Utrecht University

A lower bound on the Chromatic Number by Simplicial Complexes

Bachelor Thesis

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Contents

1 Introduction

This bachelor thesis is about two theorems on graph colorings, both providing lower bounds on the chromatic number of a graph.

In order to explain, prove and be able to use these theorems the reader will be introduced to a variety of subjects mostly having to do with graphs and topology, bordering on combinatorics.

We will start with a bit of graph theory, explaining which kinds of graphs we will be looking at and which definitions we will be using. We will also take a look at the notion of the chromatic number, explaining and relating it to graph homomorphisms and complete graphs.

Then we go into the subject of simplicial complexes, both as a more geometrical or topological object, and as a purely combinatorical object in the form of the abstract simplicial complex, and we shall show how these 'versions' of simplicial complexes relate to each other. We shall then take a look at a certain simplicial complex, the neighborhood complex, which can be built from a graph. We shall finish this introduction to simplicial complexes by taking a look at the barycentric subdivision, a way to define a finer structure on a simplicial complex.

It is then that we start looking into $\mathbb{Z}_2$-spaces, which are spaces under a $\mathbb{Z}_2$-action. This leads us into the topic of maps that respect the $\mathbb{Z}_2$-structure, so called $\mathbb{Z}_2$-maps, and we will see how we can define an ordering-like structure on $\mathbb{Z}_2$-spaces, and a label that depends on where a space is in this ordering relative to n-spheres, this is the so-called $\mathbb{Z}_2$-index.

We will then combine some of these notions by defining another simplicial complex which is also based on a graph but which is a $\mathbb{Z}_2$-space as well, called the box complex. We will take a look at the box complex of a complete graph in particular and give an upper bound on its $\mathbb{Z}_2$-index.

Then it is time to go into the first major theorem this thesis is about. We will have all the pieces by then so we will just have to put them together.

Hereafter we will go into a short introduction into the subject of homotopy, explaining homotopies, homotopy-equivalent spaces, deformation retract(ion)s and k-connectedness, whilst trying to keep our treatment of these matters at a minimum.

The time will have come by then to go into the second theorem, which will require a little extra work because a certain piece of the puzzle, called $\mathrm{L}(G)$, will have to be found first.

We shall then conclude this bachelor thesis.

2 Graphs

The major theory in this thesis, apart from topology, is that of graphs. They will be very important in this thesis for an obvious reason: The main theorems of this thesis are statements about graph theory and topology.

Graphs should probably be familiar to the reader, but they are used and described in various ways so here is the way we will use and describe them. Note that considerable inspiration has been taken from [1].

An informal definition of a graph is that it is some collection of points, connected by lines. We call the points 'vertices' and the lines 'edges'. Take care that there are many different kinds of graphs and that there are different definitions suitable for different kinds of graphs. For example, in many cases people like to have edges between a vertex and itself, or multiple edges between the same vertices. We do nothing of the sort. Our graphs are made up of a set of vertices, with a set of edges between distinct vertices. Our edges are in turn sets of the two vertices the edge is between. Because we have a set of edges an edge is either in it or not, there is no such thing as multiple edges between the same vertices in the way that we use graphs. Edges being sets themselves make them undirected, in other words: There is no difference between an edge from a to b and one from b to a. These graphs are usually called 'simple'. A formal definition would be:

Definition 1. *(Paraphrasing [1, preliminaries xii]) A (simple) graph is a tuple* (V, E) *of a set of vertices* V*, and a set of unordered pairs (sets with two elements) of elements of* V*: the edges,* E*. We say there's an edge between a and b (both elements of* V*) if* $\{ a, b \} \in E$*.*
Often, for a graph G*, we denote its vertices as* $V(G)$ *and its edges as* $E(G)$*. When it is obvious which graph we are speaking about we might even just use* V *and* E*.*

It will turn out to be useful to have something we can use to speak of all the neighbors of a vertex, that is: Every vertex between which and the original vertex there is an edge. We shall define something like this but broader. (And more useful!) It answers the question: "Given some set of vertices, what is the set of vertices between which and all of the original vertices exists an edge?" The answer is:

Definition 2. *([1, p 129]) The common neighbors: Given a set* $A \subseteq V(G)$*, we define*

$$\mathrm{CN}(A) = \{ v \in V(G) : \forall a \in A \{ a, v \} \in E(G) \}$$

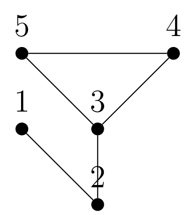
2.1 The Chromatic Number

On a graph we can define a graph coloring, the idea being that we can assign a 'color' to every vertex of a graph, most importantly in such a way that vertices that have an edge between them do not share colors. The language of colors is often used, but to provide a mathematically correct definition often some set of numbers is used, though other sets can be used as well. This concept of a graph coloring immediately leads one to an interesting question for any given graph: How many colors does one need to color the graph, such that no vertices with an edge between them share the same color? The answer to this question is called the chromatic number (sometimes called the color number) of the graph.

Definition 3. *A graph coloring is a function* $f : V(G) \to S$ *where* S *is some set. Usually (and in our case too) we are only interested in cases where* $f(a) \neq f(b)$ *for all* $\{ a, b \} \in E(G)$*. From now on we shall only consider such colorings. For* $|\operatorname{Im}(f)| = k$ *we speak of a* k*-coloring of* G*.*

Definition 4. *The chromatic number* $\chi(G)$ *is the lowest possible number* n *such that there is an* n*-coloring of* G*.*

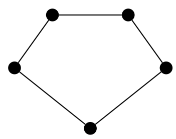
As an example, consider the following image:



This illustrates a graph with 5 nodes. Node 1 is connected to node 2; which is connected to node 3; And 3, 4 and 5 are all connected to each other. Its chromatic number is 3. Proof: Because 3, 4 and 5 are all connected to each other we need a different color for every one of these 3 vertices, and thus we need at least 3 colors. We don't need any more, because if we give 2 the same color as 4, and 1 the same color as 3, we get a coloring such that there are no edges between vertices with the same color, using 3 colors in total.

This is quite a simple example: This graph contains three nodes which are all connected to each other (3, 4 and 5) so of course you need at least 3 colors. Having assigned these vertices unique colors, the solution for 3 colors is obvious.

It is not always this simple however. A graph need not have n vertices all connected to one another to require at least n colors. For example if we have 5 vertices connected in a circle.



Nowhere are there 3 vertices all connected to each other in this graph, yet it is impossible to use fewer than 3 colors for a proper graph coloring of this graph. This is easy to see. Take a vertex and give it color 'a'. Now go round, the first vertex we meet will need a different color, 'b'. Now we can't add another color lest we use three colors. So the next one will need to be 'a' again, then 'b' again, but which color will the last vertex have? It can't be 'a' because of the next vertex, and not be 'b' because of the previous vertex. We will need a third color, and the only choice we made was to not use a third color earlier on.

At the end of this thesis, after having introduced a lot of other notions, we will return to this problem in the form of two theorems that give a lower bound on the chromatic number of a graph.

2.2 Graph Colorings and Graph Homomorphisms

There is another way to look at graph colorings than the way we looked at them just now which will be helpful to us later on. For this other way we need two rather simple concepts, that of a complete graph and that of a graph homomorphism.

Definition 5. *A complete graph is simply a graph in which every vertex is connected to every other vertex. We write the complete graph as K_n where n is the amount of vertices. More formally: $K_n = (V, E)$ where V is a set of n elements, and $E = \{\{x, y\} \in V \times V : x \neq y\}$. It is obvious that any two graphs fitting this definition are isomorphic if and only if their amount of vertices is equal.*

Definition 6. *([1, p. 128]) A graph homomorphism is a function f between* *the vertices of two graphs, say G and H, such that edges are sent to edges. That is to say:* $\{a, b\} \in E(G)$ *implies* $\{f(a), f(b)\} \in E(H)$*.*

Now we can define the a graph coloring in a different way: A graph coloring of G using n colors is simply a graph homomorphism from G to K_n, the color being defined by which vertex of the complete graph a vertex of G is sent to.

Let's check this.

Theorem 1. *(Property suggested by [1, p 128]) If a graph G can be n-colored then there exists a homomorphism $G \to K_n$ defined by the coloring such that vertices with the same color are sent to the same vertex.*

Proof. Say we have a m-coloring of G. Without loss of generality we will assume this is a function $f : V(G) \to \{1, \ldots, n\}$. We may assume $V(K_n)$ to be $\{1, \ldots, n\}$ as well, so we have $f : G \to K_n$ as a supposed graph homomorphism. To prove this is a graph homomorphism, we need that edges are sent to edges. An edge $\{a, b\} \in E(G)$ implies $f(a) \neq f(b)$ because it is a graph coloring. By definition, there is an edge between all pairs of distinct vertices in the complete graph, so $\{f(a), f(b)\} \in E(K_n)$ just like we wanted. □

2.3 The Bipartite Subgraph

Yet another concept that we'll be using is that of the bipartite subgraph. This is not a hard topic but it is useful to get our notation and terminology straight. The notation has been adopted from [1, p. 129]

Definition 7. *When we talk of a <u>bipartite subgraph</u> of G, which we denote $G[A, B]$, where A and B are subsets of the vertices of G with $A \cap B = \emptyset$, we refer to the graph with vertices from A and B, and edges all those that are in $E(G)$, but also go across, in the sense that for an edge one of the vertices is in A and the other in B. We say that $G[A, B]$ is a <u>complete bipartite subgraph</u> if and only if $\forall \{a, b\} \in A \times B : \{a, b\} \in E(G[A, B])$. That is to say, it has all the edges it can have whilst remaining a bipartite graph.*

So for example, if we have a graph with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$, then $G[\{1, 2\}, \{3\}]$ is not a complete bipartite subgraph, because there is no edge $\{1, 3\}$ even though 1 is on one side and 3 on the other. $G[\{1, 3\}, \{2\}]$ is a complete bipartite subgraph of G, because $\{1, 2\}$

and $\{3, 2\}$ are both edges of G, which is the maximum amount of edges it could have whilst remaining a bipartite subgraph.

3 Simplicial Complexes

The two central theorems that we will be using depend heavily on the concept of a simplicial complex. A simplicial complex is a space which is built out of certain building blocks, called simplices. Just like we have the n-spheres and n-dimensional real-coordinate spaces, we have a 'version' of the simplex of different dimensions, of which you might think as an inter-dimensional generalization of a filled triangle. The one-dimensional simplex is just a closed line, the two-dimensional simplex is the filled triangle, the three-dimensional simplex is the tetrahedron (the pyramid with a triangular base), to give you a sense of what we're talking about. There are two ways to speak about simplicial complexes, one which we will call the geometric way, the other we will call the abstract way. We shall start off by taking a look at the basics of these complexes.

We will then take a look at the neighborhood complex, a simplicial complex that can be constructed based on a graph, and at the barycentric subdivision, a way to build a finer version of another simplicial complex which we will use later in this thesis to prove things about the original complex.

3.1 The Geometric Approach

To make the previously mentioned idea of 'simplex' more exact, here is a rigid definition, as used in [1, p 8]:

Definition 8. *A simplex is a convex hull of a finite affinely independent set $A \subset \mathbb{R}^n$. Points of A are called vertices of the simplex. Its dimension is $|A| - 1$, when we speak of the n-dimensional simplex we often refer to it as the n-simplex.*

It seems that using this definition, every simplex consists of lower-dimensional simplices with a filled hole: The tetrahedron consists of 4 triangles arranged in such a way that would be a void in it, but with the void filled. The triangle in turn consists of 3 closed lines, arranged such that there is a hole in it, but filled once more. Here is a proper definition relevant to this idea:

Definition 9. *A face of a simplex is the convex hull of a nonempty subset of vertices of the simplex. [1, p 8]*

Now we can define what we mean be a simplicial complex:

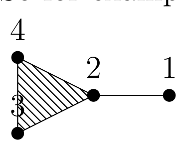
Definition 10. *A nonempty family of simplices is a simplicial complex if each face of any simplex in the complex is also part of the complex, and the intersection of any two simplices in the complex is a face of both simplices. The union of all these simplices is the polyhedron of the simplicial complex. The vertex set of the complex is the union of all vertices of simplices in the complex. [1, p 9]*

3.2 The Abstract Approach

The definition of a simplicial complex that we have just stated is not really nice to work with, at least for our purposes. That is why we will work with a different definition that defines the same structure. For the most part we will not be concerned about how a certain structure is realized geometrically, being homeomorphic is equal enough for us, with some but few exceptions. That's why we might as well just look at which simplices there are by noting which vertices every simplex consists of. This gives us a rather abstract definition of a simplicial complex, but one that is a lot easier to use in our proofs.

Definition 11. *(Paraphrasing [1, p. 13, 1.5.1]) An abstract simplicial complex is a pair (V, K) where V is some set and K is a subset of the powerset of V, such that $G \subset F \in K \implies G \in K$. The sets in K are the (abstract) simplices of the abstract simplicial complex.*

So for example, a filled triangle with a line sticking out,



is a realization of something that we might define as the following abstract simplicial complex:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{2,4\}, \{2,3,4\}\}$$

Notice that an abstract n-simplex is always encoded using $n+1$ vertices, just like a geometric n-simplex has $n+1$ vertices.

The actual realization of the space is called the polyhedron.

Definition 12. *(Paraphrased from [1, p. 14])*

The polyhedron or geometric realization of an abstract simplicial complex K is a simplicial complex $\|K\|$ which has a vertex for every vertex of K, and which has as its simplices exactly those that have as their vertices a set of vertices which corresponds to an abstract simplex of K. Based on [1, p. 14]

It is the case that any abstract simplicial complex can be realized by some polyhedron embedded in some R^n. This is nice to know because it means we won't have to check whether the abstract simplicial complexes we use actually define spaces, they always do. For a abstract simplicial complex we can without loss of generality assume its vertices to be the vertices of the $|V|-1$ dimensional simplex. We can take the $|V|-1$ dimensional simplex which trivially is a simplicial complex and define a subcomplex by having the simplices be only the convex hulls of subsets of the vertices of this simplicial complex that are simplices of the abstract simplicial complex. [1, p. 14]

We'd also like to have a structure-preserving map of some sort between simplicial complexes. For this we have the simplicial map:

Definition 13. *A simplicial map is a map f between (abstract) simplicial complexes X and Y such that if A is a simplex of X then $f(A)$ is a simplex of Y. (paraphrasing [1, p. 14])*

As we stated before we use abstract simplicial complexes mostly to more easily make statements about their polyhedra. To justify this we will show that a map which respects the structure of an abstract simplicial complex induces a map that respects the structure of the polyhedron. In other words, we will show that a simplicial map between abstract simplicial complexes induces a continuous map between their polyhedra. The proof is paraphrased from [1, p. 15].

Say we have a map between abstract simplicial complexes $f : A \to B$. We can construct $\|f\| : \|A\| \to \|B\|$ by starting at the obvious mapping between vertices that it induces, and then affinely extending this to the entire polyhedron. We know that if we have a k-simplex σ with vertices $v_0, \ldots, v_k$ (in the polyhedron these are also points within some $\mathbb{R}^d$) we can write every $x \in \sigma$ as a combination $x = \sum_{i=0}^{k} \alpha_i v_i$ for one and only one set of

$\alpha_0, \ldots, \alpha_k \geq 0$ and $\sum_{i=0}^{k} \alpha_i = 1$, because x is in the convex hull of $v_0, \ldots, v_k$ which are affinely independent. We define $\|f\|(x) = \sum_{i=0}^{k} \alpha_i f(v_i)$. Because f is a simplicial map all $f(v_i)$ together form the vertex set of some simplex of $\|B\|$, though some may be mapped to the same vertex. In any case, we still have the weighed sum of vertices of a simplex with the sum of the weights being 1 so $\|f\|(x)$ will be in this simplex. And by the way we defined it, it is obviously continuous.

We will start to omit whether we are speaking about the abstract simplicial complex or its polyhedron where this is obvious and just call it a simplicial complex. We will also label abstract simplicial complexes by the properties of their polyhedra. We might for example say that A is simply connected even though $\|A\|$ is, because abstract simplicial complexes are mostly just a tool to more easily make statements about actual simplicial complexes.

3.3 The Neighborhood Complex

The neighborhood complex (taken from [1, p. 130]) is an abstract simplicial complex that is constructed out of a graph. The idea is that the vertices of the complex are the vertices of the graph, but that instead of edges between them like in a graph, there will be simplices. The simplices are defined by vertices which in the graph share a common neighbor. To make this precise we have the following definition:

Definition 14. *(Paraphrasing [1, p. 130])*
Given a graph G, the neighborhood complex $\mathrm{N}(G)$ of this graph is an (abstract) simplicial complex with the following simplices:

$$\mathrm{N}(G) = \{ A \subseteq V(G) : \mathrm{CN}(A) \neq \emptyset \}$$

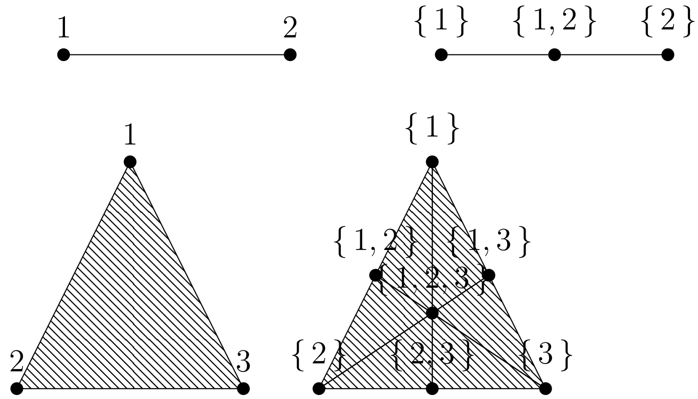
3.4 The Barycentric Subdivision

Given a simplicial complex we can define its barycentric subdivision (taken from [1, p. 19]). The idea here is that all the simplices of the original complex become the vertices of the complex we're creating. The simplices of this new complex are all sets of simplices of the previous complex, and not just sets, they're chains ordered by strict inclusion.

Definition 15. *The barycentric subdivision is defined as follows: Given an abstract simplicial complex K:*

$$\mathrm{sd}(K) = \{\, \{\, A_1, \ldots, A_n \,\} \in K : A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \,\}$$

The interesting thing about the barycentric subdivision is that it encodes the same exact space, in that $\| \mathrm{sd}(K) \| = \|K\|$ [1, p. 19]. This is because all the barycentric subdivision does, is divide simplices up into smaller simplices. For example, the simplex $\{\, v_1, v_2 \,\}$ is transformed into two simplices, $\{\, \{\, v_1 \,\}, \{\, v_1, v_2 \,\} \,\}$ and $\{\, \{\, v_1, v_2 \,\}, \{\, v_2 \,\} \,\}$, See below for an illustration of this and of the barycentric subdivision of the 2-simplex as well.



4 $\mathbb{Z}_2$-Spaces

Of particular interest to us are spaces with a notion of antipodality; spaces with an antipodal map or $\mathbb{Z}_2$-action: A map which composed with itself is the identity. These will be called $\mathbb{Z}$-spaces. (summarizing [1, p. 93])

Definition 16. *(from [1, p. 93, 5.2.1]) A $\mathbb{Z}_2$-space (X, v) is a topological space X with a homeomorphism v, the $\mathbb{Z}_2$-action on the space, such that $v^2 = \mathrm{id}_X$ A free $\mathbb{Z}_2$-space is one where $\forall x : v(x) \neq x$*

A good example of such spaces are the n-sphere and the plane under the usual definition of $-$, as in $x \to -x$, as the associated $\mathbb{Z}_2$-action. Under this $\mathbb{Z}_2$-action the circle is also a free $\mathbb{Z}_2$-space, whereas the plane is not (because $-0 = 0$).

We will start omitting what the specific $\mathbb{Z}_2$-structure on a space is when this is obvious. On n-spheres this would be $x \to -x$, and for most other spaces we will define only one $\mathbb{Z}_2$-action.

4.1 $\mathbb{Z}_2$-Maps

Of course when we define a certain structure, we are interested in maps that respect this structure:

Definition 17. *(from [1, p. 93, 5.2.1]) A* <u>$\mathbb{Z}_2$*-map*</u> $f : (X, v) \to (Y, w)$ *is a map between* $\mathbb{Z}_2$*-spaces, such that* $f \circ v = w \circ f$*. [1, p. 93]*

We just gave the example of $\mathbb{R}^n$ and S^n as $\mathbb{Z}_2$-spaces. These also give us an example of a $\mathbb{Z}_2$-map: Use $\{x \in \mathbb{R}^2 : \|x\| = 1\}$ as a model for the circle, then $f : S^1 \to \mathbb{R}^2$ by $f(x) = x$ (the inclusion) is a $\mathbb{Z}_2$-map.

4.2 The Borsuk-Ulam Theorem

Not every $\mathbb{Z}_2$-space can be $\mathbb{Z}_2$-mapped into every other $\mathbb{Z}_2$-space. Because of this we can (and will shortly hereafter) define a way to compare and somewhat order $\mathbb{Z}_2$-spaces relative to one another by looking at which spaces can be $\mathbb{Z}_2$-mapped into which spaces. The most important spaces to compare $\mathbb{Z}_2$-spaces to in this manner are the n-spheres. An example of this which will also be significant in other parts of this thesis is the following result from the Borsuk-Ulam theorem:

Theorem 2. *(Paraphrasing [1, p. 23, 2.1.1]) There is no* $\mathbb{Z}_2$*-map from the* n*-sphere to the* $(n-1)$*-sphere.*

This is not that easy to prove using just the things we already discussed. The Borsuk-Ulam theorem is a theorem which takes many forms and which has many different proofs. We shall just assume this theorem in this thesis, but a geometric proof can be found at [1, p. 30] and a discrete one at [1, p. 35]. An interesting proof from algebraic topology using homology can be found at [2, p. 174].

4.3 The Existence of $\mathbb{Z}_2$-Maps as a Relation

The relation of there being a $\mathbb{Z}_2$-map between spaces is transitive (as asserted in [1, p. 95]): If X can be $\mathbb{Z}_2$-mapped to Y, and Y can be $\mathbb{Z}_2$-mapped to Z, then X can be $\mathbb{Z}_2$-mapped to Z. The proof is simple:

Theorem 3. *The relation between two spaces that there exists a* $\mathbb{Z}_2$*-map from the first space to the second space is a transitive relation.*

Proof. Let $(X, \alpha), (Y, \beta), (Z, \gamma)$ be $\mathbb{Z}_2$-spaces. Suppose there exist $\mathbb{Z}_2$-maps $f : X \to Y$ and $g : Y \to Z$. We will prove that $g \circ f : X \to Z$ is a $\mathbb{Z}_2$-map which proves that the just-mentioned relation is transitive.
Because f and g are $\mathbb{Z}_2$-maps we know that $f \circ \alpha = \beta \circ f$, and $g \circ \beta = \gamma \circ g$. This implies $g \circ f \circ \alpha = g \circ \beta \circ f = \gamma \circ g \circ f$ which proves that $g \circ f$ is a $\mathbb{Z}_2$-map. □

We can now see that this relation is somewhat like that of a poset. It is not exactly a poset though, because this relation lacks antisymmetry, that is to say the property that if $a \leq b$ and $b \leq a$ then $a = b$. It is however a useful way to think about this relation so we will adopt notation based on this somewhat shaky intuition. We write $X \leq_{\mathbb{Z}_2} Y$ if there exists a $\mathbb{Z}_2$-map from X to Y. (This idea was taken from [1, p. 95])

4.4 The $\mathbb{Z}_2$-Index

The poset-like behavior of whether a $\mathbb{Z}_2$-map exists gives us a way to measure spaces relative to each other. We'd like to have some measure that actually gives us some numerical label, rather than just the ability to compare. This is because we will end up using which spaces a $\mathbb{Z}_2$-space can be $\mathbb{Z}_2$-mapped into as defining certain numeric values, especially in our theorems on the lower bound of the chromatic number.

The most natural measure would be that of spheres [1, p. 95]. How a space compares to spheres as far as its '$\mathbb{Z}_2$-ness' goes, will be encoded in the $\mathbb{Z}_2$-index, defined in following way:

Definition 18. *The* <u>*$\mathbb{Z}_2$-index*</u> *of the $\mathbb{Z}_2$-space (X, v) is defined to be* $\mathrm{ind}_{\mathbb{Z}_2}(X) = \min\{n \in \mathbb{N}_{\geq 0} : X \leq_{\mathbb{Z}_2} S^n\}$. *If there is no $\mathbb{Z}_2$-map to any n-sphere, the $\mathbb{Z}_2$-index of the space is ∞. [1, p. 95, 5.3.1]*

The most important property of the $\mathbb{Z}_2$-index for our purposes is that the existence of a $\mathbb{Z}_2$-map from one space to another tells us that the $\mathbb{Z}_2$-index can't be higher than that of the space that can be $\mathbb{Z}_2$-mapped into:

Theorem 4. *If $X \leq_{\mathbb{Z}_2} Y$, then $\mathrm{ind}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(Y)$. (asserted at [1, p. 96])*

Proof. Let $n = \mathrm{ind}_{\mathbb{Z}_2}(Y)$, we have $X \leq_{\mathbb{Z}_2} Y, Y \leq_{\mathbb{Z}_2} S^n$. We have shown $\leq_{\mathbb{Z}_2}$ to be transitive so $X \leq_{\mathbb{Z}_2} S^n$. It follows that $\mathrm{ind}_{\mathbb{Z}_2}(X) \leq n = \mathrm{ind}_{\mathbb{Z}_2}(Y)$. □

Like we stated before, the n-spheres function as a good standard to use our ordering-like structure on $\mathbb{Z}_2$-spaces to provide a numerical label. We will justify this now.

Theorem 5. *The n-sphere has $\mathbb{Z}_2$-index n. (Stated and hint of proof provided at [1, p. 96, 5.3.2.ii])*

Proof. It's trivial that the n-sphere has $\mathbb{Z}_2$-index at most n, the definition of the $\mathbb{Z}_2$-index makes this obvious.

The inclusion map is a $\mathbb{Z}_2$-map $S^n \to S^m$ for any $n \leq m$, and the composition of $\mathbb{Z}_2$-maps is a $\mathbb{Z}_2$-map, so if there were any $k < n$ with $S^n \leq_{\mathbb{Z}_2} S^k$, then we would have a $\mathbb{Z}_2$-map to S^{n-1} by composing this map with the inclusion of the k-sphere into the $(n-1)$-sphere, which exists because $k \leq n-1$. But we know because of Borsuk-Ulam that there is no such map. $\lightning$

And thus the smallest n-sphere into which any n-sphere $\mathbb{Z}_2$-maps is itself. Thus the n-sphere has $\mathbb{Z}_2$-index n. $\square$

4.5 The Box-Complex

There is one $\mathbb{Z}_2$-space that is of particular interest to us, which is the box complex. The reason this space is so interesting is that it is constructed based on a graph, linking graphs to $\mathbb{Z}_2$-spaces, and we can link graph-homomorphisms to $\mathbb{Z}_2$-maps between the box-complexes, turning statements about $\mathbb{Z}_2$-spaces into graph-theoretical statements. First we will take a look at the definition of the box-complex, then we will show that graph homomorphisms induce $\mathbb{Z}_2$-maps between their box complexes, and we will finish by looking more closely at the box-complex of the complete graph.

4.5.1 The Definition of the Box-Complex

The box-complex is defined as an abstract simplicial complex $B(G)$ based on a graph G. We first need another simple definition to nicely state the definition of the box complex:

Definition 19. *We define $A \uplus B$ as $A \times \{1\} \cup B \times \{2\}$. [1]*

Now we can properly define the box-complex. We start with some graph, taking two copies of its vertices. Then we define simplices which are made out of some vertices A_1 of the first copy and some vertices A_2 of the second copy. For A_1 and A_2 we can choose all subsets of $V(G)$ as long as they are disjoint, all vertices of a chosen subset have a common neighbor, and the subsets define a complete bipartite subgraph. More exactly:

Definition 20. *Given a graph G, the box-complex of G is a simplicial complex defined by the following simplices:*

$$B(G) = \{A_1 \uplus A_2 : A_1, A_2 \subseteq V(G), A_1 \cap A_2 = \emptyset, G[A_1, A_2] \text{ is complete}, \\ \mathrm{CN}(A_1) \neq \emptyset \neq \mathrm{CN}(A_2)\}$$

[1, p. 129]

The box-complex can easily be made into a $\mathbb{Z}_2$-space, because the definition has a obvious symmetry to it: If $A_1 \uplus A_2$ is a simplex of the box-complex, then so is $A_2 \uplus A_1$. So we can define a $\mathbb{Z}_2$-action on the box complex as the map that sends the top to the bottom and the bottom to the top: $(v, 1) \to (v, 2)$ and $(v, 2) \to (v, 1)$ [1, p. 129].

4.5.2 Graph Homomorphisms and Box-Complexes

Once we have a graph homomorphism $f : G \to H$, it is not hard to define a simplicial $\mathbb{Z}_2$-map between between their box complexes:

Theorem 6. *Given some graph homomophism $f : G \to H$, the following is a simplicial $\mathbb{Z}_2$-map:*

$$B(f) : B(G) \to B(H) \text{ by } (v, j) \to (f(v), j)$$

. [1, p. 129]

Proof. The map $B(f) : B(G) \to B(H)$ needs to be simplicial and a $\mathbb{Z}_2$-map.

For it to be simplicial it needs to map simplices to simplices. The conditions for being a simplex are part of our definition of the box complex, so say we have a simplex of $B(G)$ then it is some $A_1 \uplus A_2$, with $A_1 \cap A_2 = \emptyset$, and $\mathrm{CN}(A_1) \neq \emptyset \neq \mathrm{CN}(A_2)$, and $G[A_1, A_2]$ is a complete bipartite subgraph.

For $B(f)(A_1 \uplus A_2)$ to be a simplex as well, we firstly need $f(A_1) \cap f(A_2) = \emptyset$. This has to be true because if there is some $v \in f(A_1) \cap f(A_2)$ then there is some $v_1 \in A_1, v_2 \in A_2$ with $f(v_1) = f(v_2)$, even though $G[A_1, A_2]$ is complete thus $\{ v_1, v_2 \} \in E(G)$ which implies $\{ v, v \} \in E(H)$↯

We secondly need $\mathrm{CN}(f(A_1)) \neq \emptyset \neq \mathrm{CN}(f(A_2))$. We will prove this for A_1, for A_2 it then follows out of symmetry. Let $v \in \mathrm{CN}(A_1)$. We know that

$\forall v \in A_1 : (v, a) \in E(G)$. It is implied by f being a homomorphism that $\forall v \in A_1 : (f(v), f(a)) \in E(H)$. This in turn implies $v \in \mathrm{CN}(f(A_1))$ so it can't be empty.

Lastly (for it being a simplicial map), we need $H[f(A_1), f(A_2)]$ complete. This is easy. All vertices in $f(A_1)$ are $f(v_1)$ for some $v_1 \in A_1$, and all vertices in $f(A_2)$ are $f(v_2)$ for some $v_2 \in A_2$. $G[A_1, A_2]$ is complete, so $(v_1, v_2) \in E(G) \implies (f(v_1), f(v_2)) \in E(H)$. So $H[f(A_1), f(A_2)]$ is complete.

For it to be a $\mathbb{Z}_2$-map it needs to commute with the $\mathbb{Z}_2$-action, which it does, because when we have a pair (v, j), the $\mathbb{Z}_2$-action only acts on the right side, whilst $B(f)$ only acts on the left side, thus the order in which they are applied does not matter. □

4.5.3 The Box-Complex of the Complete Graph

For our proofs we need some data about the box-complex of a complete graph in particular. We will first calculate the box-complex and then give an upper bound on its $\mathbb{Z}_2$-index. It has been suggested ([1, p. 129]) that the box complex of a complete graph is isomorphic to a certain crosspolytope with opposite facets removed and we should therefore be able to $\mathbb{Z}_2$-map it into a certain n-sphere, giving a lower bound on its $\mathbb{Z}_2$-index. This we shall prove.

Theorem 7. $\mathrm{ind}_{\mathbb{Z}_2}(B(K_n)) \le n - 2$

Proof. For any two disjoint sets $A_1, A_2 \subseteq V(K_n)$ it is trivial that $K_n[A_1, A_2]$ is complete, because K_n is complete. Furthermore, $\mathrm{CN}(A)$ is non-empty for any $A \subset V(K_n)$ as long as $A \neq V(K_n)$. So the simplices of $B(K_n)$ are all disjoint sums of non-intersecting subsets that are not the whole set:

$$B(K_n) = \{\, A_1 \uplus A_2 : A_1 \cap A_2 = \emptyset, A_1 \neq V(K_n) \neq A_2 \,\}$$

We know there are $2n$ vertices of $B(K_n) = V(K_n) \uplus V(K_n)$. The definition of the simplices can be interpreted as: All sets taking i vertices of the first copy of $V(K_n)$ and j vertices of the second copy, such that $i + j \le n$, $i < n$ and $j < n$ whilst never taking any vertex of the second copy that was taken from the first copy, and vice-versa.

In order to make a statement about its $\mathbb{Z}_2$-index we'd like to relate this to what is called the boundary complex of the n-dimensional crosspolytope.

The n-dimensional crosspolytope is simply the convex hull of $\{\, e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n \,\} \subset \mathbb{R}^n$. The boundary complex is simply the simplicial complex that is formed by the boundary of this object. A subset A of these points is a face of this complex if and only if $\forall x \in A : -x \notin A$. [1, p. 11] Make sure you notice that this boundary complex is a actually a $(n-1)$-sphere. The crosspolytope itself is $\{ x \in \mathbb{R}^d : \|x\|_1 \leq 1 \}$, and its boundary is $\{ x \in \mathbb{R}^d : \|x\|_1 = 1 \}$, and one should be familiar with the fact that this is homeomorphic to S^{n-1}. [1, p. 11]

So the abstract simplicial complex which is the boundary complex of the n-dimensional crosspolytope has vertices $V = \{ e_1, -e_1, e_2, -e_2, \ldots, e_n, -e_n \}$ and the simplices $C = \{ A \subset V : \forall x \in A : -x \notin A \}$. So all sets of i units and j negative units as vertices, such that $i + j <= n$, whilst never taking any vertex of which the negative was taken, and vice-versa.

Except for labeling (which does not matter up to isomorphism) these are the exact same with just one exception, the box-complex of the complete graph does not have a simplex consisting of all elements of one of the copies, though the boundary complex of the n-dimensional crosspolytope does include a simplex of all positive and one of all negative units. This means the box-complex of the n-vertex complete graph is the boundary complex of the n-dimensional crosspolytope with two opposite (non-overlapping, all their faces remain present) facets removed.

The $(n-1)$-sphere is the suspension of the $(n-2)$-sphere (where the (-1)-sphere is regarded as the empty set). Removing two opposite facets from the $(n-1)$-sphere leaves you with a cylinder of the $(n-2)$-sphere or the $(n-2)$-sphere itself, depending on how big the facets being removed are. Whatever the case, it can trivially be $\mathbb{Z}_2$-mapped into the $(n-2)$ sphere.

In conclusion, the (polyhedron of the) box-complex of K_n can be $\mathbb{Z}_2$-mapped into the $(n-2)$ sphere, and thus we have derived the following fact:

$$\operatorname{ind}_{\mathbb{Z}_2}(B(K_n)) \leq n-2$$

□

5 The First Lower Bound

The first theorem on a lower bound on the chromatic number that we have worked up to so far relates the $\mathbb{Z}_2$-index of the box complex of a graph to a lower bound of its chromatic number. The proof is rather easy because of a lot of work we have already done. Note that the proof and theorem are from [1].

Here is the exact formulation of the first theorem.

Theorem 8. ***Graph Coloring and the $\mathbb{Z}$-index***
For every graph G, $\chi(G) \geq ind_{\mathbb{Z}_2}(B(G)) + 2$;
And moreover, if $ind_{\mathbb{Z}_2}(B(G)) > ind_{\mathbb{Z}_2}(B(H))$, then there is no homomorphism $G \to H$.

Firstly we know that if there is a m-coloring of a graph G, then there is a homomorphism $f : G \to K_m$. We have also seen that homomorphisms between graphs induce $\mathbb{Z}_2$-maps between their box complexes, we thus have a $\mathbb{Z}_2$-map $B(f) : B(G) \to B(K_m)$. We can use that in combination with the fact that if there exists a $\mathbb{Z}_2$-map between spaces, this tells us something about their respective $\mathbb{Z}_2$-indices; to be exact, recall: If there is a $\mathbb{Z}_2$-map from X to Y, then the $\mathbb{Z}_2$-index of X is smaller than or equal to that of Y. Therefore the existence of the $\mathbb{Z}_2$-map $B(f)$ implies $\text{ind}_{\mathbb{Z}_2}(B(G)) \leq \text{ind}_{\mathbb{Z}_2}(B(K_m))$. We have also previously determined something about the $\mathbb{Z}_2$-index of $B(K_m)$: It is at most $m - 2$.

Now we can finish our proof: We have $\text{ind}_{\mathbb{Z}_2}(B(G)) \leq \text{ind}_{\mathbb{Z}_2}(B(K_m))$ if G can be m-colored, and $\text{ind}_{\mathbb{Z}_2}(B(K_m)) \leq m-2$. Therefore $\text{ind}_{\mathbb{Z}_2}(B(G))+2 \leq m$ for a graph that can be m-colored. A graph G can be $\chi(G)$-colored in particular, and thus we have derived the following fact:

$$\chi(G) \geq \text{ind}_{\mathbb{Z}_2}(B(G)) + 2$$

The fact that $\text{ind}_{\mathbb{Z}_2}(B(G)) > \text{ind}_{\mathbb{Z}_2}(B(H))$ implies there is no homomorphism $G \to H$ should be clear from the proof so far, but it can be shortly summarized as follows: Were there such a homomorphism, then there would have been a $\mathbb{Z}_2$-map between the corresponding box complexes, which in turn would mean that the $\mathbb{Z}_2$-index of $B(G)$ would be at most that of $B(H)$, directly contradicting the condition of the implication.

The box complex of a space might not be the nicest thing to work with, which is why we will define another theorem which uses the first theorem by giving a certain lower bound on the $\mathbb{Z}_2$-index of the box complex in terms of a property of the neighborhood complex.

6 Homotopy

The proof of our second theorem will depend on a certain topological property called k-connectedness, so we will first go into a bit of topology to clarify some concepts and definitions. The property of k-connectedness is not only invariant under homeomorphism, but even under a weaker kind of topological 'equality', called homotopy-equivalence. We shall now define and prove these notions; please note that when we speak of maps in the context of topology, we mean continuous maps unless stated otherwise.

In topology we usually think of two spaces as equivalent if and only if they are homeomorphic. Recall that a homeomorphism is a continuous bijection with a continuous inverse. This means of course that there are (continuous) maps f, g such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Sometimes we see maps that seem to be similar to one another, in the sense that it seems like a map could be continuously transformed into another map. To make this precise we have the notion of a homotopy between maps. This transformation is defined by a map from $I \times X$ instead of X, assuming X is the domain of both of the original maps, where for $(t, x) \in I \times X$ we start at the first map, where $t = 0$, and we end up at the second map, where $t = 1$.

Definition 21. *Given maps $f_0, f_1 : X \to Y$, a homotopy between them is a map $F : I \times X \to Y$ such that $F(0, x) = f_0(x)$ and $F(1, x) = f_1(x)$. Two maps are said to be homotopic if a homotopy exists between them. For f homotopic to g we write $f \simeq g$. (paraphrased from [2, p. 3])*

We can use this to define a notion of equivalence between spaces that is weaker than being homeomorphic. For example, A line piece is homotopy-equivalent to a filled cylinder, but not homeomorphic. A möbius band is homotopy-equivalent to a cylinder, but not homeomorphic. Note the similarity of this definition to how we described the existence of homeomorphisms at the start of this section.

Definition 22. *We call spaces X and Y homotopy equivalent if there exist f, g such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. We also write $X \simeq Y$ if spaces are homotopy equivalent. (paraphrased from [2, p. 3])*

6.1 Deformation Retractions

A special though simple case of homotopy equivalent spaces are spaces where one is what we call a 'deformation retract' of the other. This means that a space can be continuously transformed into a subspace whilst keeping this subspace fixed. For example, a circle is a deformation retract of a cylinder.

Definition 23. *A* <u>*deformation retraction*</u> *from* X *to* Y *is formally a map* $F : I \times X \to X$ *such that* $F(0,x) = x$*,* $F(1,x) \in Y$*, and* $x \in Y \implies F(t,x) = x$*. (paraphrased from [2, p. 2])*

We call Y *a deformation retract of* X*.*

The example of the cylinder to the circle is simple: A cylinder is $S^1 \times I$ for example, and $S^1 \times \{ 0 \}$ is a subspace of the cylinder which is a circle; define $F : I \times S^1 \times I \to S^1 \times I$ as $F(t,x,s) = (x,(1-t)s)$. This is clearly continuous, it keeps the subspace fixed, it starts as the identity, and it ends in a map to the circle.

Now we will prove something quite useful about deformation retracts:

Theorem 9. *Whenever* Y *is a deformation retract of* X *we have* $X \simeq Y$*.*

Proof. Given a deformation retraction F from X to Y we have the map $r : X \to Y$ defined as $r(x) = F(1,x)$, and we have the inclusion map $i : Y \to X$ defined as $i(x) = x$. It is trivial that $i \circ r \simeq \mathrm{id}_X$, the homotopy is the deformation retraction from X to Y. At the same time $r \circ i = \mathrm{id}_Y$, because i is just the inclusion of Y into X, and r keeps the subspace Y fixed. Trivially $r \circ i = \mathrm{id}_Y \implies r \circ i \simeq \mathrm{id}_Y$. □

6.2 k-Connectedness

Something else that we can define using homotopy is the notion of k-connectedness. The idea here is to speak of the property of a space of having no holes up to some dimension. With a n-dimensional hole we mean a hole which prevents some n-sphere in the space to shrink to a point.

So a space which is just two path-components would have a 0-dimensional hole (by placing two points in different path components we have 'placed' a 0-sphere in it in such a way that it cannot continuously shrink to a point), a torus has a 1-dimensional hole in the middle, a sphere has a 2-dimensional hole in it, and so forth. First we need to define what we mean by shrinking to a point; the formal name is being nullhomotopic.

Definition 24. *A map* $f : X \to Y$ *is* <u>*nullhomotopic*</u> *if there is a constant map* $c : X \to Y$ *such that* $f \simeq c$*. (paraphrasing [2, p. 4])*

Now we can give a definition of k-connectedness:

Definition 25. *([1, p. 78]) A space X is k-connected, where $k \geq -1$ is an integer, if for every n-sphere where $n \in \{0, \dots, k\}$ every map $S^n \to X$ is nullhomotopic. By (-1)-connected we just mean nonempty.*

When we have a statement about whether a space is k-connected, we have a statement about which maps from n-spheres to a space are homotopic to constant maps. It seems that spaces homotopy-equivalent to a k-connected space should themselves be k-connected. This is something we can prove:

Theorem 10. *If a space X is k-connected, and $X \simeq Y$, then Y is k-connected as well.*

Proof. Say we have a k-connected space X and a homotopy-equivalent space Y, then we have f, g such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. Let $\gamma : S^n \to Y$, $n \leq k$. We can rest assured that $g \circ \gamma$ is nullhomotopic, so we have a homotopy $F : I \times S^n \to X$ between $g \circ \gamma$ and some constant map c_X. We can also take a look at $f \circ F$, which starts at $f \circ g \circ \gamma$ and ends at $c_Y = f \circ c_X$. We also have a homotopy $G : I \times Y \to Y$ between id_Y and $f \circ g$. Now we can define $H : I \times S^n \to Y$ as $H(t, x) = G(t, \gamma(x))$ to be a homotopy between $\gamma = \mathrm{id}_Y \circ \gamma$ and $f \circ g \circ \gamma$.

Now we can combine $f \circ F$ and H to build one big homotopy between γ and c_y: Define $\Phi : I \times S^n \to Y$ as $\Phi(t, y) = \begin{cases} H(2t, y) & \text{if } t \leq \frac{1}{2} \\ f \circ F(2t - 1, x) & \text{otherwise} \end{cases}$

So now we have proven that for homotopy-equivalent spaces, if every map from the n-sphere is homotopic to a constant map in one space, then the same is true in the other. This implies that if a space is k-connected, spaces homotopy-equivalent to this space are k-connected as well. □

There is a lot more to say on the topic of homotopy, but this will be everything we need in order to prove the things we'd like to prove. The most significant in this thesis is the feature of k-connectedness that it gives a lower bound on the $\mathbb{Z}_2$-index. (The proof is paraphrased from [1, p. 96].)

Theorem 11. *If X is $(n-1)$-connected, then $\mathrm{ind}_{\mathbb{Z}_2}(X) \geq n$.*

Proof. Let X be a $(n-1)$-connected space. We shall prove the existence of a $\mathbb{Z}_2$-map from S^n to X. This implies that the $\mathbb{Z}_2$-index of S^n is lower than or equal to that of X, which is what we want to prove.

By induction on k we shall define $\mathbb{Z}_2$-maps from S^k to X until we have defined the map that we need.

The $\mathbb{Z}_2$-maps from S^{-1} and S^0 are trivial to construct, so we have a base case. Now we shall formulate the induction step. Given a map $f : S^{k-1} \to X$ we shall define a map $g : S^k \to X$, provided $k \leq n$. Note that we can identify S^{n-1} with the "equator" of S^n: $\{ x \in S^k : x_{k+1} = 0 \}$. It is the case that the upper hemisphere of the n-sphere is homeomorphic to the n-ball. That is to say: $H_k^+ = \{ x \in S^k : x_{k+1} \geq 0 \} \cong B^k$. A map $f : S^{k-1} \to X$ is nullhomotopic because of the $(n-1)$-connectedness of X. So we can extend it to a map $f : B^k \to X$. This is because we have a map (the homotopy to the constant map that makes f nullhomotopic) $F : I \times S^{n-1} \to X$ where $F(1, x) = x$, and B^n is homeomorphic to the cone of S^{n-1} which can be defined as $(I \times S^{n-1})/(\{ 1 \} \times S^{n-1})$. We can also view B^n as a filled S^{n-1}, and thus we can construct a projection $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ that deletes the last coordinate, which is also a homeomorphism $H_k^+ \to B^k$

Now we can define g on H_k^+:

$$g = f \circ \pi : H_k^+ \to B^k \to X$$

And on the lower hemisphere H_k^- as $g(x) = v(g(-x))$ where v is the $\mathbb{Z}_2$-action on X. Because g is a $\mathbb{Z}_2$-map on the intersection of the hemispheres (it is equal to f there) we have a well-defined map which is a $\mathbb{Z}_2$-map by construction. We have no doubts about its continuity, because it is continuous on both hemispheres which have S^n as their union.

This completes the induction, which completes the proof of the existence of the $\mathbb{Z}_2$-map from the start of this proof. As we stated there, the existence of this map concludes the proof. □

7 The Second Lower Bound

Now for the second theorem. This theorem uses the k-connectedness of the neighborhood complex. It will take more work to prove this because there is a big lemma that we need which we have not yet proved. This is the theorem:

Theorem 12. *If* $\mathrm{N}(G)$ *is* k*-connected, then* $\chi(G) \geq k + 3$

In order to prove this we will first be defining another simplicial complex that is based on G, which we will prove to be a deformation retract of the neighborhood complex, and which $\mathbb{Z}_2$-maps into the box complex. We need this as a bridge from the k-connectedness of $\mathrm{N}(G)$ to the $\mathbb{Z}_2$-index of $\mathrm{B}(G)$, after which the rest of the proof will be very easy.

We will define this simplicial complex as $\mathrm{L}(G)$. We will first look at the definition for $\mathrm{L}(G)$, after which we will prove two features of it that make it into the bridge we just mentioned: $\mathrm{L}(G)$ is a deformation retract of $\|\,\mathrm{N}(G)\|$; and $\|\,\mathrm{L}(G)\| \leq_{\mathbb{Z}_2} \|B(G)\|$. Please note this proof is mostly taken from [1, p. 131].

7.1 The Definition of $\mathrm{L}(G)$

We will refer to $\mathrm{L}(G)$ as L, to $\mathrm{N}(G)$ as N and to $\mathrm{sd}(\mathrm{N}(G))$ (the barycentric subdivision of the neighborhood complex) as N_1. We have seen that the barycentric subdivision of a complex has the simplices of this complex as vertices. We will be defining our complex L as a subcomplex of N_1, but we want to put a constraint on which simplices are actually taken to be the vertices of L. We will later try to use the common neighbors mapping CN on G as a simplicial $\mathbb{Z}_2$-action on L, because a map that takes a set of vertices of G to a set of vertices of G also takes a vertex of N_1 to a vertex of N_1. In order for this idea of a $\mathbb{Z}_2$-map to work, we need to have CN^2 as the identity of L. So we take just those subsets of the vertices of G as the vertices of L for which this works:

$$V(L) = \{\, A \subseteq V(G) : \mathrm{CN}^2(A) = A \,\}$$

Its simplices will be the simplices of N_1 that do not contain any of these vertices.

Of course CN itself needs to be a simplicial map, so let's check this: Remember that a simplex of N_1 is a chain of simplices of N, ordered by strict inclusion. Given a vertex A_i of N, for any $A_i \supset A_j \implies \mathrm{CN}(A_i) \subset \mathrm{CN}(A_j)$ Usually we don't have strict inclusion, but CN^2 is the identity by definition, and if two different sets would have the same image under CN, we would have $\mathrm{CN}^2(A) = \mathrm{CN}^2(B)$ for $A \neq B$ so for one of them CN would not be the identity. In addition, $\mathrm{CN}(A) \neq \emptyset$, because vertices of N_1 are simplices of N and simplices of N are sets with common neighbors by definition. So $\mathrm{CN} : \{\, A_1, \ldots, A_n \,\} \to \{\, B_n, \ldots, B_1 \,\}$, where $A_1 \subset, \ldots, \subset A_n$ by definition and $\emptyset \neq B_n \subset \cdots \subset B_1$ by implication. This is again a simplex.

Lastly we would like the $\mathbb{Z}_2$-action to be free, in other words: $\|\,\mathrm{CN}\,\|$ has no fixed point on $\|L\|$. This must be true if every simplex is mapped to an entirely different simplex, in other words, if for every simplex $\mathcal{A}$ we have $\mathcal{A} \cap \mathrm{CN}(\mathcal{A}) = \emptyset$. And this is trivial if we remember how simplices are defined.

$\mathcal{A} = A_1 \subset \cdots \subset A_n$. And of course if $A_i \subset A_j$ then neither $\mathrm{CN}(A_i) = A_j$ nor $\mathrm{CN}(A_j) = A_i$ is possible.

So in conclusion, L is a simplicial subcomplex of N_1, its vertices being those for which CN^2 is the identity where $\mathrm{CN} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$, and its simplices being those simplices of N_1 which don't use any vertex for which CN^2 is not the identity. We may also conclude that L is a $\mathbb{Z}_2$-complex with CN as the $\mathbb{Z}_2$-action.

7.2 The Space $\|L\|$ is a Deformation Retract of $\|N\|$

The idea here is that if we have some vertex A of N_1 which is not in L, then $\mathrm{CN}^2(A)$ must be a superset of A. This superset is also a vertex of N_1, but it is actually in L. The way we transform N_1 into L is by having any vertex A that is in N_1 but not in L travel to $\mathrm{CN}^2(A)$. Of course (the polyhedron of the) barycentric subdivision is homeomorphic to the original, and thus on a topological level, by proving L to be a deformation retract of N_1 we have immediately proven it to be a deformation retract of $\mathrm{N}(G)$.

We will define a deformation retraction $F : I \times \|N_1\| \to \|L\|$. Of course we want to start at the identity, so we define $F(0, x) = x$. We want to end at the map we previously defined, so $F(1, x) = \|\mathrm{CN}^2\|(x)$. Simplices are convex, so we can very easily construct the rest of the deformation retraction if can show that $F(0, x)$ and $F(1, x)$ are in the same simplex of N, instead of N_1 but of course they encode the same polyhedron.

Say x is contained in the simplex of N_1 with the vertices $A_1 \subset, \ldots, A_n$. Then we know that the vertices of the simplex containing $\mathrm{CN}^2(1, x)$ are $\mathrm{CN}^2(A_1), \ldots, \mathrm{CN}^2(A_n)$. All of $A_1 \subset, \ldots, A_n$ as well as all of $\mathrm{CN}^2(A_1), \ldots, \mathrm{CN}^2(A_n)$ are vertices in the subdivision of the simplex $\mathrm{CN}^2(A_n)$ of N, so we are done. So now we can easily define all of F:

$$F(t, x) = (1 - t)x + t\|\mathrm{CN}^2\|(x)$$

7.3 There is a $\mathbb{Z}_2$-Map $\|L\| \to \|\mathrm{B}(G)\|$

We will define a $\mathbb{Z}_2$-map from $\mathrm{sd}(L)$ into $\mathrm{sd}(\mathrm{B}(G))$.

Note that the vertices of $\mathrm{sd}(L)$ are chains $\mathcal{A} = A_0, \ldots, A_k$ ordered by strict inclusion, where A_i is closed under CN^2 because we are considering $\mathrm{sd}(L)$ and not $\mathrm{sd}(N_1)$. Vertices of $\mathrm{sd}(B(G))$ are simplices of $B(G)$ so they are sets $A_1 \uplus A_2$ where A_1, A_2 both subsets of the vertices of G, $A_1 \cap A_2 = \emptyset$, $G[A_1, A_2]$ is complete, and $\mathrm{CN}(A_1) \neq \emptyset \neq \mathrm{CN}(A_2)$.

We define the $\mathbb{Z}_2$-map f by setting $f(\mathcal{A}) = A_0 \uplus \mathrm{CN}(A_k)$. It's trivial that for both sets applying CN does not yield an empty set, their intersection is empty, and the bipartite subgraph defined by them is fully connected, and thus is maps vertices to vertices. Now we just need to prove that it is a simplicial- and a $\mathbb{Z}_2$-map.

If we have two vertices $\mathcal{A}, \mathcal{A}'$ they lie in the same simplex of $\mathrm{sd}(L)$ if and only if one includes the other. If $\mathcal{A}'$ extends $\mathcal{A}$ then $f(\mathcal{A}') \subseteq f(\mathcal{A})$. So if we have a chain of vertices order by inclusion, the image is also a chain of vertices (of $\mathrm{sd}(B(G))$, they are simplices of $B(G)$) ordered by inclusion, which is a simplex of $\mathrm{sd}(B(G))$ so it is a simplicial map.

$\mathcal{A}$ under the $\mathbb{Z}_2$-action on $\mathrm{sd}(L)$ is the chain $\mathcal{B} = (\mathrm{CN}(A_k) \ldots \mathrm{CN}(A_0)$. $f(\mathcal{B}) = \mathrm{CN}(A_k) \uplus \mathrm{CN}^2(A_0) = \mathrm{CN}(A_k) \uplus A_0$. Whilst $f(\mathcal{A}) = A_0 \uplus \mathrm{CN}(A_k)$ which under the $\mathbb{Z}_2$-action on $B(G)$ is $\mathrm{CN}(A_k) \uplus A_0$ as well. Thus f is a $\mathbb{Z}_2$-map.

7.4 Finishing the Proof

Because $\mathrm{L}(G)$ is a deformation retract of $\mathrm{N}(G)$, we know that if $\mathrm{N}(G)$ is k-connected then $\mathrm{L}(G)$ must be k-connected as well. We have seen that k-connectedness gives us a lower bound on the $\mathbb{Z}_2$-index: $\mathrm{ind}_{\mathbb{Z}_2}(\mathrm{L}(G)) \geq k+1$. And we have just shown that $\|\mathrm{L}(G)\| \leq_{\mathbb{Z}_2} \|B(G)\|$, which tells us that $\mathrm{ind}_{\mathbb{Z}_2}(B(G)) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathrm{L}(G))$. We can substitute this result into our first theorem, $\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}(B(G)) + 2$, and we get:

$$\chi(G) \geq k + 3$$

8 Conclusions

We have seen that we can associate graphs with certain simplicial complexes, most notably the neighborhood complex and the box complex. We have also seen that topological properties of these complexes allow us to make statements about the graphs that they were associated to. We have concluded that $\chi(G) \geq \mathrm{ind}_{\mathbb{Z}_2}(\mathrm{B}(G)) + 2$, providing a lower bound on the chromatic number by the $\mathbb{Z}_2$-index of the box complex of the graph, and we have found that for $\mathrm{N}(G)$ k-connected: $\chi(G) \geq k + 3$, providing another lower bound on the chromatic number of a graph by the k-connectedness of its neighborhood complex.

References

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