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A Discretization Procedure in Finite Horizon Optimal Stopping Tested on AR(1) Processes.

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1 Introduction

In this research, optimal stopping problems are investigated. We will briefly and imprecisely introduce this subject here, for a formal introduction to optimal stopping, see Section 3 and Section 4.

In optimal stopping, the objective is to find an optimal stopping time, that is to say, the optimal strategy to stop a stochastic process. A stopping time is a random variable, i.e. a function on Ω which for each "state of nature" $\omega \in \Omega$ has a value which is a point in time. Loosely speaking, a stopping time has to satisfy that this time only depends on events that are observable at that time and as such does not depend on future events. A stopping time may be required to be smaller than a fixed and finite time, in which case we say we have a finite horizon stopping problem. The stopping times are ranked by the expected value of the stochastic process at the time the stopping time indicates. A stopping time which maximises this expectation is called an optimal stopping time.

Another important concept in optimal stopping is the value function. For each time, the value function is the function that maps a value of the stochastic process to the expectation of the stochastic process started at this value and stopped by the stopping time. This function is called the value function.

The theory of optimal stopping has many applications. In this research, we will highlight one particular area of applications, which is that of demand-side management (DSM), which is defined as follows. DSM is the exertion of influence on consumers of electrical power to change their demand. DSM is also called energy demand management.

In any power grid, supply and demand of electrical power have to be well matched. Otherwise, at the consumer level, appliances will not work correctly. In addition, the capacity of power grids to act as a buffer is very limited, complicating matters further. Traditionally the power suppliers have matched their supply to the demand of consumers. However, as our economy and technology changes, consumers may also get involved in the matching of supply and demand. Depending on how much control the consumer retains over his consumption, this is either called demand response or dynamic response.

One way to influence demand is to charge consumers variable electricity prices, depending for example on supply. Certain loads can be shifted in time, which allows a consumer to reduce the amount he has to pay for electricity, without consuming less power. An example is that the washing machine of a residential consumer can be turned on when the price is low. In certain scenarios, finding the optimal time to consume can be formulated as an optimal stopping problem. The relevant problems in DSM, demand response and dynamic response,

as well as their relation to the theory treated in this research will introduced in Section 2.

The price process used to influence demand may have a complicated structure. To an extent, we can only guess what the structure will be, because scenarios in which consumers pay variable electricity prices are not widely adopted. We see in current adoptions that the price can only change every 15 minutes or so. This time span is not short enough to justify modelling the price process with continuous time stochastic processes. Furthermore we see that the price can take a large range of values. We conclude that it is justified to model the price process with a continuous state stochastic process.

In Section 6, we introduce the discretization procedure. The input of this procedure is data of a discrete time stochastic process and the output of the procedure is a stopping strategy that is an approximation of the optimal stopping time of stopping the stochastic process with a finite horizon. The reason we refer to it as the discretization procedure is that it internally uses a stochastic model of original process that generates the data and that this model has finitely many states, i.e. it is discrete, while the process generating the data is not assumed to be discrete. Next, Utilities are associated with each state of this model and a policy is found that maximises the expected utility. This policy can be interpreted as a stopping time for the original problem, which is main result or output of this procedure.

We discussed that it is difficult to make a good model for the price process and the discretization procedure makes very few assumptions about the process that generates its input data. It turns out that a particular stochastic process is often used in literature to model the price process, which is the AR(1) process. In [12], Löhndorf says: "In line with literature, we assume that the stochastic processes of price and supply follow an Autoregressive (AR) process." In [7], Gonsch also takes the price process to be an AR(1) process. We can apply the discretization procedure to this particular case and see how well it performs. If it performs well for this case, the fact that the AR(1) is seen as a good model in literature then gives us confidence that the discretization procedure will perform well for realistic models.

An AR(1) process $(X_n)_{n=1}^N$ is a stochastic process that follows the equation

$$X_{n+1} = bX_n + \epsilon_n \tag{1.1}$$

Where the ϵ_n are i.i.d. In this research, we shall also assume that $0 < b < 1$ and that $\mathbb{E}[\epsilon_n] = 0$.

It turns out that under these assumptions, the optimal stopping strategy of stopping the AR(1) process (X_n) has a simple structure. There exists a sequence (d_n) such that at each time n, it is optimal to stop if $X_n \geq d_n$ and to continue otherwise. We say that the optimal stopping problem has a boundary structure, which corresponds to our Definition 4.22. The fact that we have this structure is the statement of Corollary 7.14.

Now suppose we let the input of our discretization procedure be data of an AR(1) process. As we will see in Subsection 6.5, empirical evidence suggests that as the size of our sample data goes to infinity, the probability that the stopping time the discretization procedure finds has a boundary structure goes to one. We consider the discretization procedure to "perform well" for the AR(1) process if it approximates the optimal value functions and the boundary well. Here we care most about consistency of the approximation.

1.1 Research questions

Broad question: "What problems exist in DSM that can be solved using the theory of finite horizon discrete time optimal stopping, how can we reasonably model the price process that a consumer faces who is charged variable prices for electricity, how can we (approximately) solve the resulting finite horizon optimal stopping problems and how can we evaluate the quality of an approximate solution?"

Main research question: How well does the discretization procedure perform for an AR(1) process?

Subquestion: What can we say about the structure of the value functions of an AR(1) process?

1.2 Outline

Problems in demand-side management, demand response and dynamic response that can be formulated as finite horizon optimal stopping problems are introduced in Section 2. Section 3 provides formal definitions of optimal stopping in discrete time, as well as a useful theorem. Section 4 provides definitions and a theorem for the case where the process that we need to stop is a Markov Process. Section 5 is an example of how the theory of Section 3 can be applied, in a way that is similar to how we will apply the theory in later chapters. In Section 6, we introduce the discretization procedure. In Section 7, first AR(1) processes are introduced and basic properties of their value functions are derived. In Subsection 7.1, structural properties of the value functions of AR(1) processes are derived. In Subsection 7.3 a procedure is introduced that is tailored to the case where the process to be stopped is an AR(1) process, to find approximate

solutions for the problem of optimally stopping an AR(1) process with a finite horizon. This procedure shall be referred to as the AR(1) approximation procedure. In Section 8 the results of the discretization procedure when applied to data from an AR(1) process and the results of the AR(1) approximation procedure are compared.

2 Optimal Stopping in Demand-Side Management

One economical and technological change causing demand-side management (DSM) to be more attractive is the shift towards renewable energy resources we see today. Wind and solar electricity production are both volatile, in the sense that it is hard to predict when and how much production will occur, which has a negative effect on the balance of the network. The production of solar power is concentrated around noon, whereas the production of wind power is spread out over the day. In some sense, this makes solar power relatively more predictable than wind power, but it also introduces a problem. Even if we would know perfectly how much solar power production would occur, such a concentrated peak of production around noon does not correspond to a peak in demand. It is costly to make other power plants temporarily downscale their production to compensate for this. Therefore, an increase in wind and solar power production puts pressure on the network balance, so that it is natural to look to DSM as an additional method of balancing the network.

Figure 1: Solar electricity production is volatile. It is hard to predict when and how much production wil occur.

Another technological development causing DSM to be more feasible and enforceable is the development of the smart meter. The smart meter comprises of a computer and a sensor capable of measuring electrical consumption of, for example, a household in real time. This creates the possibility to provide consumers with monetary incentives to change their demand. If such information is available, an electricity supplier can charge different prices for electricity at different moments. An alternative way to provide a monetary incentive is provided in [11], where software agents trade directly on a market for electricity on behalf of the consumer.

Smart meters become more widely available at a fast rate, for example in Italy about 85% of all households have one [6]. Furthermore, EU directives assume that in 2020, 80% of all consumers will have a smart meter [4] [5].

It is common in the Netherlands for a residential consumer to have a contract with an electricity supplier that is such that the prices for electricity are relatively low in the evening and at night and high during the day. This can be

Figure 2: The smart meter comprises of a computer and a sensor for measuring electrical consumption.

seen as a first step towards more variable electricity prices.

2.1 Relevance of this research to DSM

The only cases in demand-side management that we are aware of in which consumption problems that can neatly be formulated as optimal stopping problems are cases in which the consumer is a residential consumer. The consumer appliance that we are most interested in is the washing machine. If a consumer sees flexible energy prices, he will want to turn on his washing machine when the price is low. The washing machine may be controlled by a computer to help the consumer time his consumption. The consumer may instruct this computer to finish washing before a certain time. In this research, we explore the mathematical theory needed to design software that can perform this task.

Both the cases where the dynamics of the price process are known and unknown are investigated. In the case where the dynamics of the price process are known, we will focus on a very specific price process, which is the AR(1) process. A procedure to handle the case where these dynamics are not known is introduced in Section 6.

3 Definitions and Basic Theory of Optimal stopping

In this section we will give a basic and detailed introduction to finite horizon optimal stopping. Concepts from probability theory that are relevant and not entirely basic are also included. The motivation for including this introduction is that our research questions are formulated in terms optimal stopping times and other concepts that will be properly introduced here. Not many examples are provided, nor do we discuss the theory at length, as the theory is relatively well known and there are many books in which the theory is treated, for example [15]. We only consider optimal stopping problems where the horizon is finite and time is discrete, as that is all we need in other sections.

3.0 Basic definitions

The content in this subsection corresponds to Chapter 9 in [21].

Recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$.

Definition 3.1 (Stochastic process)**.**
Let $(\Omega, \mathcal{F})$ and $(E, \mathscr{E})$ be measurable spaces. Let $(X_n)_{n=0}^{\infty}$ be a sequence of random elements on Ω with values in E. We say that (X_n) is a stochastic process (on $(\Omega, \mathcal{F})$, with values in E). ■

Definition 3.2 (Filtration)**.**
Let $\mathcal{F}$ be a σ-algebra. A filtration on $\mathcal{F}$ is sequence $(\mathcal{F}_k)_{k=0}^{\infty}$ of sub σ-algebras of $\mathcal{F}$, such that for all $k, n \in \mathbb{N}$ with $k \leq n$, it holds that $\mathcal{F}_k \subset \mathcal{F}_n$. ■

Definition 3.3 (Filtered probability space)**.**
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration on $\mathcal{F}$. We say that $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ is a filtered probability space. ■

Definition 3.4 (Adapted process)**.**
Let $(X_n)_{n=0}^{\infty}$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration on $\mathcal{F}$. We say that (X_n) is an adapted process, if for all $n \in \mathbb{N}_0$, X_n is $\mathcal{F}_n$ measurable. ■

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space. We define a stopping time as follows.

Definition 3.5 (Stopping time)**.**
$\tau : \Omega \to \overline{\mathbb{N}}_0$ is a stopping time (w.r.t the filtration $(\mathcal{F}_n)_{n=0}^{\infty}$) iff

$$[\tau \leq n] \in \mathcal{F}_n \quad \forall n \in \mathbb{N}_0$$

■

Let $\mathfrak{M}$ be the set of all stopping times, i.e.

$$\mathfrak{M} = \{\tau : \Omega \to \overline{\mathbb{N}}_0 \,|\, \forall n \in \overline{\mathbb{N}}_0 \,:\, [\tau \leq n] \in \mathcal{F}_n\}$$

Note that in the definitions above we use the notation $[\tau \leq n]$ as the common shorthand notation for $\{\omega \in \Omega \,|\, \tau(\omega) \leq n\}$. Let us derive some basic facts about sets like these. If τ is a stopping time, then because $\mathcal{F}_n$ is a σ-algebra we have

$$[\tau > n] = [\tau \leq n]^c \in \mathcal{F}_n \tag{3.1}$$
$$[\tau = n] = [\tau \leq n] \setminus [\tau \leq n-1] \in \mathcal{F}_n \tag{3.2}$$

We will also use the terminology that a particular property holds on a subset of Ω. An example would be that on $[\tau \geq n]$, we have $\tau \geq n-1$. By this we mean that for all $\omega \in [\tau \geq n]$, we have that $\tau(\omega) \geq n-1$, or more compactly, that $[\tau \geq n] \subset [\tau \geq n-1]$.

Note that by the definition of a stopping time above, stopping times can assume the value ∞. From now on, however, we will focus on the case where we restrict the stopping time to have a finite value. We shall have an $N \in \mathbb{N}$, which we shall call our horizon or deadline.

The following definition is a bit dry and formal and defines an object which we call a stopping problem. The "problem" here is to find the value associated with this object as defined later on in Definition 3.7 (and a stopping time that attains the supremum in this definition).

Definition 3.6 (Finite horizon stopping problem)**.**
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space, let $N \in \mathbb{N}$ and let $(G_n)_{n=0}^{N}$ be a stochastic process that is adapted to this filtered probability space, that satisfies

$$\mathbb{E}[\sup_{k \in \{0,\ldots,N\}} |G_n|\,] < \infty \tag{3.3}$$

We say that the 6-tuple $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{N}, \mathbb{P}, (G_n)_{n=0}^{N}, N)$ is a (finite horizon) stopping problem. The stochastic process G shall be referred to as the utility. ■

3.1 Finite horizon

In this subsection we will first lay out some general definitions corresponding to the resource [[13]: Peskir and Shiryaev 2006]. The main result of this subsection is Theorem 3.10 which corresponds to Theorem 1.2 in [13]. The proof we present here is also an adaptation of the proof in that resource.

Throughout this subsection, let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^N, N)$ be a finite horizon stopping problem. For $n \in \{0, \ldots, N\}$ we define the following subsets of $\mathfrak{M}$.

$$\mathfrak{M}_n^N = \{\tau \in \mathfrak{M} \,|\, n \leq \tau \leq N\} \tag{3.4}$$

In this text we shall frequently encounter G_τ and other stochastic processes indexed at a stopping time. For a stochastic process Y and a finite stopping time $\tau \in \mathfrak{M}_0^N$, we define Y_τ to be the function $\omega \mapsto Y_{\tau(\omega)}(\omega)$. If Y is adapted, we find that Y_τ is $\mathcal{F}_N$ measurable. To see this, write $Y_\tau = \sum_{i=0}^N \mathbb{1}_{[\tau=i]} Y_i$. Because the underlying sets of the indicator functions are $\mathcal{F}_N$ measurable by property (3.2), all the functions that occur in this sum are $\mathcal{F}_N$ measurable. $\mathcal{F}_N$ measurability of Y_τ then follows by the fact that measurable functions for a vector space (i.e. Proposition 3.5 (i) in [21]). Note that this also justifies calling this function a random variable.

Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^N, N)$ is a stopping problem. We have, using condition (3.3), that for all $\tau \in \mathfrak{M}_0^N$

$$\mathbb{E}[|G_\tau|] = \mathbb{E}[|\sum_{k=n-1}^{N} \mathbb{1}_{\{\tau=k\}} G_k|] \leq \mathbb{E}[\sup_{k \in \{0,\ldots,N\}} |G_k|] < \infty$$

This justifies making the following definition.

Definition 3.7 (Value of a stopping problem)**.**
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^N, N)$ be a finite horizon stopping problem. We define the value W^N of this problem to be

$$W^N = \sup_{\tau \in \mathfrak{M}_0^N} \mathbb{E}[G_\tau] = \sup_{\tau \in \mathfrak{M}_0^N} \int G_{\tau(\omega)}(\omega) \mathbb{P}(d\omega) \tag{3.5}$$

and we define the subproblem value sequence $(W_n^N)_{n=0}^N$ to be the sequence defined by

$$W_n^N = \sup_{\tau \in \mathfrak{M}_n^N} \mathbb{E}[G_\tau] \tag{3.6}$$

■

Definition 3.8 (Snell envelope)**.**
Let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^\infty, N)$ be a finite horizon stopping problem. We define the Snell envelope of $\mathfrak{A}$ to be the stochastic process $(S_n^N)_{n=0}^N$ that satisfies the following recursion

$$S_N^N = G_N \tag{3.7}$$

$$S_n^N = \max(G_n, \mathbb{E}[S_{n+1}^N | \mathcal{F}_n]) \tag{3.8}$$

■

We see that (S_n^N) is well defined as follows. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^\infty, N)$ be a finite horizon stopping problem and let $(S_n^N)_{n=0}^N$ be the Snell envelope. Then using condition (3.3), we see that $\mathbb{E}[|S_N^N|] = \mathbb{E}[|G_N|] < \infty$. Now suppose that for some $n \in \{0, \ldots, N-1\}$, $\mathbb{E}[|S_{n+1}^N|] < \infty$. Then, again using condition (3.3)

$$\begin{aligned}\mathbb{E}[|S_n^N|] &= \mathbb{E}[|\max(G_n, \mathbb{E}[S_{n+1}^N|\mathcal{F}_n])|] \\ &\leq \mathbb{E}[\max(|G_n|, |\mathbb{E}[S_{n+1}^N|\mathcal{F}_n]|)] \\ &\leq \mathbb{E}[\max(|G_n|, \mathbb{E}[|S_{n+1}^N||\mathcal{F}_n])] < \infty\end{aligned}$$

It follows by induction that that for all $n \in \{0, \ldots, N\}$, $\mathbb{E}[|S_n^N|] < \infty$.

Throughout the rest of this section, let $(W_n^N)_{n=0}^N$ be the subproblem values sequence of $\mathfrak{A}$ and let $(S_n^N)_{n=0}^N$ be the Snell envelope of $\mathfrak{A}$.

G_N is $\mathcal{F}_N$ measurable because G is adapted, so that S_N^N is $\mathcal{F}_N$ measurable. Let $n \in \{0, \ldots, N-1\}$, we have that G_n is $\mathcal{F}_n$ measurable and, by the definition of conditional expectation (Definition 8.1 in [21]), $\mathbb{E}[S_{n+1}^N|\mathcal{F}_n]$ is $\mathcal{F}_n$ measurable. The maximum of two measurable functions is measurable (by Proposition 8.5 in [21]), so S_n is $\mathcal{F}_n$ measurable. We conclude that (S_n^N) is an adapted process.

The name "earliest optimal stopping time sequence" in the definition below is justified by Theorem 10.1.

Definition 3.9 (Earliest optimal stopping time sequence)**.**
Let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^N, N)$ be a finite horizon stopping problem. We define the earliest optimal stopping time sequence of $\mathfrak{A}$ to be the sequence of stopping times $(\tau_n^N) \subset \mathfrak{M}_n^N$ that satisfies that for all $n \in \{0, \ldots N\}$

$$\tau_n^N = \inf\{k \in \{0, \ldots, N\} \,|\, S_k^N = G_k\} \tag{3.9}$$

We say that τ_0^N is the earliest optimal stopping time. ■

To see that τ_n^N is well defined, note that $S_N^N = G_N$, so that on all of Ω the set inside the infimum has at least one element.

We shall now show that, for all $n \in \{0, \ldots, N\}$, τ_n^N is indeed a stopping time. Let $m \in \mathbb{N}_0$ be arbitrary. We have

$$[\tau_n^N \leq m] = \bigcup_{l=0}^{m} [\tau_n^N = l] \tag{3.10}$$

If $l \in \{0, \ldots, n-1\} \cup \{N+1 \ldots m\}$, then $[\tau_n^N = l] = \emptyset$. By manipulating its right hand side, the equation above then becomes

$$
\begin{aligned}
[\tau_n^N \leq m] &= \bigcup_{l=n}^{m\wedge N} [\tau_n^N = l] \\
&= \bigcup_{l=n}^{m\wedge N} [l = \inf\{n \leq k \leq N | G_k = S_k^N\}] \\
&= \bigcup_{l=n}^{m\wedge N} ([G_l = S_l^N] \cap \bigcap_{k=n}^{l-1} [G_k \neq S_k^N])
\end{aligned}
$$

Now let $Z_l = G_l - S_l^N$. Because of Proposition 3.5 (i) of [21] and the fact that (G_k) and (S_k^N) are adapted, Z_l is $\mathcal{F}_l$ measurable. The equation above becomes

$$
[\tau_n^N \leq m] = \bigcup_{l=n}^{m\wedge N} (Z_l^{-1}(\{0\}) \cap \bigcap_{k=n}^{l-1} Z_k^{-1}(\mathbb{R} \setminus \{0\})) \tag{3.11}
$$

Because, for all $l \in \{n, \ldots, m \vee N\}$, Z_l is $\mathcal{F}_m$ measurable and $\{0\} \in \mathcal{B}(\mathbb{R})$, we have $Z_l^{-1}(\{0\}) \in \mathcal{F}_m$ and, for all $k \in \{n, \ldots, l-1\}$, $Z_k^{-1}(\mathbb{R} \setminus \{0\})) \in \mathcal{F}_m$. So the right hand side of (3.11) is a union of intersections of elements of $\mathcal{F}_m$, which implies that it is itself an element of $\mathcal{F}_m$. Therefore $[\tau_n^N \leq m] \in \mathcal{F}_m$, which shows that τ_n^N is indeed a stopping time.

We now present a theorem which corresponds to Theorem 1.2 in [13]. The proof we present here is also an adaptation of the proof in that resource.

Theorem 3.10 (**Finite horizon**).
Let $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbb{P}, (G_n)_{n=0}^N, N)$ be a finite horizon stopping problem. Let (W_n^N) be the subproblem value sequence of $\mathfrak{A}$, let (S_n^N) be the Snell envelope of $\mathfrak{A}$ and let (τ_n^N) be the earliest optimal stopping time sequence of $\mathfrak{A}$. We have, for all $n \in \{0, \ldots N\}$

$$
S_n^N \geq \mathbb{E}[G_\tau | \mathcal{F}_n] \qquad \forall \tau \in \mathfrak{M}_n^N \tag{3.12}
$$

$$
S_n^N = \mathbb{E}[G_{\tau_n^N} | \mathcal{F}_n] \tag{3.13}
$$

$$
W_n^N = \mathbb{E}[G_{\tau_n^N}] \tag{3.14}
$$

■

Proof.
We will prove the equations (3.12) and (3.13), using two separate proofs by induction on n. Equation (3.14) follows as a result.

First we present two equations that are used in both the proofs of (3.12) and (3.13).

We have $S_N^N = G_N$ by (3.7). Furthermore, for all $\tau \in \mathfrak{M}_N^N$, $\tau = N$. Lastly, we have that $G_N = \mathbb{E}[G_N|\mathcal{F}_N]$, because G_N is $\mathcal{F}_N$ measurable. Combining these facts, we find that for all $\tau \in \mathfrak{M}_N^N$, we have

$$S_N^N = G_N = \mathbb{E}[G_N|\mathcal{F}_N] = \mathbb{E}[G_\tau|\mathcal{F}_N] \tag{3.15}$$

To derive the second equation, let $n \in \{1, \ldots, N\}$ and let $\tau \in \mathfrak{M}_{n-1}^N$. Note that, because $\tau \geq n-1$, we have

$$G_\tau = \mathbb{1}_{[\tau=n-1]}G_{n-1} + \mathbb{1}_{[\tau\geq n]}G_\tau$$

We then have (by Proposition 8.5 (iii) in [21])

$$\mathbb{E}[G_\tau|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{1}_{[\tau=n-1]}G_{n-1}|\mathcal{F}_{n-1}] + \mathbb{E}[\mathbb{1}_{[\tau\geq n]}G_\tau|\mathcal{F}_{n-1}] \tag{3.16}$$

We write $\tau \vee n = \max(\tau, n)$. Now note that on $[\tau \geq n]$, we have $G_\tau = G_{\tau\vee n}$, which gives us that on all of Ω, we have $\mathbb{1}_{[\tau\geq n]}G_\tau = \mathbb{1}_{[\tau\geq n]}G_{\tau\vee n}$. Using this, the last expression (3.16) above becomes

$$\mathbb{E}[G_\tau|\mathcal{F}_{n-1}] = \mathbb{E}[\mathbb{1}_{[\tau=n-1]}G_{n-1}|\mathcal{F}_{n-1}] + \mathbb{E}[\mathbb{1}_{[\tau\geq n]}G_{\tau\vee n}|\mathcal{F}_{n-1}] \tag{3.17}$$

Now note that the indicator functions in the equation above are $\mathcal{F}_{n-1}$ measurable, because the underlying sets are $\mathcal{F}_{n-1}$ measurable, because of results (3.1) and (3.2). This gives us, by Theorem 8.7 (ii) in [21], that the last expression above equals

$$\mathbb{E}[G_\tau|\mathcal{F}_{n-1}] = \mathbb{1}_{[\tau=n-1]}\mathbb{E}[G_{n-1}|\mathcal{F}_{n-1}] + \mathbb{1}_{[\tau\geq n]}\mathbb{E}[G_{\tau\vee n}|\mathcal{F}_{n-1}] \tag{3.18}$$

Now, by the tower property, i.e. Theorem 8.7 (i) of [21], the last expression in (3.18) above, equals

$$\mathbb{E}[G_\tau|\mathcal{F}_{n-1}] = \mathbb{1}_{[\tau=n-1]}\mathbb{E}[G_{n-1}|\mathcal{F}_{n-1}] + \mathbb{1}_{[\tau\geq n]}\mathbb{E}[\,\mathbb{E}[G_{\tau\vee n}|\mathcal{F}_n]\,|\,\mathcal{F}_{n-1}] \tag{3.19}$$

This is the second equation we use in both proofs.

Now we will prove (3.12) by induction on $n \in \{0, \ldots N\}$, starting with $n = N$ and letting n decrease. The induction hypothesis itself is (3.12). We see that (3.12) holds for $n = N$, because of equation (3.15). Now we prove that if (3.12) holds for one particular $n \in \{1, \ldots N\}$, then it holds for $n-1$.

Assume that (3.12) holds for $n \in \{1, \ldots N\}$. We have, for any $\tau \in \mathfrak{M}_{n-1}^N$,

$$S_{n-1}^N = \mathbb{1}_{[\tau=n-1]}S_{n-1}^N + \mathbb{1}_{[\tau\geq n]}S_{n-1}^N$$

Now by the definition (3.8) of S_n^N, because the maximum of a set is at least as large as each element, we obtain

$$S_{n-1}^N \geq \mathbb{1}_{[\tau=n-1]} G_{n-1}^N + \mathbb{1}_{[\tau \geq n]} \mathbb{E}[S_n^N | \mathcal{F}_{n-1}]$$

Recall the Definition 3.5 of a stopping time. On all of Ω, we have that $n \leq \tau \vee n \leq N$. For $k \geq n$, we have $[\tau \vee n \leq k] = [\tau \leq n] \in \mathcal{F}_n$. For $k < n$ we have $[\tau \vee n \leq k] = \emptyset \in \mathcal{F}_n$. Therefore $\tau \vee n \in \mathfrak{M}_n^N$. By the induction hypothesis (3.12) and Proposition 8.5 (i) in [21] we then obtain

$$S_{n-1}^N \geq \mathbb{1}_{[\tau=n-1]} G_{n-1}^N + \mathbb{1}_{[\tau \geq n]} \mathbb{E}[\mathbb{E}[G_{\tau \vee n} | \mathcal{F}_n] | \mathcal{F}_{n-1}]$$

Now (3.19) gives us that the RHS above equals

$$S_{n-1}^N \geq \mathbb{E}[G_\tau | \mathcal{F}_{n-1}]$$

This shows that (3.12) holds for $n-1$. We conclude that (3.12) holds for all $n \in \{0, \ldots, N\}$.

We will now prove (3.13) in a similar fashion, by induction on $n \in \{0, \ldots, N\}$, starting with $n = N$ and letting n decrease. Again, the induction hypothesis itself is (3.13). Again, we see that (3.13) holds for $n = N$, because of equation (3.15). Now we prove that if (3.13) holds for one particular $n \in \{1, \ldots, N\}$, then it holds for $n-1$.

Assume that (3.13) holds for $n \in \{1, \ldots, N\}$. We have

$$S_{n-1}^N = \mathbb{1}_{[\tau_{n-1}^N = n-1]} S_{n-1}^N + \mathbb{1}_{[\tau_{n-1}^N \geq n]} S_{n-1}^N$$

By the definition (3.9) of τ_n^N, we have on $[\tau_{n-1}^N = n-1]$ that $S_{n-1}^N = G_{n-1}$. Similarly, on $[\tau_{n-1}^N \geq n]$, we have $S_{n-1}^N \neq G_{n-1}$, so that here we have $S_{n-1}^N = \max(G_{n-1}, \mathbb{E}[S_n^N | \mathcal{F}_{n-1}]) = \mathbb{E}[S_n^N | \mathcal{F}_{n-1}]$. This gives us that the last expression above equals

$$S_{n-1}^N = \mathbb{1}_{[\tau_{n-1}^N = n-1]} G_{n-1}^N + \mathbb{1}_{[\tau_{n-1}^N \geq n]} \mathbb{E}[S_n^N | \mathcal{F}_{n-1}]$$

By the induction hypothesis (3.13), this equals

$$S_{n-1}^N = \mathbb{1}_{[\tau_{n-1}^N = n-1]} G_{n-1}^N + \mathbb{1}_{[\tau_{n-1}^N \geq n]} \mathbb{E}[\mathbb{E}[G_{\tau_n^N} | \mathcal{F}_n] | \mathcal{F}_{n-1}]$$

We see that, because τ_{n-1}^N is a stopping time, $[\tau_{n-1}^N \geq n] = [\tau_{n-1}^N \leq n-1]^c \in \mathcal{F}_{n-1}$. Therefore, $\mathbb{1}_{[\tau_{n-1}^N \geq n]}$ is both $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$ measurable. We then have (by Theorem 8.7 (ii) in [21]) that the RHS above equals

$$S_{n-1}^N = \mathbb{1}_{[\tau_{n-1}^N \geq n-1]} G_{n-1}^N + \mathbb{E}[\mathbb{E}[\mathbb{1}_{[\tau_{n-1}^N \geq n]} G_{\tau_n^N} | \mathcal{F}_n] | \mathcal{F}_{n-1}]$$

We see that on $[\tau_{n-1}^N \geq n]$, we have $\tau_n^N = \tau_{n-1}^N \vee n$, by the definition of τ_n^N. Therefore, on all of Ω, we have $\mathbb{1}_{[\tau_{n-1}^N \geq n]}\tau_n^N = \mathbb{1}_{[\tau_{n-1}^N \geq n]}\tau_{n-1}^N \vee n$. The RHS above then equals (using Theorem 8.7 (ii) in [21] again)

$$S_{n-1}^N = \mathbb{1}_{[\tau_{n-1}^N = n-1]}G_{n-1}^N + \mathbb{1}_{[\tau_{n-1}^N \geq n]}\mathbb{E}[\mathbb{E}[G_{\tau_{n-1}^N \vee n}|\mathcal{F}_n]|\mathcal{F}_{n-1}]$$

Equation (3.19) now gives us, because $\tau_{n-1}^N \in \mathfrak{M}_{n-1}^N$, that the RHS equals

$$S_{n-1}^N = \mathbb{E}[G_{\tau_{n-1}^N}|\mathcal{F}_{n-1}]$$

This shows that (3.13) holds for $n-1$. We conclude that (3.13) holds for all $n \in \{0, \ldots, N\}$.

Now we prove equation (3.14). For all $n \in \{0, \ldots, N\}$ and for all $\tau \in \mathfrak{M}_n^N$ inequality (3.12) holds. We can take expectations on both sides and the inequality will still hold, by Proposition 4.18 in [21], so that we obtain

$$\mathbb{E}[S_n^N] \geq \mathbb{E}[\mathbb{E}[G_\tau|\mathcal{F}_n]]$$

By the tower property, this becomes

$$\mathbb{E}[S_n^N] \geq \mathbb{E}[G_\tau] \tag{3.20}$$

We also have, that for all $n \in \{0, \ldots, N\}$ equation (3.13) holds. Taking expectations and using the tower property again, we find that

$$\mathbb{E}[S_n^N] = \mathbb{E}[G_{\tau_n^N}] \tag{3.21}$$

Combining (3.20) and (3.21) we find that for all $n \in \{1, \ldots, N\}$ and for all $\tau \in \mathfrak{M}_n^N$

$$\mathbb{E}[G_{\tau_n^N}] \geq \mathbb{E}[G_\tau] \tag{3.22}$$

From this see that τ_n^N attains the supremum below, as $\tau_n^N \in \mathfrak{M}_n^N$.

$$\mathbb{E}[G_{\tau_n^N}] = \sup_{\tau \in \mathfrak{M}_n^N} \mathbb{E}[G_\tau] \tag{3.23}$$

This proves (3.14). □

4 Theory Markovian Case

This Section is based on Subsection 4.1 and Subsection 1.2 of [13].

The main result in this section, Theorem 4.18, corresponds to Theorem 1.7 of [13]. The proof presented here follows the proof in [13] very closely, but provides more details. We do not vouch for the "efficiency" of this proof and our proofs of the details seem to lead to some "back and forth" between recursive equations and properties of the Snell envelope. This may be an argument in favour of using a direct approach which does not use a theorem like Theorem 3.10, like the approach chosen in the proof of Theorem 1 of Section 2 of [19] .

Theorem 4.18 is only formulated for a special class of discrete time Markov Processes, which are canonical Markov processes. However, the theorem can be applied to general Markov processes, by converting the Markov process to a canonical Markov process. Another reason why it can be natural to work with canonical Markov processes, is that an initial distribution and a Markov kernel generate a canonical homogeneous Markov process.

4.0 Basic results and definitions

Definition 4.1. We say that a measurable space $(E, \mathscr{E})$ is a phase space iff

$$\forall e \in E : \{e\} \in \mathscr{E}$$

■

Definition 4.2 (Cylinder σ-algebra)**.**
Let $(E, \mathscr{E})$ be a measurable space. The cylinders of $(E, \mathscr{E})$ are the sets

$$\mathcal{C} = \{B_0 \times B_1 \times \cdots \times B_n \times \mathbb{R}^\infty \,|\, n \in \mathbb{N}, \forall i \in \{0, \ldots, n\} : \; B_i \in \mathscr{E}\} \tag{4.1}$$

The cylinder σ-algebra is then $\sigma(\mathcal{C})$. ■

To see that the following definition is correct in that the X_n are measurable, so that (X_n) is indeed a stochastic process, see Lemma 4.4.

Definition 4.3 (Canonical process)**.**
Let $(E, \mathscr{E})$ be a measurable space, let $\Omega = E^\infty$ and let $\mathcal{F}$ be the cylinder σ-algebra. We define the canonical process X on $(E, \mathscr{E})$ to be the sequence of $\mathcal{F}/\mathscr{E}$ random elements (X_n) defined by, for all $n \in \mathbb{N}_0$,

$$X_n : E^\infty \to E : X_n(\omega) = \omega_n \tag{4.2}$$

That is to say, X_n takes the n-th coordinate of any sequence ω. ■

Lemma 4.4. Let $(E, \mathscr{E})$ be a measurable space and let (X_n) be the canonical process on $(E, \mathscr{E})$. We have that for all $n \in \mathbb{N}_0$, X_n is $\mathcal{F}/\mathscr{E}$ measurable.

Proof. Let $B \in \mathscr{E}$

$$\begin{aligned} X_n^{-1}(B) &= \{\omega \in \Omega \,|\, X_n(\omega) \in B\} = \{\omega \in \Omega \,|\, \omega_n \in B\} \\ &= \mathbb{R}^{n-1} \times B \times \mathbb{R}^{\infty} \end{aligned} \tag{4.3}$$

So $X_n^{-1}(B)$ is a cylinder and therefore $X_n^{-1}(B) \in \mathcal{F}$. □

Below we define a discrete time Markov process. Equation (4.4) is called the Markov property. Note that equivalent definitions exist, in particular definitions that instead of (4.4) assume a so called generalized Markov property.

Definition 4.5 (Discrete Time Markov Process)**.**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space. We say that a discrete time stochastic process $X = (X_n)_{n=0}^{\infty}$ that is adapted to this filtration, is a discrete time Markov process on $(\mathcal{F}_n)_{n=0}^{\infty}$ (on this filtered space), with values in the phase space $(E, \mathscr{E})$, if the following property, which is called the Markov Property, is satisfied for all $B \in \mathscr{E}$

$$\mathbb{P}[X_{n+1} \in B | \mathcal{F}_n] = \mathbb{P}[X_{n+1} \in B | X_n] \tag{4.4}$$

■

Next we define a Markov kernel, which in a sense generalizes the concept of transition probability matrix. A Markov kernel is also called a transition function, or probability kernel.

Definition 4.6 (Markov kernel)**.**
A Markov kernel from a measurable space $(\Omega, \mathcal{F})$ to another measurable space $(E, \mathscr{E})$ is a function $\kappa : \Omega \times \mathscr{E} \to [0, 1]$ that satisfies

1. For all $A \in \mathcal{E}$, the function $\kappa_A : \Omega \to [0, 1] : x \mapsto \kappa(x, A)$ is $\mathcal{F}/\mathcal{B}([0, 1])$ measurable.
2. For all $x \in \Omega$, the function $\kappa_x : \mathscr{E} \to [0, 1] : B \mapsto \kappa(x, B)$ is a probability measure.

■

With each finite Markov chain, we can associate a sequence of probability matrices (P_n), such that $P_{n,i,j}$ gives us the probability that the Markov chain transitions from state i at time n to state j at time $n + 1$. The existence of Markov kernels which play the same role for the (not necessarily discrete space) discrete time Markov processes we consider here, is shown in Lemma 10.29. For each $n \in \mathbb{N}_0$ this lemma gives us a Markov kernel P_n, such that for all $B \in \mathscr{E}$, $P_n(X_n, B)$ is a version of $\mathbb{P}[X_{n+1} \in B | X_n]$. Note that if (X_n) is canonical, then $X_n(\Omega) = E \in \mathscr{E}$, so that this (perhaps somewhat peculiar) condition of Lemma 10.29 is satisfied.

Definition 4.7 (Homogeneous discrete time Markov process)**.**
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space and let $(E, \mathscr{E})$ be a phase space. Let (X_n) be a discrete time Markov process on this filtered space with values in $(E, \mathscr{E})$. We say that (X_n) is homogeneous if for all $n \in \mathbb{N}_0$ and for all $B \in \mathscr{E}$

$$\mathbb{P}[X_{n+1} \in B \,|\, X_n] = \mathbb{P}[X_1 \in B \,|\, X_0] \tag{4.5}$$

■

From this point onward, in particular in Subsection 4.1, we will only work with Markov processes (X_n) that are homogeneous. In this case, we only need to work with a single Markov kernel P, because for all $n \in \mathbb{N}_0$ and all $B \in \mathscr{E}$, if $P(X_0, B)$ is version of $\mathbb{P}[X_1 \in B \,|\, X_0]$, then it is also a version of $\mathbb{P}[X_{n+1} \in B \,|\, X_n]$.

Definition 4.8 (Markov kernel of a homogeneous Markov process)**.**
Let $(E, \mathscr{E})$ be a phase space and let (X_n) be a discrete time canonical homogeneous Markov process on $(E, \mathscr{E})$. We say (X_n) has Markov kernel P, or that P is a Markov kernel of (X_n) iff P is a Markov kernel such that for all $B \in \mathscr{E}$, $P(X_0, B)$ is version of $\mathbb{P}[X_1 \in B \,|\, X_0]$. ■

Note that in a sense there are still many stochastic processes that form a Markov process with a particular Markov kernel. We can make this class smaller by specifying an initial distribution π, which is a measure on the phase space $(E, \mathscr{E})$, so that we require of our process (X_n) that for all $B \in \mathscr{E}$, $\mathbb{P}[X_0 \in B] = \pi(B)$. It is stated in [13] that the initial distribution π on $(E, \mathscr{E})$ and the Markov kernel P fix the entire distribution of the process (X_n) and we can speak of a Markov process on $(E, \mathscr{E})$ generated by (π, P). This is also stated in Corollary 3.2.2 of [20], but because the Kolmogorov consistency theorem is used, E needs to a Polish space.

4.1 Theorem for homogeneous Markov Processes

Throughout this subsection, let $(E, \mathscr{E})$ be a phase space. Let $\Omega = E^{\infty}$ and let $\mathcal{F}$ be the σ-algebra generated by the cylinders of $\mathscr{E}$. Let P be a Markov kernel, let X be the canonical process on $(E, \mathscr{E})$ and let $(\mathcal{F}_n)_{n=0}^{\infty}$ be the natural filtration of X. Let $\mathbb{P} = \{\mathbb{P}_x \,|\, x \in E\}$ be a family of probability measures so that for all $x \in E$, X is a homogeneous Markov process on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ with Markov kernel P and $\mathbb{P}_x[X_0 = x] = 1$.

For random variables Y, we shall write $\mathbb{E}_x[Y] = \int Y(\omega)\mathbb{P}_x(d\omega)$ and we stress that this makes an expression $\mathbb{E}_x[Y]$ implicitly depend on $\mathbb{P}_x$. We let $G : E \to \mathbb{R}$ be a function that satisfies, for all $x \in E$ and for all $N \in \mathbb{N}$

$$\mathbb{E}_x[\sup_{k \in \{0,\dots,N\}} |G(X_k)|] < \infty \tag{4.6}$$

Definition 4.9. Recall Definition 3.6 of a finite horizon stopping problem. Let $x \in E$ and $N \in \mathbb{N}_0$. We define $\mathfrak{A}_x^N$, the finite horizon stopping problem of starting at x with horizon N, to be

$$\mathfrak{A}_x^N = (\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P}_x, (G(X_n))_{n=0}^N, N) \tag{4.7}$$

■

Definition 4.10 (Value function)**.** Let $N \in \mathbb{N}_0$ and consider the family of stopping problems $(\mathfrak{A}_x^N)_{x\in E}$. We define that value function $V^N : E \to \mathbb{R}$ as follows. For all $x \in E$, we set $V^N(x)$ to be the value of the stopping problem $\mathfrak{A}_x^N$ in the sense of Definition 3.7. We have, for all $x \in E$.

$$V^N(x) = \sup_{0\le\tau\le N} \mathbb{E}_x[G(X_\tau)] \tag{4.8}$$

■

In the proof of the Theorem 4.18 and the statements of Lemma 4.12 and Lemma 4.13 we will use the shift operator θ, which is defined as follows.

Definition 4.11 (Shift operator)**.** We define the shift operator θ to be the function $\theta : \Omega \to \Omega$ satisfying

$$\theta(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots) \tag{4.9}$$

Furthermore, we define θ_0 to be the identity function on Ω, i.e. $\theta_0 : \Omega \to \Omega : \omega \mapsto \omega$. For $n \in \mathbb{N}$ we define

$$\theta_n = \theta_{n-1} \circ \theta \tag{4.10}$$

■

The shift operator interacts nicely with our canonical process (X_n), as illustrated by the following lemma, which is really just remark about our notation.

Lemma 4.12. Let f be a function $f : E \to \mathbb{R}$ and let $k, n \in \mathbb{N}_0$. We have

$$f \circ X_k \circ \theta_n = f(X_{k+n}) \tag{4.11}$$

Proof. Let $\omega \in \Omega$. ω is just a sequence $\omega = (\omega_0, \omega_1, \dots)$, with for all $i \in \mathbb{N}_0$, $\omega_i \in \mathbb{E}$. We have

$$\begin{aligned} f \circ X_k \circ \theta_n(\omega) &= f \circ X_k \circ \theta_n(\omega_0, \omega_1, \dots) \\ &= f \circ X_k(\omega_n, \omega_{n+1}, \dots) \\ &= f(X_{k+n}(\omega_0, \omega_1, \dots)) \end{aligned} \tag{4.12}$$

□

It is convenient to define the following function h. We will later prove that for all $x \in E$ and $N \in \mathbb{N}_0$, $h^N(x) = V^N(x)$. We define, for all $x \in E$ and all $n \in \mathbb{N}_0$

$$h^0(x) = G(x) \tag{4.13}$$

$$h^{n+1}(x) = \max(G(x), \int h^n(y) P(x, dy)) \tag{4.14}$$

Lemma 4.13. For all $x \in E$ and $N \in \mathbb{N}$, let $(S_n^{N,x})_{n=0}^N$ be the Snell Envelope of $\mathfrak{A}_x^N$ in the sense of Definition 3.8. We have, for all $x \in E$ and $N \in \mathbb{N}_0$ and $k, n \in \mathbb{N}_0$ with $k + n \leq N$,

$$S_k^{N-n,x} \circ \theta_n = S_{n+k}^{N,x} \quad P_x\text{-a.s.} \tag{4.15}$$

For all $x \in E$ and $m, N \in \mathbb{N}_0$ with $m \leq N$ we have

$$h^{N-m}(X_m) = S_m^{N,x} \quad P_x\text{-a.s.} \tag{4.16}$$

Proof. Let $N \in \mathbb{N}_0$. We have, for all $n \in \mathbb{N}$, by Lemma 10.32, because (X_n) is a Markov process under $\mathbb{P}_x$ with Markov kernel P, we have $\mathbb{P}_x$ almost surely that

$$\begin{aligned} h^{N-n}(X_n) &= \max(G(X_n), \int h^{N-n-1}(y) P(X_n, dy)) \\ &= \max(G(X_n), \mathbb{E}_x[h^{N-n-1}(X_{n+1})|X_n] \\ &= \max(G(X_n), \mathbb{E}_x[h^{N-(n+1)}(X_{n+1})|\mathcal{F}_n] \end{aligned} \tag{4.17}$$

Furthermore, we see that $h^0(X_N) = G(X_N) = S_N^{N,x}$. So we see that $h^{N-n}(X_n)$ follows the same recursion as $S_n^{N,x}$ and by induction it follows that $\mathbb{P}_x$ almost surely $h^{N-n}(X_n) = S_n^{N,x}$. Because N was arbitrary, it follows that for all $x \in E$, $N \in \mathbb{N}_0$ and all $n \in \{0, \ldots, N\}$

$$h^{N-n}(X_n) = S_n^{N,x} \quad P_x\text{-a.s.} \tag{4.18}$$

This gives us that for all $k, n, N \in \mathbb{N}_0$ with $k + n \leq N$, we have (by Lemma 4.12)

$$\begin{aligned} S_k^{N-n,x} \circ \theta_n &= h^{N-n-k}(X_k) \circ \theta_n = h^{N-(n+k)}(X_{n+k}) \\ &= S_{n+k}^{N,x} \end{aligned} \tag{4.19}$$

□

Lemma 4.13 justifies making the following definition.

Definition 4.14 (Universal Snell envelope)**.**
Let $N \in \mathbb{N}_0$ and consider the family of stopping problems $(\mathfrak{A}_x^N)_{x \in E}$. We define the universal Snell envelope of this family to be the sequence $(S_n^N)_{n=0}^N$ satisfying

$$S_n^N = h^{N-n}(X_n) \tag{4.20}$$

■

In the proof of the theorem, we use the following application of the generalized Markov property, i.e. Proposition 10.31.

Lemma 4.15. Let $N \in \mathbb{N}_0$, let $n \in \{0, \ldots, N\}$ and let $x \in E$. Let $\tau_0^{N-n,x}$ be the earliest optimal stopping time of $\mathfrak{A}_x^{N-n}$ in the sense of Definition 3.9. We have

$$E_x[G \circ X_{\tau_0^{N-n,x}} \circ \theta_n | \mathcal{F}_n^X] = E_{X_n}[G \circ X_{\tau_0^{N-n,x}}] \quad \mathbb{P}_x\text{-a.s} \tag{4.21}$$

Proof. $\max_{k\in\{0,\dots,N-n\}} |G(X_k)|$ is measurable and nonnegative. So we have

$$E_x[\max_{k\in\{0,\dots,N-n\}} |G(X_k)| \circ \theta_n | \mathcal{F}_n^X] = E_{X_n}[\max_{k\in\{0,\dots,N-n\}} |G(X_k)|] \quad \mathbb{P}_x\text{-a.s} \quad (4.22)$$

Let $Z = \max_{k\in\{0,\dots,N-n\}} |G(X_k)| - G \circ X_{\tau_0^{N-n,x}} \geq 0$. Then Z is measurable and nonnegative, so we have

$$\mathbb{E}_x[Z \circ \theta_n | \mathcal{F}_n^X] = \mathbb{E}_{X_n}[Z] \quad \mathbb{P}_x\text{-a.s} \quad (4.23)$$

For all $\omega \in \Omega$, assumption (4.6) gives us that $\mathbb{E}_{X_n(\omega)}[\max_{k\in\{0,\dots,N-n\}} |G(X_k)|] < \infty$ and $\mathbb{E}_{X_n(\omega)}[Z] < \infty$, we do not get $\infty - \infty$ when considering $\mathbb{E}_{X_n(\omega)}[\max_{k\in\{0,\dots,N-n\}} |G(X_k)|] - \mathbb{E}_{X_n(\omega)}[Z]$. We get, using (4.22) and (4.23), $\mathbb{P}_x$ almost surely

$$\begin{aligned}
\mathbb{E}_x[G \circ X_{\tau_0^{N-n,x}} \circ \theta_n | \mathcal{F}_n] &= \mathbb{E}_x[(\max_{k\in\{0,\dots,N-n\}} |G(X_k)| - Z) \circ \theta_n | \mathcal{F}_n] \\
&= \mathbb{E}_x[\max_{k\in\{0,\dots,N-n\}} |G(X_k)| \circ \theta_n | \mathcal{F}_n] - \mathbb{E}_x[Z \circ \theta_n | \mathcal{F}_n] \\
&= \mathbb{E}_{X_n}[\max_{k\in\{0,\dots,N-n\}} |G(X_k)|] - \mathbb{E}_{X_n}[Z] \\
&= \mathbb{E}_{X_n}[\max_{k\in\{0,\dots,N-n\}} |G(X_k)|] - Z] \\
&= \mathbb{E}_{X_n}[G \circ X_{\tau_0^{N-n,x}}]
\end{aligned}$$

□

For all $N \in \mathbb{N}_0$ and $n \in \{0, \dots, N\}$, we define D_n^N, which is sometimes called the stopping domain in literature, as follows. The word optimal in this definition is justified by Remark 4.19.

Definition 4.16 (Preferred optimal stopping region)**.** For all $N \in \mathbb{N}$, we define

$$D_n^N = \{x \in \mathbb{R} | V^{N-n}(x) = G(x)\} \quad (4.24)$$

We write $D^N = D_0^N$ and we will refer to $(D^N)_{N=0}^{\infty}$ as the preferred optimal stopping region. ■

The word optimal in the following definition is justified by (4.28) of Theorem 4.18 below.

Definition 4.17 (Preferred optimal stopping time)**.** For all $N \in \mathbb{N}_0$ we define τ_D^N, which plays a role in the theorem below, as follows

$$\begin{aligned}
\tau_D^N &= \inf\{n \in \{0, \dots, N\} | X_n \in D_n^N\} \\
&= \inf\{n \in \{0, \dots, N\} | X_n \in D^{N-n}\}
\end{aligned} \quad (4.25)$$

■

Theorem 4.18.
Let $N \in \mathbb{N}_0$. Note that as an initial condition, we have $V^0 = G$. If condition (4.6) holds, then for $n \in \{1, \dots, N\}$, the value function V^n satisfies the Wald-Bellman equations, i.e. for all $x \in E$

$$V^n(x) = \max(G(x), \mathbb{E}_x[V^{n-1}(X_1)]) \tag{4.26}$$

Let $x \in E$ and let $S_n^{N,x}$ be the Snell envelope of the stopping problem $\mathfrak{A}_x^N$. We have, for $n \in \{0, \dots, N\}$,

$$V^{N-n}(X_n) = S_n^{N,x} \quad \mathbb{P}_x\text{-a.s.} \tag{4.27}$$

Let $x \in E$, let $n \in \{0, \dots, N\}$ let $\tau_n^{N,x}$ be the earliest optimal stopping time of the stopping problem $\mathfrak{A}_x^N$. Then $\mathbb{P}_x[\tau_D^N = \tau_n^{N,x}] = 1$. In particular

$$V^N(x) = \mathbb{E}_x[G(X_{\tau_D^N})] \tag{4.28}$$

Proof.
First we prove (4.27). For all $x \in E$ and $N \in \mathbb{N}$, let $(S_n^{N,x})_{n=0}^N$ be the Snell Envelope of $\mathfrak{A}_x^N$ and let $(\tau_n^{N,x})_{n=0}^N$ be the earliest optimal stopping time sequence of $\mathfrak{A}_x^N$ in the sense of Definition 3.9. Let $x \in E$, $N \in \mathbb{N}_0$ and let $n \in \{0, \dots, N\}$, we have that

$$\begin{aligned}
\tau_n^{N,x} &= \inf\{n \le k \le N | S_k^{N,x} = G(X_k)\} \\
&= \inf\{0 \le k - n \le N - n | S_k^{N,x} = G(X_k)\} + n \\
&= \inf\{0 \le m \le N - n | S_{n+m}^{N,x} = G(X_{n+m})\} + n
\end{aligned}$$

Now we use Lemma 4.13, which gives us that for all $m \in \{0, \dots, N-n\}$, $S_{n+m}^{N,x} = S_m^{N-n,x} \circ \theta_n$, and Lemma 4.12, which gives us that $G(X_{n+m}) = G \circ X_m \circ \theta_n$. We obtain

$$\tau_n^{N,x} = \inf\{0 \le m \le N - n | S_m^{N-n,x} \circ \theta_n = G \circ X_m \circ \theta_n\} + n$$

Note that $\tau_0^{N-n,x}$ is the following function

$$\tau_0^{N-n,x} : \Omega \to \mathbb{N}_0 : \omega \mapsto \inf\{0 \le m \le N - n | S_m^{N-n,x}(\omega) = G \circ X_m(\omega)\}$$

We then see that

$$\tau_n^{N,x}(\omega) = \tau_0^{N-n,x} \circ \theta_n(\omega) + n \tag{4.29}$$

Using (3.13) of Theorem 3.10, we have that

$$S_n^{N,x} = \mathbb{E}_x[G(X_{\tau_n^{N,x}}) | \mathcal{F}_n]$$

This then becomes, by the result (4.29) we just found

$$S_n^{N,x} = \mathbb{E}_x[G(X_{\tau_0^{N-n,x} \circ \theta_n + n}) | \mathcal{F}_n] \tag{4.30}$$

Now note that, for all $\omega \in \Omega$, (by Lemma 4.12)

$$\begin{aligned} X_{n+\tau_0^{N-n,x}\circ\theta_n}(\omega) &= X_{n+\tau_0^{N-n,x}\circ\theta_n(\omega)}(\omega) \\ &= X_{\tau_0^{N-n,x}\circ\theta_n(\omega)}(\theta_n(\omega)) \end{aligned}$$

This gives us

$$X_{n+\tau_0^{N-n,x}\circ\theta_n} = X_{\tau_0^{N-n,x}} \circ \theta_n$$

We then see that (4.30) becomes

$$S_n^{N,x} = \mathbb{E}_x[G \circ X_{\tau_0^{N-n,x}} \circ \theta_n | \mathcal{F}_n] \tag{4.31}$$

Now, we use our application of the generalized Markov property, i.e. Lemma 4.15, noting that assumption (4.6) holds. Also note that x, N and n were arbitrary, so we have for all $x \in E$, $N \in \mathbb{N}_0$ and $n \in \{0, \dots, N\}$

$$S_n^{N,x} = E_x[G \circ X_{\tau_0^{N-n,x}} \circ \theta_n | \mathcal{F}_n^X] = E_{X_n}[G \circ X_{\tau_0^{N-n,x}}] \quad \mathbb{P}_x - a.s. \tag{4.32}$$

Because $\tau_0^{N-n,x}$ is optimal (recall (3.14) of Theorem 3.10), we have for all $x \in E$, $N \in \mathbb{N}_0$ and $n \in \{0, \dots, N\}$

$$\mathbb{E}_x[G \circ X_{\tau_0^{N-n,x}}] = \sup_{\tau \in \mathfrak{M}_0^{N-n}} \mathbb{E}_x[G \circ X_\tau] = V^{N-n}(x) \tag{4.33}$$

Now let, for all $N \in \mathbb{N}_0$, $(S_n^N)_{n=0}^N$ be the universal Snell envelope of the family of stopping problems $(\mathfrak{A}_x^N)_{x\in E}$ in the sense of Definition 4.14. Furthermore, we set, for all $N \in \mathbb{N}$ and $n \in \{0, \dots, N\}$

$$\tau_n^N = \inf\{k \in \{n, \dots, N\} \mid S_k^N = G(X_k)\} \tag{4.34}$$

Because $\mathbb{P}_x$ almost surely $S_n^N = S_n^{N,x}$, we have that $\mathbb{P}_x$ almost surely $\tau_0^{N-n} = \tau_0^{N-n,x}$. So (4.33) gives us

$$V^{N-n}(x) = \mathbb{E}_x[G \circ X_{\tau_0^{N-n,x}}] = \mathbb{E}_x[G \circ X_{\tau_0^{N-n}}] \tag{4.35}$$

Finally, we use (4.32) and fill in X_n in the equation above to find for all $N \in \mathbb{N}_0$ and $n \in \{0, \dots, N\}$

$$S_n^{N,x} = E_{X_n}[G \circ X_{\tau_0^{N-n}}] = V^{N-n}(X_n) \quad \mathbb{P}_x\text{-a.s.} \tag{4.36}$$

Which proves equation (4.27).

Now we prove the Wald-Bellman equations (4.26). Let $N \in \mathbb{N}_0$. We apply (4.36) and the definition of $S_n^{N,x}$ as the Snell envelope to find for all $n \in \{0, \dots, N\}$

$$V^{N-n}(X_n) = \max(G(X_n), E_x[V^{N-n-1}(X_{n+1}) | \mathcal{F}_n]) \quad \mathbb{P}_x\text{-a.s} \tag{4.37}$$

Let $\omega \in \{X_0 = x\}$ and note that $\mathbb{P}_x[X_0 = x] = 1$. Using the case of $n = 0$ in (4.37) above (and using Lemma 10.43), we find that

$$\begin{aligned} V^N(x) &= V^N(X_0(\omega)) = \\ &= \max(G(X_0(\omega)), \mathbb{E}_x[V^{N-1}(X_1) \,|\, \mathcal{F}_0](\omega)) \\ &= \max(G(x), \mathbb{E}_x[V^{N-1}(X_1)]) \end{aligned} \tag{4.38}$$

This shows (4.26).

Let $N \in \mathbb{N}_0$ and $x \in E$. We show that $\mathbb{P}_x[\tau_D^N = \tau_n^{N,x}] = 1$ and (4.28) as follows. Using (4.36), we have $\mathbb{P}_x$ almost surely

$$\begin{aligned} \tau_D^N &= \inf\{n \in \{0, \dots, N\} \,|\, X_n \in D_n^N\} \\ &= \inf\{n \in \{0, \dots, N\} \,|\, X_n \in \{x \in E \,|\, V^{N-n}(x) = G(x)\}\} \\ &= \inf\{n \in \{0, \dots, N\} \,|\, S_k^{N,x} = G(X_n)\} \\ &= \tau_n^{N,x} \end{aligned}$$

In particular, (recall Theorem 3.10)

$$V^N(x) = \mathbb{E}_x[G(X_{\tau_n^{N,x}})] = \mathbb{E}_x[G(X_{\tau_D})]$$

□

Remark 4.19. The word optimal in "preferred optimal stopping region" in Definition 4.16 is justified by considering that we might choose other sets $(C_n)_{n=0}^{\infty}$ in place of $(D^N)_{N=0}^{\infty}$ and construct a stopping time τ_C^N in a similar way as we constructed τ_D^N in Definition 4.17. $(D^N)_{N=0}^{\infty}$ is optimal in the sense that τ_D^N is optimal. We use the word preferred because other optimal sequences of sets $(C_n)_{n=0}^{\infty}$ may exist, but the defining property of D^N is nice.

Lemma 4.20. For all $N \in \mathbb{N}_0$, we have $D^{N+1} \subseteq D^N$.

Proof. By (4.8), because we take supremums over continually larger sets, we have, for any $x \in E$

$$G(x) = V^0(x) \leq V^1(x) \leq \cdots \leq V^N(x) \tag{4.39}$$

This gives us that for all $N \in \mathbb{N}_0$.

$$\forall x \in E : V^{N+1}(x) = G(x) \implies V^N(x) = G(x)$$

This gives us, for all for all $N \in \mathbb{N}_0$

$$D^{N+1} = \{x \in E \,|\, V^{N+1}(x) = G(x)\} \subseteq \{x \in E \,|\, V^N(x) = G(x)\} = D^N \tag{4.40}$$

□

It is convenient to explicitly translate what action τ_D dictates us to take when we observe a value x of the process at a specific time n. The concept of a policy is borrowed from the theory of Markov Decision Processes.

Definition 4.21 (Preferred optimal stopping policy)**.** For all $N \in \mathbb{N}_0$, $x \in E$, we define

$$\pi_D^N(x) = \begin{cases} \text{STOP} & \text{if} \quad x \in D^N \\ \text{CONT} & \text{otherwise} \end{cases} \tag{4.41}$$

■

This definition is justified as follows. For all $\omega \in \Omega$ and all $N \in \mathbb{N}_0$ we have

$$\begin{aligned} \tau_D^N(\omega) &= \inf\{n \in \{0, \dots, N\} \,|\, X_n(\omega) \in D^{N-n}\} \\ &= \inf\{n \in \{0, \dots, N\} \,|\, \pi_D^{N-n}(X_n(\omega)) = \text{STOP}\,\} \end{aligned} \tag{4.42}$$

So $\pi_D^N(x)$ is optimal in a similar sense as D^N (Remark preferred-remark).

Now we consider the special case where $E = \mathbb{R}$. The following definition is analogous to the concept of "optimal stopping boundary" in the theory of optimal stopping in continuous time.

Definition 4.22 (Boundary)**.** We say that $(d_n)_{n=1}^{\infty} \subset \mathbb{R}$ is the boundary of the preferred optimal stopping region $D = (D^N)_{N=0}^{\infty}$ if for all $n \in \mathbb{N}$, $D^n = [d_n, \infty)$.

■

Lemma 4.23. If (D_n) has a boundary (d_n), then (d_n) is increasing.

Proof. This follows from Lemma 4.20. □

5 Sequences of i.i.d. Random Variables

This section serves mainly as an example to get some feeling for the material of Section 3 and Section 4. In addition, we will see that when we consider independent normally distributed random variables, we cannot find a nice direct formula for the expected payoff and we have to resort to doing a recursion. Furthermore, the recursive function involves error functions. In Section 7, when stopping AR(1) processes, which generalize the process under consideration here, formulas in terms of error functions also show up. We also show the idea of doing a Monte Carlo simulation to verify our results.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space and let $(X_i)_{i=0}^{\infty}$ be an i.i.d. sequence of random variables on $(\Omega, \mathcal{F})$, with $\mathbb{E}[|X_0|] < \infty$ and for all $n \in \mathbb{N}$, let X_n be $\mathcal{F}_n$ measurable and independent of $\mathcal{F}_{n-1}$. In the special case where for all $n \in \mathbb{N}_0$, $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ note that X_n is $\mathcal{F}_n$ measurable by definition and X_n is independent of $\mathcal{F}_{n-1}$ by the assumption that the X_n are independent (and by the Grouping Lemma, i.e. Lemma 4.4.1 in [15], which says that if $\sigma(X_n)$ is independent of $\sigma(X_i)$ for all $i < n$, then $\sigma(X_n)$ is also independent of the "grouping" $\sigma(X_1, \ldots, X_{n-1})$). Note that $(X_n)_{n=0}^{\infty}$ is a stochastic process adapted to $(\mathcal{F}_n)$. Recall Definitions 3.6, 3.7 and 3.9. For all $N \in \mathbb{N}_0$ let W^N be the value of the stopping problem $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P}, (X_n)_{n=0}^{N}, N)$ and let τ_0^N be its earliest optimal stopping time. We will find a recursive formula for W^N and we will show that for all $N \in \mathbb{N}_0$

$$\tau_0^N = \inf\{k \in \{0, \ldots, N\} \mid X_k \geq W^{N-k-1}\} \tag{5.1}$$

We could put the problem in the context of Subsection 4.1, as a sequence of i.i.d. random variables is a Markov process. However, we prefer using the simpler theory of Section 3.

5.1 A recursion for i.i.d. sequences

To fully use the notation of Section 3, for all $n \in \mathbb{N}_0$, we set $G_n = X_n$. Recall Definitions 3.7, 3.8 and 3.9. For all $N \in \mathbb{N}_0$, we let $(W_n^N)_{n=0}^N$ be the subproblem value sequence, $(S_n^N)_{n=0}^N$ be the snell envelope and $(\tau_n^N)_{n=0}^N$ be the earliest stopping time sequence of the stopping problem $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P}, (G_n)_{n=0}^N, N)$.

Let $N \in \mathbb{N}$. We will first prove by induction that for all $n \in \{0, \ldots, N-1\}$,

$$\mathbb{E}[S_{n+1}^N \mid \mathcal{F}_n] = \mathbb{E}[S_{n+1}^N] \tag{5.2}$$

First we prove the case for $n = N$. By our assumptions, G_N is independent of $\mathcal{F}_{N-1}$. This gives us, by property (12) of paragraph 10.3 of [15], using the fact that by (3.7) $G_N = S_N$, that

$$\mathbb{E}[S_N \mid \mathcal{F}_{N-1}] = \mathbb{E}[S_N] \tag{5.3}$$

Now let $n \in \{1, \ldots, N-1\}$ and assume that the induction hypothesis (5.2) holds for n. Using the definition of (S_n^N) (3.8), this gives us

$$S_{n-1}^N = \max(G_{n-1}, \mathbb{E}[S_n^N \mid \mathcal{F}_{n-1}^N]) = \max(G_{n-1}, \mathbb{E}[S_n^N]) \tag{5.4}$$

Because $\mathbb{E}[S_n^N]$ is constant, this gives us that S_{n-1}^N is $\sigma(G_{n-1}^N)$ measurable, i.e. $\sigma(S_{n-1}^N) \subset \sigma(G_{n-1})$. It then follows from the definition of independent σ-algebras and the fact that $\sigma(G_{n-1})$ and $\mathcal{F}_{n-2}$ are independent, that $\sigma(S_{n-1}^N)$ is independent of $\mathcal{F}_{n-2}$. Using property (12) of paragraph 10.3 of [15] once again, we obtain the induction hypthesis for $n-1$, i.e. $\mathbb{E}[S_{n-1}^N \mid \mathcal{F}_{n-2}] = \mathbb{E}[S_{n-1}^N]$. By induction, (5.2) then holds for all $n \in \{0, \ldots, N-1\}$.

Combining (3.13) and (3.14) of Theorem 3.10 and the tower property, we find that for all $n \in \{0, \ldots, N\}$

$$W_n^N = \mathbb{E}[G_{\tau_n^N}] = \mathbb{E}[\mathbb{E}[G_{\tau_n^N} | \mathcal{F}_n]] = \mathbb{E}[S_n^N] \tag{5.5}$$

Using this and (5.2), we obtain, for all $n \in \{0, \ldots, N-1\}$

$$\begin{aligned} S_n^N &= \max(G_n, \mathbb{E}[S_{n+1}^N \mid \mathcal{F}_n]) = \max(G_n, \mathbb{E}[S_{n+1}^N]) \\ &= \max(G_n, W_{n+1}^N) \end{aligned} \tag{5.6}$$

Taking expectations and using (5.5) again, this gives us that for all $n \in \{0, \ldots, N-1\}$,

$$W_n^N = \mathbb{E}[S_n^N] = \mathbb{E}[\max(G_n, W_{n+1}^N)] \tag{5.7}$$

Using the fact that the G_i are i.i.d., we obtain, for all $n \in \{0, \ldots, N-1\}$,

$$W_n^N = \mathbb{E}[\max(G_0, W_{n+1}^N)] \tag{5.8}$$

The initial condition for this recursion is the fact that $W_N^N = \mathbb{E}[S_N^N] = \mathbb{E}[G_N] = \mathbb{E}[G_0]$. Note N only has a trivial role in the recursion. We see that for all $N \in \mathbb{N}_0$ and all $k \in \{0, \ldots, N\}$,

$$W^N = W_k^{N+k} \tag{5.9}$$

This corresponds to the general fact that we only need to use one index when developing the theory for homogeneous Markov Processes in Subsection 4.1. Our recursion is then

$$W^n = \mathbb{E}[\max(X_0, W^{n-1})] \tag{5.10}$$

$$W^0 = \mathbb{E}[X_0] \tag{5.11}$$

We can express τ_n^N in terms of W_n^N in a simple way. From (5.6) we see that for all $n \in \{0, \ldots, N\}$

$$S_n^N = G_n \iff G_n \geq W_{n+1}^N \tag{5.12}$$

The definition (3.9) of τ_n^N then gives us

$$\begin{aligned} \tau_n^N &= \inf\{k \in \{n, \ldots, N\} \,|\, G_k \geq W_{k+1}^N\} \\ &= \inf\{k \in \{n, \ldots, N\} \,|\, X_k \geq W^{N-k-1}\} \end{aligned} \tag{5.13}$$

5.2 Sequences of i.i.d. normally distributed random variables

The computations in this section correspond to an analysis in [9]. The analysis is the one done for f_2 in [9] and the numerical results correspond to Table 2 in [9].

Consider the case where $X_0 \sim N(0,1)$. Let Φ be the distribution function of the standard normal distribution. We have the following identity.

Identity 5.1. For $X \sim N(0,1)$ and $c \in \mathbb{R}$, we have

$$\mathbb{E}[\max(X, c)] = \frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi}} + c\,\Phi(c)$$

Proof. Let $X \sim N(0,1)$ and let $c \in \mathbb{R}$ be arbitrary. We have

$$\begin{aligned} \mathbb{E}[\max(X, c)] &= \mathbb{E}[c\,\mathbb{1}_{\{X \leq c\}} + \mathbb{1}_{\{X > c\}} X] \\ &= c\,\mathbb{P}[X \leq c] + \int_{-\infty}^{\infty} \mathbb{1}_{\{x > c\}}\, x\, f_X(x)\, dx \\ &= c\,\Phi(c) + \int_c^{\infty} \frac{1}{\sqrt{2\pi}}\, x\, e^{-\frac{1}{2}x^2}\, dx \\ &= c\,\Phi(c) + \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{1}{2}x^2}\right]_c^{\infty} \\ &= c\,\Phi(c) + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}c^2} \end{aligned}$$

□

Writing Identity 5.1 in terms of the error function gives

$$\mathbb{E}[\max(X, c)] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}c^2} + \frac{1}{2} c \left(\operatorname{erf}\left(\frac{c}{\sqrt{2}}\right) + 1\right) \tag{5.14}$$

Let $f(c) = \frac{e^{-\frac{c^2}{2}}}{\sqrt{2\pi}} + c\,\Phi(c)$. We then have the following recursion for W^n.

$$W^n = \mathbb{E}[\max(X, W^{n-1})] = f(W^{n-1})$$

The initial condition is

$$W^0 = \mathbb{E}[X_0] = 0$$

This gives us

$$W^n = f^{(n)}(0)$$

Where $f^{(n)} = f \circ f \circ \cdots \circ f$ denotes the n-fold composition of f. A plot of W^n as a function of n is shown in Figure 3.

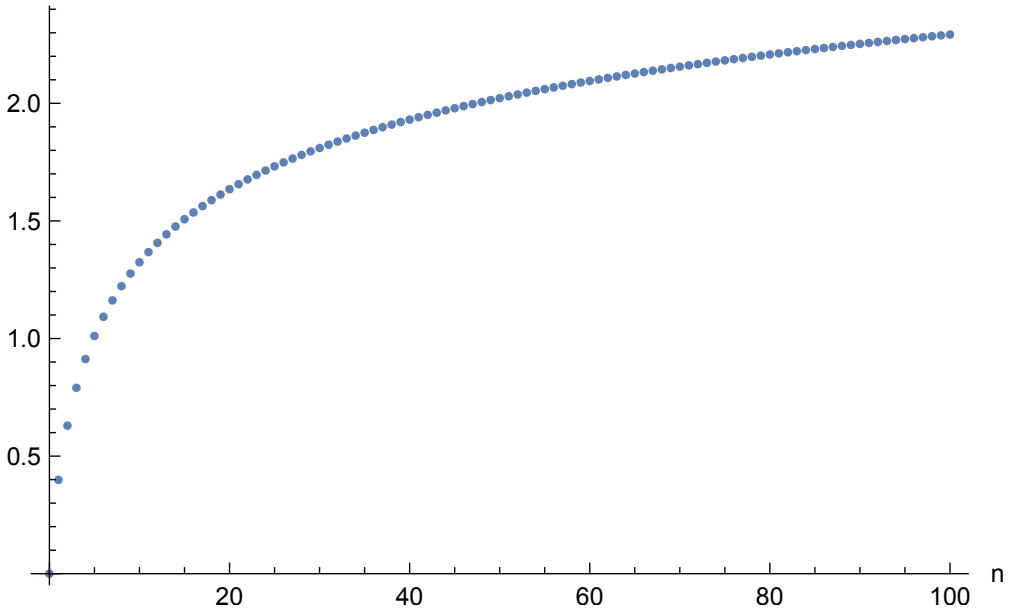

Figure 3: W^n as a function of n for standard normally distributed i.i.d. random variables.

We find $W^1 = f^{(1)}(0) = \frac{1}{\sqrt{2\pi}} = 0.398942$. The number 0.39916388 from Table 2 of [9] corresponds to this. We have calculated 0.398942 in a few different ways. Furthermore, we have verified it using a Monte Carlo simulation, as we will see further on, so we suspect a numerical error has been made in the calculation of 0.39916388 in [9].

We have done a simulation, by executing our optimal stopping rule. We have done this simulation ten million times, for $N = 100$. In Figure 4 we display the relative frequency in this simulation of each moment the process can be stopped, including the last moment $N = 100$ at which the only option is to stop. The frequency of stopping at a particular time decreases as the time gets closer to N. The relative frequency for the moments $N - 1 = 99$ and $N = 100$ almost identical. That they should be identical can be derived as follows. We

have

$$\mathbb{P}[X_{N-1} \geq W_N^N] = \mathbb{P}[X_{N-1} \geq 0] = \frac{1}{2} = \mathbb{P}[X_{N-1} < W_N^N] \tag{5.15}$$

Therefore we have, by independence of the X_i, by (5.13) and by (5.15)

$$\begin{aligned} \mathbb{P}[\tau_0^N = N-1] &= \mathbb{P}[\bigwedge_{k=0}^{N-2} X_k < W_{k+1}^N]\, \mathbb{P}[X_{N-1} \geq W_N^N] \\ &= \mathbb{P}[\bigwedge_{k=0}^{N-2} X_k < W_{k+1}^N]\, \mathbb{P}[X_{N-1} < W_N^N] \\ &= \mathbb{P}[\tau_0^N = N] \end{aligned} \tag{5.16}$$

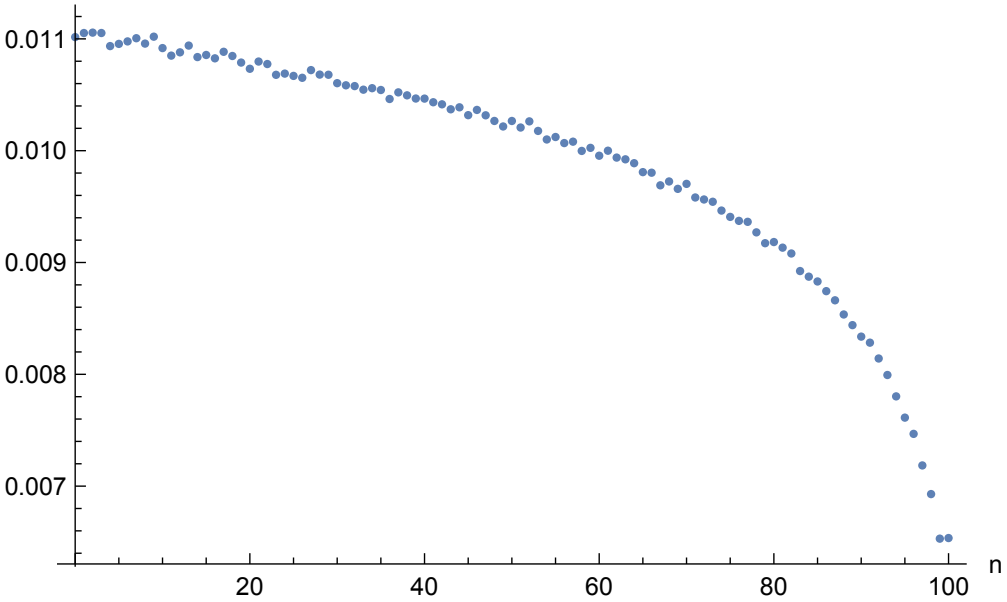

Figure 4: A sequence of i.i.d. standard normal random variables is stopped with $N = 100$ and the relative frequency that the process is stopped at a particular time is displayed for each moment at which the process can be stopped.

In Figure 5 we show a histogram of the realizations of $X_{\tau_0^N}$, i.e. payoffs.

The average payoff is 2.29274. This closely corresponds to $W_0^{100} = f^{(100)}(0) = 2.29257$.

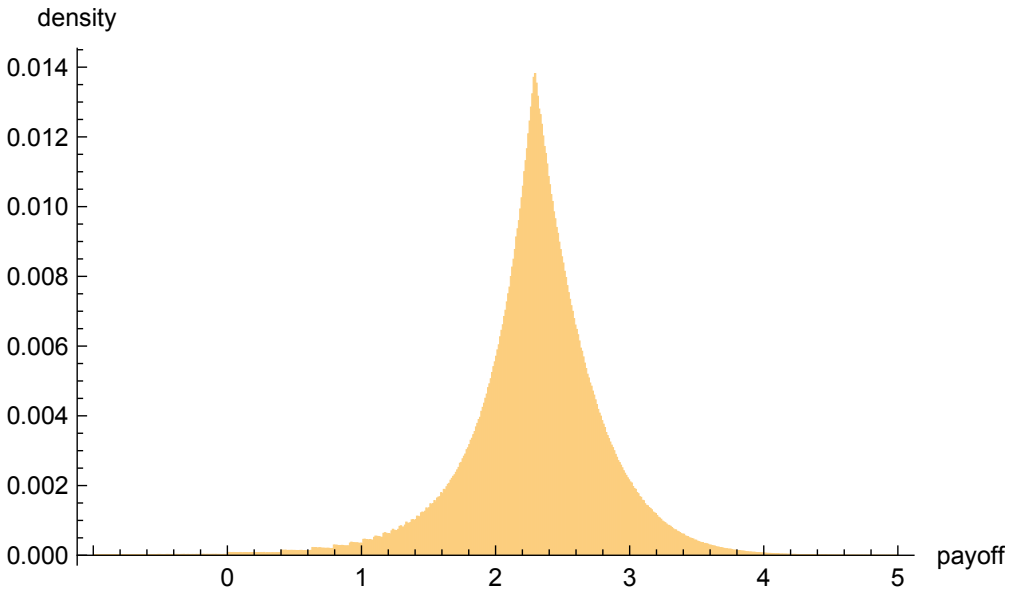

Figure 5: A histogram of the payoffs, i.e. realizations of $X_{\tau_0^N}$

6 The Discretization Procedure

Again our goal is to optimally stop the discrete time stochastic process (X_n) on $\mathbb{R}$. We include no utility function in our analysis, that is to say our utility function is the identity $G(x) = x$. In the previous section, we assumed we knew the probability law governing the process (X_n), which is something we will also do in sections further on. In this section however, we will present a procedure that does not assume that we know this law. Instead, we assume we are given data of the process.

We will also assume that (X_n) is stationary. The assumption of stationarity is required for the quantiles, probabilities etc defined in the following definitions to converge to their true values. Rather than providing theoretical arguments, we will provide numerical evidence in Subsections 6.5 and 8.2 to show that the procedure described in this section works correctly.

The data we have of our process is, for some $K \in \mathbb{N}$, a sequence $Y = (y_i)_{i=1}^K \subset \mathbb{R}$. First we shall use the data to make a discrete model of the dynamics of the process. The result will be a Discrete Time Discrete State (DTDS) Markov Chain.

6.1 Construction DTDS Markov Chain

Below, we shall make definitions that are needed to describe the process of discretization. The concepts of vector and sequence will be somewhat conflated. This is because it is more natural to think about a sequence of digits, rather than a vector of digits and we have defined a discrete time stochastic process as a sequence. Further on in this section however, we will manipulate our data in a way that is associated with vectors, e.g. by taking matrix-vector products.

For a sequence or vector of random variables X, we shall use the usual notation $X_{(i)}$ to mean the i-th order statistic of X. In particular $X_{(1)} = \min_i X_i$ and $X_{(N)} = \max_i X_i$. Here, N is the number of elements in the sequence $(X_i)_{i=1}^N$ or the vector $(X_1, \ldots, X_N)$. Similarly, for a vector or sequence x that is the realization of a random vector, we shall write $x_{(i)}$ for the i-th order statistic.

Definition 6.1.
Let, for some $K \in \mathbb{N}$, $A = (a_i)_{i=1}^K$ be a sequence of real numbers. Let $(c_i)_{i=1}^K$ be the sequence satisfying, for all $i \in \{1, \ldots, K\}$, $c_i = a_{(i)}$. For $0 < \alpha < 1$, we define the α-quantile of this sequence to be the number $c_{\lfloor \alpha N \rfloor}$. In particular, we shall write q_α for the α-quantile of Y and we define $q_0 = -\infty$ and $q_1 = \infty$.

■

Definition 6.2.
Let $i \in \mathbb{N}_0$ and $m \in \mathbb{N}$, with $i \leq m-1$. We define the i/m-level (of Y) to be the

interval $L_i^m := [q_{i/m}, q_{(i+1)/m})$. In addition, for a fixed m, we define the level partition $\mathscr{K} := \{L_i^m \mid i \in \{0, \ldots, m-1\}\}$.

■

In the procedure of discretising the process, we shall fix a parameter m which we shall refer to as the fineness of the discretization. If we pick a fineness of m, we work with m levels and "cut the process into m slices". More precisely, having fixed m, we will associate a level with every element y_i in the realization, using the level function defined below.

Definition 6.3.
Let $m \in \mathbb{N}$. We define the level function $\rho : \mathbb{R} \to \mathbb{N}_0$ to be the function that maps any $x \in \mathbb{R}$ to the unique level $L \in \mathscr{K}$ such that $x \in L$.

■

The result of our discretization will be a discrete state discrete time Markov Chain Z_i on $\mathscr{S} := \{0, \ldots, m-1\}^n$. We define the state function as follows, which coincides with ρ for $n = 1$.

Definition 6.4.
Let $m \in \mathbb{N}$. We define the state function σ to be

$$\sigma : \mathbb{R}^n \to \mathscr{S} : (x_1, \ldots, x_n) \mapsto (\rho(x_1), \ldots, \rho(x_n))$$

■

Using the state function, we discretise our data.

Definition 6.5.
Let $m, n \in \mathbb{N}$. We define the discretization (of Y) to be the sequence $(u_i)_{i=1}^{K-n+1} \subset \mathscr{S}$ satisfying, for all $1 \leq i \leq K - n + 1$

$$u_i = \sigma(y_i, y_{i+1}, \ldots, y_{i+n})$$

and we write $U := (u_i)_i^{K-n-1}$

■

Definition 6.6.
Let $i, m, n \in \mathbb{N}$, with $i \leq m$. Let $(c_i)_{i=1}^K$ be the sequence satisfying $c_i = y_{(i)}$. Let $J = \lfloor K/m \rfloor$. We define the i/m-slice to be the sequence $R^i = (r_j^{(i)})_{j=1}^J := (c_j)_{j=(i-1)J+1}^{iJ}$

■

With every two states $s_1, s_2 \in \mathscr{S}$ we will associate a probability. This idea is captured by the empirical probability function $\bar{p}$, defined below.

Definition 6.7.
Let $m, n \in \mathbb{N}$. We define the empirical probability function $\bar{p}$ to be

$$\bar{p} : \mathscr{S} \times \mathscr{S} \to [0,1] : (s_1, s_2) \mapsto \frac{|\{i \in \mathbb{N} \,|\, u_i = s_1 \wedge u_{i+1} = s_2\}|}{|\{i \in \mathbb{N} \,|\, u_i = s_1\}|}$$

■

Note that $\bar{p}$ must be 0 for many combinations s_1, s_2, as there are no transitions from s_1 to s_2 if the last $n-1$ coordinates of s_1 are not equal to the first $n-1$ coordinates of s_2. Furthermore, if $n = 1$, $|\{i \in \mathbb{N} \,|\, u_i = s_1\}| \approx K/m$ and $\bar{p}(i,j)$ becomes proportional to the number of transitions from slice i to slice j.

With the states of Z_i we shall associate values, which are the averages of the slices.

Definition 6.8.
Let $m, n \in \mathbb{N}$. We define the level utility function vector to be $g := (\overline{R^1}, \ldots, \overline{R^m})$, where for a sequence A, $\overline{A}$ denotes its average. The utility function considered here will be $G(i_1, i_2, \ldots, i_n) := g_{i_n+1}$.

■

In order to formally define a Markov Chain, we will order the state space, so that each state can be associated with a row of the probability matrix. We will do so by interpreting a vector $s \in \mathscr{S}$ as a decimal representation in base m.

Definition 6.9.
Let $k, m, n \in \mathbb{N}$. We define the function ξ to be the function

$$\xi : \mathscr{S} \to \{1, \ldots, m^n\} : (i_1, i_2, \ldots, i_n) \mapsto 1 + \sum_{j=1}^{n} i_j m^{n-j}$$

ξ is a bijection and we define χ to its inverse function. ■

Now we will construct the matrix P by $P_{i,j} = \bar{p}(\chi(i), \chi(j))$. P is a stochastic matrix. At last we can define Z_i to be the Markov Chain on $\mathscr{S}$ with transition probability matrix P. The process Z_i is in a sense a discretization of X_i, based on Y. Note that U could be a realization of Z_i.

6.2 Stopping the Markov Chain and the original process

Next, we will present a stopping rule for X_i based on the optimal stopping rule of Z_i.

In order to do efficient calculations, we define $\hat{P}$, which is much like P, but without the zeroes corresponding to infeasible transitions. We define $\hat{p}$ by

$$\hat{p}((i_1, i_2, \ldots, i_n), j) = \bar{p}((i_1, \ldots, i_n), (i_2, \ldots, i_n, j))$$

We then define the entries of $\hat{P}$ by $\hat{P}_{i,j} = \hat{p}(\chi(i), j)$.

To find the optimal stopping rule of Z_n, note that it is a Markov Chain, so that we can use the Wald-Bellman equations (4.26). Note that our utility function is $\hat{G}(i_1, i_2, \ldots, i_n) = g_{i_n}$. We define $\hat{V}$ to be the value function of optimally stopping Z_n. From (4.26) we then obtain

$$\hat{V}^{k+1}(s_1, \ldots s_n) = \max(g_{s_n+1}, \sum_{j=0}^{m-1} \hat{P}_{\xi(s_1,\ldots,s_n),j}\, \hat{V}^k(s_2, \ldots s_n, j)) \tag{6.1}$$

The initial condition is

$$\hat{V}^0(s_1, \ldots s_n) = g_{s_n+1} \tag{6.2}$$

This recursion is straightforward to compute. A more detailed procedure for resolving the recursion is provided in Subsection 6.4.

The preferred optimal stopping policy of this problem, in the sense of Definition 4.21 is

$$\hat{\pi}^k(s_1, \ldots, s_n) = \begin{cases} \text{STOP} & \text{if} \quad g_{s_n+1} = \hat{V}^k(s_1, \ldots s_n) \\ \text{CONT} & \text{otherwise} \end{cases} \tag{6.3}$$

We can transform $\hat{\pi}$ and $\hat{V}$ into approximations of the optimal policy and value function of stopping X_i. We shall call these $\bar{\pi}$ and $\bar{V}$ respectively and these are the main outputs of the procedure. Recall Definition 6.4, which defines the state function σ. We set, for $x \in \mathbb{R}^n$

$$\bar{\pi}^k(x) = \hat{\pi}^k(\sigma(x)) \tag{6.4}$$

$$\bar{V}^k(x) = \hat{V}^k(\sigma(x)) \tag{6.5}$$

6.3 The Markovian case

Now consider the special case in which we know that (X_i) is a Markov process and that in addition it is known that the preferred optimal stopping region D of the original problem of stopping (X_i) must have boundary structure in the sense of Definition 4.22. In this case, we set $n = 1$ in our discretization procedure. Consider the following conjecture, which we state without proof.

Conjecture 6.10. Let (X_i) be a stationary Markov process which satisfies that its preferred optimal stopping region D has boundary structure. For each $K \in \mathbb{N}$, we apply the discretization procedure of Subsection 6.1 with $n = 1$ on the sample $(X_i)_{i=1}^K$, resulting in a discrete time discrete state Markov Chain (Z_i^K). Then the probability that the problem of stopping (Z_i^K) has boundary structure goes to 1 as K goes to infinity.

The conjecture can be seen as consisting of two parts. Firstly, the quantiles, levels, transition probabilities etc all have theoretical analogues, corresponding to expected values rather than approximations by data. This leads to a theoretical discretization of (Z_i) of (X_i). The first part of the conjecture is then that this theoretical discretization (Z_i) has boundary structure whenever (X_i) does.

The second part of the conjecture is that as our sample size K grows larger, (Z_i^K) becomes sufficiently similar to (Z_i) for the problem of stopping (Z_i^K) to also have boundary structure. We argue that the assumption of stationarity of (X_i) is needed for the convergence in the second part, but it is outside the scope of this research to provide a proof.

Now we assume that Conjecture 6.10 is true and let $N \in \mathbb{N}$. We then have that there exists an $i^* \in \{0, \dots, m-1\}$ such that for all $i \in \{i^*, \dots, m-1\}$, we have $\hat{\pi}^N(i) = \text{STOP}$. This gives us a direct way to approximate d_N. Recall Definition 6.1 which defines q_α to be the α-quantile of our input data Y. We have

$$\begin{aligned} d_N &= \inf\{x \in \mathbb{R} \,|\, \pi^N(x) = \text{STOP}\} \\ &\approx \inf\{x \in \mathbb{R} \,|\, \hat{\pi}^N(\sigma(x)) = \text{STOP}\} \\ &= \inf \bigcup \{L_i \,|\, i \in \{1, \dots, m-1\},\, \hat{\pi}^N(i) = \text{STOP}\} \\ &= \inf \bigcup \{L_i \,|\, i \in \{i^*, \dots, m-1\}\} \\ &= \inf L_{i^*} \\ &= q_{i^*/m} \end{aligned} \tag{6.6}$$

6.4 Procedure in terms of matrices and vectors

For any fixed k, we wish to calculate the right hand side of (6.1) for all $s = (s_1, \dots, s_n)$. Calculating the right hand side directly is straightforward, but regardless we present here a so called vectorized approach, that is to say we make use of array programming. The benefit of such an approach is that it may be more efficient and more elegant. We find a procedure for resolving this expression, involving a matrix-vector product. Apart from a step of taking a pairwise maximum and a step in which matrix vector products are taken, all the steps in the procedure are rearrangements of data structures. All rearrangements can be thought of as combinations of splitting or gluing-together rows or

columns, or taking the transpose of a matrix, where a matrix may have vectors as its entries (i.e. it is an order 3 tensor).

In order to express the rearrangements, we make the following definitions

$$\begin{aligned}
&\phi : \mathbb{N} \to \mathbb{N} : i \mapsto (i-1 \mod m) + 1 \\
&\psi : \mathbb{N} \to \mathbb{N} : i \mapsto \lfloor \frac{i-1}{m^{n-1}} \rfloor + 1 \\
&\zeta : \mathbb{N} \to \mathbb{N} : i \mapsto (i-1 \mod m^{n-1}) + 1
\end{aligned}$$

In addition, for a matrix A, we shall write A_i for the i-th row of A and likewise, for a vector v we shall write v_i for its i-th entry. If an index was already present, we shall use an extra pair of parenthesis, so that the i-th entry of $w^{(k)}$ will be written as $(w^{(k)})_i$.

Let, for all $1 \leq k \leq N$, $w^{(k)}$ be the vector satisfying, that its i-th entry $(w^{(k)})_i$ for all $i \in \{1, \ldots, m^n\}$ satisfies

$$(w^{(k)})_i = \hat{V}^{(k)}(\chi(i)) \tag{6.7}$$

We can retrieve the value function $\hat{V}^{(k)}$ from $w^{(k)}$ in a straightforward way as follows

$$\hat{V}^{(k)}(s_1, s_2, \ldots, s_n) = (w^{(k)})_{\xi(s_1, s_2, \ldots, s_n)} \tag{6.8}$$

So we only need to find all the $w^{(k)}$ to know the value functions $V^{(k)}$. Therefore this procedure will recursively find the $w^{(k)}$.

As an initialization for the recursion, for $j \in \{1, \ldots, m^{n-1}\}$, let L_j be the $m \times m$ matrix, that satisfies that for each $i \in \{1, \ldots, m\}$, the i-th row $(L_j)_i$ of L_j satisfies

$$(L_j)_i = \hat{P}_{(i-1)*m^{n-1}+j}$$

The recursive procedure is then as follows. Given, for a fixed k, a vector $w^{(k)}$, we construct the $m^{n-1} \times m$ matrix $B^{(k)}$ from $w^{(k)}$ by the following rearrangement. We set $B^{(k)}$ to be the $m^{n-1} \times m$ matrix whose entries $(B^{(k)})_{i,j}$ satisfy, for $i \in \{1, \ldots, m^{n-1}\}$, $j \in \{1, \ldots, m\}$

$$(B^{(k)})_{i,j} = (w^{(k)})_{(i-1)m+j}$$

We let $M^{(k)}$ be the $m \times m$ matrix which satisfies that its i-th row $(M^{(k)})_i$, for $i \in \{1, \ldots, m^{n-1}\}$, satisfies

$$(M^{(k)})_i = L_i \cdot (B^{(k)})_i \tag{6.9}$$

and let $q^{(k+1)}$ be the vector, satisfying, for $i \in \{1, \dots, m^n\}$

$$(q^{(k+1)})_i = (M^{(k)})_{\zeta(i),\psi(i)} \tag{6.10}$$

We then set $w^{(k+1)}$ as follows, which is justified by Lemma 6.11. We then obtain, for $i \in \{1, \dots, m^n\}$

$$(w^{(k+1)})_i = \max(g_{\phi(i)}, (q^{(k+1)})_i) \tag{6.11}$$

This concludes the procedure.

Lemma 6.11.
The procedure sets $w^{(k+1)}$ correctly.

Proof. First note that, for all $i \in \{1, \dots, m^n\}$

$$\begin{aligned}(\psi(i)-1)m^{n-1} + \zeta(i) &= \lfloor \frac{i-1}{m^{n-1}} \rfloor + (i-1 \mod m^{n-1}) + 1 \\ &= i\end{aligned} \tag{6.12}$$

Now let $i \in \{1, \dots, m^n\}$ and let (s_i) be such that $\xi(s_1, \dots, s_n) = i$. We have

$$(i-1 \mod m^{n-1}) = (\sum_{j=1}^{n} s_i m^{n-j} \mod m^{n-1}) = \sum_{j=2}^{n} s_i m^{n-j} \tag{6.13}$$

This gives us, for all $k \in \{1, \dots, m\}$

$$\begin{aligned}\xi(s_2, \dots s_n, k-1) &= k + \sum_{j=1}^{n-1} s_{j+1} m^{n-j} \\ &= k + m \sum_{j=2}^{n} s_j m^{n-j} \\ &= k + m\,(i-1 \mod m^{n-1}) \\ &= k + m\,(\zeta(i)-1)\end{aligned} \tag{6.14}$$

This gives us

$$\begin{aligned}q_i^{(k+1)} &= (M^{(k)})_{\zeta(i),\psi(i)} \\ &= (L_{\zeta(i)} \cdot (B^{(k)})_{\zeta(i)})_{\psi(i)} \\ &= (L_{\zeta(i)})_{\psi(i)} \cdot (B^{(k)})_{\zeta(i)} \\ &= \hat{P}_{(\psi(i)-1)m^{n-1}+\zeta(i)} \cdot (w^{(k)}_{(\zeta(i)-1)m+j})_{j=1}^{m}\end{aligned} \tag{6.15}$$

Now we use the equations (6.12) and (6.14) above to find

$$\begin{aligned}q_i^{(k+1)} &= \hat{P}_i \cdot (w^{(k)}_{\xi(s_2,\dots,s_n,j-1)})_{j=1}^{m} \\ &= \sum_{j=0}^{m-1} \hat{P}_{\xi(s_1,\dots,s_n),j}\,(\hat{V}^{(k)}(s_2, \dots, s_n, j))\end{aligned} \tag{6.16}$$

Because $w_i^{(k+1)}$ is simply the maximum of $q_i^{(k+1)}$ and $g_{\phi(i)}$, we see that this corresponds to the Wald Bellman equations (6.1).

□

6.5 Example

We present an example where $n = 1$, so that our discrete model is a proper Markov Chain in the sense that we do not care about the history of the process beyond one value (for larger n we also construct a Markov chain, but we have to include "history" in the state). The process used to generate the data is an AR(1) process with regression parameter $b = \frac{1}{2}$, which is a Markov process, so that $n = 1$ is a good choice. We generate $K = 10^6$ data points using this process. We distinguish $m = 10$ levels and the horizon is $N = 11$. We have run the procedure, including the generation of the data 20 times.

The quantiles the discretization procedure finds in each of the 20 runs corresponds closely to quantiles of the stationary distribution, which has a normal distribution with mean 0 and standard deviation $\frac{2}{\sqrt{3}}$. The quantiles found by our procedure in the last run are listed below, with the theoretical quantiles found by using the stationary distribution alongside them.

Algorithm	Theoretical
-1.490	-1.480
-0.977	-0.972
-0.607	-0.606
-0.295	-0.293
-0.003	0.000
0.289	0.293
0.601	0.606
0.969	0.972
1.480	1.480

Figure 6: Quantiles found by the discretization procedure compared with the theoretical quantiles of the stationary distribution.

The quantiles the procedure finds are associated with levels in the sense of Definition 6.2. The procedure distinguishes $m = 10$ levels, which are the intervals $L_0^{10} = (-\infty, -1.49)$, $L_1^{10} = [-1.49, -0.98)$, $L_2^{10} = [-0.98, -0.61)$ and so on, with the last level $L_{10}^{10} = [1.48, \infty)$.

Because $n = 1$, the level function ρ of Definition 6.3 coincides with the state function σ of Definition 6.4. As described in Subsection 6.1, using this function

transitions between levels are tallied and probabilities are associated with each pair of states. Because $n = 1$, the matrix P and the matrix $\hat{P}$ again coincide. By this process, we construct the stochastic matrix shown in Figure 7.

$$\begin{pmatrix}
0.32 & 0.19 & 0.14 & 0.10 & 0.08 & 0.06 & 0.04 & 0.03 & 0.02 & 0.01 \\
0.19 & 0.17 & 0.14 & 0.12 & 0.10 & 0.09 & 0.07 & 0.05 & 0.04 & 0.02 \\
0.14 & 0.14 & 0.14 & 0.12 & 0.11 & 0.10 & 0.09 & 0.07 & 0.06 & 0.03 \\
0.11 & 0.12 & 0.12 & 0.12 & 0.12 & 0.11 & 0.10 & 0.09 & 0.07 & 0.04 \\
0.08 & 0.10 & 0.11 & 0.12 & 0.12 & 0.11 & 0.11 & 0.10 & 0.09 & 0.06 \\
0.06 & 0.09 & 0.10 & 0.11 & 0.11 & 0.12 & 0.12 & 0.11 & 0.10 & 0.08 \\
0.04 & 0.07 & 0.09 & 0.10 & 0.11 & 0.11 & 0.12 & 0.12 & 0.12 & 0.11 \\
0.03 & 0.06 & 0.07 & 0.09 & 0.10 & 0.11 & 0.12 & 0.14 & 0.14 & 0.14 \\
0.02 & 0.04 & 0.06 & 0.07 & 0.09 & 0.10 & 0.12 & 0.14 & 0.17 & 0.19 \\
0.01 & 0.02 & 0.03 & 0.05 & 0.06 & 0.08 & 0.10 & 0.14 & 0.19 & 0.32
\end{pmatrix}$$

Figure 7: Stochastic matrix P found by the discretization approach

In the matrix shown in Figure 8, a 1 represents the decision to continue and a 0 represents the decision to stop. The matrix is $(\hat{\pi}^k(i))_{k,i}$, where each column corresponds to a state i with the first column corresponding to a state associated with the lowest values. Each row corresponds essentially to how much time is left before the deadline and with the top row corresponding to the maximum amount of time left. In this way, the last row corresponds to the last decision, which is taken at time $N - 1$. If at this time we do not decide to stop and obtain X_{N-1}, we will always get X_N, so that there is no decision at time N.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 8: Matrix of decisions for each state as we approach the deadline

As described in Subsection 6.2, using the function $\hat{\pi}^k$ which is completely determined by the matrix above, we can find a policy $\bar{\pi}$ which is a stopping policy for the original problem.

We see that all the zeroes appear to the right of the ones. This corresponds to a policy that stops whenever the observed value of the process is higher than a fixed value, depending on the time left until the deadline. We will show the optimal stopping policy of an AR(1) process has boundary structure in Section 7.1, in Proposition 7.13. So this is a case where the original preferred optimal stopping region (D_n) of the original Markov process has boundary structure in the sense of Definition 4.22 and the corresponding discretization has this same structure. This supports our Conjecture 6.10.

The fact that the zeroes only appear below ones is a consequence of (4.40). It immediately follows that the "boundary sequence" (d_k) of our discrete time discrete state Markov Chain is increasing (as stated in Lemma 4.23), which can be seen in our matrix from the fact that as we go down the rows, the column number of the leftmost zero is decreasing.

7 AR(1) Processes

7.0 Basic results and definitions

Let $(X_n)_{n=1}^N$ be an AR(1) process. That is, X_n follows the equation

$$X_{n+1} = bX_n + \epsilon_n \tag{7.1}$$

Here the ϵ_n are i.i.d. and are called innovations. Various distributions of ϵ_n are of interest, but the most usual distribution it has in models is a normal distribution. In [8], Grunwald et al. give an overview of what distributions of innovations are generally used in models. We will assume that ϵ_n has zero mean. In addition we assume that $0 < b < 1$. We shall consider the problem of optimally stopping such a process before a finite horizon N, when the distribution of ϵ and the value of b are known.

Here by optimally stopping a process, we mean that we consider the utility function G of Subsection 4.1 to be the identity $G(x) = x$ and we try to optimise the supremum in (4.8). The infinite horizon case, where payoffs are discounted so they do not grow to infinity, bears some similarity to the finite horizon case and is treated in [3] and [2].

The fact that $0 < b < 1$ implies that the process is stationary. Note that the process becomes a Martingale for $b = 1$. In this case, Theorem 9.15 of [21] then tells us, because τ is bounded, that our expected payoff is always X_0. It is therefore natural to dismiss this case.

For all $1 \leq n \leq N$ we define the value of continuing, $\mathcal{Q}^n$, to be the following function

$$\mathcal{Q}^n : \mathbb{R} \to \mathbb{R} : x \mapsto \mathbb{E}_x[V^{n-1}(X_1)]$$

This function is analogous to the Q function in the theory of Markov Decision Processes, which maps pairs (s, a) of a state and an action to a real number, which is the expected reward. We can then think of $\mathcal{Q}^n$ as $\mathcal{Q}^n(x) = Q^n(x, c)$, where c is the name of the action "continue".

In our notation, the Wald-Bellman equations (4.26) are, for $n \in \mathbb{N}$

$$V^n(x) = \max(x, \mathcal{Q}^n(x)) \tag{7.2}$$

We know by the definition (4.8) of V, that

$$V^0(x) = \sup_{0 \leq \tau \leq 0} \mathbb{E}_x[G(X_\tau)] = \mathbb{E}_x[X_0] = x$$

Therefore we see that, because the ϵ_n are assumed to have zero mean,

$$\mathcal{Q}^1(x) = \mathbb{E}_x[V^0(X_1)] = \mathbb{E}_x[bX_0 + \epsilon_0] = bx$$

Because of our assumption that $0 < b < 1$, we find

$$D_1 = \{x \geq Q^1(x)\} = \{x \geq bx\} = [0, \infty) \tag{7.3}$$

Therefore we also find $d_1 = \inf D_1 = 0$.

Note that d_1 is the unique solution of $x = Q^1(x)$. It will be proven later that for each $1 \leq n \leq N$, d_n is the unique solution of $\mathcal{Q}^n(d_n) = d_n$ and that D has a boundary in the sense of Definition 4.22. It is then useful to find these d_n because we then have

$$\begin{aligned} \tau_D &= \inf \{n \in \{0, \ldots, N\} \,|\, X_n \in D_{N-n}\} \\ &= \inf \{n \in \{0, \ldots, N\} \,|\, X_n \geq d_{N-n}\} \end{aligned} \tag{7.4}$$

So that in this case this optimal stopping strategy takes a simple form. The d_n are the points at which we are indifferent between continuing or stopping.

For $n = 1$ we find by (7.2)

$$V^1(x) = \max(x, \mathcal{Q}^1(x)) = \begin{cases} x & x \geq 0 \\ b\,x & x < 0 \end{cases}$$

For higher n, $\mathcal{Q}^n(x)$, $V^n(x)$ and d_n will depend on the distribution of ϵ_1. The derivations above hint at a way to use (4.26) to recursively find $V^n(x)$, $\mathcal{Q}^n(x)$ and d_n. In this way, for the distributions we consider here, we can find some expression for $\mathcal{Q}^n(x)$ and $V^n(x)$ for $n = 2$, although for the normal distribution $\mathcal{Q}^2(x)$ will be expressed in terms of the error function.

After doing a few steps of the recursion, it becomes impossible to compute the expectations exactly, for any distribution of the innovations of interest. We list some value functions for various distributions here, to convince the reader that these integrals can become quite difficult.

Let erf be the error function.

For $\epsilon_i \sim N[0, 1]$, we have

$$\mathcal{Q}_N^2(x) = \frac{(1-b)e^{-\frac{1}{2}b^2x^2}}{\sqrt{2\pi}} + \frac{1}{2}bx\left((1-b)\text{erf}\left(\frac{bx}{\sqrt{2}}\right) + b + 1\right) \tag{7.5}$$

For $\epsilon_i \sim U[-1,1]$, we have

$$\mathcal{Q}^2_U(x) = \begin{cases} b^2 x & b\,x \leq -1 \\ \frac{1}{4}(1+b(-1+x(2+b(2+x-b\,x)))) & -1 < b\,x \leq 1 \\ b\,x & 1 < b\,x \end{cases}$$

7.1 Structure of Q^n

In this subsection, we shall prove lemmas about the structure of Q^n and V^n and D_n. We shall show that Q^n has linear asymptotes, provide conditions for Q^n to be differentiable and strictly convex and show that D_n has "boundary structure" in the sense of Definition 4.22.

Throughout this subsection, we shall let Z be a random variable that has the same distribution as the innovations ϵ_i of our AR(1) process. We let F be the distribution function of Z and μ the measure to which F corresponds (in the sense of Theorem 1 of §2 of Chapter II of [18], μ is also called the law of Z). Recall that we made following two assumptions

$$(\mathbb{E}[Z] =) \int y\,\mu(dy) = 0 \tag{7.6}$$

$$0 < b < 1 \tag{7.7}$$

We collect here some lemmas that are needed to prove that the value functions have a certain structure.

Throughout this subsection, let $l_n^{(1)}$ and $l_n^{(2)}$ be the functions

$$l_n^{(1)} : \mathbb{R} \to \mathbb{R} : x \mapsto b^n x$$
$$l_n^{(2)} : \mathbb{R} \to \mathbb{R} : x \mapsto bx.$$

Definition 7.1 (**Asymptote**)**.**
A linear function l is an asymptote of f if for some $T \in \{-\infty, \infty\}$

$$\lim_{x \to T} f(x) - l(x) = 0$$

If $T = -\infty$ we call l a left asymptote and if $T = \infty$ we call it a right asymptote.

Definition 7.2 (Convex function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is called convex if for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \tag{7.8}$$

■

Definition 7.3 (Contraction Mapping)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is a contraction mapping if there exists a constant $k \in [0, 1)$ such that for all $x, y \in \mathbb{R}$

$$|f(y) - f(x)| \leq k|y - x| \tag{7.9}$$

■

Consider the following assertion. We will indirectly show that if Z admits a density, the assertion holds for all $n \in \{1, \ldots, N\}$, but the purpose of including the assertion is only to enable compact formulations of the lemmas that follow, so that a proof is contained in the proofs of those lemmas.

Assertion 7.4. Let $n \in \{1, \ldots, N\}$. $l_n^{(1)}$ is a left asymptote of Q^n, $l_n^{(2)}$ is a right asymptote of Q^n. Furthermore, there exists a unique solution of $Q^n(x) = x$, called d_n and $D_n = [d_n, \infty)$.

Lemma 7.5. Assertion 7.4 holds for $n = 1$, Q^1 is differentiable and convex and Q^1 is a contraction mapping.

Proof. We have

$$Q^1(x) = l_1^{(1)}(x) = l_1^{(2)}(x) = bx$$

So that indeed $l_1^{(1)}$ is a left asymptote of Q^1, $l_1^{(2)}$ is a right asymptote, Q^1 is differentiable and Q^1 is convex. Q^1 is a contraction mapping for $k = \frac{1}{2}(b + 1)$. There exists a unique solution $d_1 = 0$ of $Q^1(x) = x$. Lastly, we have

$$D_1 = \{x \in \mathbb{R} | V^1(x) = x\} = \{x \in \mathbb{R} | \mathbb{1}_{x \geq d_1} x + \mathbb{1}_{x < d_1} bx = x\} = [d_1, \infty)$$

□

Lemma 7.6. Assume Assertion 7.4 holds and Q^n is convex. Then V^n has left asymptote $l^{(1)}$ and right asymptote $l : x \mapsto x$ and V^n is convex. Furthermore

$$V^n(x) = \begin{cases} x & x \geq d_n \\ Q^n(x) & x < d_n \end{cases} \tag{7.10}$$

Proof. Note that $V^n(x) = \max(x, Q^n(x))$, which are our Wald-Bellman equations (7.2). Equation (7.10) follows directly from the fact that $D_n = [d_n, \infty)$.

To see that V^n is convex, note that by assumption, Q^n has right asymptote bx. So we have, by Lemma 10.12, that for all $x < z$

$$\frac{Q^n(z) - Q^n(x)}{z - x} \leq b \tag{7.11}$$

This gives us

$$\sup_{x \in (-\infty, d_n)} \frac{Q^n(d_n) - Q^n(x)}{d_n - x} \leq b < 1 = \inf_{x \in (d_n, \infty)} \frac{x - d_n}{x - d_n} \tag{7.12}$$

Also note that by assumption, $Q^n(d_n) = d_n$. Lemma 10.15 then gives us that V^n is convex.

The fact that fact that V^n has $l^{(1)}$ and l as asymptotes follows directly from (7.10) and the fact that Q^n has $l^{(1)}$ a left asymptote.

□

Lemma 7.7. Assume Assertion 7.4 holds. Let $x \in \mathbb{R}$. Then $V^n(bx + Z) \in L^1(\mu)$.

Proof. For all $y \in \mathbb{R}$, we have $V^n(y) \leq \max(d_{N-n}, y)$. Therefore, for a fixed x, because $Z \in L^1$, we have

$$\mathbb{E}[(V^n(bx + Z))^+] \leq \mathbb{E}[\max(d_{N-n}, bx + Z)] < \infty \tag{7.13}$$

Furthermore, we have that $b^n x$ is a left asymptote of V^n, so that $b^n x$ is a lower bound of V^n, which gives us that $(b^n x)^-$ is an upper bound of $(V^n)^-$,

$$\begin{aligned}\mathbb{E}[(V^n(bx + Z))^-] &\leq \mathbb{E}[(b^n(bx + Z))^-] \leq \mathbb{E}[(bx + Z)^-] \\ &\leq \mathbb{E}[(bx)^- + (Z)^-] < \infty\end{aligned}$$

□

Using the notation of this subsection, we have, for all $x \in \mathbb{R}$,

$$\begin{aligned}Q^{n+1}(x) &= \mathbb{E}_x[V^n(X_1)] \\ &= \mathbb{E}_x[V^n(bX_0 + \epsilon_1)] \\ &= \int V^n(bx + y)\mu(dy)\end{aligned} \tag{7.14}$$

Lemma 7.8. Assume Assertion 7.4 holds and that Q^n is convex. Then Q^{n+1} is convex.

Proof. Consider $h(x, y) = V^n(bx + y)$. Because the composition of convex functions is convex (i.e. Lemma 10.10), for all $y \in \mathbb{R}$, $x \mapsto h(x, y)$ is convex. By Lemma 7.7, for all $x \in \mathbb{R}$, $\int |h(x, y)|\mu(dy) < \infty$. It follows from Lemma 10.21 that $x \mapsto \int h(x, y)\mu(dy)$ is convex, that is $Q^{n+1}(x) = \int V^n(bx + y)\mu(dy)$ is convex. □

Lemma 7.9. Let $n \in \{1, \ldots, N\}$. If Assertion 7.4 holds for n and Q^n is convex, then Q^{n+1} has left asymptote $l_n^{(1)} : x \mapsto b^{n+1}x$ and right asymptote $l_n^{(2)} : x \mapsto bx$.

Proof. Let

$$\theta(x) = \begin{cases} bx & x \geq 0 \\ b^{n+1}x & x < 0 \end{cases}$$

Let $y \in \mathbb{R}$ be arbitrary. Because $b^n x$ is a left asymptote of V^n, it follows from Lemma 10.24 that $b^n(bx+y)$ is a left asymptote of $x \mapsto V^n(bx+y)$. Similarly, $x \mapsto bx+y$ is a right asymptote of $x \mapsto V^n(bx+y)$. Indeed we have the following limits

$$\lim_{x \to -\infty} V^n(bx+y) - \theta(x) = b^n y \tag{7.15}$$

$$\lim_{x \to \infty} V^n(bx+y) - \theta(x) = y \tag{7.16}$$

Because by assumption Assertion 7.4 holds for n and Q^n is convex, it follows from Lemma 7.6 that V^n is convex. Because V^n is convex, we have by Lemma 10.10, that $x \mapsto V^n(bx+y)$ is convex. Because a convex function lies above its asymptotes (by Lemma 10.14), we have for $x < 0$,

$$V^n(bx+y) - \theta(x) \geq b^{n+1}x + b^n y - b^{n+1}x \geq -|y|$$

For $x \geq 0$, we have

$$V^n(bx+y) - \theta(x) \geq bx + y - bx \geq -|y|$$

So for all $x \in \mathbb{R}$, we have

$$V^n(bx+y) - \theta(x) \geq -|y| \tag{7.17}$$

Now we shall show that $V^n(bx+y) - \theta(x)$ attains a maximum $V^n(y)$ at $x = 0$. Let $x < z \leq 0$. Because $x \mapsto V^n(bx+y)$ is convex with left asymptote $b^n(bx+y)$, we have, by Lemma 10.13,

$$\frac{V^n(bz+y) - V^n(bx+y)}{z-x} \geq b^{n+1} \tag{7.18}$$

This gives us

$$\begin{aligned} &V^n(bz+y) - \theta(z) - (V^n(bx+y) - \theta(x)) \\ =& V^n(bz+y) - V^n(bx+y) - (\theta(z) - \theta(x)) \\ \geq& (z-x)b^{n+1} - (b^{n+1}z - b^{n+1}x) \\ =& 0 \end{aligned} \tag{7.19}$$

So that on $(-\infty, 0]$, $x \mapsto V^n(bx+y) - \theta(x)$ is increasing. Similarly, on $[0, \infty)$, $x \mapsto V^n(bx+y) - \theta(x)$ is decreasing.

Because V^n is convex, it is continuous. θ is continuous, so $V^n(bx+y) - \theta(x)$ is also continuous in x. Note that $V^n(y) = V^n(b\,0+y) - \theta(0)$, We have (by Theorem 3.27 of [1])

$$V^n(y) = \sup\{V^n(bx+y) - \theta(x) \,|\, x < 0\} \tag{7.20}$$

Similarly $V^n(y) = \sup\{V^n(bx+y) - \theta(x) \,|\, x > 0\}$. So indeed we have shown that $V^n(y)$ is the maximum, i.e. we have for all $x \in \mathbb{R}$,

$$V^n(bx+y) - \theta(x) \leq V^n(y) \tag{7.21}$$

Let $w(y) = \max(|y|, |V^n(y)|)$. Combining (7.21) with (7.17), we obtain, for all $x \in \mathbb{R}$,

$$|V^n(bx+y) - \theta(x)| \leq w(y) \tag{7.22}$$

Because of our assumption (7.6), (and part (iii) of Theorem 10.3 from [16]), we have $\int |y|\mu(dy) < \infty$. By Lemma 7.7, we have $\int V^n(y)\mu(dy) < \infty$. Therefore (by part (iii) of Theorem 10.4 from [16]), we have

$$\int w(y)\mu(dy) < \infty \tag{7.23}$$

Now consider Lemma 10.7 with its named conditions A, B, and C. Because A follows Lemma 7.7, and B is (7.15), and C follows from (7.23) and (7.22), it follows from Lemma 10.7, that

$$\lim_{x\to-\infty} \int V^n(bx+y) - \theta(x)\mu(dy) = \int \lim_{x\to-\infty} V^n(bx+y) - \theta(x)\mu(dy) \tag{7.24}$$

Similarly it follows from Lemma 10.7, because B is (7.16) that

$$\lim_{x\to\infty} \int V^n(bx+y) - \theta(x)\mu(dy) = \int \lim_{x\to\infty} V^n(bx+y) - \theta(x)\mu(dy) \tag{7.25}$$

By (7.24) and (7.15), we get

$$\begin{aligned}
\lim_{x\to-\infty} Q^{n+1}(x) - b^{n+1}x &= \lim_{x\to-\infty} \int V^{n+1}(bx+y) - \theta(x)\mu(dy) \\
&= \int \lim_{x\to-\infty} V^n(bx+y) - \theta(x)\mu(dy) \\
&= \int b^n y \,\mu(dy) = 0
\end{aligned}$$

Similarly, by (7.25) and (7.16), we get

$$\begin{aligned}
\lim_{x\to-\infty} Q^{n+1}(x) - bx &= \lim_{x\to\infty} \int V^{n+1}(bx+y) - \theta(x)\mu(dy) \\
&= \int \lim_{x\to\infty} V^n(bx+y) - \theta(x)\,\mu(dy) \\
&= \int by\mu(dy) = 0
\end{aligned}$$

□

Lemma 7.10. Let $n \in \{1, \ldots, N\}$. If Assertion 7.4 holds for n and Q^n is convex, then Q^{n+1} is a contraction mapping and Q^{n+1} is strictly increasing. $Q^{n+1}(x) = x$ has d_{n+1} as its unique solution and $D_{n+1} = [d_{n+1}, \infty)$.

Proof. It follows from Lemma 7.9 that Q^{n+1} has $b^{n+1}x$ as a left asymptote and bx as a right asymptote and it follows from Lemma 7.8 that Q^{n+1} is convex. The fact that Q^{n+1} is a contraction mapping then follows from the fact that $-1 < b^{n+1}$ and $b < 1$ and Lemma 10.22. The fact that Q^{n+1} is strictly increasing follows from $0 < b^{n+1}$ and Lemma 10.23. Because Q^{n+1} is a contraction mapping, it follows from Theorem 10.26 that $Q^{n+1}(x) = x$ has a unique solution γ. $x \mapsto Q^{n+1}(x) - x$ has right asymptote $x \mapsto -bx$ and it follows from $-b < 0$ and Lemma 10.23 that $x \mapsto Q^{n+1}(x) - x$ is strictly decreasing. So for all $x \in [\gamma, \infty)$ we have $Q^{n+1}(x) \leq x$, so that

$$D_{n+1} = \{x \in \mathbb{R} \,|\, V^{n+1}(x) = x\} = \{x \in \mathbb{R} \,|\, Q^{n+1}(x) \leq x\} = [\gamma, \infty) \tag{7.26}$$

We then have $d_{n+1} = \inf D_{n+1} = \gamma$, so that d_{n+1} is the unique solution of $Q(x) = x$ and $D_{n+1} = [d_{n+1}, \infty)$. □

The following Lemma is almost an application of Theorem 11.5 in [16] and the proof of this lemma is similar to the proof of this theorem.

Lemma 7.11. Assume that Assertion 7.4 holds and that Q^n is convex and differentiable. Furthermore, assume that μ is absolutely continuous w.r.t λ, with density $f = \frac{d\mu}{d\lambda}$ and distribution function F. Then Q^{n+1} is differentiable with derivative

$$\begin{aligned} \dot{Q}^{n+1}(x) &= b \int_{\mathbb{R} \setminus \{d_n - bx\}} \dot{V}^n(bx + y)\mu(dy) \\ &= b(1 - F(d_n - bx) + \int_{(-\infty, d_n - bx)} \dot{Q}^n(bx + y) f(y) \lambda(dy)) \end{aligned} \tag{7.27}$$

Proof. Consider $Q_x = \mathbb{R} \setminus \{d_n - bx\}$ and let μ_x^* be the restriction of μ to Q_x. We have (by the remarks at the start of Subsection 4.3 in [21]) that $(Q_x, \mathcal{B}(Q_x), \mu_x^*)$ is a probability space. Let, for all $x, y \in \mathbb{R}$, $\phi(x, y) = V^n(bx + y)$. Let $x \in \mathbb{R}$ be arbitrary and let, for all $z \neq x$

$$\rho_x(z) = \frac{Q^{n+1}(z) - Q^{n+1}(x)}{z - x} \tag{7.28}$$

We have

$$\begin{aligned}\rho_x(z) &= \frac{Q^{n+1}(z) - Q^{n+1}(x)}{z-x} \\ &= \frac{\int \phi(z,y)\mu(dy) - \int \phi(x,y)\mu(dy)}{z-x} \\ &= \int \frac{\phi(z,y)-\phi(x,y)}{z-x}\mu(dy)\end{aligned}$$

Now because μ is absolutely continuous w.r.t. λ and $\lambda(\{d_n - bx\}) = 0$, we have that $\mu(\{d_n - bx\}) = 0$, so that (by Proposition 4.9 and Proposition 4.21 in [21])

$$\begin{aligned}\rho_x(z) &= \int \mathbb{1}_{Q_x}(y)\frac{\phi(z,y)-\phi(x,y)}{z-x}\mu(dy) \\ &= \int \frac{\phi(z,y)-\phi(x,y)}{z-x}\mu_x^*(dy)\end{aligned} \tag{7.29}$$

We therefore have, if the limit exists

$$\begin{aligned}\dot{Q}^{n+1}(x) &= \lim_{z\to x}\rho_x(z) \\ &= \lim_{z\to x}\int \frac{\phi(z,y)-\phi(x,y)}{z-x}\mu_x^*(dy)\end{aligned} \tag{7.30}$$

We shall show that the limit exists and show that the limit and the integral can be interchanged. We will apply Theorem 10.2 so that we can apply the Lebesgue Dominated Convergence theorem to the resulting sequences.

Let, for all $x, y \in \mathbb{R}$, $\sigma_y(x) = \phi(x,y) = V^n(bx+y)$. Let $y \in \mathbb{R}$ be arbitrary. Again, because $x \mapsto b^n x$ is a left asymptote of V^n, it follows from Lemma 10.24 that $b^n(bx+y)$ is a left asymptote of σ_y. Similarly, $x \mapsto bx + y$ is a right asymptote of σ_y. Because σ_y is the composition of two convex functions, it follows from Lemma 10.10 that σ_y is convex. Lemma 10.12 gives us, that for all $x, z \in \mathbb{R}$ with $x < z$, we have

$$b^n \leq \frac{\sigma_y(z)-\sigma_y(x)}{z-x} \leq b \tag{7.31}$$

This gives us, for all $x, z \in \mathbb{R}$ with $x \neq z$

$$|\frac{\phi(z,y)-\phi(x,y)}{z-x} = |\frac{\sigma_y(z)-\sigma_y(x)}{z-x}| \leq b \tag{7.32}$$

Recall the piecewise structure of V^n found in Lemma 7.6. Because Q^n is differentiable, we have, for all $x \in (-\infty, \frac{d_n - y}{b})$

$$\frac{d}{dx}\sigma_y(x) = \frac{d}{dx}Q^n(bx+y) = b\dot{Q}^n(bx+y) \tag{7.33}$$

For all $x \in (\frac{d_n - y}{b}, \infty)$, we have

$$\frac{d}{dx}\sigma_y(x) = \frac{d}{dx}bx + y = b \tag{7.34}$$

Now let $x \in \mathbb{R}$ be arbitrary. We have, because of (7.33), for all $y \in (-\infty, d_n - bx)$

$$\lim_{z \to x} \frac{\sigma_y(z) - \sigma_y(x)}{z - x} = b\dot{Q}^n(bx + y) \tag{7.35}$$

Similarly, for all $y \in (d_n - bx, \infty)$, we have

$$\lim_{z \to x} \frac{\sigma_y(z) - \sigma_y(x)}{z - x} = b \tag{7.36}$$

Now let (a_j) be a sequence with $\lim_{j\to\infty} a_j = x$ and $a_j \neq x$ and let, for all $y \in Q_x$

$$u_{j,x}(y) = \frac{\phi(a_j, y) - \phi(x, y)}{a_j - x} = \frac{\sigma_y(a_j) - \sigma_y(x)}{a_j - x} \tag{7.37}$$

We have, by Theorem 10.2 and (7.35), for all $y \in (-\infty, d_n - bx)$

$$\lim_{j \to \infty} u_{j,x}(y) = \lim_{z \to x} \frac{\sigma_y(z) - \sigma_y(x)}{z - x} = b\dot{Q}^n(bx + y) \tag{7.38}$$

Similarly, for all $y \in (d_{N-n} - bx, \infty)$

$$\lim_{j \to \infty} u_{j,x}(y) = b \tag{7.39}$$

So,

$$\forall y \in Q_x : \lim_{j \to \infty} u_{j,x}(y) \in \mathbb{R} \tag{7.40}$$

Furthermore, by a triangle inequality and Lemma 7.7, we have

$$\begin{aligned}
&\int |u_{j,x}(y)|\mu_x^*(dy) = \int |u_{j,x}(y)|\mu_x^*(dy) \\
=&|\frac{1}{a_j - x}| \int |\phi(a_j, y) - \phi(x, y)|\mu_x^*(dy) \\
\leq&|\frac{1}{a_j - x}|(\int |\phi(a_j, y)|\mu_x^*(dy) + \int |\phi(x, y)|\mu_x^*(dy)) \\
=&|\frac{1}{a_j - x}|(\int |V^n(a_j x + y)|\mu(dy) + \int |V^n(bx + y)|\mu(dy)) \\
<&\infty
\end{aligned} \tag{7.41}$$

So we have that $u_{j,x} \in L^1(Q_x, \mathcal{B}(Q_x), \mu_x^*)$. Now let $w(y) = b$, we have, by (7.32)

$$|u_{j,x}(y)| = |\frac{\phi(a_j, y) - \phi(x, y)}{a_j - x}| \leq b = w(y) \tag{7.42}$$

Lastly, we have

$$\int |w(y)|\mu_x^*(dy) = \int b\mu_x^*(dy) = b \tag{7.43}$$

It now follows by the Lebesgue Dominated Convergence theorem, i.e. Theorem 11.2 in [16], using (7.40), (7.41), (7.42) and (7.43), that

$$\begin{aligned} &\lim_{j\to\infty} \int \frac{\phi(a_j, y) - \phi(x, y)}{a_j - x} \mu_x^*(dy) = \lim_{j\to\infty} \int u_{j,x}(y)\mu_x^*(dy) \\ &= \int \lim_{j\to\infty} u_{j,x}(y)\mu_x^*(dy) = \int \lim_{j\to\infty} \frac{\phi(a_j, y) - \phi(x, y)}{a_j - x} \mu_x^*(dy) \end{aligned} \tag{7.44}$$

Because (a_j) was an arbitrary sequence with $\lim_{j\to\infty} a_j = x$ and $\forall j \in \mathbb{N} : a_j \neq x$, the equation above holds for all such sequences. It then follows from Theorem 10.2 that

$$\lim_{z\to x} \int \frac{\phi(z, y) - \phi(x, y)}{z - x} \mu_x^*(dy) = \int \lim_{z\to x} \frac{\phi(z, y) - \phi(x, y)}{z - x} \mu_x^*(dy) \tag{7.45}$$

Note that $x \in \mathbb{R}$ was arbitrary, so the equation above holds for all $x \in \mathbb{R}$. So indeed the limit in (7.30) exists and the limit and integral can be interchanged. We find for all $x \in \mathbb{R}$

$$\begin{aligned} \dot{Q}^{n+1}(x) &= \int \lim_{z\to x} \frac{\phi(z, y) - \phi(x, y)}{z - x} \mu_x^*(dy) \\ &= b \int_{\mathbb{R}\setminus\{d_n - bx\}} \dot{V}^n(bx + y)\mu(dy) \end{aligned}$$

We find, for all $x \in \mathbb{R}$

$$\begin{aligned} \dot{Q}^{n+1}(x) &= b \int_{(-\infty, d_n - bx)} \dot{Q}^n(bx + y)\mu(dy) \\ &\quad + b \int_{(d_n - bx, \infty)} 1 \, \mu(dy) \\ &= b\,\mu(d_n - bx, \infty) + b \int_{(-\infty, d_n - bx)} \dot{Q}^n(bx + y)\mu(dy) \end{aligned}$$

Writing this in terms of the density function and the distribution function gives the result, i.e. for all $x \in \mathbb{R}$

$$\dot{Q}^{n+1}(x) = b(1 - F(d_{N-n} - bx) + \int_{(-\infty, d_n - bx)} \dot{Q}^n(bx + y)f(y)\lambda(dy)) \tag{7.46}$$

□

Lemma 7.12. Let $n \in \{1, \dots, N\}$. If Assertion 7.4 holds for n and Q^n is differentiable and convex and μ is absolutely continuous w.r.t λ, then Q^{n+1} is strictly convex.

Proof. Let $x, z \in \mathbb{R}$, $z > x$. We have, by Lemma 7.11 and the fact that for all $y \in \mathbb{R}$, $\mu(\{y\}) = 0$,

$$\begin{aligned}
&\dot{Q}^{n+1}(z) - \dot{Q}^{n+1}(x) \\
&= b \int_{\mathbb{R}\setminus\{d_n - bz\}} \dot{V}^n(bz + y)\mu(dy) - b \int_{\mathbb{R}\setminus\{d_n - bx\}} \dot{V}^n(bx + y)\mu(dy) \\
&= b \int_{\mathbb{R}\setminus\{d_n - bx,\, d_n - bz\}} \dot{V}^n(bz + y) - \dot{V}^n(bx + y)\mu(dy)
\end{aligned}$$

Because, by Lemma 7.6, V^n is convex, so that $\dot{V}^n$ is increasing, which gives us that for all $y \notin \{d_n - bx,\, d_n - bz\}$, $\dot{V}^n(bz + y) - \dot{V}^n(bx + y) \geq 0$. Therefore, because the right hand side above is greater than or equal to the right hand side below, we obtain

$$\dot{Q}^{n+1}(z) - \dot{Q}^{n+1}(x) \geq b \int_{(d_n - bz, d_n - bx)} \dot{V}^n(bz + y) - \dot{V}^n(bx + y)\mu(dy) \quad (7.47)$$

For $y \in (d_n - bz, d_n - bx)$, we have $bz + y > d_n$ and therefore $\dot{V}^n(bz + y) = 1$. Furthermore $bx + y < d_n$, so $\dot{V}^n(bx + y) = \dot{Q}^n(bx + y)$. Because Q^n is convex with right asymptote $l^{(2)}(x) = bx$, we have, by Lemma 10.13, $\dot{Q}^n \leq b < 1$, so that $1 - \dot{Q}^n(bx + y) > 0$. Therefore

$$\begin{aligned}
\dot{Q}^{n+1}(z) - \dot{Q}^{n+1}(x) &\geq b \int_{(d_n - bz, d_n - bx)} 1 - \dot{Q}^n(bx + y)\mu(dy) \\
&> 0
\end{aligned}$$

This shows that $\dot{Q}^{n+1}$ is strictly increasing, so that Q^{n+1} is strictly convex.

□

Proposition 7.13. Assertion 7.4 holds for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, Q^n is convex and strictly increasing and Q^n is a contraction mapping.

Proof. Assertion 7.4 holds for $n = 1$ by Lemma 7.5 and by the same lemma Q^1 is convex and a contraction mapping. It follows from Lemma 7.8, Lemma 7.9 and Lemma 7.10 by induction that for all $n \in \mathbb{N}$ Assertion 7.4 holds and that Q^n is convex and strictly increasing and Q^n is a contraction mapping. □

Corollary 7.14. The sequence stopping of stopping regions (D_n) has a boundary in the sense of Definition 4.22.

Proof. This follows directly from Proposition 7.13, as it this states that for all $n \in \mathbb{N}$ Assertion 7.4 holds. □

Proposition 7.15. If μ is absolutely continuous w.r.t. λ, then for all $n \in \{2, \dots\}$, Q^n is strictly convex and differentiable.

Proof. This follows from Proposition 7.13, Lemma 7.11 Lemma 7.12 and by induction. □

7.2 More calculations for normal innovations

First we formally derive the formula found for Q_N^2, which is equation (7.5). Let $\mu \in \mathbb{R}$. Let f_N be the density function of the standard normal distribution. Then

$$
\begin{aligned}
&\int_{-\infty}^{\infty} V^1(\mu + y) f_N(y) dy = \\
&\int_{-\infty}^{-\mu} b(\mu + y) f_N(y) dy + \int_{-\mu}^{\infty} (\mu + y) f_N(y) dy = \\
&b\mu \int_{-\infty}^{-\mu} f_N(y) dy + b \int_{-\infty}^{-\mu} y f_N(y) dy + \\
&\mu \int_{-\mu}^{\infty} f_N(y) dy + \int_{-\mu}^{\infty} y f_N(y) dy = \\
&\mu(b\Phi(-bx) + 1 - \Phi(-\mu)) + (b-1) \int_{-\infty}^{-\mu} y f_N(y) dy = \\
&\mu(1 + (b-1)\Phi(-\mu)) + (b-1)[-\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}]_{-\infty}^{-\mu} = \\
&\mu + \mu(b-1)\Phi(-\mu) + (1-b)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2}
\end{aligned}
$$

This gives us

$$
\begin{aligned}
\mathcal{Q}_N^2(x) &= \int_{-\infty}^{\infty} V^1(bx + y) f_N(y) dy \\
&= bx + bx(b-1)\Phi(-bx) + (1-b)\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}b^2x^2}
\end{aligned}
\tag{7.48}
$$

From this, (7.5), which is the representation preferred by Mathematica, follows by using $\operatorname{erf}(x) = -\operatorname{erf}(-x)$ and the identity

$$\Phi(x) = \frac{1}{2}(1 + \operatorname{erf}(\frac{x}{\sqrt{2}}))$$

We see that if we set $b = 0$ in (7.48) we obtain the constant function $\mathcal{Q}_N^2(x) = \frac{1}{\sqrt{2\pi}}$. This corresponds to $W^1 = \frac{1}{\sqrt{2\pi}}$ in Subsection 5.2, because for $b = 0$, the AR process becomes a sequence of independent standard normally distributed random variables.

For normally distributed innovations, for $n \geq 2$, we cannot find an exact expression for d_n. For $n = 2$ we can exploit the fact that $\mathcal{Q}_N^2$ is a contraction mapping that has d_2 as its fixed point. Given any starting point $x_0 \in \mathbb{R}$, a fixed point iteration then converges relatively quickly to d_2, for example we can easily get the first 20 digits.

$$d_2 \approx 0.32331501349170887327$$

The formula for $\mathcal{Q}_N^2$ found in Section 7 is relatively well suited for numerical integration, as it is an exact formula which allows us to calculate a reasonable number of digits per second. Using numerical integration, we can estimate values of $\mathcal{Q}_N^3$. Therefore, to estimate d_3 we can resort to a combination of numerical integration and fixed point iteration. We can again find the first 20 digits relatively easily

$$d_3 \approx 0.54397956922321062520$$

Note that the formula of $\mathcal{Q}_N^2$ contains an error function, which is defined as an integral. Numerical integration may not be the most efficient way to calculate values of the error function, but it should be clear that having an expression that requires numerical integration to evaluate as the integrand of another numerical integration is not necessarily a problem. However, it already takes us 80 seconds to approximate a single value of $\mathcal{Q}_N^3$ to a precision of 10 in this way. This justifies approximating d_n and values of $\mathcal{Q}_N^n$ in a different way, for example using the approach in Subsection 7.3.

7.3 Description of the AR(1) approximation procedure

In this section we present a procedure that solves the Wald-Bellman equations (4.26) approximately, by means of numerical integration and interpolation.

Here we will first describe the procedure. The inputs of the procedure are the distribution of Z and a parameter k that indicates how precisely we will work. The procedure computes functions $\overline{Q}^n$ which are approximations of $\mathcal{Q}^n$. It also computes $\overline{d}^n$ which are approximations of the d^n. Combining $\overline{Q}^n$ and $\overline{d}^n$ in a direct way gives us an approximation of V^n, $\overline{V}^n$. With the fact that $\overline{V}^n$ can be reconstructed in mind, the outputs of the procedure will be the functions $\overline{Q}^n$ and the $\overline{d}^n$.

We will now define $\overline{V}^{\,n}$ and H^{n+1} in terms of $\overline{Q}^{\,n}$ and $\overline{d}^{\,n}$. Note that this leads to an implicit dependency of $\overline{V}^{\,n}$ and H^{n+1} on $\overline{Q}^{\,n}$ and $\overline{d}^{\,n}$.

Let, for all $n \geq 1$, $\overline{Q}^{\,n} : \mathbb{R} \to \mathbb{R}$ be a contraction mapping with the same asymptotes as Q^n, that is left asymptote $b^n x$ and right asymptote bx. Let $\overline{d}^{\,n} \in \mathbb{R}$ and suppose that Z admits a density f.

For all $n \geq 1$, we define $\overline{V}^{\,n}$ as

$$\overline{V}^{\,n}(x) := \begin{cases} x & \text{if } x \geq \overline{d}^{\,n} \\ \overline{Q}^{\,n}(x) & \text{otherwise} \end{cases} \tag{7.49}$$

We have the relation

$$\mathbb{E}_x[\bar{V}^{n-1}(X_1)] = \mathbb{E}_x[\bar{V}^{n-1}(bX_0 + \epsilon)] = \int_\infty^\infty \bar{V}^{n-1}(bx + y) f(y) dy \tag{7.50}$$

We shall now define H^n to be an approximation of $\mathcal{Q}^n$ based on (7.50). That is, for all $n \geq 2$ we define H^n as

$$H^n(x) := \int_\infty^\infty \bar{V}^{n-1}(bx + y) f(y) dy \tag{7.51}$$

To see that H^n is well defined note that $\overline{Q}^{\,n-1}$ was assumed to be a contraction mapping, which is continuous, and was assumed to have linear asymptotes $b^{n-1}x$ and bx, so that integrability follows from Lemma 10.27. In our procedure we will need a subroutine to numerically approximate values of H^n.

By Lemma 10.28, H^n is a contraction mapping. This guarantees the uniqueness of the fixed point in step S2 of the procedure below.

We now show the pseudocode for the AR(1) approximation procedure. The pseudocode contains very few details, but details are provided below the pseudocode.

Pseudocode.

Subroutine (R1)

Calculates values of H^n approximately.

Initialization

For $n = 1$, set

$$\overline{Q}^{\,1}(x) = bx$$
$$\overline{d}^{\,1} = 0$$

Recursion

For $n = 2, \ldots, N$

Step 1 (S1)

Set $\overline{Q}^{\,n}$ to be a function that is an approximation of H^n. Use R1 to approximate the values of H^n needed to construct $\overline{Q}^{\,n}$.

Step 2 (S2)

Set $\overline{d}^{\,n}$ to be a numerical approximation of the unique solution of $H^n(x) = x$. Use R1 to compute values of H^n.

■

We can specify more details for R1, S1 and S2. In S1, for example, we can use a piecewise linear approximation. Figure 9 is a plot of what the approximations $\overline{Q}^{\,n}$ of H^n might look like.

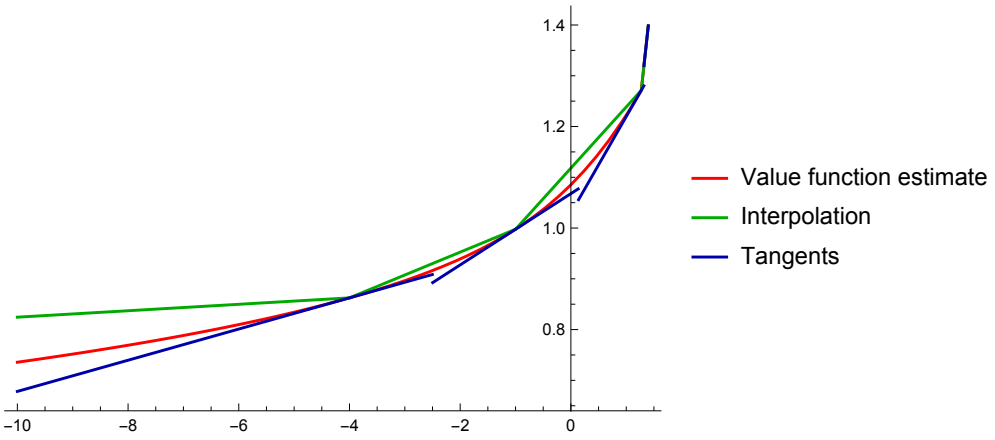

Figure 9: The value function estimate H^n and two functions that approximate it, one by linear interpolation and one by using tangents.

Typically we will use interpolation with polynomials of order greater than one in S1. In such cases, even though the function we are interpolating is convex,

the interpolation function may not be convex. There is then also no guarantee that integrating the interpolation function will yield a convex function. The property of Q^n that it is a contraction mapping is easier to preserve than its convexity and it is more relevant because it guarantees the existence of a unique fixed point.

We have to make sure we deal with asymptotes of $\bar{Q}^n$ and $\bar{V}^n$ correctly. Because we know from our analysis in Subsection 7.1 what the asymptotes have to be, we can glue together these asymptotes with an interpolation to construct an approximation $\bar{Q}^n$ of H^n on all of $\mathbb{R}$ is step S1.

In step S2 we will use fixed point iteration. Theorem 10.26 tells us that a fixed point iteration converges quickly, i.e. it is typically much faster than other numerical equation solving techniques like Newton's method and it does not require computation of derivatives.

In the subroutine R1 we will use numerical integration. We have chosen to do some special handling of the asymptotes and the routine is aware of the bounds of the interval on which our interpolation is valid.

8 Results

We have implemented both the AR(1) approximation procedure of Subsection 7.3 and the vectorized version of the discretization procedure of Subsection 6.4 in Mathematica.

8.1 Testing the AR(1) approximation procedure

In order to test the AR(1) approximation procedure using interpolation, numerical integration and fixed point iteration, we focus on a relatively easy case. That is, we estimate d_3 for normally distributed innovations and $b = 1/2$, which we also approximated in a slightly different way in Subsection 7.2, which was more efficient because we can efficiently get digits of $Q^2(x)$ using the formula given there. Because we know many digits of d_3, we say which of the digits the AR(1) approximation procedure are correct.

We wish to do only a single iteration of the procedure to give as much insight into how much accuracy is lost in each step as possible. However, the interpolation constructed for iteration n of the procedure is only used in iteration $n+1$. In order to do a representative calculation, we do the interpolation (S1) for $n = 2$ by directly setting $\bar{Q}^2$ to be an interpolation of Q^2, where instead of using R1 to compute values, we use the formula of Q^2 found in Subsection 7.2. Then, for $n = 3$, we use numerical integration in R1 to approximate values of H^3 while doing a fixed point iteration to approximate the unique solution of $H^3(x) = x$, which is our estimate $\bar{d}_3$ of d_3.

Below is a table of the accuracy to which we estimate d_3 for normally distributed innovations and $b = 1/2$. k is the number of points we sample using the formula for Q^2 to create our interpolation and m is the order of the interpolating polynomials that we use.

k / m	1	2	3	4
30	7.3	9.5	11.9	14.6
60	8.4	11.3	14.3	17.6
100	9.3	12.6	16.0	19.7

Figure 10: Table of Accuracies of estimations of d_3

Now use the AR(1) approximation procedure described in Section 7 to approximate the optimal thresholds (d_n) for $b = 1/2$ for three different distributions of innovations. The distributions used for the innovations are $2(B-1)$ where B is a beta distribution with parameters $\alpha = 2$ and $\beta = 2$, so that we have shifted and stretched B to have support $[-1, 1]$. The second distribution is the uniform distribution on $[-1, 1]$ and the third is the standard normal distribution. Again we vary the number k of points at which we sample H^n in each

iteration. For our finite horizon N, we choose $N = 10$. We use quadratic interpolating polynomials. We test the resulting threshold policy using a Monte Carlo simulation.

In Figure 11, we have listed our estimated d_{10} for each of see that a Monte Carlo simulation agrees with our estimations of the boundaries. Our estimates also seem to converge, which also gives us confidence that we can get more correct digits.

k / Distribution	Beta	Uniform	Normal
10	0.57410534	0.74212773	1.262234
20	0.57423373	0.74221178	1.268813
30	0.57424418	0.74221982	1.269742
40	0.57424674	0.74222116	1.269998
50	0.57424780	0.74222168	1.270097
60	0.57424828	0.74222188	1.270143
70	0.57424851	0.74222202	1.270167
80	0.57424864	0.74222209	1.270182
90	0.57424875	0.74222213	1.270190
100	0.57424879	0.74222216	1.270195
Monte Carlo	0.57429818	0.74227152	1.270566

Figure 11: Table showing the convergence of the AR(1) approximation procedure and a Monte Carlo estimations to validate the results.

8.2 Comparing the procedures

We compare the AR(1) approximation procedure with the discretization procedure. We compare both the estimates of the value function and the estimates of the (d_n).

We consider innovations following a standard normal distribution and regression parameter $b = 1/2$ and we consider a horizon of $N = 4$. We approximate V^N using both the discretization procedure and the AR(1) approximation procedure. For the discretization procedure we use $K = 10^7$ data points, $m = 100$ levels and we choose $n = 1$ because we are modelling a Markov process. For the AR(1) approximation procedure, for each $i \in \{2, \ldots, 4\}$ in Step 1 of the procedure, we approximately sample H^i using numerical integration 60 times and we use interpolating polynomials of order 4.

The discretization procedure gives us an estimate of V^k by (6.5). The function $x \mapsto \hat{V}^k(\sigma(x))$ is a step function, which is constant on each level (see Definition 6.2). In Figure 12, we have plotted this function in red dots with the label discretization estimate. We do not show the entire step function, but instead for each $i \in \{0, \ldots, m-1\}$ we only show $\hat{V}^k(\sigma(x))$ for $x = g_{i+1} \in L_i$ so that

we draw a red dot at $(g_{i+1}, \hat{V}^k(g_{i+1}))$, see Definition 6.8. In a sense g_{i+1} is the most representative element of L_i and this representation makes the plot less cluttered, while at the same time highlighting that the g_{i+1} are the points where the two estimates of the value function are closest.

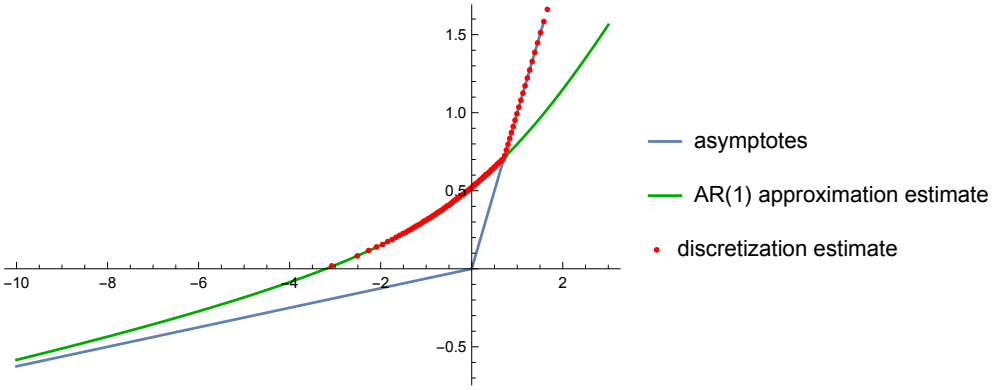

Figure 12: Comparison of the value functions for $N = 4$

In the figure (Figure 12), the green line labeled "AR(1) approximation estimate" represents the approximation $\bar{Q}^k$ of Q^k found by the AR(1) approximation procedure. The approximate location of the intersection of $\bar{Q}^k$ with the line $x \mapsto x$ would be the approximation of d_k by this procedure. The approximation of the value function V^k by this procedure, defined by (7.49), coincides with the the green line to the left of the intersection and with the blue line to its right.

Considering these remarks about the approximation $\bar{V}^k$ of the value function V^k by the AR(1) approximation procedure, we see in the figure (Figure 12) that the red dots lie on $\bar{V}^k$. As discussed in Subsection 6.3, we conjectured that the probability is high that the discretization has boundary structure in the sense of Definition 4.22. In the realisation corresponding to this figure, this is indeed the case, as was it the case in 50 other experiments. The estimation of d_k by the discretization procedure is then the quantile $q_{i^*/m}$, where $i^* \in \{1, \ldots, m-1\}$ is smallest i such that the red dot at $(g_{i^*+1}, \hat{V}^k(g_{i^*+1})))$ lies exactly on the blue line. We see that the point $(q_{i^*/m}, q_{i^*/m})$ lies close to the intersection of the green and the blue line, so that $q_{i^*/m}$ is close the approximation of d_k by the AR(1) approximation procedure.

Figure 13 shows the same plot, but for horizon $N = 16$, so that it is much harder to generate. The left asymptote is less clear here, as the slope of the asymptote is $(1/2)^{16}$.

In Figure 14 we show the estimates of the (d_k) found by both procedures in the same experiment.

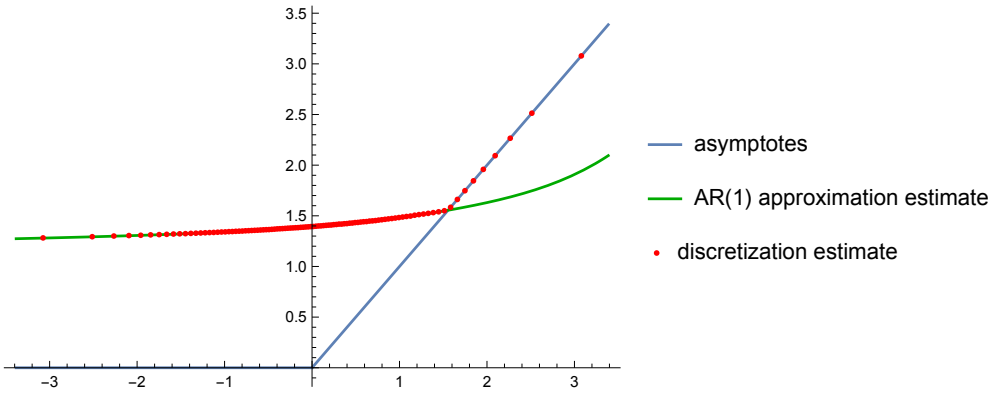

Figure 13: Comparison of the value functions for $N = 16$

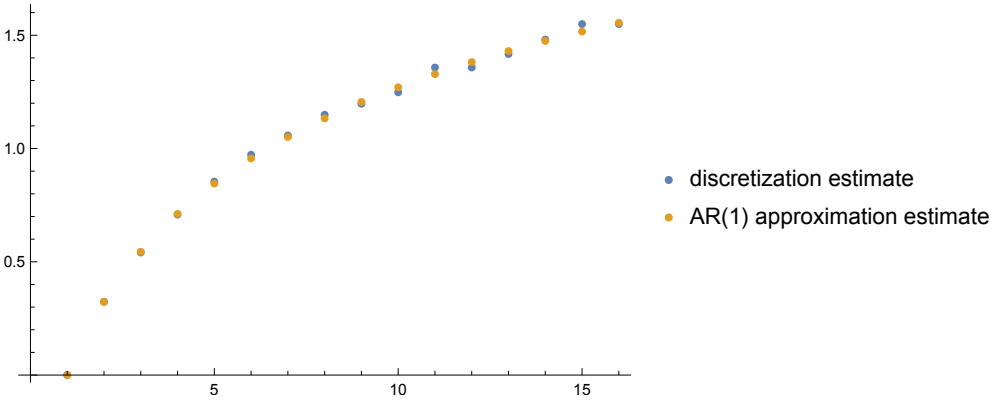

Figure 14: Comparison of estimates of the (d_k) found by the procedures

9 Conclusions and Future Work

9.1 Answers to research questions

Our first research question is: *How well does the discretization procedure perform for an AR(1) process?*

We have found empirical evidence that the discretization procedure is consistent. We conclude this because for large input data, the output of the discretization procedure coincides with the AR(1) approximation procedure of Subsection 7.3, as evidenced by Figure 12. It must be said that the discretization needs to be set increasingly "fine", i.e. the number of levels in the sense of Definition 6.2 also needs to go to infinity as the length of our data goes to infinity, in order to really have convergence.

The correctness of the AR(1) approximation procedure of Subsection 7.3 was itself tested by doing a Monte Carlo simulation, establishing its correctness.

Secondly, we asked: *What can we say about the structure of the value functions of an AR(1) process?*

The results about the structure of the value functions we have found are summarised in Proposition 7.13 and Proposition 7.15.

9.2 Future work

There is a number of directions this research can take from here. These directions encompass both methods to assess the procedure's performance more accurately and ways to extend it.

For example, one way to further investigate the performance of the discretization procedure is to test it on smaller amounts of data. So far, our discretization procedure has only been tested on large amounts of data. It would be very useful to see how our procedure compares to other procedures for different amounts of data. Furthermore it would be interesting to see how well the discretization procedure performs for processes that are not Markov processes. It would be interesting to look at higher order Markov processes, for example ARMA processes, as the procedure is capable of approximating the process from which it has data by a higher order Markov chain of order n and this research we have only considered the case where $n = 1$.

Moreover, a possible extension of the discretization procedure could be to integrate factors like time of day and weather data. This will make the procedure more suitable for the demand-side management use-case. Despite the fact that it is often suggested that the price process of electricity should follow an AR(1) process, we believe that assuming that price process has a richer structure, that

factors in weather conditions and time of day, can be beneficial. In this case it can be considered to discretise the state space of the weather conditions as well as the values of the price process.

10 Appendix

10.1 Complete statement of the finite horizon theorem

We present here Theorem 1.2 of [13], of which Theorem 3.10 in the current text is only the first half. We also present proofs here of the statements that comprise the second half. The proofs follow the proofs in [13] very closely, but more details are provided.

Let, for some index set $M \subset \mathbb{N}_0$, $(X_k)_{k\in M}$ and $(Y_k)_{k\in M}$ be two stochastic processes on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$. We say that (X_k) dominates (Y_k), if, for all $k \in M$, $X_k \geq Y_k$ almost surely.

The statement (10.5) below asserts that S_k^N is the smallest supermartingale satisfying some property. By this we mean that this process is the smallest in the sense of domination, so that any other process satisfying the property dominates S_k^N.

Theorem 10.1 (**Finite horizon complete**)**.**
Suppose condition (3.3) holds. We then have, for all $n \in \{0, \dots, N\}$

$$S_n^N \geq \mathbb{E}[G_\tau | \mathcal{F}_n] \qquad \forall \tau \in \mathfrak{M}_n^N \tag{10.1}$$

$$S_n^N = \mathbb{E}[G_{\tau_n^N} | \mathcal{F}_n] \tag{10.2}$$

$$W_n^N = \mathbb{E}[G_{\tau_n^N}] \tag{10.3}$$

$$\mathbb{E}[G_\tau] = W_n^N \implies \mathbb{P}[\tau_n^N \leq \tau] = 1 \qquad \forall \tau \in \mathfrak{M}_n^N \tag{10.4}$$

The sequence $(S_k^N)_{n\leq k\leq N}$ is the smallest supermartingale which dominates $(G_k)_{n\leq k\leq N}$.

(10.5)

The stopped sequence $(S^N_{k\wedge\tau_n^N})_{n\leq k\leq N}$ is a martingale.

(10.6)

■

Proof. For a proof of the first three statements, see the proof of Theorem 3.10 of the current text. We will prove the rest of the statements here.

First we we will prove (10.5). By (3.8) we find that, for all $n \in \{0, \dots, N\}$

$$S_n^N \geq \mathbb{E}[S_{n-1}^N | \mathcal{F}_{n-1}] \tag{10.7}$$

This shows that (S_k) is a supermartingale. From (3.7) and (3.8) we see that, for all $k \in \{0, \dots, N\}$, $S_k \geq G_k$ almost surely, so that (S_k^N) dominates (G_k).

Now let $(\tilde{S}_k)$ be another supermartingale that dominates (G_k). We will show that $(\tilde{S}_k)$ also dominates (S_k^N). We will prove this by induction on $k \in \{n, \ldots, N\}$. The induction hypothesis is that $S_k^N \leq \tilde{S}_k$ almost surely. We will show that the induction hypothesis holds for $k = N$ and then prove that if it holds for $k \in \{n+1, \ldots, N\}$, then it holds for $k-1$.

We see that the induction hypothesis holds for $k = N$, by (3.7), as $S_N^N = G_N \leq \tilde{S}_N$. Now suppose that the induction hypothesis holds for a certain $k \in \{n+1, \ldots, N\}$. We have, by the definition (3.8) of S_k^N and by Proposition 8.5 in [21], that

$$S_{k-1}^N = \max(G_{k-1}, \mathbb{E}[S_k^N | \mathcal{F}_{k-1}]) \leq \max(G_{k-1}, \mathbb{E}[\tilde{S}_k | \mathcal{F}_{k-1}]) \qquad \text{a.s.} \tag{10.8}$$

Now, because $(\tilde{S}_k)$ dominates (G_k) and $\tilde{S}$ is a supermartingale, we find that

$$\max(G_{k-1}, \mathbb{E}[\tilde{S}_k | \mathcal{F}_{k-1}]) \leq \tilde{S}_{k-1} \qquad \text{a.s.} \tag{10.9}$$

Combining the two inequalities above, we find that $S_{k-1}^N \leq \tilde{S}_{k-1}$, which is the induction hypothesis for $k-1$.

So indeed $(\tilde{S}_k)$ also dominates (S_k^N), so that we have proven (10.5).

Now we shall prove (10.4). Let $n \in \{0, \ldots, N\}$ and suppose that $\tau_* \in \mathfrak{M}_n^N$ is such that

$$\mathbb{E}[G_{\tau_*}] = W_n^N \tag{10.10}$$

For all $k \in \{n, \ldots, N\}$, by the definition (3.8) of S_n^N, we have $G_k \leq S_k^N$. Therefore, we have

$$G_{\tau_*} = \sum_{k=n}^{N} \mathbb{1}_{[\tau_*=k]} G_k \leq \sum_{k=n}^{N} \mathbb{1}_{[\tau_*=k]} S_k = S_{\tau_*} \tag{10.11}$$

We then have, by Proposition 4.7 of [21], that

$$\mathbb{E}[G_{\tau_*}] \leq \mathbb{E}[S_{\tau_*}^N] \tag{10.12}$$

By Theorem 9.15 in [21], which is an optimal stopping theorem, because (S_n) is a supermartingale, which we have shown in the proof of (10.5), and because τ_* is a stopping time with $\tau_* \geq n$ and because τ_* is bounded so that condition (ii) of the Theorem is satisfied, we obtain

$$\mathbb{E}[S_{\tau_*}^N] \leq \mathbb{E}[S_n^N] \tag{10.13}$$

Furthermore, (10.2), Proposition 8.5 (ii) in [21] and (10.3) give us

$$\mathbb{E}[S_n^N] = \mathbb{E}[\mathbb{E}[G_{\tau_n^N}|\mathcal{F}_n]] = \mathbb{E}[G_{\tau_n^N}] = W_n^N \tag{10.14}$$

Combining (10.10), (10.12), (10.13) and (10.14), we obtain

$$W_n^N = \mathbb{E}[G_{\tau_*}] \leq \mathbb{E}[S_{\tau_*}^N] \leq \mathbb{E}[S_n^N] = W_n^N \tag{10.15}$$

We conclude that

$$\mathbb{E}[G_{\tau_*}] = \mathbb{E}[S_{\tau_*}^N] \tag{10.16}$$

Now let $Z = S_{\tau_*}^N - G_{\tau_*}$. We see by (10.11) that $Z \geq 0$ and by the equation above that $\mathbb{E}[Z] = 0$. We then have, by Lemma 4.11 in [21], that $S_{\tau_*}^N - G_{\tau_*} = 0$ almost surely, i.e. $S_{\tau_*}^N = G_{\tau_*}$ almost surely.

Let $F \subset \mathcal{F}$ be such that $\mathbb{P}(F) = 1$ and such on F it holds that $S_{\tau_*}^N = G_{\tau_*}$. Let $\omega \in F$ be arbitrary and let $m := \tau_*(\omega)$, so that $n \leq m \leq N$ and $S_m(\omega) = G_m(\omega)$. By the definition (3.9) of τ_n^N and because the infimum is a lower bound, we have that

$$\tau_n^N(\omega) = \inf\{n \leq k \leq N | S_k(\omega) = G_k(\omega)\} \leq m = \tau_*(\omega) \tag{10.17}$$

So we have that $\tau_n^N(\omega) \leq \tau_*(\omega)$ almost surely, which proves (10.4).

Now we shall prove (10.6). We have

$$\mathbb{E}[S_{k+1\wedge\tau_n^N}^N|\mathcal{F}_k] = \mathbb{E}[\mathbb{1}_{[\tau_n^N \leq k]} S_{k+1\wedge\tau_n^N}^N|\mathcal{F}_k] + \mathbb{E}[\mathbb{1}_{[\tau_n^N \geq k+1]} S_{k+1\wedge\tau_n^N}^N|\mathcal{F}_k] \tag{10.18}$$

On $[\tau_n^N \leq k]$, $k+1\wedge\tau_n^N = k\wedge\tau_n^N$ and on $[\tau_n^N \geq k+1]$, $k+1\wedge\tau_n^N = k+1$, so this becomes

$$\mathbb{E}[S_{k+1\wedge\tau_n^N}^N|\mathcal{F}_k] = \mathbb{E}[\mathbb{1}_{[\tau_n^N \leq k]} S_{k\wedge\tau_n^N}^N|\mathcal{F}_k] + \mathbb{E}[\mathbb{1}_{[\tau_n^N \geq k+1]} S_{k+1}^N|\mathcal{F}_k] \tag{10.19}$$

We have $\mathbb{1}_{[\tau_n^N \leq k]} S_{k\wedge\tau_n^N}^N = \sum_{m=0}^k \mathbb{1}_{[\tau_n^N = m]} S_m^N$. As we have seen in Subsection 3.1, S_m is $\mathcal{F}_m$ measurable. Because τ_n^N is a stopping time, each indicator function $\mathbb{1}_{[\tau_n^N = m]}$ is $\mathcal{F}_m$ measurable. Because $(\mathcal{F}_k)$ is a filtration and because we always had $m \leq k$, all these functions are also $\mathcal{F}_k$ measurable. Proposition 3.5 (i) in [21] now gives us that $\sum_{m=0}^k \mathbb{1}_{[\tau_n^N = m]} S_m^N$ is $\mathcal{F}_k$ measurable. So $\mathbb{1}_{[\tau_n^N \leq k]} S_{k\wedge\tau_n^N}^N \in \mathcal{F}_k$. This gives us that

$$\mathbb{E}[\mathbb{1}_{[\tau_n^N \leq k]} S_{k\wedge\tau_n^N}^N|\mathcal{F}_k] = \mathbb{1}_{[\tau_n^N \leq k]} S_{k\wedge\tau_n^N}^N \tag{10.20}$$

Because τ_n^N is a stopping time, $\mathbb{1}_{[\tau_n^N \geq k+1]}$ is $\mathcal{F}_k$ measurable. By Proposition 8.5 (iii) of [21] and the equation above, by transforming its right hand side, we see that (10.19) becomes

$$\mathbb{E}[S^N_{k+1\wedge\tau_n^N}|\mathcal{F}_k] = \mathbb{1}_{[\tau_n^N \leq k]} S^N_{k\wedge\tau_n^N} + \mathbb{1}_{[\tau_n^N \geq k+1]} \mathbb{E}[S^N_{k+1}|\mathcal{F}_k] \tag{10.21}$$

On $[\tau_n^N \geq k+1]$ we have that $S_k^N = \mathbb{E}[S_{k+1}^N|\mathcal{F}_k]$, so the equation above becomes

$$\begin{aligned} \mathbb{E}[S^N_{k+1\wedge\tau_n^N}|\mathcal{F}_k] &= \mathbb{1}_{[\tau_n^N \leq k]} S^N_{k\wedge\tau_n^N} + \mathbb{1}_{[\tau_n^N \geq k+1]} S_k^N \\ &= \mathbb{1}_{[\tau_n^N \leq k]} S^N_{k\wedge\tau_n^N} + \mathbb{1}_{[\tau_n^N \geq k+1]} S^N_{k\wedge\tau_n^N} = S^N_{k\wedge\tau_n^N} \end{aligned} \tag{10.22}$$

This proves (10.6). □

10.2 Lebesgue Dominated Convergence and limits in $\bar{\mathbb{R}}$

The following theorem is a minor reformulation of Theorem 4.1.1. of [22], see also [22] for a proof.

Theorem 10.2 (Sequential Criterion for the Limit of a Function)**.** Let $D \subseteq \mathbb{R}$, let $y \in \mathbb{R}$ be a limit point of D and let $f : D \to \mathbb{R}$ be a function. Then $\lim_{x\to y} f(x)$ exists and is finite iff for every sequence $(x_i)_{i=1}^\infty \subseteq D \setminus \{y\}$ that converges to y, $(f(x_i))_{i=0}^\infty$ converges. If either of the statements of this equivalence holds, then the limit $\lim_{x\to y} f(x)$ and the sequential limits $\lim_{i\to\infty} f(x_i)$ are all equal.

We wish to extend Theorem 10.2 to cases where $y = \pm\infty$ and to cases where the limit exists but is infinite. To do so, we need the following theorem, which is an adaptation of Theorem 4.15 of [1], called the substitution theorem, including Remark 4.16.

Theorem 10.3. Let $c \in \bar{\mathbb{R}}$ and $z \in \mathbb{R}$. Let $V \subseteq \mathbb{R}$ and let $f : V \to \mathbb{R}$ be a function, with

$$\lim_{x\to c} f(x) = z \tag{10.23}$$

Let $r \in \mathbb{R}$, $D \subset \mathbb{R}$, $(z-r, z+r) \setminus \{z\} \subseteq D$ and let $g : D \to \mathbb{R}$ be a function, so that g defined on a neighbourhood of z, except possibly at the point z. If any of the following conditions holds

a g is continuous at z.

b There exists a neighbourhood I_c of c such that for all $x \in I_c \setminus \{c\}$, $f(x) \neq z$ and the limit $\lim_{y\to z} g(y)$ exists

Then

$$\lim_{x\to\infty} g(f(x)) = \lim_{y\to z} g(y) \tag{10.24}$$

The purpose of the following lemma is to extend Theorem 10.2 to cases where $y = \pm\infty$.

Lemma 10.4. Let $s > 0$ and let $h : (s, \infty) \to \mathbb{R}$. Let $L \in \mathbb{R}$. We have

$$\lim_{x\to\infty} h(x) = L \Leftrightarrow$$
$$\text{For all sequences } (a_n) \subset \mathbb{R} : \lim_{n\to\infty} a_n = \infty \implies \lim_{n\to\infty} h(a_n) = L$$

Proof.
$\Rightarrow$) Let $\epsilon > 0$. Let N be such that $\forall x > N : |h(x) - L| < \epsilon$ and let M be such that $\forall n > M : a_n > N$. Then $\forall n > M : |h(a_n) - L| < \epsilon$.

$\Leftarrow$) Let $f(x) = |\frac{1}{x}|$ and let

$$g : (-\frac{1}{s}, \frac{1}{s}) \to \mathbb{R} : x \mapsto \begin{cases} h(f(x)) & x \neq 0 \\ L & x = 0 \end{cases}$$

Let (b_n) be any sequence in $(-\frac{1}{s}, \frac{1}{s}) \setminus \{0\}$ that satisfies $\lim_{n\to\infty} b_n = 0$. We construct the sequence (a_n) with $a_n = f(b_n)$. It follows that $\lim_{n\to\infty} a_n = \infty$. By assumption $\lim_{n\to\infty} h(a_n) = L$. Therefore

$$\lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} h(f(c_n)) = \lim_{n\to\infty} h(a_n) = L$$

We now have by Theorem 10.2, that

$$\lim_{x\to 0} g(x) = L$$

In the notation of Theorem 10.3 (the substitution theorem), we have $\lim_{x\to\infty} f(x) = 0 = z$. g is defined in a neighbourhood of 0 and g is continuous at 0, so the conditions of the theorem are satisfied. We conclude that

$$\lim_{x\to\infty} g(f(x)) = g(0) = L$$

We therefore have

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} h(f(f(x))) = \lim_{x\to\infty} g(f(x)) = L$$

□

The purpose of the following lemma is to extend Theorem 10.2 to cases where the limit is infinite.

Lemma 10.5. Let $s > 0$ and let $h : (s, \infty) \to \mathbb{R}$. We have

$$\lim_{x\to\infty} h(x) = \infty \Leftrightarrow$$
$$\text{For all sequences } (a_n) \subset \mathbb{R} : \lim_{n\to\infty} a_n = \infty \implies \lim_{n\to\infty} h(a_n) = \infty$$

Proof.
$\Rightarrow$) Let $K > 0$. Let $N \in \mathbb{N}$ be such that $\forall x > N : h(x) > K$ and let $M \in \mathbb{N}$ be such that $\forall n > M : a_n > N$. Then $\forall n > M : h(a_n) > K$.

$\Leftarrow$)

Let $g(x) = |\frac{1}{x}|$ and let

$$f : (s, \infty) \to \mathbb{R} : x \mapsto g(h(x))$$

Let (a_n) be any sequence with $\lim_{n\to\infty} a_n = \infty$. We have $\lim_{n\to\infty} h(a_n) = \infty$ so that

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} g(h(a_n)) = 0 \tag{10.25}$$

Because (a_n) was an arbitrary sequence with $\lim_{n\to\infty} a_n = \infty$, we find by Lemma 10.4 that

$$\lim_{x\to\infty} f(x) = 0 \tag{10.26}$$

It then follows from Theorem 10.3 (the substitution theorem), because g is defined on $(-1, 1) \setminus \{0\}$ and (s, ∞) is a neighbourhood of ∞ on which f does not equal 0 and $\lim_{y\to 0} g(x) = \infty$ exists, we get

$$\lim_{x\to\infty} g(f(x)) = \lim_{y\to 0} g(y) = \infty$$

We conclude that

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} g(f(x)) = \infty \tag{10.27}$$

□

The extended version of Theorem 10.2 is then the following, where we consider a topology on $\bar{\mathbb{R}}$ that makes it homeomorphic to $[-1, 1]$, so that e.g. ∞ is a limit point of $\mathbb{R}$.

Theorem 10.6. Let $D \subseteq \mathbb{R}$, let $y \in \bar{\mathbb{R}}$ be a limit point of D in $\bar{\mathbb{R}}$ and let $f : D \to \mathbb{R}$ be a function. $\lim_{x\to y} f(x)$ exists iff for every sequence $(x_i)_{i=1}^{\infty} \subseteq D \setminus \{y\}$ with $\lim_{i\to\infty} x_i = y$, $\lim_{i\to\infty} f(x_i)$ exists. If either of the statements of this equivalence holds, then the limit $\lim_{x\to y} f(x)$ and the sequential limits $\lim_{i\to\infty} f(x_i)$ are all equal.

Proof. This follows from Theorem 10.2, Lemma 10.4 and Lemma 10.5. □

Lemma 10.7. Let $(X, \mathcal{F}, \mu)$ be a measure space, let $s \in \mathbb{R}$ and $u : (s, \infty) \times X \to \mathbb{R}$. If

$$A : \text{For every } t \in (s, \infty),\ h_t : x \mapsto u(t, x) \text{ has } h_t \in L^1 \tag{10.28}$$

$$B : \text{For any } x \in X,\ \lim_{t\to\infty} u(t, x) = L \text{ exists and } L \in \mathbb{R} \tag{10.29}$$

$$C : \text{For some } w \in L^1, \text{ for all } (t, x) \in (s, \infty) \times X,\ |u(t, x)| \leq w(x) \tag{10.30}$$

Then

$$\lim_{t\to\infty}\int u(t,x)\mu(dx) = \int \lim_{t\to\infty} u(t,x)\mu(dx) \tag{10.31}$$

Proof. By dominated convergence, we have that for any sequence $(t_j) \subset (s,\infty)$ with $\lim_{j\to\infty} t_j = \infty$, we have

$$\lim_{j\to\infty}\int u(t_j,x)\mu(dx) = \int \lim_{j\to\infty} u(t_j,x)\mu(dx) \tag{10.32}$$

The result now follows from Theorem 10.6. □

10.3 Convex functions

We collect here some lemmas that are needed to prove statements about the structure of V^n and Q^n in Subsection 7.1.

Lemma 10.8. Let $r_1, r_2, r_3 \in \mathbb{R}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and $r_2 = \alpha r_1 + \beta r_2$. Then we have the following equivalence

$$r_1 \le r_2 \Longleftrightarrow r_2 \le r_3 \Longleftrightarrow r_1 \le r_3 \tag{10.33}$$

Proof.

$$\begin{aligned} r_1 \le r_2 &\Longleftrightarrow r_1 \le \alpha r_1 + \beta r_3 \Longleftrightarrow \\ \beta r_1 \le \beta r_3 &\Longleftrightarrow r_1 \le r_3 \Longleftrightarrow \\ \alpha r_1 \le \alpha r_3 &\Longleftrightarrow \alpha r_1 + \beta r_3 \le r_3 \Longleftrightarrow \\ r_2 &\le r_3 \end{aligned} \tag{10.34}$$

□

The following well known proposition and its proof are an adaptation of the following web page [14].

Proposition 10.9. For a function $f : \mathbb{R} \to \mathbb{R}$ and an interval I, the following definitions of a convex function are equivalent to Definition 7.2.

Alternative definition: f is convex on I, iff for all $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_1)}{x_3 - x_1} \tag{10.35}$$

Alternative definition: f is convex iff for all $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{10.36}$$

Alternative definition: f is convex iff for all $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{10.37}$$

Proof. For all $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$ we have

$$\begin{aligned} &\frac{f(x_3) - f(x_1)}{x_3 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_3 - x_1} + \frac{f(x_3) - f(x_2)}{x_3 - x_1} \\ &= \frac{x_2 - x_1}{x_3 - x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} + \frac{x_3 - x_2}{x_3 - x_1} \frac{f(x_3) - f(x_2)}{x_3 - x_2} \end{aligned} \tag{10.38}$$

Using Lemma 10.8, this gives us the following equivalence. For all $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$

$$\begin{aligned} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} &\iff \\ \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2} &\iff \\ \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2} &\iff \end{aligned} \tag{10.39}$$

The equivalence of Definition (10.35), Definition (10.36) and Definition (10.37) follows directly from (10.39).

Because it simplifies our notation and our argument, in this proof we consider yet another equivalent definition of a convex function.

Alternative definition: f is convex on I, iff for all $x_1, x_3 \in I$ with $x_1 < x_3$ and any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$

$$f(\alpha x_1 + \beta x_3) \leq \alpha f(x_1) + \beta f(x_3) \tag{10.40}$$

To see that this definition is equivalent to Definition 7.2, note that the case of $x = y$ holds for any function f. In the case $x \neq y$, we see that the definitions are equivalent by identifying $x_1 = \min(x, y)$, $x_3 = \max(x, y)$, $\alpha = \lambda$ and $\beta = (1 - \lambda)$.

Let

$$S = \{(x_1, x_2, x_3, \frac{x_3 - x_2}{x_3 - x_1}, \frac{x_2 - x_1}{x_3 - x_1}) \,|\, x_1, x_2, x_3 \in I : \, x_1 < x_2 < x_3\} \tag{10.41}$$

We show that for any $x_1, x_3 \in I$ with $x_1 < x_3$ and any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, we can find an x_2 such that $(x_1, x_2, x_3, \alpha, \beta) \in S$, by showing that $x_2 = \alpha x_1 + \beta x_3$ satisfies the requirements. Clearly $x_1 < x_2 < x_3$ so that $x_2 \in I$.

Furthermore, we have

$$\begin{aligned}\frac{x_3 - x_2}{x_3 - x_1} &= \frac{x_3 - (\alpha x_1 + \beta x_3)}{x_3 - x_1} = \frac{x_3 - (\alpha x_1 + (1-\alpha)x_3)}{x_3 - x_1} \\ &= \frac{(1-(1-\alpha))x_3 - \alpha x_1}{x_3 - x_1} = \alpha \frac{x_3 - x_1}{x_3 - x_1} \\ &= \alpha\end{aligned} \tag{10.42}$$

Likewise, for this choice of x_2, we have

$$\begin{aligned}\frac{x_2 - x_1}{x_3 - x_1} &= \frac{(1-\beta)x_1 + \beta x_3 - x_1}{x_3 - x_1} = \frac{\beta x_3 - \beta x_1}{x_3 - x_1} \\ &= \beta\end{aligned} \tag{10.43}$$

For all $(x_1, x_2, x_3, \alpha, \beta) \in S$, we have the following equivalence

$$\begin{aligned} f(\alpha x_1 + \beta x_3) &\leq \alpha f(x_1) + \beta f(x_3) \iff \\ f(x_2) &\leq \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3) \iff \\ (x_3 - x_1) f(x_2) &\leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \iff \\ (x_3 - x_1) f(x_2) &\leq (x_3 - x_1 + x_1 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \iff \\ (x_3 - x_1)(f(x_2) - f(x_1)) &\leq (x_2 - x_1)(f(x_3) - f(x_1)) \iff \\ \frac{f(x_2) - f(x_1)}{x_2 - x_1} &\leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \end{aligned} \tag{10.44}$$

Now suppose that on I, f satisfies Definition (10.35) of a convex function. Then for all $x_1, x_3 \in I$ with $x_1 < x_3$ and any $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, we can again set $x_2 = \alpha x_1 + \beta x_3$. Equivalence (10.44) then gives us

$$f(\alpha x_1 + \beta x_3) \leq \alpha f(x_1) + \beta f(x_3)$$

Therefore Definition (10.40) is satisfied. Conversely, suppose that Definition (10.40) holds. Then for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$, we can set $\alpha = \frac{x_2 - x_1}{x_3 - x_1}$ and $\beta = \frac{x_3 - x_2}{x_3 - x_1}$, so that $(x_1, x_2, x_3, \alpha, \beta) \in S$. Using equivalence (10.44) we see that Definition (10.35) is satisfied. We conclude that Definition (10.35) and (10.40) are equivalent.

We saw that Definition (10.35), Definition (10.36) and Definition (10.37) are equivalent, that Definition (10.40) and Definition 7.2 are equivalent and that Definition (10.40) and Definition (10.35) are equivalent. So the definitions are all equivalent.

□

Lemma 10.10. The composition of two convex functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is convex.

Proof. Let $0 < \lambda < 1$, $x, y \in \mathbb{R}$. We have

$$\begin{aligned} &\lambda f(g(x)) + (1-\lambda) f(g(y)) \\ =& f(\lambda g(x) + (1-\lambda) g(y)) \\ =& f(g(\lambda x + (1-\lambda) y)) \end{aligned}$$

□

The following Lemma is the first equation in (5.14) in [1].

Lemma 10.11. Let a function f have asymptote $l(x) = rx + q$. Then

$$\lim_{x \to \infty} f(x)/x = r \tag{10.45}$$

Proof. This follows from

$$0 = \lim_{x \to \infty} \frac{f(x) - rx - q}{x} = \lim_{x \to \infty} f(x)/x - r$$

□

Lemma 10.12. Let f be a convex function and let $l(x) = q + rx$ be a right asymptote of f. Then for all $x \in \mathbb{R}$, we have

$$\lim_{y \to \infty} \frac{f(y) - f(x)}{y - x} = \sup_{y \in (x, \infty)} \frac{f(y) - f(x)}{y - x} = r \tag{10.46}$$

Proof. Let

$$\rho(y) = \frac{f(y) - f(x)}{y - x}$$

By Proposition 10.9, ρ is increasing, so $\lim_{y \to \infty} \rho(y)$ exists and the limit equals the supremum. We have (by Corollary 5.6 in [1]) that

$$\lim_{y \to \infty} \rho(y) = \lim_{y \to \infty} \frac{f(y)}{y} \tag{10.47}$$

Lemma 10.11 then gives us

$$\lim_{y \to \infty} \rho(y) = r \tag{10.48}$$

□

Lemma 10.13. If f is differentiable and convex and $l(x) = rx + q$ is a right asymptote of f. We have

$$\lim_{y \to \infty} f'(y) = \sup_{y \in \mathbb{R}} f'(y) = r \tag{10.49}$$

Proof. By Theorem 6.37 in [1], f' is increasing, so that the limit equals the supremum. We have, by Lemma 10.12 and the fact that suprema can be interchanged,

$$\begin{aligned}\sup_{y\in\mathbb{R}} f'(y) &= \sup_{y\in\mathbb{R}} \sup_{x\in(-\infty,y)} \frac{f(y)-f(x)}{y-x} \\ &= \sup_{x\in\mathbb{R}} \sup_{y\in(x,\infty)} \frac{f(y)-f(x)}{y-x} = \sup_{x\in\mathbb{R}} r \\ &= r\end{aligned} \tag{10.50}$$

□

Lemma 10.14. Let f be a convex function that has a linear right asymptote l. Then $f \geq l$.

Proof. Let $r, q \in \mathbb{R}$ be such that $l(x) = rx + q$. The proof will be done by contradiction. Assume that at $\bar{x} \in \mathbb{R}$ we have $f(\bar{x}) < l(\bar{x})$. Let $x \in \mathbb{R}$ with $x > \bar{x}$. Because f is convex, we have by Lemma 10.12 that $f(x) \leq f(\bar{x}) + (x-\bar{x})r$. So

$$l(x) - f(x) \geq l(\bar{x}) + (x-\bar{x})r - (f(\bar{x}) + (x-\bar{x})r) = l(\bar{x}) - f(\bar{x}) \tag{10.51}$$

So for $\epsilon = l(\bar{x}) - f(\bar{x}) > 0$ it is not true that for x large enough $|l(x) - f(x)| < \epsilon$. We conclude that $\lim_{x\to\infty} f(x) - l(x) \neq 0$, which contradicts the fact that l is a right asymptote of f. We conclude that such a $\bar{x}$ does not exist and the result follows. □

Lemma 10.15. Let f_1 be convex on $(-\infty, d]$ and let f_2 be convex on $[d, \infty)$ and $f_1(d) = f_2(d)$. Let

$$g(x) = \begin{cases} f_2(x) & x \geq d \\ f_1(x) & x < d \end{cases} \tag{10.52}$$

Then g is convex iff

$$\sup_{x\in(-\infty,d)} \frac{f_1(d) - f_1(x)}{d-x} \leq \inf_{y\in(d,\infty)} \frac{f_2(y) - f_2(d)}{y-d} \tag{10.53}$$

Proof. $\Rightarrow$) Assume g is convex. Let $x \in (-\infty, d)$ and $y \in (d, \infty)$, then

$$\frac{f_1(d) - f_1(x)}{d-x} = \frac{g(d) - g(x)}{d-x} \leq \frac{g(y) - g(d)}{y-d} = \frac{f_2(y) - f_2(d)}{y-d} \tag{10.54}$$

So for all $x \in (-\infty, d)$, $\frac{f_1(d)-f_1(x)}{d-x}$ is a lower bound for $\{\frac{f_2(y)-f_2(d)}{y-d} \mid y \in (d, \infty)\}$. Because the infimum is the largest lower bound, it follows that for all $x \in (-\infty, d)$,

$$\frac{f_1(d) - f_1(x)}{d-x} \leq \inf_{y\in(d,\infty)} \frac{f_2(y) - f_2(d)}{y-d} \tag{10.55}$$

Because this infimum is an upper bound for the expression on the LHS and the supremum is the smallest upper bound, (10.53) follows.

$\Leftarrow$) Assume that (10.53) holds. We shall show that g satisfies the equivalent definition (10.37) of a convex function, by showing that the inequality (10.37) holds in each of five cases. First we derive an inequality that will be useful in cases 3 and 4.

Let $u < d < v$. We have

$$\frac{g(v)-g(u)}{v-u} = \frac{d-u}{v-u}\,\frac{g(d)-g(u)}{d-u} + \frac{v-d}{v-u}\,\frac{g(v)-g(d)}{v-u} \tag{10.56}$$

We see from (10.53) that for all $x \in (-\infty, d)$ and $y \in (d, \infty)$ (10.54) holds. By applying this to the right hand side of the equation above, we obtain

$$\begin{aligned}\frac{g(v)-g(u)}{v-u} &\leq \frac{d-u}{v-u}\,\frac{g(v)-g(d)}{v-d} + \frac{v-d}{v-u}\,\frac{g(v)-g(d)}{v-d} \\ &= \frac{g(v)-g(d)}{v-d}\end{aligned} \tag{10.57}$$

Case 1: Let $x < y < z \leq d$, we have, because $g(x) = f_1(x)$, $g(y) = f_1(y)$ and $g(z) = f_1(z)$ and f_1 is convex that (10.37) is satisfied.

Case 2: Let $d \leq x < y < z$, then (10.37) is again satisfied because g coincides with f_2.

Case 3: Let $x < d < y < z$. We have, using (10.57), (10.37) and (10.35) respectively,

$$\begin{aligned}\frac{g(y)-g(x)}{y-x} &\leq \frac{g(y)-g(d)}{y-d} \leq \frac{g(z)-g(d)}{z-d} \\ &\leq \frac{g(z)-g(y)}{y-d}\end{aligned} \tag{10.58}$$

Case 4: Let $x < d < y < z$. Similarly to Case 3, we find using (10.57),

$$\begin{aligned}\frac{g(y)-g(x)}{y-x} &\leq \frac{g(d)-g(x)}{d-x} \leq \frac{g(d)-g(y)}{d-y} \\ &\leq \frac{g(z)-g(y)}{y-d}\end{aligned} \tag{10.59}$$

Case 5: Let $x < d = y < z$. Because of the assumption (10.37), (10.53) follows directly.

□

Lemma 10.16. Let f be convex and differentiable and $r, d \in \mathbb{R}$. Let

$$g(x) = \begin{cases} r(x-d) + f(d) & x \geq d \\ f(x) & x < d \end{cases} \tag{10.60}$$

Then g is convex iff $r \geq f'(d)$.

Proof. This follows directly from Lemma 10.15.

□

Lemma 10.17. Let $a, b, c \in \mathbb{R}$ with $a < c < b$ and let f be a convex function on (a, b) that is differentiable on $(a, b) \setminus \{c\}$. Then

$$\lim_{x \uparrow c} f'(x) = f'_-(c) \leq f'_+(c) = \lim_{x \downarrow c} f'(x) \tag{10.61}$$

and for every $x, y \in (a, b) \setminus \{c\}$, $x < y \implies f'(x) \leq f'(y)$.

Proof. By Theorem 6.37 of [1], because f is convex, we have that f' is increasing on both (a, c) and (c, b). Using this and the fact that difference quotients of a convex function are increasing, we have

$$\begin{aligned} f'_-(c) &= \lim_{x \uparrow c} \frac{f(c) - f(x)}{c - x} \\ &= \sup_{a < x < c} \frac{f(c) - f(x)}{c - x} \\ &= \sup_{a < x < c} \sup_{a < y < c} \frac{f(y) - f(x)}{y - x} \\ &= \sup_{a < y < c} \sup_{a < x < c} \frac{f(y) - f(x)}{y - x} \\ &= \sup_{a < y < c} f'(y) \\ &= \lim_{y \uparrow c} f'(y) \end{aligned} \tag{10.62}$$

Similarly,

$$f'_+(c) = \lim_{y \downarrow c} f'(y) \tag{10.63}$$

Again because the difference quotients of a convex function are increasing, we have for all $a < x < c < y < b$,

$$\frac{f(c) - f(x)}{c - x} \leq \frac{f(y) - f(c)}{y - c} \tag{10.64}$$

So that

$$f'_-(c) = \lim_{x \uparrow c} \frac{f(c) - f(x)}{c - x} \leq \lim_{y \downarrow c} \frac{f(y) - f(c)}{y - c} = f'_+(c) \tag{10.65}$$

Equations (10.65), (10.62) and (10.63) show (10.61). Now because f' is increasing on both (a, c) and (c, b), we have that for every $x, y \in (a, b) \setminus \{c\}$

$$f'(x) \leq f'_-(c) \leq f'_+(c) \leq f(y) \tag{10.66}$$

□

Lemma 10.18. Let $c, r, q \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R} \setminus \{c\}$ and f is convex and f has right asymptote $x \mapsto rx + q$, then for all $x \in \mathbb{R} \setminus \{c\}$, $f'(x) \leq r$.

Proof. This follows from Lemma 10.17 and Lemma 10.13.

□

The lemma below is taken from [10].

Lemma 10.19. Let f be a continuously differentiable function on an open convex set $\mathcal{C} \subseteq \mathbb{R}^n$. Then f is convex if and only if for any two vectors $x, \bar{x}$

$$\nabla f(x)^T(\bar{x} - x) \leq f(\bar{x}) - f(x) \leq \nabla f(\bar{x})^T(\bar{x} - x) \tag{10.67}$$

Proof. For a proof, see Lemma 1.49 of [10] □

Corollary 10.20. A convex continuously differentiable function lies above all its tangents.

Proof. This follow from (10.67). □

Lemma 10.21. Let Z be a random variable. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any function that satisfies that for all $y \in \mathbb{R}$ the function $h_y : x \mapsto h(x, y)$ is convex and for all $x \in \mathbb{R}$, $h(x, Z) \in L^1$. Let g be the function $g : \mathbb{R} \to \mathbb{R} : x \mapsto \mathbb{E}[h(x, Z)]$. Then g is convex.

Proof. Let $0 \leq \lambda \leq 1$ and $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$. We have

$$\begin{aligned}
&\lambda g(x_1) + (1 - \lambda)g(x_2) \\
=& \lambda \mathbb{E}[h(x_1, Z)] + (1 - \lambda)\mathbb{E}[h(x_2, Z)] \\
=& \mathbb{E}[\lambda h(x_1, Z) + (1 - \lambda)h(x_2, Z)] \\
\geq& \mathbb{E}[h(\lambda x_1 + (1 - \lambda)x_2, Z)] \\
=& g(\lambda x_1 + (1 - \lambda)x_2)
\end{aligned}$$

□

Lemma 10.22. Let f be a convex function that has left asymptote $rx + a$ and right asymptote $tx + b$. If $-1 < r$ and $t < 1$, then f is a contraction mapping.

Proof. By Lemma 10.12, we have, for $x, y \in \mathbb{R}$ with $x < y$

$$r \leq \frac{f(y) - f(x)}{y - x} \leq t \tag{10.68}$$

So that indeed $\max(|r|, |t|) < 1$ and

$$|f(y) - f(x)| \leq \max(|r|, |t|)\,|y - x| \tag{10.69}$$

□

Lemma 10.23. Let f be a convex function that has left asymptote $rx + a$. If $0 < r$ then f is strictly increasing.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$. We have, by Lemma 10.12

$$r \leq \frac{f(y) - f(x)}{y - x} \tag{10.70}$$

So

$$f(y) - f(x) \geq r(y - x) > 0 \tag{10.71}$$

□

10.4 Results from analysis

Lemma 10.24. Let f have right asymptote $r_1 + q_1$ and let g have right asymptote $r_2 x + q_2$. Then $g \circ f$ has right asymptote $r_2(r_1 x + q_1) + q_2$.

Proof. We have

$$\lim_{x \to \infty} f(x) - r_1 x = q_1$$

and (by Theorem 4.15 in [1], i.e. the substitution theorem)

$$\lim_{x \to \infty} g(f(x)) - r_2 f(x) = q_2$$

Combining these two equations gives

$$\begin{aligned} \lim_{x \to \infty} g(f(x)) - r_1 r_2 x &= \lim_{x \to \infty} g(f(x)) - r_2 f(x) + r_2 f(x) - r_1 r_2 x \\ &= q_2 + r_2 q_1 \end{aligned} \tag{10.72}$$

It follows that

$$\begin{aligned} &\lim_{x \to \infty} g(f(x)) - (r_2(r_1 x + q_1) + q_2) = \\ &\lim_{x \to \infty} g(f(x)) - (r_1 r_2 x + q_2 + r_2 q_1) = 0 \end{aligned} \tag{10.73}$$

□

Definition 10.25. Let f be a contraction mapping on $\mathbb{R}$. The best Lipschitz constant of f is the smallest $k \in (0,1)$ such that for all $x, y \in \mathbb{R}$

$$d(f(x), f(y)) \leq kd(x, y) \tag{10.74}$$

■

Theorem 10.26 (Banach fixed point theorem)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction mapping and let $f^{(n)}$ denote the n-fold composition of f. Let r be the best Lipschitz constant of f. We have that $f(x) = x$ has a unique solution d, for all $x \in \mathbb{R}$, $\lim_{n\to\infty} f^{(n)}(x) = d$ and for all $x \in \mathbb{R}$ there is a $c \in \mathbb{R}$ such that $|f^{(n)}(x) - d| \leq cr^n$.

Proof. See Paragraph 9.3.4 of [22]. □

Lemma 10.27. Let h be continuous on $\mathbb{R}$ and let $r, t \in \mathbb{R}$ be such that h has left asymptote rx and right asymptote tx. Let μ be a probability measure on $\mathbb{R}$ with $\int |y|\mu(dy) < \infty$. We have that

$$\int |h(y)|\mu(dy) < \infty \tag{10.75}$$

Proof. There are $y_1, y_2 \in \mathbb{R}$, so that for all $y \in (y_1, \infty)$, $|h(y)| < |y|$ and for all $y \in (-\infty, y_2)$, $|h(y)| < |y|$. Furthermore, because h is continuous it assumes a maximum M on $[x_1, x_2]$. Let

$$w(y) = \begin{cases} M & \text{if} \quad y \in [x_1, x_2] \\ |y| & \text{otherwise} \end{cases} \tag{10.76}$$

We have, for all $y \in \mathbb{R}$, $h(y) \leq w(y)$. So

$$\int |h(y)|\mu(dy) \leq \int w(y)\mu(dy) \leq \int |y|\mu(dy) + \int M\mu(dy) < \infty \tag{10.77}$$

□

Lemma 10.28. Let μ be a probability measure on $\mathbb{R}$ and let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function that satisfies that for all $y \in \mathbb{R}$, $h_y : x \mapsto h(x, y)$ is a contraction mapping with best Lipschitz constant k_y and $\kappa \sup_{y\in\mathbb{R}} k_y \in (0, 1)$. Furthermore let h satisfy for all $x \in \mathbb{R}$, $\int |h(x, y)|\mu(dy) < \infty$. Let g be the function defined by

$$g : x \mapsto \int h(x, y)\mu(dy) \tag{10.78}$$

We have that g is a contraction mapping.

Proof. Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. We have

$$\begin{aligned} |\frac{g(x_2 - g(x_1)}{x_2 - x_1}| &= |\int \frac{h(x_2, y) - h(x_1, y)}{x_2 - x_1}\mu(dy)| \\ &\leq \int |\frac{h(x_2, y) - h(x_1, y)}{x_2 - x_1}|\mu(dy) \leq \int \kappa\mu(dy) \\ &= \kappa \end{aligned} \tag{10.79}$$

□

10.5 Discrete time Markov processes

This subsection is based on Subsection 3.1 of [20] and provides more details for some results in that subsection for the special case of discrete time Markov processes. A difference with [20] is that we shall consider nonnegative measurable functions f is results known as generalised Markov properties, while in [20] bounded functions are considered. Another difference is that we shall use compositions of Markov kernels, like in Chapter 3 and Chapter 9 of [17].

Throughout this subsection, let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ be a filtered probability space, let $(E, \mathscr{E})$ be a phase space and let (X_n) be a discrete time Markov process on $(\mathcal{F}_n)$, with values in $(E, \mathscr{E})$.

With each finite Markov chain (i.e. E is finite), we can associate a sequence of probability matrices (P_n), such that $P_{n,i,j}$ gives us the probability that the Markov chain transitions from state i at time n to state j at time $n+1$. An analogue for the more general discrete time Markov processes is given by the following lemma.

Lemma 10.29. Suppose that the Markov process (X_n) satisfies for all $n \in \mathbb{N}$, $X_n(\Omega) \in \mathscr{E}$. For every $n \in \mathbb{N}$, there exists a Markov kernel P_n that satisfies

$$\forall B \in \mathscr{E} : P_n(X_n, B) = \mathbb{P}[X_{n+1} \in B | X_n] \quad \text{a.s.} \tag{10.80}$$

Proof.
Let $n \in \mathbb{N}$. Theorem 4 of §7 of Chapter II of [18] gives us that there exists a regular conditional distribution Q of X_n w.r.t. $\sigma(X_{n-1})$. That is, there exists a function $Q : \Omega \times \mathscr{E} \to [0,1]$, that satisfies that for all $\omega \in \Omega$, $Q(\omega, \cdot)$ is a probability measure on $(E, \mathscr{E})$ and for all $B \in \mathscr{E}$, $Q(\cdot, B)$ is a version of $\mathbb{P}[X_{n+1} \in B | X_n]$.

Let $B \in \mathscr{E}$. Consider the function $\eta_B = Q(\cdot, B)$. Because η_B is a version of $\mathbb{P}[X_{n+1} \in B | X_n]$, η_B is $\sigma(X_n)$ measurable. Now Theorem 3 of §4 of Chapter II of [18] gives us that there exists a $\mathscr{E}/\mathcal{B}([0,1])$ measurable function ϕ_B such that $\eta_B = \phi_B \circ X_n$. We remark that ϕ_B is in fact the conditional probability $\phi_B = \mathbb{P}[X_{n+1} \in B | X_n = \cdot]$.

Because we can construct such a ϕ_B for any $B \in \mathscr{E}$, we define ϕ_B accordingly, and for any $B \in \mathscr{E}$, we define $\tilde{P}$ to be the function $\tilde{P} : E \times \mathscr{E} : (e, B) \mapsto \phi_B(e)$. Then $\tilde{P}(\cdot, B) = \phi_B$ is $\mathcal{F}/\mathcal{B}([0,1])$ measurable and $\tilde{P}$ satisfies that for all $B \in \mathscr{E}$ and for all $\omega \in \Omega$,

$$\tilde{P}(X_n(\omega), B) = \phi_B \circ X_n(\omega) = \eta_B(\omega) = Q(\omega, B) \tag{10.81}$$

Let $e \in E$ be fixed and let δ_e be Dirac measure. Now let P be the following function

$$P : E \times \mathscr{E} \to [0,1] : (x,B) \mapsto \begin{cases} \tilde{P}(x,B) & \text{if } x \in X_n(\Omega) \\ \delta_x(B) & \text{otherwise} \end{cases}$$

By construction, for all $x \in X_n(\Omega)^c$, $P(x,\cdot) = \delta_x$ is a measure. If $x \in X_n(\Omega)$, there is an $\omega \in \Omega$ such that $X_n(\omega) = x$. For this ω we have that $\tilde{P}(x,\cdot) = \tilde{P}(X_n(\omega),\cdot) = Q(\omega,\cdot)$. By the definition of a regular conditional distribution, $Q(\omega,\cdot)$ is a measure, so that $\tilde{P}(x,\cdot)$ is a measure.

Let $B \in \mathscr{E}$ and let $A \in \mathcal{B}([0,1])$ and let $P_B = P(\cdot, B)$. Furthermore, let

$$\begin{aligned} B_{A,0} &= \begin{cases} B^c & \text{if } \quad 0 \in A \\ \emptyset & \text{otherwise} \end{cases} \\ B_{A,1} &= \begin{cases} B & \text{if } \quad 1 \in A \\ \emptyset & \text{otherwise} \end{cases} \end{aligned} \tag{10.82}$$

We have

$$\begin{aligned} P_B^{-1}(A) &= \{x \in X_n(\Omega) \,|\, \tilde{P}(x,B) \in A\} \cup \{x \in X_n(\Omega)^c \,|\, \delta_x(B) \in A\} \\ &= X_n(\Omega) \cap \phi_B^{-1}(A) \cup X_n(\Omega)^c \cap B_{A,0} \cap B_{A,1} \end{aligned} \tag{10.83}$$

By our assumption that $X_n(\Omega) \in \mathscr{E}$ and the fact that ϕ_B is measurable, we see that in each of the cases $\{0,1\} \cap A = \{1\}, \{0\}, \{0,1\}, \emptyset$, we have $P_B^{-1}(A) \in \mathscr{E}$. So P_B is $\mathscr{E}/\mathcal{B}([0,1])$ measurable.

Because P is $\mathscr{E}/\mathcal{B}([0,1])$ measurable in its first argument and a measure in its second argument, P is a Markov kernel. Furthermore we have, for all $\omega \in \Omega$

$$\forall B \in \mathscr{E} : P(X_n(\omega), B) = Q(\omega, B) \tag{10.84}$$

For all $B \in \mathscr{E}$, $Q(\cdot, B)$ is a version of $\mathbb{P}[X_{n+1} \in B | X_n]$, so for all $B \in \mathscr{E}$, $P(X_n, B)$ is a version of $\mathbb{P}[X_{n+1} \in B | X_n]$, i.e. (10.80) holds.

□

For the rest of this subsection, we shall only consider homogeneous Markov processes (X_n).

Definition 10.30 (Power of a Markov kernel)**.**
Let P be a Markov kernel. We set $P^1 = P$ and for $k \in \mathbb{N}$ with $k \geq 2$, we set P^k to be the function

$$P^k : (x,B) \mapsto \int P^{k-1}(y,B) P(x,dy) \tag{10.85}$$

■

We claim without proof that for any $k \in \mathbb{N}$, P^k is a Markov kernel. The following lemma is only interesting as a tool to prove other lemmas, as we will see further on that the condition always holds, so that the conclusion also always holds.

Lemma 10.31. Let (X_n) be a homogeneous Markov process with Markov kernel P. For all $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have that if for all $B \in \mathscr{E}$

$$P^k(X_n, B) = \mathbb{P}[X_{n+k} \in B | \mathcal{F}_n] \quad \text{a.s.} \tag{10.86}$$

then for all nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable functions f, we have

$$\int f(x) P^k(X_n, dx) = \mathbb{E}[f(X_{n+k}) | \mathcal{F}_n] \quad \text{a.s.} \tag{10.87}$$

Proof. We use the standard machine. Let $B \in \mathcal{F}$ and let $u = \mathbb{1}_B$. Then by (10.86),

$$\begin{aligned} \mathbb{E}[u(X_{n+k}) \,|\, \mathcal{F}_n^X] &= \mathbb{E}[\mathbb{1}_B(X_{n+k}) \,|\, \mathcal{F}_n^X] = \mathbb{P}[X_{n+k} \in B \,|\, \mathcal{F}_n^X] \\ &= P^k(X_n, B) = \int \mathbb{1}_B(x) P^k(X_n, dx) \\ &= \int u(x) P^k(X_n, dx) \end{aligned} \tag{10.88}$$

Let u be a nonnegative simple function with standard representation $u = \sum_{i=0}^n b_i \mathbb{1}_{A_i}(X_n)$. Then

$$\mathbb{E}[u(X_{n+k}) \,|\, \mathcal{F}_n^X] = \sum_{i=0}^m b_i \mathbb{E}[\mathbb{1}_{A_i}(X_{n_k}) \,|\, \mathcal{F}_n^X] \tag{10.89}$$

Using (10.88), we get

$$\begin{aligned} \mathbb{E}[u(X_{n+k}) \,|\, \mathcal{F}_n^X] &= \sum_{i=0}^m b_i \int \mathbb{1}_{A_i}(x) P(X_{n+k}, dx) \\ &= \int \sum_{i=0}^m b_i \mathbb{1}_{A_i}(x) P^k(X_n, dx) \\ &= \int u(x) P^k(X_n, dx) \end{aligned} \tag{10.90}$$

Let u be a nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function and let u_m be an increasing sequence of nonnegative simple functions so that $u = \lim_{m\to\infty} u_m$. We have, by Theorem 2 of §7 of Chapter II of [18] (i.e. monotone convergence for conditional expectations) and by (10.90),

$$\begin{aligned} \mathbb{E}[u(X_{n+k}) \,|\, \mathcal{F}_n^X] &= \lim_{m\to\infty} \mathbb{E}[u_m(X_{n+k}) \,|\, \mathcal{F}_n^X] \\ &= \lim_{m\to\infty} \int u_m(x) P^k(X_n, dx) \\ &= \int u(x) P^k(X_n, dx) \end{aligned} \tag{10.91}$$

□

Lemma 10.32. Let X be a homogeneous discrete time Markov process with Markov kernel P and let f be a nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function. Then for all $n \in \mathbb{N}_0$

$$\int f(x)P(X_n, dx) = \mathbb{E}[f(X_{n+1})|\mathcal{F}_n^X] \quad \text{a.s.} \tag{10.92}$$

Proof. This follows from Lemma 10.31 with $k = 1$, as for $k = 1$ the condition (10.86) is the equation (10.80) and by definition P is a kernel that satisfies this equation. □

Lemma 10.33. Let X be a homogeneous discrete time Markov process with Markov kernel P. For all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ and for all $B \in \mathscr{E}$, we have

$$P^k(X_n, B) = \mathbb{P}[X_{n+k} \in B|\mathcal{F}_n] \quad \text{a.s.} \tag{10.93}$$

Proof. We prove this by induction. Our induction hypothesis is that for some $k \in \mathbb{N}$, we have that for all $m \in \mathbb{N}_0$ and for all $B \in \mathscr{E}$

$$\mathbb{P}[X_{m+k} \in B \,|\, \mathcal{F}_m] = P^k(X_m, B) \tag{10.94}$$

We will now show that this yields the same statement for $k+1$. Let $n \in \mathbb{N}$, we have

$$\mathbb{P}[X_{n+k+1} \in B \,|\, \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[\mathbb{1}_B(X_{(n+1)+k}) \,|\, \mathcal{F}_{n+1}] \,|\, \mathcal{F}_n] \tag{10.95}$$

By equation (10.94), where we consider $m = n + 1$, this is

$$\mathbb{P}[X_{n+k+1} \in B \,|\, \mathcal{F}_n] = \mathbb{E}[P^k(X_{n+1}, B) \,|\, \mathcal{F}_n] \tag{10.96}$$

Now by Lemma 10.32, this is

$$\begin{aligned} \mathbb{P}[X_{n+k+1} \in B \,|\, \mathcal{F}_n] &= \int P^k(x, B)P(X_n, dx) \\ &= P^{k+1}(X_n, B) \end{aligned} \tag{10.97}$$

Because $n \in \mathbb{N}_0$ was arbitrary, we have indeed proven that the statement of the induction hypothesis holds for $k+1$. The statement holds for $k = 1$ by the definition of P as a Markov kernel satisfying this relation. So the result follows from the principle of induction. □

Lemma 10.34. Let X be a discrete time homogeneous Markov process with Markov kernel P and let f be a nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function. For all $k \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have

$$\mathbb{E}[f(X_{n+k})|\mathcal{F}_n^X] = \int f(x)P^k(X_n, dx) \quad \text{a.s.} \tag{10.98}$$

Proof. This follows from Lemma 10.33 and Lemma 10.31. □

The following definition corresponds to the definition of compositions of Markov kernels in [17], though here we define the composition for a smaller class of pairs of Markov kernels. The more general definition is somewhat less direct for this simple subset of pairs of Markov kernels, which is why I have introduced this equivalent definition and called this the simple composition of Markov kernels.

Definition 10.35 (**Simple composition of Markov kernels**)**.**

Let κ_1 be a kernel from $(\Xi, \mathcal{X})$ to $(\Upsilon, \mathcal{Y})$ and κ_2 a kernel from $(\Upsilon, \mathcal{Y})$ to $(\Gamma, \mathcal{Z})$. Then we define $\kappa_1 \otimes \kappa_2$ to be the function with domain $(\Xi, \mathcal{Y} \times \mathcal{Z})$ and range $\mathcal{B}([0,1])$, that maps elements in the following way

$$(x, A) \mapsto \int \mathbb{1}_A(y, z)\kappa_1(x, dy) \int \kappa_2(y, dz) \tag{10.99}$$

Let κ_i be a sequence of Markov kernels. We shall also write

$$\bigotimes_{i=1}^{n} \kappa_i = \kappa_1 \otimes \cdots \otimes \kappa_n$$

Let μ be a measure and ρ a Markov kernel. We can construct another kernel κ_μ by letting, for all x and B, $\kappa_\mu(x, B) = \mu(B)$. We then define

$$\mu \otimes \rho = \kappa_\mu \otimes \rho$$

■

Lemma 10.36 and Lemma 10.37 below are claimed to hold without proof. Note that Lemma 10.37 justifies the notation $\bigotimes_{i=1}^{n} \kappa_i$.

Lemma 10.36. Let κ_1 be a kernel from $(\Xi, \mathcal{X})$ to $(\Upsilon, \mathcal{Y})$ and κ_2 a kernel from $(\Xi \times \Upsilon, \mathcal{X} \times \mathcal{Y})$ to $(\Gamma, \mathcal{Z})$. Then their simple composition $\kappa_1 \otimes \kappa_2$ is a Markov kernel.

Lemma 10.37. The simple composition of Markov kernels is associative.

Lemma 10.38. Let (Ξ, χ), $(\Upsilon, \mathcal{Y})$ and $(\Gamma, \mathcal{Z})$ be measurable spaces. Let $f : \Upsilon \times \Gamma \to \mathbb{R}$ be bounded and measurable. Let κ be a Markov kernel from (Ξ, χ) to $(\Upsilon, \mathcal{Y})$ and ρ be a Markov kernel from $(\Upsilon, \mathcal{Y})$ to $(\Gamma, \mathcal{Z})$. For all $x \in \Xi$

$$\int f(u)(\kappa \otimes \rho)(x, du) = \int \int f(y, z)\rho(y, dz)\kappa(x, dy) \tag{10.100}$$

Proof. Let $x \in \Xi$ and $A \in \mathcal{Y} \otimes \mathcal{Z}$. We consider the equality for $\mathbb{1}_A$. We have

$$\begin{aligned} \int \mathbb{1}_A(u)(\kappa \otimes \rho)(x, du) &= (\kappa \otimes \rho)(x, A) \\ &= \int \int \mathbb{1}_A(x, y)\rho(y, dz)\kappa(x, dy) \end{aligned} \tag{10.101}$$

Now let $(A_i)_{i=0}^n \subset \mathcal{Y} \otimes \mathcal{Z}$ and let $(b_i)_{i=0}^n \subset \mathbb{R}$. We consider the simple function $\sum_{i=1}^n b_i \mathbb{1}_{A_i}$. We have, by (10.101)

$$\begin{aligned}\int \sum_{i=1}^n b_i \mathbb{1}_{A_i}(u)(\kappa \otimes \rho)(x, du) &= \sum_{i=1}^n b_i \int \mathbb{1}_{A_i}(u)(\kappa \otimes \rho)(x, du) \\ &= \sum_{i=1}^n b_i \int \int \mathbb{1}_{A_i}(x, y)\rho(y, dz)\kappa(x, dy) \\ &= \int \int \sum_{i=1}^n b_i \mathbb{1}_{A_i}(x, y)\rho(y, dz)\kappa(x, dy)\end{aligned} \tag{10.102}$$

Now we consider a nonnegative measurable bounded function g. We know that we can find an increasing sequence $(g_k)_{k=0}^\infty$ of simple functions such that $g = \lim_{k\to\infty} g_k$. We have, by monotone convergence

$$\int \lim_{k\to\infty} g_k(u)(\kappa \otimes \rho)(x, du) = \lim_{k\to\infty} \int g_k(u)(\kappa \otimes \rho)(x, du) \tag{10.103}$$

By (10.102), this is

$$\int \lim_{k\to\infty} g_k(u)(\kappa \otimes \rho)(x, du) = \lim_{k\to\infty} \int \int g_k(x, y)\rho(y, dz)\kappa(x, dy) \tag{10.104}$$

Now, because its integrand is increasing, $\int g_k(x, y)\rho(y, dz)$ is increasing in k. Applying monotone convergence twice, we find

$$\int \lim_{k\to\infty} g_k(u)(\kappa \otimes \rho)(x, du) = \int \int \lim_{k\to\infty} g_k(x, y)\rho(y, dz)\kappa(x, dy) \tag{10.105}$$

Finally, we have that $f = f^+ - f^-$. This gives us, by (10.105),

$$\begin{aligned}&\int f(u)(\kappa \otimes \rho)(x, du) \\ &\int f^+(u)(\kappa \otimes \rho)(x, du) - \int f^-(u)(\kappa \otimes \rho)(x, du) \\ &= \int \int f^+(x, y)\rho(y, dz)\kappa(x, dy) - \int \int f^-(x, y)\rho(y, dz)\kappa(x, dy) \\ &= \int \int f(x, y)\rho(y, dz)\kappa(x, dy)\end{aligned} \tag{10.106}$$

□

Lemma 10.39. Let (X_n) be a discrete time homogeneous Markov process with Markov kernel P. Let $m \in \mathbb{N}$ and let for all $i \in \{1, \dots, m\}$, g_i be a nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function. Let $k \in \mathbb{N}_0$, let $(n_i)_{i=0}^m \subset \mathbb{N}$ be strictly increasing with $n_0 = 0$ and $n_1 \geq k$. We have, almost surely,

$$\mathbb{E}[\prod_{j=1}^m g_j(X_{n_j+k})|\mathcal{F}_k] = \int \prod_{j=1}^m g_j(x_j) \bigotimes_{i=1}^m P^{n_i - n_{i-1}}(X_k, (dx_1, \dots dx_m)) \tag{10.107}$$

Proof. We prove this by induction. For $m = 1$ this holds by Lemma 10.34. Our induction hypothesis is that (10.107) holds for $m - 1$. We have

$$\mathbb{E}[\prod_{j=1}^{m} g_j(X_{n_j+k})|\mathcal{F}_k] = \mathbb{E}[g_1(X_{n_1+k})\mathbb{E}[\prod_{j=2}^{m} g_j(X_{n_j+k})|\mathcal{F}_{n_1}]|\mathcal{F}_k] \tag{10.108}$$

By the induction hypothesis, we have that this is

$$\mathbb{E}[\prod_{i=j}^{m} g_j(X_{n_j})|\mathcal{F}_k] =$$
$$\mathbb{E}[g_1(X_{n_1+k}) \int \prod_{j=2}^{m} g_j(x_j) \bigotimes_{i=2}^{m} P^{n_i-n_{i-1}}(X_{n_1+k}, (dx_2, \dots dx_m))|\mathcal{F}_k] \tag{10.109}$$

Again by Lemma 10.34, this is, almost surely

$$\begin{aligned}
&\mathbb{E}[\prod_{j=1}^{m} g_j(X_{n_j+k})|\mathcal{F}_k] \\
&= \int g_1(x_1) \int \prod_{j=2}^{m} g_j(x_j) \bigotimes_{i=2}^{m} P^{n_i-n_{i-1}}(x_1, (dx_2, \dots dx_m)) P^{n_1}(X_k, dx_1) \\
&= \int \int \prod_{j=1}^{m} g_j(x_j) \bigotimes_{i=2}^{m} P^{n_i-n_{i-1}}(x_1, (dx_2, \dots dx_m)) P^{n_1}(X_k, dx_1)
\end{aligned}$$

It now follows from Lemma 10.38 that this is, almost surely

$$\begin{aligned}
&\mathbb{E}[\prod_{j=1}^{m} g_j(X_{n_i+k})|\mathcal{F}_k] \\
&= \int \prod_{j=1}^{m} g_j(x_j)(P^{n_1-0} \otimes \bigotimes_{i=2}^{m} P^{n_i-n_{i-1}})(X_k, (dx_1, dx_2, \dots dx_m)) \\
&= \int \prod_{j=1}^{m} g_j(x_j) \bigotimes_{i=1}^{m} P^{n_i-n_{i-1}}(X_k, (dx_1, dx_2, \dots dx_m))
\end{aligned}$$

So we have proven that if (10.107) holds for $m - 1$, then it holds for m. We also saw that (10.107) holds for $m = 1$, so the result follows by induction. □

Corollary 10.40. Let (X_n) be a discrete time homogeneous Markov process with Markov kernel P. Let $m \in \mathbb{N}$ and let for all $i \in \{1, \dots, m\}$, $B_i \in \mathscr{E}$. Let $k \in \mathbb{N}_0$, let $(n_i)_{i=0}^m \subset \mathbb{N}$ be strictly increasing with $n_0 = 0$ and $n_1 \geq k$. We have, almost surely

$$\mathbb{P}[\bigwedge_{i=1}^{m} X_{n_i+k} \in B_i \,|\, X_k] = \bigotimes_{i=1}^{m} P^{n_i-n_{i-1}}(X_k, B_1 \times \cdots \times B_m)$$

Proof.

$$\begin{aligned}
&\mathbb{P}[\bigwedge_{i=1}^{m} X_{n_i+k} \in B \,|\, X_k] \\
=&\mathbb{E}[\mathbb{1}_{B_1\times\cdots\times B_m}(X_{n_1+k}\dots X_{n_m+k}) \,|\, X_k] \\
=&\int \mathbb{1}_{B_1\times\cdots\times B_m}(u) \bigotimes_{i=1}^{m} P^{n_i-n_{i-1}}(X_k, du) \\
=&\bigotimes_{i=1}^{m} P^{n_i-n_{i-1}}(X_k, B_1\times\cdots\times B_m)
\end{aligned} \tag{10.110}$$

□

10.6 Generalized Markov property

Here we prove a discrete time version of Theorem 3.2.4 of [20].

Throughout this subsection, let $(E, \mathscr{E})$ be a phase space, let $\Omega = E^{\mathbb{N}_0}$, let $\mathcal{F}$ be the cylinder σ-algebra on Ω generated by $(E, \mathscr{E})$ and let X be the canonical process on $(E, \mathscr{E})$ and let $(\mathcal{F}_n)_{n=0}^{\infty}$ be the natural filtration of X. Let $\mathbb{P} = \{\mathbb{P}_x \,|\, x \in E\}$ be a family of probability measures so that for all $x \in E$, X is a homogeneous Markov process on $(\Omega, \mathcal{F}, \mathbb{P}_x)$ with Markov kernel P and $\mathbb{P}_x[X_0 = x] = 1$.

Lemma 10.41. Let f be a nonnegative $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function. For all $x \in E$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}_0$, we have that

$$\mathbb{E}_{X_n}[f(X_k)] = \int f(y) P^k(X_n, dy) \tag{10.111}$$

Proof. Let $k \in \mathbb{N}_0$. It follows from Lemma 10.34 that for all $x \in E$, we have

$$\begin{aligned}
\mathbb{E}_x[f(X_k)] = \mathbb{E}_x[\mathbb{E}_x[f(X_k) \,|\, X_0]] = \mathbb{E}_x[\int f(y) P^k(X_0, dy)] \\
= \int f(y) P^k(x, dy)
\end{aligned} \tag{10.112}$$

For all $x \in E$ and for all $\omega \in \Omega$ we have, substituting $x = X_n(\omega)$ in (10.112)

$$\mathbb{E}_{X_n(\omega)}[f(X_k)] = \int f(y) P^k(X_n(\omega), dy) \tag{10.113}$$

□

Combining Lemma 10.41 with Lemma 10.34, we find

$$\mathbb{E}_{X_n}[f(X_k)] = \mathbb{E}_x[f(X_{n+k}) \,|\, \mathcal{F}_n] \quad \mathbb{P}_x\text{-a.s.} \tag{10.114}$$

Note that to arrive at (10.114) we did not have to use compositions of Markov kernels. Using compositions of Markov kernels however, we can generalize this statement to the following Proposition, which is a generalized Markov property.

Proposition 10.42.
For all $x \in E$ and $n \in \mathbb{N}$ and all nonnegative $\mathcal{F}$ measurable random variables H, we have

$$\mathbb{E}_x[H \circ \theta_n \,|\, \mathcal{F}_n] = \mathbb{E}_{X_n}[H] \qquad \mathbb{P}_x\text{-a.s.} \tag{10.115}$$

Proof.
We will use the standard machine, where by far the most work is in proving that the result holds for indicator functions. We first show that the result holds for indicator functions. Let

$$S = \{B \in \mathcal{F} \,|\, \mathbb{E}_x[\mathbb{1}_B \circ \theta_n \,|\, \mathcal{F}_n] = \mathbb{E}_{X_n}[\mathbb{1}_B] \quad \mathbb{P}_x\text{-a.s.}\} \tag{10.116}$$

We show that S is a Dynkin system.

i) Note that $\mathbb{1}_\Omega = 1$ so that $\mathbb{1}_\Omega \circ \theta_n = 1$, so that

$$\mathbb{E}_x[\mathbb{1}_\Omega \circ \theta_n \,|\, \mathcal{F}_n] = 1 = \mathbb{E}_{X_n}[\mathbb{1}_\Omega] \tag{10.117}$$

We also see that $\Omega \in \mathcal{F}_\infty^X$, so $\Omega \in S$.

ii) Let $B \in S$. We have that $B \in \mathcal{F}_\infty^X$ and

$$\mathbb{E}_x[\mathbb{1}_B \circ \theta_n \,|\, \mathcal{F}_n] = \mathbb{E}_{X_n}[\mathbb{1}_B] \tag{10.118}$$

Because $\mathcal{F}_\infty^X$ is a σ-algebra, we have that $B^c \in \mathcal{F}_\infty^X$. We have

$$\begin{aligned}
\mathbb{E}_x[\mathbb{1}_{B^c} \circ \theta_n \,|\, \mathcal{F}_n] &= \mathbb{E}_x[\mathbb{1}_{\theta_n^{-1}(\Omega \setminus B)} \,|\, \mathcal{F}_n] \\
&= \mathbb{E}_x[\mathbb{1}_{\Omega \setminus \theta_n^{-1}(B)} \,|\, \mathcal{F}_n] \\
&= 1 - \mathbb{E}_x[\mathbb{1}_{\theta_n^{-1}(B)} \,|\, \mathcal{F}_n] \\
&= 1 - \mathbb{E}_x[\theta_n \circ \mathbb{1}_B \,|\, \mathcal{F}_n]
\end{aligned} \tag{10.119}$$

This gives us, by using (10.118) on the right hand side

$$\begin{aligned}
\mathbb{E}_x[\mathbb{1}_{B^c} \circ \theta_n \,|\, \mathcal{F}_n] &= 1 - \mathbb{E}_{X_n}[\mathbb{1}_B] \\
&= \mathbb{E}_{X_n}[\mathbb{1}_{B^c}]
\end{aligned} \tag{10.120}$$

iii) Let $(B_i)_{i \in \mathbb{N}} \subset S$ be a disjoint sequence. We have for all $i \in \mathbb{N}$

$$\mathbb{E}_x[\mathbb{1}_{B_i} \circ \theta_n \,|\, \mathcal{F}_n] = \mathbb{E}_{X_n}[\mathbb{1}_{B_i}] \tag{10.121}$$

We have, by monotone convergence for conditional expectations

$$\begin{aligned}
\mathbb{E}_x[\mathbb{1}_{\bigcup_{i \in \mathbb{N}} B_i} \circ \theta_n \,|\, \mathcal{F}_n] &= \mathbb{E}_x[\mathbb{1}_{\bigcup_{i \in \mathbb{N}} \theta_n^{-1}(B_i)} \,|\, \mathcal{F}_n] \\
&= \mathbb{E}_x[\sum_{i \in \mathbb{N}} \mathbb{1}_{\theta_n^{-1}(B_i)} \,|\, \mathcal{F}_n] \\
&= \sum_{i \in \mathbb{N}} \mathbb{E}_x[\mathbb{1}_{\theta_n^{-1}(B_i)} \,|\, \mathcal{F}_n] \\
&= \sum_{i \in \mathbb{N}} \mathbb{E}_x[\mathbb{1}_{B_i} \circ \theta_n \,|\, \mathcal{F}_n]
\end{aligned} \tag{10.122}$$

This gives us, by using (10.121) on the right hand side

$$\mathbb{E}_x[\mathbb{1}_{\bigcup_{i\in\mathbb{N}} B_i} \circ \theta_n \,|\, \mathcal{F}_n] = \sum_{i\in\mathbb{N}} \mathbb{E}_{X_n}[\mathbb{1}_{B_i}] \tag{10.123}$$

Indeed i), ii) and iii) are satisfied, so that S is a Dynkin system.

We define the set of rectangles $\mathcal{C}$ to be the following

$$\mathcal{C} = \{C \in \mathcal{F}_\infty^X \,|\, (s_i)_{i=0}^m \subset \mathbb{N}_0^m\,, (C_i)_{i=0}^m \subset \mathscr{E}, C = \{X_{s_0} \in C_0, \dots X_{s_m} \in C_m\}\}$$

We consider $\mathcal{C}$ because it is a π-system generating $\mathcal{F}_\infty^X$. Note that restricting the (s_i) to be strictly increasing yields the same set. Let $C \in \mathcal{C}$ and let $(s_i)_{i=0}^m \subset \mathbb{N}_0$ be strictly increasing with $s_0 = 0$ and let $(C_i)_{i=1}^m \subset \mathscr{E}$ be such that

$$C = \{X_{s_1} \in C_1, \dots X_{s_m} \in C_m\} \tag{10.124}$$

We have

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_C \circ \theta_n \,|\, \mathcal{F}_n] &= \mathbb{E}_x[\mathbb{1}_{\{X_{s_1}\in C_1,\dots X_{s_m}\in C_m\}} \circ \theta_n \,|\, \mathcal{F}_n] \\ &= \mathbb{E}_x[\mathbb{1}_{\{X_{s_1+n}\in C_1,\dots X_{s_m+n}\in C_m\}} \,|\, \mathcal{F}_n] \\ &\mathbb{E}_x[\prod_{i=2}^m \mathbb{1}_{C_i}(X_{s_i}) \,|\, \mathcal{F}_n]\end{aligned}$$

By Lemma 10.39, this is, $\mathbb{P}_x$ almost surely,

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_C \circ \theta_n \,|\, \mathcal{F}_n] &= \int \prod_{i=1}^m \mathbb{1}_{C_i}(x_i) \bigotimes_{i=1}^m P^{s_i - s_{i-1}}(X_n, dx_1, x_2, \dots x_n) \\ &= \int \mathbb{1}_{C_1}(x_1) \prod_{i=2}^m \mathbb{1}_{C_i}(x_i) (P^{s_1} \otimes \bigotimes_{i=2}^m P^{t_i - t_{i-1}})(X_n, dx_1, x_2, \dots x_n)\end{aligned}$$

By Lemma 10.38, this is, $\mathbb{P}_x$ almost surely

$$\mathbb{E}_x[\mathbb{1}_C \circ \theta_n \,|\, \mathcal{F}_n] = \int \mathbb{1}_{C_1}(y) \int \prod_{i=2}^m \mathbb{1}_{C_i}(x_i) \bigotimes_{i=2}^m P^{s_i - s_{i-1}}(y, dx_2, \dots dx_m) P^{s_1}(X_n, dy)$$

By Lemma 10.41, this is, $\mathbb{P}_x$ almost surely

$$\begin{aligned}&\mathbb{E}_x[\mathbb{1}_C \circ \theta_n \,|\, \mathcal{F}_n] \\ =&\mathbb{E}_{X_n}[\mathbb{1}_{C_1}(X_{s_1}) \int \prod_{i=2}^m \mathbb{1}_{C_i}(x_i) \bigotimes_{i=2}^m P^{s_i - s_{i-1}}(X_{s_1}, dx_2, \dots dx_m)] \\ =&\mathbb{E}_{X_n}[\mathbb{1}_{C_1}(X_{s_1}) \int \prod_{i=2}^m \mathbb{1}_{C_i}(x_i) \bigotimes_{i=2}^m P^{(s_i - s_1) - (s_{i-1} - s_1)}(X_{s_1}, dx_2, \dots dx_m)]\end{aligned}$$

Lemma 10.39 holds for any $\mathbb{P} \in \{\mathbb{P}_x \,|\, x \in E\}$, so for every $\omega \in \Omega$, it holds for $\mathbb{P}_{X_n}$, so we get, $\mathbb{P}_x$ almost surely

$$\begin{aligned}\mathbb{E}_x[\mathbb{1}_C \circ \theta_n \,|\, \mathcal{F}_n] &= \mathbb{E}_{X_n}[\mathbb{1}_{C_1}(X_{s_1})\mathbb{E}_{X_n}[\prod_{i=2}^{m} \mathbb{1}_{C_i}(X_{s_i}) \,|\, X_{s_1}]] \\ &= \mathbb{E}_{X_n}[\mathbb{E}_{X_n}[\mathbb{1}_{C_1\times\cdots\times C_m}(X_{s_1},\ldots,X_{s_m}) \,|\, X_{s_1}]] \\ &= \mathbb{E}_{X_n}[\mathbb{1}_{C_1\times\cdots\times C_m}(X_{s_1},\ldots,X_{s_m})] \\ &= \mathbb{E}_{X_n}[\mathbb{1}_C]\end{aligned}$$

So we have for all $C \in \mathcal{C}$ that $C \in S$, i.e. $\mathcal{C} \subset S$. We know that $\sigma(\mathcal{C}) = \mathcal{F}$ and by its definition $S \subset \mathcal{F}$, so by the π-λ theorem, i.e. Theorem 5.5 in [16], we have that $S = \mathcal{F}$. So the result holds for indicator functions, that is to say, for all $B \in \mathcal{F}$

$$\mathbb{E}_x[\mathbb{1}_B \circ \theta_n \,|\, \mathcal{F}_n] = \mathbb{E}_{X_n}[\mathbb{1}_B] \quad \mathbb{P}_x\text{-a.s.} \tag{10.125}$$

Now u be a simple function and let $m \in \mathbb{N}$, $(c_i)_{i=0}^m \subset \mathbb{R}$, $(B_i)_{i=0}^m \subset \mathscr{E}$ be such that $u = \sum_{i=0}^m \mathbb{1}_{B_i} c_i$. We have $\mathbb{P}_x$ almost surely

$$\begin{aligned}\mathbb{E}_x[u \circ \theta_n \,|\, \mathcal{F}_n] &= \sum_{i=0}^{m} c_i \mathbb{E}_x[\mathbb{1}_{B_i} \circ \theta_n \,|\, \mathcal{F}_n] \\ &= \sum_{i=0}^{m} c_i \mathbb{E}_{X_n}[\mathbb{1}_{B_i}] \\ &= \mathbb{E}_{X_n}[u]\end{aligned} \tag{10.126}$$

This shows that the result holds for simple functions. Now let u be a nonnegative measurable function and let $(u_k)_{k=0}^\infty$ be a sequence of simple functions such that $u = \lim_{k\to\infty} u_k$. We have, applying monotone convergence and the usual monotone convergence, $\mathbb{P}_x$ almost surely

$$\begin{aligned}\mathbb{E}_x[u \circ \theta_n \,|\, \mathcal{F}_n] &= \lim_{k\to\infty} \mathbb{E}_x[u_k \circ \theta_n \,|\, \mathcal{F}_n] \\ &= \lim_{k\to\infty} \mathbb{E}_{X_n}[u_k] \\ &= \mathbb{E}_{X_n}[u]\end{aligned} \tag{10.127}$$

□

10.7 Other

Lemma 10.43. Let $(E, \mathscr{E})$ be a measure space, let $x \in E$, let X be a random element with values in $(E, \mathscr{E})$ and $Y \in L^1$, let $F = \{X = x\}$ and suppose that $\mathbb{P}[F] = 1$. Let ξ be any version of $\mathbb{E}[Y \,|X]$. We have, for all $\omega \in F$

$$\mathbb{E}[Y] = \xi(\omega) \tag{10.128}$$

So we write that we have, for all $\omega \in F$

$$\mathbb{E}[Y] = \mathbb{E}[Y \,|X](\omega) \tag{10.129}$$

In particular $\mathbb{E}[Y]$ is a version of $\mathbb{E}[Y\,|X]$, which we write as

$$\mathbb{E}[Y] = \mathbb{E}[Y\,|X] \quad \text{a.s.} \tag{10.130}$$

Proof. Let ξ be any version of $\mathbb{E}[Y\,|X]$. ξ is $\sigma(X)/\mathcal{B}(\mathbb{R})$ measurable, so Theorem 3 of §4 of Chapter II of [18] gives us that that there exists a $\mathscr{E}/\mathcal{B}(\mathbb{R})$ measurable function ϕ such that $\xi = \phi(X)$. The fact that $\mathbb{P}[F] = 1$ and the tower property then give us, for all $\omega \in F$

$$\mathbb{E}[Y] = \mathbb{E}[\xi] = \mathbb{E}[\phi(X)] = \phi(x) = \phi(X(\omega)) = \xi(\omega) \tag{10.131}$$

Note that this means that $\mathbb{E}[Y] = \xi$ almost surely. One way we can then see that $\mathbb{E}[Y]$ is a version of $\mathbb{E}[Y\,|\,X]$ is to use that the versions of $\mathbb{E}[Y\,|\,X]$ form an equivalence class of $L^1(\mathbb{P})$ under the equivalence relation $\approx$ given by $Z_1 \approx Z_2$ iff $Z_1 = Z_2$ a.s. Alternatively, one can directly show that the definition of a version holds as follows, using the fact that ξ is a version. For any $G \in \sigma(X)$, we have

$$\mathbb{E}[\mathbb{1}_G \mathbb{E}[Y]] = \mathbb{E}[\mathbb{1}_G \xi] = \mathbb{E}[\mathbb{1}_G Y] \tag{10.132}$$

□

Note that in the proof of Lemma 10.43, the function $y \mapsto \mathbb{E}[Y\,|\,X = y]$ is such a function ϕ, so we could also have written, that for all $\omega \in F$

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y\,|\,X]] = \mathbb{E}[Y\,|\,X = x] \\ &= \mathbb{E}[Y\,|X = X(\omega)] = \mathbb{E}[Y\,|X](\omega) \end{aligned} \tag{10.133}$$

This is however slightly harder to justify.

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