Utrecht University

Faculteit Bètawetenschappen

Percolation

Bachelor Thesis TWIN

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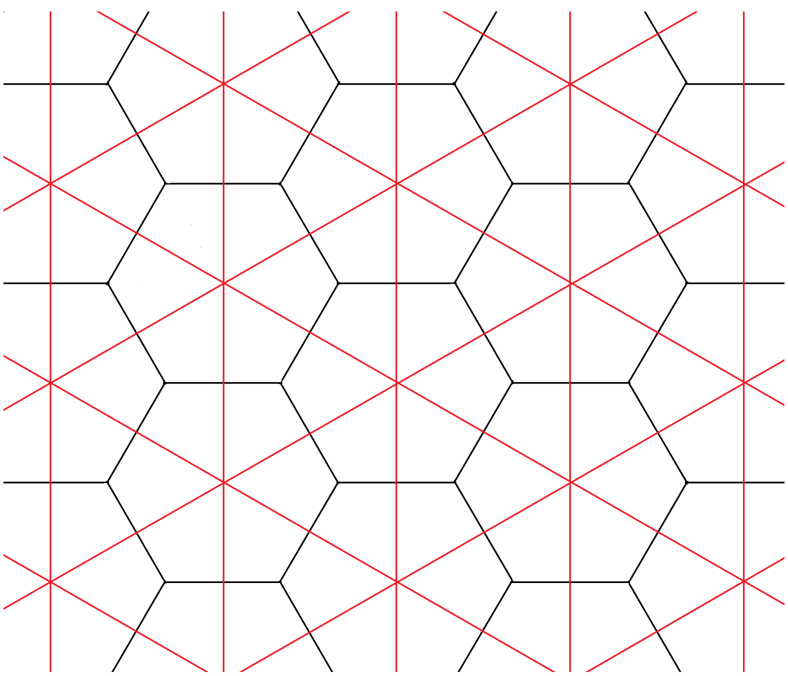
Wiskunde



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1 Introduction

Imagine that you are in a big city and some streets are closed off because reparations are carried out there. Only the streets that are not closed off are accesable to you. This impairs your movement; if you want to move from some point A to some point B in the city, maybe you will have to travel a long way around. If a lot of streets are closed off, maybe it will be impossible to ever reach point B if you start at some point A.

If you are a mathematician and you let your imagination go free for some time, then maybe some interesting questions about this city will pop up. A question you could be thinking about might be something like:

"If we suppose that the city we are in has some regular street patterns and the probability that any street is closed off is equal and independent of the probability that any other street is closed off - say, the probability is equal to some $p \in [0, 1]$ for all streets - then there is some probability P that point B can be reached from point A. If we make this probability P a function of the distance between points A and B, what happens at the limit at infinity?"

Questions like these are dealt with in percolation theory. Percolation theory is about the effects that randomly opening or closing parts of a lattice have on long-range phenomena on this lattice. Most answers to problems in percolation theory are far from trivial. The question in the above paragraph, for example, will not even be fully anwered in this thesis, though we will answer some related questions.

The probability P as discussed before either converges to 0 or converges to a number greater than 0 at the limit at infinity, dependent on the value of p. There appears to exist a critical probability p_c for which P converges to 0 at infinity if $p < p_c$, and P does not do so if $p > p_c$. The value of this number p_c depends on what kind of street pattern the city we are in has. If the street has a triangular-like pattern, the value of p_c is different than in the situation where the treet would have a hexagonal, or honeycomb-like, pattern. In this thesis, we will try to show that the sum of these different critical probabilities is equal to 1! This is our main theorem.

In chapter 2 we introduce some different lattice spaces: the square, triangular and hexagonal lattice. You could compare these to street patterns in infinitely large cities. We will also discuss a concept called duality.
In chapter 3 we introduce the concepts of percolation theory in a mathematical way. It is here that we state our main theorem that we want to proof, too.
In chapter 4 we show that the critical probabilities, as discussed before, are not trivial; we show that they are unequal to 0 and how our main theorem implies that they are unequal to 1, too.
In chapter 5 we prove some important theories in probability theory that are useful in percolation theory.
In chapter 6 we solely work in the triangular lattice. We discuss a formula that, in the city discussed before, can be interpreted as the average number of places that are connected by streets that are not closed off.
In chapter 7 we return to the concept of the dual space. We prove a theorem that relates the formula discussed in the previous chapter to the same formula for the hexagonal lattice. With help of an unproven conjecture, this would be sufficient to prove our main theorem.
In chapter 8 we prove a weaker version of our main theorem without using this unproven conjecture.
In chapter 9 we prove our main theorem with help of the theorem proven in the previous chapter.
In chapter 10 we look back and make some more remarks.

In this thesis, we were inspired a lot by the book *Percolation* of Geoffrey Grimmett (printed in the year 1989). All chapters are in some part based on this book. Grimmett focused on the square lattice, not the triangular or hexagonal lattice, and we have often modified his arguments to prove theorems about these other lattices. In each chapter we will explain what part is based on the book of Grimmett.

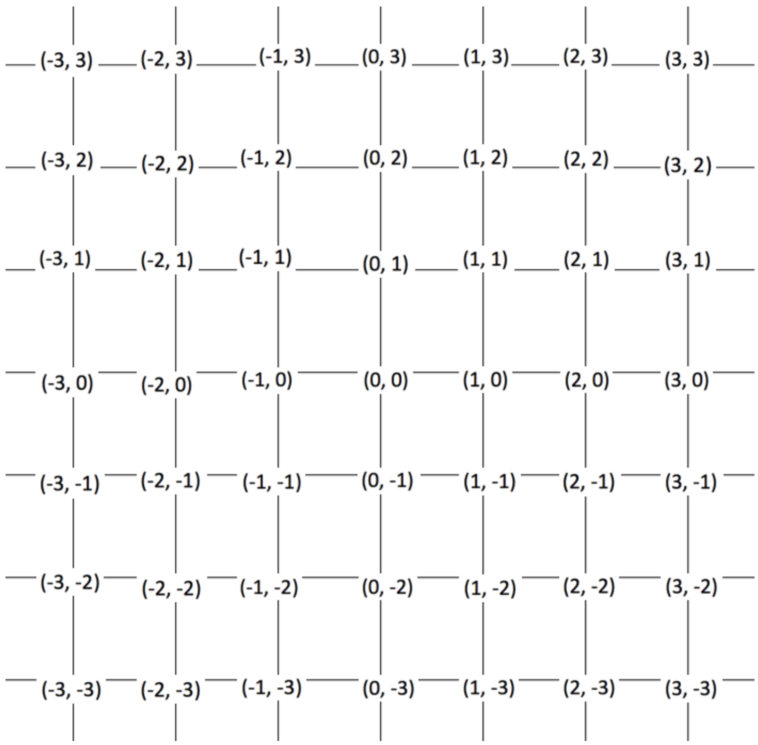

Figure 1: S^2, the square lattice

2 Some definitions of lattice spaces

In this section, we define the square, triangular and hexagonal lattices. The definition of the square lattice is borrowed from Grimmetts book, altough we invented our own notation. We have defined the other lattices ourselves. The notion of duality is also borrowed from Grimmett.

2.1 Definition of the square lattice

In Figure (1) part of the infinite square lattice is drawn. As can be seen in the figure, every vertex x can be represented by an x_1-coordinate and an x_2-coordinate, both of which are integers. We write Z^2 for the set of all vertices of this lattice. If $q \in Z^2$, then $q \equiv (q_1, q_2)$, q_1 and q_2 being the coordinates of q.

For all $x, y \in Z^2$, we define the square distance $\delta_t(x, y)$ between them in the following way:

Definition 2.1.1. $\delta_s(x, y) =\mid (x_1) - y_1 \mid + \mid (x_2) - y_2 \mid$

For every $x, y \in Z^2$ for which $\delta_t(x, y) = 1$, we define an edge $< x, y >$. The set of all these edges is denoted by E_t^2 and is drawn in the picture. Together, Z^2 and E_s^2 form the square lattice S^2, and we write $S^2 \equiv (Z^2, E_s^2)$.

2.2 Definition of the triangular lattice

In Figure (2) part of the infinite triangular lattice is drawn. As can be seen in the figure, every vertex x can again be represented by an x_1-coordinate and an x_2-coordinate, both of which are integers. We write Z^2 for the set of all vertices of this lattice. Again, if $q \in Z^2$), then $q \equiv (q_1, q_2)$, q_1 and q_2 being the coordinates of q.

For all $x, y \in Z^2$, we define the triangular distance $\delta_t(x, y)$ between them in the following way:

Definition 2.2.1. $\delta_t(x, y) =\mid x_1 - y_1 \mid + \mid x_2 - y_2 \mid$ if $(x_1 - x_2)(y_1 - y_2) \leq 0$ and
$\delta_t(x, y) = max\mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid$ if $(x_1 - x_2)(y_1 - y_2) > 0$.

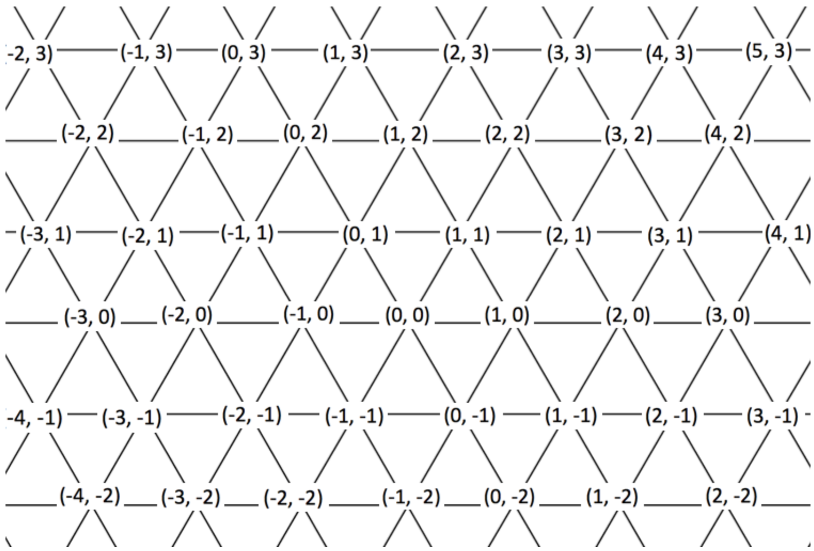

Figure 2: T^2, the triangular lattice

For every $x, y \in Z^2$ for which $\delta_t(x, y) = 1$, we define again an edge $< x, y >$. The set of all these edges is denoted by E_t^2. As can be verified, these are exactly the edges drawn in the picture. Together, Z^2 and E_t^2 form the triangular lattice T^2, and we write $T^2 \equiv (Z^2, E_t^2)$.

2.3 Definition of the hexagonal lattice

In Figure (3) part of the infinite hexagonal lattice is drawn. Again, as can be seen in the figure, every vertex x can be represented by an x_1-coordinate and an x_2-coordinate, both of which are integers. We therefore again write Z^2 for the set of all vertices of this lattice.

For all $x, y \in Z^2$, we can define the hexagonal distance $\delta_h(x, y)$ between them such that:

Definition 2.3.1. $\delta_h(x, y) = 1$ if and only if $\mid (x_1) - y_1 \mid + \mid (x_2) - y_2 \mid = 1$ and it is not the case that x_1 is even and $x_1 = y_1 + 1$ or y_1 is even and $y_1 = x_1 + 1$.

For every $x, y \epsilon Z^2$ for which $\delta_h(x, y) = 1$, we define again an edge $< x, y >$. The set of all these edges is denoted by E_h^2 and as can be verified, these are exactly the edges drawn in the picture. Together, Z^2 and E_h^2 form the hexagonal lattice H^2, and we write $H^2 \equiv (Z^2, E_h^2)$. The precise definition of $\delta_h(x, y)$ is not relevant, since we will make no further use of it.

2.4 The dual space

When drawing a lattice, the edges of that lattice will make faces. A face is what you intuitively expect it to be: two points in a face can always be connected by a line that does not cross an edge, and two points in two different faces can only be connected by lines that do cross one or more edges. This only works if the edges are drawn such that edges only cross at points where there is a vertex. Faces are important for defining the so-called dual space:

Definition 2.4.1. The dual space of a lattice L is a lattice obtained by placing a vertex in each face of L and by joining two such vertices by an edge whenever the corresponding faces of L share a boundary edge of L.

As a result, every edge of L crosses a unique edge of its dual and vice versa.
The following propositions are important:

Proposition 2.4.2. *The square lattice is its own dual space.*

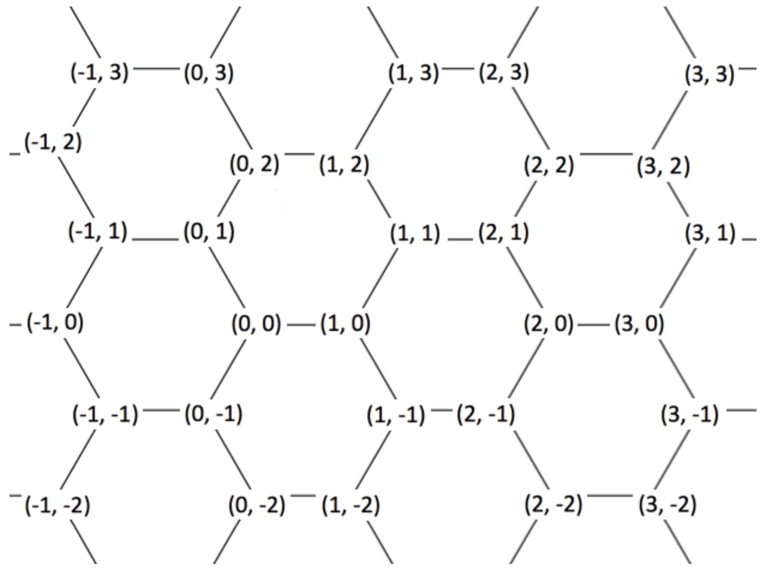

Figure 3: H^2, the hexagonal lattice

Proposition (2.4.2) is a claim that Grimmett makes at page 16 of his book.

Proposition 2.4.3. *The triangular lattice and hexagonal lattice are each others dual space.*

Proposition (2.4.3) does not appear in Grimmetts book. The proposition is true according to an article that the following link leads to (or lead to at June 1st, 2017):
http://www.cambridge.org/core/journals/advances-in-applied-probability-article/
bond-percolation-on-honeycomb-and-triangular-lattices/D2F18A5392DEFC352B9C89CACG21FDAO

Propositions 2.4.2 and 2.4.3 are illustrated in Figure (4) and Figure (5), respectively. The way the two square lattices are drawn in Figure (4) demonstrates that they are each others dual space. Similarly, the way the triangular and hexagonal lattices are drawn in Figure (5) demonstrates that they are each others dual space. Propositions 2.4.2 and 2.4.3 can be proven topologically, but since that is beyond the scope of this thesis, we do not provide proofs here.

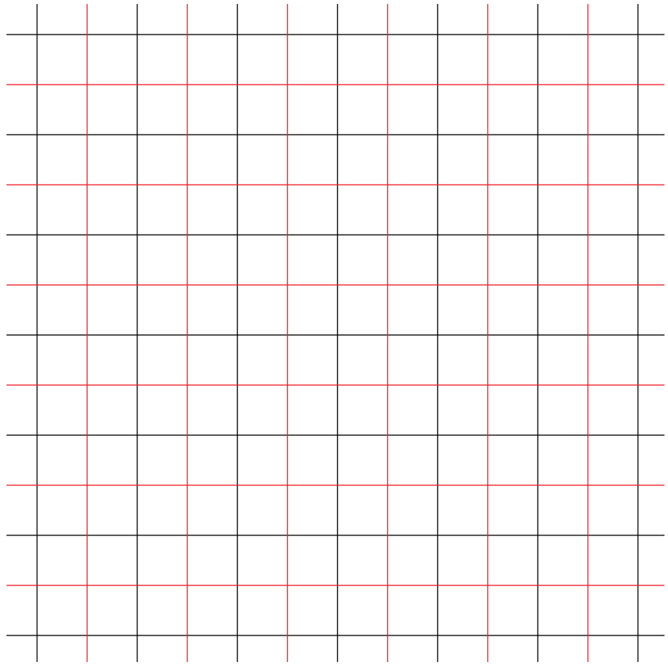

Figure 4: Two square lattices, one colored black and the other colored red, as each others dual.

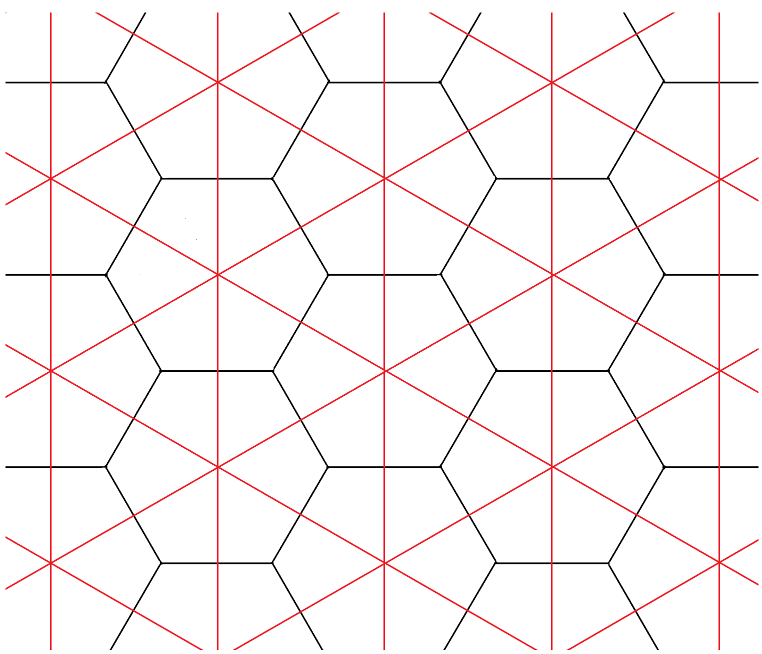

Figure 5: The triangular lattice (in red) and hexagonal lattice (in black), as each others dual.

3 Introduction to percolation theory

The disussion in this chapter is based on Chapter 1 of the book of Grimmett. We however sometimes use our own notation. We also skip quite large sections of the chapter, but we add the triangular and hexagonal lattices to the discussion. Grimmett does not talk about these lattices.

Imagine an infinitely large lattice L that consists of edges and vertices that connect edges to other edges. Examples of such lattices are S^2, T^2 and H^2. Suppose that we declare each edge open with probability p and closed with probability $1-p$, with $p \in [0,1]$. An equivalent way of saying this is that for all e that are edges in L, we define a random variable $X(e)$ that is uniformly distributed on $[0,1]$. Then we define $\eta_p(e) = 1$ if $X(e) < p$ and $\eta_p(e) = 0$ if $X(e) \geq p$, $p \in [0,1]$, and call each e open if $\eta_p(e) = 1$, and closed if $\eta_p(e) = 0$.

We define an open path as follows:

Definition 3.0.4. A path of a lattice L is a sequence $x_0, e_0, x_1, e_1, x_2, ...e_{n-1}, x_n$ for which all $e_n \equiv < x_j, x_{j+1} >$ are edges in L and all x_n are vertices in L. If all the edges in a path are open, then we speak of an open path. Paths can be infinite, too, and on both sides. For example, $..., x_{-2}, e_{-2}, x_{-1}, e_{-1}, x_0, e_0, x_1, e_1, x_2, ...$ is a path that is infinite on both sides.
An (open) path can also bedefined on a subset S of a lattice L by replacing L with $S \subset L$ in the above definition.

We define an open cluster as a set of vertices such that you can get from any vertex in the set to each other vertex in the set by moving over open edges only. Open clusters can be of any size greater than 0. They could contain just 1 element, but infinitely many, too. There are two possible formal definitions (we will not show the equality):

Definition 3.0.5. An open cluster U of (a subset of) a lattice L is a set of vertices such that $x, y \in U$ if and only if there exists an open path V on (a subset of) L such that $x, y \in V$.
This is equivalent to saying that U is a set of vertices for which the following conditions hold:
1. If $x \in U$ and $< x, y >$ is an open edge, then $y \in U$ as well.
2. U is not empty and does not contain any nonempty proper subset for which property (1) holds.
An open cluster is also referred to as a connected component.

Property (2) of the second definition may be the only propery that doesn't come off as intuitive. It is needed to ensure that U is connected and not merely a collection of open clusters.

Percolation theory is concerned with the behaviour of open clusters, and specifically how this behaviour relates to the value of p.

Because we declared edges to be open with probability p, p is called the edge probability and we are concerned with bond percolation. One could also declare each vertex open with probability p and study open clusters this way. In this case, p is referred to as the site probability and we would be concerned with site percolation. In this thesis, we focus soloy on bond percolation and we will only use the letter p to refer to edge probability. For other probabilities that depend on the edge probability (and the type of lattice we are looking at), we use the following notation:

Definition 3.0.6. $P_{p,s}(X) \equiv P(X \mid$ we work on the square lattice with edge probability $p)$;
$P_{p,t}(X) \equiv P(X \mid$ we work on the triangular lattice with edge probability $p)$;
$P_{p,h}(X) \equiv P(X \mid$ we work on the hexagonal lattice with edge probability $p)$.
Expected values are noted using the same kind of notation. So for example:
$E_{p,t}(X) \equiv E(X \mid$ we work on the triangular lattice with edge probability $p)$.

Notice that every vertex is an element of only one open cluster. Thus, every element can be associated with an open cluster in the following definitions:

Definition 3.0.7. Let L be a lattice and x an edge in L. Then $C(x)$ is the open cluster of L containing x.
Also, $C \equiv C(0)$ is the open cluster containing the origin.

The cardinality $\mid U \mid$ of an open cluster U is just the number of vertices in U. The cardinality of U could be infinite. For example, if $p = 1$ then the only open cluster that exists consists of all the edges and has therefore infinite cardinality. However, if $p = 0$ then we know for sure that no open cluster has cardinality greater than 1. An important question is whether there could be an infinite open cluster if $p \in (0, 1)$. Another important question is what would be the probability that any arbitrary vertex is an element of an infinite open cluster. In the spaces we have encountered, any arbitrary vertex can be translated onto the origin because of symmetry, which means that the probability that any arbitrary vertex is an element of an infinite open cluster is just the probability that $\mid C \mid = \infty$.
We define the following functions:

Definition 3.0.8. $\theta_s(p) = P_{p,s}(\mid C \mid = \infty)$;
$\theta_t(p) = P_{p,t}(\mid C \mid = \infty)$;
$\theta_h(p) = P_{p,h}(\mid C \mid = \infty)$.

The different functions $\theta_s(p)$, $\theta_t(p)$ and $\theta_h(p)$ are of great importance and very challenging to find. We do know that they are monotonely increasing; after all, increasing the probability that an edge is open can only increase the probability that $\mid C = \infty$. The following numbers are of interest:

Definition 3.0.9. $p_{c,s} \equiv \sup\limits_p\{\theta_s(p) = 0\}$;
$p_{c,t} \equiv \sup\limits_p\{\theta_t(p) = 0\}$;
$p_{c,h} \equiv \sup\limits_p\{\theta_h(p) = 0\}$.

These numbers are the so-called *critical probabilities* of their respective lattices. They are of interest because their values are not trivial:

Theorem 3.0.10. $0 < p_{c,s}, p_{c,t}, p_{c,h} < 1$

Part of the proof of this Theorem is written in the next chapter. Finding the exact values of the critical probabilities of specific lattices is the focus of much research in percolation theory.

For any lattice, the situation where $p < p_c$ is referred to as the subcritical phase, the case where $p = p_c$ is referred to as the critical phase, and the case where $p > p_c$ is referred to as the supercritical phase. The different phases all show distinct interesting properties. In this thesis, we will not focus that much on these properties; rather, we will try to find out for which values of p we are in which phase. We are thus concerned in finding out what the value for p_c is for different lattices.
Specifically, in this thesis, we will try to proof the following Theorem:

Theorem 3.0.11. $p_{c,t} + p_{c,h} = 1$

There are a few important properties that we will use, but not take the time to prove. Before we introduce these properties, first we will clarify a term that we will, now and then, use in this thesis:

Definition 3.0.12. 'Almost surely' means 'with probability 1'. If something is almost surely true, there could exist cases that it is not true, but the probability to be in one of these cases is 0. For example, if X is uniformly distributed on $[0, 1]$, then a realisation x from this distribution is almost surely an irrational number.

In the following proposition, important consequences in the case $\theta(p) > 0$ are shown. The proposition is valid for the square, triangular and hexagonal lattice, and so $\theta(p)$ can be interpreted as any one of $\theta_s(p)$, $\theta_t(p)$ and $\theta_h(p)$, dependent on the lattice we are in.

Proposition 3.0.13. *If $\theta(p) > 0$, then*
** If S is a set of infinitely many vertices, then almost surely there exists an $x \in S$ such that x is in an infinite open cluster;*
** If S_n are sets of vertices and for $n \longmapsto \infty$, $\mid S \mid \longmapsto \infty$ as well, then the probability that there exists an $x \in S_n$ such that x is in an infinite open cluster converges to 1;*
** If x and y are in an infinite open cluster, then x and y are in the same cluster almost surely.*

This proposition means that whenever $\theta(p) > 0$, then there exists one unique infinite open cluster almost surely. The proposition was, for the square lattice, proven by Grimmett.

4 The critical probabilities are nontrivial

The proof of the lemma in this section is essentially the same proof that Grimmett gave at pages 15 and 16 of his book. However, we show more intermediate steps and go into somewhat more detail. Also, Grimmett only shows that $0 < p_{c,s}$; we slightly modify his proof in order to show that $0 < p_{c,t}, p_{c,h}$, too.

Recall Theorem (3.0.10) that $0 < p_{c,s}, p_{c,t}, p_{c,h} < 1$. We will prove a weaker version, namely:

Lemma 4.0.14. $0 < p_{c,s}, p_{c,t}, p_{c,h}$

To do this, we will make use of the following extra definitions. A non-selfintersecting path is a path in which no vertex or edge appears twice, and a path with length n is a path that contains n edges. We define $\sigma(n)$ as the number of non-selfintersecting paths with length n starting at the origin. Similarly, we define $N(n)$ as the number of non-selfintersecting **open** paths with length n starting at the origin.

We are now equipped to prove Lemma (4.0.14).

Proof. If the origin lies in an infinite open cluster, then an infinite non-selfintersecting open path exists, too, and hence for any n, there exists at least one non-selfintersecting open path starting at the origin with length n. Thus, if $\mid C \mid = \infty$, then $N(n) > 1$ for all $n \in \mathbb{N}$. Because of this implication, we have, for all $n \in \mathbb{N}$, that

$$\theta(p) \leq P_p(N(n) \geq 1) \tag{1}$$

where we could be in any of the lattices S^2, T^2 and H^2, and $\theta(p)$ could be any of $\theta_s(p)$, $\theta_t(p)$ and $\theta_h(p)$. Since

$$E_p(N(n)) = P_p(N(n) = 1) + 2P_p(N(n) = 2) + 3P_p(N(n) = 3) + ... \geq P_p(N(n) \geq 1) \tag{2}$$

we can conclude that

$$\theta(p) \leq E_p(N(n)) \tag{3}$$

for all $n \in \mathbb{N}$. Now, realise that the average number of non-intersection open paths starting at the origin is the sum of all those paths times the probability that those paths are open. Since every non-intersecting path with length n starting at the origin contains n edges and every edge is open with probability p, the probability that any non-intersecting path with length n is open, is p^n. It is for this reason that

$$E_p(N(n)) = p^n \sigma(n) \tag{4}$$

and thus

$$\begin{aligned} \theta(p) &\leq p^n \sigma(n) \\ &= (p\sigma(n)^{\frac{1}{n}})^n \end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$. Now we will try to find an upper bound for $\sigma(n)^{\frac{1}{n}}$ on all lattices. Let k be the number of edges connecting the origin. Then $k = 4$ on S^2, $k = 6$ on T^2 and $k = 3$ on H^2. Then by definition, on these lattices

$$\sigma(1) = k \tag{6}$$

Let $n \in \mathbb{N}$. Any nonintersecting path with length $n + 1$ that starts at the origin is a path with length n that starts at the origin, with an extra edge and corresponding vertex added to it. In the lattices S^2, T^2 and H^2, there are k choises for the extension of a path starting at the origin. Since a non-intersecting path can never go back, one such choise is eliminated for sure, and thus there are only $k - 1$ choises left for the extension of a non-intersecting open path starting at the origin. Thus, on S^2, T^2 and H^2,

$$\frac{\sigma(n+1)}{\sigma(n)} \leq k - 1 \tag{7}$$

for all $n \in \mathbb{N}$. We combine equations (6) and (7) to conclude that

$$\begin{aligned} \sigma(n) &= \sigma(1) \prod_{j=2}^{n} \frac{\sigma(j)}{\sigma(j-1)} \\ &\leq k(k-1)^{n-1} \end{aligned} \tag{8}$$

Hence,

$$\sigma(n)^{\frac{1}{n}} \leq k^{\frac{1}{n}} (k-1)^{\frac{n-1}{n}} \tag{9}$$

Because

$$\begin{aligned} \lim_{n\to\infty} k^{\frac{1}{n}} (k-1)^{\frac{n-1}{n}} &= k^0 (k-1)^1 \\ &= k-1 \end{aligned} \tag{10}$$

we can conclude

$$\limsup_{n\to\infty} \sigma(n)^{\frac{1}{n}} \leq k-1 \tag{11}$$

Thus, there exists an $N_k \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n > N$, we have $\sigma(n)^{\frac{1}{n}} < k - \frac{1}{2}$. Let $p_\epsilon \equiv \frac{1}{k}$. Then for all $n > N$,

$$\begin{aligned} p_\epsilon \sigma(n)^{\frac{1}{n}} &\leq \frac{k - \frac{1}{2}}{k} \\ &= 1 - \frac{1}{2k} \\ &< 1 \end{aligned} \tag{12}$$

This means that

$$\lim_{n\to\infty} (p_\epsilon \sigma(n)^{\frac{1}{n}})^n = 0 \tag{13}$$

Now we combine this result with equation (5) to conclude that

$$\sigma(p_\epsilon) = 0 \tag{14}$$

on the lattices S^2, T^2 and H^2. Because of equation (14), the critical probabilities have to be greater than p_ϵ on these lattices. Since $p_\epsilon \equiv \frac{1}{k}$ is strictly positive on all of these lattices, the critical probabilities have to be so as well.

□

To prove Theorem (3.0.10), we would need to show that $p_{c,s}, p_{c,t}, p_{c,h} < 1$, too. Grimmett uses duality to show that $p_{c,s} < 1$; for the exact proof I recommend reading Chapter 1.4 of Grimmetts book. The fact that $p_{c,t}, p_{c,h} < 1$ is a direct consequence from Theorem (3.0.11), that says that $p_{c,t} + p_{c,h} = 1$, and from Lemma (4.0.14), that says that $0 < p_{c,t}, p_{c,h}$. The reason for this is that if either $p_{c,t}$ or $p_{c,h}$ would be equal to 1, then according to Theorem (3.0.11), the other would be equal to 0, which is impossible according to Lemma (4.0.14). This is a contradication, which proves that neither of them can be equal to 1. Hence, when we prove Theorem (3.0.11), we will automatucally prove Theorem (3.0.10), too.

5 Some probability theory

In percolation theory, several results from probability theory are used. In this section we will introduce a few useful theorems. Basically, these are Chapters 2.1 and 2.1 of Grimmetts book, but then with somewhat more detail added to it. Also, this section contains a proof that is based on the proof of the 'square root trick' by Grimmet on page 194. We think that proof is wrong, though, so we modified it.

5.1 Increasing random variables and events

Recall that the percolation process is equivalent to defining a random variable $X(e)$ for all e that are edges, that are uniformly distributed on $[0,1]$. Then we define $\eta_p(e) = 1$ if $X(e) < p$ and $\eta_p(e) = 0$ if $X(e) \geq p$, $p \in [0,1]$, and call each e open if $\eta_p(e) = 1$, and closed if $\eta_p(e) = 0$. We will continue this section with some more definitions.

Definition 5.1.1. When declaring an edge (or vertex, in the case of site percolation) of a lattice to be open with some probability, one ends up with a particular arrangement of open edges (or vertices). One calls each one of such particular arrangements configurations. The set of all configurations is called Ω. For ω a configuration and e an edge, we define $\omega(e) = 1$ if e is open in ω, and $\omega(e) = 0$ if e is closed in ω. This means that η_p can be interpreted as a configuration as well.

Definition 5.1.2. An event on a lattice is a set of different configurations, typically a set for which some statement holds true for all elements. We say that F occurs if an element of F occurs and so $P_p(F) \equiv P_p(\omega \in F)$ is the probability that you will end up with an element of F after you declare each edge (or vertex) open with probability p.

Definition 5.1.3. Let ω and ω' be configurations on some lattice. We say $\omega \leq \omega'$ if every open edge (or vertex, in the case of site percolation) in ω is open in ω' as well, or equivalently, if $\omega(e) \leq \omega'(e)$ for all edges e. We say $\omega < \omega'$ if $\omega \leq \omega'$ and $\omega \neq \omega'$.

Definition 5.1.4. Let F be an event on some lattice. F is an increasing event if, for $\omega \leq \omega'$, $\omega \in F \Rightarrow \omega' \in F$.

Definition 5.1.5. Let N be a random variable. N is an increasing random variable if, for $\omega \leq \omega'$, $N(\omega) \leq N(\omega')$.

For example, the event $\Psi \equiv$ 'there exists an infinite open cluster' is an event for which all the elements are configurations in which there is an infinite open cluster. Ψ is even an increasing event, for if one takes a configuration which is in Ψ and makes more edges open, there will still be an infinite open cluster and hence that new configuration will be in Ψ as well.

Theorem 5.1.6. *Let N be an increasing random variable on the set of configurations, Ω, and let $p_1 \leq p_2$. Then $E_{p_1}(N) \leq E_{p_2}(N)$.*

Proof. Let $p_1 \leq p_2$ and e an edge. If $\eta_{p_1} = 1$ then $X(e) < p_1$ and since $p_1 \leq p_2$ it then follows that $X(e) < p_2$ so then $\eta_{p_2} = 1$ as well. If, on the other hand, $\eta_{p_1} = 0$ then $\eta_{p_1} \leq \eta_{p_2}$ trivially. This means that always $\eta_{p_1} \leq \eta_{p_2}$. Let N be an increasing random variable, then $N(\eta_{p_1}) \leq N(\eta_{p_2})$. Because $E(N(\eta_p)) = E_p(N)$, it follows that $E_{p_1}(N) \leq E_{p_2}(N)$.

□

Theorem 5.1.7. *Let F be an increasing event and $p_1 \leq p_2$. Then $P_{p_1}(F) \leq P_{p_2}(F)$.*

Proof. An indicator funtion $I_F(\omega)$, ω being a configuration and F being an event, is defined as $I_F(\omega) = 0$ if $\omega \notin F$ and $I_F(\omega) = 1$ if $\omega \in F$.
Let F be an increasing event. Since in the case that $\omega \leq \omega'$, we have $I_F(\omega) = 1 \Leftrightarrow \omega \in F \Rightarrow \omega' \in F \Leftrightarrow I_F(\omega')$, so $I_F(\omega) = 1$ implies that $I_F(\omega) \leq I_F(\omega')$, and from $I_F(\omega) = 0$ follows $I_F(\omega) \leq I_F(\omega')$ trivially. So I_F is an example of an increasing random variable.
Let $p_1 \leq p_2$. According to theorem 5.1.6, then, $E_{p_1}(I_F) \leq E_{p_2}(I_F)$. Because $E_p(I_F) = P_p(I_F = 1) = P_p(F)$, it follows that $P_{p_1}(F) \leq P_{p_2}(F)$.

□

5.2 The FKG inequalities

Theorem 5.2.1. *Let X and Y be increasing random variables on Ω such that $E_p(X^2)$ and $E_p(Y^2)$ are finite, then $E_p(XY) \geq E_p(X)E_p(Y)$.*

Proof. We will use induction. Suppose that X and Y are increasing random variables on Ω which depend only on the state of the edges e_i, $i \in 1, 2, ...n$. Our induction hypothesis is that $E_p(XY) \geq E_p(X)E_p(Y)$.

Suppose that $n = 1$. Then $X(\omega)$ and $Y(\omega)$ depend only on the state of $\omega(e_1) \in \{0, 1\}$.Let $\omega_0, \omega_1 \in \Omega$ such that $\omega_0(e_1) = 0$ and $\omega_1(e_1) = 1$. Let $k \in \{0, 1\}$. Then we define $X(k) \equiv X(\omega_k)$ and $Y(k) \equiv Y(\omega_k)$. Since X and Y are increasing random variables, $X(1) \geq X(0)$ and $Y(1) \geq Y(0)$. Thus for $i, j \in \{0, 1\}$, $X(i) - X(j)$ and $Y(i) - X(j)$ have the same sign (if they are not zero), so then $(X(i) - X(j))(Y(i) - Y(j)) \geq 0$. Since probabilities are nonnegative as well, each elements in the following sum is nonnegative, so the sum is nonnegative as well:

$$
\begin{aligned}
0 &\leq \sum_{i\in 0,1}\sum_{j\in 0,1}(X(i) - X(j)(Y(i) - Y(j)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) \\
&= \sum_{i\in 0,1}\sum_{j\in 0,1} X(i)Y(i)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) + \sum_{i\in 0,1}\sum_{j\in 0,1} X(j)Y(j)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) \\
&+ \sum_{i\in 0,1}\sum_{j\in 0,1} -X(i)Y(j)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) + \sum_{i\in 0,1}\sum_{j\in 0,1} -X(j)Y(i)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) \\
&= \sum_{i\in 0,1} X(i)(Y(i)P_p(\omega(e_1) = i) + \sum_{i\in 0,1} X(j)(Y(j)P_p(\omega(e_1) = j) \\
&+ 2\sum_{i\in 0,1}\sum_{j\in 0,1} -X(i)Y(j)P_p(\omega(e_1) = i)P_p(\omega(e_1) = j) \\
&= 2E_p(XY) - 2E_p(X)E_p(Y)
\end{aligned}
\tag{15}
$$

so $E_p(XY) - E_p(X)E_p(Y) \geq 0$ so $E_p(XY) \geq E_p(X)E_p(Y)$. We have proven the induction hypothesis for $n = 1$.

Suppose that the induction hypothesis is true for all $n \leq k$, k being a natural number. Let X and Y be increasing functions that depend only on the states e_i, $i \in 1, 2, ...k, k + 1$. Then $E_p(X \mid \omega(1), \omega(2), ...\omega(k))$ and $E_p(Y \mid \omega(1), \omega(2), ...\omega(k))$ are increasing functions that depend on the states e_i, $i \in 1, 2, ...k$ and for which the induction hypothesis is therefore true. Thus we can write

$$
\begin{aligned}
E_p(XY) &= E_p(E_p(XY \mid \omega(1), \omega(2), ...\omega(k))) \\
&\geq E_p(E_p(X \mid \omega(1), \omega(2), ...\omega(k))E_p(Y \mid \omega(1), \omega(2), ...\omega(k))) \\
&= E_p(E_p(X \mid \omega(1), \omega(2), ...\omega(k)))E_p(E_p(Y \mid \omega(1), \omega(2), ...\omega(k))) \\
&= E_p(X)E_p(Y)
\end{aligned}
\tag{16}
$$

We have proven the induction hypothesis for all natural numbers n.

Suppose again that X and Y are increasing random variables, but this time they can depend on an infinite number of edges. Also suppose that $E_p(X^2)$ and $E_p(Y^2)$ are finite. Since we are working on a lattice with countably many elements, there exists a list $\{e_i\}_{\{}i \in \mathbb{N}\}$ that contains all edges of the lattice. Define $X_n \equiv E_p(X \mid \omega(e_1), \omega(e_2), ...\omega(e_n))$ and $Y_n \equiv E_p(X \mid \omega(e_1), \omega(e_2), ...\omega(e_n))$. Since X_n and Y_n are increasing functions of n states, we have, according to what was shown before:

$$
E_p(X_nY_n) \geq E_p(X_n)E_p(Y_n) \tag{17}
$$

Now we will use the Martingale Convergence Theorem, which we will not proof in this thesis. The Martingale Convergence Theorem tells us that $\lim_{n\to\infty} X_n = X$ and $\lim_{n\to\infty} Y_n = Y$ almost surely, so

$$E(X)E(Y) = \lim_{n\to\infty} E_p(X_n)E_p(Y_n) \tag{18}$$

By the triangle inequality and Cauchy-Schwarz inequality,

$$\begin{aligned}\lim_{n\to\infty} E_p(\mid X_nY_n - XY \mid) &\leq \lim_{n\to\infty} E_p(\mid X_n - X)Y_n \mid + \mid X(Y_n - y) \mid \\ &\leq \lim_{n\to\infty} \sqrt{E_p((X_n - X)^2)E_p(Y_n)^2} + \sqrt{E_p((Y_n - Y)^2)E_p(X_n)^2} \\ &= 0 + 0 = 0\end{aligned} \tag{19}$$

Combining 17, 18 and 19 gives

$$E_p(XY) = \lim_{n\to\infty} E_p(X_nY_n) \geq \lim_{n\to\infty} E_p(X_n)E_p(Y_n) = E_p(X)E_p(Y) \tag{20}$$

which is what we wanted to proof.

□

Theorem 5.2.2. *Let A and B be increasing events. Then $P_p(A \cap B) \geq P_p(A)P_p(B)$*

Proof. Like we saw before, the indicator function of an increasing event is an increasing random variable. So I_A and I_B are increasing random variables, and we have, according to 5.2.1, that $E_p(I_AI_B) \geq E_p(I_A)E_p(I_B)$. Since I_A and I_B are only nonzero if A respectively B ocurs, and they are in this case equal to 1, we have $E_p(I_AI_B) = P_p(A)P_p(B)$ and $E_p(I_A) = P_p(A)$ and $E_p(I_B) = P_p(B)$, so $P_p(A \cap B) \geq P_p(A)P_p(B)$.

□

5.3 The square root trick

Theorem 5.3.1. *Let all A_i, $i \in 1, 2, ...n$ be increasing random events with equal probability. Then*

$P_p(A_1)^c \geq 1 - (1 - P_p(\bigcup_{i=1}^n A_i^c))^{\frac{1}{n}}$.

Proof. We have

$$\begin{aligned}1 - P_p(\bigcup_{i=1}^n A_i^c) &= P_p(\bigcap_{i=1}^n A_i) \\ &\geq \prod_{i=1}^n P_p(A_i) \\ &= (1 - P_p(A_1^c))^n\end{aligned} \tag{21}$$

where we repeatedly applied the FKG inequality (5.2.2) for increasing events in the second step.

From this follows that

$$(1 - P_p(\bigcup_{i=1}^n A_i^c))^{\frac{1}{n}} \geq 1 - P_p(A_1^c) \text{ so } P_p(A_1^c) \geq 1 - (1 - P_p(\bigcup_{i=1}^n A_i^c))^{\frac{1}{n}} \tag{22}$$

from which the theorem follows.

□

6 Number of open clusters per vertex

In this section, we only work in the triangular lattice. Like the name suggests, the section is dedicated to a theorem about the number of open clusters per vertex. Our theorem, and the proof of it, is a modified and somewhat more detailed version of the theorem and proof as discussed in Chapter 4.1 of Grimmetts book. In his book, Grimmett shows the same as we will show, but then for the square lattice, not for the triangular lattice.
We start with a lemma.

Lemma 6.0.2. *Let $U \subset T^2$. Let $C_U(x)$ be the open cluster in U containing x. Then $\sum_{x \in U} | C_U(x) |^{-1}$ is the number of open clusters in U.*

Proof. Let Q_i, $i \in S \subset N$ be the open clusters in U. Then the number of open clusters in U is given by $| S |$. Since $Q_i i \in S$ form a partition of U, we have

$$\sum_{x \in U} | C_U(x) |^{-1} = \sum_{i \in S} \sum_{x \in Q_i} | C_U(x) |^{-1} \tag{23}$$

When $x \in Q_i$, then $Q_i = C_U(x)$ by definition. So

$$\sum_{x \in Q_i} | C_U(x) |^{-1} = \sum_{x \in Q_i} | Q_i |^{-1} = 1 \tag{24}$$

When we combine equations (23) and (24), we obtain

$$\sum_{x \in U} | C_U(x) |^{-1} = \sum_{i \in S} 1 = | S | \tag{25}$$

which proves the lemma.

□

Definition 6.0.3. On the triangular lattice, $\kappa(p) \equiv E_p(| C |^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n} P_p(| C |= n)$.

Definition 6.0.4. We define $B_n \subset Z^2$ in the following way: If $x \in Z^2$, then $x \in B_n$ if and only if $\delta_t(0, x) \leq n$. We define $G_n \subset T^2$ as the collection of all $x \in B_n$ and all $< x, y > \in E_t^2$ for which $x, y \in B_n$.
For $x \in B_n$, we define $C_n(x)$ to be the open cluster of B_n containing x.

Theorem 6.0.5. *Let K_n be the number of open clusters in B_n, or equivalently, the number of connected components in B_n when all closed edges are deleted. We consider bond percolation on T^2. Let $p \in [0, 1]$. Then $\lim_{n \to \infty} \frac{K_n}{|B_n|} = \kappa(p)$ almost surely.*

The above theorem implies that $\kappa(p)$ can be interpreted as the number of open clusters per vertex in the triangular lattice.

Proof. Let $x \in B_n$ and $y \in C_n(x)$. Then $y \in T^2$ and y is connected to x; so $y \in C(x)$. Thus, $C_n(x) \subseteq C(x)$, so $| C_n(x) | \leq | C(x) |$. Using the convention that if $C(x)$ has infinite cardinality, we write $| C(x) |^{-1} \equiv 0$, it follows that

$$| C(x) |^{-1} \leq | C_n(x) |^{-1} . \tag{26}$$

Let C be some open cluster inside B_n. Then

$$\sum_{x \in C} | C_n(x) |^{-1} = \sum_{x \in C} | C(x) |^{-1} = \frac{| C(x) |}{| C(x) |} = 1 \tag{27}$$

Now let $C_1, C_2, ..., C_j$ be all the open clusters inside B_n. Then

$$\sum_{x \in B_n} | C_n(x) |^{-1} = \sum_{i=1}^{j} \sum_{x \in C_i} | C_i |^{-1} = j \tag{28}$$

Since j is exactly the number of open clusters inside B_n,

$$\sum_{x \in B_n} \mid C_n(x) \mid^{-1} = K_n. \tag{29}$$

Combining (26) and (29) results in the following equation:

$$\frac{K_n}{\mid B_n \mid} \geq \frac{\sum\limits_{x \in B_n} \mid C(x) \mid^{-1}}{\mid B_n \mid} \tag{30}$$

According to the symmetry of the lattice, we have

$$E_p(C(x)) = E_p(C(0)) \equiv E_p(C) \tag{31}$$

and since

$$\lim_{n \to \infty} B_n = Z^2 \tag{32}$$

we know that

$$\lim_{n \to \infty} \frac{\sum\limits_{x \in B_n} \mid C(x) \mid^{-1}}{\mid B_n \mid} = E_p(\mid C \mid^{-1}) \equiv \kappa(p)) \tag{33}$$

almost surely, according to the strong law of large numbers. Using (30) and (33), we can conclude that as n goes to infinity, $\frac{K_n}{|B_n|} \geq \kappa(p)$, so

$$\liminf_{n \to \infty} \frac{K_n}{\mid B_n \mid} \geq \kappa(p) \tag{34}$$

In order to obtain an equality, we have to find an upper bound for the left side of equation (34).
We'll define $U_n \equiv \{x \in B_n | x \text{ is in the same cluster as some } y \in B_n \text{ for which } d_t(0, y) = n\}$. In other words, U_n is the collection of all edges in B_n that are connected to the surface of B_n by some open cluster. We'll define $V(n) \equiv B(n) - U(n) = \{x \in B_n | \text{ there is no } y \in B_n \text{for which } d_t(0, y) = n \text{ and } y \text{ is on the same cluster as } x\}$. Obviously, U_n and V_n form a disjoint union of B_n. We also notice that if $x \in V_n$ and $z \in C(x)$, then it must be that $z \in B_n$ and thus $z \in C_n(x)$; for else, x would be connected to some edge outside of B_n, hence be connected to the surface of B_n, and hence x would not lie in V_n. So if $x \in V_n$, then $C(x) \subseteq C_n(x)$. Since always $C(x) \supseteq C_n(x)$, we have that $C(x) = C_n(x)$ if $x \in V_n$.
Now we will rewrite equation (29) to obtain an upper bound of K_n:

$$\begin{aligned}
K_n &= \sum_{x \in B_n} \mid C_n(x) \mid^{-1} \\
&= \sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \in B_n} \mid C_n(x) \mid^{-1} - \mid C(x) \mid^{-1} \\
&= \sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \in U_n} \mid C_n(x) \mid^{-1} - \mid C(x) \mid^{-1} + \sum_{x \epsilon V_n} \mid C_n(x) \mid^{-1} - \mid C(x) \mid^{-1} \\
&= \sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \in U_n} \mid C_n(x) \mid^{-1} - \mid C(x) \mid^{-1} + \sum_{x \in V_n} \mid C(x) \mid^{-1} - \mid C(x) \mid^{-1} \\
&= \sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \in U_n} \mid C_n(x) \mid^{-1} - \mid C(x) \mid^{-1} \\
&\leq \sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \in U_n} \mid C_n(x) \mid^{-1}
\end{aligned} \tag{35}$$

We now use lemma 6.0.2 to notice that $\sum\limits_{x \in U_n} \mid C_n(x) \mid^{-1}$ is simply the number of open clusters in U_n. Since each element of U_n can be in only one cluster, we have

$$\sum_{x \in U_n} \mid C_n(x) \mid^{-1} \leq \mid U_n \mid \tag{36}$$

Combining (35) and (36), we obtain

$$\frac{K_n}{\mid B_n \mid} \leq \frac{\sum_{x \in B_n} \mid C(x) \mid^{-1} + \sum_{x \epsilon U_n} \mid C_n(x) \mid^{-1}}{\mid B_n \mid} \leq \frac{\sum_{x \in B_n} \mid C(x) \mid^{-1}}{\mid B_n \mid} + \frac{\mid U_n \mid}{\mid B_n \mid} \tag{37}$$

Since B_n grows at a faster rate than its surface U_n, we have

$$\lim_{n \to \infty} \frac{\mid U_n \mid}{\mid B_n \mid} = 0 \tag{38}$$

If we combine this with equation (33), then, we can see that the right hand side of equation (37) converges to $\kappa(p)$ by just adding the limits. Therefore we have

$$\limsup_{n \to \infty} \frac{K_n}{\mid B_n \mid} \leq \kappa(p) \tag{39}$$

When we combine equation (34) and equation (35), we can see that

$$\lim_{n \to \infty} \frac{K_n}{\mid B_n \mid} = \kappa(p) \tag{40}$$

□

From now on, we call $\kappa(p) \equiv \kappa_t(p)$ to specify that we are working on the triangular lattice. When we are working on the square, respectively hexagonal lattice, we will call the corresponding functions $\kappa_s(p)$ and $\kappa_h(p)$, respectively.

7 The dual space

7.1 Dual percolation process

This section is based on Chapter 9.1 of Grimmetts book. In the second half of this chapter, Grimmett (who uses slightly different notation) shows that $\kappa_s(p) = \kappa_s(1-p) + 1 - 2p$. We modify his proof to show the relationship between $\kappa_t(p)$ and $\kappa_h(p)$.

We recall that if L is a lattice, then L_d, the dual space of L, is a lattice such that every edge of L_d crosses one unique edge of L and vice versa.

Definition 7.1.1. Let L be a lattice and L_d be its dual. Let every edge in L be open with probability p. Then in the dual percolation process, an edge $e \in L_d$ is declared open if and only if e crosses a closed edge in L.

We can immediately see that if an edge in L is open with probaility p, then an edge in L_d, if we use the dual percolation process, is open with probability $1-p$. Also we can see that whether different edges in L_d are declared open are completely independent of each other, because this is the case in L as well. Thus, the dual percolation process described a different way of having a normal percolation process in L_d, with edge-probability $1-p$.

This is very useful. In our big theorem, what we need to show is equivalent with showing that if $p < p_{c,t}$ then $1-p > p_{c,h}$ and if $p > p_{c,t}$ then $1-p < p_{c,h}$. Since T^2 and H^2 are each others dual space, the dual percolation process can be very useful in showing this.

The theorems proven in the rest of this chapter will not be needed for the understanding of the proof of our main theorem that $p_{c,t} + p_{c,h} = 1$.

7.2 A theorem about the number of open clusters in the dual

Theorem 7.2.1. *The numbers $\kappa_t(p)$ and $\kappa_h(p)$ of open clusters per vertex in T^2 and H^2, respectively, satisfy $\kappa_t(p) = 2\kappa_h(1-p) + 1 - 3p$.*

For the proof of this theorem, we will, surprisingly, need the well-known Euler's Theorem. We won't proof Euler's Theorem here, but leave it as a proposition:

Proposition 7.2.2. *Let G be a finite planar graph, drawn in the plane with $v(G)$ vertices, $e(G)$ edges, $f(g)$ finite faces, and $c(G)$ connected components. Then*

$$c(G) = v(G) - e(G) + f(G) \tag{41}$$

Now we can start with the proof of Theorem (7.2.1).

Proof. We will consider percolation on T^2 with edge-probability p, $p \in [0,1]$ and we will study G_n (see Definition (6.0.4)).

When we apply Euler's Theorem (7.2.2) to the open part of G_n and we take expectation values, we obtain

$$E_p(c(G_n)) = E_p(v(G_n)) - E_p(e(G_n)) + E_p(f(G_n)) \tag{42}$$

We notice that $E_p(v(G_n)) = |B_n|$ and divide by $|B_n|$ to obtain

$$\frac{E_p(c(G_n))}{|B_n|} = 1 - \frac{E_p(e(G_n))}{|B_n|} + \frac{E_p(f(G_n))}{|B_n|} \tag{43}$$

Also we notice that $E_p(c(G_n)) = K_n$ by definition. Then by theorem 6.0.5, we have

$$\lim_{n\to\infty} \frac{E_p(c(G_n))}{\mid B_n \mid} = \kappa_t(p) \tag{44}$$

Next, we will make use of the following proposition.

Proposition 7.2.3.

$$\lim_{n\to\infty} E_p(e(G_n)) = 3p|B_n| \tag{45}$$

Although we will not proof this proposition, we will give an intuitive reasoning. Notice that in T^2, every edge is connected to 2 vertices, and every vertex is connected to 6 edges; so for every edge there will be three $\frac{6}{2} = 3$ vertices. The same is true for B_n, except on the boundary ΔB_n. But since for large n, $\frac{|\Delta B_n|}{|B_n|} \mapsto 0$, we can ignore this boundary effect for large n. Then the proposition follows from the fact that $E_p(e(G_n)) = pe(B_n)$.

When we rewrite proposition 7.2.3, we get

$$\lim_{n\to\infty} \frac{E_p(e(G_n))}{\mid B_n \mid} = 3p \tag{46}$$

Now consider the following lemma.

Proposition 7.2.4. *Define $G_{n,d}$ as the part of the dual of T^2 that only contains the edges that cross an (open or closed) edge of G_n, and that only contains the vertices that lie inside a finite face of G_n. Then define every edge in $G_{n,d}$ to be open according to the dual percolation process: so an edge in $G_{n,d}$ is open if and only if it crosses a closed edge of G_n. Define an open face to be a face when all closed edges are removed. Then every open face of G_n contains a unique connected component of $G_{n,d}$.*

The proposition requires a quite rigorous proof; we will give intuitive reasoning for it here.
Because of the way the dual space is defined, every face of G_n contains one vertex of $G_{n,d}$. Since an open face contains one or more faces, any open face of G_n must contain at least one vertex of G_n.
Let $q \in G_{n,d}$ be a vertex inside some face of G_n and let $r \in G_{n,d}$ be some vertex outside of it, then there is no open path from q to r, since any path from q to r must contain an edge that crosses the boundary of the face, and in the dual percolation process, such an edge is closed. Thus, every face of G_n contains a connected component of $G_{n,d}$.
Now let $q, s \in G_{n,d}$ both be vertices inside some open face of G_n. Then there is an open path from q to s. Thus, every face of G_n contains no more than one connected component of $G_{n,d}$. This shows that the proposition is true, but not rigorously since the existence of an open path from q to s was not properly shown.

Now we have to find out how many connected components of $G_{n,d}$ do not lie inside the finite face of G_n. There can be no more than $\mid \Delta B_n \mid$ of these, ΔB_n being the set of boundary vertices of B_n. This is the case because every vertex in $G_{n,d}$ is connected to an (open or closed) edge that crosses an edge in G_n, so vertices in the outside have to be connected to an edge that crosses a unique edge in G_n. We know that $\frac{|\Delta B_n|}{|B_n|} \mapsto 0$, and by (2.4.3), we know that H^2 is the dual of T^2. Any edge in $G_{n,d}$ has a probability of $1-p$ to be open (because every open edge of $G_{n,d}$ corresponds to a closed edge in G_n, which is closed with probability $1-p$). Thus, given that $\kappa_h(q)$ is the number of open clusters per vertex in H^2 with edge-probability q, the number of open clusters in $G_{n,d}$ per vertex should, excluding possibly boundary effects, converge to $\kappa_h(1-p)$ for large enough n. In conclusion, we have

$$\mid \lim_{n\to\infty} \frac{E_p(f(G_n))}{\mid e(G_{n,d}) \mid} - \kappa_h(1-p) \mid = \mid \lim_{n\to\infty} \frac{E_p(c(G_{n,d}))}{\mid e(G_{n,d}) \mid} - \kappa_h(1-p) \mid \leq \lim_{n\to\infty} \frac{\mid \delta B_n \mid}{\mid B_n \mid} \mapsto 0 = 0 \tag{47}$$

so

$$\lim_{n\to\infty} \frac{E_p(f(G_n))}{\mid e(G_{n,d}) \mid} = \kappa_h(1-p) \tag{48}$$

Next, consider the following proposition:

Proposition 7.2.5. $\lim_{n\to\infty} \frac{|e(G_{n,d})|}{|B_n|} = 2$

This is again a proposition we will not proof, but can be made intuitive. In H^2, every vertex has 3 edges and in T^2, every vertex has 6 edges, which is twice as much. Since $G_{n,d} \subset H^2$ and $G_n \subset T^2$ have the same number of edges, you expect $G_{n,d}$ to have twice the amount of vertices as G_n, apart from boundary conditions. But because boundary conditions become neglectible when $n \mapsto \infty$, you expect $\frac{|e(G_{n,d})|}{|B_n|} \mapsto 2$ when $n \mapsto \infty$.

When we combine equation (48) and Proposition (7.2.5), we obtain

$$\lim_{n\to\infty} \frac{E_p(f(G_n))}{\mid B_n \mid} = 2\kappa_h(1-p) \tag{49}$$

Filling equations (44), (46) and (49) in in equation (42) results in

$$\kappa_t(p) = 1 - 3p + 2\kappa_h(1-p) = 2\kappa_h(1-p) + 1 - 3p \tag{50}$$

which is what we wanted to proof.

□

7.3 Remark

In the proof of Theorem 7.2.1, we looked primarily at the triangular lattice and used the hexagonal lattice as dual space. We could have reversed this by primarily looking at the hexagonal lattice and by using the triangular lattice as its dual space. Doing the same kind of steps as above, we would then have been able to prove

$$\kappa_h(p) = \frac{1}{2}\kappa_t(1-p) + 1 - \frac{3}{2}p \tag{51}$$

As it turns out, Theorem (7.2.1) could be rewritten as (51) and vice versa. One could, as a 'sanity check', calculate equation 51 by doing the same kind of steps as were used in the proof of Theorem (7.2.1), and then verify that the found expression can indeed be rewritten as the first one.

7.4 Applications

For most lattices, an as yet unproven conjecture is that $\kappa(p)$ is infinitely many times differentiable everywhere except at $p = p_c$. We can use this to derive that $p_{c,t} + p_{c,h} = 1$ easily.

Suppose that the only point where $\kappa_t(p)$ fails to be infinitely many times differentiable is at $p = p_{c,t}$. According to Theorem (7.2.1), $\kappa_t(p_{c,t}) = 2\kappa_h(1 - p_{c,t}) + 1 - 3p$, and the only part at the right hand side that could be responsible for $\kappa_t(p_{c,t})$ being not infinitely many times differentiable, is $\kappa_h(1 - p_{c,t})$. Thus $\kappa_h(1 - p_{c,t})$ is not infinitely times differentiable. If we further assume that $\kappa_h(p)$ only fails to be infinitely many times differentiable if $p = p_{c,h}$, then it follows that $1 - p_{c,t} = p_{c,h}$, so then $p_{c,t} + p_{c,h} = 1$.
The only problem with this is that our assumptions about the differentiability of $\kappa_t(p)$ and $\kappa_h(p)$ are not proven. We will later prove that $p_{c,t} + p_{c,h} = 1$ without making these assumptions.

Another possible application of Theorem 7.2.1 is in finding direct estimates of the critical points. If we were to estimate $\kappa_t(p)$ and $\kappa_h(p)$ around the critical points, then this also leads to estimates of the critical points themselves.

8 $p_{c,t} + p_{c,h} \geq 1$

In this section, we will show that $p_{c,t} + p_{c,h} \geq 1$. We base ourselves largely on part of Chapter 9.3 of Grimmetts book, where he shows that $p_{c,s} + p_{c,s} \geq 1$ so that $p_{c,s} \geq \frac{1}{2}$.
Intuitively, $p_{c,t} + p_{c,h} \geq 1$ because otherwise, there would be values of p for which both T^2 with edge-probability p and H^2 with edge-probability $1-p$, in which every edge is declared open if and only if the edge in T^2 that it crosses, is closed, would contain an infinite open cluster. But because every edge of T^2 that crosses an open edge in its dual H^2, must by definition be closed, no open path in T^2 can cross an open path in its dual. That makes the coexistance of these two infinite open clusters unlikely. We will now get more formal.

Theorem 8.0.1. $p_{c,t} + p_{c,h} \geq 1$

Proof. Suppose, to the contrary, that $p_{c,t} + p_{c,h} < 1$. Then there exists an $\epsilon > 0$ such that $p_{c,t} = 1 - p_{c,h} - \epsilon$. Let's now define $p_\epsilon \equiv 1 - p_{c,h} - \frac{\epsilon}{2}$. Then $p_\epsilon > p_{c,t}$. Recall that $p_{c,s} \equiv \sup_p\{\theta_s(p) = 0\}$. So then by definition, $\theta_t(p_\epsilon) > 0$. Also, $1 - p_\epsilon = p_{c,h} + \frac{\epsilon}{2} > p_{c,h}$, so by definition, $\theta_h(1 - p_\epsilon) > 0$.

We will consider percolation on T^2 with edge-probability p_ϵ and we will study B_n as given in Definition (6.0.4); specifically, the boundary ΔB_n. We will define four subsets of ΔB_n with corresponding events:

$$B_n^a \equiv \{q \equiv (q_1, q_2) \in \Delta B_n \mid q_1 > 0, q_2 \geq 0\} \tag{52}$$

$$B_n^b \equiv \{q \equiv (q_1, q_2) \in \Delta B_n \mid q_1 \leq 0, q_2 > 0\} \tag{53}$$

$$B_n^c \equiv \{q \equiv (q_1, q_2) \in \Delta B_n \mid q_1 < 0, q_2 \leq 0\} \tag{54}$$

$$B_n^d \equiv \{q \equiv (q_1, q_2) \in \Delta B_n \mid q_1 \geq 0, q_2 < 0\} \tag{55}$$

Let $A_n^u, u \in \{a, b, c, d\}$, be the event that some $q \in B_n^u$ is in an infinite open path in T^2 that uses no other element of B_n. (56)

Since $B_n^a \cup B_n^b \cup B_n^c \cup B_n^d = \Delta B_n$, we know that $A_n^a \cup A_n^b \cup A_n^c \cup A_n^d$ implies that some $q \in \Delta B_n$ is in an infinite open path in T^2 that uses no other element of B_n, which in turn implies the weaker statement that some $q \in \Delta B_n$ is in an infinite open cluster.
If some $q \in B_n$ is in an infinite open cluster, this cluster will contain elements that will still be in an infinite open cluster when all elements of B_n are declared closed. Let $r \notin B_n$ be such an element, then r must be in an infinite open path that does not use elements of B_n. Since q and r lie in the same open cluster, there must be an open path from q to r. This open path has to cross ΔB_n at least once. Let s be the element in ΔB_n that will be visited last by moving from q to r, then there is an open path from s to r that does not use other elements of B_n (namely the remainder of the open path from q to r); and because r is in an infinite open path that does not use elements of B_n, s is in this same path with itself added to it. Since s is in an infinite open path in T^2 that uses no other element of B_n and $s \in \Delta B_n = B_{n,a} \cup B_{n,b} \cup B_{n,c} \cup B_{n,d}$, we have met the condition for $A_n^a \cup A_n^b \cup A_n^c \cup A_n^d$.
In conclusion, $A_n^a \cup A_n^b \cup A_n^c \cup A_n^d$ if and only if some $q \in B_n$ is in an infinite open cluster.
Since $\theta_t(p_\epsilon) > 0$ and $\lim_{n\to\infty} \mid B_n \mid= \infty$,we have

$$\lim_{n\to\infty} P_{p_\epsilon}(A_n^a \cup A_n^b \cup A_n^c \cup A_n^d) = 1 \tag{57}$$

I claim that follows that

Proposition 8.0.2. $P_{p_\epsilon}(A_n^u) \geq 1 - (1 - P_p(A_n^a \cup A_n^b \cup A_n^c \cup A_n^d))^{\frac{1}{n}}$

Grimmett made an analogous claim about the square lattice and said that the square root trick was sufficient to show that his claim was correct. Though we think the claim is correct, we think Grimmett made a mistake stating the square root trick and we think furthermore that using the square root trick is insufficient to prove the claim. As such, we do not want to borrow Grimmetts arguments this time. Unfortunately, we have not yet found a neat way to prove the claim, or to find a way around it. Nonetheless, we continue the proof.

Because of equation (57),

$$\lim_{n\to\infty} P_{p_\epsilon}(A_n^u) = 1, u \in a, b, c, d \tag{58}$$

This means that we can choose N in such a way that

$$P_{p_\epsilon}(A_{N-1}^u) > \frac{7}{8}, u \in a, b, c, d \tag{59}$$

and hence also

$$P_{p_\epsilon}(A_N^u) > \frac{7}{8}, \in a, b, c, d \tag{60}$$

Now recall Definition (6.0.4) of $G_{n,d}$. We will use $G_{n,d}$ in the following. Let again $u \in a, b, c, d$. Then:

We define $B_{n,d}^u$ as the collection of vertices of G_n that connect to an edge that crosses an edge of B_n^d. (61)

We define $A_{n,d}^u$ as the event that some $q \in B_{n,d}^u$ is in an infinite open path in T^2 that uses no other element of G_n. (62)

Every edge in the dual is open with probability $1 - p_\epsilon$. Since $\theta_h(1 - p_\epsilon) > 0$ and $\lim_{n\to\infty} \mid G_{n,d} \mid = \infty$, $\lim_{n\to\infty} P_{p_\epsilon}(A_{n,d}^a \cup A_{n,d}^b \cup A_{n,d}^c \cup A_{n,d}^d) = 1$. This means that we can repeat the same analysis as was applied before on A_n^u, $u \in a, b, c, d$, to eventually obtain the result

$$P_{p_\epsilon}(A_{N_d-1,d}^u), P_{p_\epsilon}(A_{N_d,d}^u) > \frac{7}{8}, u \in a, b, c, d \tag{63}$$

Choose $M = max(N, N_d)$ so that equations (59), (60) and (63) all hold when M is used instead of N or N_d.

We define the event A as follows:

$$A \equiv A_M^a \cap A_M^c \cap A_{M,d}^b \cap A_{M,d}^d \tag{64}$$

If A does not occur, then A_M^a, A_M^c, $A_{M,d}^b$ and $A_{M,d}^d$ do all not occur. So we have, by equations (60) and (63), that

$$\begin{aligned} P_{p(\epsilon)}(A)a &= 1 - P_{p(\epsilon)}(A^c) \\ &\geq 1 - P_{p(\epsilon)}(A_M^a) + P_{p(\epsilon)}(A_M^c) + P_{p(\epsilon)}(A_{M,d}^b) + P_{p(\epsilon)}(A_{M,d}^d) \\ &\geq 1 - 4\frac{1}{8} = \frac{1}{2} \end{aligned} \tag{65}$$

Suppose that A occurs. Then there would be some $a \in A_M^a$ and some $c \in A_M^c$ that are both in an infinite open cluster; since there is almost surely one open cluster, a and b would be in the same infinite open cluster and hence there will be an open path from a to b. We also have some $b \in A_{M,d}^b$ and some $d \in A_{M,d}^d$ that are in an infinite open path that contains no other element of $B_{n,d}$. Since an open path cannot cross an open path in its dual space, the open path from a to c must use vertices that do not cross vertices outside $B_{n,d}$ to go through B_n. So there must be an open path through B_n. But since a is in an infinite open path, c is in another infinite open path and there is an open path from a to c, any open path from b to d in the dual would cross an open vertex in T^2, which is impassible. So b and d cannot lie in the same cluster, despite both lying in an infinite cluster. This happens with probability 0 (3.0.13). So we must have that $P_{p(\epsilon)}(A) = 0$. This is in contradiction with our earlier result (65) that $P_{p(\epsilon)}(A) = \frac{1}{2}$.

The conclusion is that our assumption that $p_{c,t} + p_{c,h} < 1$ is incorrect. This proves the theorem.

□

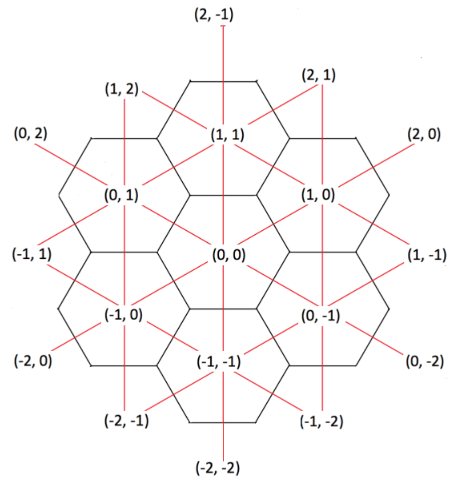

Figure 6: J_2 on the triangular lattice in red; $G_{2,d}$ on the triangular lattice in black.

9 $p_{c,t} + p_{c,h} \leq 1$

We still base ourselves on part of Chapter 9.3 of Grimmetts book; specifically the second proof he gave that showed that $p_{c,s} + p_{c,s} \geq 1$ so that $p_{c,s} \geq \frac{1}{2}$.
To prove Theorem (3.0.11), that says that $p_{c,t} + p_{c,h} = 1$, we only need to show that $p_{c,t} + p_{c,h} \leq 1$ since we already know that $p_{c,t} + p_{c,h} \geq 1$ according to Theorem (8.0.1). To do this, we will make use of the following proposition:

Proposition 9.0.3. *We work on the triangular lattice. Let W_n be the event that there is an edge x such that $\delta_t(0, x) = n$ and there exists an open path from the origin to x. Then:*
If $p < p_{c,t}$ then there exists $\psi > 0$ such that $P_p(W_n) < e^{-n\psi}$ for all n.

Grimmett proved this proposition for the square lattice. We are fairly certain that his proof can be extended to the triangular lattice. To write a full proof of Proposition (9.0.3) would take a lot of time, though, because we would need to prove a number of lemma's about probability theory, too. That is why we choose to omit a proof. With this proposition, we can start the proof of Theorem (3.0.11).

Proof. Recall the definitions of B_n, G_n (Definition (6.0.4)) and $G_{n,d}$ (Proposition (7.2.4)) on the triangular lattice. We define J_n in the following way:

J_n is the set of vertices B_n together with the set of edges $\{e \equiv < x, y > \in E_t^2 \mid x, y \in B_n, \delta_t(0, x) < n or \delta_t(0, y) < n\}$.
Figure (6) shows an example of such a J_n and $G_{n,d}$ for $n = 2$.

Now consider the following events. A_n is the event that some vertex on the lower left side of J_n is connected to a vertex on the upper right side of J_n, and D_n is the event that some vertex on the upper left side of $G_{n,d}$ is connected to some vertex on the bottom right side of $G_{n,d}$. More formally, A_n is said to occur if there exist vertices $< -n, y >, < n, b > \in Z^2$ between which an open path in J_n exists. Let $X \subset G_{n,d}$ be the set of vertices that connect to an edge in $G_{n,d}$ that crosses an edge in J_n that connects to an edge in J_n that is **not** of the form $< -n, y > \in Z^2$ or $< n, b > \in Z^2$, but does have a distance n to the origin in δ_t. D_n is said to occur if there exist vertices $v, w \in X$ that have no path between them in X, but that do have an open path between them in $G_{n,d}$.

It is the case that A_n occurs if and only if D_n does not occur. We are, unfortunately, not able to give an exact proof, but we will reason why we suppose that this is the case.
It can be seen that any open path for which A_n occurs must cross any open path for which D_n occurs. Since

it is not possible for open paths to cross open paths of its dual in the dual percolation process, A_n and D_n cannot occur simultaneously and we have $A_n \cap D_n = \emptyset$.
Assume that A_n does not occur. Define Q as the set of edges and vertices that can occur in open paths in J_n that contain a vertex of the form $< -n, y > \in Z^2$, and define R to be the set of edges and vertices that can occur in open paths in J_n that contain a vertex of the form $< n, b > \in Z^2$. If there would be an open path from some vertex $q \in Q$ to some vertex $r \in R$ through J_n, then there would be an open path from some $< -n, y > \in Z^2$ to q to r to some $< n, b > \in Z^2$, which is impossible because A_n cannot occur. Therefore, such connections do not exist. The edges between Q and R are closed and thus correspond to open edges in $J_{n,d}$, that form a path between Q and R. This implies that D_n occurs. Using similar arguments, if D_n does not occur, then A_n occurs.

Since A_n occurs if and only if D_n does not occur, we have:

$$P_p(A_n) + P_p(D_n) = 1 \tag{66}$$

Using equation (refjaja), we will show that $p_{c,t} + p_{c,h} \leq 1$ using contradiction.

Suppose, on the other hand, that $p_{c,t} + p_{c,h} > 1$. Then there exists an $\epsilon > 0$ such that $p_{c,t} = 1 + \epsilon - p_{c,h}$. Let's now define $p_\epsilon \equiv 1 + \frac{\epsilon}{2} - p_{c,h}$. Then $p_\epsilon < p_{c,t}$. Also, $p_{c,h} = 1 - p_{c,t} > p_\epsilon$. We can use Theorem (9.0.3) to estimate $P_{p_\epsilon}(A_n)$. Given a vertex $(-n, y)$, the probability that there is a path to some vertex (n, b) is no bigger than the probability that there is a path to any vertex with a distance of $2n$ (since the former event implies the latter event). Because of symmetry, we can move $(-n, y)$ to the origin and conclude that this probability is no bigger than $P_{p_\epsilon}(W_n)$. There are n different vertices for which A_n occurs if there is an open path from this vertex to some vertex in the form of (n, b). All these probabilities are no bigger than $P_{p_\epsilon}(W_n)$ and they cannot negatively affect each other. Thus, $P_{p_\epsilon}(A_n) \leq n P_{p_\epsilon}(W_n)$. Since $p < p_{c,t}$, we can use Theorem (9.0.3) to see that there is a $\psi > 0$ such that

$$P_{p_\epsilon}(A_n) < n e^{-n\psi} \tag{67}$$

for all n. We have

$$\lim_{n \to \infty} P_{p_\epsilon}(A_n) \leq \lim_{n \to \infty} n e^{-n\psi} = 0 \tag{68}$$

According to equation (66), we then have

$$\lim_{n \to \infty} P_{p_\epsilon}(B_n) = 1 \tag{69}$$

But $p_\epsilon < p_{c,h}$. This means that there is no infinite open cluster in the dual space. Yet, equation 69 implies that there is. The conclusion is that our assertion that $p_{c,t} + p_{c,h} > 1$. Thus, $p_{c,t} + p_{c,h} \leq 1$. Since we already proved that $p_{c,t} + p_{c,h} \geq 1$, we have

$$p_{c,t} + p_{c,h} = 1 \tag{70}$$

and our main theorem is proven.

□

10 Final remarks

10.1 Exact values

We have proven - despite using some unproven corollaries - that $p_{c,h} + p_{c,t} = 1$. A more interesting question would be what the exact values of $p_{c,h}$ and $p_{c,t}$ are. This is - again - beyond the scope of the thesis, but I won't deny the curious reader the answers. It is proven that

Theorem 10.1.1. $p_{c,t} = 2sin(\frac{\pi}{18})$

This was proven in the article that the following link leads to (or lead to at June 1st, 2017):
http://www.cambridge.org/core/journals/advances-in-applied-probability-article/
bond-percolation-on-honeycomb-and-triangular-lattices/D2F18A5392DEFC352B9C89CACG21FDAO

(This is the same article that mentioned the duality of the triangular and hexagonal lattices.) Thanks to our theorem, we can use this result to conclude

Theorem 10.1.2. $p_{c,h} = 1 - 2sin(\frac{\pi}{18})$

For most percolation processes, no exact values for the critical probabilities are known, though. This is the case for bond percolation on more complicated lattices, but it is true for site percolation on virtually all lattices, too.

10.2 The square lattice

As we mentioned many times, we based the proof of our main theorem on part of the book written by Grimmett that were about the square lattice. One may wonder what Grimmett managed to prove.

We have proved our main theorem that $p_{c,s} + p_{c,s} = 1$, which combines the critical probabilities of the hexagonal and triangular lattice. You may have noticed that we did not use many specific properties of these lattices. You may wonder whether it is true in general that the critical probability is equal to 1 minus the critical probability of its dual space. The answer is that this is indeed the case for simple lattices.

Take, for instance, the square lattice. Since the square lattice is its own dual space, it would follow that

$$p_{c,s} + p_{c,s} = 1 \tag{71}$$

Then it immediately follows that

Theorem 10.2.1. $p_{c,s} = \frac{1}{2}$

This theorem is proven by Grimmett in much the same way as our big theorem, theorem 3.0.11. In fact, the proof is simpler because there is only one space involved instead of two.

As mentioned earlier, there is a variant of Theorem (7.2.1) for the square space, too:

Theorem 10.2.2. *The numbers $\kappa_s(p)$ of open clusters per vertex in S^2 satisfies $\kappa_s(p) = \kappa_s(1-p) + 1 - 2p$.*

This theorem looks a bit nicer than Theorem (7.2.1) and can be proven in much the same way, too.