_0, h_1 : M \to M'$ are smoothly isotopic if there exists a map $f : M \times [0,1] \to M'$ such that f is smooth, each f_t is a diffeomorphism, $f_0 = h_0$ and $f_1 = h_1$). We have the following result, which is due to Whitehead:[51]

Proposition 7.5: If M is an n-dimensional submanifold of S^{n+k} $(k > n)$ then M has trivial normal bundle if and only if M is S-parallelizable. Furthermore, a connected manifold with non-empty boundary is S-parallelizable if and only if it is parallelizable.

The proof will be based on the following lemma.

Lemma 7.6: Let ξ be a k-dimensional vector bundle over an n-dimensional base space, and assume $k > n$. If the Whitney sum of ξ with the trivial bundle ε^r is trivial, then ξ itself is trivial as well.

Proof: By induction, we may set $r = 1$. We may assume ξ is oriented. An isomorphism $\xi \oplus \varepsilon^1 \xrightarrow{\sim} \varepsilon^{k+1}$ induces a bundle map f from ξ to the bundle γ^k of oriented k-planes in $\mathbb{R}^{k+1}$. Since the base space of ξ has dimension n, and since that of γ^k is S^k with $k > n$ it follows that f must be null-homotopic, hence ξ must be trivial. ■

Proof (of Proposition 7.5): We start by proving the first part of the proposition. Let TM, NM denote the tangent and normal bundles of M. We know that $TM \oplus NM$ must be trivial, hence so is $(TM \oplus \varepsilon^1) \oplus NM$. Assume that M is S-parallelizable. Then $TM \oplus \varepsilon^1$ is trivial as well, so that $(TM \oplus \varepsilon^1) \oplus NM \cong \varepsilon^{n+1} \oplus NM$. Now apply the above lemma to see that NM is trivial. Conversely, suppose NM is trivial. Then $TM \oplus NM \cong TM \oplus \varepsilon^k$, so that we may apply Lemma 7.6 to see that TM is trivial.

The second part follows by a similar argument. The hypothesis on the manifold guarantees that every map into a sphere of the same dimension is null-homotopic. ■

Define the subgroup $bP_{n+1} \subseteq \Theta_n$ as follows. A homotopy n-sphere M represents an element of bP_{n+1} if and only if M is the boundary of a parallelizable manifold. We must, of course, verify that this does indeed form a subgroup, and that the condition depends only the H-cobordism class of M.

Theorem 7.7: The quotient group Θ_n/bP_{n+1} is finite.

Proof: Let M be an S-parallelizable manifold of dimension n, and let $i : M \to S^{n+k}$ be an embedding with $k > n+1$. As mentioned, such an embedding is unique up to smooth isotopy. By Proposition 7.5 the normal bundle of M is trivial. Let φ be a field of normal k-frames. Then the Pontryagin-Thom construction (construction of the Thom space; see Section 5.2) yields a map $p(M,\varphi) : S^{n+k} \to S^k$. The homotopy class of $p(M,\varphi)$ is a well-defined element of the homotopy group $\pi_{n+k}(S^k)$. Allowing φ to vary, we obtain a set of elements $p(M) \subseteq \pi_{n+k}(S^k)$. We present several lemmas which will help us with the proof.

Lemma 7.8: The subset $p(M)$ contains the zero element of $\pi_{n+k}(S^k)$ if and only if M bounds a parallelizable manifold.

Proof: If M bounds a parallelizable manifold W then the embedding i can be extended to an embedding of W into D^{n+k+1}. Now let ψ be a field of k-frames for W and let φ be its restriction to M. The Pontryagin-Thom map $p(M,\varphi)$ now extends over D^{n+k+1} hence it is null-homotopic.

Conversely, suppose $p(M,\varphi)$ is null-homotopic, then M bounds a manifold which is S-parallelizable and hence parallelizable by Proposition 7.5. ■

Lemma 7.9: If M_0 is H-cobordant to M_1 then $p(M_0) = p(M_1)$.

Proof: If $M_0 \sqcup (-M_1)$ is the boundary of W, then choose an embedding of W into $S^{n+k} \times [0,1]$ so that M_0, M_1 correspond to restricting this embedding to $S^{n+k} \times \{0,1\}$. A normal frame field φ_0 on M_0 extends to a normal frame field on W which then restricts to a normal frame field φ_1 on M_1; by symmetry, this holds the other way around too and hence (W,ψ) gives rise to a homotopy between $p(M_0,\varphi_0)$ and $p(M_1,\varphi_1)$. ■

Lemma 7.10: If M, M' are S-parallelizable then $p(M) + p(M') \subseteq p(M\#M') \subseteq \pi_{n+k}(S^k)$.

Proof: (Sketch) Given the disjoint union $M \times [0,1] \sqcup M' \times [0,1]$ we can basically glue the boundary components $M \times \{1\}$ and $M' \times \{1\}$ together to obtain a manifold W bounded by the disjoint union $(M\#M') \sqcup (-M) \sqcup (-M')$. We can embed W into $S^{n+k} \times [0,1]$ in such a way that $-M$ and $-M'$ go into $S^{n+k} \times \{0\}$ and $M\#M'$ goes into $S^{n+k} \times \{1\}$. Given fields of normal k-frames φ, φ' on $-M$ and $-M'$, it turns out we can define an extension of these fields throughout W. As in the previous lemma, restricting this extension to $M\#M'$ gives the required homotopy. ■

Lemma 7.11: The set $p(S^n)$ is a subgroup of $\pi_{n+k}(S^k)$. For any homotopy sphere Σ the set $p(\Sigma)$ is a coset of the subgroup $p(S^n)$. Thus the correspondence $\Sigma \mapsto p(\Sigma)$ defines a homomorphism p' from Θ_n into the quotient group $\pi_{n+k}(S^k)/p(S^n)$.

Proof: Apply Lemma 7.10 to the identities $S^n \# S^n = S^n$, $S^n \# \Sigma = \Sigma$ and $\Sigma \# (-\Sigma) = S^n$. ■

By Lemma 7.8, then, the kernel of p' must be precisely the set of H-cobordism classes of homotopy n-spheres which bound parallelizable manifolds. Thus these elements form a group which we denote by $bP_{n+1} \subseteq \Theta_n$. It follows that Θ_n/bP_{n+1} must be isomorphic to a subgroup of $\pi_{n+k}(S^k)/p(S^n)$. A result in [43] ensures that $\pi_{n+k}(S^k)$ is finite, so that the proof of Theorem 7.7 follows. ■

The proof of Theorem 7.3 reduces to showing that bP_{n+1} is finite for $n \geq 3$. Milnor and Kervaire consider two cases. They first prove that bP_{n+1} is zero for n even, and then prove that it is finite cyclic for n odd. Since we are particularly interested in $n = 7$, we restrict our attention to the second case.

7.3 The groups bP_{2k}

We now take a closer look at the groups bP_{2k}. Milnor and Kervaire show that bP_{2k} are finite cyclic for $k \neq 2$; for k odd, the group bP_{2k} has at most two elements. In fact for $k = 2m \neq 2$, we have

$$|bP_{4m}| = \varepsilon_m 2^{2m-2}(2^{2m-1} - 1) \cdot \text{numerator}(4B_m/m),$$

where B_m denotes the m-th Bernoulli number and ε_m is 1 or 2.

Recall the following definition from Section 6.1. Let M be an $(p + q + 1)$-manifold, and let $\varphi : S^p \times D^{q+1} \to M$ be an embedding. A new manifold $M' = \chi(M, \varphi)$ is formed by taking the disjoint sum

$$\big(M \setminus \varphi(S^p \times \{0\})\big) \sqcup D^{p+1} \times S^q$$

and identifying $\varphi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$ and $t \in]0, 1]$. We say that M' is obtained by **surgery** from M. Note that the boundaries of M and M' coincide.

Theorem 7.12: Let M be a compact, connected, S-parallelizable manifold of dimension $n \geq 2k$. Then by a sequence of surgery operations one can obtain an S-parallelizable manifold M_1 which is $(k - 1)$-connected.

Proof: (Sketch) Choose a suitable embedding $\varphi : S^1 \times D^{n-1} \to M$ to obtain an S-parallelizable manifold M' by surgery in such a way that $\pi_1(M')$ is generated by fewer elements than $\pi_1(M)$. After a finite-number of steps, we obtain a 1-connected manifold. Next, repeat this process to get a 2-connected manifold, and so on until we find a $(k-1)$-connected manifold. ■

Proposition 7.13: Let M be a $(k-1)$-connected $2k$-manifold ($k \geq 3$) and suppose $H_k(M)$ is free abelian with basis $\{\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_r\}$ where $\lambda_i\lambda_j = 0$ and $\lambda_i\mu_j = \delta_{ij}$. Suppose further that every embedded sphere in M which represents a homology class in the subgroup generated by $\lambda_1, \ldots, \lambda_r$ has trivial normal bundle. Then $H_k(M)$ can be killed by a sequence of surgery modifications.

Proof: (Sketch) We assume for now that any homology class in $H_k(M)$ can be represented by an embedded sphere. The reader can find a proof of this in [29, Lem. 6]. Here the hypothesis $k \geq 3$ turns out to be necessary.

Let $\varphi_0 : S^k \to M$ be an embedding so as to represent the homology class λ_r. Since the normal bundle is trivial, we can extend φ_0 to an embedding $\varphi : S^k \times D^k \to M$. Denote by $M' = \chi(M, \varphi)$ the manifold obtained by surgery, and let $M_0 = M \setminus \operatorname{Int}\varphi(S^k \times D^k) = M' \setminus \operatorname{Int}\varphi'(D^{k+1} \times S^{k-1})$. In [33, Lem. 4.1] it is proved that $H_k(M')$ is isomorphic to a quotient group of $H_k(M \setminus \operatorname{Int}\varphi(S^k \times D^k))$, with basis $\{\lambda'_1, \ldots, \lambda'_{r-1}, \mu'_1, \ldots, \mu'_{r-1}\}$, where each λ'_i corresponds to a coset $\lambda_i + \lambda_r\mathbb{Z} \subseteq H_k(M)$, and each μ'_i to $\mu_i + \lambda_r\mathbb{Z}$. Furthermore, M' also satisfies the hypothesis of the proposition, so that we can iterate this contraction until we obtain a k-connected manifold. ■

Consider an S-parallelizable manifold M of dimension $2k$, bounded by a homology sphere. By Theorem 7.12 we can assume M is $(k-1)$-connected. By the Poincaré Duality Theorem, it follows that $H_k(M)$ is free abelian, and that the intersection pairing $H_k(M) \otimes H_k(M) \to \mathbb{Z}$ has determinant ± 1. Milnor and Kervaire now consider several cases for k. Since we are specifically looking for 7-spheres, we will only concern ourselves with the case in which $k = 4m$. We then have the following result, as found in [29]:

Lemma 7.14: Let M be a parallelizable $4m$-manifold bounded by a homology sphere. The homotopy groups of M can be killed by a sequence of surgery operations is and only if the signature $\tau(M)$ is zero.

Lemma 7.15: For all $k = 4m$ there exists a parallelizable manifold M_0 whose boundary ∂M_0 is the ordinary $(4m-1)$-sphere, such that the signature $\tau(M_0)$ is non-zero.

Proof: (Sketch) The proof is based on a result found in [32, p. 457], where it is shown that there exists a closed 'almost parallelizable' $4m$-manifold whose signature is non-zero. Removing the interior of an embedded $4m$-disk from this manifold, we obtain the required manifold M_0. ■

Next, Milnor and Kervaire consider the collection of *all* $4m$-manifolds M_0 which are S-parallelizable and are bounded by the $(4m-1)$-sphere. The corresponding signatures $\tau(M_0) \in \mathbb{Z}$ then form a group under addition; let σ_m denote the positive generator of this group.

Theorem 7.16: Let Σ_1, Σ_2 be homotopy spheres of dimension $4m-1$ $(m > 1)$ which bound S-parallelizable manifolds M_1, M_2. Then Σ_1 is H-cobordant to Σ_2 if and only if $\tau(M_1) \equiv \tau(M_2)$ mod σ_m.

Proof: (Sketch) Suppose first that we can write $\tau(M_1) = \tau(M_2) + \tau(M_0)$, where M_0 is a parallelizable manifold whose boundary is the ordinary $(4m-1)$-sphere. By a connected sum of $-M_1, M_2, M_0$ along the boundary, we obtain a manifold M with boundary given by $\partial M = -\Sigma_1 \# \Sigma_2 \# S^{4m-1} \cong -\Sigma_1 \# \Sigma_2$. Since τ is additive under this operation, $\tau(M) = 0$ and hence by Lemma 7.14, $\partial M = -\Sigma_1 \# \Sigma_2$ must belong to a trivial H-cobordism class.

Conversely, suppose W is an H-cobordism between $-\Sigma_1 \# \Sigma_2$ and S^{4m-1}. Consider the connected sum along the boundary of $-M_1$ and M_2, which has boundary $-\Sigma_1 \# \Sigma_2$. We can glue W to this connected sum along their common boundary $-\Sigma_1 \# \Sigma_2$ in the obvious way, and obtain a manifold M bounded by a sphere S^{4m-1}. By definition, then $\tau(M) \equiv 0 \mod \sigma_m$, so that $\tau(M_1) \equiv \tau(M_2)$ mod σ_m, as was to be shown. ■

As a consequence, we see that bP_{4m} $(m > 1)$ is isomorphic to a subgroup of the cyclic group of order σ_m, and hence bP_{4m} must be a finite cyclic group. In particular, combining this result with Theorem 7.7 gives us a partial proof of Theorem 7.3. To summarize, we now have:

Theorem 7.17: If $n \equiv 3 \mod 4$ then the group Θ_n is finite.

In particular, this holds if $n = 7$; thus, there are finitely many differentiable structures on the 7-sphere.

7.4 How many? A rough estimate

Let us take a look at the order of Θ_n / bP_{n+1}. As shown in the proof of Theorem 7.7, the group Θ_n / bP_{n+1} must be isomorphic to some subgroup of the group $\pi_{n+k}(S^k)/p(S^n)$. Now in [19,

p. 349], an explicit description of $p(S^n)$ can be found: it is the image of the so-called Hopf-Whitehead homomorphism

$$J_n : \pi_n(\mathrm{SO}(k)) \to \pi_{n+k}(S^k).$$

This gives us a method to explicitly compute $\pi_{n+k}(S^k)/p(S^k)$. The results are as follows.

n	1	2	3	4	5	6	7	8
$\pi_{n+k}(S^k)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\pi_{n+k}(S^k)/p(S^n)$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$

For an explicit computation of the groups Θ_n/bP_{n+1}, Milnor and Kervaire refer to Part II of their paper. Unfortunately, the second part has never been published. Luckily for us, the case $n = 7$ is easy: since $\pi_{n+k}(S^k)/p(S^7)$ is trivial for $n = 7$, we conclude that Θ_7/bP_8 is trivial too. Thus we conclude that the number of smooth structures on the 7-sphere is precisely $|bP_8|$.

Recall from Theorem 7.16 that bP_{4m} is a subgroup of the cyclic group of σ_m elements. It is proven in [32, p. 457] that

$$\sigma_m = 2^{2m-1}(2^{2m-1} - 1)B_m j_m a_m/m,$$

where B_m denotes the m-th Bernoulli number[2], j_m denotes the order of the cyclic group

$$J_k(\pi_{4m-1}(\mathrm{SO})) \subseteq \pi_{4m+k-1}(S^k)$$

and a_m is either 1 or 2 if m is either even or odd. An explicit computation then gives us an upper bound on the number of differentiable structures on the 7-sphere. Let us take $n = 7$. Then the only difficult part is the value of j_k. In principle, this can be calculated directly. Let us instead invoke a powerful theorem by Adams.[3]

Theorem 7.18: The value of j_k is precisely the denominator of $B_k/(4k)$.

Thus if $n = 7$ a quick calculation shows that

$$\sigma_m = \sigma_2 = 224,$$

and thus $|bP_8| \leq 224$. We thus conclude the section with the following.

Theorem 7.19: There are at most 224 differentiable structures on the 7-sphere.

[2] We will give a rough sketch of the proof in the next section.

7.5 An explicit computation of bP_{4k}

This is still not quite close to the actual answer. As before, Milnor and Kervaire refer to a non-existent Part II for the explicit computation of bP_{n+1}. A paper by Levine[23] intends to fill up this gap. In particular, §3 gives an explicit computation for bP_{4k}. This section will give an overview of the methods that are used in this paper.

Theorem 7.20: Let M be an S-parallelizable $(2k-1)$-connected $4k$-manifold whose boundary is a homotopy sphere. Then the signature $\tau(M)$ is a multiple of 8.

Proof: (Sketch) Pick $\alpha \in H_{2k}(M)$ and let $\alpha' \in H^{2k}(M, \partial M)$ be its Poincaré dual. We have $\alpha' \smile \alpha' \equiv 0 \mod 2$ because M is S-parallelizable; it follows that $\alpha \smile \alpha$ is even, and hence the intersection pairing is an even quadratic form. It is a known result[44] that the signature of an even unimodular quadratic form is a multiple of 8. Thus we are done if we show the following.

Lemma 7.21: The intersection pairing is unimodular.

Proof: We have the following diagram.

$$\begin{tikzcd} H^{2k}(M,\partial M) \arrow[r, "\sim", "i^*"'] & H^{2k}(M) \arrow[rr, "\sim", "\mathrm{PD}"'] \arrow[dr] & & H_{2k}(M,\partial M) \arrow[r, "\sim", "i_*"'] \arrow[dl] & H_{2k}(M) \\ & & \mathrm{Hom}(H_{2k}(M);\mathbb{Z}) & & \end{tikzcd}$$

The pairing is unimodular if and only if the map $H_{2k}(M) \to \mathrm{Hom}(H_{2k}(M);\mathbb{Z})$ is an isomorphism. But the above diagram factors this map into the composition of isomorphisms. ■

This concludes the proof sketch of Theorem 7.20 ■

Theorem 7.22: Let $k > 1$ and $t \in \mathbb{Z}$. Then there exists an S-parallelizable $4k$-manifold M with ∂M a homotopy sphere and signature $\tau(M) = 8t$.

The proof can be found in [8]. Using the notation above, we now define $b_k : \mathbb{Z} \to bP_{4k}$ by setting $b_k(t) = [\partial M^{4k}]$.

Lemma 7.23: If M_1 and M_2 are S-parallelizable $4k$-manifolds then ∂M_1 is cobordant to ∂M_2. Thus the map b_k is well-defined. Furthermore, the map is surjective.

Proof: It suffices to show that the connected sum $\partial M_1 \# \partial M_2$ is cobordant to zero. We know that if $W = M_1 \# M_2$ then $\partial W = \partial M_1 \# \partial M_2$. By Proposition 3.2, $\tau(W) = 0$ so by Proposition 7.13, W can be surgered into a contractible manifold. Surjectivity is an immediate consequence of Theorem 7.22. ■

It follows, then, that $bP_{4k} \cong \mathbb{Z}/\operatorname{Ker} b_k$. Levine then goes on to determine $\operatorname{Ker} b_k$. We say a manifold M is **almost s-parallelizable** if there exists a frame of $TN|_{N\setminus\{x\}}$ for some $x \in N$.

Suppose $t \in \operatorname{Ker} b_k$. Then we have an s-parallelizable manifold M with signature $8t$ whose boundary Σ is a homotopy sphere that bounds a contractible manifold D. Attaching D to M by identifying ∂M with ∂D gives an almost s-parallelizable manifold with signature $8t$. Conversely, given an almost s-parallelizable closed $4k$-manifold N with signature $8t$, let $D \subseteq N$ be any embedded disc. Then $N \setminus \operatorname{Int}(D)$ is s-parallelizable with signature $8t$ and boundary S^{4k-1}. Thus we conclude the following.

Theorem 7.24: We have $t \in \operatorname{Ker} b_k$ if and only if there exists an almost s-parallelizable closed $4k$-manifold with signature $8t$.

Levine determines $\operatorname{Ker} b_k$ by switching attention to the signature of almost s-parallelizable manifolds. It turns out that the obstruction to extending an almost framing F to a stable framing F' of M provides clues towards determining $\tau(M)$. More precisely, it is shown that the Pontryagin classes of almost s-parallelizable $4k$-manifolds can be determined by examining the k-th Pontryagin class of stable vector bundles over S^{4k}.

Theorem 7.25: If ξ is a stable vector bundle over S^{4k} then $p_k(\xi) = \pm a_k(2k-1)!\mu_\xi$, where a_k is either 1 or 2 if k is either even or odd.

Levine goes on to determine some basic properties of the Hopf-Whitehead homomorphism which we already mentioned without definition in the previous paragraph. Loosely speaking, this homomorphism provides a connection between the possible obstructions to the extension of the almost framings on the one hand, and the order j_k of the image of the Hopf-Whitehead homomorphism J_k on the other. From this analysis, we conclude the following, as already mentioned in the previous paragraph.

Theorem 7.26: The possible values for $\tau(M)$ are precisely the multiples of

$$\frac{a_k 2^{2k-1}(2^{2k-1}-1)B_k j_k}{k}.$$

We may now combine Theorem 7.24 and Theorem 7.26 to find the explicit order of bP_{4k}. In fact, we see that $|bP_{4k}|$ is precisely $1/8$ times the value of σ_k. In particular, if $n = 7$ this gives us the required strengthening of Theorem 7.19:

Theorem 7.27 (The Theorem to End All Theorems II): There are precisely 28 differentiable structures on the 7-sphere.

7.6 Addendum

I think it is safe to say we have completed our analysis of the structures of S^7. The details of the techniques involved in [33] and [23] were quite beyond the scope of this thesis (not to mention beyond my own understanding). Nevertheless I hope that I at least managed to give a rough overview of the approaches that were taken to get to the results.

Although we have mostly restricted our attention to the dimension which is relevant for this thesis, the papers above give relevant techniques for pretty much any dimension apart from $n = 3$. For the sake of completeness, the numbers are as follows.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert bP_{n+1}\rvert$	1	1	?	1	1	1	28	1	2	1	992	1	1	1	8128	1
$\lvert\Theta_n\rvert$	1	1	?	1	1	1	28	2	8	6	992	1	3	2	16.256	2

n	17	18	19	20	21	22	23	24	25	26	27
$\lvert bP_{n+1}\rvert$	2	1	261.632	1	2	1	1.448.424.448	1	2	1	67.100.672
$\lvert\Theta_n\rvert$	16	16	523.264	24	8	4	69.524.373.504	2	4	12	67.100.672

n	28	29	30	31	32	33	34	35	36
$\lvert bP_{n+1}\rvert$	1	1	1	1.941.802.827.776	1	2	1	753.623.571.759.104	1
$\lvert\Theta_n\rvert$	2	3	3	7.767.211.311.104	8	32	32	3.014.494.287.036.416	6

The recent proof the Poincaré Conjecture implies that $|\Theta_3| = 1$ so that the table is complete. For large n, precise computations are impossible because stable homotopy groups of spheres have only been computed completely up to dimension ca. 64. Apart from dimension $1, 2, 3, 5, 6, 12$, the group Θ_n is also trivial for $n = 61$. The largest calculated value of $|\Theta_n|$ occurs at $n = 63$: there exist precisely 142.211.872.163.171.481.167.115.958.878.208 differentiable structures on the 63-sphere.

Finally, it should be noted that this table does *not* give the size of the set of smooth structures when $n = 4$ (even though $|\Theta_4|$ is known). The question whether there are exotic smooth structures on the 4-sphere is known as the **Smooth Poincaré Conjecture** which is, to this very day, wide-open.

Chapter 8

Brieskorn fibrations

The classification theorem is non-constructive, in the sense that Milnor and Kervaire don't actually give an explicit construction of all exotic 7-spheres. In 1966, Egbert Brieskorn provided such an explicit construction. It relies on a completely different method: rather than considering fibre bundles, he constructed the spaces by considering zero-sets of non-trivial polynomials, then intersecting this zero-set with a sufficiently small sphere. The resulting sets can then be endowed with a natural smooth structure. Brieskorn gives twenty-eight such sets, each of homeomorphic to S^7, and each of them giving rise to a different smooth structure. By the results of Milnor and Kervaire, it follows that these must in fact be all of them.

In Chapter 9 we will give an exposition of Brieskorn's construction. We first review some basic facts related to varieties and hyper-surfaces. The contents of this chapter are entirely based on [31].

8.1 Preliminaries

Let F be an infinite field. A subset $V \subseteq F^n$ is an **algebraic set** if V is the common zero locus for some collection of polynomial functions on F^n. Let $I(V) \subseteq F[x_1, \ldots, x_n]$ be the ideal of those polynomials vanishing on V. An algebraic set is called a **variety** if it cannot be expressed as the union of two proper algebraic subsets (in other words: if $I(V)$ is a prime ideal).

Let V be an algebraic set, and let $f_1, \ldots, f_k$ span $I(V)$. For each $x \in V$, consider the matrix $\left(\partial f_i / \partial x^j|_x\right)_{i,j}$. Let ρ be the largest rank which this matrix attains at any point of V. A point $x \in V$ is **non-singular** if the matrix attains its maximal rank at x; otherwise, x is **singular**.

Proposition 8.1: The set $\Sigma(V)$ of singular points of V forms a proper algebraic subset of V.

In the special case where F is $\mathbb{R}$ or $\mathbb{C}$, we have the following wonderful result, whose proof can be found in [53].

Theorem 8.2: If F is the field of real resp. complex numbers, then the set $V \setminus \Sigma(V)$ forms a smooth, non-empty manifold. This manifold is real analytic resp. complex analytic, and has dimension $n-\rho$ over F. For any pair $V \supseteq W$ of algebraic sets, the difference $V \setminus W$ has finitely many topological components.

Let V be a real or complex variety, and let x_0 be either a non-singular point of V or an isolated point of $\Sigma(V)$. The following result will be of great importance for the construction of Brieskorn spheres.

Theorem 8.3: Every sufficiently small sphere S_ε centred at x_0 intersects V in a (possibly empty) smooth manifold K. This intersection is transverse.

Before proving this result, we need a useful lemma. Let $M_1 = V \setminus \Sigma(V)$ be the manifold of non-singular points, and let g be a polynomial on F^n.

Lemma 8.4: The set of critical points of $g|_{M_1}$ is the intersection of M_1 with the algebraic set W consisting of all points $x \in V$ at which the matrix

$$\begin{pmatrix} \partial g/\partial x^1 & \cdots & \partial g/\partial x^n \\ \partial f/\partial x^1 & \cdots & \partial f_1/\partial x^n \\ \vdots & & \vdots \\ \partial f_k/\partial x^1 & \cdots & \partial f_k/\partial x^n \end{pmatrix} \tag{$\star$}$$

has rank at most ρ.

Proof: Near any point of M_1 we can choose a local analytic system of coordinates $u^1, \dots, u^n$ for K^n so that M_1 corresponds to the locus $u^1 = \cdots = u^\rho = 0$; then $u^{\rho+1}, \dots, u^n$ can be taken as local coordinates on M_1. Since f_i is assumed to vanish on M_1, we know that $\partial f_i/\partial u^j$ evaluated at a point on M_1 is zero for all $j > \rho$. The matrix $\left(\partial f_i/\partial u^j\right)$ is column equivalent to $\left(\partial f_i/\partial x^j\right)$ and hence has rank ρ; it follows that the first ρ columns of $\left(\partial f_i/\partial u^j\right)$ must be linearly independent.

Next, we take a look at the enlarged matrix

$$\begin{pmatrix} \partial g/\partial u^1 & \cdots & \partial g/\partial u^n \\ \partial f/\partial u^1 & \cdots & \partial f_1/\partial u^n \\ \vdots & & \vdots \\ \partial f_k/\partial u^1 & \cdots & \partial f_k/\partial u^n \end{pmatrix}.$$

We see that this matrix has rank ρ if and only if $\partial g/\partial u^j = 0$ for all $j > \rho$; in other words, if and only if the point is a critical point of $g|_{M_1}$. But this new matrix is column equivalent to matrix $(\star)$, thus completing the proof. ■

Notice that it follows that polynomials on M_1 have at most a finite number of critical values. For the set of critical points of $g|_{M_1}$ can be expressed as a difference $W \setminus \Sigma(V)$ of algebraic sets, and hence as a finite union of smooth manifolds $M_1' \cup \cdots \cup M_p'$, where each M_i' has only finitely many components.

Proof (of Theorem 8.3): In the real case, consider the polynomial function $g(x) = ||x - x_0||^2$. By the discussion above, it has finitely many critical values. Let ε^2 be smaller than any critical value of $g|_{V\setminus\Sigma(V)}$. Then ε^2 will be a regular value, hence its inverse image $g^{-1}(\varepsilon^2) \cap \big(V \setminus \Sigma(V)\big) = S_\varepsilon \cap \big(V \setminus \Sigma(V)\big)$ will be a smooth manifold. We may assume ε is small enough that $S_\varepsilon \cap \Sigma(V) = \varnothing$, in which case K will be $S_\varepsilon \cap V$. The statement in the complex case is proven in the same way. ■

Let $V \subseteq \mathbb{R}^n$ be a real algebraic set, and let $U \subseteq \mathbb{R}^n$ be an open set defined by finitely many polynomial inequalities:

$$U = \big\{x \in \mathbb{R}^m : g_1(x) > 0, \ldots, g_l(x) > 0\big\}.$$

For reference, we have the following technical result, which will be used in the next sections. The proof can be found in [31, §3].

Theorem 8.5 (Curve Selection Lemma): If $U \cap V$ contains points arbitrarily close to the origin, then there exists a real analytic curve $p : [0, \varepsilon[\to \mathbb{R}^n$ with $p(0) = 0$ and $p(t) \in U \cap V$ for $t > 0$.

8.2 The Fibration Theorem

Given an analytic function f of n complex variables, we set its **complex gradient** to be $\operatorname{grad} f = \big(\overline{\partial f/\partial z_1}, \ldots, \overline{\partial f/\partial z_n}\big)$, so that the directional derivative of f along v at the point z is equal to the inner product $\langle v, \operatorname{grad} f(z)\rangle$.

First, let us define K to be the intersection of S_ε with $f^{-1}(0)$. We will always assume ε to be "sufficiently small", so that we leave it out of the notation. K is very important; as to stress its importance, we endow it with a name: the **Brieskorn variety** of f.

We define $\varphi : S_\varepsilon \setminus K \to S^1$ by sending z to $f(z)/|f(z)|$. This turns out to be an important map, maybe not as important as K, but still sufficiently important to deserve its own (non-standard) name: the **Brieskorn fibration**.

We will assume throughout this section that f vanishes at the origin, even if some results (like the next one) hold more generally.

Lemma 8.6: The critical points of φ are those points $z \in S_\varepsilon \setminus K$ for which the vector z is a real multiple of $i\,\mathrm{grad}\log f(z)$.

It should be noted that while log attains infinitely many values, its gradient is well-defined, since for all z, we have

$$\mathrm{grad}\log f(z) = (\mathrm{grad}\, f(z))/\overline{f(z)} \tag{8.7}$$

Proof: Let us set $f(z)/|f(z)| = e^{i\theta(z)}$. Note that θ can be described as the real part of $-i\log f(z)$. Differentiate $\theta = \mathrm{Re}(-i\log f)$ along some curve $z = p(t)$:

$$\begin{aligned} \frac{d\theta(p(t))}{dt} &= \mathrm{Re}\left(\frac{d(-i\log f)}{dt}\right) \\ &= \mathrm{Re}\left\langle \frac{dp}{dt}, \mathrm{grad}(-i\log f)\right\rangle \\ &= \mathrm{Re}\left\langle \frac{dp}{dt}, i\,\mathrm{grad}\log f\right\rangle \end{aligned} \tag{8.8}$$

You can already see where the $i\,\mathrm{grad}\log$ is coming from. We can think of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, with the Euclidean inner product defined to be the real part of the Hermitian one. Now if $i\,\mathrm{grad}\log f(z)$ is a real multiple of z, then by Equation 8.8, for every v tangent to S_ε at z the directional derivative of θ in the direction v will be zero. Conversely, if the vectors $i\,\mathrm{grad}\log f(z)$ and z are linearly independent over the reals, then there exists v such that $\mathrm{Re}\langle v, z\rangle = 0$ and $\mathrm{Re}\langle v, i\,\mathrm{grad}\log f(z)\rangle = 1$, and so the directional derivative of θ along v is non-zero. ■

We are going to use the above lemma for the following claim.

Lemma 8.9: If ε is sufficiently small then the map $\varphi : S_\varepsilon \setminus K \to S^1$ has no critical points at all.

Once we have shown this, it follows that, for each $e^{i\theta} \in S^1$, the inverse image

$$F_\theta = \varphi^{-1}(e^{i\theta}) \subseteq S_\varepsilon \setminus K$$

fits the hypotheses of the Regular Level Set Theorem[1], and hence is a smooth $(2n-2)$-dimensional manifold.

[1] A variation of the Implicit Function Theorem; see [22, Cor. 5.14]

Proof (of Lemma 8.9): Since this is going to take us a while, we will divide up the work in sub-statements.

Lemma 8.10: If $p : [0, \varepsilon[\to \mathbb{C}^n$ is a real-analytic path with $p(0) = 0$ such that, for each $t > 0$, the number $f(p(t))$ is non-zero, and the vector $\operatorname{grad}\log f(p(t))$ is a complex multiple $\lambda(t)p(t)$, then the argument of $\lambda(t)$ tends to zero as $t \to 0$.

Proof: Every semi-competent mathematician should, upon hearing the terms 'analytic' and 'tends to', think of Taylor expansions. Indeed, that's exactly what we are going to need.

$$\begin{aligned} p(t) &= at^\alpha + a_1 t^{\alpha+1} + \cdots, \\ f(p(t)) &= bt^\beta + b_1 t^{\beta+1} + \cdots, \\ \operatorname{grad} f(p(t)) &= ct^\gamma + c_1 t^{\gamma+1} + \cdots. \end{aligned}$$

The leading coefficients are non-zero, and by assumption, we know $\alpha, \beta \geq 1$. These series are convergent for sufficiently small t — say, $|t| < \varepsilon'$. For each $t > 0$ we have $\operatorname{grad}\log f(p(t)) = \lambda(t)p(t)$, and hence by Equation 8.7, $\operatorname{grad} f(p(t)) = \lambda(t)p(t)\overline{f(p(t))}$. In other words,

$$ct^\gamma + c_1 t^{\gamma+1} + \cdots = \lambda(t)\big(a\bar{b}t^{\alpha+\beta} + \cdots\big).$$

We see that $\lambda(t)$ is itself a quotient of real-analytic functions, and as such it has an expansion like

$$\lambda(t) = \lambda_0 t^{\gamma-\alpha-\beta}(1 + k_1 t + \cdots),$$

with the leading coefficient $\lambda_0 = c/a\bar{b}$. We substitute what we know in the identity $df(p(t))/dt = \langle dp(t)/dt, \operatorname{grad} f(t)\rangle$ to obtain

$$\begin{aligned} \beta b_1 t^{\beta-1} + \cdots &= \langle \alpha a t^{\alpha-1} + \cdots, ct^\gamma + \cdots \rangle \\ &= \alpha|a|^2\overline{\lambda_0} b t^{\alpha-1+\gamma} + \cdots. \end{aligned}$$

Comparing the leading coefficients, it follows that $\beta = \alpha|a|^2\overline{\lambda_0}$; it follows that λ_0 is a positive real number, and so we win. ■

Lemma 8.11: With the notation as before, there exists $\varepsilon_0 > 0$ so that for all $z \in \mathbb{C}^n \backslash f^{-1}(0)$ with $|z| \leq \varepsilon_0$, the vectors z and $\operatorname{grad}\log f(z)$ are either linearly independent over $\mathbb{C}$ *or* $\operatorname{grad}\log f(z) = \lambda z$, with $|\arg(\lambda)| < \pi/4$.

The choice of $\pi/4$ in the above lemma, by the way, is arbitrary. In fact the entire lemma may seem pretty arbitrary to the uninitiated reader. The choice of $\pi/4$ is made so that $\mathrm{Re}(\lambda) > 0$ so that in particular λ cannot be purely imaginary. Why this is relevant will follow soon; we first prove the lemma.

Proof (of Lemma 8.11): Suppose, to the contrary, that there are points $z \in \mathbb{C}^n \setminus f^{-1}(0)$ arbitrarily close to the origin with $\operatorname{grad} \log f(z) = \lambda z \neq 0$, with $\arg \lambda$ *greater* than $\pi/4$. Let W be the set of $z \in \mathbb{C}^n$ for which $\operatorname{grad} f(z)$ and z are linearly dependent. Notice that W is a real algebraic set, since $z \in W$ if and only if $z_j \overline{\partial f/\partial z_k} = z_k \overline{\partial f/\partial z_j}$ for all j, k.

Next, note that a point $z \in \mathbb{C}^n \setminus f^{-1}(0)$ belongs to W if and only if $(\operatorname{grad} f(z))/\overline{f(z)} = \lambda z$ for some complex λ. Multiplying with $\overline{f(z)}$ and taking the inner product with $\overline{f(z)}z$, this yields

$$\langle \operatorname{grad} f(z), \overline{f(z)}z \rangle = \lambda |\overline{f(z)}z|^2.$$

What's great is that we see that $\arg \lambda = \arg \langle \operatorname{grad} f(z), \overline{f(z)}z \rangle$. Now let U_+ resp. U_- denote the set of all z such that

$$\operatorname{Re}\big((1+i)\langle \operatorname{grad} f(z), \overline{f(z)}z \rangle\big) < 0 \quad \text{resp.} \quad \operatorname{Re}\big((1-i)\langle \operatorname{grad} f(z), \overline{f(z)}z \rangle\big) < 0. \tag{$\star$}$$

The assumption that points $z \in \mathbb{C}^n \setminus f^{-1}(0)$ with $\operatorname{grad} \log f(z) = \lambda z \neq 0$ and $\arg \lambda$ greater than $\pi/4$ get arbitrarily close to the origin is *precisely* the assumption that points $z \in W \cap (U_+ \cup U_-)$ get arbitrarily close to the origin. This allows us to invoke the machinery of Theorem 8.5 (the Curve Selection Lemma). By this result, there must exist an analytic path $p : [0, \varepsilon[\to \mathbb{C}^n$ with $p(0) = 0$ and with either $p(t) \in W \cap U_+$ for $t > 0$ or $p(t) \in W \cap U_-$ for all $t > 0$. In either case, for $t > 0$ we find that $\operatorname{grad} \log f(p(t)) = \lambda(t)p(t)$ with $|\arg \lambda(t)| > \pi/4$, which cannot happen by Lemma 8.10. We conclude that our initial assumptions are impossible.

There is another case to be considered. Suppose now that there are points $z \in \mathbb{C}^n \setminus f^{-1}(0)$ arbitrarily close to the origin with $\operatorname{grad} \log f(z) = \lambda z \neq 0$ and $\arg \lambda$ *precisely equal* to $\pi/4$. In this case we go through the same argument, replacing the inequalities in ($\star$) by an '='. ■

The statement we just showed is somewhat stronger than what we really need:

Corollary 8.12: For every $z \in \mathbb{C}^n \setminus f^{-1}(0)$ which is sufficiently close to the origin, the two vectors z and $i \operatorname{grad} \log f(z)$ are linearly independent over $\mathbb{R}$.

Finally, we conclude the proof of Lemma 8.9, which is a combination of Lemma 8.6 and Corollary 8.12. ■

As mentioned before, Lemma 8.9 enables us to conclude that $F_\theta = \varphi^{-1}(e^{i\theta})$ is a smooth $(2n-2)$-manifold. While not particularly interesting on its own, it is the behaviour of φ as θ varies that we are interested in; the crux of this section is the following theorem, which also summarizes this behaviour quite nicely.

Theorem 8.13 (Fibration Theorem): For ε sufficiently small the space $S_\varepsilon \setminus K$ is a smooth fibre bundle over S^1, with projection mapping φ.

In other words, the Brieskorn fibration is not just any map with manifold pre-images: it is the projection map of a locally trivial fibration, which explains the name we gave. We start with a lemma again.

Lemma 8.14: If ε is sufficiently small then there exists a smooth tangential vector field $v(z)$ on $S_\varepsilon \setminus K$ such that for all z, $\langle v(z), i \operatorname{grad} \log f(z)\rangle \neq 0$ and $|\arg(v(z))| < \pi/4$.

Proof: First note that we can prove this theorem locally; the result follows by a partition-of-unity argument. So, let's pick an arbitrary $z \in S_\varepsilon \setminus K$. By Lemma 8.11 we see that there are only two cases to consider.

Case I. The vectors z and $\operatorname{grad} \log f(z)$ are linearly independent. Then there exists a simultaneous solution to the linear equations $\langle v, z\rangle = 0$, $\langle v, i \operatorname{grad} \log f(z)\rangle = 1$.

Case II. We have $\operatorname{grad} \log f(z) = \lambda z$ with $|\arg(\lambda)| < \pi/4$. In this case we set $v(z) = iz$, and a straightforward calculation shows that this choice suits our demands.

In either case, we can choose a local tangential vector field in a neighbourhood of z. By continuity, we know that the requirement that $|\arg\langle v(z), i \operatorname{grad} \log f(z)\rangle| < \pi/4$ will hold throughout a sufficiently small neighbourhood. ■

Proof (of Theorem 8.13): Notice that we can normalize v to make sure that at all points, we have $\operatorname{Re}\left(\langle v(z), i \operatorname{grad} \log f(z)\rangle\right) = 1$; since we required that $|\arg(v(z))| < \pi/4$, we also know that $\operatorname{Im}\left(\langle v(z), i \operatorname{grad} \log f(z)\rangle\right) < 1$. The joy that v will bring us will become apparent when we consider the trajectories of the differential equation $dz/dt = v(z)$. Given $z \in S_\varepsilon \setminus K$, we would like to find a smooth path $p_z : \mathbb{R} \to S_\varepsilon \setminus K$ which satisfies the differential equation $dp/dt = v(p_z(t))$ with $p_z(0) = z$. Certainly, a solution exists locally, and can be extended over some maximum open interval.

There is one thing that we have to be careful with: it could happen that $p_z(t)$ tends towards K as t tends to some finite limit t_0. Since $K = S_\varepsilon \cap f^{-1}(0)$, this amounts to saying that $f(p_z(t))$ might tend towards 0 as t tends to t_0, or, again equivalently, that $\operatorname{Re}(\log f(p_z(t))) \to -\infty$ as $t \to t_0$. But this will not happen because

$$\begin{aligned}\frac{d \operatorname{Re}(\log f(p_z(t)))}{dt} &= \operatorname{Re}\left(\left\langle \frac{dp_z(t)}{dt}, \operatorname{grad} \log f(p_z(t))\right\rangle\right)\\ &= \operatorname{Re}(\langle v(p_z(t)), \operatorname{grad} \log f(p_z(t))\rangle) > -1,\end{aligned}$$

where we use the fact that we normalized v.

We now apply this knowledge to a particular case, which will be used to prove the main result. By Equation 8.8 applied to our situation, we find that $\theta(p_z(t)) = t$, modulo some constant. In other words, $p_z(t)$ projects under φ to a path which winds around the unit circle with unit velocity. We can then define $h_t(z) = p_z(t)$. For each t, h_t is a diffeomorphism mapping $S_\varepsilon \setminus K$ to itself. Furthermore, each fibre F_θ is mapped diffeomorphically onto the fibre $F_{\theta+t}$.

We can now give the explicit proof of the theorem. Let $e^{i\theta} \in S^1$ be arbitrary, and let U be a small open neighbourhood of $e^{i\theta}$. An explicit diffeomorphism $U \times F_\theta \to \varphi^{-1}(U)$ is then given by sending $\left(e^{i(\theta+t)}, z\right)$ to $h_t(z)$. ■

8.3 Morse-theoretical analysis of the topology of K

In this section, we will continue our study of the Brieskorn fibration, which we have just shown to indeed be a locally trivial fibration over S^1. We still assume $f(z_1, \dots, z_n)$ vanishes at the origin. We will expose some nice results using Morse theory associated with the real-valued function $|f|$. We first introduce some notation. We define $a_\theta : F_\theta \to \mathbb{R}$ and $a : S_\varepsilon \setminus K \to \mathbb{R}$ by

$$a_\theta(z) = a(z) = \log |f(z)|.$$

The critical points of a are precisely those of $|f|$, and the critical points of a_θ are those of a, restricted to F_θ.

We start with an alternative description of the critical points of a_θ. Note that by Equation 8.8, the directional derivative of a_θ in the direction v is precisely $\mathrm{Re}\left(\langle v, \operatorname{grad} \log f(z)\rangle\right)$, thus z is a critical point if and only if $\operatorname{grad} \log f(z)$ is normal to F_θ. But F_θ has (real) co-dimension 2 in $\mathbb{C}^n$, and its normal bundle is everywhere spanned by z and $i \operatorname{grad} \log f(z)$. Thus z is a critical point if and only if $\operatorname{grad} \log f(z)$ is a linear combination of z and $i \operatorname{grad} \log f(z)$. We summarize this with the following lemma.

Lemma 8.15: The critical points of a_θ are precisely those points z for which $\operatorname{grad} \log f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$.

Recall from Section 4.2, that at a critical point of a_θ, we can define the Hessian.

Lemma 8.16: At $t = 0$, we have

$$\ddot{a}_\theta(p(t)) = \sum_{j,k} \mathrm{Re}(b_{jk} v_j v_k) - c|v|^2,$$

with (b_{jk}) some complex-valued matrix and c is a positive real number.

Proof: Since p is required to lie on F_θ, we have

$$a_\theta(p(t)) = \log|f(p(t))| = \log f(p(t)) - i\theta.$$

From this point on, the proof is a matter of straightforward algebraic manipulations, whose details we will omit, but can be found in [31, Lem. 5.5]. ■

In order to apply Morse theory in its usual form, we need a non-degenerate mapping $g : F_\theta \to \mathbb{R}$ such that the set of z with $g(z) \leq c$ is compact for every constant c. The second property is attained if we can find a g which is both proper and bounded from below. This motivates the following lemma.

Lemma 8.17: There exists $\eta_\theta > 0$ so that all the critical points of a_θ lie within the compact subset $\{z \in F_\theta : |f(z)| \geq \eta_\theta\}$ of F_θ.

Proof: If there were critical points of a_θ with $|f(z)|$ arbitrarily close to zero, then these critical points would have a limit point z_0 on S_ε. By the Curve Selection Lemma, there would exist a smooth path $p : [0, \varepsilon[\to F_\theta$ consisting of critical points, with $p(t) \to z_0$ as $t \to 0$. But since a_θ is, by its definition, constant along this path, so is $|f|$, and as such, $|f|$ cannot tend to zero. This gives a contradiction. ■

Note that from this lemma, there exists $\eta > 0$ so that the critical points z of a satisfy $|f(z)| \geq \eta$. We can use the above lemma to give a bound on the index of the quadratic function

$$\ddot{a}_\theta(z)(v) = \sum_{j,k} \mathrm{Re}(b_{jk} v_j v_k) - c|v|^2, \tag{$\star$}$$

where $v \in T_z F_\theta$. If $\ddot{a}_\theta(v) \geq 0$ for any non-zero v, a quick glance at ($\star$) shows that we have $\ddot{a}_\theta(iv) < 0$. This motivates us to split the tangent space of F_θ at z into a direct sum $T_+ \oplus T_-$, where $\ddot{a}_\theta$ is positive semi-definite on T_+ and negative definite on T_-. By what we just mentioned, $\ddot{a}_\theta$ is also negative definite on iT_+, and therefore $\dim T_0 \geq \dim(iT_1) = \dim T_1 = 2(n-1) - \dim T_0$; in other words, $\dim T_0 \geq n - 1$. Now by definition, $\dim T_0$ is precisely the index at z. Thus we can summarize the discussion as follows.

Lemma 8.18: The index of a_θ at a critical point is $\geq n - 1$.

This information will be sufficient to find the function g we have been longing for. First off, we are going to find a map $s_\theta : F_\theta \to \mathbb{R}_{\geq 0}$ so that all critical points of s_θ are non-degenerate with index $\neq n-1$, and so that $s_\theta(z) = |f(z)|$ whenever $|f(z)|$ is sufficiently close to zero. This can be done as follows. By Lemma 8.17, the critical points of $|f|$ lie in a compact set, and by Lemma 8.18, these critical points all have index $\geq n-1$. We construct s_θ from $|f|$ by *altering* it on a compact neighbourhood of the set of critical points, so as to make the critical points non-degenerate, and so that the first and second derivatives of s_θ uniformly approximate those of $|f|$. Furthermore, if the approximation is sufficiently close, the critical points of s_θ also have index $n-1$. By Theorem 4.2 such an approximation is allowed.

The function g is then found by simply setting $g(z) = -\log s_\theta(z)$. Clearly this function is now bounded from below, and for every c, the set of z with $g(z) \leq c$ is compact. The index of g is that of $-\log s_\theta$, which is precisely $2(n-1) - \text{index}(\log s_\theta) \leq n-1$. This concludes our construction. The reason for constructing g is because it enables us to prove the following results. Recall from Chapter 7 that a manifold is parallelizable if its tangent bundle is trivial.

Theorem 8.19: Each fibre F_θ is parallelizable, and has the homotopy type of a CW complex of dimension $n-1$.

Theorem 8.20: The Brieskorn variety K of f is $(n-3)$-connected. Thus for $n \geq 3$ the space is connected, and for $n \geq 4$ it is simply-connected.

Proof (of Theorem 8.19): We need the following result, as found in [30, Thm. 3.5].

Theorem 8.21: Let f be a smooth function on a manifold M with no degenerate critical points, and suppose each $M^a = f^{-1}(]-\infty, a[)$ is compact. Then M has the homotopy type of a CW complex, with one cell of dimension λ for each critical point of index λ.

According to this result, the manifold F_θ has the homotopy type of a CW complex of dimension $\leq n-1$, made up of one cell for each critical point of g. This proves the second part of the theorem.

As a consequence, if $n-1 \geq 1$, the homology group $H_{2n}(F_\theta; \mathbb{Z}/2\mathbb{Z})$ will be zero, so the $2n$-dimensional manifold F_θ cannot have any compact component. This follows from Proposition 7.5. The result can be applied here because F_θ is embedded in the sphere S_ε (and hence in $\mathbb{C}^n$) with trivial normal bundle. ■

Proof (of Theorem 8.20): (Sketch) Let $N_\eta(K)$ denote the neighbourhood of K consisting of $z \in S_\varepsilon$ with $|f(z)| \leq \eta$. By Lemma 8.17, all critical points of a lie outside this region, hence $N_\eta(K)$ is a manifold (with boundary). We can define s on $S_\varepsilon \setminus K$ in the same way that we defined s_θ on

F_θ. If we restrict this function s to $S_\varepsilon \setminus \operatorname{Int}(N_\eta(K))$, we see that the sphere S_ε has the homotopy type of a complex built up from $N_\eta(K)$ by adjoining cells of dimension $\geq n-1$, one cell for each critical point of s.

The adjunction of a cell of dimension $\geq n-1$ does not alter the homotopy groups of dimension $\leq n-3$, and therefore $\pi_i(N_\eta(K)) \cong \pi_i(S_\varepsilon) = 0$ for $i \leq n-3$. The proof follows because K is a retract of the neighbourhood of $N_\eta(K)$, provided that η is sufficiently small. ■

8.4 The case of an isolated critical point

From now on, we make the additional assumption that the origin is either a non-singular point, or an isolated singular point. In this case, Theorem 8.3 tells us that K is a smooth $(2n-3)$-dimensional manifold. The result does not give us any interesting details about it. Let us therefore improve the result.

Lemma 8.22: The closure $\overline{F_\theta}$ of each fibre F_θ in S_ε is a smooth $(2n-2)$-dimensional manifold with boundary, the interior being F_θ and the boundary being K.

Proof: By Theorem 8.3, the mapping $f|_{S_\varepsilon}$ has no critical points on K provided ε is sufficiently small. Let $z_0 \in K$ be an arbitrary point. Choose coordinates $u_1, \ldots, u_{2n-1}$ for S_ε in a neighbourhood U of z_0 so that $f(z) = u_1(z) + iu_2(z)$ for all $z \in U$. This is allowed since we can locally approximate our function f to a linear one. A point of U then belongs to F_0 if and only if $u_1 > 0$ and $u_2 = 0$. Therefore $\overline{F_0}$ intersects U in the set $u_1 \geq 0$, $u_2 = 0$. This is a smooth $(2n-2)$-dimensional manifold, with $F_0 \cap U$ as interior and $K \cap U$ as its boundary. We can do this for every $z_0 \in K$; thus, the result follows for $\theta = 0$. Since the result does not depend on θ, we can repeat this discussion for the other fibres. ■

Corollary 8.23: If $n \geq 4$ then the fibre F_θ is $(n-2)$-connected.

Proof: The manifold $\overline{F_\theta}$ is embedded in S_ε in such a way as to have the same homotopy type as its complement $S_\varepsilon \setminus \overline{F_\theta}$. For the complement is a locally trivial fibre space over the contractible manifold $S^1 \setminus \{e^{i\theta}\}$. Hence $S_\varepsilon \setminus \overline{F_\theta}$ has any other fibre $F_{\theta'}$ as deformation retract, and in particular the fibre F_θ.

Next, apply Alexander Duality (2.7) to $S_\varepsilon \setminus \overline{F_\theta}$, and use the above info to find that the fibre F_θ has the homology of a point in dimensions less that $n-1$. Thus to prove the main result we need only verify that F_θ is simply-connected when $n \geq 2$. To this end we need the following lemma, which is a result from Morse theory.

Lemma 8.24: There exists a smooth mapping $s_\theta : F_\theta \to \mathbb{R}_+$ so that all critical points of s_θ are non-degenerate, with Morse index $\leq n-1$, and so that $s_\theta(z) = -|f(z)|$ whenever $|f(z)|$ is close to zero.

Using the Morse function s_θ on $\overline{F_\theta}$, we see that $\overline{F_\theta}$ can be built up by starting with a $(2n-2)$-disk, and successively adjoining handles of index $\leq n-1$. All of these handles can be attached within the space S_ε. Now $S_\varepsilon \setminus D^{2n-2}$ certainly is simply-connected, and the adjunction of handles cannot change that, since their index does not exceed $\dim S_\varepsilon - 3$. ∎

Theorem 8.25: If $n \geq 3$, each fibre has the homotopy type of a bouquet of $(n-1)$-spheres.

Proof: The homology group $H_{n-1}(F_\theta)$ must be free abelian, since any torsion elements would give rise to n-dimensional cohomology classes, which, by Theorem 8.19, cannot happen. By the Hurewicz Theorem (2.8), there exist finitely many pointed maps from S^{n-1} to F_θ which represent a basis of $\pi_{n-1}(F_\theta)$. These combine to yield a map $S^{n-1} \vee \cdots \vee S^{n-1} \to F_\theta$. By the Whitehead Theorem (2.9) this must be a homotopy equivalence, since it induces an isomorphism on homology groups. ∎

Chapter 9

Determining the smooth structures

We go on to take a look at the smooth structures of the different Brieskorn varieties. In Section 9.1 we will give necessary and sufficient conditions for a specific class of Brieskorn varieties to be a topological sphere. Then in Section 9.2 we determine their smooth structures by constructing a suitable 8-dimensional co-boundary and determining its signature.

This chapter is loosely based on several sources. The main source is [7], the original paper by Brieskorn. Unforturnately, the paper is written in German, so I have occasionally borrowed information from the English expositions [16] and [31].

9.1 Brieskorn varieties and topological spheres

Throughout this chapter, we restrict our attention to the function

$$f(z_1, \ldots, z_n) = f_a(z_1, \ldots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n},$$

even though many results hold for arbitrary functions with an isolated critical point at the origin. The corresponding Brieskorn variety $K = K_a = f_a^{-1}(0) \cap S_\varepsilon$ is a smooth $(2n-3)$-manifold. Let $\varphi : S_\varepsilon \setminus K \to S^1$ be the associated Brieskorn fibration, with fibres F_θ of dimension $2n-2$.

An interesting question to ask, at least in view of the topic of this thesis, is the following. Let $f(z_1, \ldots, z_n)$ be a polynomial which attains an isolated critical point at the origin. How can we decide whether or not its $(2n-3)$-dimensional Brieskorn variety K is a topological sphere? In a moment, we summarize the answer with Theorem 9.1. Let us first establish some notation.

Choose an orientation for the $(2n-2)$-dimensional manifold $\overline{F_\theta}$, and note that any two homology classes $\alpha, \beta \in H_{n-1}(\overline{F_\theta})$ have a well-defined intersection number which we denote by $s(\alpha, \beta)$.

Next, we consider an arbitrary fibre bundle $\varphi : E \to S^1$. The natural action of a generator $\pi_1(S^1)$ induces a homeomorphism h of the fibre F_0. Let $h_* : H_*(F_0) \to H_*(F_0)$ be the induced isomorphism on homology. We denote by Δ the characteristic polynomial $\det(t\ \mathrm{id}_* - h_*)$ of the linear transformation $h_* : H_{n-1}(F_0) \to H_{n-1}(F_0)$.

Finally, given $a = (a_1, \ldots, a_n)$ we consider the graph Γ_a with n vertices $a_1, \ldots, a_n$ and an edge between a_i, a_j if and only if $\gcd(a_i, a_j) > 1$.

Theorem 9.1: If $n \geq 4$ then the following are equivalent.

(1) K is homeomorphic to S^{2n-3}.
(2) K has the homology of a sphere.
(3) $H_{n-2}(K)$ is trivial.
(4) $\det s = \pm 1$.
(5) $\Delta(1) = \pm 1$.
(6) Γ_a has at least two isolated points, or it has one isolated point and at least one connected component K with an odd number of vertices such that $\gcd(\gamma_i, \gamma_j) = 2$ for $\gamma_i, \gamma_j \in K$, $i \neq j$.

The proof of this result will take up the rest of this section.

Proof: If $n \geq 4$ then K is simply-connected by Theorem 8.20, hence the hypotheses of the Generalized Poincaré Conjecture (6.21) are satisfied. From this we see that **(2) implies (1)**. The converse, **(1) implies (2)**, is obvious.

This criterion can be somewhat strengthened as follows. By Theorem 8.20, together with Poincaré Duality, it immediately follows that K is a homology sphere once we have that $H_{n-2}(K)$ is trivial. Thus **(3) implies (2)**; once again, the converse, **(2) implies (3)**, is trivial.

Next, consider the long exact sequence of the pair $(\overline{F_\theta}, K)$:

$$H_{n-1}(\overline{F_\theta}) \xrightarrow{j_*} H_{n-1}(\overline{F_\theta}, K) \xrightarrow{\partial} H_{n-2}(K) \longrightarrow 0.$$

From Theorem 8.25 we know that the first group is free abelian of rank μ (we will give an explicit description of μ in Lemma 9.3). It follows by Poincaré Duality that $H_{n-1}(\overline{F_\theta}, K)$ is also free abelian of rank μ, and that the intersection pairing $s' : H_{n-1}(\overline{F_\theta}, K) \otimes H_{n-1}(\overline{F_\theta}) \to \mathbb{Z}$ has determinant ± 1. Now, using the identity $s(\alpha, \beta) = s'(j_*\alpha, \beta)$, we find that j_* is an isomorphism if and only if $\det s = \pm 1$. Thus we find **(3) implies (4)** and **(4) implies (3)**.

Next, we will need the following lemma.

Lemma 9.2: To any fibre bundle $\varphi : E \to S^1$ there exist an exact sequence (the **Wang sequence**) of the following form

$$\cdots \longrightarrow H_{j+1}(E) \longrightarrow H_j(F_0) \xrightarrow{h_* - \mathrm{id}_*} H_j(F_0) \longrightarrow H_j(E) \longrightarrow \cdots .$$

Proof: (Sketch) The Covering Homotopy Theorem induces a mapping $F_0 \times [0, 2\pi] \to E$ which gives rise to an isomorphism

$$H_j(F_0 \times [0, 2\pi], F_0 \times \{0, 2\pi\}) \xrightarrow{\sim} H_j(E, F_0).$$

The result follows by substituting this into the exact sequence associated with the pair (E, F_0).■

We now apply this sequence to our situation; we find

$$H_{n-1}(F_0) \xrightarrow{h_* - \mathrm{id}_*} H_{n-1}(F_0) \longrightarrow H_{n-1}(S_\varepsilon \setminus K) \longrightarrow 0.$$

The result immediately follows upon applying Alexander Duality (2.7) to $H_{n-1}(S_\varepsilon \setminus K) \cong H^{n-1}(K)$, then applying Poincaré Duality to the latter space. We find that **(3) implies (5)** and **(5) implies (3)**.

It remains to be shown that (6) is equivalent with the other statements; this part is based on [7, Satz 1]. We will prove this at the end of this section. ■

We consider the group

$$G_a = \prod_{k=1}^{n} \mathbb{Z}/a_k\mathbb{Z}.$$

If we denote by ω_k the generator $\exp(2\pi i/a_k)$ of $\mathbb{Z}/a_k\mathbb{Z}$, then G_a has a natural action on $f_a^{-1}(1)$ given by

$$\left(\omega_1^{k_1}, \ldots, \omega_n^{k_n}\right) : (z_1, \ldots, z_n) \mapsto \left(\omega_1^{k_1} z_1, \ldots, \omega_n^{k_n} z_n\right).$$

Let J_a be the group ring $\mathbb{Z}[G_a]$, and let I_a be the ideal of J_a generated by the elements of the form $1 + \omega_k + \cdots + \omega_k^{a_k - 1}$.

Lemma 9.3: The group $H_{n-1}(F_\theta)$ is free abelian of rank

$$\mu = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1).$$

Proof: As mentioned in the proof of Theorem 8.25, we already know that the group must be free abelian. The proof is found in [16, p. 15]. We prove the result for F_0; for other θ, the result follows from the Fibration Theorem (8.13). First, note that we have

$$\begin{aligned} F_0 &= \{z \in S_\varepsilon : \varphi(z) = 1\} \\ &= \left\{(z_1, \dots, z_n) \in \mathbb{C} : \frac{z_1^{a_1} + \cdots + z_n^{a_n}}{|z_1^{a_1} + \cdots + z_n^{a_n}|} = 1 \text{ and } \left|z_1^{a_1} + \cdots + z_n^{a_n}\right| = \varepsilon\right\} \\ &= f_a^{-1}(\varepsilon). \end{aligned}$$

There exists a diffeomorphism between $f_a^{-1}(1)$ and $f_a^{-1}(\varepsilon)$: namely, just send every $(z_1, \dots, z_n)$ to $\left(\varepsilon^{1/a_1} z_1, \dots, \varepsilon^{1/a_n} z_n\right)$. We may thus restrict our attention to $f_a^{-1}(1)$, as to make the notation a bit easier. Consider the subspace U_a of $f_a^{-1}(1)$ given by

$$U_a = \left\{(z_1, \dots, z_n) \in f_a^{-1}(1) : z_j^{a_j} \in \mathbb{R}_{\geq 0} \text{ for all } j = 1, \dots, n\right\}.$$

Lemma 9.4: The subspace U_a is a deformation retract of $f_a^{-1}(1)$ by a deformation compatible with the operations of G_a.

The proof of the lemma can be found in [39, p. 338]. Next, we note that we can describe U_a by the conditions that $z_j = u_j|z_j|$ for some a_j-th root of unity $u_j \in \mathbb{Z}/a_j\mathbb{Z}$. If we put $t_j = |z_j|^{a_j}$, then U_a becomes the space of n-tuples of complex numbers

$$t_1 u_1 \oplus \cdots \oplus t_n u_n \quad \text{with} \quad u_j \in \mathbb{Z}/a_j\mathbb{Z},\, t_j \geq 0,\, \sum_{j=1}^{n} t_j = 1.$$

In other words, we can identify U_a with the join $J = *_j\, \mathbb{Z}/a_j\mathbb{Z}$ when the elements $\mathbb{Z}/a_j\mathbb{Z}$ are interpreted as roots of unity in $\mathbb{C}$.

Lemma 9.5: Let A, B be spaces with torsion-free integral homology. The reduced singular homology groups of the join $A * B$ with coefficients in a principal ideal domain are given by

$$\widetilde{H}_{r+1}(A * B) \cong \sum_{i+j=r} \widetilde{H}_i(A) \otimes \widetilde{H}_j(B)$$

The proof can be found in [27, Lem. 2.1] and relies on the Künneth Theorem (2.5). As a result, we find that

$$H_{n-1}(f_a^{-1}(1)) = \widetilde{H}_{n-1}(f_a^{-1}(1)) \cong \bigoplus_{k=1}^{n} \widetilde{H}_0(\mathbb{Z}/a_k\mathbb{Z}).$$

Note that the proof of this result also implies that all lower homology groups vanish; in particular, this provides us with an alternative proof of Corollary 8.23. In any case, this completes the proof of Lemma 9.3. ■

Since U_a can be seen as a join of the $\mathbb{Z}/a_j\mathbb{Z}$, it is an n-dimensional simplicial complex which has an $(n-1)$-simplex for each element of G_a. Let us denote by e the $(n-1)$-simplex belonging to the unit of G_a. All other $(n-1)$-simplices are then obtained from e by the action of G_a on U_a. Thus we find that n-dimensional chain group $C_{n-1}(U_a)$ coincides with $J_a e$. The homology group $H_{n-1}(U_a)$ is then an additive subgroup of J_a

Recall from Section 2.1 the definition of the j-th face map, henceforth denoted ∂_j. It is easily verified that it commutes with the action of G_a on $C_{n-1}(U_a)$, and furthermore that it satisfies $\partial_j = \omega_j \circ \partial_j$. From this it follows that

$$\mathfrak{e} := (1-\omega_1)\cdots(1-\omega_n)e$$

is an $(n-1)$-cycle, hence it represents an element of $H_{n-1}(U_a)$, which we also denote by $\mathfrak{e}$. It follows that $H_{n-1}(f_a^{-1}(1)) = J_a\mathfrak{e}$. We summarize what we have found so far with the following lemma; the result was first proved in [39].

Lemma 9.6: The map $G_a \to H_{n-1}(f_a^{-1}(1))$ given by $g \mapsto g\mathfrak{e}$ induces an isomorphism $J_a/I_a \xrightarrow{\sim} H_n(f_a^{-1}(1))$. Therefore, a basis of $H_{n-1}(f_a^{-1}(1);\mathbb{C}) \cong H_{n-1}(f_a^{-1}(1)) \otimes \mathbb{C}$ is given by the vectors

$$\left(\sum_{r=0}^{a_1-1} u_1^r\omega_1^r, \ldots, \sum_{r=0}^{a_n-1} u_n^r\omega_n^r\right),$$

ranging over all a_i-th roots of unity u_i $(i = 1, \ldots, n)$.

The results above provide us with an easy expression for the characteristic polynomial.

Corollary 9.7: The characteristic roots of the linear transformation h_* are the products $u_1 \cdots u_n$, where each u_j ranges over all a_j-th roots of unity other than 1. Hence the characteristic polynomial is given by

$$\Delta(t) = \prod (t - u_1 \cdots u_n).$$

Proof: The fibration $\varphi : S_\varepsilon \setminus K \to S^1$ extends to a map $\psi : \mathbb{C}^n \setminus f^{-1}(0) \to S^1$, which is defined by the same formula. Using the family of diffeomorphisms $h_t : \mathbb{C}^n \setminus f^{-1}(0) \to \mathbb{C}^n \setminus f^{-1}(0)$ defined by

$$h_t(z_1, \ldots, z_n) = \left(\omega_1^t z_1, \ldots, \omega_n^t z_n\right),$$

we see that each h_t carries the fibre $\psi^{-1}(y)$ diffeomorphically onto the fibre $\psi^{-1}(e^{2\pi i t}y)$. It follows that ψ is locally trivial.

Each fibre $\psi^{-1}(y)$ is diffeomorphic to $\varphi^{-1}(y) \times \mathbb{R}$ under the correspondence

$$(z, r) \mapsto \left(e^{r/a_1} z_1, \ldots, e^{r/a_n} z_n\right)$$

for all $z \in S_\varepsilon \setminus K$, $r \in \mathbb{R}$. Clearly the homeomorphism $h_{2\pi}$ carries the join $J = *_j \, \mathbb{Z}/a_j\mathbb{Z}$ to itself, and $h_{2\pi}|_J$ can be described as the 'join' $*_j \, r_{a_j} : J \to J$, where $r_{a_j} : \mathbb{Z}/a_j\mathbb{Z} \to \mathbb{Z}/a_j\mathbb{Z}$ is obtained by $x \mapsto \omega_j x$.

The map r_{a_j} induces a map $(r_{a_j})_* : \widetilde{H}_0(\mathbb{Z}/a_j\mathbb{Z}; \mathbb{C}) \to \widetilde{H}_0(\mathbb{Z}/a_j\mathbb{Z}; \mathbb{C})$. The eigenvalues of this map are the a_j-th roots of unity, other than 1. To see this, let $n \in \{1, \ldots, a_j - 1\}$ and consider the homology class in $\widetilde{H}_0(\mathbb{Z}/a_j\mathbb{Z}; \mathbb{C})$ which associates the coefficient ξ^n to each point $\xi \in \mathbb{Z}/a_j\mathbb{Z}$. This is an eigenvector of $(r_{a_j})_*$ with eigenvalue $\exp(-2\pi i n/a)$. Hence the eigenvalues of the homomorphism $(h_{2\pi}|_J)_* = (r_{a_1})_* \otimes \cdots \otimes (r_{a_n})_*$ are the products $u_1 \cdots u_n$, where u_j ranges over all a_j-th roots of unity other than 1. ■

Finally, we are ready to prove the last of equivalence of Theorem 9.1.

Proof (of Theorem 9.1; continued): We prove **(5) implies (6)** and **(6) implies (5)**. First note that we can strengthen statement (5); by Corollary 9.7, we must have $\Delta(1) = +1$. From this corollary, we also know that $\Delta(1)$ is precisely the product of cyclotomic polynomials $\Phi_d(t)$, where d runs over the orders of the products $\omega_1^{i_1} \cdots \omega_n^{i_n}$. It is a known result that $\Phi_{p^n}(1) = p$ for all primes p, and $\Phi_d(1) = 1$ otherwise. Thus, $\Delta(1) = 1$ if and only if for all $(i_1, \ldots, i_n)$ $(0 < i_k < a_k)$ the element $\omega_1^{i_1} \cdots \omega_n^{i_n}$ is not a prime power.

It remains to be shown that this is equivalent to (6). Let K be a component of Γ_a; we label the vertices of Γ_a so that $K = \{\gamma_1, \ldots, \gamma_r\}$. Let $\kappa(K)$ be the number of r-tuples $(i_1, \ldots, i_r)$ $(0 < i_k < a_k)$ so that $\omega_1^{r_1} \cdots \omega_r^{i_r} = 1$. Then $\kappa(K) = 0$ if and only if either $\gcd(\gamma_i, \gamma_j) = 2$ for all $\gamma_i, \gamma_j \in K$, $i \neq j$, or K consists of only one point.

Clearly, at least *two* components K are such that $\kappa(K) = 0$ if and only if for all $(i_1, \ldots, i_n)$ $(0 < i_k < a_k)$ the element $\omega_1^{i_1} \cdots \omega_n^{i_n}$ is not a prime power. Thus, it follows that (5) and (6) are indeed equivalent. ■

We will apply this knowledge to the specific polynomial that is used to construct exotic 7-spheres. Theorem 9.1 immediately shows that the resulting Brieskorn variety is a topological sphere.

9.2 Differentiable structures on Brieskorn varieties

We first recall some notation. By S_ε we denoted the $(2n-1)$-sphere of sufficiently small radius ε. Let us denote by D_ε the corresponding disk. The Brieskorn variety $K_a = f^{-1}(0) \cap S_\varepsilon$ is then clearly bounded by the manifold $M_a = f^{-1}(0) \cap S_\varepsilon$.

A known result by Ehresmann[10] tells us that for sufficiently small δ, the manifold $K_a(\delta) = f^{-1}(\delta) \cap S_\varepsilon$ is diffeomorphic to K_a. Furthermore, if δ is sufficiently small then $M_a(\delta) = f^{-1}(\delta) \cap D_\varepsilon$ has trivial normal bundle in $\mathbb{C}^n$ and is thus parallelizable (Proposition 7.5). Finally, if δ is sufficiently small, then $M_a(\delta)$ has interior diffeomorphic to $f_a^{-1}(1)$.[7, p. 9]

From now on we assume that n is odd, and that δ is sufficiently small to meet the above requirements. Our goal is to calculate the signature $\tau_a = \tau(M_a(\delta))$. We recall some results from Chapter 7. If M' and M'' are orientable parallelizable $4n$-dimensional manifolds with boundary homotopy spheres Σ' and Σ'', then by Theorem 7.16, $\tau(M')$ and $\tau(M'')$ differ by a factor $\sigma_n = 8 \cdot bP_{4n}$. In this thesis, the last equality has been proved only in the case $n = 2$ (see Section 7.5), but it holds more generally.

We now prove the following result. It will be the main tool for finding the exotic spheres; as such, I have given it a dramatic name to stress its importance.

Theorem 9.8 (Brieskorn's Epiphany): If $n \geq 5$ is odd, and $K_a(\delta)$ is a topological sphere, then the diffeomorphism type of $K_a(\delta)$ is completely determined by the signature τ_a. We have $\tau_a = \tau_a^+ - \tau_a^-$, where τ_a^+ is precisely the number of n-tuples of integers $(x_1, \ldots, x_n)$, with

$$0 < x_j < a_j \quad \text{and} \quad 0 < \sum_{j=1}^{n} \frac{x_j}{a_j} < 1 \mod 2,$$

and τ_a^- is precisely the number of n-tuples with

$$0 < x_j < a_j \quad \text{and} \quad -1 < \sum_{j=1}^{n} \frac{x_j}{a_j} < 0 \mod 2.$$

Proof: (Sketch) A basis for $H_{n-1}(f_a^{-1}(1); \mathbb{C}) \cong J_a/I_a \otimes \mathbb{C}$ is given in Lemma 9.6. A basis element

$$\left(\sum_{r=0}^{a_1-1} u_1^r \omega_1^r, \ldots, \sum_{r=0}^{a_n-1} u_n^r \omega_n^r \right)$$

depends only on the choice of roots of unity $u_1, \ldots, u_n$, and can thus be denoted by v_j, where $j = (j_1, \ldots, j_n)$ and j_k is precisely the number such that $u_k = \exp(2\pi i j_k / a_k)$.

Given v_j, v_i, a tedious calculation shows that applying the signature form to v_j, v_i gives

$$\langle v_j, v_i\rangle = (-1)^{\frac{(n-1)(n-2)}{2}}\left(1 - u_1^{j_1}\cdots u_n^{j_n}\right)$$
$$\times \prod_{k=1}^{n}\left(1-\left(u_k^{j_k}\right)^{-1}\right)\prod_{k=1}^{n}\left(1+u_k^{j_k+i_k}+\cdots+\left(u_k^{j_k+i_k}\right)^{a_k-1}\right).$$

An equally tedious verification shows that $\langle v_j, v_i\rangle \neq 0$ if and only if $i_k = a_k - j_k$ for all k. Thus a basis for $J_a/I_a \otimes \mathbb{R}$ is given by the vectors $v_j + v_{a-j}$ and $i(v_j - v_{a-j})$. With respect to this basis, the matrix corresponding to the quadratic form is diagonal. We have

$$\langle v_j + v_{a-j}, v_j + v_{a-j}\rangle = \langle i(v_j - v_{a-j}), i(v_j - v_{a-j})\rangle = 2\langle v_j, v_{a-j}\rangle.$$

Clearly $\langle v_j, v_{a-j}\rangle > 0$ if and only if the diagonal element corresponding to $v_j + v_{a-j}$ is positive. Thus it remains to be shown that $\langle v_j, v_{a-j}\rangle > 0$ if and only if $0 < \sum_{k=1}^{n}\frac{x_k}{a_k} < 1 \mod 2$, as well as that $\langle v_j, v_{a-j}\rangle < 0$ if and only if $-1 < \sum_{k=1}^{n}\frac{x_k}{a_k} < 0 \mod 2$.

The result follows by a bunch of straightforward manipulations. We have

$$\begin{aligned}
\left(\prod_{k=1}^{n} a_k^{-1}\right)\langle v_j, v_{a-j}\rangle &= (-1)^{\frac{n-1}{2}}\left(\prod_{k=1}^{n}\left(1-\left(u_k^{j_k}\right)^{-1}\right)+\prod_{k=1}^{n}\left(1-\left(u_k^{j_k}\right)\right)\right)\\
&= \mathrm{Re}(-1)^{\frac{n-1}{2}}\prod_{k=1}^{n}\left(1-\left(u_k^{j_k}\right)\right)\\
&= \mathrm{Re}(-1)^{\frac{n-1}{2}}\prod_{k=1}^{n}\left(-2ie^{\frac{\pi i j_k}{a_k}}\sin\frac{\pi j_k}{a_k}\right)\\
&= \mathrm{Re}\left(-e^{\pi i\left(\frac{1}{2}+\sum_k \frac{j_k}{a_k}\right)}\prod_{k=1}^{n}2\sin\frac{\pi j_k}{a_k}\right)
\end{aligned}$$

The product $\prod_k 2\sin\frac{\pi j_k}{a_k}$ is always positive so $\langle v_j, v_{a-j}\rangle > 0$ if and only if $\mathrm{Re}\exp\left(\pi i(1/2 + \sum_k j_k/a_k)\right) < 0$, in other words if and only if $0 < \sum_k j_k/a_k < 1 \mod 2$. The calculation when $\langle v_j, v_{a_j}\rangle < 0$ is analogous. ■

Example 9.9: Let us take n odd and $a = (3, 6k-1, 2, \ldots, 2)$. Then $K_a(\delta)$ is a topological sphere, and $\tau_a = (-1)^{(n-1)/2}8k$.

Proof: The first part easily follows from statement (6) of Theorem 9.1. In order to find the signature, we apply Theorem 9.8 as follows. Using the notation of this theorem, we need $x_3, \ldots, x_n = 1$, while we have $x_1 \in \{1, 2\}$ and $x_2 \in \{1, 2, \ldots, 6k-2\}$; thus the total number of n-tuples is $12k - 4$. We now consider two cases.

CASE I. $n = 4m + 1$. Let us find τ_a^+. Note that we have

$$\sum_{j=1}^{n} \frac{x_j}{a_j} = \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{4m-1}{2} = \frac{x_1}{a_1} + \frac{x_2}{a_2} + (2m-1) + \frac{1}{2}$$

$$\equiv \frac{x_1}{3} + \frac{x_2}{6k-1} - \frac{1}{2} \mod 2.$$

Let us first consider the case where $x_1 = 1$. We find that

$$\sum_{j=1}^{n} \frac{x_j}{a_j} \equiv -\frac{1}{6} + \frac{x_2}{6k-1} \mod 2.$$

We now see that $0 < \sum_j \frac{x_j}{a_j} < 1 \mod 2$ if and only if $x_2 \in \{k, \ldots, 6k-2\}$; thus we find a total number of $5k - 1$ solutions. Next consider the case where $x_1 = 2$. We then find that

$$\sum_{j=1}^{n} \frac{x_j}{a_j} \equiv \frac{1}{6} + \frac{x_2}{6k-1} \mod 2.$$

By symmetry, we find the same amount of solutions; we conclude that $\tau_a^+ = 10k - 2$, while $\tau_a^- = (12k - 4) - (10k - 2) = 2k - 2$, hence $\tau_a = (10k - 2) - (2k - 2) = 8k$.

CASE II. $n = 4m + 3$. We do the same as in the previous case. This time we have

$$\sum_{j=1}^{n} \frac{x_j}{a_j} = \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{4m+1}{2} = \frac{x_1}{a_1} + \frac{x_2}{a_2} + (2m) + \frac{1}{2}$$

$$\equiv \frac{x_1}{3} + \frac{x_2}{6k-1} + \frac{1}{2} \mod 2.$$

Again, first take $x_1 = 1$ to find that

$$\sum_{j=1}^{n} \frac{x_j}{a_j} \equiv \frac{5}{6} + \frac{x_2}{6k-1} \mod 2;$$

we have $0 < \sum_j \frac{x_j}{a_j} < 1 \mod 2$ if and only if $x_2 \in \{1, \ldots, k-1\}$. Next, take $x_1 = 2$ to find that

$$\sum_{j=1}^{n} \frac{x_j}{a_j} \equiv \frac{7}{6} + \frac{x_2}{6k-1} \mod 2;$$

by symmetry, this gives an equal amount of solutions. Thus $\tau_a^+ = 2k - 2$ and so $\tau_a = -8k$. ■

This concludes our quest. For each k, we have determined the signature of the parallelizable manifold $M_a(\delta)$. As we saw in Chapter 7, this is enough information to determine to which element of Θ_7 the Brieskorn variety belongs: if $a = (3, 6k-1, 2, 2, 2)$, then K_a is precisely the k-th element of the group Θ_7.

Chapter 10

Epilogue

10.1 Overview and comparison

In Chapter 4 we constructed three exotic manifolds M_k ($k = 3, 5, 7$) by taking two copies of $D^4 \times S^3$, then gluing them along their boundary with the map $(x, y) \mapsto (x, x^k y x^l)$. We constructed 8-manifolds B_k bounded by M_k and showed that $2q(B_k) - \tau(B_k) \equiv 1, 3, 6 \mod 7$ (where $k = 3, 5, 7$).

In Chapter 7 we concluded that there are precisely 28 structures on the 7-sphere by determining the group of homotopy 7-spheres modulo H-cobordism; in particular, we found that every such homotopy 7-sphere bounds a parallelizable 8-manifold. The H-cobordism class is uniquely determined by the signature of this 8-manifold, which can admit any value in $\{8, 16, \ldots, 8 \cdot 28 = 224\}$, modulo 224.

We constructed explicit examples of such manifolds in Chapter 9. We considered the polynomial $f_a : \mathbb{C}^5 \to \mathbb{C}$ given by $f(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} + x_4^{a_4} + x_5^{a_5}$, with $a = (3, 6k - 1, 2, 2, 2)$ (where $k = 1, \ldots, 28$), then looked at the manifold $K_a = f_a^{-1}(0) \cap S_\varepsilon$, for sufficiently small ε. To determine its smooth structure, we constructed a parallelizable 8-manifold M_a bounded by K_a, and we determined its signature, which we found to be $8k$.

A natural question to ask, as well as the question which has been asked to me as an *exercise*, is as follows: "To which Brieskorn varieties do the Milnor manifolds M_3, M_5, M_7 correspond? Furthermore, is there a natural diffeomorphism between the Milnor manifolds and their corresponding Brieskorn varieties?" Prior writing this thesis, I was confident that finding the correspondence would be easy, since in both cases, the smooth structures are determined by finding the signature of an 8-dimensional co-boundary. It wasn't until taking a closer look that I realized this does not work; the manifold B_k used by Milnor is not parallelizable since its signature is not divisible by 8, thus contradicting Theorem 7.20.

I am ashamed to admit that I found a partial solution by accident; while mindlessly browsing Wikipedia I came across the definition of the $\hat{A}$-genus (`https://archive.fo/QSeHb`), where I was referred to the following result.

Theorem 10.1: Let M be a closed oriented manifold with vanishing second Stiefel-Whitney class. Let (A_k) be the multiplicative sequence belonging to the power series

$$f(t) = \frac{\sqrt{t}}{2\sinh\sqrt{t}/2} = 1 - \frac{1}{24}t + \frac{7}{5760}t^2 + \cdots .$$

Then the $\hat{A}$-genus $\hat{A}[M]$ is an integer.[17, Cor. 3.2]

This seems to give a promising way to construct some new invariant. For a closed 8-manifold B, the $\hat{A}$-genus is given by

$$\hat{A}[B] = \left\langle \nu, \frac{1}{5760}\big(- 4p_2(B) + 7p_1^2(B)\big) \right\rangle.$$

Using the Hirzebruch Signature Theorem (Equation 4.8), we can rewrite this as

$$\hat{A}[B] = \left\langle \nu, \frac{1}{896}p_1^2(B) \right\rangle - \frac{1}{224}\tau(B), \tag{10.2}$$

where $\tau(B)$ is, as usual, the signature of B. We can use this to define the invariant we are looking for.

Let M be a closed, oriented 7-manifold satisfying the hypothesis $(\star)$ that $H^3(M) = H^4(M) = 0$. As we saw in Section 3.1, every closed oriented 7-manifold is the boundary of an 8-manifold B. By hypothesis $(\star)$, the inclusion homomorphism $i : H^4(B, M) \to H^4(B)$ is an isomorphism; thus, we can define a 'Pontryagin number'

$$q(B) = \big\langle \nu, \big(i^{-1}p_1(TB)\big)^2 \big\rangle.$$

We define the invariant $\lambda'(M)$ by setting

$$\lambda'(M) = \frac{1}{896}q(B) - \frac{1}{224}\tau(B) \mod 1,$$

where B is any 8-manifold bounded by M *with vanishing second cohomology* (we show in a moment why we need the last requirement). It turns out that this invariant may *not* necessarily defined for *all* 7-manifolds, but in the case of both the Milnor manifolds M_k, and the Brieskorn varieties K_k, it turns out that it is.[1]

Lemma 10.3: In the case of the Milnor and Brieskorn varieties, this is a well-defined definition; i.e., the value does not depend on the choice of the manifold B.

Proof: Suppose B_1, B_2 are two manifolds with boundary M. Then consider the closed 8-manifold C by taking the disjoint union of B_1 and B_2, and gluing them along their boundaries in the obvious way. Choose an orientation ν for C that is consistent with the orientation ν_1 of B_1. By Equation 10.2 we have that

$$\hat{A}[C] = \left\langle \nu, \frac{1}{896} p_1^2(C) \right\rangle - \frac{1}{224}\tau(C).$$

Using Lemma 4.9, we see that this is precisely the difference

$$\hat{A}[C] = \left(\frac{1}{896} q(B_1) - \frac{1}{224}\tau(B_1) \right) - \left(\frac{1}{896} q(B_2) - \frac{1}{224}\tau(B_2) \right).$$

Our goal is to apply Theorem 10.1 to conclude that this difference is integer-valued. For this, we need the second Stiefel-Whitney class of C to vanish. By a Mayer-Vietoris argument, we have the long exact sequence

$$\cdots \longrightarrow H^1(M_k) \longrightarrow H^2(B_1) \oplus H^2(B_2) \longrightarrow H^2(C) \longrightarrow H^2(M_k) \longrightarrow \cdots$$

By assumption, $H^2(B_1) = H^2(B_2) = 0$. Furthermore, $H^1(M_k) = H^2(M_k) = 0$, since M_k is homeomorphic to the 7-sphere (this is the part which may fail for general 7-manifolds).

For the Brieskorn varieties, we can do the exact same thing. ■

We go on to compute the invariant for both the Milnor manifolds and the Brieskorn varieties, and compare them.

The 8-manifold B_k bounded by M_k which was constructed by Milnor can be used to compute λ'; to see this, note that B_k can be interpreted as a disk bundle, hence its cohomology coincides with that of its base space S^4. From this, it follows that B_k has vanishing second cohomology.

Using the results found in Lemma 4.14, we have $q(B_k) = \langle \nu, i^{-1}(\pm 2k\alpha))^2 \rangle = 4k^2$, $\tau(B_k) = 1$, and hence

$$\lambda'(M_k) = \frac{k^2 - 1}{224} \mod 1.$$

[1] The reason is that I don't know (and don't care) if every 7-manifold bounds an 8-manifold with vanishing second cohomology. But we will soon see that such a co-boundary exists for both the M_k and the K_k.

We move on to take a look at the Brieskorn varieties $K_k = K_a(\delta) = f_a^{-1}(\delta) \cap S_\varepsilon$ $(k = 1, \dots, 28)$. It bounds the 8-manifold $B_k = f_a^{-1}(\delta) \cap D_\varepsilon$. The second cohomology of this manifold vanishes. Here's why. By [7, Lemma 7], for sufficiently small δ, the manifold B_k has interior diffeomorphic to $f_a^{-1}(1)$, and boundary diffeomorphic to $K_a(\delta)$. We saw in the proof of Lemma 9.3 that $f_a^{-1}(1)$ can be identified with the fibre F_θ, which, by Theorem 8.25, has the homotopy type of a bouquet of $(n-1)$-spheres. As for the boundary, we know it to be homeomorphic to S^7. Writing out the Mayer-Vietoris sequence for $U, V =$ the interior and boundary of our manifold (which is allowed by the Collar Neighbourhood Theorem 6.3) we see that the second cohomology group of B_k must vanish.

What remains to be done is actually computing the invariant for the Brieskorn varieties. By Example 9.9, we know that the signature $\tau(B_k)$ is given by $8k$. Furthermore, B_k is parallelizable since its normal bundle is trivial, and thus its Pontryagin classes vanish. We conclude, then, that

$$\lambda'(K_k) = \frac{28 - k}{28} \mod 1.$$

The fact that this gives us 28 different values is particularly nice (not to mention sheer luck); it leads to the following result, that the value of λ' completely determines the diffeomorphism type of a given manifold, provided they satisfy the hypotheses needed to compute λ'.

Let's see some values. Take the Milnor manifold M_k, and consider the values $k = 1, 3, 5, 7, \dots$. We find **16** different values. Thus, the Milnor manifolds produce exactly 16 different smooth structures, which significantly strengthens the results found in Section 4.3.

In his paper, Milnor mentions that it is not known whether there are non-trivial k such that M_k is diffeomorphic to S^7. This result provides an answer to his question. The answer is *yes*: a manifold M_k is diffeomorphic to the standard sphere S^7 if and only if $k^2 - 1$ is a multiple of 224. The first non-trivial example occurs when $k = 15$. On the other hand, when $k = 13$, the original Milnor invariant became zero, yet $\lambda'(M_{13}) \neq 0$.

In particular, if $k = 3, 5, 7$, we find the values 1/28, 3/28, 6/28. Looking at the invariants of the Brieskorn varieties, we conclude that M_3, M_5, M_7 correspond to the Brieskorn varieties K_{27}, K_{25} and K_{22}. We could apply the same ideas to larger k.

This still does not provide us with an explicit mapping. Whatever the mapping is, I highly doubt such a mapping would be 'natural' in any way, since the correspondence between the Milnor and Brieskorn varieties is seemingly arbitrary.

10.2 Recent results

We have come a long way from where we have started. We have found a classification theorem for the exotic structures on the n-sphere ($n \neq 4$) which, in principle, gives the number of smooth structures on the n-sphere, provided the stable homotopy groups $\pi_{n+k}(S^n)$ of the n-sphere are known; in practice, these stable homotopy groups have been computed for values up to $k = 64$, so that the number exotic structures can be found for all n-spheres up to $n = 64$. The first 36 values were given at the end of Chapter 7. In this section, we will give (without proof) some more recent results concerning the exotic structures, not in any particular order.

Theorem 10.4: The only odd-dimensional spheres with a unique smooth structure are S^1, S^3, S^5 and S^{61}.[50]

The result above implicitly concludes the existence of exotic structures in many high dimensions. This raises the following question: for which dimension n do there exist exotic n-manifolds? The answer is as follows.

Theorem 10.5: There exist exotic n-manifolds if and only if $n \geq 4$.

Proof: It is known that the 4-manifold $\mathbb{R}^4$ has exotic structures — in fact, it has uncountably many structures.[13] For $n \neq 4$, however, $\mathbb{R}^n$ has a unique smooth structure.[46] If $n \geq 5$ then there are always exotic tori. In fact the PL-structures on T^n are in correspondence with the group $H^3(T^n; \mathbb{Z}/2\mathbb{Z})$,[18] and every one of these is smoothable.[49, Ch. 15A] Since any smooth manifold admits a unique PL-structure up to PL-isomorphism,[2] it follows that there are many manifolds homeomorphic but not diffeomorphic to the standard torus. Finally, no exotic structures exist if $n = 1$ (by the classical classification of 1-manifolds), $n = 2$,[40] and $n = 3$.[35] ■

Personally, I find the first part in particular quite interesting. I do not know of any manifolds other than $\mathbb{R}^4$ which admit uncountably many exotic structures. Let's give some answers on whether other such manifolds can exist.

Theorem 10.6: Let M be a closed topological n-manifold, and suppose $n \geq 5$. Then M has finitely many smooth structures.[20, Essay IV] Furthermore, any 4-manifold having uncountably many smooth structures would necessarily have to be non-compact.[38]

Most manifolds we have mentioned in this thesis (in fact, all of them) have a smooth structure, but it is *not* true that all topological manifolds admit such a structure. A counterexample is given by the E_8 manifold. Here's why.

Theorem 10.7: Let M be a smooth, compact 4-manifold. If its second Stiefel-Whitney class $w_2(M)$ vanishes, then the signature of its intersection form is divisible by 16.[41]

The E_8-manifold is a simply-connected compact topological manifold with vanishing w_2 and intersection form of signature 8, and as such, it cannot have a smooth structure.[11]

There are many other non-smoothable manifolds. Here's another one, which has a more explicit description. Consider the homogeneous equation in $\mathbb{CP}^5$ given by

$$z_1^5 + z_2^5 + z_3^2 + z_4^2 + z_5^2 + \sum_{j=1}^{5} e^{j-1} z_j^6 = 0,$$

where e is the e you saw in high school (though any transcendental number would suffice). Then the solution set is a 4-dimensional PL-manifold which is not homeomorphic to a smooth manifold.[21]

It is also known that the E_8 manifold is not even triangulable. This itself raises some more questions.

In dimensions up to three, it is well-known that every manifold is triangulable. As we just mentioned, in dimension 4, the E_8-manifold is a non-triangulable manifold. In fact, it is a known result that a closed 4-manifold is triangulable if and only if it is smoothable. To see this, pick a triangulation; the links of vertices are always both homology and homotopy spheres, so they are copies of S^3 by the Poincaré Conjecture. This gives a PL structure, and is therefore smoothable.[1] Note that this reasoning also shows that closed 4-manifolds have at most countably many exotic structures.

What about higher dimensions? A construction found in [12] implies there are non-triangulable manifolds in every dimension $n \geq 5$. If $n \geq 6$ their construction can also be used to find non-triangulable *orientable* manifolds.

An obvious question to ask is whether we can generalize this thesis, which is solely concerned with C^∞-manifolds, to arbitrary C^k-manifolds (manifolds with C^k transition maps). It turns out this is not an interesting generalization. It was proved by Whitney that every C^1-structure can be uniquely smoothed to a smooth structure.[52]

An **analytic manifold** is a topological manifold such that the transition maps are analytic; **complex manifolds** are topological manifolds with holomorphic transition maps. Similar questions arise when we consider these manifolds. Whitney's aforementioned paper is, in fact, slightly stronger: C^1 manifolds admit not only compatible smooth structures, but also compatible analytic structures. This result was later strengthened by Morrey and Grauert, who proved that not only does every smooth manifold have a real-analytic structure, the real-analytic structure is unique.[14][36]

The answer to the question whether a complex structure exists on a real-analytic (or smooth) manifold, however, is largely unknown. Here's what we know in the case of spheres.

Theorem 10.8: If $k \neq 1, 3$ then the sphere S^{2k} does not admit any complex structures.

We actually have the machinery to prove a specific case. Let us take $k = 2$. Suppose S^4 has a complex structure, making its tangent bundle into a complex vector bundle. By Proposition 3.25 together with the fact that the first Chern class of the tangent bundle of S^4 (after ignoring the complex structure) vanishes, we have $p_1(S^4) = -2c_2(S^4)$, the latter being precisely -2 times the Euler class of the tangent bundle. By Theorem 3.11, we know that $\langle e(S^4), \mu \rangle$ is precisely the Euler characteristic of S^4, i.e. the alternating sum of the cohomology dimensions, which is 2. By the Hirzebruch Signature Theorem, then,

$$\tau(S^4) = \frac{1}{3}\langle p_1(S^4), \mu \rangle = \frac{1}{3}\langle -2e(S^4), \mu \rangle = -\frac{4}{3}.$$

But by Proposition 3.2, the signature is of S^4 is clearly zero, hence we have found ourselves a contradiction.

What about $k = 1, 3$? Case $k = 1$ clear since S^1 is homeomorphic to $\mathbb{CP}^2$. As for $k = 3$, a recent paper by Atiyah[4] asserts that the sphere S^6 does not admit any complex structure, though the consensus seems to be that the proof is incorrect. To complicate the matter even more, there exist a number of papers (several of which are known to be incorrect by now) which seemingly prove the affirmative. It is safe to say that the problem is still open... I think I'll just leave this case as an exercise to the reader.

Bibliography

[1] Found on `https://archive.fo/lPLsW`.

[2] Whitehead triangulations (lecture 3), 2009. Found on `http://www-math.mit.edu/~lurie/937notes/937Lecture3.pdf`. Accessed 14-05-2017.

[3] J. F. Adams. On the groups $J(X)$ IV. *Topology*, 5:21–71, 1966.

[4] M. Atiyah. The non-existent complex 6-sphere. arXiv:1610.09366.

[5] M. Audin and M. Damian. *Morse Theory and Floer Homology*. Universitext. Springer-Verlag, London, 2014.

[6] R. Bott. The stable homotopy of the classical groups. *Annals of Math.*, 70:313–337, 1959.

[7] E. Brieskorn. Beispiele zur Differentialtopologie von Singularitäten. *Inventiones Math.*, 2:1–14, 1966.

[8] W. Browder. *Surgery on Simply-Connected Manifolds*. Springer-Verlag, Princeton, NJ, 1972.

[9] J. Cerf. Topologie de certains espaces de plongements. *Bull. Soc. Math. France*, 89:227–380, 1961.

[10] C. Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. *Séminaire Bourbaki*, No. 1:153–168, 1948–51.

[11] M. H. Freedman. The topology of four-dimensional manifolds. *Diff. Geom.*, 17:357–453, 1982.

[12] D. E. Galewski and R. J. Stern. A universal 5-manifold with respect to simplicial triangulations. *Geometric Topology*, pages 345–350, 1979.

[13] R. Gompf. An infinite set of exotic $\mathbb{R}^4$'s. *Diff. Geom.*, 21:283–300, 1985.

[14] H. Grauert. On Levi's problem and the imbedding of real analytic manifolds. *Annals of Math.*, 68:460–472, 1958.

[15] A. E. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002.

[16] F. E. P. Hirzebruch. Singularities and Exotic Spheres. *Séminaire Bourbaki*, No. 314:13–32, 1966–68.

[17] F. E. P. Hirzebruch and A. Borel. Characteristic classes and homogeneous spaces III.

[18] W. C. Hsiang and J. L. Shaneson. *Proc. Natl. Acad. Sci. U.S.A.*

[19] M. Kervaire. An interpretation of G. Whitehead's generalization of the Hopf invariant. *Annals of Math.*, 69:345–364, 1959.

[20] R. C. Kirby and L. C. Siebenmann. *Foundational essays on topological manifolds, smoothings and triangulations*, volume 88 of *Annals of Math. Studies.* Princeton University Press, 1977.

[21] N. Kuiper. Algebraic equations for nonsmoothable 8-dimensional manifolds. *Math. Publ. of IHES*, 33:139–155, 1967.

[22] J. M. Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Math., Vol. 218. Springer-Verlag, New York, NY, 2nd edition, 2013.

[23] J. P. Levine. Lectures on groups of homotopy spheres. *Algebraic and Geometric Topology*, pages 62–95, 1985.

[24] A. A. Markov. Insolubility of the problem of homeomorphy. *Proc. ICM Cambridge*, 1958, *Edinburgh*, Cambridge University Press, Cambridge, 1960.

[25] Y. Matsumoto. *An Introduction to Morse Theory.*

[26] J. P. May. *A Concise Course in Algebraic Topology.*

[27] J. W. Milnor. Construction of Universal Bundles II. *Annals of Math.*, 63:430–436, 1956.

[28] J. W. Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Math.*, 64:399–405, 1956.

[29] J. W. Milnor. A procedure for killing the homotopy groups of differentiable manifolds. *Symposia in Pure Math. A.M.S.*, 3:39–55, 1961.

[30] J. W. Milnor. *Morse Theory.* Annals of Math. Studies, Vol. 51. Princeton University Press, Princeton, NJ, 5th edition, 1963. Based on notes by M. Spivak and R. Wells.

[31] J. W. Milnor. *Singular Points of Complex Hypersurfaces.* Princeton University Press, Princeton, NJ, 1968.

[32] J. W. Milnor and M. Kervaire. Bernoulli numbers, homotopy groups and a theorem of Rohlin. *Proc. Int. Congress of Math., Edinburgh*, 1958.

[33] J. W. Milnor and M. A. Kervaire. Groups of Homotopy Spheres I. *Annals of Math.*, 77:504–537, 1963.

[34] J. W. Milnor and J. D. Stasheff. *Characteristic Classes.* Annals of Math. Studies, Vol. 76. Princeton University Press, Princeton, NJ, 1974.

[35] E. E. Moise. Affine structures on 3-manifolds. *Annals of Math.*, pages 96–114, 1952.

[36] C. B. Morrey. The analytic embedding of abstract real analytic manifolds. *Annals of Math.*, 68:159–201, 1958.

[37] P. S. Novikov. On the algorithmic unsolvability of the word problem in group theory. *Proc. Steklov Inst. Math.*, 44:1–143, 1955.

[38] S. Peters. Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds. *J. Reine Angew. Math.*, 349:77–82, 1984.

[39] F. Pham. Formules de Picard-Lefschetz généralisées et ramification des intégrales. *Bull. Soc. Math. France*, 93:333–367, 1965.

[40] T. Rado. Über den Begriff der Riemannsche Fläche. *Acta Litt. Scient. Univ. Szegd*, 2:101–121, 1925.

[41] V. A. Rokhlin. New results in the theory of four-dimensional manifolds. *Doklady Acad. Nauk. SSSR (N.S.)*, 84, 1952.

[42] T. L. Saaty. *Lectures on Modern Mathematics, Volume II, Part 6.* 1964.

[43] J.-P. Serre. Homologie singulière des espaces fibrés. *Annals of Math.*, 54:425–505, 1951.

[44] J.-P. Serre. Formes bilinéaires symétriques entières à discriminant ± 1. *Séminaire Cartan*, 14, 1961–62.

[45] L. Siebenmann and J. Sondow. *Lectures on the h-Cobordism Theorem.* Princeton University Press, Princeton, NJ, 1965. Based on lectures by John Milnor.

[46] J. R. Stallings and E. C. Zeeman. The piecewise-linear structure of Euclidean space. *Math. Proc. Cambridge Philos. Soc.*, 58:481–488, 1962.

[47] N. E. Steenrod. *The Topology of Fibre Bundles.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 3rd edition, 1951.

[48] R. E. Stong. *Notes on Cobordism Theory.* Princeton Math. Notes. Princeton University Press, 1968.

[49] C. T. C. Wall. *Surgery on Compact Manifolds.*

[50] G. Wang and Z. Xu. The triviality of the 61-stem in the stable homotopy groups of spheres. arXiv:1601.02184.

[51] J. H. C. Whitehead. On the homotopy type of manifolds. *Annals of Math.*, 41:825–832, 1940.

[52] H. Whitney. Differentiable manifolds. *Annals of Math.*, 37:645–680, 1936.

[53] H. Whitney. Elementary structure of real algebraic varieties. *Annals of Math.*, 66:545–556, 1957.