Universiteit Utrecht

Faculteit Bètawetenschappen

The Gelfand–Naimark theorem for commutative Banach star algebras

BACHELOR THESIS

Robin C. Verstraten
4262840

Wiskunde en Natuurkunde (TWIN)

Supervisor :

Prof. Dr. E.P. van den Ban
Mathematical Institute

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Abstract

We start by studying properties of several kinds of algebras, taking a look at the spectrum, ideals and abelian algebras. Then we prove the Gelfand–Naimark Theorem for commutative Banach star algebras (also known as abelian C^*-algebras). After that, we prove some basic theorems about Riemann integration of Banach valued functions. We then study some applications of the Gelfand–Naimark Theorem where we start by studying the functional calculus, in particular the Riesz functional calculus and its extension to C^*-algebras. We then take a look at positive elements, representations of C^*-algebras and in particular the Gelfand–Naimark–Segal construction. Lastly, we study spectral measures and, using representations, we prove the spectral theorem for bounded normal operators on a Hilbert space.

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1 Introduction

A theorem that is often discussed in an introductory topology course is the Gelfand–Naimark theorem. It states that any compact Hausdorff space X is homeomorphic to the topological spectrum of its algebra $C(X)$ of continuous functions. In the proof we start with a topological space X. We then construct the algebra $C(X)$. Then we look at the topological spectrum of $C(X)$ which is the space of nonzero characters on $C(X)$. Using the theory of maximal ideals we can then show that the structure of these characters is equivalent to the space X and with that we prove the theorem.

This raises a question if we could go though the same procedure, but start with the algebra rather than the topology. In particular for which algebras this procedure will give back an algebra which is equivalent to the original one. Our goal, the Gelfand–Naimark theorem, will answer that question.

The Gelfand–Naimark theorem is actually quite strong and has many applications. One in particular is in spectral analysis, where we try to find the connection between the structure of the spectrum of elements, and the structure of the element itself.

In the process we will assume the reader has a Bachelor level understanding of topology, linear algebra, real analysis, complex analysis, measure theory and in particular linear functional analysis. Though one can read up on particular subjects whilst we use them.

We will begin by studying what objects we are dealing with and what properties they have. Then we will think about how we can link the object that we have, and finally prove the Gelfand–Naimark theorem. Then we look at some applications of the Gelfand–Naimark theorem to get an understanding how important it is and what it really means.

The results and methods in this thesis are mostly based on the book by John B. Conway, A Course in Functional Analysis [1]. The book has been written to teach about the whole subject area of functional analysis. For this thesis we have organized the material necessary for a detailed treatment of the Gelfand–Naimark theorem and some of its applications. In addition we have written a brief treatment on Riemann integration of Banach valued continuous functions. This in turn has been applied to a discussion of Banach valued holomorphic functions.

I want to thank my supervisor Prof. Dr. E.P. van den Ban for guiding me in this learning journey. Even though he is so busy he really took the time for me where it was necessary. It was a great experience!

I would also like to thank my family and friends, but most of all, I want to thank my mother for all the support throughout my school and study years. Without her I would probably not have made it to university and would not have been able to write this thesis.

2 Banach Algebras

To get to our main goal we will need to know what the objects are that we will be studying. So the aim of this section is to get a good understanding of what Banach algebras are and what properties they have. A small reminder: we use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.0.1. An *algebra* over $\mathbb{F}$ is a vector space $\mathscr{A}$ over $\mathbb{F}$ that has a multiplication that makes $\mathscr{A}$ into a ring and such that if $\alpha \in \mathbb{F}$ and $a, b \in \mathscr{A}$ then $\alpha(ab) = (\alpha a)b = a(\alpha b) \in \mathscr{A}$.

Recall that a Banach space is a complete normed vector space, where completeness means that for every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ there is an $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Definition 2.0.2. A *Banach algebra* is an algebra $\mathscr{A}$ over $\mathbb{F}$ with a norm $\|\cdot\|$ that makes $\mathscr{A}$ a Banach space and such that for all $a, b \in \mathscr{A}$,

$$\|ab\| \leq \|a\| \; \|b\|.$$

If $\mathscr{A}$ has an identity element we denote it by 1 and assume that $\|1\| = 1$.

Example 2.0.3. Some easy examples of Banach algebras are $\mathbb{R}$ and $\mathbb{C}$. If X is a compact space, then $C(X)$, the space of all continuous functions $f : X \to \mathbb{F}$, with pointwise multiplication and supremum norm also is a Banach algebra. △

Definition 2.0.4. Let X and Y be normed linear vector spaces and let $T \in L(X, Y)$. If $\|T(x)\| = \|x\|$ for all $x \in X$, then T is called an *isometry*.

Let X, Y be Banach algebras, then a *homomorphism* from X to Y is a map $h : X \to Y$ such that $h(xy) = h(x)h(y)$ and $h(x + \lambda y) + h(x) + \lambda h(y)$. In essence, h a map that preserves all structure.

An *isomorphism* is a homomorphism that is bijective.

Observe that if a function h is an isometry and a homomorphism, then $\|h(x)-h(y)\| = \|h(x-y)\| = \|x-y\|$, so h is automatically injective. Hence to prove that h is an isometric isomorphism we only need to prove that it is a surjective isometric homomorphism.

Proposition 2.0.5. *[1, p.188 Proposition 1.3] Let $\mathscr{A}$ be a Banach algebra without identity. Then define $\mathscr{A}_1 := \mathscr{A} \times \mathbb{F}$ with algebraic operations*

$$\begin{aligned} i) \quad & (a, \alpha) + (b, \beta) = (a + b, \alpha + \beta) \\ ii) \quad & \beta(a, \alpha) = (\beta a, \beta\alpha) \\ iii) \quad & (a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta) \end{aligned}$$

and norm $\|(a, \alpha)\| = \|a\| + |\alpha|$. Then $\mathscr{A}_1$ is a Banach algebra with identity $(0, 1)$. The map $\phi : \mathscr{A} \to \mathscr{A}_1, a \mapsto (a, 0)$ is an isometric isomorphism of $\mathscr{A}$ onto its image.

Proof. It is easy to see that $\mathscr{A}_1$ is an algebra and that it is still a Banach space, to check if it is a Banach algebra we have to check that $\|ab\| \leq \|a\| \; \|b\|$. So $\|(a, \alpha)(b, \beta)\| = \|(ab + \alpha b + \beta a, \alpha\beta)\| = \|ab+\alpha b+\beta a\|+|\alpha\beta| \leq \|a\| \; \|b\|+|\alpha| \; \|b\|+|\beta| \; \|a\|+|\alpha\beta| = (\|a\|+|\alpha|)(\|b\|+|\beta|) = \|(a, \alpha)\| \; \|(b, \beta)\|$. Furthermore we can easily check that ϕ is a linear isometry and it is surjective to its image. □

This proposition can be useful, since we can add the identity to any Banach algebra by only adding one dimension to it. This also means that a lot of results that need the identity element to work can still be made valid for Banach algebras without an identity.

2.1 Ideals

Definition 2.1.1. A *subalgebra* $\mathscr{M}$ is a subset of an algebra $\mathscr{A}$ such that for any $a, b \in \mathscr{M},\ \alpha \in \mathbb{F}$, we have $a + b \in \mathscr{M},\ ab \in \mathscr{M}$ and $\alpha a \in \mathscr{M}$.

Definition 2.1.2. Let $\mathscr{A}$ be an algebra:

A *left ideal* of $\mathscr{A}$ is a subalgebra $\mathscr{M}$ of $\mathscr{A}$ such that $ax \in \mathscr{M}$ for any $a \in \mathscr{A}$, $x \in \mathscr{M}$.

A *right ideal* of $\mathscr{A}$ is a subalgebra $\mathscr{M}$ of $\mathscr{A}$ such that $xa \in \mathscr{M}$ for any $a \in \mathscr{A}$, $x \in \mathscr{M}$.

An *ideal* of $\mathscr{A}$ is a subalgebra that is both a left and a right ideal.

A *proper ideal* is an ideal that is strictly smaller than $\mathscr{A}$.

A *maximal ideal* is a proper ideal that is not contained in any other proper ideal.

If $a \in \mathscr{A}$ and $\mathscr{A}$ has an identity 1, then a is *left invertible* if there is an $x \in \mathscr{A}$ such that $xa = 1$, and a is *right invertible* if there is an $x \in \mathscr{A}$ such that $ax = 1$.

Notice that if a is invertible and $x, y \in \mathscr{A}$ are such that $xa = 1 = ay$, then

$$y = 1y = (xa)y = x(ay) = x1 = x.$$

So there is a unique element $a^{-1} \in \mathscr{A}$ such that $aa^{-1} = a^{-1}a = 1$. If $\mathscr{M}$ is an ideal and $a \in \mathscr{M}$ is an invertible element, then $1 = aa^{-1} \in \mathscr{M}$, so we get that $\mathscr{M} = \mathscr{A}$. This gives us a very important relation between ideals and invertibility, because no elements in an ideal that is not the whole space can be invertible. Since we can not talk of invertibility without an identity, we will come across many statements that require an identity in the algebra.

The following proof is based on the fact that if $x \in \mathbb{R}$ and $|x| < 1$ then $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$ and roughly says that any element close to the identity element is invertible.

Lemma 2.1.3. *If $\mathscr{A}$ is a Banach algebra with identity and $x \in \mathscr{A}$ such that $\|x - 1\| < 1$, then x is invertible with inverse $x^{-1} = \sum_{n=0}^{\infty}(1-x)^n$.*

Proof. Define $y = 1 - x$ such that $\|y\| = r < 1$. Since

$$\|y^n\| \leq \|y^{n-1}\|\ \|y\| \leq \|y^{n-2}\|\ \|y\|^2 \leq \cdots \leq \|y\|^n = r^n$$

we know that

$$\|\sum_{n=0}^{\infty} y^n\| \leq \sum_{n=0}^{\infty} \|y\|^n = \frac{1}{1-r} < \infty.$$

Since $\mathscr{A}$ is complete, we know that $z = \sum_{n=0}^{\infty} y^n$ is convergent in $\mathscr{A}$. Now define $z_k = \sum_{n=0}^{k} y^n$ then $z_k(1-y) = 1 - y^{k+1}$. But since $\|y^{n+1}\|$ goes to 0, we see that $y^{n+1} \to 0$ and thus

$$z(1-y) = \lim_{n\to\infty} z_n(1-y) = \lim_{n\to\infty} 1 - y^{n+1} = 1.$$

So $1 = z(1-y) = z(1-(1-x)) = zx$. Since we can do the same for the right inverse we can conclude that x is invertible with inverse $x^{-1} = z = \sum_{n=0}^{\infty} y^n = \sum_{n=0}^{\infty}(1-x)^n$. □

Theorem 2.1.4. *[1, p.192] If $\mathscr{A}$ is a Banach algebra with identity, $G_l = \{a \in \mathscr{A} | a \text{ is left invertible}\}$, $G_r = \{a \in \mathscr{A} | a \text{ is right invertible}\}$ and $G = \{a \in \mathscr{A} | a \text{ is invertible}\}$, then $G_l,\ G_r$ and G are open in $\mathscr{A}$.*

Proof. Let $a_0 \in G_l$ and let $b_0 \in \mathscr{A}$ such that $a_0 b_0 = 1$. if $\|a - a_0\| < \|b_0\|^{-1}$ then $\|b_0 a - 1\| = \|b_0(a - a_0)\| \leq \|b_0\|\ \|a - a_0\| < 1$. So by Lemma 2.1.3 we get that $b_0 a$ is invertible. If $b = (b_0 a)^{-1} b_0$ then $ba = 1$, so $G_l \supseteq \{a \in \mathscr{A} |\ \|a - a_0\| < \|b_0\|^{-1}\}$. Since this is valid for any $a_0 \in G_l$ and $b_0 \in \mathscr{A}$, the set G_l must be open. Likewise we find that G_r must be open. Since $G = G_l \cap G_r$, G is open. □

In this proof we also used a trick that is useful to remark: If $b_0 a_0 = 1$ and $\|a - a_0\| < \|b_0\|^{-1}$, then a is left invertible.

Corollary 2.1.5. *If $\mathscr{A}$ is a Banach algebra with identity, then the closure of a proper ideal is a proper ideal, also all maximal ideals are closed.*

Proof. Let $\mathscr{M}$ be a proper ideal and let G as in the preceding lemma. Since there are no invertible elements in $\mathscr{M}$ it follows that $\mathscr{M} \cap G = \emptyset$ and $\mathscr{M} \subseteq \mathscr{A} \backslash G$. Since G is open, it follows that $\mathscr{A} \backslash G$ is closed and thus also that $cl\mathscr{M} \subseteq \mathscr{A} \backslash G$. So the closure of $\mathscr{M}$ is not equal to the whole algebra. Since $cl\mathscr{M}$ is still an ideal, it follows that it is a proper ideal.

If $\mathscr{M}$ is a maximal ideal, then by the first part the closure of $\mathscr{M}$ is a closed proper ideal, but since $\mathscr{M}$ is maximal, it follows that $cl\mathscr{M} = \mathscr{M}$, hence $\mathscr{M}$ is closed. □

2.2 The Spectrum

Definition 2.2.1. If $\mathscr{A}$ is a Banach algebra over $\mathbb{F}$ with identity and $a \in \mathscr{A}$, then *the spectrum of a* is defined by

$$\sigma(a) := \{\lambda \in \mathbb{F} |\ (a - \lambda) \text{ is not invertible}\}.$$

The resolvent set of a is defined to be $\rho(a) := \mathbb{F} \backslash \sigma(a)$.

Notice that formally speaking we should write $(a - \lambda 1)$, but since its usually clear that there should be a 1 there, it is often ignored in the notation.

Example 2.2.2. If X is a compact space, and $f \in C(X)$, then $\sigma(f) = f(X)$. This because if $\alpha = f(x_0)$, then $f - \alpha$ has a zero and thus cannot be invertible, hence $f(X) \subseteq \sigma(f)$. On the other hand, if $\alpha \notin F(X)$ then $f - \alpha$ has no zeros, so it is invertible (with the pointwise inversion). △

As a reminder to complex analysis. We call a function f analytic if for every point x_0 in the domain of f there is an open neighborhood U of x_0 such that f is given by a locally convergent power series. In other words, for all $x_0 \in U$ there is an open neighborhood $V \subset U$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$ for all $z \in V$. In particular, in complex analysis a big theorem is that all analytic functions are holomorphic and all holomorphic functions are analytic on $\mathbb{C}$. We will take a closer look at this later on for $\mathscr{A}$ valued functions. But for now, we will assume $f : U \to \mathscr{A}$ is analytic on $U \subset \mathbb{C}$ if the derivative $f'(z) := \lim_{h \to 0} h^{-1}[f(z + h) - f(z)]$ exists for every $z \in U$ and is continuous on U.

Theorem 2.2.3. *[1, p.196] If $\mathscr{A}$ is a Banach algebra over $\mathbb{C}$ with identity, then for each $a \in \mathscr{A}$, $\sigma(a)$ is a nonempty subset of $\mathbb{C}$. Moreover, $f : \rho(a) \to \mathscr{A}$ defined by $f(z) = (z - a)^{-1}$ is an $\mathscr{A}$ valued analytic function on $\rho(a)$ and if $|\alpha| > \|a\|$, then $\alpha \notin \sigma(a)$.*

Proof. If $|\alpha| > \|a\|$, then $\alpha - a = \alpha(1 - \alpha^{-1}a)$ with $\|\alpha^{-1}a\| < 1$. So using Lemma 2.1.3 we see that $\|(1 - \alpha^{-1}a) - 1\| = \|\alpha^{-1}a\| < 1$, hence $1 - \alpha^{-1}a$ is invertible. Since $\alpha - a = \alpha(1 - \alpha^{-1}a)$, we see that $\alpha - a$ is invertible, so $\alpha \notin \sigma(a)$. This implies that $\sigma(a) \subset \{\alpha \in \mathbb{C} |\ |\alpha| \leq \|a\|\}$, so we see that $\sigma(a)$ is bounded.

Let G be the set of invertible elements of $\mathscr{A}$. The map $f : \mathbb{C} \to \mathscr{A};\ \alpha \mapsto (\alpha - a)$ is continuous. Since G is open and $\rho(a) = f^{-1}(G)$ we find that $\rho(a)$ must be open. So $\sigma(a) = \mathbb{C} \backslash \rho(a)$ is closed. Since $\sigma(a)$ is closed and bounded in $\mathbb{C}$, it is compact.

Now define $g : \rho(a) \to \mathscr{A}$ by $g(z) = (z - a)^{-1}$. Now we use the identity

$$x^{-1} - y^{-1} = x^{-1}yy^{-1} - x^{-1}xy^{-1} = x^{-1}(y - x)y^{-1}.$$

Letting $x = \alpha + h - a$ and $y = \alpha - a$ then we get that

$$\frac{g(\alpha + h) - g(\alpha)}{h} = \frac{(\alpha + h - a)^{-1}(-h)(\alpha - a)^{-1}}{h} = -(\alpha + h - a)^{-1}(\alpha - a)^{-1}.$$

So $g'(\alpha) = -(\alpha - a)^{-2}$ and since g is also continuous we see that g is analytic on $\rho(a)$.

Now to show that $\sigma(a)$ is non-empty we look at $g(z)$ when $|z| > \|a\|$. Then $g(z) = z^{-1}(1-a/z)^{-1}$. As $z \to \infty$ we see that $(1 - a/z) \to 1$ so also $(1 - a/z)^{-1} \to 1$, hence $g(z) \to 0$. If we assume g is an entire function, then Liouville's theorem implies that $g' \equiv 0$, since $g \neq 0$ we have a contradiction. Since g is defined on all of $\rho(a)$, this means that $\rho(a) \neq \mathbb{C}$, so $\sigma(a) = \mathbb{C}\backslash\rho(a)$ is non-empty. □

Remark 2.2.4. Since all following results will be about Banach algebras over $\mathbb{C}$, we will from now on assume all Banach algebras are over $\mathbb{C}$.

Definition 2.2.5. If $\mathscr{A}$ is a Banach algebra with identity and $a \in \mathscr{A}$, then the *spectral radius of* a is defined by

$$r(a) = \sup_{\alpha \in \sigma(a)} |\alpha|.$$

Observe that this definition only makes sense because of the previous theorem.

Proposition 2.2.6. *[1, p197 Proposition 3.8] If $\mathscr{A}$ is a Banach algebra with identity and $a \in \mathscr{A}$, then $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. Let $G = \{z \in \mathbb{C}|\ z = 0 \text{ or } z^{-1} \in \rho(a)\}$ and define $f : G \to \mathscr{A}$ by $f(0) = 0$ and $f(z) = (z^{-1} - a)^{-1}$ for $z \neq 0$. Remark that f is analytic on G, so f has a power series expansion. From complex analysis this power series converges for $R = d(0, \sigma(a)^{-1})$ where $\sigma(a)^{-1} = \{z^{-1}|z \in \sigma(a)\}$. So $R = \inf_{\alpha^{-1}\in\sigma(a)} |\alpha| = r(a)^{-1}$. We also know from complex analysis that $R^{-1} = \limsup \|a^n\|^{1/n}$, so we find that

$$r(a) = \limsup \|a^n\|^{1/n}.$$

Now let $\alpha \in \mathbb{C}$ and $n \geq 1$, then $\alpha^n - a^n = (\alpha - a)(\alpha^{n-1} + \alpha^{n-2}a + \cdots + a^{n-1}) = (\alpha^{n-1} + \alpha^{n-2}a + \cdots + a^{n-1})(\alpha - a)$. So if $\alpha^n - a^n$ is invertible then so is $(\alpha - a)$ and $(\alpha - a)^{-1} = (\alpha^n - a^n)^{-1}(\alpha^{n-1} + \alpha^{n-2}a + \cdots + a^{n-1})$. So for $\alpha \in \sigma(a)$ we find that $\alpha^n - a^n$ is not invertible for every $n \geq 1$. So by Theorem 2.2.3 we find that $|\alpha|^n \leq \|a^n\|$. Hence $|\alpha| \leq \|a^n\|^{1/n}$ for all $n \geq 1$ and $\alpha \in \sigma(a)$. So if $\alpha \in \sigma(a)$ then $|\alpha| \leq \liminf \|a^n\|^{1/n}$. Hence

$$r(a) = \sup_{\alpha\in\sigma(a)} |\alpha| \leq \liminf \|a^n\|^{1/n} \leq \limsup \|a^n\|^{1/n} = r(a).$$

So the limit exists and $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. □

3 Abelian Banach Algebras

In light of our main goal, the Gelfand–Naimark theorem, we will need to make sure we use all the information we can get. An important part of this information is that we know that algebras that are made of continuous functions on a compact space are abelian. In this section we will take a look at what kind of properties we can find. Once we know some details about these properties we will look at the Gelfand transform, a function from an abelian Banach algebra to the space of continuous functions on its maximal ideal space. This function will be the key in understanding the Gelfand–Naimark theorem.

3.1 Maximal Ideal Space

Definition 3.1.1. A *division algebra* is an algebra with identity such that every nonzero element has a multiplicative inverse.

These might seem like abstract spaces, but the following theorem proven by Gelfand and Mazur makes these spaces a bit more clear.

Theorem 3.1.2 (Gelfand–Mazur). *[1, p.218] If $\mathscr{A}$ is a Banach division algebra with identity element 1, then $\mathscr{A} = \{\lambda 1 |\ \lambda \in \mathbb{C}\}$.*

Proof. If $a \in \mathscr{A}$ then $\sigma(a) \neq \emptyset$. If $\lambda \in \sigma(a)$, then $(a - \lambda 1)$ has no inverse. Since $\mathscr{A}$ is a division algebra, $a - \lambda 1 = 0$, so $a = \lambda 1$. □

By contraposition we see that if a Banach algebra $\mathscr{A}$ is not isomorphic to the complex numbers, then there is a nonzero element in $\mathscr{A}$ that is not invertible.

Another remarkable consequence of this theorem is that there is no norm that makes the quaternions $\mathbb{H}$ into a Banach space. This because the quaternions form a division algebra but are not isomorphic to $\mathbb{C}$.

Another very strong and important consequence of the Gelfand–Mazur theorem is the following proposition. This is an important step towards the Gelfand–Naimark theorem since it links ideals to homomorphisms.

Proposition 3.1.3. *[1, p.218] If $\mathscr{A}$ is an abelian Banach algebra and $\mathscr{M}$ is a maximal ideal, then there is a homomorphism $h : \mathscr{A} \to \mathbb{C}$ such that $\mathscr{M} = \ker h$. Conversely, if $h : \mathscr{A} \to \mathbb{C}$ is a nonzero homomorphism, then $\ker h$ is a maximal ideal. Moreover, the map $h \mapsto \ker h$ is a bijection.*

Proof. If $\mathscr{M}$ is a maximal ideal, then by Corollary 2.1.5, $\mathscr{M}$ is closed. Hence $\mathscr{A}/\mathscr{M}$ is a Banach algebra with identity. Let $\pi : \mathscr{A} \to \mathscr{A}/\mathscr{M}$ be the quotient map. If $a \in \mathscr{A}$ and $\pi(a)$ is not invertible in $\mathscr{A}/\mathscr{M}$, then $\pi(\mathscr{A}a) = \pi(a)[\mathscr{A}/\mathscr{M}]$ is a proper ideal in $\mathscr{A}/\mathscr{M}$. Now let

$$I = \{b \in \mathscr{A} |\ \pi(b) \in \pi(\mathscr{A}a)\} = \pi^{-1}(\pi(\mathscr{A}a)).$$

Then I is a proper ideal of $\mathscr{A}$ and $\mathscr{M} \subseteq I$. Since $\mathscr{M}$ is maximal, $\mathscr{M} = I$. So $\pi(a\mathscr{A}) \subseteq \pi(I) = \pi(\mathscr{M}) = \{0\}$, so $\pi(a) = 0$. So we find that if $\pi(a)$ is not invertible, then $\pi(a) = 0$. In other words, $\mathscr{A}/\mathscr{M}$ is a Banach division algebra. So by the previous theorem $\mathscr{A}/\mathscr{M} = \mathbb{C} = \{\lambda + \mathscr{M} |\ \lambda \in \mathbb{C}\}$. Define $\tilde{h} : \mathscr{A}/\mathscr{M} \to \mathbb{C}$ by $\tilde{h}(\lambda + \mathscr{M}) = \lambda$ and define $h : \mathscr{A} \to \mathbb{C}$ by $h = \tilde{h} \circ \pi$. Then h is a homomorphism and $\ker h = \mathscr{M}$.

Now consider $h : \mathscr{A} \to \mathbb{C}$ a nonzero homomorphism, then $\ker h = \mathscr{M}$ is a non trivial ideal and $\mathscr{A}/\mathscr{M}$ has the structure of $\mathbb{C}$. So $\mathscr{M}$ is a maximal ideal.

Lastly, if h, h' are two nonzero homomorphisms and $\ker h = \ker h'$. Then, since homomorphisms are linear, there is an $\alpha \in \mathbb{C}$ such that $h = \alpha h'$. So $1 = h(1) = \alpha h'(1) = \alpha$, so $\alpha = 1$ and $h = h'$. □

Corollary 3.1.4. *If $\mathscr{A}$ is an abelian Banach algebra and $h : \mathscr{A} \to \mathbb{C}$ is a homomorphism, then h is continuous.*

Proof. h is linear and $\ker h$ is a maximal ideal by the theorem. By Corollary 2.1.5, maximal ideals are closed, so $\ker h$ is closed, so h is continuous. □

Observe that if $h : \mathscr{A} \to \mathbb{C}$ is a homomorphism, then $h \in \mathscr{A}^*$. We can use this property in the following proposition.

Proposition 3.1.5. *[1, p.219] If $\mathscr{A}$ is an abelian Banach algebra and $h : \mathscr{A} \to \mathbb{C}$ is a nonzero homomorphism, then $\|h\| = 1$.*

Proof. Let $a \in \mathscr{A}$ and define $\lambda = h(a)$. If $|\lambda| > \|a\|$ then $\|a/\lambda\| < 1$, so $1 - a/\lambda$ is invertible. Let $b = (a - a/\lambda)^{-1}$ then $1 = b(a - a/\lambda) = b - ba/\lambda$. Since $h(1) = 1$ we find that

$$1 = h(1) = h(b - ba/\lambda) = h(b) - h(b)h(a)/\lambda = h(b) - h(b) = 0$$

by the definition of λ. So we have a contradiction. Hence $\|a\| > |\lambda| = |h(a)|$ and $\|h\| < 1$. Since $h(1) = 1$ we find that $\|h\| = 1$. □

Definition 3.1.6. Let X be a normed space and X^* its dual space, then the *weak* topology* on X^* is the topology defined by the family of seminorms $\{p_x | \, x \in X\}$ where $p_x(x^*) = |x^*(x)|$.

Definition 3.1.7. A *directed set* is a partially ordered set $(I, \leq)$ such that if $i_1, i_2 \in I$ then there exists an i_3 such that $i_1 \leq i_3$ and $i_2 \leq i_3$.

A *net* in X is a pair $((I, \leq), x)$ where $(I, \leq)$ is a directed set and x is a function from I onto X. We usually write x_i instead of $x(i)$ and say "let $\{x_i\}$ be a net in X".

Observe that a normal sequence $(x_i)_i \in \mathbb{N}$ is also a net if we use the ordering of $\mathbb{N}$. We will use nets instead of sequences because we can only prove the statements with sequences if they are second countable. But as we can see in the following definition, Σ is not second countable. Fortunately the statements can be proven using these nets.

Definition 3.1.8. If $\mathscr{A}$ is an abelian Banach algebra, let $\Sigma = \{h : \mathscr{A} \to \mathbb{C} | \text{ h is a nonzero homomorphism}\}$. Give Σ the relative weak* topology that it has as a subset of $\mathscr{A}^*$. Then Σ with this topology is called the *maximal ideal space* of $\mathscr{A}$.

Remark 3.1.9. When the Gelfand–Naimark theorem is proved from the perspective of topology, these nonzero homomorphisms are called the *characters* of $\mathscr{A}$ and Σ is called the topological spectrum of $\mathscr{A}$.

We will need one more big Theorem from functional analysis.

Theorem 3.1.10 (Alaoglu‘s Theorem)**.** *[1, p.130] Let X be a normed space, then the closed unit ball in X^* is compact in the weak* topology.*

Theorem 3.1.11. *[1, p219] If $\mathscr{A}$ is an abelian Banach algebra, then its maximal ideal space Σ is a compact Hausdorff space. Moreover, if $a \in \mathscr{A}$, then $\sigma(a) = \Sigma(a) := \{h(a) | \, h \in \Sigma\}$.*

Proof. Since $\mathscr{A}^*$ is a Hausdorff space and $\Sigma \subseteq B_1^{\mathscr{A}^*}(0)$, the unit ball in $\mathscr{A}^*$, we only need to show that Σ is weak* closed due to Alaoglu‘s Theorem. For this, let $\{h_i\}$ be a net in Σ and let $h \in B_1^{\mathscr{A}^*}(0)$ such that $h_i \to h$ for the weak* topology. If $a, b \in \mathscr{A}$, then $h(ab) = \lim h_i(ab) = \lim h_i(a)h_i(b) = h(a)h(b)$. So h is a homomorphism. Since $h(1) = \lim h_i(1) = 1$, we conclude that $h \in \Sigma$. Thus Σ is compact.

If $h \in \Sigma$ and $\lambda = h(a)$ then $a - \lambda \in \ker h$. So $a - \lambda$ is not invertible, hence $\lambda \in \sigma(a)$ and $\Sigma(a) \subseteq \sigma(a)$. Now let $\lambda \in \sigma(a)$ then $a - \lambda$ is not invertible so we find that $I = (a - \lambda)\mathscr{A}$ is a proper ideal. Now let $\mathscr{M}$ be a maximal ideal that contains I. By Proposition 3.1.3 we know there is an $h \in \Sigma$ such that $\mathscr{M} = \ker h$. thus $0 = h(a - \lambda) = h(a) - \lambda$ so that $h(a) = \lambda \in \sigma(a)$. So $\sigma(a) \subseteq \Sigma(a)$ and we find that $\Sigma(a) = \sigma(a)$. □

3.2 The Gelfand Transform

Definition 3.2.1. Let $\mathscr{A}$ be an abelian algebra with maximal ideal space Σ. If $a \in \mathscr{A}$, then the *Gelfand transform* of a is the function $\hat{a} : \Sigma \to \mathbb{C}$ defined by $\hat{a}(h) = h(a)$. The *Gelfand transform* of $\mathscr{A}$ is the function $\gamma : \mathscr{A} \to C(\Sigma)$ defined by

$$\gamma(a) = \hat{a}.$$

It is not yet clear that $\hat{a} \in C(\Sigma)$ since we do not know if $\hat{a}$ is continuous, however, the following theorem ensures this.

Theorem 3.2.2. *[1, p220] If $\mathscr{A}$ is an abelian algebra with maximal ideal space Σ and $a \in \mathscr{A}$, then the Gelfand transform $\hat{a}$ of a, belongs to $C(\Sigma)$. Furthermore, the Gelfand transform of $\mathscr{A}$ is a continuous homomorphism of norm 1 and its kernel is given by*

$$\bigcap\{\mathscr{M} \mid \mathscr{M} \text{ is a maximal ideal of } \mathscr{A}\}.$$

Moreover, for all $a \in \mathscr{A}$ we have

$$\|\hat{a}\|_\infty = \lim_{n\to\infty} \|a^n\|^{1/n} = r(a).$$

Proof. If $h_i \to h$ in Σ, then $h_i \to h$ weak* in $\mathscr{A}^*$. So if $a \in \mathscr{A}$, then $\hat{a}(h_i) = h_i(a) \to h(a) = \hat{a}(h)$. So $\hat{a} \in C(\Sigma)$.

Now let γ be the Gelfand transform of $\mathscr{A}$. If $a, b \in \mathscr{A}$ then $\gamma(ab)(h) = \widehat{ab}(h) = h(ab) = h(a)h(b) = \hat{a}(h)\hat{b}(h) = \gamma(a)(h)\gamma(b)(h)$. So $\gamma(ab) = \gamma(a)\gamma(b)$. Since all $h \in \Sigma$ are linear it is easy to see that γ must be linear, so γ is a homomorphism.

By Proposition 3.1.5 we see that if $a \in \mathscr{A}$ then $|\hat{a}(h)| = |h(a)| \le \|a\|$, so $\|\gamma(a)\|_\infty = \|\hat{a}\|_\infty \le \|a\|$. So γ is continuous and $\|\gamma\| \le 1$. Since $\gamma(1)(h) = \hat{1}(h) = h(1) = 1$ for all $h \in \Sigma$ we find that $\gamma(1) = 1$ and so $\|\gamma\| = 1$.

Since $a \in \ker\gamma$ if and only if $\hat{a} \equiv 0$ (that is, $h(a) = 0$ for all $h \in \Sigma$), we can see that $a \in \ker\gamma$ if and only if a belongs to every maximal ideal in $\mathscr{A}$.

Finally, by Theorem 3.1.11 we get that if $a \in \mathscr{A}$ then $\|\hat{a}\|_\infty = \sup\{|\lambda| \mid \lambda \in \sigma(a)\} = r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$ by Proposition 2.2.6.

□

The Gelfand transform is a function that links algebras and the space of continuous functions on their maximal ideal space. So one might expect the Gelfand transform to be bijective. As it will turn out this is indeed the case for a certain type of algebra.

Definition 3.2.3. If $\mathscr{A}$ is an abelian Banach algebra and $a \in \mathscr{A}$, then we call a a *generator* of $\mathscr{A}$ if the set $\{p(a) \mid p \text{ is a polynomial}\}$ is dense in $\mathscr{A}$.

Recall from topology that if $\tau : X \to Y$ is a homeomorphism, then $A : C(Y) \to C(X)$ defined by $Af = f \circ \tau$ is an isometric isomorphism. We denote this relation between A and τ by $A = \tau^\#$.

Proposition 3.2.4. *[1, p221] Let $\mathscr{A}$ be an abelian Banach algebra with identity and maximal ideal space Σ and let $a \in \mathscr{A}$ be a generator of $\mathscr{A}$. Then there is a homeomorphism $\tau : \Sigma \to \sigma(a)$ such that if $\gamma : \mathscr{A} \to C(\Sigma)$ is the Gelfand transform and p is a polynomial, then $\gamma(p(a)) = \tau^\#(p)$.*

Proof. Define $\tau : \Sigma \to \sigma(a)$ by $\tau(h) = h(a)$. Then by Theorem 3.1.11 we see that τ is surjective. We can also see that τ is continuous. Now suppose $\tau(h_1) = \tau(h_2)$, then $h_1(a) = h_2(a)$ and hence $h_1(a^n) = h_2(a^n)$ for all $n \in \mathbb{N}$. Since h_1, h_2 are linear we see that $h_1(p(a)) = h_2(p(a))$ for all polynomials p. Since a is a generator of $\mathscr{A}$ and h_1, h_2 are continuous, we see that $h_1(x) = h_2(x)$ for all $x \in \mathscr{A}$. Hence $h_1 = h_2$ and τ is injective. Since Σ is compact we conclude that τ is a homeomorphism. Now, since γ and τ are both homomorphisms, we see that

$$\gamma(p(a))(h) = p(\gamma(a))(h) = p(\hat{a})(h) = p(\hat{a}(h)) = p(h(a)) = p(\tau(h)) = \tau^\#(p)(h).$$

□

Corollary 3.2.5. *If $\mathscr{A}$ has two elements a_1 and a_2 that are both a generator of $\mathscr{A}$, then $\sigma(a_1)$ and $\sigma(a_2)$ are homeomorphic*

Proof. From the last proposition we see that $\sigma(a_1)$ and $\sigma(a_2)$ are both homeomorphic to Σ. □

4 C^*-Algebras

The algebras that we have studied so far have a lot of good properties. However, there are some properties that we are still missing if we want to make a link between an algebra and a topological space. This is because the link between the topological space and an algebra is the space of continuous functions on the topological space. Since we want to make a bijection between the two, we cannot leave out any information that is given by the functions. Hence we need to add a condition to our algebras which says something about the complex part of the functions on the topological space. In particular we will add an involution to the algebra. The involution looks a lot like the conjugation on $\mathbb{C}$, but is slightly different since a general algebra is not commutative.

4.1 Properties of C^*-Algebras

Definition 4.1.1. Let $\mathscr{A}$ be a Banach algebra, an *involution* is a map $a \mapsto a^*$ from $\mathscr{A}$ to $\mathscr{A}$ such that the following properties hold for all $a, b \in \mathscr{A}$ and $\alpha \in \mathbb{C}$:

1. $(a^*)^* = a$
2. $(ab)^* = b^* a^*$
3. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$.

Observe that if $\mathscr{A}$ has an involution and an identity then $1^* a = (1^* a)^{**} = (a^* 1)^* = (a^*)^* = a$ and similarly $a1^* = a$, so 1^* is an identity. Since the identity is unique we see that $1^* = 1$. Furthermore, observe that if $\alpha \in \mathbb{C}$ then $\alpha^* = \bar{\alpha}$.

Definition 4.1.2. A *C^*-algebra* is a Banach algebra $\mathscr{A}$ with an involution such that $\|a^* a\| = \|a\|^2$ for all $a \in \mathscr{A}$.

Example 4.1.3. $\mathbb{C}$ is a C^*-algebra where we let the involution be complex conjugation. Another example are the complex $n \times n$ matrices where the involution is the complex transpose of a matrix. △

Example 4.1.4. Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{A} = B(\mathscr{H})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$ together with the operator norm. Then $\mathscr{A}$ is a C^*-algebra where if $A \in B(\mathscr{H})$ we let A^* be the adjoint of A, where the adjoint of A is the operator A^* such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x, y \in \mathscr{H}$. △

Proposition 4.1.5. *If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$ then $\|a^*\| = \|a\|$.*

Proof. $\|a\|^2 = \|a^* a\| \leq \|a^*\| \, \|a\|$ so $\|a\| \leq \|a^*\|$ and since $a^{**} = a$ we also find that

$$\|a^*\|^2 = \|a^{**} a^*\| \leq \|a^{**}\| \, \|a^*\| = \|a\| \, \|a^*\|$$

so that $\|a^*\| \leq \|a\|$. So we find that $\|a^*\| = \|a\|$. □

Proposition 4.1.6. *If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$, then*

$$\|a\| = \sup\{\|ax\| \mid x \in \mathscr{A}, \|x\| \leq 1\} = \sup\{\|xa\| \mid x \in \mathscr{A}, \|x\| \leq 1\}.$$

Proof. Let $\alpha = \sup\{\|ax\| \mid x \in \mathscr{A}, \|x\| \leq 1\}$. Then, since $\|ax\| \leq \|a\| \, \|x\|$, we see that $\alpha \leq \|a\|$. Now let $x = a^*/\|a\|$, then $\|x\| = 1$ because of Proposition 4.1.5. So

$$\|ax\| = \|aa^*/\|a\| \, \| = \|a\|^2/\|a\| = \|a\|.$$

Hence $\alpha \geq \|a\|$ and thus $\alpha = \|a\|$. The proof for the second equality is similar. □

Corollary 4.1.7. *If $\mathscr{A}$ is a C^*-algebra, then $\mathscr{A}$ is isometrically isomorphic to a subalgebra of $B(\mathscr{A})$.*

Proof. For $a \in \mathscr{A}$ define the function $l_a : \mathscr{A} \to \mathscr{A}$ by $L_a(x) = ax$, then by Proposition 4.1.6 we see that $\|L_a\| = \|a\|$ and since L_a is clearly linear we find that $L_a \in B(\mathscr{A})$. Now define $\rho : \mathscr{A} \to B(\mathscr{A})$ by $\rho(a) = L_a$, then ρ is a homomorphism and an isometry. Since the inverse is also clearly a homomorphism we find that ρ is an isometric isomorphism from $\mathscr{A}$ to $\rho(\mathscr{A}) \subset B(\mathscr{A})$. □

Definition 4.1.8. If $\mathscr{A}$ and $\mathscr{B}$ are two C^*-algebras, then a **-homomorphism* is a homomorphism $\rho : \mathscr{A} \to \mathscr{B}$ such that $\rho(a^*) = \rho(a)^*$ for all $a \in \mathscr{A}$.

Proposition 4.1.9. *[1, p.233 Proposition 1.9] If $\mathscr{A}$ is a C^*-algebra, then there is a C^*-algebra $\mathscr{A}_1$ with identity such that $\mathscr{A}_1$ contains $\mathscr{A}$ as an ideal.*

If $\mathscr{A}$ does not have an identity, then we can require $\mathscr{A}$ to be a maximal ideal such that $\mathscr{A}/\mathscr{A}_1$ is one dimensional and, if we require $\mathscr{A}$ to be a maximal ideal, then $\mathscr{A}_1$ is unique up to *-isomorphism.

If $\mathscr{B}$ is a C^*-algebra with identity and $\nu : \mathscr{A} \to \mathscr{B}$ is a *-homomorphism, then $\nu_1 : \mathscr{A}_1 \to \mathscr{B}$ defined by $\nu_1(a + \alpha) = \nu(a) + \alpha$ for $\alpha \in \mathbb{C}$ and $a \in \mathscr{A}$, is a *-homomorphism.

Proof. If $\mathscr{A}$ has an identity then the proposition is trivial, so assume $\mathscr{A}$ does not have an identity. Define $\mathscr{A}_1 := \{a + \alpha |\ a \in \mathscr{A},\ \alpha \in \mathbb{C}\}$ where $a + \alpha$ is just a formal sum. Define multiplication and addition in the obvious way, define $(a + \alpha)^* = a^* + \bar{\alpha}$, and define the norm on $\mathscr{A}_1$ to be

$$\|a + \alpha\| = \sup\{\|ax + \alpha x\| \mid x \in \mathscr{A},\ \|x\| \leq 1\}.$$

This is a complete norm on $\mathscr{A}_1$ so we only have to show that $\|y^*y\| = \|y\|^2$ for all $y \in \mathscr{A}_1$. So let $y = a + \alpha$ and $\epsilon > 0$ then there is an $x \in \mathscr{A}$ such that

$$\begin{aligned}\|a + \alpha\|^2 - \epsilon < \|ax + \alpha x\|^2 &= \|(x^*a^* + \bar{\alpha}x^*)(ax + \alpha x)\| = \|x^*(a + \alpha)^*(a + \alpha)x\| \\ &\leq \|(a + \alpha)^*(a + \alpha)\|.\end{aligned}$$

So $\|a + \alpha\|^2 \leq \|(a + \alpha)^*(a + \alpha)\|$.

For the other inequality, observe that $\|(a + \alpha)^*(a + \alpha)\| \leq \|(a + \alpha)^*\|\ \|a + \alpha\|$. Hence we only need to show that $\|(a + \alpha)^*\| \leq \|a + \alpha\|$.

Now let $x, z \in \mathscr{A}$ and $\|x\|, \|z\| \leq 1$, then

$$\|z(a + \alpha)^*x\| = \|za^*x + \bar{\alpha}zx\| = \|x^*az^* + \alpha x^*z^*\| = \|x^*(a + \alpha)z^*\| \leq \|a + \alpha\|.$$

Thus taking the supremum over all x and z with norm less than one gives $\|(a + \alpha)^*\| \leq \|(a + \alpha)\|$. Hence $\|y^*y\| = \|y\|^2$.

Furthermore, by the construction of $\mathscr{A}_1$ it is easy to see that $\mathscr{A}$ is an ideal in $\mathscr{A}_1$ and that $\mathscr{A}_1/\mathscr{A}$ has dimension one.

It is also clear that ν_1 is still a homomorphism. It is also a *-homomorphism since $\nu_1((a + \alpha)^*) = \nu_1(a^* + \bar{\alpha}) = \nu(\alpha^*) + \bar{\alpha} = \nu(\alpha)^* + \bar{\alpha} = \nu_1(a + \alpha)^*$.

To prove the uniqueness up to *-isomorphism, let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two C^*-algebras that hold to the requirements the proposition. Then both $\mathscr{A}_1/\mathscr{A}$ and $\mathscr{A}_2/\mathscr{A}$ are one dimensional, hence $\mathscr{A}_1 = \mathscr{A} \times \mathbb{C} \times \{0_1\} \times \cdots \times \{0_n\}$ and $\mathscr{A}_2 = \mathscr{A} \times \mathbb{C} \times \{0_1\} \times \cdots \times \{0_k\}$. Hence the projections that leave $\mathscr{A} \times \mathbb{C}$ fixed and change the number of zeros to the correct amount form a *-isomorphism. □

Remark 4.1.10. If $\mathscr{A}$ is a C^*-algebra with identity and $a \in \mathscr{A}$, then the spectrum of a is well defined. If $\mathscr{A}$ does not have an identity, then $\sigma(a)$ is defined as the spectrum of a as an element of $\mathscr{A}_1$ as defined in the last proposition.

4.2 Hermitian, Normal and Unitary elements

Definition 4.2.1. If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$, then:

- a is called *hermitian* if $a = a^*$.
- a is called *normal* if $a^*a = aa^*$.
- If $\mathscr{A}$ has an identity then a is called *unitary* if $a^*a = aa^* = 1$.

Proposition 4.2.2. *[1, p.234] Let $\mathscr{A}$ be a C^*-algebra and $a \in \mathscr{A}$.*

1. *If a is invertible, then a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$.*
2. *$a = x + iy$ where x, y are hermitian elements of $\mathscr{A}$.*
3. *If u is a unitary element of $\mathscr{A}$, then $\|u\| = 1$.*
4. *If a is hermitian, then $\|a\| = r(a)$.*
5. *If $\mathscr{B}$ is a C^*-algebra and $\rho : \mathscr{A} \to \mathscr{B}$ is a *-homomorphism, then $\|\rho(a)\| \leq \|a\|$.*

Proof. 1. If a is invertible then $a^{-1}a = 1$, hence $1 = 1^* = (a^{-1}a)^* = a^*(a^{-1})^*$. So we see that a^* is invertible with inverse $(a^*)^{-1} = (a^{-1})^*$.

2. Define $x = \frac{a+a^*}{2}$ and $y = \frac{a-a^*}{2i}$, then $a = x + iy$. Furthermore, we see that

$$x^* = \frac{(a+a^*)^*}{2} = \frac{a^* + a^{**}}{2} = \frac{a + a^*}{2} = x$$

and

$$y^* = \frac{(a-a^*)^*}{(2i)^*} = \frac{a^* - a^{**}}{-2i} = \frac{a^* - a}{-2i} = y.$$

So x and y are indeed hermitian elements on $\mathscr{A}$.

3. $\|u\|^2 = \|u^*u\| = \|1\| = 1$. Hence $\|u\| = 1$.

4. Since a is hermitian, $a^* = a$. Hence $\|a\|^2 = \|a^*a\| = \|a^2\|$. By induction we see that $\|a^{2n}\| = \|a\|^{2n}$. so by Proposition 2.2.6 we have

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^{2n}\|^{1/2n} = \lim_{n\to\infty} \|a\| = \|a\|.$$

5. Let $\rho : \mathscr{A} \to \mathscr{B}$ be a *-homomorphism. If $\mathscr{A}$ does not have an identity then using Proposition 4.1.9 we may assume that $\mathscr{A}$ and $\mathscr{B}$ have identities and that $\rho(1) = 1$. If $x \in \mathscr{A}$, then $\sigma(\rho(x)) \subseteq \sigma(x)$ and hence $r(\rho(x)) \leq r(x)$. Now using part 4. and the fact that a^*a is hermitian, we get

$$\|\rho(a)\|^2 = \|\rho(a^*a)\| = r(\rho(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

□

Proposition 4.2.3. *[1, p.235] If $\mathscr{A}$ is an abelian C^*-algebra with identity, $a \in \mathscr{A}$ and $h : \mathscr{A} \to \mathbb{C}$ is a nonzero homomorphism, then:*

1. *If a is hermitian, then $h(a) \in \mathbb{R}$.*
2. *$h(a^*) = \overline{h(a)}$.*

3. $h(a^*a) \geq 0$.

4. *If* $u \in \mathscr{A}$ *is unitary, then* $|h(u)| = 1$.

Proof. Since $\mathscr{A}$ is abelian, by Proposition 3.1.5 we have $\|h\| = 1$ and hence $h(1) = 1$.

1. If $a = a^*$ and $t \in \mathbb{R}$, then

$$|h(a+it)|^2 \leq \|a+it\|^2 = \|(a+it)(a-it)\| = \|a^2+t^2\| \leq \|a\|^2 + |t|^2.$$

So if $h(a) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, then

$$\|a\|^2 + t^2 \geq |h(a+it)|^2 = |\alpha + i(\beta+t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2.$$

Hence $\|a\|^2 \geq \alpha^2 + \beta^2 + 2\beta t$ for all $t \in \mathbb{R}$. Letting t go to $\pm\infty$, we get a contradiction if $\beta \neq 0$. So we see that $\beta = 0$ and $h(a) \in \mathbb{R}$.

2. Let $a = x + iy$ where x and y are hermitian as in Proposition 4.2.2 point 2. Then by 1. we find that $h(x), h(y) \in \mathbb{R}$. So

$$h(a^*) = h(x-iy) = h(x) - ih(y) = \overline{h(x)+ih(y)} = \overline{h(x+iy)} = \overline{h(a)}.$$

3. By 2. we find that

$$h(a^*a) = h(a^*)h(a) = \overline{h(a)}h(a) = |h(a)|^2 \geq 0.$$

4. By 2. we find that if u is unitary, then

$$|h(u)|^2 = h(u^*)h(u) = h(u^*u) = h(1) = 1.$$

□

Observe that point 2. implies that any homomorphism $h : \mathscr{A} \to \mathbb{C}$ is a *-homomorphism.

Corollary 4.2.4. *If* $\mathscr{A}$ *is an abelian* C^**-algebra with identity and* $a \in \mathscr{A}$ *is a hermitian element, then* $\sigma(a) \in \mathbb{R}$.

Proof. From Theorem 3.1.11 we know that $\sigma(a) = \{h(a)|\ h \in \Sigma\}$. Since $h(a) \in \mathbb{R}$ for all $h \in \Sigma$ by the last proposition, we find that $\sigma(a) \subset \mathbb{R}$. □

We now miss one ingredient to get to our main theorem. Since this theorem is proven in most basic topology courses we will only state it here without a proof.

Theorem 4.2.5 (Stone–Weierstrass theorem)**.** *[1, p.145] Suppose* X *is compact and* $\mathscr{A}$ *is a closed subalgebra of* $C(X)$ *such that the following conditions hold.*

1. $1 \in \mathscr{A}$

2. *If* $x, y \in X$ *and* $x \neq y$*, then there is an* $f \in \mathscr{A}$ *such that* $f(x) \neq f(y)$

3. *If* $f \in \mathscr{A}$*, then* $\bar{f} \in \mathscr{A}$ *(the pointwise complex conjugation)*

Then $\mathscr{A} = C(X)$.

Theorem 4.2.6 (The Gelfand–Naimark theorem for commutative Banach star algebras.)**.** *[1, p. 236 Theorem 2.1] If* $\mathscr{A}$ *is an abelian* C^**-algebra with identity and* Σ *is its maximal ideal space, then the Gelfand transform* $\gamma : \mathscr{A} \to C(\Sigma)$ *is an isometric* **-isomorphism.*

Proof. By Theorem 3.2.2, we see that $\|\hat{x}\|_\infty \leq \|x\|$ for all $x \in \mathscr{A}$. But we also know that

$$\|\hat{x}\|_\infty = \lim_{n\to\infty} \|a^n\|^{1/n} = r(x)$$

by Proposition 2.2.6. Now by Proposition 4.2.2 point 4 we find that if x is hermitian, then $\|x\| = r(x) = \|\hat{x}\|_\infty$. So in particular we find that

$$\|x^*x\| = \|\widehat{x^*x}\|_\infty \quad \forall x \in \mathscr{A}.$$

Now if $a \in \mathscr{A}$ and $h \in \Sigma$ then by Proposition 4.2.3 point 2 we find that

$$\widehat{a^*}(h) = h(a^*) = \overline{h(a)} = \overline{\hat{a}(h)}.$$

So we see that $\widehat{a^*} = \overline{\hat{a}}$. Since the involution in $\mathbb{C}$ is just complex conjugation, we find that

$$\gamma(a^*) = \widehat{a^*} = \overline{\hat{a}} = \hat{a}^* = \gamma(a)^*.$$

Since we know from Theorem 3.2.2 that γ is a continuous homomorphism, we find that γ is a *-homomorphism. Also observe that

$$\|a\|^2 = \|a^*a\| = \|\widehat{a^*a}\|_\infty = \|\ |\hat{a}|^2\ \|_\infty = \|\hat{a}\|_\infty^2.$$

Therefore we find that $\|a\| = \|\hat{a}\|_\infty$ and hence γ is an isometry.

Since $\mathscr{A}$ is a Banach space, we find that the range of γ is closed. So to show that γ is surjective, we only need to show it has a dense range. For this we use the Stone–Weierstrass theorem. Observe that $\hat{1} = 1$, so $\gamma(\mathscr{A})$ is a subalgebra of $C(\Sigma)$ containing 1. Because γ preserves involution, it is clear that $\gamma(\mathscr{A})$ is closed under complex conjugation. Now let $h_1, h_2 \in \Sigma$ such that $h_1 \neq h_2$ then there is an $a \in \mathscr{A}$ such that $h_a(a) \neq h_2(a)$. Hence $\hat{a}(h_1) \neq \hat{a}(h_2)$.

So the Stone–Weierstrass theorem applies and γ is surjective, hence γ is an isometric *-isomorphism. □

Example 4.2.7. Lets take a look at the easiest example: Let $\mathscr{A} = \mathbb{C}$, then Σ is the space of all nonzero homomorphisms from $\mathbb{C}$ to $\mathbb{C}$. The reader may check that this is only the identity function. Hence we have that $(\Sigma, \mathcal{T}) = (\{Id\}, \{\emptyset, \{Id\}\})$. So $C(\Sigma)$ are the continuous functions from a point to $\mathbb{C}$. We can clearly see that these are just the functions which map to a point in $\mathbb{C}$. Hence $\mathbb{C} \simeq C(\Sigma)$. △

5 Riemann integration of functions with values in a Banach space

In this section consider in the Riemann integration of functions with values in a Banach space. The integration of these functions is very useful in many applications. In particular the integration of elements of $B(X)$, the space of bounded operators on a Banach space X, will be very useful. However, since $B(X)$ is automatically a Banach space when X is a Banach space, these results are immediate and we will only look at functions from $\mathbb{C}$ to X.

5.1 The Real Case

Although the results in this section may not be new, no literature other than common knowledge about one dimensional Riemann integration was used in the making. In this part we will take a look at the integration of functions of the form $f : [a, b] \to X$ where f is continuous and X is a Banach space. Fortunately many of the steps in the Riemann integration for real functions can be done for these functions too. Let us start by defining some things that will be useful.

Definition 5.1.1. A *partition* of $[a, b]$ is a finite subset $V \subset [a, b]$ such that $a, b \in V$. Let V be a partition, then we write $V = \{a = a_0 < a_1 < \cdots < a_n = b\}$ to indicate its elements in increasing order. The refinement of two partitions is defined to be the union of the two partitions and is denoted by $V \cup W$.

The *mesh* of V is defined to be $\|V\| := \max\{|a_i - a_{i-1}| \, |i = 1, 2, ..., n\}$.

A *tag of V* is an n-tuple $\Xi = (\xi_1, ..., \xi_n) \in [a, b]^n$ such that $\xi_j \in [a_{j-1}, a_j]$ for all $j = 1, ..., n$. The set of all tags of a partition V is denoted by $X(V)$.

We define the *Riemann sum* of f associated with V and Ξ to be

$$S(f, V, \Xi) := \sum_{j=1}^{n} f(\xi_j)(a_j - a_{j-1}).$$

Definition 5.1.2. The *variation of f over V* is defined to be

$$\mathrm{var}_V(f) := \max_{1 \le i \le n} \mathrm{var}_{[a_{i-1}-a_i]} f \quad \text{where} \quad \mathrm{var}_{[a_{i-1}-a_i]} f = \sup_{x,y \in [a_{i-1},a_i]} \|f(x) - f(y)\|.$$

Observe that for V, W two partitions, $\mathrm{var}_{V \cap W}(f) \le \min(\mathrm{var}_V(f), \mathrm{var}_W(f))$. Furthermore we can also show that

$$\begin{aligned}
&\|S(f, V, \Xi_1) - S(f, V, \Xi_2)\| = \| \sum_{j=1}^{n} (f(\xi_j^1) - f(\xi_j^2)(a_j - a_{j-1})\| \\
&\le \sum_{j=1}^{n} \|(f(\xi_j^1) - f(\xi_j^2)\|(a_j - a_{j-1}) \le \sum_{j=1}^{n} \mathrm{var}_V(f)(a_j - a_{j-1}) = \mathrm{var}_V(f) \sum_{j=1}^{n} (a_j - a_{j-1}) \\
&= \mathrm{var}_V(f)(b - a).
\end{aligned}$$

So we see that $\|S(f, V, \Xi_1) - S(f, V, \Xi_2)\| \le \mathrm{var}_V(f)(b - a)$.

Definition 5.1.3. The Δ of a function f over V is defined to be

$$\Delta(f, V) := \sup_{\Xi_1, \Xi_2 \in X(V)} \|S(f, V, \Xi_1) - S(f, V, \Xi_2)\|.$$

Definition 5.1.4. A function $f : [a, b] \to X$ is called *Riemann integrable* if f is bounded and there exists an element $I \in X$ such that for every $\epsilon > 0$ there exists a partition V of $[a, b]$ such that for all $\Xi \in X(V)$ we have

$$\|S(f, V, \Xi) - I\| < \epsilon.$$

If f is Riemann integrable then we denote the element I in the definition as the *integral of f over* $[a, b]$ and write

$$I = \int_a^b f(x)\mathrm{d}x.$$

We will now need some results from functional analysis.

Definition 5.1.5. Let X be a Banach space, then the the *dual of X*: X^* is defined as the space of all bounded linear functionals from X to $\mathbb{F}$ with the operator norm $\|x^*\| = \sup_{x \in X; \|x\|=1} \|x^*(x)\|$.

We will also need a result from Hahn-Banach ([2, Cor 5.22, p.136]) that we will not prove here.

Proposition 5.1.6 (Corollary of the Hahn-Banach theorem)**.** *If X is a Banach space and $x \in X$ then*

$$\|x\| = \sup_{x^* \in X^*; \|x^*\|=1} |x^*(x)|.$$

Since x^* is linear we can get the following lemma:

Lemma 5.1.7. *If X is a Banach space and $x^* \in X^*$ and $f : [a, b] \to X$ is a Riemann integrable function, then*

$$x^*[S(f, V, \Xi)] = S(x^*(f), V, \Xi)$$

,

$$\mathrm{var}_V(x^*(f)) \leq \|x^*\| \, \mathrm{var}_V(f)$$

and

$$\Delta(x^*(f), V) \leq \|x^*\|\Delta(f, V).$$

Proof.

$$x^*[S(f, V, \Xi)] = x^*\left[\sum_{j=1}^{n} f(\xi_j)(a_j - a_{j-1})\right] = \sum_{j=1}^{n} x^*[f(\xi_j)](a_j - a_{j-1}) = S(x^*(f), V, \Xi)$$

secondly,

$$\begin{aligned} \mathrm{var}_V(x^*(f) &= \max_{1\leq j\leq n} \sup_{x,y\in[a_{j-1},a_j]} |x^*(f(x)) - x^*(f(y))| \\ &= \max_{1\leq j\leq n} \sup_{x,y\in[a_{j-1},a_j]} |x^*(f(x) - f(y))| \\ &\leq \max_{1\leq j\leq n} \sup_{x,y\in[a_{j-1},a_j]} \|x^*\| \, \|f(x) - f(y)\| \\ &= \|x^*\| \, \mathrm{var}_V(f) \end{aligned}$$

and lastly

$$\begin{aligned} \Delta(x^*(f), V) &= \sup_{\Xi_1,\Xi_2\in X(V)} |S(x^*(f), V, \Xi_1) - S(x^*(f), V, \Xi_2)| \\ \text{(by the first equality)} \quad &= \sup_{\Xi_1,\Xi_2\in X(V)} |x^*[S(f, V, \Xi_1)] - x^*[S(f, V, \Xi_2)]| \\ &= \sup_{\Xi_1,\Xi_2\in X(V)} |x^*[S(f, V, \Xi_1) - S(f, V, \Xi_2)]| \\ &\leq \sup_{\Xi_1,\Xi_2\in X(V)} \|x^*\| \, \|S(f, V, \Xi_1) - S(f, V, \Xi_2)\| \\ &= \|x^*\|\Delta(f, V). \end{aligned}$$

□

This lemma is very useful since we can now look at the sums and variations of the function $x^*(f) : [a, b] \to \mathbb{F}$ which we know a lot about. In particular we know the following theorem for real valued functions.

Lemma 5.1.8. *Let $f : [a, b] \to X$ be a bounded function, V, W be two partitions of $[a, b]$ and $\Xi_V \in X(V)$ and $\Xi_W \in X(W)$. Then*

$$\|S(f, V, \Xi_V) - S(f, W, \Xi_W)\| \leq 2(\Delta(f, V) + \Delta(f, W)).$$

Proof. First let $X = R$, then we can use the traditional upper and lower Riemann sums to see that

$$\begin{aligned}
&|S(f,V,\Xi_V) - S(f,W,\Xi_W)| \\
&\leq |S(f,V,\Xi_V) - \overline{S}(f,V)| + |\overline{S}(f,V) - \overline{S}(f,W)| + |\overline{S}(f,W) - S(f,W,\Xi_W) \\
&\leq \Delta(f,V) + |\overline{S}(f,V) - \overline{S}(f,W)| + \Delta(f,W) \\
&\leq \Delta(f,V) + \Delta(f,W) + |\overline{S}(f,V) - \overline{S}(f,V\cup W)| + |\overline{S}(f,V\cup W) - \overline{S}(f,W)| \\
&\leq \Delta(f,V) + \Delta(f,W) + |\overline{S}(f,V) - \underline{S}(f,V)| + |\overline{S}(f,W) - \underline{S}(f,W)| \\
&\leq 2(\Delta(f,V) + \Delta(f,W)).
\end{aligned}$$

Now more generally we use Proposition 5.1.6 and Lemma 5.1.7 to see that

$$\begin{aligned}
&\|S(f,V,\Xi_V) - S(f,W,\Xi_W)\| \\
&= \sup_{x^*\in X^*,\|x^*\|=1} |x^*(S(f,V,\Xi_V) - S(f,W,\Xi_W))| \\
&= \sup_{x^*\in X^*,\|x^*\|=1} |S(x^*(f),V,\Xi_V) - S(x^*(f),W,\Xi_W)| \\
&\leq \sup_{x^*\in X^*,\|x^*\|=1} 2(\Delta(x^*(f),V) + \Delta(x^*(f),W)) \\
&\leq \sup_{x^*\in X^*,\|x^*\|=1} 2\|x^*\|(\Delta(f,V) + \Delta(f,W)) \\
&= 2(\Delta(f,V) + \Delta(f,W)).
\end{aligned}$$

□

We now recall the following theorem from one dimensional analysis.

Theorem 5.1.9 (Criterion for one dimensional Riemann integrability)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function, then the following are equivalent:*

1. *f is Riemann integrable.*
2. *For all $\epsilon > 0$ there exists a V such that for all $\Xi_1, \Xi_2 \in X(V)$ we have*

$$|S(f,V,\Xi_1) - S(f,V,\Xi_2)| < \epsilon.$$

Knowing this theorem, one can ask the question if this is still true for Banach valued functions. The following theorem answers that question.

Theorem 5.1.10 (Criterion for Riemann integrability)**.** *Let $f : [a,b] \to X$ be a bounded function, then the following are equivalent:*

1. *f is Riemann integrable.*
2. *For all $\epsilon > 0$ there exists a partition V such that $\Delta(f,V) < \epsilon$.*

Proof. $1) \Rightarrow 2)$:
Let $\epsilon > 0$, since f is Riemann integrable there exists a partition V such that for all $\Xi \in X(V)$ we have $\|S(f,V,\Xi) - I\| < \frac{\epsilon}{2}$. So

$$\begin{aligned}
\Delta(f,V) &= \sup_{\Xi_1,\Xi_2\in X(V)} \|S(f,V,\Xi_1) - S(f,V,\Xi_2)\| \\
&\leq \sup_{\Xi_1,\Xi_2\in X(V)} \|S(f,V,\Xi_1) - I\| + \|S(f,V,\Xi_2) - I\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

2) $\Rightarrow$ 1):
Choose V_n such that $\Delta(f, V_n) < \frac{1}{n}$ and let $\Xi_n \in X(V_n)$. Then by Lemma 5.1.8 we have

$$\begin{aligned}\|S(f, V_n, \Xi_n) - S(f, V_m, \Xi_m)\| &\leq 2(\Delta(f, V) + \Delta(f, W)) \\ &< 2(\frac{1}{n} + \frac{1}{m}).\end{aligned}$$

So $S(f, V_n, \Xi_n)$ is a Cauchy sequence in X. Since X is complete we know this sequence converges in X. Let I be its limit, then we find that $\|S(f, V_n, \Xi_n) - I\| = \lim_{m\to\infty} \|S(f, V_n, \Xi_n) - S(f, V_m, \Xi_m)\| < 2(\frac{1}{n} + \frac{1}{m}) = \frac{2}{n}$ So f is Riemann integrable with integral I.

□

Proposition 5.1.11. *Let $f : [a, b] \to X$ be a bounded continuous function, then f is Riemann integrable.*

Proof. Since f is continuous on $[a, b]$ and $[a, b]$ is compact, we know that f is uniformly continuous. Now letting $\epsilon > 0$ and letting $\epsilon' = \frac{\epsilon}{b-a}$ we find that there exists a $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$ then $\|f(x) - f(y)\| < \epsilon'$. Now choose V a partition of $[a, b]$ with $\mathrm{mesh}(V) < \delta$. Observe that such a partition always exists since we can choose a uniform distribution of n points with $n > \frac{|b-a|}{\delta}$ so that $\mathrm{mesh}(V) = \frac{|b-a|}{n} < \delta$.

Now we find that

$$\begin{aligned}\Delta(f, V) &= \sup_{\Xi_1,\Xi_2 \in X(V)} \|S(f, V, \Xi_1) - S(f, V, \Xi_2)\| \\ &= \sup_{\Xi_1,\Xi_2 \in X(V)} \|\sum_{j=1}^{n} (f(\xi_j^1) - f(\xi_j^2))(a_j - a_{j-1})\| \\ &\leq \sup_{\Xi_1,\Xi_2 \in X(V)} \sum_{j=1}^{n} \|f(\xi_j^1) - f(\xi_j^2)\|(a_j - a_{j-1}) \\ &\leq \sup_{\Xi_1,\Xi_2 \in X(V)} \sum_{j=1}^{n} \epsilon(a_j - a_{j-1}) \\ &= \epsilon'(b - a) \\ &= \epsilon.\end{aligned}$$

So for all $\epsilon > 0$ there exists a partition V such that $\Delta(f, V) < \epsilon$. Now using Theorem 5.1.10 we find that f is Riemann integrable. □

5.2 The Complex Case

So now that we have defined what the integral of a Banach valued function is, we can look at some functions that go from $\mathbb{C}$ to X that we want to integrate over a path $\gamma : [a, b] \to \mathbb{C}$. From complex analysis we know that the integral of a function along a curve can be expressed by $\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$. Since $\gamma'(t)$ is just a scalar we can still do all the things that we did in the real case.

Let's give ourselves a reminder of some of the main theorems from complex analysis.[3, Chapter III]

Definition 5.2.1. A *curve* in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}$ that is C^1. We call $\gamma(a)$ the *begin point* and $\gamma(b)$ the *end point* of γ and we call γ a curve from $\gamma(a)$ to $\gamma(b)$.

A *path* is a sequence of curves $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ such that the end point of γ_j is equal to the begin point of γ_{j+1}. If $\gamma_j : [a_j, b_j] \to \mathbb{C}$ then this means that $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$. We call $\gamma_1(a_1)$ the *begin point* and $\gamma_n(b_n)$ the *end point* of γ and we call γ a path from $\gamma(a)$ to $\gamma(b)$. We define the integral of f over a path $\gamma = \{\gamma_1, \dots, \gamma_n\}$ to be

$$\int_\gamma f(z)\mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\mathrm{d}z.$$

We call a path or curve $\gamma : [a, b] \to \mathbb{C}$ *closed* if its begin point is equal to its end point.

Let γ, η be two paths in an open $U \subset \mathbb{C}$ that are defined over the same interval $[a, b]$ and have the same begin and end points, then γ is *homotopic* to η in U if there exists a continuous function

$$\psi : [a, b] \times [0, 1] \to U$$

such that $\psi(t, 0) = \gamma(t)$, $\psi(t, 1) = \eta(t)$ and ψ keeps the begin and end points fixed. We often write $\psi_s(t) = \psi(t, s)$ and we may view it as a continuous curve for every $s \in [0, 1]$.

We define the winding number of a closed curve or path γ at a point $z_0 \in \mathbb{C}\setminus \operatorname{im}(\gamma)$ to be

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0}\mathrm{d}z.$$

We call two closed curves $\gamma, \eta : [a, b] \to U$ *homologous in U* if $W(\gamma, z_0) = W(\eta, z_0)$ for all $z_0 \in \mathbb{C}\setminus U$.

We call a closed curve $\gamma : [a, b] \to U$ *homologous to 0 in U* if $W(\gamma, z_0) = 0$ for all $z_0 \in \mathbb{C}\setminus U$.

Theorem 5.2.2. *[3, p.116 theorem 5.1] Let γ, η be two paths on $U \subset \mathbb{C}$ that have the same begin and end points. Assume that γ and η are homotopic in U and let f be holomorphic on U, then*

$$\int_\gamma f(z)\mathrm{d}z = \int_\eta f(z)\mathrm{d}z.$$

This theorem is the important step in many proofs of big theorems. For instance Cauchy's theorem. If one is interested in reading more about this I suggest to read chapters three and four in the book of Lang [3].

Theorem 5.2.3. *[3, p.143 theorem 2.2] Let $\gamma : [a, b] \to U$ be closed path that is homologous to 0 and let f be a holomorphic function on U, then*

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

In particular we also have the Cauchy formula:

Theorem 5.2.4. *[3, p. 145 theorem 2.5] Let $\gamma : [a, b] \to U$ be a closed path that is homologous to 0 in U. Let f be a holomorphic function on U and let $z_0 \in U\setminus \operatorname{im}(\gamma)$, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\mathrm{d}z = W(\gamma, z_0)f(z_0).$$

Now we will consider these theorems with respect to Banach valued complex functions. For this we will use complex linear functionals from X to $\mathbb{C}$ where X is a Banach space. Let us first look at what being holomorphic means for Banach valued functions. For this we will use a theorem that states the equivalence of holomorphic and analytic functions. With this we can just consider analytic functions and don't have to worry about derivatives in Banach spaces. A small remark for the interested reader is that derivatives can be well defined in Banach spaces, although we will not discuss that here.

Theorem 5.2.5. *Let $\gamma : [a, b] \to U$ be closed path that is homologous to 0 in U. Let $f : U \to X$ be an analytic function on U, then*

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

Proof. Let $\xi : X \to \mathbb{C}$ be a complex continuous linear functional, Then

$$\begin{aligned}\xi(\int_\gamma f(z)\mathrm{d}z) &= \xi(\int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t) \\ &= \int_a^b \xi(f(\gamma(t))\gamma'(t))\mathrm{d}t \\ &= \int_a^b \xi(f(\gamma(t)))\gamma'(t)\mathrm{d}t \\ &= \int_\gamma \xi(f(z))\mathrm{d}z.\end{aligned}$$

Since ξ is continuous and linear and f is analytic, we find that $\xi(f)$ is analytic. Since $\xi(f) : U \to \mathbb{C}$ we can use the theorem that states that $\xi(f)$ is holomorphic on U. So using Cauchy's theorem we find that

$$\int_\gamma \xi(f(z))\mathrm{d}z = 0.$$

In particular we can see now that for all complex continuous linear functionals ξ on X we have

$$\xi(\int_\gamma f(z)\mathrm{d}z) = 0.$$

Thus by the Hahn-Banach Theorem (5.1.6) we have

$$\int_\gamma f(z)\mathrm{d}z = 0.$$

□

6 Functional Calculus

In this section we are going to look at the Riesz functional calculus. This calculus gives a link between an algebra and the spectrum of elements in the algebra. In particular we will define how we can put Banach values in regular holomorphic functions in $\mathbb{C}$. After that we can make an algebra homomorphism from an algebra $\mathscr{A}$ and the algebra of holomorphic functions on the spectrum of elements of $\mathscr{A}$. From these results we will get the spectral mapping theorem which will be very useful in our study. After that we will look at the functional calculus for C^*-algebras.

We will also make the assumption from now on that all Banach algebras have an identity.

6.1 The Riesz Functional Calculus

We first start with some reminders from complex analysis.

Definition 6.1.1. Let γ be a closed path and let $U \subset \mathbb{C}$ be open.

- γ is *positively oriented in U* if for every $a \in \mathbb{C}\setminus \operatorname{im}(\gamma)$ we have that $W(\gamma, a)$ is either 0 or 1.

- If γ is positively oriented then the *inside* of γ is defined by

$$\text{ins}\,\gamma := \{a \in \mathbb{C} \mid W(\gamma, a) = 1\}.$$

- If γ is positively oriented then the *outside* of γ is defined by

$$\text{out}\,\gamma := \{a \in \mathbb{C} \mid W(\gamma, a) = 0\}.$$

If $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ where γ_i are paths in an open $U \subset \mathbb{C}$, then we call Γ a *chain* in U. We define the *integral of f over* Γ to be $\int_\Gamma f(z)\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\mathrm{d}z$. We define the *image* of Γ to be $\operatorname{im}\Gamma = \cup_{j=1}^n \operatorname{im}\gamma_j$.

If $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ where γ_i are closed paths in an open $U \subset \mathbb{C}$, then we call Γ a *closed chain* in U. We define the *winding number* of a closed chain Γ in U at a point $z_0 \in \mathbb{C}\setminus \operatorname{im}\Gamma$ to be

$$W(\Gamma, z_0) = \sum_{j=1}^n W(\gamma_j, z_0).$$

Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a closed chain in an open set $U \subset \mathbb{C}$.

- Γ is *positively oriented* if $\operatorname{im}\gamma_i \cap \operatorname{im}\gamma_j = \emptyset$ for $i \neq j$ and for every $a \in \mathbb{C}\setminus \operatorname{im}(\Gamma)$ we have that $W(\Gamma, a)$ is either 0 or 1.
- If Γ is positively oriented then the *inside* of Γ is defined by

$$\text{ins}\,\Gamma := \{a \in \mathbb{C} \mid W(\Gamma, a) = 1\}.$$

- If Γ is positively oriented then the *outside* of Γ is defined by

$$\text{out}\,\Gamma := \{a \in \mathbb{C} \mid W(\Gamma, a) = 0\}.$$

Observe that with these definitions, we can still apply the Theorems from complex Riemann integration like Cauchy's formula.

We will use the following proposition without a proof since it is necessary, but not the focus of what we want to do.

Proposition 6.1.2. *[1, p.200 Proposition 4.4] If G is an open subset of $\mathbb{C}$ and K is a compact subset of G, then there exists a positively oriented closed chain $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ such that $K \subset$ ins Γ and $\mathbb{C}\setminus G \subset$ out Γ. The paths $\{\gamma_1, \dots, \gamma_n\}$ can be found such that they are smooth.*

The basic idea of the proof is that $\inf\{d(x, y) \mid x \in K,\ y \in \mathbb{C}\setminus G\} > 0$, so we can use this "ring" between K and $\mathbb{C}\setminus G$ to place a path that meets the conditions. It is however possible that there are holes in G and K, so that is why we need to use a chain rather than a path.

Remark 6.1.3. If f is an analytic function to $\mathbb{C}$ on an open $U \subset \mathbb{C}$ and $z_0 \in U$, then there is a radius of convergence (roc) $r > 0$ such that $f(z) = \sum_{i=0}^\infty a_i(z - z_0)^i$ for all $z \in B_r(z_0)$, the ball of radius r around z_0. With this power series it is actually well defined when we plug in a value straight from a Banach algebra. However, this method is less convenient since it depends on the input where you have to take the power series.

Proposition 6.1.4. *[1, p.201 Proposition 4.6] Let $\mathscr{A}$ be a Banach algebra, $a \in \mathscr{A}$, and G an open subset of $\mathbb{C}$ such that $\sigma(a) \subset G$. If Γ and Λ are two positively oriented paths that satisfy the requirements of Proposition 6.1.2 with $K = \sigma(a)$ and $f : G \to \mathbb{C}$ is analytic, then*

$$\int_\Gamma f(z)(z - a)^{-1}\mathrm{d}z = \int_\Lambda f(z)(z - a)^{-1}\mathrm{d}z.$$

Proof. Let ζ be a path in $G\backslash\sigma(a)$ from the begin point of Γ to the begin point of Λ Define the path Ω by

- $\Omega(t) = \Gamma(4t)$ for $0 \leq t \leq 1/4$,
- $\Omega(t) = \zeta(-1 + 4t)$ for $1/4 \leq t \leq 1/2$,
- $\Omega(t) = \Lambda(3 - 4t)$ for $1/2 \leq t \leq 3/4$ and
- $\Omega(t) = \zeta(4 - 4t)$ for $3/4 \leq t \leq 1$.

Then for $z \in \sigma(a)$ we have that

$$W(z, \Omega) = W(z, \Gamma) - W(z, \Lambda) = 1 - 1 = 0.$$

For $z \in \mathbb{C}\backslash G$ we have that $W(z, \Omega) = W(z, \Gamma) - W(z, \Lambda) = 0 - 0 = 0$. Hence Ω is a closed path in $U := G\backslash\sigma(a)$. Now, since $z \mapsto f(z)(z-a)^{-1}$ is analytic on U, we find that

$$0 = \int_\Omega f(z)(z-a)^{-1}\mathrm{d}z = \int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z - \int_\Lambda f(z)(z-a)^{-1}\mathrm{d}z.$$

□

As we can see, the idea of the proof is to make one closed integral out of the two closed integrals, we did this by first going over Γ, then integrating over a path ζ towards Λ, going over Λ and then going back over ζ to Γ. This way the integral goes back and forth over ζ and hence ζ doesn't contribute to the integral over Ω.

Definition 6.1.5. If $\mathscr{A}$ is a Banach algebra, $a \in \mathscr{A}$, $G \subset \mathbb{C}$ open such that $\sigma(a) \subset G$, and $f : G \to \mathbb{C}$ is an analytic function, then we define

$$f(a) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z$$

where Γ is as in Proposition 6.1.2.

Observe this if well-defined because of the previous proposition. We call this method of obtaining the value $f(a)$ the *Riesz functional calculus*.

Definition 6.1.6. Let $\mathscr{A}$ be a Banach algebra and $a \in \mathscr{A}$, then we define $\widetilde{hol}(a)$ as the set of all pairs (f, U) where U is an open neighborhood of $\sigma(a)$ and f is an analytic function on U. If $(f, U), (g, V) \in \widetilde{hol}(a)$ then we define $(f, U) + (g, V) = (f + g, U \cap V)$ and $(f, U)(g, V) = (fg, U \cap V)$.

Observe that $\widetilde{hol}(a)$ is not a vector space, since $(f, B_r(0)) + (-f, \mathbb{C}) = (0, B_r(0)) \neq (0, \mathbb{C})$. We can however look at the equivalence class $\sim$ where we say that $(f, U), (g, V) \in \widetilde{hol}(a)$ are equivalent if there is an open neighborhood $W \subseteq U \cap V$ of $\sigma(a)$ such that $f = g$ on W.

Now we can see that $\mathrm{Hol}(a) := \widetilde{hol}(a)/\sim$ actually forms an algebra. However, this is not a Banach algebra since the space is not complete. Since the domain doesn't really matter anymore in this quotient algebra, we often only say "let $f \in \mathrm{Hol}(a)$".

Theorem 6.1.7 (The Riesz Functional Calculus)**.** *[1, p.201] Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$.*

1. *The map $f \mapsto f(a)$, $\mathrm{Hol}(a) \to \mathscr{A}$ is an algebra homomorphism.*
2. *If $f(z) = 1$ for all $z \in \mathbb{C}$, then $f(a) = 1$.*
3. *If $f(z) = z$ for all $z \in \mathbb{C}$, then $f(a) = a$.*

4. *If $f, f_1, f_2, \ldots$ are all analytic on an open G, $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\lim_{n\to\infty} \|f_n(a) - f(a)\| = 0$.*

5. *If $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ has radius of convergence greater than $r(a)$, then $f \in \mathrm{Hol}(a)$ and $f(a) = \sum_{k=0}^{\infty} \alpha_k a^k$.*

Proof. 1. Let $f, g \in \mathrm{Hol}(a)$ and let G be an open neighborhood of $\sigma(a)$ such that f and g are analytic on G. Let Γ be a positively oriented closed chain such that $\sigma(a) \subset$ ins Γ, and let Λ be a positively oriented closed chain in G such that (ins Γ) $\cup \, \mathrm{im}(\Gamma) = \mathrm{cl}$(ins Γ) $\subseteq$ ins Λ. Then

$$\begin{aligned}
f(a)g(a) &= \frac{-1}{4\pi^2}\left[\int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z\right]\left[\int_\Lambda g(\zeta)(\zeta-a)^{-1}\mathrm{d}\zeta\right] \\
&= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(\zeta)(z-a)^{-1}(\zeta-a)^{-1}\mathrm{d}\zeta\mathrm{d}z \\
\left(\text{using } \frac{1}{x}-\frac{1}{y}=\frac{y-x}{xy}\right) \quad &= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(\zeta)\left[\frac{(z-a)^{-1}-(\zeta-a)^{-1}}{\zeta-z}\right]\mathrm{d}\zeta\mathrm{d}z \\
&= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(\zeta)\left[\frac{(z-a)^{-1}-(\zeta-a)^{-1}}{\zeta-z}\right]\mathrm{d}\zeta\mathrm{d}z \\
&= \frac{-1}{4\pi^2}\int_\Gamma f(z)\left[\int_\Lambda \frac{g(\zeta)}{\zeta-z}\mathrm{d}\zeta\right](z-a)^{-1}\mathrm{d}z \\
&+ \frac{1}{4\pi^2}\int_\Lambda g(\zeta)\left[\int_\Gamma \frac{f(z)}{\zeta-z}\mathrm{d}\zeta\right](\zeta-a)^{-1}\mathrm{d}\zeta.
\end{aligned}$$

Since $\zeta \in \Lambda$ we find that $\zeta \in$ out Γ, hence $\int_\Gamma \frac{f(z)}{\zeta - z}\mathrm{d}\zeta = 0$ by Cauchy's theorem. Since $z \in \Gamma$ we find that $z \in$ ins Λ, so $\int_\Lambda \frac{g(\zeta)}{\zeta-z}\mathrm{d}\zeta = 2\pi i g(z)$. Thus we find that

$$f(a)g(a) = \frac{1}{2\pi i}\int_\Gamma f(z)g(z)(z-a)^{-1}\mathrm{d}z = (fg)(a).$$

The functional is also linear since integrals are linear. Hence $f \mapsto f(a)$ is an algebra homomorphism.

2. If $f(z) \equiv 1$ then

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma (z-a)^{-1}\mathrm{d}z \\
&= W(a,\Gamma) \\
&= 1.
\end{aligned}$$

3. If $f(z) = z$, then

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\Gamma z(z-a)^{-1}\mathrm{d}z \\
&= W(a,\Gamma)a \\
&= a.
\end{aligned}$$

4. If Γ is a positively oriented closed chain in G such that $\sigma(a) \subseteq \text{ins}\,\Gamma$ and Γ is homologous to 0 in G, then

$$\begin{aligned}\|f_n(a) - f(a)\| &= \| \int_\Gamma f_n(z)(z-a)^{-1}\mathrm{d}z - \int_\Gamma f(z)(z-a)^{-1}\mathrm{d}z\| \\ &= \| \int_\Gamma (f_n(z) - f(z))(z-a)^{-1}\mathrm{d}z\| \\ &\leq \int_\Gamma |f_n(z) - f(z)|\; \|(z-a)^{-1}\|\mathrm{d}z.\end{aligned}$$

Since $\|(z-a)^{-1}\|$ is a continuous function on the compact chain Γ, it is bounded by some constant $M > 0$. So

$$\begin{aligned}\|f_n(a) - f(a)\| &\leq \int_\Gamma |f_n(z) - f(z)|\; \|(z-a)^{-1}\|\mathrm{d}z \\ &\leq M\|\Gamma\| \sup\{f_n(z) - f(z)|z \in \Gamma\}\end{aligned}$$

where $\|\Gamma\|$ is the length of the chain Γ. Since $f_n \to f$ uniformly on compact subsets, and $\mathrm{im}(\Gamma)$ is a compact subset, we find that $\|f_n(a) - f(a)\| \to 0$ as $n \to \infty$.

5. let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ have radius of convergence greater than $r(a)$, and let G be an open such that $\sigma(a) \subset G$ and f is convergent on G (for instance, $G = B_R(0)$ where $r(a) < R <$roc). Now define $p_n(z) = \sum_{k=0}^{n} \alpha_k z^k$, then combining 1, 2 and 3 we find that $p_n(a) = \sum_{k=0}^{n} \alpha_k a^k$. We also see that $p_n(z) \to f(z)$ on compact subsets of G. So by 4 we find that $\|p_n(a) - f(a)\| \to 0$, hence $f(a) = \lim_{n\to\infty} p_n(a) = \sum_{k=0}^{\infty} \alpha_k a^k$

□

The most important part of this theorem is that it tells us that we can link the space of holomorphic functions on the spectrum with the algebra. This makes it quite useful for multiple applications, but especially for linear operators on spaces. We want to make sure that we have a unique homomorphism between our spaces, for this we will need another result from complex analysis that we won't prove here:

Theorem 6.1.8 (Runge's Theorem)**.** *[1, p.83] Let $K \subset \mathbb{C}$ be a compact subset and let $E \subset \mathbb{C}\backslash K$ be a subset that meets each component of $\mathbb{C}\backslash K$. If f is analytic in a neighborhood of K, then there are rational functions f_n whose poles all lie in E such that $f_n \to f$ uniformly on K.*

Here rational functions means functions that are of the form $\frac{p(z)}{q(z)}$ where p and q are polynomials. Now we can prove the uniqueness of the Riesz Functional Calculus.

Proposition 6.1.9. *[1, p.203] Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$. Let $\tau : \mathrm{Hol}(a) \to \mathscr{A}$ be a homomorphism such that $\tau(1) = 1$, $\tau(z) = a$ and if $\{f_n\}$ is a sequence of analytic functions on an open G such that $\sigma(a) \subset G$ and $f_n(z) \to f(z)$ uniformly on compact subsets of G, then $\tau(f_n) \to \tau(f)$. Then $\tau(f) = f(a)$ for all $f \in \mathrm{Hol}(a)$.*

Proof. First let $p_n(z) = \sum_{k=1}^{n} \alpha_n z^n$, then p_n is a polynomial function and hence analytic. In particular we see that

$$\tau(p_n(z)) = \tau\left(\sum_{k=1}^{n} \alpha_n z^n\right) = \sum_{k=1}^{n} \alpha_n \tau(z)^n = \sum_{k=1}^{n} \alpha_n a^n = p_n(a).$$

Now let q be a polynomial such that $q(z) \neq 0$ whenever $z \in \sigma(a)$, then $q^{-1} \in \mathrm{Hol}(a)$ and

$$a = \tau(qq^{-1}) = \tau(q)\tau(q^{-1}) = q(a)\tau(q^{-1})$$

so $q(a)$ is invertible and its inverse is $q(a)^{-1} = \tau(q^{-1})$. However, using the Riesz Functional Calculus we can show that $q(a)^{-1} = q^{-1}(a)$, so $\tau(q^{-1}) = q^{-1}(a)$. Hence if $f = \frac{p}{q}$ is a rational function, then

$$\tau(f) = \tau(pq^{-1}) = \tau(p)\tau(q^{-1}) = p(a)q^{-1}(a) = f(a).$$

Now let $f \in \text{Hol}(a)$ and suppose that f is analytic on an open set G such that $\sigma(a) \subset G$. Then by Runge's theorem there are rational functions $\{f_n\}$ in $\text{Hol}(a)$ such that $f_n \to f$ uniformly on compact subsets of G. So

$$\tau(f) = \lim_{n\to\infty} \tau(f_n) = \lim_{n\to\infty} f_n(a) = f(a).$$

□

If we define the Riesz Functional Calculus by $\tau(f) = f(a)$ then

$$f(a)g(a) = \tau(fg) = \tau(gf) = g(a)f(a).$$

So these always commute. We can also take this a step further.

Proposition 6.1.10. *If $a, b \in \mathscr{A}$, $ab = ba$ and $f \in \text{Hol}(a)$, then $f(a)b = bf(a)$*

Proof. Let $f \in \text{Hol}(a)$ then by Runge's theorem there are rational functions $\{f_n\}$ that converge uniformly to f on compact subsets. Since these rational functions are only finite powers of a, we see that b commutes with them. So we get that $f(a)b = \lim_{n\to\infty} f_n b = \lim_{n\to\infty} bf_n = bf(a)$. □

Theorem 6.1.11 (The Spectral Mapping Theorem)**.** *[1, p.204] If $a \in \mathscr{A}$ and $f \in \text{Hol}(a)$, then*

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof. If $\alpha \in \sigma(a)$, let $g \in \text{Hol}(a)$ such that $f(z) - f(\alpha) = (z - \alpha)g(z)$. If we assume by contradiction that $f(\alpha) \notin \sigma(f(a))$, then $(a - \alpha)$ would be invertible with inverse $g(a)(f(a) - f(\alpha))^{-1}$. But $\alpha \in \sigma(a)$ so $(\alpha - a)$ is not invertible. Hence $f(\alpha) \in \sigma(f(a))$ and

$$f(\sigma(a)) \subseteq \sigma(f(a)).$$

In the other direction, if $\beta \notin f(\sigma(a))$, then $g(z) = (f(z) - \beta)^{-1} \in \text{Hol}(a)$. So $g(a)(f(a) - \beta) = 1$. Hence $\beta \notin \sigma(f(a))$ and

$$\sigma(f(a)) \subseteq f(\sigma(a)).$$

□

6.2 The Functional Calculus for C^*-Algebras

If $\mathscr{B}$ is a C^*-algebra and $a \in \mathscr{B}$ is a normal element, then define $\mathscr{A} = C^*(a)$, the C^*-algebra generated by a and 1, then $\mathscr{A}$ is abelian. Let $\mathscr{A}$ have maximal ideal space Σ, then we know from the Gelfand–Naimark theorem that $\mathscr{A} \cong C(\Sigma)$. So if $f \in C(\Sigma)$ then there is a unique element $x \in \mathscr{A}$ such that $\hat{x} = f$. We would like to think that $x = f(a)$ using a functional calculus. For this to be useful, we would like a clear way of identifying Σ. The idea of Proposition 3.2.4 is the key here.

Proposition 6.2.1. *[1, p.237] If $\mathscr{A}$ is an abelian C^*-algebra with maximal ideal space Σ and $a \in \mathscr{A}$ such that $\mathscr{A} = C^*(a)$, then the map $\tau : \Sigma \to \sigma(a)$ defined by $\tau(h) = h(a)$ is a homeomorphism. If $p(z, \bar{z}) = \sum_{k=1}^{n} \sum_{l=1}^{m} C_{k,l} z^k \ \bar{z}^l$ is a polynomial and $\gamma : \mathscr{A} \to C(\Sigma)$ is the Gelfand transform, then $\gamma(p(a, a^*)) = p \circ \tau$.*

Proof. The proof is along the lines of Proposition 3.2.4.

We can still use all the same steps to see that τ is a homeomorphism. Now if $p(z, \bar{z}) = \sum_{k=1}^{n} \sum_{l=1}^{m} C_{k,l} z^k \ \bar{z}^l$ is a polynomial, define

$$p(a, a^*) = \sum_{k=1}^{n} \sum_{l=1}^{m} C_{k,l} a^k \ (a^*)^l.$$

Then for all $h \in \Sigma$

$$\gamma(p(a, a^*))(h) = p(\gamma(a), \gamma(a^*))(h) = p(\hat{a}, \widehat{a^*})(h) = p(\hat{a}, \overline{\hat{a}})(h) = p(\tau(h), \overline{\tau(h)}) = p \circ \tau(h).$$

□

Now we can do something interesting. If we define $\tau : \Sigma \to \sigma(a)$ by $\tau(h) = h(a)$, and $\tau^{\#} : C(\sigma(a)) \to C(\Sigma)$ by $\tau^{\#}(f) = f \circ \tau$. By the previous proposition we see that $\tau^{\#}$ is an isometric *-isomorphism since τ is a homeomorphism. Now the last part of the proposition says that $\gamma(p(a, a^*)) = \tau^{\#}(p)$. Since these polynomials are just functions in $C(\sigma(a))$, we can define a map $\rho : C(\sigma(a)) \to C^*(a)$ by $\rho = \gamma^{-1} \circ \tau^{\#}$ such that the following diagram commutes.

$$\begin{array}{ccc} C(\sigma(a)) & \xrightarrow{\tau^{\#}} & C(\Sigma) \\ \downarrow \rho & \nearrow \gamma & \\ C^*(a) & & \end{array}$$

By what we just observed, we see that if $p(z, \bar{z}) \in C(\sigma(a))$, then $\rho(p) = p(a, a^*)$.

Definition 6.2.2. If $\mathscr{B}$ is a C^*-algebra and $a \in \mathscr{B}$ is a normal element, then let $\rho : C(\sigma(a)) \to C^*(a) \subset \mathscr{B}$ as in the previous diagram. If $f \in C(\sigma(a))$ then define

$$f(a) := \rho(f).$$

We call the map ρ the *functional calculus for a*.

Theorem 6.2.3. *[1, p.238] If $\mathscr{B}$ is a C^*-algebra and $a \in \mathscr{B}$ is a normal element, then the functional calculus ρ has the following properties.*

1. *ρ is an injective *-homomorphism from $C(\sigma(a))$ to $C^*(a)$ (otherwise known as a *-monomorphism).*
2. $\|\rho(f)\| = \|f\|_\infty$.
3. *ρ is an extension of the Riesz Functional Calculus.*

Moreover, the functional calculus is unique in the sense that if $\tau : C(\sigma(a)) \to C^*(a)$ is a *-homomorphism that extends the Riesz Functional Calculus, then $\tau(f) = \rho(f)$ for all $f \in C(\sigma(a))$.

Proof. Since $\gamma \circ \rho = \tau^{\#}$ and γ and $\tau^{\#}$ are both *-isomorphisms, we see that ρ is a *-monomorphism. We also know that both γ and $\tau^{\#}$ are isometries, hence ρ must be too.

Now let $\pi : \mathrm{Hol}(a) \to C^*(a)$ denote map defined by the Riesz Functional Calculus. Then $\rho(z) = \pi(z) = a$, hence doing some algebra gives us that $\rho(f) = \pi(f)$ for all rational functions with poles off $\sigma(a)$. Now let $f \in \mathrm{Hol}(a)$, then by Runge's theorem there is a sequence of rational functions $\{f_n\}$ that converge uniformly to f in a neighborhood of $\sigma(a)$. Thus $\pi(f_n) \to \pi(f)$ and $\rho(f_n) \to \rho(f)$. Since $\pi(f_n) = \rho(f_n)$ for all $n \in \mathbb{N}$ we see that $\pi(f) = \rho(f)$.

Now let $\tau : C(\sigma(a)) \to C^*(a)$ be a *-homomorphism that extends the Riesz functional Calculus. If $f \in C(\sigma(a))$, then by the Stone–Weierstrass Theorem there is a sequence $\{p_n\}$ of polynomials in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to f(z)$ uniformly on $\sigma(a)$. But we know that $\tau(p_n) = p_n(a, a^*)$ and $\tau(p_n) \to \tau(f)$ and since $p_n(a, a^*) \to f(a)$ we find that $\tau(f) = f(a)$. □

Due to the uniqueness in this theorem we come to the conclusion that the functional calculus is uniquely defined by the property that $f \mapsto f(a)$ is an isometric *-monomorphism such that if $f(z) \equiv 1$ then $f(a) = 1$, and if $f(z) = z$ then $f(a) = a$.

Theorem 6.2.4 (Spectral Mapping Theorem)**.** *[1, p.239] If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$ is a normal element, then*

$$\sigma(f(a)) = f(\sigma(a))$$

for all $f \in C(\sigma(a))$.

Proof. Let $\rho : C(\sigma(a)) \to C^*(a)$ be defined by $\rho(f) = f(a)$, then ρ is a *-isomorphism. Hence $\sigma(f(a)) = \sigma(\rho(f)) = \sigma(f)$. But $\sigma(f) = f(\sigma(a))$, hence $\sigma(f(a)) = f(\sigma(a))$. Here we used that if $\mathscr{A}$ and $\mathscr{B}$ are two C^*-algebras with common identity and norm such that $\mathscr{A} \subseteq \mathscr{B}$ and $a \in \mathscr{A}$, then the spectrum of a relative to $\mathscr{A}$ is equal to the spectrum of a relative to $\mathscr{B}$. [1, p.235, Proposition 1.14] □

7 Positive Elements and Representations of a C^*-Algebra

As it will turn out, there is a nice link between algebras and bounded operators on a Hilbert space using representations. In order to do so we will first look at positive elements. These elements give some extra structure that will be necessary for the Gelfand–Naimark–Segal construction that will give us the link we are looking for.

7.1 Positive Elements in a C^*-Algebra

In this part we are taking a look at some properties of hermitian elements that are very useful in the study of operators on a Hilbert space. We will use many results from the functional calculus to prove statements here.

Definition 7.1.1. If $\mathscr{A}$ is a C^*-algebra, then we write Re $(\mathscr{A})$ for the set of hermitian elements in $\mathscr{A}$.

Notice that this notation might seem odd at first glance, but we use this notation because the spectrum of a hermitian element is always real. In light of the C^*-algebra $\mathbb{C}$, these are the real elements.

Definition 7.1.2. If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$, then a is *positive* if $a \in$ Re $(\mathscr{A})$ and $\sigma(a) \subset [0, \infty)$. We denote $a \geq 0$ to say that a is a positive element and we denote $\mathscr{A}_+$ for the set of all positive elements in $\mathscr{A}$.

Example 7.1.3. If $\mathscr{A} = C(X)$ for a compact space X then f is positive in $\mathscr{A}$ if and only if $f(x) \geq 0$ for all $x \in X$. △

Proposition 7.1.4. *If $\mathscr{A}$ is a C^*-algebra and $a \in$ Re $(\mathscr{A})$, then there are unique positive elements $u, v \in \mathscr{A}$ such that $a = u - v$ and $uv = vu = 0$.*

Proof. Let $f(t) = \max(t, 0)$ and $g(t) = \min(t, 0)$, then $f, g \in C(\mathbb{R})$ and $f(t) - g(t) = t$ and $f(t)g(t) = 0$. Using the functional calculus, let $u = f(a)$ and $v = g(a)$. Then u and$v$ are hermitian and by the spectral mapping Theorem (6.2.4) we find that $u, v \geq 0$. We also see that

$$u - v = f(a) - g(a) = a \text{ and } uv = vu = f(a)g(a) = 0.$$

For uniqueness, let $u_1, v_1 \in \mathscr{A}_+$ such that $a = u_1 - v_1$ and $u_1v_1 = v_1u_1 = 0$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n(0) = 0$ for all $n \in \mathbb{N}$ and $p_n(t) \to f(t)$ uniformly on $\sigma(a)$. Then $p_n(a) \to u$ in $\mathscr{A}$. Since $u_1a = au_1$, we find that $u_1p_n(a) = p_n(a)u_1$ for all $n \in \mathbb{N}$. Hence $u_1u = uu_1$. Similarly we find that a, u, v, u_1 and v_1 are pairwise commuting hermitian elements of $\mathscr{A}$.

Now let $\mathscr{B}$ be the C^*-algebra generated by a, u, v, u_1 and v_1, then $\mathscr{B}$ is abelian. Hence $\mathscr{B} \cong C(\Sigma)$ by the Gelfand–Naimark Theorem (4.2.6).

So we find that $\hat{a} = \hat{u} - \hat{v}$ and $\hat{u}\hat{v} = 0$. Since $\hat{u}, \hat{v}$ are positive functions to $\mathbb{C}$, we find that for any $h \in \Sigma$ either $\hat{v}(h) = 0$ or $\hat{u}(h) = 0$. Assume that $\hat{v}(h) = 0$, then $\hat{a}(h) = \hat{u}(h) - 0 \geq 0$. Hence $\hat{u}(h) = \max(\hat{a}(h), 0)$. Now assume that $\hat{u}(h) = 0$, then $\hat{a}(h) = -\hat{v}(h) \leq 0$. Hence $\hat{u}(h) = 0 = \max(\hat{a}(h), 0)$. Thus $\hat{u}(h) = \max(\hat{a}(h), 0)$ and $\hat{v}(h) = \hat{u}(h) - \hat{a}(h) = \min(\hat{a}(h), 0)$. Since we can do the same for $\hat{u}_1$ and $\hat{v}_1$ we can conclude that $\hat{u} = \hat{u}_1$ and $\hat{v} = \hat{v}_1$. So $u = u_1$ and $v = v_1$, hence they are unique.

□

Proposition 7.1.5. *If $a \in \mathscr{A}_+$ and $n \geq 1$, then there is a unique element $b \in \mathscr{A}_+$ such that $a = b^n$.*

Proof. Let $f(t) = |t|^{1/n}$, then $f \in C(\mathbb{R})$. Now define $b = f(a)$ using the functional calculus. By the spectral mapping Theorem (6.2.4) we find that $b \geq 0$. We also see that $b^n = f(a)^n = a$.

For the uniqueness we only need to remark that f is strictly monotone on the spectrum of a, hence it is injective. □

We call the decomposition $a = u - v$ of a hermitian element a the orthogonal decomposition of a and we call u the positive part of a and v the negative part of a and denote them by $u = a_+$ and $v = a_-$. We call the unique b such that $b^n = a$ the nth root of a and denote it with $b = a^{1/n}$. Also observe that b is only unique because we assume b to be positive, otherwise we have the trivial counterexample on the real number line that $(\pm 2)^2 = 4$.

Definition 7.1.6. If $\mathscr{A}$ is a C^*-algebra and $a, b \in$ Re $(\mathscr{A})$, then $a \leq b$ if $b - a \geq 0$.

For the next theorem we will need one proposition that we will not prove here.

Proposition 7.1.7. *[1, p.241 Proposition 3.7] If $\mathscr{A}$ is a C^*-algebra, then $\mathscr{A}_+$ is a closed cone.*

Theorem 7.1.8. *[1, p.241] If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$, then the following statements are equivalent.*

1. $a \geq 0$.
2. $a = b^2$ *for some* $b \in$ *Re* $(\mathscr{A})$.
3. $a = x^*x$ *for some* $x \in \mathscr{A}$.
4. $a = a^*$ *and* $\|t - a\| \leq t$ *for all* $t \geq \|a\|$.
5. $a = a^*$ *and* $\|t - a\| \leq t$ *for some* $t \geq \|a\|$.

Proof. By the last proposition it is clear that $1 \Rightarrow 2$. We also see that $2 \Rightarrow 1$ by the spectral theorem. Since an element $b \in \mathscr{A}$ is hermitian if $b^* = b$, we see that $2 \Rightarrow 3$. And $4 \Rightarrow 5$ is trivial.

$5 \Rightarrow 1$: Since $a = a^*$, we see that $C^*(a)$ is abelian. Let $X = \sigma(a)$ then $X \subset \mathbb{R}$ and $f \mapsto f(a)$ is a *-isomorphism from $C(X)$ to $C^*(a)$. Since $\|t - a\| \leq t$ for some $t \geq \|a\|$, we find that, using this *-isomorphism, $|t - x| \leq t$ for some $t \geq \|a\| = r(a) = \sup_{s \in X} |s|$ and all $x \in X$. Since $t \geq 0$, we find that $x \geq 0$ for all $x \in X$ or in other words that $X \subset [0, \infty)$. Hence $\sigma(a) = X \subset [0, \infty)$ and $a \geq 0$.

$1 \Rightarrow 4$: If $a \geq 0$ then $a = a^*$ and $\sigma(a) \subset [0, \infty)$. Now let $f : \sigma(a) \to [0, \infty)$ be defined by $f(x) = x$, then $\|f\| = \sup_{s \in \sigma(a)} |f(s)| = \sup_{s \in \sigma(a)} |s| = r(a) = \|a\|$. So for $t \geq \|a\|$ we see that $|f(x) - t| \leq t$ for all $x \in \sigma(a)$. Now using the functional calculus on f we see that $f(a) = a$, hence $\|a - t\| = \|f(a) - t\| \leq t$ for all $t \geq \|a\|$.

$3 \Rightarrow 1$: If $a = x^*x$ for some $x \in \mathscr{A}$ then $a^* = (x^*x)^* = x^*x = a$. Hence by Proposition 7.1.4 we let $a = u - v$ where $u, v \geq 0$ and $uv = vu = 0$. We want to show now that $v = 0$. Let $b + ic = xv^{1/2}$ with $b, c \in$ Re $(\mathscr{A})$, then

$$(xv^{1/2})^*(xv^{1/2}) = (b - ic)(b + ic) = b^2 + c^2 + i(bc - cb)$$

and

$$(xv^{1/2})^*(xv^{1/2}) = v^{1/2}x^*xv^{1/2} = v^{1/2}av^{1/2} = v^{1/2}(u-v)v^{1/2} = -v^2.$$

Hence $i(bc-cb) = -(v^2+b^2+c^2) \leq 0$ by Proposition 7.1.7. And because $(xv^{1/2})^*(xv^{1/2}) = -v^2$ we also see that $(xv^{1/2})^*(xv^{1/2}) \leq 0$ due to the fact that 1 and 2 are equivalent. Let $y = -(xv^{1/2})(xv^{1/2})^*$, then $y \in \mathscr{A}_+$. So $-y = (b+ic)(b-ic) = b^2+c^2-i(bc-cb)$ and thus $i(bc-cb) = b^2+c^2+y \geq 0$. So

$$0 \leq i(bc-cb) \leq 0.$$

Hence $i(bc-cb) = 0$ and $-v^2 = b^2+c^2 \geq 0$. But by 2 we see that $-v^2 \leq 0$. Hence $v = 0$ and $a = u \geq 0$. □

7.2 Representations of C^*-Algebras

Definition 7.2.1. A *representation* of a C^*-algebra $\mathscr{A}$ is a pair $(\pi, \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space and $\pi : \mathscr{A} \to B(\mathscr{H})$ is a *-homomorphism. If $\mathscr{A}$ has an identity then we assume that $\pi(1) = 1$. It is common not to mention $\mathscr{H}$ and just say that π is a representation.

Example 7.2.2. If $\mathscr{H}$ is a Hilbert space and $\mathscr{A}$ is a C*-subalgebra of $B(\mathscr{H})$, then the identity map from $\mathscr{A}$ to $\mathscr{B}(\mathscr{H})$ is a representation. △

Definition 7.2.3. A representation π of a C^*-algebra $\mathscr{A}$ is called *cyclic* if there is a vector $e \in \mathscr{H}$ such that $\mathrm{cl}(\pi(\mathscr{A})e) = \mathscr{H}$. If such a vector e exists then we call e a *cyclic vector* for the representation π.

Definition 7.2.4. If $\{(\pi_i, \mathscr{H}_i) \ |i \in I\}$ is a family of representations of $\mathscr{A}$, then the direct sum of this family is the representation $(\pi, \mathscr{H})$, where $\mathscr{H} = \oplus_i \mathscr{H}_i$ is the direct sum and $\pi(a) = \oplus_i \pi_i(a)$ for all $a \in \mathscr{A}$.

Observe that $\|\pi_i(a)\| \leq \|a\|$ for all $i \in I$ due to Proposition 4.2.2 point 5, so $\pi(a)$ is a bounded operator on $\mathscr{H}$. It is easy to see that $\pi(a)$ is a *-homomorphism, hence π is a representation.

Definition 7.2.5. If $\mathscr{A}$ is a C^*-algebra and $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ are two representations of $\mathscr{A}$, then we call π_1 and π_2 equivalent if there is an isomorphism $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that $U\pi_1(a)U^{-1} = \pi_2(a)$ for all $a \in \mathscr{A}$.

Observe that

$$\pi_2(a^*) = \pi_2(a)^* = (U\pi_1(a)U^{-1})^* = U^{-1*}\pi_1(a^*)U^* = U\pi_1(a^*)U^{-1}.$$

Hence $UU^* = U^*U = 1$ and U is naturally unitary.

Theorem 7.2.6. *[1, p.249] If π is a representation of the C^*-algebra $\mathscr{A}$, then there is a family of cyclic representations $\{\pi_i\}$ of $\mathscr{A}$ such that π and $\oplus_i \pi_i$ are equivalent.*

Proof. Let S be the collection of all subsets $E \subset \mathscr{H}$ of nonzero vectors such that $\pi(\mathscr{A})e \perp \pi(\mathscr{A})f$ for $e, f \in E$ with $e \neq f$. Order S by inclusion, then an application of Zorn's Lemma implies that S has a maximal element E_0. Now let $\mathscr{H}_0 = \oplus_{e \in E_0} cl(\pi(\mathscr{A})e) \subseteq \mathscr{H}$. If $h \in \mathscr{H} \ominus \mathscr{H}_0 (\equiv \mathscr{H} \cap \mathscr{H}_0^\perp)$, then $0 = \langle \pi(a)e, h\rangle$ for all $a \in \mathscr{A}$ and $e \in E_0$. So if $a, b \in \mathscr{A}$ and $e \in E_0$, then

$$0 = \langle \pi(b^*a)e, h\rangle = \langle \pi(b)^*\pi(a)e, h\rangle = \langle \pi(a)e, \pi(b)h\rangle.$$

Hence $\pi(a)e \perp \pi(b)h$ for all $e \in E_0$ and $E_0 \cup \{h\} \in S$. Since E_0 is maximal this implies that $h = 0$ and thus $\mathscr{H} = \mathscr{H}_0$.

For $e \in E_0$, define $\mathscr{H}_e = \mathrm{cl}(\pi(\mathscr{A})e)$. If $a \in \mathscr{A}$, then $\pi(a)\mathscr{H}_e \subseteq \mathscr{H}_e$. Since $a^* \in \mathscr{A}$ and $\pi(a^*) = \pi(a)^*$, $\mathscr{H}_e$ reduces $\pi(a)$. If $\pi_e : \mathscr{A} \to B(\mathscr{H}_e)$ is defined by $\pi_e(a) = \pi(a)|_{\mathscr{H}_e}$, then π_e is a cyclic representation of $\mathscr{A}$ and $\pi = \oplus_{e \in E_0} \pi_e$. Hence they are equivalent.

□

This theorem tels us that we only have to know about cyclic representations, since we can make a decomposition of a non-cyclic representation into cyclic representations. We can also look at the function $f : \mathscr{A} \to \mathbb{C}$ defined by $f(a) = \langle \pi(a)e, e\rangle$ where $\pi : \mathscr{A} \to B(\mathscr{H})$ is a cyclic representation with cyclic vector e. Then f is a bounded linear functional on $\mathscr{A}$ with the property that $\|f\| \leq \|e\|^2$. Since $\|f(1)\| = \langle 1e, e\rangle = \|e\|^2$, we find that $\|f\| = \|e\|^2$. Moreover, $f(a^*a) = \langle \pi(a^*a)e, e\rangle = \langle \pi(a)e, \pi(a)e\rangle = \|\pi(a)e\|^2$.

Definition 7.2.7. If $\mathscr{A}$ is a C^*-algebra, then a linear functional $f : \mathscr{A} \to \mathbb{C}$ is called *positive* if $f(a) \geq 0$ whenever $a \in \mathscr{A}_+$. A *state* on $\mathscr{A}$ is a positive linear functional of norm 1.

Observe that, by Theorem 7.1.8, the function $f(a) = \langle \pi(a)e, e\rangle$ is positive.

We will now need a result that we will not prove here since it requires some steps that are not further relevant for what we want to look at.

Proposition 7.2.8. *[1, p. 250 Proposition 5.11] If f is a positive linear functional on a C^*-algebra $\mathscr{A}$, then for all $x, y \in \mathscr{A}$ we have*

$$|f(y^*x)|^2 \leq f(y^*y)f(x^*x).$$

The idea of the proof is that one can show that $[x, y] = f(y^*x)$ is a semi-inner product on $\mathscr{A}$. Then there is a theorem by Cauchy-Bunyakowsky-Schwarz [1, p.3] that precisely gives this inequality.

A result that follows from this proposition is that any positive linear functional f on $\mathscr{A}$ is bounded and has norm $\|f\| = f(1)$.

As we just saw, any cyclic representation gives rise to a positive linear functional. The following theorem shows that we can also get a cyclic representation from any positive functional and that there is a clear correspondence between them.

Theorem 7.2.9 (Gelfand–Naimark–Segal Construction)**.** *[1, p.250] Let $\mathscr{A}$ be a C^*-algebra with identity.*

1. *If f is a positive linear functional on $\mathscr{A}$, then there is a cyclic representation $(\pi_f, \mathscr{H}_f)$ of $\mathscr{A}$ with cyclic vector e such that $f(a) = \langle \pi_f(a)e, e\rangle$.*
2. *If $(\pi, \mathscr{H})$ is a cyclic representation of $\mathscr{A}$ with cyclic vector e and $f(a) \equiv \langle \pi(a)e, e\rangle$ and $(\pi_f, \mathscr{H}_f)$ is constructed as in 1, then π and π_f are equivalent.*

Proof. Let f be a positive linear functional on $\mathscr{A}$ and define $\mathscr{L} = \{x \in \mathscr{A} \mid f(x^*x) = 0\}$. Clearly $\mathscr{L}$ is closed in $\mathscr{A}$. Also observe that if $a \in \mathscr{A}$ and $x \in \mathscr{L}$, then Proposition 7.2.8 implies that

$$f((ax)^*(ax))^2 = f(x^*a^*ax)^2 \leq f(x^*x)f(x^*a^*aa^*ax) = 0.$$

Hence $\mathscr{L}$ is a closed left ideal in $\mathscr{A}$ and $\mathscr{A}/\mathscr{L}$ is a vector space (since $\mathscr{L}$ is just a left ideal, it is not an algebra). For $x, y \in \mathscr{A}$, define

$$\langle x + \mathscr{L}, y + \mathscr{L}\rangle = f(y^*x),$$

then $\langle \cdot, \cdot\rangle$ is an inner product on $\mathscr{A}/\mathscr{L}$. This is relatively easy to prove using what we know about positive linear functionals. Now define $\mathscr{H}_f$ to be the completion of $\mathscr{A}/\mathscr{L}$ with respect to this inner product.

Observe that, since $\mathscr{L}$ is a left ideal of $\mathscr{A}$, the map $x + \mathscr{L} \mapsto ax + \mathscr{L}$ with $a \in \mathscr{A}$ is a well defined linear transformation on $\mathscr{A}/\mathscr{L}$. Furthermore we observe that

$$\|ax + \mathscr{L}\|^2 = \langle ax + \mathscr{L}, ax + \mathscr{L}\rangle = f(x^*a^*ax).$$

If we now consider $\|a^*a\| = \|a^*a\|1$ as an element in $\mathscr{A}$, then using the functional calculus we can see that $\|a^*a\| - a^*a \geq 0$.

Now we need one more fact: if $a \in \mathscr{A}_+$ and $x \in \mathscr{A}$, then using Theorem 7.1.8 we see that $a = y^*y$ for some $y \in \text{Re}\,(\mathscr{A})$. So

$$x^*ax = x^*y^*yx = (xy)^*(xy)$$

and by the same theorem we see that $x^*ax \geq 0$.

Now, observe that $0 \leq x^*(\|a^*a\| - a^*a)x = \|a\|^2x^*x - x^*a^*ax$, hence $x^*a^*ax \leq \|a\|^2x^*x$. Going back to what we had earlier, we can now see that

$$\|ax + \mathscr{L}\|^2 = f(x^*a^*ax) \leq \|a\|^2 f(x^*x) = \|a\|^2\|x + \mathscr{L}\|^2.$$

Hence the map $\pi_f(a) : \mathscr{A}/\mathscr{L} \to \mathscr{A}/\mathscr{L}$ defined by $\pi_f(a)(x + \mathscr{L}) = ax + \mathscr{L}$ is a bounded linear operator on $\mathscr{A}/\mathscr{L}$ with $\|\pi_f(a)\| \leq \|a\|$. So $\pi_f(a) \in B(\mathscr{H}_f)$ for all $a \in \mathscr{A}$. By the definition of π_f it is not difficult to see that π_f is a *-homomorphism. Hence π_f is a representation of $\mathscr{A}$.

Let $e = 1 + \mathscr{L} \in \mathscr{H}_f$, then $\pi_f(\mathscr{A})e = \{a + \mathscr{L} \mid a \in \mathscr{A}\} = \mathscr{A}/\mathscr{L}$ which, by definition of $\mathscr{H}_f$, is dense in $\mathscr{H}_f$. So e is a cyclic vector for π_f. The last observation to make for part 1 is that

$$\langle \pi_f(a)e, e\rangle = \langle a + \mathscr{L}, 1 + \mathscr{L}\rangle = f(1^*a) = f(a).$$

Now let $(\pi, \mathscr{H}), e, f$ and $(\pi_f, \mathscr{H}_f)$ be as said in 2. Let e_f be the cyclic vector for π_f so that $f(a) = \langle \pi_f(a)e_f, e_f\rangle$ for all $a \in \mathscr{A}$, Then $\langle \pi(a)e, e\rangle = f(a) = \langle \pi_f(a)e_f, e_f\rangle$ for all $a \in \mathscr{A}$. Define $U : \pi_f(\mathscr{A})e_f \to \mathscr{H}$ by $U\pi_f(a)e_f = \pi(a)e$, where $\pi_f(\mathscr{A})e_f$ is dense in $\mathscr{H}_f$. Then, since

$$\|\pi(a)e\|^2 = \langle \pi(a)e, \pi(a)e\rangle = \langle \pi(a^*a)e, e\rangle = \langle \pi_f(a^*a)e_f, e_f\rangle = \|\pi(a)e_f\|^2,$$

U is well defined and an isometry. So U extends to an isomorphism from $\mathscr{H}_f$ to $\mathscr{H}$. If we let $x, a \in \mathscr{A}$, then

$$U\pi_f(a)\pi_f(x)e_f = U\pi_f(ax)e_f = \pi(ax)e = \pi(a)\pi(x)e = \pi(a)U\pi_f(x)e_f.$$

Thus $\pi(a)U = U\pi_f(a)$ and hence π and π_f are equivalent. □

We often call this theorem the GNS construction. Lastly, it is not difficult to see that we could scale the function that we made with a constant such that π_f and $\pi_{\alpha f}$ are equivalent. So we often assume that f is a state.

8 The Spectral Theorem

In this section we are going to take a look at an important result that follows from the Gelfand–Naimark theorem: the Spectral Theorem. This is about combining spectral measures and normal operators on a Hilbert space. In the process to proving the spectral theorem we will require the knowledge about measures and σ-algebras.

Remember that a σ-algebra Ω of a set X is a family of subsets such that $X \in \Omega$, if $\Delta \in \Omega$ then also the complement $\Delta^c \in \Omega$ and if $(\Delta_n)_{n\in\mathbb{N}} \subset \Omega$, then also $\cup_{n\in\mathbb{N}}\Delta_n \in \Omega$. If X is a topological space then we call the σ-algebra generated by the open sets in X the *Borel σ-algebra* of X.

8.1 Spectral Measures

For our definition of a spectral measure we need to start by introducing some topologies.

Definition 8.1.1. If $\mathscr{H}$ is a Hilbert space, then the *weak operator topology* (WOT) on $B(\mathscr{H})$ is the locally convex topology defined by the seminorms $\{p_{h,k} \mid h, k \in \mathscr{H}\}$ where $p_{h,k}(A) = |\langle Ah, k\rangle|$.

The *strong operator topology* (SOT) is the topology on $B(\mathscr{H})$ defined by the the seminorms $\{p_h \mid h \in \mathscr{H}\}$ where $p_h(A) = \|Ah\|$.

Some useful properties to know are the following. But since the proof does not give much insight towards the spectral theorem, we will not prove these statements here.

Definition 8.1.2. If X is a Banach space, then we call a subset $F \subset X$ a *total subset* of X if $\overline{\operatorname{span} F} = X$.

Proposition 8.1.3. *[1, p.256 Proposition 1.3] Let $\mathscr{H}$ be a Hilbert space and let $\{A_i\}$ be a net in $B(\mathscr{H})$.*

1. *$A_i \to A$ (WOT) if and only if $\langle A_i h, k\rangle \to \langle Ah, k\rangle$ for all $h, k \in \mathscr{H}$.*
2. *If $\sup_i \|A_i\| < \infty$ and F is a total subset of $\mathscr{H}$, then $A_i \to A$ (WOT) if and only if $\langle A_i h, k\rangle \to \langle Ah, k\rangle$ for all $h, k \in F$.*
3. *$A_i \to A$ (SOT) if and only if $\|A_i h - Ah\| \to 0$ for all $h \in \mathscr{H}$.*
4. *If $\sup_i \|A_i\| < \infty$ and F is a total subset of $\mathscr{H}$, then $A_i \to A$ (SOT) if and only if $\|A_i h - Ah\| \to 0$ for all $h \in F$.*
5. *If $\mathscr{H}$ is separable, then the WOT and the SOT are metrizable on bounded subsets of $B(\mathscr{H})$.*

Definition 8.1.4. If X is a set, Ω is a σ-algebra of subsets of X, and $\mathscr{H}$ is a Hilbert space, then a *spectral measure for* $(X, \Omega, \mathscr{H})$ is a function $E : \Omega \to B(\mathscr{H})$ such that the following properties hold.

1. $E(\Delta)$ is a projection for all $\Delta \in \Omega$.
2. $E(\emptyset) = 0$ and $E(X) = 1$.
3. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for all $\Delta_1, \Delta_2 \in \Omega$.
4. If $\{\Delta_n\}_{n\in\mathbb{N}}$ are pairwise disjoint sets from Ω, then
$$E(\cup_{n\in\mathbb{N}} \Delta_n) = \sum_{n\in\mathbb{N}} E(\Delta_n).$$

Remark 8.1.5. If $\{P_n\}_{n\in\mathbb{N}}$ are orthogonal projections with pairwise orthogonal images on a Hilbert space, then we can define the sum of these projections P as the projection on the space $\overline{\oplus_{n\in\mathbb{N}} \operatorname{im}(P_n)}$. We will show that $\sum_{k=0}^{n} P_k \to P = \sum_{k=0}^{\infty} P_k$ for the strong operator topology. Fix $v \in \mathscr{H}$, because $\|Pv\|^2 + \|(I-P)v\|^2 = \|v\|^2$, we find that
$$\sum_{k=0}^{\infty} \|P_k v\|^2 = \|Pv\|^2 \leq \|v\|^2.$$
Thus
$$\|\sum_{k=0}^{n} P_k v - Pv\|^2 = \| - \sum_{k=n+1}^{\infty} P_k v\|^2 = \sum_{k=n+1}^{\infty} \|P_k v\|^2$$
goes to 0 as $n \to \infty$ because the sum converges. Thus using this definition, the sum of orthogonal projections always converges with respect to the strong operator topology.

Now if Δ_1, Δ_2 are two disjoint sets, then $\Delta_1 \cap \Delta_2 = \emptyset$. So $E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1) = E(\Delta_1 \cap \Delta_2) = E(\emptyset) = 0$. Hence we see that $E(\Delta_1)$ and $E(\Delta_2)$ are orthogonal if Δ_1 and Δ_2 are disjoint. So point 4 of the definition is unambiguous.

Example 8.1.6. Let X be any set, let Ω be the set of all subsets of X, let $\mathscr{H}$ be separable Hilbert space and let $\{x_n\}$ be a sequence in X. Since $\mathscr{H}$ is separable we know there is an orthonormal basis $\{e_1, e_2, \ldots\}$ of $\mathscr{H}$. For $\Delta \in \Omega$, define $E(\Delta)$ as the projection onto $\overline{\operatorname{span}}\{e_n \mid x_n \in \Delta\}$. Then E is a spectral measure for $(X, \Omega, \mathscr{H})$. $\triangle$

Example 8.1.7. Let X be a compact set, Ω the collection of all Borel subsets of X, μ a measure on Ω and let $\mathscr{H} = L^2(X,\mu)$. Define $E(\Delta) = \mathbb{1}_\Delta$, then E is a spectral measure for $(X,\Omega,\mathscr{H})$. △

Definition 8.1.8. Let μ be a complex measure on (X,Ω), then the *variation* of μ over $\Delta \in \Omega$ is defined as

$$|\mu|(\Delta) = \sup_\pi \sum_{A\in\pi} |\mu(A)|$$

where the supremum is taken over all partitions π of Δ into a countable number of disjoint measurable subsets.

The *total variation* is defined as $\|\mu\| := |\mu|(X)$.

Lemma 8.1.9. *[1, p.257] If E is a spectral measure for $(X,\Omega,\mathscr{H})$ and $g,h\in\mathscr{H}$, then*

$$E_{g,h}(\Delta) := \langle E(\Delta)g, h\rangle$$

defines a measure on Ω with total variation $\le \|g\|\,\|h\|$.

Proof. Let $\{\Delta_n\}$ be a countable collection of pairwise disjoint sets in Ω, then $E_{g,h}(\cup_{n\in\mathbb{N}}\Delta_n) = \langle E(\cup_{n\in\mathbb{N}}\Delta_n)g,h\rangle = \langle\sum_{n\in\mathbb{N}} E(\Delta_n)g,h\rangle = \sum_{n\in\mathbb{N}}\langle E(\Delta_n)h,g\rangle = \sum_{n\in\mathbb{N}} E_{g,h}(\Delta_n)$. So $E_{g,h}$ is a measure on Ω.

Now let $\Delta_1,\dots,\Delta_n$ be pairwise disjoint sets in Ω and let $\alpha_j\in\mathbb{C}$ such that $|\alpha_j|=1$ and $|\langle E(\Delta_j)g,h\rangle| = \alpha_j\langle E(\Delta_j)g,h\rangle$. Then

$$\sum_j |\mu(\Delta_j)| = \sum_j \alpha_j\langle E(\Delta_j)g,h\rangle = \langle\sum_j E(\Delta_j)\alpha_j g,h\rangle \le \|\sum_j E(\Delta_j)\alpha_j g\|\,\|h\|.$$

Now $\{E(\Delta_j)\alpha_j g\}_{1\le j\le n}$ is a finite sequence of pairwise orthogonal vectors so that

$$\|\sum_j E(\Delta_j)\alpha_j g\|^2 = \sum_j \|E(\Delta_j)g\|^2 = \|E(\cup_j\Delta_j)g\|^2 \le \|g\|^2.$$

Hence $\sum_j |\mu(\Delta_j)| \le \|g\|\,\|h\|$ and thus $\|\mu\| \le \|g\|\,\|h\|$. □

We know from measure theory how to integrate with respect to a measure. So in particular we know how to integrate with respect to $E_{g,h}$. We can use this to derive what we need to do in order to integrate with respect to a spectral measure. But first we will need one more result from functional analysis. This requires one more property of functions.

Definition 8.1.10. if $\mathscr{H}_1,\mathscr{H}_2$ are two Hilbert spaces, then a function $u : \mathscr{H}_1\times\mathscr{H}_2\to\mathbb{F}$ is called *sesquilinear* if for $g,h\in\mathscr{H}_1$, $k,f\in\mathscr{H}_2$ and $\alpha,\beta\in\mathbb{F}$ we have

$$u(\alpha h+\beta g,k) = \alpha u(h,k)+\beta u(g,k)$$

and

$$u(h,\alpha k+\beta f) = \bar{\alpha}u(h,k)+\bar{\beta}u(h,f).$$

We call a sesquilinear function u bounded if $\sup_{\|h\|=\|k\|=1}|u(h,k)| < \infty$. If u is a bounded sesquilinear function then we say it has bound $M = \sup_{\|h\|=\|k\|=1}|u(h,k)|$.

The proof uses the Riesz Representation Theorem, but we will not prove this here:

Theorem 8.1.11 (The Riesz Representation Theorem)**.** *[1, p.13] If $L:\mathscr{H}\to\mathbb{F}$ is a bounded linear functional, then there is a unique vector $h_0\in\mathscr{H}$ such that $L(h) = \langle h,h_0\rangle$ for all $h\in\mathscr{H}$. Moreover, h_0 has the property that $\|L\| = \|h_0\|$.*

Theorem 8.1.12. *[1, p.31 Theorem 2.2] If $u : \mathscr{H}_1 \times \mathscr{H}_2 \to \mathbb{F}$ is a bounded sesquilinear function with bound M, then there are unique operators $A \in B(\mathscr{H}_1, \mathscr{H}_2)$ and $B \in B(\mathscr{H}_2, \mathscr{H}_1)$ such that*

$$u(h,k) = \langle Ah, k\rangle_2 = \langle h, Bk\rangle_1$$

for all $h \in \mathscr{H}_1$, $k \in \mathscr{H}_2$, Moreover, $\|A\|, \|B\| \leq M$.

Proof. Fix $k \in \mathscr{H}_2$, then $f : \mathscr{H}_1 \to \mathbb{C}$, $f(h) = u(h,k)$ is a bounded linear functional on $\mathscr{H}_1$. So by the Riesz Representation Theorem there is a unique vector $B(k) \in \mathscr{H}_1$ such that $f(h) = \langle h, B(k)\rangle$ with the property that $\|f\| = \|B(k)\|$. Due to the uniqueness, we see that $B : \mathscr{H}_2 \to \mathscr{H}_1$ is linear. Furthermore,

$$\|B\| = \sup_{\|k\|=1, k\in\mathscr{H}_2} \|B(k)\| \leq \sup_{\|h\|=1, h\in\mathscr{H}_1} \sup_{\|k\|=1, k\in\mathscr{H}_2} |\langle h, B(k)\rangle| = \sup_{\|h\|=\|k\|=1} |u(h,k)| = M.$$

The proof for A is equivalent. □

Proposition 8.1.13. *[1, p.258] If E is a spectral measure for $(X, \Omega, \mathscr{H})$ and $\phi : X \to \mathbb{C}$ is a bounded Ω-measurable function, then there is a unique operator $A \in B(\mathscr{H})$ such that if $\epsilon > 0$ and $\{\Delta_1, \dots, \Delta_n\}$ is an Ω-partition of X with $\sup\{|\phi(x) - \phi(x')| \mid x, x' \in \Delta_k\} < \epsilon$ for all $1 \leq k \leq n$, then for any $x_k \in \Delta_k$,*

$$\|A - \sum_{k=1}^{n} \phi(x_k) E(\Delta_k)\| < \epsilon.$$

Proof. Define $B(g,h) = \int \phi \, \mathrm{d}E_{g,h}$ for $g, h \in \mathscr{H}$. By the last lemma we can see that B is a sesquilinear form with $|B(g,h)| \leq \|\phi\|_\infty \|g\| \, \|h\|$. So by the preceding theorem there is a unique operator $A \in B(\mathscr{H})$ such that $B(g,h) = \langle Ag, h\rangle$ for all $g, h \in \mathscr{H}$.

Let $\{\Delta_1, \dots, \Delta_n\}$ be an Ω-partition of X with $\sup\{|\phi(x) - \phi(x')| \mid x, x' \in \Delta_k\} < \epsilon$ for all $1 \leq k \leq n$. If $g, h \in \mathscr{H}$ and $x_k \in \Delta_k$ for $1 \leq k \leq n$ are arbitrary, then

$$\begin{aligned} |\langle Ag, h\rangle - \sum_{k=1}^{n} \phi(x_k)\langle E(\Delta_k)g, h\rangle| &= |\sum_{k=1}^{n} \int_{\Delta_k} (\phi(x) - \phi(x_k) \mathrm{d}\langle E(x)g, h\rangle| \\ &\leq \sum_{k=1}^{n} \int_{\Delta_k} |(\phi(x) - \phi(x_k)| \mathrm{d}|\langle E(x)g, h\rangle| \\ &\leq \epsilon \int \mathrm{d}|\langle E(x)g, h\rangle| \\ &\leq \epsilon \, \|g\| \, \|h\|. \end{aligned}$$

Hence $\|A - \sum_{k=1}^{n} \phi(x_k) E(\Delta_k)\| < \epsilon$. □

We call the operator obtained in this proposition the *integral of ϕ with respect to E* and denote it by $\int \phi \, \mathrm{d}E$. The proof also implies that for $g, h \in \mathscr{H}$ we have

$$\langle \left(\int \phi \, \mathrm{d}E \right) g, h\rangle = \int \phi \, \mathrm{d}E_{g,h}.$$

It is also good to notice the analogy here to Riemann sums.

Definition 8.1.14. Let X be a set and let Ω be a σ-algebra on X, then we define $B(X, \Omega)$ as the set of all bounded Ω-measurable functions $\phi : X \to \mathbb{C}$ and define the norm on $B(X, \Omega)$ to be $\|\phi\| = \sup_{x\in X} |\phi(x)|$.

Observe that this norm makes $B(X, \Omega)$ into a Banach algebra with identity. If we also define $\phi^*(x) = \overline{\phi(x)}$ then $B(X, \Omega)$ becomes an abelian C^*-algebra.

8.2 Representations of Abelian C^*-Algebras

Proposition 8.2.1. *[1, p.258] If E is a spectral measure for $(X, \Omega, \mathscr{H})$ and $\rho : B(X, \Omega) \to B(\mathscr{H})$ is defined by $\rho(\phi) = \int \phi \, \mathrm{d}E$, then ρ is a representation of $B(X, \Omega)$ and $\rho(\phi)$ is a normal operator on $\mathscr{H}$ for all $\phi \in B(X, \Omega)$.*

Proof. ρ is linear since we know that

$$\rho(\phi) = \int \phi \, \mathrm{d}E = \langle \left(\int \phi \, \mathrm{d}E \right) g, h \rangle$$

and measure integrals are linear and inner products are linear in the first argument. Let $\phi, \psi \in B(X, \Omega)$ and let $\epsilon > 0$. Now choose a Borel partition $\{\Delta_1, \dots, \Delta_n\}$ of X such that for $\omega = \phi, \psi$ or $\phi\psi$ we have $\sup\{|\omega(x) - \omega(x')| \mid x, x' \in \Delta_k\} < \epsilon$ for all $1 \leq k \leq n$.

Remark that such a sequence always exists, because we know that ω is Ω-measurable and bounded. So for any $\epsilon > 0$ we can find Borel subsets $\{b_1, \dots b_n\}$ of $\mathbb{C}$ such that they form a partition of $\operatorname{im} \omega$ and that for all $1 \leq i \leq n$ we have $|\omega(x) - \omega(x')| < \epsilon$ for all $x, x' \in X$ such that $\omega(x), \omega(x') \in b_i$. Hence we can choose $\{\Delta_1^\omega, \dots, \Delta_n^\omega\}$ such that $\Delta_i^\omega = \omega^{-1}(b_i)$ which are elements of Ω since ω is Ω-measurable. We can do this for all the three functions ϕ,ψ and $\phi\psi$, so we can choose the refinement of these three partitions as our partition.

Hence, if $x_k \in \Delta_k$ for all $1 \leq k \leq n$,

$$\left\| \int \omega \, \mathrm{d}E - \sum_{k=1}^n \omega(x_k) E(\Delta_k) \right\| < \epsilon$$

for $\omega = \phi, \psi$ or $\phi\psi$. So using the triangle inequality gives

$$\begin{aligned}
&\left\| \int \phi\psi \, \mathrm{d}E - \left(\int \phi \, \mathrm{d}E \right) \left(\int \psi \, \mathrm{d}E \right) \right\| \\
&\leq \epsilon + \left\| \sum_{k=1}^n \phi(x_k)\psi(x_k)E(\Delta_k) - \left(\sum_{i=1}^n \phi(x_i)E(\Delta_i) \right) \left(\sum_{j=1}^n \psi(x_j)E(\Delta_j) \right) \right\| \\
&+ \left\| \left(\sum_{i=1}^n \phi(x_i)E(\Delta_i) \right) \left(\sum_{j=1}^n \psi(x_j)E(\Delta_j) \right) - \left(\int \phi \, \mathrm{d}E \right) \left(\int \psi \, \mathrm{d}E \right) \right\|.
\end{aligned}$$

Now, since $\{\Delta_1, \dots, \Delta_n\}$ is a partition we see that $E(\Delta_i)E(\Delta_j) = E(\Delta_i \cap \Delta_j) = 0$ if $i \neq j$. Hence the second term in the equation above is 0 and we get that

$$\begin{aligned}
&\left\| \int \phi\psi \, \mathrm{d}E - \left(\int \phi \, \mathrm{d}E \right) \left(\int \psi \, \mathrm{d}E \right) \right\| \\
&\leq \epsilon + \left\| \left(\sum_{i=1}^n \phi(x_i)E(\Delta_i) \right) \left(\sum_{j=1}^n \psi(x_j)E(\Delta_j) \right) - \left(\int \phi \, \mathrm{d}E \right) \left(\int \psi \, \mathrm{d}E \right) \right\| \\
&\leq \epsilon + \left\| \left(\sum_{i=1}^n \phi(x_i)E(\Delta_i) \right) \left(\sum_{j=1}^n \psi(x_j)E(\Delta_j) - \int \psi \, \mathrm{d}E \right) \right\| \\
&+ \left\| \left(\sum_{i=1}^n \phi(x_i)E(\Delta_i) - \int \phi \, \mathrm{d}E \right) \left(\int \psi \, \mathrm{d}E \right) \right\| \\
&\leq \epsilon(1 + \|\phi\| + \|\psi\|).
\end{aligned}$$

For the second to last inequality we used that $ab-cd = a(b-d)+(a-c)d$. Since ϵ was chosen arbitrary,

$$\int \phi\psi \, \mathrm{d}E = \left(\int \phi \, \mathrm{d}E\right)\left(\int \psi \, \mathrm{d}E\right).$$

□

Corollary 8.2.2. *Let X be a compact Hausdorff space and let E be a spectral measure defined on the Borel subsets of X. Then $\rho : C(X) \to B(\mathscr{H})$ defined by $\rho(u) = \int u \, \mathrm{d}E$ is a representation of $C(X)$.*

It is clear now that we can make a representation that is expressed by a spectral measure. The following theorem tells us the converse of this, that all representations can be expressed using a spectral measure. But first we another proposition and an important theorem. Unfortunately the theorem is also called the Riesz Representation Theorem, even tough it says something totally different from the Riesz Representation Theorem that we saw earlier. For this reason, we will call it the second Riesz Representation Theorem.

Proposition 8.2.3. *[1, p.131. Proposition 4.1] If X is a normed space, then the unit ball in X is weak* dense in the unit ball in X^{**}.*

Theorem 8.2.4 (The Second Riesz Representation Theorem)**.** *[1, p.383] If X is a locally compact space and $\mu \in M(X)$, define $F_\mu : C_0(X) \to \mathbb{C}$ by*

$$F_\mu(f) = \int f \mathrm{d}\mu.$$

Then $F_\mu \in C_0(X)^*$ and the map $\mu \mapsto F_\mu$ is an isometric isomorphism of $M(X)$ onto $C_0(X)^*$.

Here $M(X)$ is the space of all complex valued regular Borel measures on X. Observe that this is indeed a vector space over $\mathbb{C}$ with the total variation as norm. Also, $C_0(X)$ is the space of all continuous functions $f : X \to \mathbb{C}$ such that the space $K = \{x \in X \mid |f(x)| > \epsilon\}$ is compact for all $\epsilon > 0$.

Theorem 8.2.5. *[1, p.259] If $\rho : C(X) \to B(\mathscr{H})$ is a representation, then there is a unique spectral measure E defined on the Borel σ-algebra of X such that for all $g, h \in \mathscr{H}$, $E_{g,h}$ is a regular measure and*

$$\rho(u) = \int u \, \mathrm{d}E$$

for all $u \in C(X)$.

Proof. If $g, h \in \mathscr{H}$, then $u \mapsto \langle \rho(u)g, h\rangle$ is a linear functional on $C(X)$ with norm $\leq \|g\|\|h\|$. So by the second Riesz Representation Theorem there is a unique measure $\mu_{g,h}$ in $M(X)$ such that

$$\langle \rho(u)g, h\rangle = \int u \mathrm{d}\mu_{g,h}$$

for all $u \in \mathbb{C}(X)$. We can check that the map $(g, h) \mapsto \mu_{g,h}$ is sesquilinear using the uniqueness of $\mu_{g,h}$. and we can also see that $\|\mu_{g,h}\| \leq \|g\|\|h\|$. Let Ω be the Borel σ-algebra of X. Fix $\phi \in B(X, \Omega)$ and define $[g, h] = \int \phi \mathrm{d}\mu_{g,h}$. Then we can check that $[\cdot, \cdot]$ is a sesquilinear map and that $|[g, h]| \leq \|\phi\|\|g\|\|h\|$. So by Theorem 8.1.12 there is a unique bounded operator A such that $[g, h] = \langle Ag, h\rangle$ with $\|a\| \leq \|\phi\|$. Let us denote this operator by $\tilde{\rho}(\phi)$. Now $\tilde{\rho} : B(X, \Omega) \to B(\mathscr{H})$ is a well-defined function with $\|\tilde{\rho}(\phi)\| \leq \|\phi\|$ and for all $g, h \in \mathscr{H}$,

$$\langle \tilde{\rho}(\phi)g, h\rangle = \int \phi \mathrm{d}\mu_{g,h}.$$

Now observe that $\tilde{\rho}(u)|_{C(X)} = \rho$. We would now like to prove that $\tilde{\rho} : B(X, \Omega) \to B(\mathscr{H})$ is a representation. If $\phi \in B(X, \Omega)$, we can consider it as an element in $M(X)^*$ $(= C(X)^{**})$ in the sense that ϕ corresponds to the linear functional $\mu \mapsto \int \phi \mathrm{d}\mu_{g,h}$. By Proposition 8.2.3 we find that $\{u \in C(X) \mid \|u\| \le \|\phi\|\}$ is weak* dense in $\{L \in M(X)^* \mid \|L\| \le \|\phi\|\}$. So there is a net $\{u_i\} \subset C(X)$ such that $\|u_i\| \le \|\phi\|$ for all u_i and $\int u_i \mathrm{d}\mu \to \int \phi \mathrm{d}\mu$ for all $\mu \in M(X)$. Now let $\psi \in B(X, \Omega)$, then $\psi\mu \in M(X)$ whenever $\mu \in M(X)$. Hence $\int u_i \psi \mathrm{d}\mu \to \int \phi\psi \mathrm{d}\mu$ for all $\psi \in B(X, \Omega)$ and $\mu \in M(X)$. So we see that $\tilde{\rho}(u_i\psi) \to \tilde{\rho}(\phi\psi)$ in the weak operator topology for all $\psi \in B(X, \Omega)$. Hence if $\psi \in C(X)$, then

$$\tilde{\rho}(\phi\psi) = \lim_{WOT} \tilde{\rho}(u_i\psi) = \lim_{WOT} \rho(u_i)\rho(\psi) = \tilde{\rho}(\phi)\rho(\psi)$$

for all $\phi \in B(X, \Omega)$. Hence $\tilde{\rho}(u_i\psi) = \rho(u_i)\tilde{\rho}(\psi)$ for all $\psi \in B(X, \Omega)$ and $u_i \in C(X)$. Because $\tilde{\rho}(u_i) \to \rho(\phi)$ and $\tilde{\rho}(u_i\psi) \to \tilde{\rho}(\phi)\tilde{\rho}(\psi)$ in the weak operator topology we find that

$$\tilde{\rho}(\phi\psi) = \tilde{\rho}(\phi)\tilde{\rho}(\psi)$$

for all $\phi, \psi \in B(X, \Omega)$. Linearity is easy to see since we know how to express $\tilde{\rho}$ as an integral.

To see that $\tilde{\rho}(\phi)^* = \tilde{\rho}(\bar{\phi})$, we let $\{u_i\}$ be the net that we had earlier. If $\mu \in M(X)$, let $\bar{\mu}$ be the measure defined by $\bar{\mu}(\Delta) = \overline{\mu(\Delta)}$. Then $\rho(u_i) \to \tilde{\rho}(\phi)$ (WOT) and thus $\rho(u_i)^* \to \tilde{\rho}(\phi)^*$ (WOT). Now, since

$$\int \bar{u}_i \mathrm{d}\mu = \overline{\int u_i \mathrm{d}\bar{\mu}} \to \overline{\int \phi \mathrm{d}\bar{\mu}} = \int \bar{\phi} \mathrm{d}\mu$$

for all $\mu \in M(X)$, we find that $\rho(\bar{u}_i) \to \tilde{\rho}(\bar{\phi})$. But $\rho(u_i)^* = \rho(\bar{u}_i)$ since ρ is a *-homomorphism. So $\tilde{\rho}(\phi)^* = \tilde{\rho}(\bar{\phi})$ and $\tilde{\rho}$ is a representation of $B(X, \Omega)$.

Now define $E(\Delta) = \tilde{\rho}(\mathbb{1}_\Delta)$ for $\Delta \in \Omega$. We will prove this is a spectral measure. Since $\mathbb{1}_\Delta$ is a hermitian idempotent ($\mathbb{1}_\Delta^2 = \mathbb{1}_\Delta$) in $B(X, \Omega)$, we find that $E(\Delta)$ is a projection since $\tilde{\rho}$ is a representation. Since $\mathbb{1}_\emptyset = 0$ and $\mathbb{1}_X = 1$, we find that $E(\emptyset) = 0$ and $E(X) = 1$. Furthermore,

$$E(\Delta_1 \cap \Delta_2) = \tilde{\rho}(\mathbb{1}_{\Delta_1 \cap \Delta_2}) = \tilde{\rho}(\mathbb{1}_{\Delta_1}\mathbb{1}_{\Delta_2}) = \tilde{\rho}(\mathbb{1}_{\Delta_1})\tilde{\rho}(\mathbb{1}_{\Delta_2}) = E(\Delta_1)E(\Delta_2).$$

Now let $\{\Delta_n\}$ be a pairwise disjoint sequence of Borel sets and let $\Lambda_n = \bigcup_{k=n+1}^\infty \Delta_k$. By induction we can see that E is finitely additive. So if $h \in \mathscr{H}$, then

$$\begin{aligned}
\left\| E(\bigcup_{k=1}^\infty \Delta_k)h - \sum_{k=1}^n E(\Delta_k)h \right\|^2 &= \langle E(\Lambda_n)h, E(\Lambda_n)h \rangle \\
&= \langle E(\Lambda_n)h, h \rangle \\
&= \langle \tilde{\rho}(\mathbb{1}_{\Lambda_n})h, h \rangle \\
&= \int \mathbb{1}_{\Lambda_n} \mathrm{d}\mu_{h,h} \\
&= \sum_{k=n+1}^\infty \mu_{h,h}(\Delta_k)
\end{aligned}$$

which clearly goes to 0 for $n \to \infty$. Hence $E(\bigcup_{k=1}^\infty \Delta_k) = \sum_{k=1}^\infty E(\Delta_k)$ for disjoint sets. Thus E is a spectral measure.

We still need to show that $\rho(u) = \int u \, \mathrm{d}E$ for $u \in C(X)$. Fix $\phi \in B(X, \Omega)$ and let $\epsilon > 0$. If $\{\Delta_1, \ldots, \Delta_n\}$ is any Borel partition of X such that $\sup\{|\phi(x) - \phi(x')| \mid x, x' \in \Delta_k\} < \epsilon$ for all $1 \le k \le n$, then $\|\phi - \sum_{k=1}^n \phi(x_k)\mathbb{1}_{\Delta_k}\|_\infty \le \epsilon$ for any choice of $x_k \in \Delta_k$. Since $\|\tilde{\rho}\| = 1$, we see that $\|\tilde{\rho}(\phi) - \sum_{k=1}^n \phi(x_k)E(\Delta_k)\| \le \epsilon$. So $\tilde{\rho}(\phi) = \int \phi \, \mathrm{d}E$ for any $\phi \in B(X, \Omega)$. Thus $\rho(u) = \int u \, \mathrm{d}E$ for all $u \in C(X)$.

By the construction of E it is clear that E must be unique for all the measurable functions since we can just plug in any Borel subset and show the value is unique. But we want to show it is unique for just the continuous functions. Let F be another spectral measure that meets the criteria of the theorem, then we know that $\int u \ \mathrm{d}E = \int u \ \mathrm{d}F$ for all $u \in C(X)$. If we let Δ be a Borel subset, then by Urysohn's Lemma we can approximate $\mathbb{1}_\Delta$ by a sequence of continuous functions f_n. Thus we find that

$$\int \mathbb{1}_\Delta \mathrm{d}E = \lim_{n\to\infty} \int f_n \ \mathrm{d}E = \lim_{n\to\infty} \int f_n \ \mathrm{d}F = \int \mathbb{1}_\Delta \mathrm{d}F$$

for any Borel subset Δ. So it follows that $F(\Delta) = E(\Delta)$ for all Borel subsets Δ. □

8.3 The Spectral Theorem

Now we can come to an important theorem in the theory of operators on Hilbert spaces. It shows us what the structure is of normal bounded operators. This theorem needs the Gelfand–Naimark theorem to be proven since it heavily depends on representations.

Theorem 8.3.1 (The Spectral Theorem)**.** *[1, p. 263] If N is a normal bounded operator on a Hilbert space $\mathscr{H}$, then there is a unique spectral measure on the Borel σ-algebra of $\sigma(N)$ such that the following hold.*

1. $N = \int z \ \mathrm{d}E(z)$.
2. *If G is a nonempty relatively open subset of $\sigma(N)$, then $E(G) \neq 0$.*
3. *If $A \in B(\mathscr{H})$, then $AN = NA$ and $AN^* = N^*A$ if and only if $AE(\Delta) = E(\Delta)A$ for every Borel subset Δ.*

Proof. Let $\mathscr{A} = C^*(N)$, then $\mathscr{A}$ is the closure of all polynomials in N and N^*. By Theorem 6.2.3, there is an isometric isomorphism $\rho : C(\sigma(N)) \to \mathscr{A} \subset B(\mathscr{H})$ that is given by $\rho(u) = u(N)$ (the functional calculus). By Theorem 8.2.5 there is a unique spectral measure E defined on the Borel subsets of $\sigma(N)$ such that $\rho(u) = \int u \ \mathrm{d}E$ for all $u \in C(\sigma(N))$. Hence, point 1 holds since $N = \rho(z)$.

If G is a nonempty relatively open subset of $\sigma(N)$, then there is a nonzero continuous function u on $\sigma(N)$ such that $0 \leq u \leq \mathbb{1}_G$. Now using a part of the proof from the previous theorem, we see that $E(G) = \tilde{\rho}(\mathbb{1}_G) \geq \rho(u) \neq 0$. Thus point 2 holds.

Now let $A \in B(\mathscr{H})$ such that $AN = NA$ and $AN^* = N^*A$. Then we extend this so that $p(N, N^*)A = Ap(N, N^*)$ for any polynomial $p(N, N^*)$ of two variables. So by the Stone–Weierstrass Theorem for these polynomials $p(N, N^*)$ we can see that $A\rho(u) = \rho(u)A$ for all $u \in C(\sigma(N))$. Hence $Au(N) = u(N)A$ for all $u \in C(\sigma(N))$. Now let

$$\Omega := \{\Delta \mid \Delta \text{ is a Borel set and } AE(\Delta) = E(\Delta)A\}$$

then we can show that Ω is a σ-algebra. If G is an open set in $\sigma(N)$, then there is a sequence $\{u_n\}$ of positive continuous functions on $\sigma(N)$ such that $u_n(z) \to \mathbb{1}_G(z)$ from below for all $z \in \sigma(N)$. Thus

$$\begin{aligned}
\langle AE(G)g, h\rangle &= \langle E(G)g, A^*h\rangle \\
&= E_{g,A^*h}(G) \\
&= \lim \int u_n \ \mathrm{d}E_{g,A^*h} \\
&= \lim \langle u_n(N)g, A^*h\rangle \\
&= \lim \langle Au_n(N)g, h\rangle \\
&= \lim \langle u_n(N)Ag, h\rangle \\
&= \langle E(G)Ag, h\rangle.
\end{aligned}$$

So Ω contains any open set and thus it is the whole Borel set of $\sigma(N)$.

If $AE(\Delta) = E(\Delta)A$ for all Δ in the Borel subsets, then $AN = A\int z \, \mathrm{d}E = \int z \, \mathrm{d}E \; A = NA$ and $AN^* = A\int \bar{z} \, \mathrm{d}E = \int \bar{z} \, \mathrm{d}E \; A = N^*A$. Thus point 3 holds.

Lastly we need to show the uniqueness of E. We saw earlier in the proof that E is unique for the condition that $\rho(u) = \int u \, \mathrm{d}E$ for all $u \in C(\sigma(N))$. But this is not yet enough. If we assume that F is another spectral measure such that $N = \int z \, \mathrm{d}F(z)$, then we can easily check that $\int p(z, \bar{z}) \, \mathrm{d}E = \int p(z, \bar{z}) \, \mathrm{d}F$ for any polynomial in two variables. So by the Stone–Weierstrass Theorem we find that

$$\int u \, \mathrm{d}E = \int u \, \mathrm{d}F$$

for all $u \in C(\sigma(N))$. Since we know that E must be unique for $C(\sigma(N))$, it follows that $F = E$. $\square$

We call this spectral measure E the "spectral measure for N".

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