Constructions of symplectic foliations

A new view on the five-sphere

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Contents

Introduction

In this thesis we study the existence of *codimension-one symplectic foliations*. Roughly speaking these are decompositions of manifolds into hypersurfaces endowed with symplectic structures, varying smoothly along the hypersurfaces. Proving the existence of these structures is far from trivial. The main goal of this text is to reprove the following result:

Theorem ([Mit11])**.** *There exists a codimension-one symplectic foliation on* S^5*.*

We will do so by proving a general theorem ensuring the existence of symplectic foliations on certain manifolds, from which we obtain:

Theorem (5.2.5)**.** *There exists a codimension-one symplectic foliation on* $S^5/\mathbb{Z}_3$*, and consequently also on* S^5*.*

We begin with a small outline of the background and history of (symplectic) foliations.

History and background

A *foliation* on a manifold is a decomposition of that manifold into connected components of lower dimension. The theory of foliations began in earnest when, in 1952, Georges Reeb constructed a foliation on the three-dimensional sphere S^3. Later, Lawson [Law71] used the theory of Milnor fibrations to construct a foliation on S^5 and later, on all spheres S^{2^k+3}, using so called *open book decompositions*. Finally, Thurston [Thu76] answered the question of existence in full generality by proving the following:

Theorem. *A compact manifold admits a codimension-one foliation if and only if its Euler characteristic vanishes.*

Symplectic foliations arise from Poisson geometry as every Poisson manifold naturally carries a (singular) symplectic foliation. *Regular* Poisson manifolds are in one-to-one correspondence with symplectic foliations. The existence of these structures has turned out to be difficult to establish. Even for the spheres there was no understanding until Mitsumatsu proved the existence of a symplectic foliation on S^5. In his PhD thesis [Tor15] Osorno Torres exhibited the proof of Mitsumatsu and gave it a more geometric interpretation. His thesis served as a starting point to this thesis.

Structure of this thesis

Most of the theory in the first three chapters of this thesis can be found in some form in [Tor15].

In Chapter 1 we discuss the basic definitions regarding symplectic foliations and their relation to Poisson geometry.

In Chapter 2 we study how symplectic foliations behave under glueing. Most foliations on closed manifolds are constructed by cutting the manifold into two pieces, foliating these pieces separately and finally glueing the pieces together. To do as such we will need to define symplectic foliations which have the property that they can be glued; these will be symplectic foliations *tame near the boundary.* The main result of this chapter is:

Theorem (2.2.7)**.** *Let* $(M_i, \mathcal{F}_i, \omega_i)$ *be two manifolds with symplectic foliations tame near the boundary such that the symplectic structures on the boundaries coincide. Then the symplectic foliations* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *glue together to a symplectic foliation on* $M_1 \cup M_2$*.*

In Chapter 3 we describe a method of constructing symplectic foliations which are tame near the boundary. To do this we will study the behaviour of symplectic manifolds around their boundary. We will be interested in two types of boundaries: *cosymplectic type* and *contact type.* Symplectic structures of cosymplectic type at the boundary will induce *cosymplectic structures* on the boundary. These cosymplectic structures are particularly well-behaved examples of symplectic foliations. Symplectic structures of contact type at the boundary will induce *contact structures* on the boundary. These are maximally non-integrable hyperplane distributions and can be thought of as opposites of foliations. Using a local form of the symplectic structures near the boundary we obtain the following result:

Theorem (3.4.7)**.** *Let* (M, ω) *be a symplectic manifold with boundary of cosymplectic type, then* $M \times S^1$ *admits a codimension-one symplectic foliation which is tame near the boundary.*

This theorem will be one of our main tools in constructing symplectic foliations. Finally we will generalize this result to *cosymplectic manifolds with boundary of s-type.*

In Chapter 4 we discuss the definitions and constructions of *open book decompositions.* A manifold with an open book decomposition will carry a foliation outside of a codimension-two submanifold, this points towards a use of open books in constructing foliations. We describe how to construct open book decompositions by interpreting them as generalized angular functions. We also recall the definition of *open book decompositions supporting a contact structure*, which are open books which behave nicely with respect to a contact structure on the manifold. These open books have the property that they admit a symplectic foliation outside a codimension-two submanifold and will be used extensively in proving our final result. The chapter ends with the following result:

Theorem (4.4.13)**.** *Let M be a principal S^1-bundle over an integral symplectic manifold. Then under certain conditions M admits an open book decomposition supporting a contact form.*

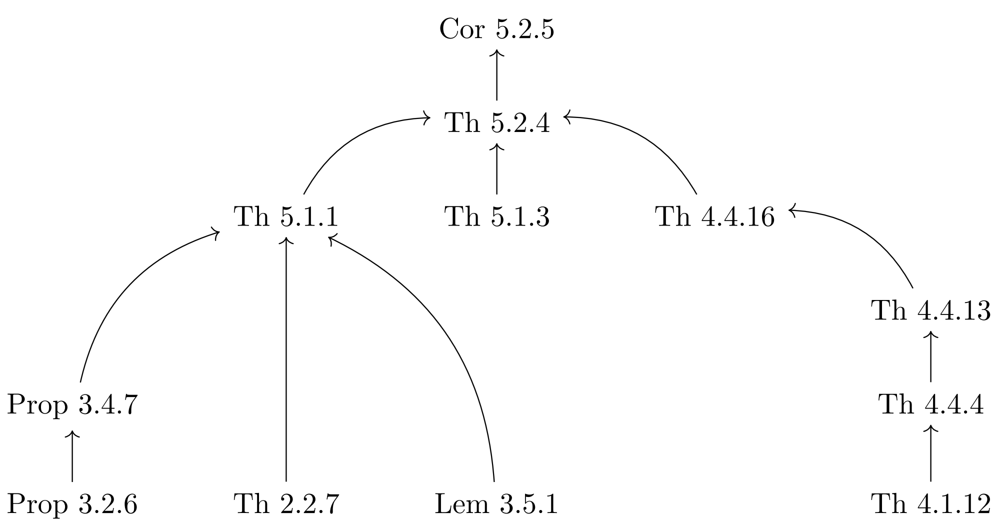
We call the open book decompositions arising from this theorem *Donaldson open book decompositions.*

In Chapter 5 we will show that under certain extra conditions a Donaldson open book gives rise to a codimension-one symplectic foliation. The further assumptions will be on properties of the symplectic manifold, basically reducing the problem to a problem in symplectic geometry. Finally we will use this result to prove the existence of a symplectic foliation on the lens space $S^5/\mathbb{Z}_3$ and consequently on S^5.

In Chapter 6 we will move away from symplectic foliation and study *complex foliations*. These are foliations together with a complex structure on each leaf, such that these structures vary smoothly from leaf to leaf. We first discuss some of the background concerning constructing complex foliations, namely describing them from an infintesimal point of view and glueing complex foliations on manifolds with boundary. Then we will describe a plan for constructing a complex foliation on S^5, motivated by our results in earlier chapters.

In Chapter 7 we give an outlook on further research.

The following diagram gives an overview of the important ingredients of proving the existence of a codimension-one symplectic foliation on S^5.



Chapter 1

Symplectic foliations

In this chapter we study the basic notions concerning symplectic foliations. In Section 1.1 we recall the basic definitions from foliation theory. In Section 1.2 we give the definition of a symplectic foliation. Finally, in Section 1.3 we recall the relation between Poisson geometry and symplectic foliations.

1.1 Foliations

The main objects of interest in this thesis are *foliations*:

Definition 1.1.1. A **foliation** $\mathcal{F}$ of dimension k on a manifold M^n is a partition $M = \cup_x L_x$ into connected immersed submanifolds of dimension k. Furthermore, the partition is required to satisfy the following local model: for every $x \in M$ there is a neighbourhood U of x and local coordinates $(x_1, \ldots, x_n)$ on U such that, for each element of the decomposition L_y, each connected component of $L_y \cap U$ is described by the relations $x_{k+1} = c_{k+1}, \ldots, x_n = c_n$.

For every $x \in M$, we call the submanifold which contains x the **leaf** though x, which we denote by L_x. The **codimension** of a foliation is defined as $n - k$.

Definition 1.1.2. Let $(M_i, \mathcal{F}_i)$ be two foliated manifolds. An **isomorphism of foliations** $\varphi : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a diffeomorphism $\varphi : M_1 \to M_2$, with the property that $\varphi^{-1}(\mathcal{F}_2) = \mathcal{F}_1$. Here $\varphi^{-1}(\mathcal{F}_2)$ is the foliation on M_1 with leaves given by $\{\varphi^{-1}(L) : L \text{ a leaf of } \mathcal{F}_2\}$.

We define the tangent bundle of the foliation by $T\mathcal{F} = \bigsqcup_{x \in M} T_x L_x$. This forms a subbundle of TM of rank k. We also define the **normal bundle** ν of the foliation as the vector bundle (unique up to isomorphism) such that $TM = T\mathcal{F} \oplus \nu$ and the **co-normal bundle** as the dual bundle ν^*.

General foliations can be very wild, so we prefer to restrict ourselves to some special types:

Definition 1.1.3. A foliation $\mathcal{F}$ is called **orientable** if $T\mathcal{F}$ is an orientable vector bundle; it is called **co-orientable** if the normal bundle of the foliation is orientable.

These two notions are equivalent on orientable manifolds:

Lemma 1.1.4. *Let $\mathcal{F}$ be a foliation on an orientable manifold M. Then $\mathcal{F}$ is orientable if and only if $\mathcal{F}$ is co-orientable.*

Proof. Let k be the codimension of the foliation, and suppose that $T\mathcal{F}$ is oriented. Consider any local frame $X_1, \ldots, X_k$ of the normal bundle ν. We define an orientation on ν by declaring this frame to be positive if and only if for every positive local frame $X_{k+1}, \ldots, X_n$ of $T\mathcal{F}$ the local frame on M given by $X_1, \ldots, X_n$ is positive. We thus conclude that the normal bundle is orientable. The converse is proven similarly. □

Distributions

The definition of a foliation as given in Definition 1.1.1 is very intuitive, but in practice it is not very convenient. It turns out that it is much easier to consider foliations from an infinitesimal point of view. For any foliation we can produce a subbundle of the tangent bundle, namely $T\mathcal{F}$. Subbundles of the tangent bundle are called **distributions**. To go back from distributions to foliations we will need the following notion:

Definition 1.1.5. Let $\xi \subset TM$ be a distribution on a smooth manifold M. A non-empty immersed submanifold $N \subset M$ is called an **integral submanifold** of ξ if $T_pN = \xi_p$ for every $p \in N$. A distribution is called **integrable** if each point in M is contained in an integrable submanifold.

By definition we have that $T\mathcal{F}$ is an integrable subbundle, hence we see that there is a one-to-one correspondence between foliations and integrable distributions. Checking the integrability condition is in general difficult to do directly, so we often use the following:

Theorem 1.1.6 (Frobenius)**.** *A distribution $\xi \subset TM$ is integrable if and only if it is* ***involutive****, that is, for any two sections $X, Y \in \Gamma(\xi)$ we have $[X, Y] \in \Gamma(\xi)$.*

This provides us with the following 1-1 correspondence:

$$\{\text{Foliations } \mathcal{F} \text{ on } M\} \xleftrightarrow{1:1} \{\text{Involutive distribtions on } M\}.$$

We prefer to work with involutive distributions, hence from now on we will often identify foliations with their corresponding distributions.

Example 1.1.7. Let $\theta \in \Omega^1(M)$ be a nowhere vanishing closed one-form and consider the distribution $\ker\theta \subset TM$. This distribution has corank one, and by an easy application of Koszul's formula we have that it is involutive. Hence we obtain a foliation of codimension one. It is also co-orientable because θ gives a trivialization for the co-normal bundle. We will call these foliations **unimodular**.

Example 1.1.8. (Products) The manifold $M \times N$ carries two natural foliations both of which we will call the *product foliation*.

Example 1.1.9. (Fibrations) Let $\pi : M \to B$ be a surjective submersion. Then the decomposition of M in the fibres of π forms a foliation on M. Indeed by naturality of the Lie bracket it follows that $\ker d\pi$ is involutive.

Definition 1.1.10. The complex of **foliated differential forms** is defined as $(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$ with

$$\Omega^\bullet(\mathcal{F}) := \Gamma(\Lambda^\bullet T^*\mathcal{F}),$$

and differential $d_{\mathcal{F}} : \Omega^\bullet(\mathcal{F}) \to \Omega^{\bullet+1}(\mathcal{F})$, given by the Koszul formula: For $\alpha \in \Omega^k(\mathcal{F})$ and $X_1, \dots, X_{k+1} \in \Gamma(T\mathcal{F})$ we define

$$\begin{aligned}(d_{\mathcal{F}}\alpha)(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}X_i(\alpha(X_1, \dots, \hat{X}_i, \dots, X_{k+1}))+\\ &\sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1}).\end{aligned}$$

1.1.1 Codimension-one foliations

In this thesis we will only consider foliations of codimension one. The existence of these foliations has been established in the following classical result:

Theorem 1.1.11 ([Thu76])**.** *A compact manifold admits a codimension-one foliation if and only if its Euler characteristic vanishes.*

We are particular interested in codimension-one co-orientable foliations. Firstly because when we consider manifolds with boundary we can study foliations for which the boundary is a leaf. Secondly they are easy to describe, as is shown by the following proposition:

Proposition 1.1.12. *There is a 1:1 correspondence between*

- *Codimension-one co-orientable foliations $\mathcal{F}$ on M.*
- *Equivalence classes of nowhere vanishing $\theta \in \Omega^1(M)$ satisfying $\theta \wedge d\theta = 0$.*

Where two one-forms $\theta, \theta' \in \Omega^1(M)$ are equivalent if there exists a nowhere vanishing function f such that $\theta = f\theta'$.

Proof. Let $\theta \in \Omega^1(M)$ be such that $\theta \wedge d\theta = 0$. By Koszul's formula we have that $T\mathcal{F} := \ker(\theta)$ is involutive, hence by Frobenius' theorem we obtain a foliation on M. We define $\phi : TM/T\mathcal{F} \to M \times \mathbb{R}$ by $\phi(x, [v]) = (x, \theta_x([v]))$. We easily check that this is well-defined and gives a trivialization of the normal bundle. In particular ν is orientable and we thus conclude that $\mathcal{F}$ is co-orientable.
Conversely, let $\mathcal{F}$ be a codimension-one co-orientable foliation. Because orientable line bundles are trivial we have that ν is globally trivializable by some map $\varphi : \nu \to M \times \mathbb{R}$. Define θ as the composition:

$$\theta : T\mathcal{F} \oplus \nu \longrightarrow \nu \xrightarrow{\varphi} M \times \mathbb{R}.$$

We see that θ is nowhere vanishing and as clearly $T\mathcal{F} \subset \ker\theta$, we conclude that $\ker\theta = T\mathcal{F}$. We have that $\ker\theta$ is involutive if and only if $d\theta|_{\ker\theta} = 0$ which implies that $\theta \wedge d\theta = 0$. □

Definition 1.1.13. For a codimension-one co-orientable foliation $\mathcal{F}$ on M, we call a one-form $\theta \in \Omega^1(M)$ such that $\ker\theta = T\mathcal{F}$ a **form defining** $\mathcal{F}$.

Although codimension-one co-orientable foliations are quite abundant, unimodular foliations are much rarer:

Theorem 1.1.14 (Tischler, [Tis70])**.** *If a compact manifold admits a unimodular foliation, then it fibres over S^1.*

Let $d\varphi$ denote the angular form on S^1. For a fibration $f : M \to S^1$ we have that $f^*(d\varphi)$ defines a unimodular foliation on M. However, not all unimodular foliations are of this form. Nonetheless, Tischler proves that any unimodular foliation can be approximated by these foliations. Unimodular foliations therefore behave quite similar to foliations induced by circle fibrations.

1.2 Symplectic foliations

Now, we have recalled the definition of a foliation we are ready to give the definition of a symplectic foliation. Intuitively, this should be a foliation together with a symplectic structure on each leaf, such that the symplectic structure vary smoothly from leaf to leaf. This is made precise in the following definition:

Definition 1.2.1. A **symplectic foliation** on a manifold M is a pair $(\mathcal{F}, \omega_{\mathcal{F}})$, where $\mathcal{F}$ is a foliation on M and $\omega_{\mathcal{F}} \in \Omega^2(\mathcal{F})$ is a foliated differential two-form for which $d_{\mathcal{F}}\omega_{\mathcal{F}} = 0$ and $\omega_{\mathcal{F}}$ restricts to a non-degenerate form on each leaf.

An **isomorphism** of symplectic foliations $(M_1, \mathcal{F}_1, \omega_1)$ and $(M_2, \mathcal{F}_2, \omega_2)$ is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^{-1}(\mathcal{F}_2) = \mathcal{F}_1$ and $\varphi^*(\omega_2) = \omega_1$.

Example 1.2.2. Let (ω, θ) with $\omega \in \Omega^2(M)$ and $\theta \in \Omega^1(M)$ be such that:

$$\theta \wedge d\theta = 0, \quad d\omega \wedge \theta = 0, \quad \omega^n \wedge \theta \neq 0.$$

Then $(\ker\theta, \omega|_{\mathcal{F}})$ defines a codimension-one symplectic foliation. Indeed the first condition ensures involutivity of $\ker\theta$ as before, the second closedness of $\omega|_{\mathcal{F}}$ and the third ensures that the restriction of $\omega|_{\mathcal{F}}$ to each leaf is non-degenerate.

We note that the foliated form $\omega_{\mathcal{F}}$ of a symplectic foliation induces an orientation on $T\mathcal{F}$. Hence symplectic foliations are always oriented. If we assume that the symplectic foliation lives on an oriented manifold, using Lemma 1.1.4 we have the following:

Proposition 1.2.3. *Let $(\mathcal{F}, \omega)$ be a codimension-one symplectic foliation on an orientable manifold M. Then the foliation $\mathcal{F}$ is co-orientable.*

This lemma shows that if we are looking for symplectic foliations on orientable manifolds, it is no restriction to begin with co-orientable foliations.

Although the existence of codimension-one foliations on compact manifolds is completely solved by Theorem 1.1.11, the corresponding problem for symplectic foliations is wide open. In this thesis we will reprove the following result:

Theorem 1.2.4 ([Mit11])**.** *There exists a codimension-one symplectic foliation on S^5.*

Our method will however deviate from Mitsumatsu's. We will first establish the existence of a symplectic foliation on the lens space $S^5/\mathbb{Z}_3$ and then use this to obtain a symplectic foliation on S^5. This approach is different then Mitsumatsu's who constructed the foliation directly on S^5.

1.3 Symplectic foliations as Poisson structures

In this section we will recall some basic definitions from Poisson geometry and consider its relation with symplectic foliations. We will only give brief proofs and often refer to [FM15], from which the contents of this section have been adapted. For completeness we will begin with defining the correspondence between symplectic structures and Poisson structures. Then we will generalise this to a correspondence between symplectic foliations and regular Poisson structures.

Basic definitions

We denote the set of all multivector fields on a smooth manifold M by $\mathfrak{X}^k(M) := \Gamma(\wedge^k TM)$.

Definition 1.3.1. Let $\nu \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$ be multivector fields. The **Schouten-Nijenhuis bracket** of ν and ζ is the multivector field $[\nu, \zeta] \in \mathfrak{X}^{k+l-1}(M)$ defined by:

$$[\nu, \zeta] = \nu \circ \zeta - (-1)^{(k-1)(l-1)} \zeta \circ \nu,$$

where we define for $f_i \in C^\infty(M)$:

$$\zeta \circ \nu(df_1, \dots, df_{k+l-1}) := \sum_\sigma (-1)^\sigma \zeta(d(\nu(df_{\sigma(1)}, \dots, df_{\sigma(k)})), df_{\sigma(k+1)}, \dots, df_{\sigma(k+l-1)})$$

with sum taken over all $(k, l-1)$-shuffles.

This definition might seem a bit peculiar, the following proposition however gives a more insightful description of the Schouten-Nijenhuis bracket. For a proof we refer to [FM15].

Proposition 1.3.2. *The Schouten-Nijenhuis bracket is the unique bilinear operation* $[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l}(M)$ *that satisfies:*

- *For any* $X_1, \dots, X_k, Y_1, \dots, Y_k \in \mathfrak{X}(M)$ *we have:*

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l.$$

- *For any multivector field* $V \in \mathfrak{X}^k(M)$ *and any* $f \in C^\infty(M)$ *we have:*

$$[V, f] = \iota_{df} V.$$

Definition 1.3.3. A bivector field $\pi \in \mathfrak{X}^2(M)$ is called a **Poisson structure** if $[\pi, \pi] = 0$.

Definition 1.3.4. A **Poisson bracket** on a manifold M is a bilinear operation

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f, g) \mapsto \{f, g\}$$

satisfying:

- **Skew-symmetry**: $\{f, g\} = -\{g, f\}$;
- **Jacobi identity**: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.
- **Leibniz identity**: $\{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h$.

The following is well-known:

Proposition 1.3.5. *Let* $\{\cdot,\cdot\}$ *be a Poisson bracket on* M*. Define a Poisson structure* π *on* M*, by* $\pi(df, dg) := \{f, g\}$*. The assignment* $\pi \to \{\cdot,\cdot\}$ *defines a 1:1 correspondence between Poisson brackets and Poisson structures.*

Poisson structures give rise to the following class of vector fields:

Definition 1.3.6. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. The **Hamiltonian vector field** of $H \in C^\infty(M)$ is the vector field $X_H \in \mathfrak{X}(M)$ defined by:

$$X_H(f) := \{H, f\}, \quad f \in C^\infty(M).$$

Note that X_H is a vector field precisely because $\{\cdot,\cdot\}$ satisfies the Leibniz identity. The proof of the following fact follows directly from the definitions:

Lemma 1.3.7. *Let $f, g \in C^\infty(M)$, then*

$$X_{\{f,g\}} = [X_f, X_g].$$

Relation between Poisson and symplectic geometry

Recall that a two-form $\omega \in \Omega^2(M)$ is called **non-degenerate** if the map

$$\omega^\flat : T_xM \to T_x^*M, \quad v \mapsto \iota_v\omega$$

is an isomorphism for all $x \in M$. If we think of two-forms as collections of skew-symmetric linear maps

$$\omega_x : T_xM \times T_xM \to \mathbb{R},$$

we see that ω is non-degenerate if and only if the map ω_x is non-degenerate for all x.
A bivector field $\pi \in \mathfrak{X}^2(M)$ induces a map

$$\pi^\sharp : \Omega^1(M) \to \mathfrak{X}^1(M), \quad \alpha \mapsto \iota_\alpha\pi$$

A bivector field π is called **non-degenerate** if the map $\pi_x^\sharp : T_x^*M \to T_xM$ is an isomoprhism for all $x \in M$. If we think of bivector fields as collections of skew-symmetric linear maps

$$\pi_x : T_x^*M \times T_x^*M \to \mathbb{R},$$

then non-degeneracy of π is equivalent to π_x being non-degenerate for all $x \in M$.

Proposition 1.3.8. *There is a $1:1$ correspondence between non-degenerate two-forms and non-degenerate bivector fields, given by:*

$$\omega^\flat = (\pi^\sharp)^{-1}, \quad \pi^\sharp = (\omega^\flat)^{-1}$$

Under this correspondence we have

$$[\pi,\pi](\alpha,\beta,\gamma) = -d\omega(\pi^\sharp(\alpha), \pi^\sharp(\beta), \pi^\sharp(\gamma)). \tag{1.1}$$

In particular we have a correspondence between symplectic forms and non-degenerate Poisson structures.

Proof. Observing that π and ω are completely determined by $\pi^\sharp$ and $\omega^\flat$, the first part of the proposition is clear. We note that it suffices to prove (1.1) for exact one-forms. We use the definition of the Schouten-Nijenhuis bracket to find for any $f_1, f_2, f_3 \in C^\infty(M)$ that

$$[\pi,\pi](df_1, df_2, df_3) = 2\left(\{\{f_1,f_2\},f_3\} + \{\{f_2,f_3\},f_1\} + \{\{f_3,f_1\},f_2\}\right).$$

Applying Koszul's formula, we find:

$$\begin{aligned} d\omega(\pi^\sharp(df_1), \pi^\sharp(df_2), \pi^\sharp(df_3)) = \pi^\sharp(df_1)\omega(\pi^\sharp(df_2), \pi^\sharp(df_3)) + \text{cycl. perm.} \\ - \omega([\pi^\sharp(df_1), \pi^\sharp(df_2)], \pi^\sharp(df_3)) - \text{cycl. perm.} \end{aligned}$$

We have

$$\begin{aligned} \pi^\sharp(df_1)\omega(\pi^\sharp(df_2), \pi^\sharp(df_3)) &= \{f_1, \omega(\pi^\sharp(df_2), \pi^\sharp(df_3))\} \\ &= \{f_1, \{f_2, f_3\}\} \\ &= -\{\{f_2, f_3\}, f_1\}, \end{aligned}$$

and

$$\begin{aligned} -\omega([\pi^\sharp(df_1), \pi^\sharp(df_2)], \pi^\sharp(df_3)) &= -\omega(\pi^\sharp(d\{f_1, f_2\}), \pi^\sharp(df_3)) \\ &= -\{\{f_1, f_2\}, f_3\}, \end{aligned}$$

where we used Lemma 1.3.7. Combining the above we find

$$d\omega(\pi^\sharp(df_1), \pi^\sharp(df_2), \pi^\sharp(df_3)) = -2\left(\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}\right),$$

which completes the proof. □

Relation between Poisson geometry and symplectic foliations

The class of non-degenerate Poisson structures is relatively small. Therefore we consider the following generalisation:

Definition 1.3.9. A Poisson structure π on M is called **regular** if the space $\pi_x^\sharp(T_x^*M)$ has the same dimension for all $x \in M$. This dimension is then called the **rank** of the Poisson structure.

Given a regular Poisson structure we see that $\mathcal{F} := \pi^\sharp(T^*M)$ forms a distribution on M, which is smooth because it is spanned by vector fields of the form $\pi^\sharp(df)$. We will show that this distribution is in fact involutive and can be given the structure of a symplectic foliation. To do as such we will need the following result:

Theorem 1.3.10. *Let (M, π) be a Poisson manifold. Given an immersed submanifold $N \hookrightarrow M$ there is at most one Poisson structure π_N on N that makes N into a Poisson submanifold. This structure exists if and only if*

$$\operatorname{Im}(\pi_x)^\sharp \subset T_xN, \text{ for all } x \in N.$$

For a proof we again refer to [FM15]. Now we are ready to begin with constructing the correspondence.

Theorem 1.3.11. *Let $\pi \in \mathfrak{X}^2(M)$ be a regular Poisson structure. Then the distribution $\pi^\sharp(T^*M)$ is integrable, and each leaf S of $\pi^\sharp(T^*M)$ is a Poisson submanifold. Furthermore, the induced Poisson structure $\pi_S \in \mathfrak{X}^2(S)$ is non-degenerate.*

Proof. We note that $\pi^\sharp(T^*M)$ is spanned by the Hamiltonian vector fields. By Lemma 1.3.7 we have that the Lie bracket of two Hamiltonian vector fields is again Hamiltonian, which shows that $\pi^\sharp(T^*M)$ is integrable.
Let S be a leaf such that $T_xS = \mathrm{Im}\pi^\sharp_x$ for every $x \in S$. By Theorem 1.3.10 we have that S is a Poisson submanifold. The induced Poisson structure satisfies

$$\pi^\sharp(\alpha) = \pi^\sharp_S(\alpha|_S), \quad \text{for all } \alpha \in T^*_SM.$$

Because $T_xS = \mathrm{Im}\pi^\sharp_x$, we see that this implies that $\pi^\sharp_S$ is surjective. Hence π_S is non-degenerate, which finishes the proof. □

We denote $\mathcal{F}_\pi := \pi^\sharp(T^*M)$, and we define $\omega_\pi := (\pi|_{T^*\mathcal{F}_\pi})^{-1}$. One easily verifies that $d_{\mathcal{F}_\pi}\omega_\pi = 0$ if and only if $[\pi,\pi]|_{T^*\mathcal{F}} = 0$, hence $(\mathcal{F}_\pi, \omega_\pi)$ is a symplectic foliation. In conclusion:

Proposition 1.3.12. *The assignment $\pi \mapsto (\mathcal{F}_\pi, \omega_\pi)$ gives a 1-1 correspondence between regular Poisson structures and symplectic foliations.*

Proof. Let $(\mathcal{F}, \omega_\mathcal{F})$ be a symplectic foliation and let S be one of its leaves. Define $\pi_S = \omega_S^{-1}$ and set $\pi \in \mathfrak{X}^2(M)$ to be

$$\pi(\alpha,\beta)|_S := \pi_S(\alpha|_S, \beta|_S).$$

We first remark that π has constant rank. Because $[\pi,\pi]|_S = [\pi_S,\pi_S]$ for all leaves S, we conclude that $[\pi,\pi] = 0$. Hence π is a regular Poisson structure. Clearly this procedure is inverse to $\pi \mapsto (\mathcal{F}_\pi, \omega_\pi)$, which finishes the proof. □

Chapter 2

Glueing symplectic foliations

Most foliations on closed manifolds are constructed in the following manner. Given a closed manifold one first decomposes the manifold into two manifolds with boundary. Secondly, one constructs a foliation on both manifolds with boundary and finally glues the foliations to obtain a foliation on the original manifold. To this end one needs to ensure that the partition obtained by glueing the two foliations is again a foliation. In this chapter we will study under which conditions it is possible to glue (symplectic) foliations.

As before we will restrict ourselves to studying codimension-one foliations. The main reason for this is that we can now consider foliations for which the boundary is a leaf. The class of (symplectic) foliations for which we will show that they can be glued are the (symplectic) foliations *tame near the boundary*. These are defined such that the (symplectic) foliations have a particular local form near the boundary. This local form will be key in proving that these foliations can be glued. In Section 2.1 we will introduce this notion for foliations and prove that foliations tame near the boundary can be glued. In Section 2.2 we will introduce the corresponding notion for symplectic foliations and prove that these can be glued.

2.1 Glueing foliations

In this section we will give the definition of foliations tame near the boundary, and prove that any two of such foliations can be glued to obtain a new foliation. We will restrict ourselves to the case of co-orientable foliations in this section. Although this assumption is not strictly needed it will make the proofs somewhat easier. In particular because co-orientable foliations are globally defined by a one-form, which allows to write global expressions. Because we are interested in symplectic foliations, and by Proposition 1.2.3 symplectic foliations on orientable manifolds are always co-orientable, we see that this assumption is not really a restriction.

2.1.1 Foliations tame near the boundary

To glue foliations it is not sufficient that the foliations are tangent to the boundary, i.e. the connected components of the boundary are leaves. We also need that the foliation can be extended smoothly. To capture this property we first have to extend the manifold with boundary to a manifold without boundary. We define

$$M_\infty := \partial M \times (-\infty, 0] \cup_{\partial M} M.$$

To endow M_∞ with a smooth structure we need to pick a collar neighbourhood $k : \partial M \times [0,1) \to U$ of the boundary. We then define a unique smooth structure on M_∞, by requiring that the inclusion of Int$M$ into M, as well as the map

$$k_\infty : \partial M \times (-\infty, 1) \to M_\infty, \quad k_\infty(x,t) = \begin{cases} (x,t) & \text{if } t \in (-\infty, 0) \\ k(x,t) & \text{if } t \in [0,1) \end{cases} \tag{2.1}$$

are smooth and open embeddings. If we want to stress which smooth structure we use we denote M_∞ endowed by this smooth structure by M_∞^k.

Definition 2.1.1. Let $\mathcal{F}$ be a codimension-one foliation which is tangent to the boundary, i.e. every connected component of the boundary is a leaf. We extend $\mathcal{F}$ to a partition on $\mathcal{F}_\infty$ by taking as leaves $\partial M \times \{t\}$ in $\partial M \times (-\infty, 0]$. We call this the **trivial extension of** $\mathcal{F}$ to M_∞.

Definition 2.1.2. Let $\mathcal{F}$ be a foliation on a manifold with boundary. We then call $\mathcal{F}$ **tame near the boundary** if:

- The foliation $\mathcal{F}$ is tangent to the boundary, i.e. the connected components of the boundary are leaves.
- For some collar neighbourhood k of ∂M the trivial extension of $\mathcal{F}$ to M_∞^k is a smooth foliation.

Normal forms

We will show that foliations tame near the boundary admit a normal form near the boundary.

Lemma 2.1.3. *Let $\mathcal{F}$ be a codimension-one co-orientable foliation on a manifold M with compact boundary, and let $\alpha \in \Omega^1(M)$ be a one-form defining the foliation. Then $\mathcal{F}$ is tame near the boundary if and only if there exists a collar neighbourhood $k : \partial M \times [0,1) \to U$ such that*

$$k^*(\alpha|_U) = \zeta_t + dt,$$

with $\zeta_t \in \Omega^1(\partial M)$ which varies smoothly with t and vanishes up to infinite order at $t = 0$. We will call such a collar neighbourhood ***adapted to*** $\mathcal{F}$.

Proof. "⇐": Let $k : \partial M \times [0,1) \to U$ be a collar neighbourhood adapted to $\mathcal{F}$. Because $\zeta_0 = 0$ we see that the connected components of the boundary are leaves of the foliation. Define a one-form on M^k_∞ by

$$\beta := \begin{cases} \alpha & \text{on } M \setminus U \\ \zeta_t + dt & \text{on } \partial M \times (-\infty, 1) \end{cases}$$

where we extended ζ_t by zero for $t < 0$. Because U is adapted to α and ζ_t vanishes up to infinite order at $t = 0$ we see that this form is smooth. We notice that $\ker(\beta) = \mathcal{F}_\infty|_{\partial M \times (-\infty,1)}$. We conclude that $\mathcal{F}_\infty$ is smooth which proves that the foliation $\mathcal{F}$ is tame near the boundary.
"⇒": Now we assume that $\mathcal{F}$ is tame near the boundary. Let $k : \partial M \times [0,1) \to U$ be the collar neighbourhood from the definition of tameness. We have

$$k^*(\alpha) = \zeta_t + f dt, \quad \text{for suibtable } \zeta_t \in \Omega^1(\partial M), \quad f \in C^\infty(\partial M \times [0,1))$$

Because the foliation is tame near the boundary we have that $k^*(\alpha)$ can be extended smoothly to a form on M^k_∞ given by

$$\theta = \begin{cases} \zeta_t + f dt & \text{on } U, \\ dt & \text{on } \partial M \times (-\infty, 0). \end{cases}$$

Hence ζ_t must vanish up to infinite order at $t = 0$. Due to the fact that the foliation is tame near the boundary we have $f|_{\partial M \times \{0\}} \neq 0$. By continuity of f, and compactness of the boundary, we find an $\varepsilon > 0$ such that $f|_{\partial M \times [0,\varepsilon)} \neq 0$. Let $\widetilde{f} \in C^\infty(U)$ be a nowhere vanishing function with the property that $\widetilde{f} \circ k = f$ on $\partial M \times [0, \varepsilon)$. Because $\widetilde{f}$ is nowhere vanishing we see that $\alpha' := \frac{1}{\varepsilon \widetilde{f}}\alpha$ defines $\mathcal{F}$. Now define a new collar neighbourhood $\widetilde{k} : \partial M \times [0, \varepsilon) \to U$, by restricting k to $\partial M \times [0, \varepsilon)$. By construction we have

$$\begin{aligned} \widetilde{k}^*(\alpha') &= \frac{1}{\varepsilon \widetilde{f} \circ k}(\zeta_t + f dt), \\ &= \frac{1}{\varepsilon f}\zeta_t + \frac{1}{\varepsilon} dt. \end{aligned}$$

After rescaling $\widetilde{k}$ we see that we have obtained the required collar neighbourhood. □

One disadvantage of the definition of foliations tame near the boundary is that it makes reference to a particular collar neighbourhood. The following lemma gives a more intrinsic way of checking whether a foliation is tame near the boundary.

Lemma 2.1.4. *Let $\mathcal{F}$ be a codimension-one co-orientable foliation on a manifold with compact boundary, such that $\mathcal{F}$ is tangent to the boundary. Then $\mathcal{F}$ is tame near the boundary if and only if there exists a one-form $\alpha \in \Omega^1(M)$ defining the foliation, with the property that $d\alpha$ vanishes up to infinite order at the boundary.*

Proof. If $\mathcal{F}$ is tame near the boundary, the form α from Lemma 2.1.3 has the property that $k^*(d\alpha) = d\zeta_t$. Because ζ_t vanishes up to infinite order at the boundary, so does $k^*(d\alpha)$, and thus $d\alpha$ vanishes up to infinite order at the boundary.
Now assume that there exists a one-form α defining the foliation such that $d\alpha$ vanishes up to infinite order at the boundary. By assumption $\mathcal{F}$ is tangent to the boundary, hence we can find a vector field near the boundary, transverse to the boundary satisfying $\alpha(X) = 1$. Let $k : \partial M \times [0, 1) \to U$ be the collar neighbourhood induced by X. Because $\alpha(X) = 1$ we have

$$k^*(\alpha|_U) = \zeta_t + dt, \quad \text{for some } \zeta_t \in \Omega^1(\partial M),$$
$$k^*(d\alpha) = d^\partial \zeta_t + \frac{d}{dt}\zeta_t \wedge dt.$$

Because $d\alpha$ vanishes up to infinite order at ∂M, the above shows that $\frac{d}{dt}\zeta_t$ vanishes up to infinite order at $t = 0$. Since $\zeta_0 = 0$ as $\mathcal{F}$ is tangent to the boundary, we conclude that ζ_t vanishes up to infinite order at the boundary. Hence the collar neighbourhood k is adapted to the foliation and we can apply Lemma 2.1.3 to finish the proof. □

Glueing foliations

We can use the normal form of foliations introduced in Lemma 2.1.3 to prove the foliations tame near the boundary can be glued. First, let M_1, M_2 be two smooth manifolds and let $\varphi : \partial M_1 \to \partial M_2$ a diffeomorphism of the boundaries. Consider the glued space $M := M_1 \cup_\varphi M_2$, and endow it with a smooth structure in the following way. Given two collar neighbourhoods

$$k_1 : \partial M_1 \times [0, 1) \to M_1, k_2 : \partial M_2 \times (-1, 0] \to M_2,$$

we can define a smooth structure on $M_1 \cup_\varphi M_2$ by requiring the inclusions $M_i \hookrightarrow M$ as well as

$$k : \partial M_1 \times (-1, 1) \to M, \quad (x, t) \mapsto \begin{cases} k_1(x, t) & \text{if } t \geq 0 \\ k_2(\varphi(x), t) & \text{if } t < 0, \end{cases} \tag{2.2}$$

to be smooth open embeddings. Now if M_i admits a foliation $\mathcal{F}_i$ tangent to the boundary, we can take the union of $\mathcal{F}_1$ and $\mathcal{F}_2$ to obtain a partition $\mathcal{F}_1 \cup_\varphi \mathcal{F}_2$ of $M_1 \cup_\varphi M_2$. The following theorem ensures that this partition is smooth provided that the foliations are tame near the boundary.

Theorem 2.1.5. *Let $(M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2)$ be two manifolds with compact boundary endowed with codimension-one co-orientable foliations which are tame near the boundary. Furthermore, let $\varphi : \partial M_1 \to \partial M_2$ be a diffeomorphism. Then $M_1 \cup_\varphi M_2$ admits a smooth structure such that $\mathcal{F}_1 \cup_\varphi \mathcal{F}_2$ is a smooth foliation. Moreover, the inclusions $M_i \hookrightarrow M$ are embeddings, such that $(\mathcal{F}_1 \cup_\varphi \mathcal{F}_2)|_{M_i} = \mathcal{F}_i$.*

Proof. Let α_i be defining forms for $\mathcal{F}_i$ and choose collar neighbourhoods $k_i : \partial M_i \times [0,1) \to U_i$ to be adapted to $\mathcal{F}_i$ in the sense of Lemma 2.1.3. Then

$$k_i^*(\alpha_i) = \zeta_t^i + dt,$$

with $\zeta_t^i \in \Omega^1(\partial M_i)$ vanishing up to infinite order at $t = 0$. Define a one-form $\alpha \in \Omega^1(M)$ by $\alpha|_{M_i} = \alpha_i$ for $i = 1, 2$. We check that this form is smooth. Let k be as in (2.2). Then $k^*(\alpha) = \zeta_t + dt$ with

$$\zeta_t = \begin{cases} \zeta_t^1 & 0 \leq t \leq 1 \\ \varphi^*(\zeta_t^2) & -1 \leq t \leq 0. \end{cases}$$

Because ζ_t^i vanishes up to infinite order at $t = 0$, we deduce that ζ_t is smooth and hence that α is smooth. Furthermore, as $\alpha_i \wedge d\alpha_i = 0$ we find that $\alpha \wedge d\alpha = 0$, which shows that α defines a foliation. Since by construction $\ker(\alpha) = \mathcal{F}_1 \cup_\varphi \mathcal{F}_2$ we have shown what was required. □

Glueing hyperplane distributions

For future use we generalise Theorem 2.1.5 to the case of hyperplane distributions.

Definition 2.1.6. Let $\mathcal{H}$ be a hyperplane distribution on a manifold M with boundary. We then call $\mathcal{H}$ **tame near the boundary** if:

- The hyperplane distribution $\mathcal{H}$ is tangent to the boundary, i.e. the tangent spaces of the connected components of the boundary are elements of $\mathcal{H}$.
- For some collar neighbourhood k of ∂M the trivial extension of $\mathcal{H}$ to M_∞^k, defined just as in Definition 2.1.1, is smooth.

Just as with foliations, we say that a hyperplane distribution $\mathcal{H} \subset TM$ is **orientable** if $\mathcal{H}$ is an orientable vector bundle; we say that $\mathcal{H}$ is **co-orientable** if the normal bundle of $\mathcal{H}$ is orientable. As in Lemma 1.1.4 one can prove that for co-orientable $\mathcal{H}$ there exists some globally defined one-form $\theta \in \Omega^1(M)$ such that $\mathcal{H} = \ker \theta$. This observation gives rise to the following generalisation of Theorem 2.1.5.

Theorem 2.1.7. *Let $(M_1, \mathcal{H}_1), (M_2, \mathcal{H}_2)$ be two manifolds with compact boundary endowed with co-orientable hyperplane distributions which are tame near the boundary. Furthermore, let $\varphi : \partial M_1 \to \partial M_2$ be a diffeomorphism. Then $M_1 \cup_\varphi M_2$ admits a smooth structure such that $\mathcal{H}_1 \cup_\varphi \mathcal{H}_2$ is a smooth distribution. Moreover, the inclusions $M_i \hookrightarrow M$ are embeddings, such that $(\mathcal{H}_1 \cup_\varphi \mathcal{H}_2)|_{M_i} = \mathcal{H}_i$.*

Proof. Inspecting the proof of Lemma 2.1.3, we see that we have not used the integrability of the form α. So we see that Lemma 2.1.3 also holds in the more general setting of co-orientable hyperplane distributions. Now the proof carries over from the proof of Theorem 2.1.7. □

2.2 Glueing symplectic foliations

In this section we will extend the notion of foliations tame near the boundary to symplectic foliations and give a version of Theorem 2.1.5 for symplectic foliations tame near the boundary. For simplicity we will always assume that all manifolds have connected boundary.

2.2.1 Leafwise symplectic structures tame near the boundary

Intuitively a symplectic foliation which is tame near the boundary is a foliation such that the symplectic structure does not vary to much near the boundary. Just as with foliations we want our definition to imply a particular local form of the symplectic structure.

Definition 2.2.1. Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M which is tangent to the boundary. The leafwise symplectic form ω is called **M-tame near the boundary** if there exists a collar neighbourhood $k : \partial M \times [0, 1) \to U$ of the boundary such that

$$\omega_L|_{L\cap U} = (k^{-1})^*(\omega_{\partial M})|_{L\cap U},$$

for any leaf L of $\mathcal{F}$.

We note that the above definition in particular gives that there exists a closed extension near the boundary of the foliated differential form. This motivates the following definition:

Definition 2.2.2. Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M which is tangent to the boundary. The leafwise symplectic form ω is called **tame near the boundary** if there exists a neighbourhood U of ∂M and a closed form $\widetilde{\omega} \in \Omega^2(U)$ such that $\widetilde{\omega}|_{L\cap U} = \omega_L|_{L\cap U}$ for all leaves L of $\mathcal{F}$.

Although the first definition is stronger than the second, in the codimension-one case they coincide.

Proposition 2.2.3. *Let M be a manifold with compact boundary and let $(\mathcal{F}, \omega)$ be a codimension-one symplectic foliation which is tangent to the boundary. Then ω is M-tame near the boundary if and only if it is tame near the boundary.*

Proof. Let ω be tame near the boundary. Let $\widetilde{\omega}$ be a closed extension of ω to some neighbourhood U around ∂M. Because $\widetilde{\omega}|_{\mathcal{F}}$ is non-degenerate we have

$$TM|_U = T\mathcal{F}|_U \oplus \ker \widetilde{\omega},$$

hence $\ker \widetilde{\omega}$ can be identified with the normal bundle of $\mathcal{F}$. Because M is orientable and $\mathcal{F}$ is orientable we have by Lemma 1.1.4 that $\mathcal{F}$ is co-orientable. Because $\ker \widetilde{\omega}$ is a line bundle we thus find that it is trivialisable. We let $X \in \Gamma(\ker \widetilde{\omega})$ be a nowhere vanishing section, which we interpret as a vector field on M. Because ∂M is a leaf and $\widetilde{\omega}|_{\partial M}$ is non-degenerate we have that X is transverse to the boundary. So, if we ensure that X points inwards we have that the flow of X exists on some neighbourhood $V \subset \partial M \times \mathbb{R}$ of $\partial M \times \{0\}$. We now consider the map induced by the flow of X:

$$k : \partial M \times [0, \varepsilon) \to k(V) : (x, t) \mapsto \varphi_X^t(x).$$

One easily shows that $(dk)_{(x,0)}(Y + \partial_t) = X + Y$, hence k is a local diffeomorphism around $(x, 0)$. We can thus shrink V such that k becomes a diffeomorphism onto its image, and is thus a collar neighbourhood. We will now show that it has the required properties. On this collar neighbourhood we write

$$\widetilde{\omega} = \eta_t + \theta_t \wedge dt, \quad \text{for some } \eta_t \in \Omega^2(\partial M), \theta_t \in \Omega^1(\partial M)$$

By definition, we have $X \in \Gamma(\ker \widetilde{\omega})$, hence $\iota_X \widetilde{\omega} = 0$. As X takes the form ∂_t on $k(V)$ we find $\iota_{\partial_t} k^*(\widetilde{\omega}) = 0$, hence $\theta_t = 0$. Let d^∂ denote the differential on ∂M. Because $k^*(\widetilde{\omega})$ is closed we have

$$0 = d^\partial \eta_t + \left(\frac{\mathrm{d}}{\mathrm{d}t} \eta_t\right) \wedge dt,$$

hence $\frac{\mathrm{d}}{\mathrm{d}t} \eta_t = 0$. We conclude that $k^*(\widetilde{\omega}) = \eta_0$. Hence this finishes the proof that ω is M-tame near the boundary. □

Remark 2.2.4. The definition of tame near the boundary for a foliated symplectic form can also be adapted to leaves on the interior of M. To do as such one has to replace all collar neighbourhoods in the above with tubular neighbourhoods. This then gives a definition for a foliated symplectic form to be *(M-)tame around a leaf* L. The proof of Proposition 2.2.3 in this setting is almost identical to the above.

2.2.2 Symplectic foliations tame near the boundary

We now combine the two notions of tameness in the following definition:

Definition 2.2.5. We call a symplectic foliation $(\mathcal{F}, \omega)$ on M **tame near the boundary** if

- The foliation $\mathcal{F}$ is tame near the boundary.
- The leafwise symplectic form ω is tame near the boundary.

Because the leafwise symplectic form is tame near the boundary, by Proposition 2.2.3 we have that there exists a preferred collar neighbourhood on which ω becomes constant. We now adapt this collar neighbourhood further, giving the required local normal form of symplectic foliations:

Proposition 2.2.6. *Let M be an orientable manifold with compact boundary and let $(\mathcal{F},\omega)$ be a codimension-one symplectic foliation on M which is tame near the boundary. Then there exists a collar neighbourhood $k : \partial M \times [0,1) \to U$ such that:*

- *k is adapted to $\mathcal{F}$,*
- *there exists a closed extension $\widetilde{\omega} \in \Omega^2(U)$ of ω such that $k^*(\widetilde{\omega}) = \omega|_{\partial M}$.*

We call a collar neighbourhood with these properties ***adapted to*** *$(\mathcal{F},\omega)$.*

Proof. We basically have to combine the proofs of Proposition 2.2.3 and Lemma 2.1.3. Let α be a one-form defining the foliation coming from Lemma 2.1.3. As in the proof of Proposition 2.2.3 we let $\widetilde{\omega}$ be a closed extension of ω, and note that $\ker(\widetilde{\omega})$ is trivialisable. We let $X \in \Gamma(\ker(\widetilde{\omega}))$ be a trivialising section such that $\alpha(X) = 1$, and let $\varepsilon > 0$ be the time for which the flow exists. We consider the collar neighbourhood induced by this vector field. Continuing along the same lines as in the proof of Proposition 2.2.3 we find $k^*(\widetilde{\omega}) = \omega|_{\partial M}$. Since $\alpha(X) = 1$, we have

$$k^*\alpha = \zeta_t + dt, \quad \text{for some } \zeta_t \in \Omega^1(\partial M).$$

Just as in the proof of Lemma 2.1.3 we conclude that ζ_t vanishes up to infinite order at $t = 0$. Hence after rescaling k and α we obtain the required collar neighbourhood. □

Now we can use the obtained local form to glue symplectic foliations. If $(\mathcal{F}_1,\omega_1)$ and $(\mathcal{F}_2,\omega_2)$ are two symplectic foliations tangent to the boundary, and if φ is a symplectomorphism of the boundaries we let $\omega_1 \cup_\varphi \omega_2$ denote the union of the collection of leafwise differential forms. The question is whether this is a smooth form, which is answered by the following:

Theorem 2.2.7. *Let $(M_1,\mathcal{F}_1,\omega_1)$ and $(M_2,\mathcal{F}_2,\omega_2)$ be two orientable manifolds with compact boundaries endowed with codimension-one symplectic foliations tame near the boundary and let*

$$\varphi : (\partial M_1, \omega_1|_{\partial M_1}) \to (\partial M_2, \omega_2|_{\partial M_2})$$

be a symplectomorphism. Then $M := M_1 \cup_\varphi M_2$ admits a smooth structure such that

$$(\mathcal{F},\omega) := (\mathcal{F}_1 \cup_\varphi \mathcal{F}_2, \omega_1 \cup_\varphi \omega_2)$$

is a codimension-one symplectic foliation on $M_1 \cup_\varphi M_2$. Moreover, the inclusions $M_i \hookrightarrow M$ are embeddings, satisfying $(\mathcal{F},\omega)|_{M_i} = (\mathcal{F}_i,\omega_i)$.

Proof. Let k_i be collar neighbourhoods adapted to $(\mathcal{F}_i,\omega_i)$, and endow $M_1 \cup_\varphi M_2$ with the smooth structure induced by these collar neighbourhoods. By Theorem 2.1.5 we know that $\mathcal{F}_1 \cup_\varphi \mathcal{F}_2$ is smooth, so we are left to check the smoothness of $\omega_1 \cup_\varphi \omega_2$. Let $k : \partial M_1 \times (-1,1) \to M$ be as in (2.2), then $k^*(\omega_1 \cup_\varphi \omega_2) = \mathrm{pr}_1^*(\omega_1|_{\partial M_1})$, where $\mathrm{pr}_1 : \partial M_1 \times (-1,1) \to \partial M_1$ is given by projection onto the first factor. This shows that $\omega_1 \cup_\varphi \omega_2$ is smooth concludes the proof. □

Chapter 3

Symplectic turbulisation

In the previous chapter we saw how symplectic foliations which are tame near the boundary can be glued to obtain symplectic foliations on closed manifolds. This chapter is devoted to studying a technique to construct symplectic foliations which are tame near the boundary. This technique is called *symplectic turbulisation* and will change symplectic foliations with certain properties into symplectic foliations that are tame near the boundary. The symplectic foliations which we start with will arise from *cosymplectic structures*.

In Section 3.1 we will define cosymplectic as well as *contact structures*. Both of these structures arise naturally when studying symplectic manifolds with certain behaviour near the boundary. Studying how cosymplectic and contact structures are induced on the boundary is the object of study in Section 3.2. In Section 3.3 we will consider a generalisation of cosymplectic structures, as well as corresponding Poisson structures. In Section 3.4 we will study turbulisation on manifolds of the form $M \times S^1$, with M symplectic. In Section 3.5 we will give the definition of *cosymplectic structures with boundary of s-type*, which is inspired by the results in Section 3.4. We will then prove the symplectic turbulisation theorem, which will be our main tool in creating symplectic foliations tame near the boundary.

3.1 Presymplectic structures

When studying symplectic manifolds with boundary it is natural to ask what structure the symplectic form induces on the boundary. The restriction of the symplectic form to the boundary is no longer non-degenerate, however its kernel is only one-dimensional. Differential two-forms for which the kernel is one-dimensional are called *presymplectic forms*. If the symplectic structure satisfies particular behaviour near the boundary, the structure on the boundary can be more then only a presymplectic structure. We will consider two cases, symplectic structures of *cosymplectic type* and *contact type* at the boundary. In these cases we get a cosymplectic respectively contact structure induced at the boundary. Finally we will study the normal form around the boundary of symplectic manifolds with boundary of cosymplectic type. This will be key in proving the symplectic turbulisation theorem in later sections.

Definition 3.1.1. Let N^{2n+1} be a smooth manifold. A **presymplectic structure** $\eta \in \Omega^2(N)$ on N is a closed two-form of maximal rank, i.e. $\eta^n \neq 0$. We call a pair (N, η) a **presymplectic manifold**.

Remark 3.1.2. Somewhat trivially, we define a presymplectic form on a one-dimensional manifold to be the zero-form.

On symplectic manifolds the symplectic form induces a natural volume form, however to obtain a volume form on a presymplectic manifold we need more data then just the presymplectic form itself.

Definition 3.1.3. Let η be a presymplectic structure on N. A one-form $\theta \in \Omega^1(N)$ is called **admissible** for η if $\eta^n \wedge \theta \neq 0$. If N is oriented, it is called **+-admissable** if $\eta^n \wedge \theta > 0$.

Remark 3.1.4. For a one-dimensional presymplectic manifold $(N, 0)$ we define an admissible form to be any nowhere vanishing one-form.

The following lemma ensures the existence of admissible forms, and studies their uniqueness. The proof of the following lemma has been suggested to the author by B. Janssen.

Lemma 3.1.5. *Let η be a presymplectic form on an orientable manifold N^{2n+1}. Then*

i. There exist +-admissible forms for η.

ii. If θ is an admissible form, then so is $f\theta + \iota_X \eta$ for all $f \in C^\infty(N)$ nowhere zero, and $X \in \mathfrak{X}(N)$.

iii. If θ and β are two admissible forms for η, then there exists $f \in C^\infty(N)$ nowhere zero, $X \in \mathfrak{X}(N)$ such that $\beta = f\theta + \iota_X \eta$.

Proof. **i):** Let $(v^1, \dots, v^{2n+1})$ be local frame for T^*N, and let ν be a volume form on N. For every $x \in N$ we define the map $(\eta^n)_x \wedge \cdot : T^*_x N \to \mathbb{R}$, where we identified $\nu_x \cdot \mathbb{R}$ with $\mathbb{R}$. We see that there must exists a covector $v \in T^*_x N$ such that $(\eta^n)_x \wedge v > 0$. Indeed if not, then η^n_x would be zero when restricted to $\ker(v)$, for any $v \in T^*N$, implying that $\eta^n_x = 0$. Without loss of generality assume that $(v^1)_x = v$, then there exists a neighbourhood U of x such that $\eta^n \wedge v > 0$. The statement now follows from a partition of unity argument.
ii): Clearly if θ is admissible, then so is $f\theta$. Using the fact that interior multiplication is an anti-deriviation we have that

$$0 = \iota_X(\eta^{n+1}) = (n+1)(\iota_X \eta \wedge \eta^n),$$

from which the result follows.
iii): Let ν be a volume form on N. We define $h, g \in C^\infty(N)$ by $h\nu = \eta^n \wedge \beta$, $g\nu = \eta^n \wedge \theta$ Then

if we define $f = h/g$, we see that $\eta^n \wedge (f\theta - \beta) = 0$. So we are left to show that for any one-form γ such that $\eta^n \wedge \gamma = 0$, there exists $X \in \mathfrak{X}(N)$ such that $\gamma = \iota_X \eta$. We once again consider the linear map $(\eta^n)_x \wedge \cdot : T_x^* N \to \mathbb{R}$ induced by ν. As there exists at least one admissible one-form, the kernel of this map is $2n$-dimensional. The rank of η^{2n} is $2n$, hence there exists local sections $X_1, \ldots, X_{2n}$ such that the $\iota_{X_i}\eta$'s are all linearly independent. This proves that any one-form for which the wedge product with η^n vanishes, must be of the form $\iota_X \eta$. Hence $f\theta - \beta = \iota_X \eta$, which finishes the proof. □

The following lemma will be used tacitly in the below:

Lemma 3.1.6. *Let $N \subset M^n$ be a hypersurface and consider a volume form $\gamma \in \Omega^n(M)$. Then for $X \in \mathfrak{X}(M)$ we have that $\iota_X \gamma|_N$ is a volume form on N if and only if X transverse to N.*

Proof. Assume that X is transverse to N. Because γ is a volume form and X is transverse to N, for every $x \in N$ we can find linearly independent vectors $X_1, \ldots, X_{n-1} \in T_x N$ such that $\gamma_x(X, X_1, \ldots, X_{n-1}) \neq 0$. This proves that $\iota_X \gamma$ is a volume form on N. Now assume that $\iota_X \gamma$ is a volume form on N. Then there exists linearly independent vectors $Y_1, \ldots, Y_{n-1} \in T_x N$ such that $(\iota_X \gamma)_x(Y_1, \ldots, Y_{n-1}) \neq 0$. Hence $X, Y_1, \ldots, Y_{n-1}$ are linearly independent, from which we conclude that X is transverse to N. □

Presymplectic structures naturally arise from restricting symplectic structures to hypersurfaces:

Lemma 3.1.7. *Let $N \subset M^{2n}$ be a hypersurface in a symplectic manifold (M, ω). Then*

i. The restriction $\eta := \omega|_N$ is a presymplectic structure on N.

ii. Furthermore, assume that N is orientable and embedded. For any vector field $X \in \mathfrak{X}(M)$ transverse to N, $(\iota_X \omega)|_N$ is an admissible form for η. Conversely any admissible form is of this form.

Proof. It is clear that η is a presymplectic structure. By Lemma 3.1.6 we have that $\iota_X \omega^n|_N$ is a volume form. Since $(\iota_X \omega)^n|_N = n\iota_X \omega|_N \wedge \omega^{n-1}|_N \neq 0$, we see that $\iota_X \omega|_N$ is an admissible form for $\omega|_N$. Now let any admissible form θ be given. Because N is an orientable submanifold of an orientable manifold, we have by an argument similar to Lemma 1.1.4 that the normal bundle of N is orientable. Because this is a line bundle, there thus exists a nowhere vanishing section. Because N is embedded we can extend this section to a vector field on M, which we denote by X, and we observe that X is transverse to N. By Lemma 3.1.5 we then have that $\theta = f(\iota_X \omega)|_N + (\iota_Y \omega|_N)$, for some $f \in C^\infty(N)$ and $Y \in \mathfrak{X}(N)$. Now because N is embedded there exists extensions $\widetilde{f}, \widetilde{Y}$ of f and Y to M. Now we see that

$$\theta = (\iota_{\widetilde{f}X + \widetilde{Y}} \omega)|_N.$$

Because $\widetilde{f}$ is non-zero at N and X is transverse to N we have that $\widetilde{f}X + \widetilde{Y}$ is transverse to N. □

Although for presymplectic forms there always exist admissible forms, the existence of closed admissible forms is non-trivial.

Definition 3.1.8. A presymplectic form $\eta \in \Omega^2(N)$ is called **cosymplectic** if it admits a closed admissible form. A pair $(\eta, \theta) \in \Omega^2(N) \times \Omega^1(N)$ is called a **cosymplectic structure** if η is a cosymplectic form and θ is a closed admissible one-form.

The following proposition, which follows directly from the definition, shows that cosymplectic structures can be seen as very well-behaved symplectic foliations.

Proposition 3.1.9. *Let (M, η, θ) be a cosymplectic manifold, then $(\ker\theta, \eta|_{\ker\theta})$ defines a symplectic foliation.*

Somewhat opposite to cosymplectic structures are *contact forms*.

Definition 3.1.10. A **contact structure** on a manifold M^{2n+1} is a codimension-one maximally non-integrable distribution ξ on M, that is, a distribution locally given by the kernel of a one-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. The contact structure is said to be **co-orientable** if the normal bundle of the distribution is trivial.

Recall that in Proposition 1.1.12 we have proven that co-orientable codimension-one foliations are defined by the kernel of a global one-form. Completely similar one can prove that co-orientable contact structures are globally defined by the kernel of a one-form $\alpha \in \Omega^1(M)$ satisfying $\alpha \wedge (d\alpha)^n \neq 0$. We call such a globally defined one-form a **contact form**.

Remark 3.1.11. Somewhat trivially we define a contact structure on a one-dimensional manifold to be the zero-distribution and a contact form to be a nowhere-vanishing one-form.

Let $H \subset TM$ be a hyperplane distribution and consider the map

$$\begin{aligned} \varphi : \Gamma(H) \times \Gamma(H) &\to \Gamma(TM/H) \\ (X, Y) &\mapsto [X, Y] \bmod H. \end{aligned}$$

For a foliation $\mathcal{F}$ we have $H = T\mathcal{F}$ and see that this map vanishes identically. The following lemma states that for contact structures this is precisely the opposite. This fact illustrates that foliations and contact structures are, in some sense, opposites of each other.

Lemma 3.1.12. *A hyperplane distribution H is contact if and only if φ is non-degenerate.*

Proof. When H is a contact structure defined locally by $\ker\alpha$ we have that $d\alpha|_H$ is non-degenerate. Hence for every $X \in H$ non-zero, there exists $Y \in H$ such that $(d\alpha)(X, Y) \neq 0$. Applying the Koszul formula we find that $(d\alpha)(X, Y) = -\alpha([X, Y]) \neq 0$. This shows that $[X, Y] \notin H$, hence φ is non-degenerate. The converse is proven in a similar manner. □

Because $d\alpha$ is non-degenerate on a distribution of codimension one, its kernel is one-dimensional and thus spanned by a single vector field.

Definition 3.1.13. Let $\alpha \in \Omega^1(M)$ be a contact form. The **Reeb vector field** of α is defined to be the unique vector field $R \in \mathfrak{X}(M)$ satisfying

$$\iota_R \alpha = 1, \quad \iota_R d\alpha = 0.$$

The existence and uniqueness of the Reeb vector field is easily established, see for instance [Gei08].

3.2 Symplectic structures near the boundary

In the end, we are interested in obtaining symplectic foliations from glueing manifolds with boundary. In the construction of these symplectic foliations we will make extensive use of normal forms around boundaries. It is thus interesting to have a general normal form for symplectic structures near the boundary.

Proposition 3.2.1. *Let (M, ω) be a symplectic manifold with compact boundary. Let $\eta := \omega|_{\partial M}$ and let $\theta \in \Omega^1(\partial M)$ be any $+$-admissible form for η. Then there exists a neighbourhood U of the boundary and a diffeomorphism $\varphi : \partial M \times [0, c) \to U$ such that*

$$\varphi^*(\omega|_U) = \eta - d(t\theta),$$

for some $c \in [0, 1)$, where t denotes the coordinate in $[0, c)$.

The proof of this proposition uses a Moser type argument. We will however only need the statement in a more specialized setting in which another proof can be given, this will be addressed in Proposition 3.2.6

One way to obtain contact and cosymplectic structures is from symplectic manifolds with certain behaviour near the boundary:

Definition 3.2.2. Let (M, ω) be a symplectic manifold with boundary. Then ∂M is called of:

- **Cosymplectic type** if there exists a vector field $X \pitchfork \partial M$ near the boundary such that $(\mathcal{L}_X \omega)|_{\partial M} = 0$.
- **Contact type** if there exists a vector field $X \pitchfork \partial M$ near the boundary such that $(\mathcal{L}_X \omega)|_{\partial M} = \omega|_{\partial M}$.

As claimed these structures induce cosymplectic and contact structures on the boundary.

Lemma 3.2.3. *Let (M, ω) be a symplectic manifold with boundary of contact type, and let X be a vector field transverse to the boundary such that $(\mathcal{L}_X \omega)|_{\partial M} = \omega|_{\partial M}$. Then $(\iota_X \omega)|_{\partial M}$ is a contact form on ∂M.*

Proof. By Cartan's formula we have $(d\iota_X\omega)|_{\partial M} = \omega|_{\partial M}$. Hence we find $(d\iota_X\omega)|^n_{\partial M} = (\omega|_{\partial M})^n$. By Lemma 3.1.7 we have that $\omega|_{\partial M}$ is presymplectic with admissible form $\iota_X\omega|_{\partial M}$. From this we conclude that $(\iota_X\omega)|_{\partial M}$ is a contact form. □

Lemma 3.2.4. *Let (M, ω) be a symplectic manifold with boundary of cosymplectic type and let X be a vector field transverse to the boundary such that $\mathcal{L}_X\omega|_{\partial M} = 0$. Then $(\omega|_{\partial M}, (\iota_X\omega)|_{\partial M})$ is a cosymplectic structure.*

Proof. By Lemma 3.1.7 we have that $(\iota_X\omega)|_{\partial M}$ is admissible for $\omega|_{\partial M}$. We can apply Cartan's formula to conclude that $(\iota_X\omega)|_{\partial M}$ is closed. This finishes the proof. □

Remark 3.2.5. We remark that the cosymplectic and contact structures induced on the boundaries depend on the choice of vector field. Nevertheless we will call the cosymplectic/contact structures defined in the above lemmata **a cosymplectic/contact structure induced by** ω.

We can characterise symplectic manifolds with boundary of cosymplectic type in different ways:

Proposition 3.2.6. *Let (M, ω) be a symplectic manifold with compact boundary. The following are equivalent:*

i. The form $\omega|_{\partial M}$ is a cosymplectic form on ∂M.

ii. The boundary of M is of cosymplectic type, i.e. there exists a vector field X near the boundary on M transverse to ∂M for which $\mathcal{L}_X\omega|_{\partial M} = 0$.

iii. There exists a vector field X on a neighbourhood U of ∂M, transverse to ∂M for which $\mathcal{L}_X\omega|_U = 0$.

iv. In a collar neighbourhood U of the boundary ω has the following local form:

$$\omega|_U = \omega|_{\partial M} + \theta \wedge dr,$$

where θ is a closed one-form admissible for $\omega|_{\partial M}$.

Proof. The implication $iii) \Rightarrow ii)$ is trivial, and $ii) \Rightarrow i)$ is proven in Lemma 3.2.4.
i) $\Rightarrow$ ii): Let $\theta \in \Omega^1(\partial M)$ be a closed admissible form for $\omega|_{\partial M}$. By Lemma 3.1.7 there exists a vector field X near the boundary, transverse to the boundary such that $\theta = (\iota_X\omega)|_{\partial M}$. Because θ is closed, we find that $(d\iota_X\omega)|_{\partial M} = 0$, hence we conclude that $(\mathcal{L}_X\omega)|_{\partial M} = 0$, which finishes the proof.
ii) $\Rightarrow$ iii): Let $\gamma \in \Omega^1(U)$ be a closed extension of $(\iota_X\omega)|_{\partial M}$. Then by non-degeneracy of ω there exists $Y \in \mathfrak{X}(U)$ such that $\gamma = \iota_Y\omega|_U$. Hence by closedness of γ we see that $\mathcal{L}_Y\omega|_U = 0$.

We are thus left to prove that Y is transverse to ∂M. Because X is transverse to ∂M we have that $(\iota_X\omega^n)|_{\partial M}$ is a volume form. We have

$$\begin{aligned}(\iota_X\omega^n)|_{\partial M} &= n\iota_X|_{\partial M}\wedge\omega^{n-1}|_{\partial M}\\ &= n\iota_Y|_{\partial M}\wedge\omega^{n-1}|_{\partial M}\\ &= (\iota_Y\omega^n)|_{\partial M},\end{aligned}$$

and so $(\iota_Y\omega^n)|_{\partial M}$ is a volume form. We conclude that Y is transverse to ∂M as required.
iii) $\Leftrightarrow$ iv): Let X be a vector field defined on a neighbourhood U of the boundary, which is transverse to the boundary, such that $\mathcal{L}_X\omega|_U = 0$. Consider the collar neighbourhood $p : \partial M\times[0,1)\to V$ defined by the flow of this vector field. Let t denote the second coordinate, then on this neighbourhood we can write

$$\omega|_V = \omega_t + \theta_t\wedge dt,$$

with $\omega_t\in\Omega^2(\partial M)$ and $\theta_t\in\Omega^1(\partial M)$. On V we have that X takes the form ∂_t. Hence $\mathcal{L}_X\omega|_V = d\theta_t = 0$. Let d^∂ denote the differential along the boundary. Because $d\theta_t = 0$, we find that

$$d^\partial\theta_t + \frac{\mathrm{d}}{\mathrm{d}t}\theta_t\wedge dt = 0.$$

Hence $\frac{\mathrm{d}}{\mathrm{d}t}\theta_t = 0$ and thus $\theta_t = \theta_0 := \theta$, and also $d^\partial\theta = 0$. Now because $d\omega = 0$ we find

$$d^\partial\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\wedge dt = 0,$$

and as before we find $\omega_t = \omega_0$ and $d^\partial\omega_0 = 0$. Because $\omega|_{\partial M} = \omega_0$, we thus conclude that

$$\omega|_V = \omega|_{\partial M} + \theta\wedge dt.$$

Finally,

$$\begin{aligned}\omega|_V^n &= \omega|_{\partial M}^{n-1}\wedge\theta\wedge dt\\ &\neq 0,\end{aligned}$$

hence θ is indeed a closed admissible one-form for $\omega|_{\partial M}$, which finishes the proof. For the converse one simply takes the vector field ∂_t on the collar neighbourhood. □

In the below we will encounter exact symplectic structures $\omega = d\alpha$ on manifolds with boundary M, with the property that $\alpha|_{\partial M}$ is contact.

Lemma 3.2.7. *Let $\omega = d\alpha$ be a symplectic structure on a manifold with boundary M. When $\alpha|_{\partial M}$ is a contact form, we have that ∂M is of contact type.*

Proof. Because $\alpha|_{\partial M}$ is contact we have that $d\alpha|_{\partial M}^n\wedge\alpha|_{\partial M}\neq 0$, i.e. $\alpha|_{\partial M}$ is an admissable form for the presymplectic structure $d\alpha|_{\partial M}$. Hence by Lemma 3.1.7 there exists a vector field X transverse to ∂M, such that $\iota_X(d\alpha)|_{\partial M} = \alpha|_{\partial M}$. Using Cartan's formula we find $d\alpha|_{\partial M} = (\mathcal{L}_X d\alpha)|_{\partial M}$ which proves that ω is of contact type at the boundary. □

3.3 Generalised cosymplectic structures

Presymplectic forms are defined to have maximal rank. One could weaken this assumption, by assuming that they only have constant rank. We will define such generalised cosymplectic structures and study some of their properties. Most notably we will show that these structures can be inverted to particular types of Poisson structures, just as symplectic forms can be inverted to non-degenerate Poisson structures. We begin with defining the almost versions:

Definition 3.3.1. An **almost k-cosymplectic structure** on M^n is a tuple $(\omega, \theta_1, \ldots, \theta_k)$ where ω is a two-form of constant rank $2l$ and $\theta_1, \ldots, \theta_k$ are one-forms such that

$$\omega^l \wedge \theta_1 \wedge \cdots \wedge \theta_k \neq 0.$$

Here $n - k = 2l$.

Definition 3.3.2. An **almost k-Poisson structure** on M^n is a tuple $(\pi, X_1, \ldots, X_k)$ where π is a bivector of constant rank $2l$ and $X_1, \ldots, X_k$ are vector fields such that

$$\pi^l \wedge X_1 \wedge \cdots \wedge X_k \neq 0.$$

Here $n - k = 2l$.

We first invert these almost versions into each other.

Lemma 3.3.3. *There is a 1:1 correspondence between almost k-cosymplectic structures and almost k-Poisson structures.*

Proof. Let $(\omega, \theta_1, \ldots, \theta_k)$ be an almost k-cosymplectic structure. Because $\omega^l \wedge \theta_1 \wedge \ldots \wedge \theta_k \neq 0$, we have that $\omega^l|_{\cap_i \ker \theta_i} \neq 0$. This implies that we can split the tangent bundle in the following way: $TM = \ker \omega \oplus \cap_i \ker \theta_i$. We define vector fields X_i, in the following way: $X_i(\theta_j) = \delta_{ij}$, and X_i is defined to be zero on the complement of span θ_i. This gives a decomposition of the co-tangent bundle $T^*M = \text{span } \theta_i \oplus \cap_i \ker X_i$. Now we consider $\omega^\flat : TM \to T^*M$, and we restrict to $\cap_i \ker \theta_i$. Let any element of $\omega^\flat(\cap_i \ker \theta_i)$ be given, say $\omega(V, \cdot)$. As $X_i \in \ker \omega$ by the decomposition of the tangent space we find $\omega(V, X_i) = -\omega(X_i, V) = 0$. Hence we conclude that $\omega^\flat(\cap_i \ker \theta_i) \subset \cap_i \ker X_i$. Because ω has maximal rank on $\cap_i \ker \theta_i$, its image must have dimension $2l$, hence $\omega^\flat(\cap_i \ker \theta_i) = \cap_i \ker X_i$. So we conclude that the map

$$\omega^\flat : \cap_i \ker \theta_i \to \cap_i \ker X_i$$

is an isomorphism. We invert $\omega^\flat$ and define π on $\cap_i \ker X_i$ by this inverse and 0 on span θ_i. Because $\pi^l|_{\cap_i \ker X_i} = (\omega^\flat)^{-1}$ we have that $\pi^l|_{\cap_i \ker X_i} \neq 0$. This implies that $\pi^l \wedge X_1 \wedge \ldots \wedge X_k \neq 0$ which shows that $(\pi, X_1, \ldots, X_l)$ is an almost k-Poisson structure. The other direction of the correspondence is defined completely analogous. □

Definition 3.3.4. A **k-cosymplectic structure** on M^n is an almost k-cosymplectic structure $(\eta, \theta_1, \dots, \theta_k)$ for which all forms are closed.

Definition 3.3.5. A **k-Poisson structure** on M^n is an almost k-Poisson structure $(\pi, X_1, \dots, X_k)$ for which π is Poisson and $X_1, \dots, X_k$ commute pairwise.

Example 3.3.6. A zero-cosymplectic structure is simply a symplectic structure. A one-cosymplectic structure is a cosymplectic structure.

Just as in the almost case we have a 1:1 correspondence:

Proposition 3.3.7. *There is a 1:1 correspondence between k-cosymplectic structures and k-Poisson structures.*

Proof. Let $(\omega, \theta_1, \dots, \theta_k)$ be an almost k-cosymplectic structure on M^n and let $(\pi, X_1, \dots, X_k)$ be the corresponding almost k-Poisson structure. We define

$$\Omega = \omega + \sum_{i=1}^{k} \theta_i \wedge dt_i \in \Omega^2(M \times \mathbb{R}^k).$$

Note that $\dim(M \times \mathbb{R}^k) = n + k = (n - k + 2k) = 2(l + k)$. We have

$$\begin{aligned} \Omega^{l+k} &= C\omega^l \wedge \left(\sum_{i=1}^{k} \theta_i \wedge dt_i \right)^k \\ &= \widetilde{C}\omega^l \wedge \theta_1 \wedge \dots \wedge \theta_l \wedge dt_1 \wedge \dots dt_k \\ &\neq 0, \end{aligned}$$

with C and $\widetilde{C}$ some combinatorial factors. Hence we can consider $\Pi = \Omega^{-1} \in \mathfrak{X}^2(M \times \mathbb{R}^k)$, this element is given by:

$$\Pi = \pi + \sum_{i=1}^{k} \partial_{t_i} \wedge X_i.$$

Recall that Π is Poisson if and only if Ω is closed. Because

$$d\Omega = d\omega + \sum_{i=1}^{k} d\theta_i \wedge dt_i \in \Omega^2(M \times \mathbb{R}^k),$$

we have that Ω is closed if and only if $\omega, \theta_1, \dots, \theta_k$ are closed. Hence we conclude that Ω is closed if and only if $(\omega, \theta_1, \dots, \theta_k)$ is k-cosymplectic. On the other hand, a straightforward but tedious calculation yields that:

$$[\Pi, \Pi] = [\pi, \pi] + \sum_{i>j} (-1)^{n(i,j)} \partial_{t_i} \wedge \partial_{t_j} \wedge [X_i, X_j], \quad \text{for some } n(i,j) \in \mathbb{N}$$

It follows that Π is Poisson if and only if $[\pi, \pi] = 0$ and $[X_i, X_j] = 0$ for all i, j. We conclude that Π is Poisson if and only if $(\pi, X_1, \dots, X_k)$ is a k-Poisson structure. This shows that $(\omega, \theta_1, \dots, \theta_k)$ is a k-cosymplectic structure if and only if $(\pi, X_1, \dots, X_k)$ is a k-Poisson structure. $\square$

Definition 3.3.8. Let (η, θ) be a cosymplectic structure and let (π, X) be the corresponding 1-Poisson structure. We will call X the **Reeb vector field** of (η, θ).

Remark 3.3.9. Recall from Secion 1.3 that symplectic foliations can be inverted into Poisson structures. So given a cosymplectic structure (η, θ) we could invert the corresponding symplectic foliation into a Poisson structure. However, there exist several cosymplectic structures which give rise to the same Poisson structure. Indeed the Poisson structure only depends on the foliation, and not on the defining one-form. The above proposition can be seen as a refinement of this procedure.

3.4 Symplectic turbulisation: $M \times S^1$

Let $f : C \to S^1$ be a fibration on a manifold with boundary. The fibres of f induce a foliation on C. However this foliation is not tame near the boundary. There is a procedure called *turbulisation* to change the foliation into a foliation which is tame near the boundary. We will first recall this procedure, first for the case of trivial circle fibrations, then for the locally trivial case. Afterwards we will construct a leafwise symplectic structure on the turbulised foliation, such that the resulting symplectic foliation becomes tame near the boundary. We will first do this for the trivial case and in the next section in more generality.

For notational purposes we introduce the following notion:

Definition 3.4.1. Let $f, g : [0, 1] \to [0, 1]$ be smooth functions. We say that (f, g) is a **good pair** if

- $f(0) = g(1) = 0$,
- $f(1) = g(0) = 1$,
- f' and g' vanish up to infinite order at 0 and 1.

Lemma 3.4.2 (Turbulisation)**.** *Let M be a manifold with compact boundary. Then $M \times S^1$ admits a codimension-one co-orientable foliation tame near the boundary.*

Proof. Consider a collar neighbourhood $k : \partial M \times [0, 1) \to U$ of the boundary of M. Let (f, g) be a good pair of functions such that $f(x) > 0$ for all $x \in (0, 1)$. Consider the following form on M:

$$\alpha = \begin{cases} d\varphi & \text{on } (M \setminus U) \times S^1, \\ f(t)d\varphi + g(t)dt & \text{on } U \times S^1. \end{cases}$$

By choice of f and g this form is smooth. Also $\alpha \wedge d\alpha = 0$, and α thus defines a foliation on M. Because on $U \times S^1$ we have $d\alpha = f'(t)dt \wedge d\varphi$, and $f'(t)$ vanishes up to infinite order at $t = 0$ we have by Lemma 2.1.4 that the foliation is tame near the boundary. □

In Figure 3.1 we will give a visualisation of the above procedure. But first we study how the leaves of the new foliation relate to the old one.

Lemma 3.4.3. *Let $\mathcal{F}$ be the foliation from Lemma 3.4.2. The restriction of $\mathcal{F}$ to* $\operatorname{Int} M \times S^1$ *is diffeomorphic to the product foliation.*

Proof. Define the function $\widetilde{h} : (0, \infty) \to \mathbb{R}$ by the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{h}(t) = \frac{g}{f}, \quad \widetilde{h}(1) = 0,$$

where f and g are as in the proof of Lemma 3.4.2. Define $h : \operatorname{Int} M \to \mathbb{R}$ by

$$h(z) = \begin{cases} \widetilde{h}(t) & \text{if } z = (x,t) \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Because $dh = \frac{g}{f}dt$, we find that $\alpha = f(d\varphi + dh)$. Because $f > 0$ on $S^1 \times (0,1)$ we see that $\ker \alpha = \ker(d\varphi + dh)$. Consider

$$\widetilde{\psi} : \operatorname{Int} M \times \mathbb{R} \to \operatorname{Int} M \times \mathbb{R}, \quad (x,t) \mapsto (x, t - h(x)),$$

and let $\psi : \operatorname{Int} M \times S^1 \to \operatorname{Int} M \times S^1$ denote the induced map on the quotient. Because $\widetilde{\psi}^*(dt + dh) = dt - dh + dh = dt$ we find that $\psi^*(dh + d\varphi) = d\varphi$. We see that ψ gives the required diffeomoprhism. □

Remark 3.4.4. We can describe the leaves of the foliation obtained in Lemma 3.4.2 in the following way. Let $h : \operatorname{Int} M \to \mathbb{R}$ be as in the proof Lemma 3.4.3. We consider the map $\varphi : \operatorname{Int} M \times \mathbb{R} \to \mathbb{R}$, given by $\varphi(x,t) = h(x) + t$. We see that the fibers of this map form a foliation given by $\ker(dt + dh)$. Hence the foliation on $\operatorname{Int} M \times S^1$ can be viewed as the quotient of this foliation. Morally speaking we see that as $h(x) \overset{x \to \partial M}{\longrightarrow} \infty$, that the leaves of the foliation are turning around faster and faster as the foliation approaches the boundary. This is visualized in Picture 3.1.

Locally trivial case

A natural generalisation of Lemma 3.4.2 is to work with S^1-fibrations.

Lemma 3.4.5 ([Law71])**.** *Let M be a manifold with compact boundary. Let $\pi : M \to S^1$ be a smooth submersion with the property that $\pi|_{\partial M}$ is also a submersion. Then M admits a codimension-one co-orientable foliation tame near the boundary.*

Proof. By Lemma A.2.4 there exists a collar neighbourhood $U \simeq \partial M \times [0,1)$ with the property that the following diagram commutes

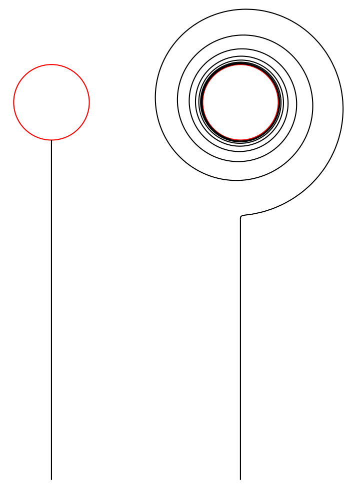


Figure 3.1: A visualisation of the turbulisation procedure. The cylinder $S^1 \times [1, \infty)$ has been visualized as a subset of the plane, which is given as the complement of the compact set bounded by the circle. Of the product foliation on $S^1 \times [1, \infty)$ there is one leaf drawn on the left and next to it a resulting leaf after turbulisation.

$$\begin{array}{ccc} \partial M \times [0,1) & \xrightarrow{\simeq} & U \\ {\scriptstyle \pi|_{\partial M} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \pi} \\ S^1 \times [0,1) & \xrightarrow{\mathrm{pr}_1} & S^1 \end{array}$$

To construct the foliation we can now proceed completely analogous to the trivial case. We define

$$\alpha := \begin{cases} \pi^* d\varphi & \text{on } M \setminus U, \\ f(t)\pi^* d\varphi + g(t)dt & \text{on } U, \end{cases}$$

with f and g as in Lemma 3.4.2. It is easily verified that $\alpha \wedge d\alpha = 0$. By the local form of π on U we see that $\alpha|_U = f(t)\pi|_{\partial M}^* d\varphi + g(t)dt$. Thus $d\alpha|_U = f'(t)\pi|_{\partial M}^* d\varphi \wedge dt$. Because f' vanishes up to infinite order at $t = 0$ we see by Lemma 2.1.4 that the foliation is tame near the boundary, which finishes the proof. □

Again we study how the turbulised foliation relates to the old one. The proof is almost identical to that of the trivial case.

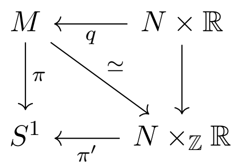
Lemma 3.4.6. *Let $\mathcal{F}$ be the foliation obtained in Lemma 3.4.5, then the foliation $\mathcal{F}|_{\mathrm{Int}\, M}$ is diffeomorphic to the foliation defined by the fibers of π.*

Proof. Let $h : \mathrm{Int}\ M \to \mathbb{R}$ be as defined in the proof of Lemma 3.4.3. Let $N = \pi^{-1}([0])$, and let

$\varphi : N \to N$ be a monodromy of $\pi : M \to S^1$ (see Theorem A.2.6). Define

$$\widetilde{\psi} : \operatorname{Int} N \times \mathbb{R} \to \operatorname{Int} N \times \mathbb{R},$$
$$(x, t) \mapsto (\varphi(x), t - h(x)).$$

It is easily verified that $\widetilde{\psi}$ is equivariant with respect to the action defined in Theorem A.2.6. It thus induces a diffeomorphism $\psi : \operatorname{Int} M \to \operatorname{Int} M$. We have $\widetilde{\psi}^* dt = dt - dh$, and $\widetilde{\psi}^* dh = \varphi^* dh$. Because h depends only on the t-coordinate of the tubular neighbourhood, and is defined to be zero outside the tubular neighbourhood, we see that $\varphi^* dh = dh$. In conclusion $\widetilde{\psi}^*(dt + dh) = dt$. Let $q : N \times \mathbb{R} \to M$ denote the composition of the quotient map and the diffeomorphism $M \simeq N \times_{\mathbb{Z}} \mathbb{R}$. By definition we have that the diagram,



with $\pi'([x, t]) = [t]$, commutes. Hence $q^*(\pi^* d\varphi) = dt$, and from combining this with $\widetilde{\psi}^*(dt + dh) = dt$ it follows that $\psi^*(\pi^* d\varphi + dh) = \pi^* d\varphi$. This finishes the proof. □

Leafwise symplectic structure

Given a symplectic manifold M with boundary we see that $M \times S^1$ admits a sympletic foliation. We would like to modify the symplectic structure such that the foliation obtained in Lemma 3.4.2 can be endowed with a leafwise symplectic form tame near the boundary. We do however have to impose some extra conditions on the symplectic manifold.

Proposition 3.4.7. *Let (M^{2n}, ω) be a symplectic manifold with compact boundary of cosymplectic type. Then the foliation on $M \times S^1$ as constructed in Lemma 3.4.2 admits a leafwise symplectic structure tame near the boundary. The symplectic structure on the boundary leaf is given by*

$$\omega|_{\partial M \times S^1} = \eta - \theta \wedge d\varphi,$$

where (η, θ) is a cosymplectic structure on ∂M induced by ω.

Proof. Let (f, g) be the pair of good functions from the proof of Lemma 3.4.2, and recall form defining the foliation

$$\alpha = \begin{cases} d\varphi & \text{on } (M \setminus U) \times S^1, \\ f(t)d\varphi + g(t)dt & \text{on } U \times S^1. \end{cases}$$

Because ω is of cosymplectic type at the boundary, by Proposition 3.2.6 there exists a collar neighbourhood, U, of the boundary on which ω takes the following form:

$$\omega|_U = \eta + \theta \wedge dt, \quad \text{where } \eta = \omega|_{\partial M}, \text{ and } \theta \text{ is admissible for } \eta.$$

Let (a, b) be a good pair of functions satisfying the following assumptions:

- a and b are constant near 0,
- $af + bg > 0$.

Define

$$\Omega = \begin{cases} \omega & \text{on } (M \setminus U) \times S^1, \\ \eta - b\theta \wedge d\varphi + a\theta \wedge dr & \text{on } U \times S^1. \end{cases}$$

We have

$$d\Omega = \begin{cases} d\omega & \text{on } (M \setminus U) \times S^1, \\ -db \wedge \theta \wedge d\varphi & \text{on } U \times S^1, \end{cases}$$

hence $d\Omega \wedge \alpha = 0$. Also

$$\Omega^n = \begin{cases} \omega^n & \text{on } (M \setminus U) \times S^1, \\ (n-1)\eta^{n-1} \wedge (a\theta \wedge dt - b\theta \wedge d\varphi) & \text{on } U \times S^1, \end{cases}$$

hence

$$\Omega^n \wedge \alpha = \begin{cases} \omega^n \wedge d\varphi & \text{on } (M \setminus U) \times S^1, \\ (n-1)(af + bg)\eta^{n-1} \wedge \theta \wedge dt \wedge d\varphi & \text{on } U \times S^1. \end{cases}$$

Because $af + bg > 0$ we find that Ω is a leafwise symplectic form on $M \times S^1$. Because we have chosen a and b constant near 0, we get that Ω is constant near the boundary. We conclude that Ω is a symplectic form tame near the boundary, which finishes the proof. □

Using this procedure we can construct the famous Reeb foliation:

Example 3.4.8. Consider the following decomposition of S^3,

$$S^3 = D_1^2 \times S_1^1 \cup_{T^2,\text{flip}} D_2^2 \times S_2^1,$$

where the glueing is done by identifying ∂D_1^2 with S_2^1 and ∂D_2^2 with S_1^1. We now apply the turbulisation procedure to $D_1^2 \times S_1^1$ and $D_2^2 \times S_2^1$. This will result in a foliation where the torus is a leaf and, using Lemma 3.4.3, with leaves in the interior diffeomorphic to $\mathbb{R}^2$. This foliation is called the Reeb foliation.

For the symplectic structure we note that the (signed) standard area form $\omega_i := (-1)^{i+1} r_i d\varphi_i \wedge dr_i$ on D_i^2 is cosymplectic at the boundary. Indeed the standard angular forms $d\theta_i \in \Omega^1(S_i^1)$ form admissible forms. Hence we can apply Proposition 3.4.7 to obtain a leafwise symplectic structure on both $D_1^2 \times S_1^1$ and $D_2^2 \times S_2^1$. The symplectic structures at the boundary leaves are given by $d\varphi_1 \wedge d\theta_1$ and $-d\varphi_2 \wedge d\theta_2$. The diffeomorphism of the boundaries is a symplectomorphism with respect to these structures hence we can apply Theorem 2.2.7 to obtain a symplectic foliation on S^3.

Example 3.4.9. Because S^2 is symplectic, the manifold $S^2 \times S^1$ admits a symplectic foliation. We will use these techniques to obtain a non-trivial symplectic foliation on $S^2 \times S^1$. Consider the decomposition of $S^2 \times S^1$

$$S^2 \times S^1 = D^2 \times S^1 \cup_{T^2,\mathrm{id}} D^2 \times S^1,$$

where the glueing is now done via the identity map. We now proceed as in the previous example, but now we turbulise both tori in the same direction.

3.5 Turbulisation: general case

In the previous section we constructed a leafwise symplectic structure on the foliation obtained by applying the turbulisation procedure to a trivial fibration. A natural way to continue would be to endow the foliation obtained in Lemma 3.4.5 with a leafwise symplectic form. However we are going to do something more general.

Recall that Tischler's theorem states that a manifold admitting a unimodular foliation fibres over S^1. Because cosymplectic manifolds admit a natural unimodular foliation, they also admit a circle fibration. So instead of generalising the turbulisation procedure to locally trivial fibrations, we will proceed to cosymplectic manifolds. We will define a certain class of cosymplectic manifolds with boundary, and then prove that we can produce symplectic foliations tame near the boundary on these manifolds. First we will consider manifolds $B \times D^2$, where B is cosymplectic. These are the manifolds in which we are mainly interested, and act as a motivating example.

3.5.1 Turbulisation $B \times D^2$

When (B, η, θ) is a cosymplectic manifold, then $(B \times D^2, \eta + \omega_{D^2}, \theta)$ is a cosymplectic manifold with boundary. Here $\omega_{D^2} = r d\varphi \wedge dr$ denotes the standard area form on D^2. As before the induced symplectic foliation on $B \times D^2$ is not tame near the boundary, so we are going to modify it.

Lemma 3.5.1. *Let (B, η, θ) be a cosympletic manifold without boundary. Then the manifold $B \times D^2$ admits a codimension-one symplectic foliation tame near the boundary, with symplectic structure on the boundary leaf given by*

$$\eta - \theta \wedge d\varphi.$$

Proof. Let (f, g) and (a, b) be good pairs of functions satisfying:

- $f(t) > 0$ for all $t \in (0, 1)$,
- a, b are constant near 1,
- $af + bg > 0$.

We denote $C = B \times D^2$ and consider the tubular neighbourhood U of ∂C induced by $V = -\frac{1}{r}\frac{\partial}{\partial r}$. On U we have that $\omega_{D^2}|_U = dt \wedge d\varphi$, where t denotes the coordinate on the tubular neighbourhood. We define

$$\theta' = \begin{cases} \theta & \text{on } C \setminus U, \\ f\theta + gdt & \text{on } U, \end{cases} \qquad \eta' = \begin{cases} \eta + \omega_{D^2} & \text{on } C \setminus U \\ \eta + adt \wedge d\varphi + bd\varphi \wedge \theta & \text{on } U \end{cases}$$

By the choice of the functions we see that these forms are smooth. It is now easy to verify that $\theta' \wedge d\theta' = 0$, $d\eta' \wedge \theta = 0$. Finally

$$(\eta')^{n+1} \wedge \theta' = \begin{cases} n\eta^n \wedge \omega_{D^2} \wedge \theta & \text{on } C \setminus U, \\ n(afr + gb)\eta^n \wedge dt \wedge d\varphi \wedge \theta & \text{on } U, \end{cases}$$

which is nowhere vanishing. Hence (η', θ') defines a symplectic foliation. Because f and g are chosen to vanish up to infinite order at $t = 0$, we have by Lemma 2.1.4 that the foliation is tame near the boundary. Because a and b are chosen to be constant near $t = 0$, we see that η' is closed near the boundary. We conclude that the symplectic foliation is tame near the boundary which finishes the proof. □

We note that the cosymplectic manifold $(B \times D^2, \eta + \omega_{D^2}, \theta)$ has the property that on the tubular neighbourhood defined by $V = -\frac{1}{r}\partial_r$, the cosymplectic structure takes the form $(\eta - d\varphi \wedge dt, \theta)$. In the following we will study cosymplectic manifolds which admit similar normal forms around the boundary.

3.5.2 Cosymplectic manifolds with boundary of s-type

The following definition is somewhat similar to the definition of symplectic manifolds with boundary of cosymplectic type.

Definition 3.5.2. Let (C^{2n+1}, η, θ) be a cosymplectic manifold with boundary. We say that C has **boundary of s-type** if there exists a nowhere vanishing vector field V in a neighbourhood, U, of the boundary such that:

- V is transverse to the boundary,
- $\mathcal{L}_V(\eta|_U) = 0$,
- $\mathcal{L}_V(\theta|_U) = 0$,
- $\eta|_{\partial C}$ is nowhere symplectic, that is $\eta|_{\partial C}^n = 0$.

Remark 3.5.3. We first give an interpretation for the requirement $\eta|_{\partial C}^n = 0$. Because $\eta^n \wedge \theta \neq 0$, we have that η is non-degenerate along the leaves of the foliation defined by θ. By requiring the assumption $\eta|_{\partial C}^n = 0$, we see that the leaves of $\ker \theta$ and the boundary of C are transverse.

Example 3.5.4. Let (M, ω) be a symplectic manifold with boundary of cosymplectic type, then $(M \times S^1, \omega, d\varphi)$ is a cosymplectic manifold with boundary of s-type. Indeed the required vector field V is obtained from Proposition 3.2.6.

The following lemma gives an alternative description of the fourth assumption of Definition 3.5.2.

Lemma 3.5.5. *Let (C, η, θ) be a cosymplectic manifold which satisfies the first three assumptions of Definition 3.5.2 and let (π, X) be the corresponding one-Poisson structure from Proposition 3.3.7. Then $\eta|_{\partial C}$ is nowhere symplectic if and only if X is tangent to the boundary.*

Proof. Let U be a collar neighbourhood of ∂C and write

$$\theta|_U = \alpha_t + f_t dt, \quad \eta|_U = \beta_t \wedge dt + \gamma_t, \quad X|_U = Z_t + g_t \partial_t,$$

for some $Z_t \in \mathfrak{X}(\partial C)$, $\alpha_t, \beta_t \in \Omega^1(\partial C)$, $\gamma_t \in \Omega^2(\partial C)$ and $f_t, g_t \in C^\infty(\partial C)$. Recall that X is defined by the equations $\iota_X \eta = 0$ and $\iota_X \theta = 1$, which on ∂C give

$$\iota_{Z_0} \gamma_0 = g_0 \beta_0, \quad \iota_{Z_0} \alpha_0 + f_0 g_0 = 1.$$

We have to prove that $\gamma^n = 0$ if and only if X is tangent to the boundary which happens precisely when $g_0 = 0$:
"$\Rightarrow$": If $g_0 = 0$, then $X|_{\partial C} = Z_0$. Because $\iota_X \theta = 1$ we must have that Z_0 is nowhere vanishing. But as $\iota_{Z_0} \gamma_0 = 0$ we see that γ_0 has non-trivial kernel and we conclude that $\gamma_0^n = 0$.
"$\Leftarrow$": Because $\mathcal{L}_V \eta|_U = 0$ and $V|_U = \partial_t$ we find $d(\iota_{\partial_t} \beta_t \wedge dt) = -d\beta_t = 0$. Then $d^\partial \beta_t + \frac{\mathrm{d}}{\mathrm{d}t} \beta_t \wedge dt = 0$ which implies that $\beta_t = \beta_0$. Using the fact that η is closed, a similar argument shows that $\gamma_t = \gamma_0$. As $\gamma_0^n = 0$ we have,

$$(\eta^n \wedge \theta)|_U = n\gamma_0^{n-1} \wedge \beta_t \wedge dt \wedge \alpha_t \neq 0,$$

also

$$0 = \iota_{Z_0}(\gamma_0^n \wedge \alpha_t) = n g_0 \beta_0 \wedge \gamma_0^{n-1} \wedge \alpha_0.$$

Combining this gives $g_0 = 0$, which finishes the proof. □

We are now ready to produce the required local form corresponding to the definition of boundary of s-type.

Proposition 3.5.6. *Let (C^{2n+1}, η, θ) be a cosymplectic manifold. Then the following are equivalent:*

i. C has boundary of s-type.

ii. There exits a nowhere vanishing vector field V in a neighbourhood U of the boundary such that

- *V is transverse to the boundary,*
- *$(\mathcal{L}_V \eta)|_U = 0$,*
- *$(\iota_V \theta)|_U = 0$,*
- *$\eta|_{\partial C}$ is nowhere symplectic.*

iii. The tuple $(\gamma, \beta, \alpha) := (\eta|_{\partial C}, (\iota_V \eta)|_{\partial C}, \theta|_{\partial C})$ on is a two-cosymplectic structure on ∂C, such that on a collar neighbourhood U we have:

$$\theta|_U = \alpha, \quad \eta|_U = \beta \wedge dt + \gamma$$

Proof. **i)** $\Leftrightarrow$ **ii)**: The implication *ii)* $\Rightarrow$ *i)* is trivial so we are left to prove the other implication. Let W be the vector field from the definition of boundary of s-type, and let (π, X) be the one-Poisson structure corresponding to (η, θ). By Lemma 3.5.5 we have that X is tangent to the boundary, hence the vector field $V = W - \theta(W)X$ is transverse to the boundary. By definition we have that $\iota_X \theta = 1$, hence $\iota_V \theta|_U = \theta(W) - \theta(W) = 0$. Also by definition we have $\iota_X \eta = 0$, hence $\mathcal{L}_V \eta|_U = 0$.
iii) $\Rightarrow$ **ii)**: Let t denote the coordinate on the collar neighbourhood, and define $V = \partial_t$. We have $\iota_V \theta|_U = 0$ and $\mathcal{L}_V \eta|_U = d\iota_V \eta|_U = 0$. Because (γ, β, α) is a two-cosymplectic structure and $\gamma = \eta|_{\partial M}$ we have that $\eta|_{\partial M}$ is nowhere symplectic.
ii) $\Rightarrow$ **iii)**: Let V be the vector field of ii), which we assume to point inwards. And let U be the the collar neighbourhood defined by the flow of this vector field. We decompose the forms θ and η on this neighbourhood as

$$\theta|_U = \alpha_t + f dt, \quad \eta|_U = \beta_t \wedge dt + \gamma_t,$$

for some forms $\alpha_t, \beta_t \in \Omega^1(\partial C)$, $\gamma_t \in \Omega^2(\partial C)$ depending smoothly on t and a smooth function f on U. Now as $\iota_V(\theta|_U) = 0$, and V has the form ∂_t on U we see that $f = 0$. Furthermore, as $\theta|_U$

is closed and $d\theta|_U = d^\partial \alpha_t + \frac{\mathrm{d}}{\mathrm{d}t}\alpha_t \wedge dt$ we see that $\frac{\mathrm{d}}{\mathrm{d}t}\alpha_t = 0$, hence $\alpha_t = \alpha_0$. Similarly because $\mathcal{L}_V \eta = 0$, we have that $d(\iota_{\partial_t}\eta) = 0$. Hence $d\beta_t = 0$, which shows that $\beta_t = \beta_0$ and $d^\partial \beta_0 = 0$. Using the fact that η is closed, we find $d\gamma_t = 0$. Therefore $\gamma_t = \gamma_0$ and $d^\partial \gamma_0 = 0$. We note that $\gamma_0 = \eta|_{\partial C}$ and hence $\gamma_0^n = 0$. Because (η, θ) is cosymplectic we thus have $\gamma_0^{n-1} \wedge \beta \wedge \alpha \wedge dt \neq 0$. This shows that $(\gamma_0, \alpha_0, \beta_0)$ is a two-cosymplectic structure and thus finishes the proof. $\square$

We are now ready to state the main theorem of this chapter. The local form we have obtained in the previous proposition will allow us to mimic the proofs in the previous section.

Theorem 3.5.7. *Let (C, η, θ) be a cosymplectic manifold with boundary of s-type. Then there exists a symplectic foliation $(\mathcal{F}, \omega)$ on C which is tame near the boundary. Let V be a vector field which satisfies the assumptions in Proposition 3.5.6.ii), then the symplectic structure at the boundary leaf is given by:*

$$\omega|_{\partial C} = \eta|_{\partial C} + (\iota_V \eta)|_{\partial C} \wedge \theta|_{\partial C}.$$

Proof. Let U be the collar neighbourhood from Proposition 3.5.6.iii), and let (γ, α, β) be the corresponding two-cosymplectic structure. Let (f, g) and (a, b) be good pairs of functions satisfying the following properties:

- $f(t) > 0$ for all $t \in (0, 1)$,
- a, b are constant near 0,
- $af + bg > 0$.

Define

$$\theta' = \begin{cases} \theta & \text{on } C \setminus U, \\ f\alpha - g dt & \text{on } U, \end{cases} \qquad \eta' = \begin{cases} \eta & \text{on } C \setminus U, \\ \gamma + \beta \wedge (a dt + b\alpha) & \text{on } U. \end{cases}$$

Note that these forms are smooth because of the local form of (η, θ) on U. We have

$$\theta' \wedge d\theta' = \begin{cases} \theta \wedge d\theta & \text{on } C \setminus U, \\ (f\alpha + g dt) \wedge (df \wedge \alpha + f d\alpha + dg \wedge dt) & \text{on } U, \end{cases}$$

which vanishes identically. Similarly $d\eta'|_U = \beta \wedge db \wedge \alpha$, hence

$$d\eta' \wedge \theta' = \begin{cases} d\eta \wedge \theta & \text{on } C \setminus U, \\ \beta \wedge db \wedge \alpha \wedge (f\alpha - g dt) & \text{on } U, \end{cases}$$

also vanishes. Finally because $\gamma^n = 0$ we have

$$\eta'^n \wedge \theta' = \begin{cases} \eta^n \wedge \theta & \text{on } C \setminus U, \\ -(n-1)(af + bg)\gamma^{n-1} \wedge \beta \wedge \alpha \wedge dt & \text{on } U, \end{cases}$$

which is nowhere vanishing because (γ, α, β) is a two-cosymplectic structure and by choice of a, b, f and g. Hence (η', θ') defines a codimension-one symplectic foliation. Because f and g are chosen to vanish up to infinite order at $t = 0$, we have by Lemma 2.1.4 that the foliation is tame near the boundary. Because a and b are chosen to be constant near $t = 0$, we see that η' is closed near the boundary. We conclude that the symplectic foliation is tame near the boundary which finishes the proof. □

Remark 3.5.8. When the cosymplectic structure (η, θ) is such that $\theta = f^*d\varphi$ for some fibration $f : C \to S^1$, we see that the foliation constructed in Theorem 3.5.7, using $(\eta, -f^*d\varphi)$, coincides with the foliation constructed in Lemma 3.4.5. Hence the leaves in the interior are diffeomorphic to the leaves of the fibration.

Using Theorem 3.5.7 we are able to reprove Lemma 3.5.1:

Proof of 3.5.1 using Theorem 3.5.7. We show that $B \times D^2$ admits a cosymplectic structure with boundary of s-type. Consider $\omega_{D^2} = r d\varphi \wedge dr$ and the vector field V, defined on a neighbourhood U of the boundary, given by $V = -(1/r)\partial_r$. Then we have $\iota_V \omega_{D^2} = d\varphi$. Extend V trivially to a vector field V' on $B \times U$. Now we extend η and θ trivially to $B \times D^2$ to obtain a cosymplectic structure $(\eta + \omega_{D^2}, \theta)$. We check that this boundary is of s-type.
Because V is nowhere vanishing, so if V'. Because V is clearly transverse to ∂D^2 we also have that V' is transverse to $B \times \partial D^2$. Clearly $\iota_{V'}\theta = 0$, and $\mathcal{L}_{V'}(\eta + \omega_{D^2}|_{B\times U}) = \mathcal{L}_V \omega_{D^2}|_U = 0$. Finally because of dimension reasons $\eta + \omega_{D^2}|_{M\times \partial D^2}$ is nowhere symplectic. We conclude that $(\eta + \omega_{D^2}, \theta)$ has boundary of s-type, hence we apply symplectic turbulization to obtain the required foliation. □

S-type at the boundary

For symplectic manifolds with boundary (M, ω) we saw in Proposition 3.5.6 that the existence of a vector field X, defined near the boundary, which satisfies $\mathcal{L}_X \omega|_{\partial M} = 0$ implied the existence of another vector field $Y \in \mathfrak{X}(U)$ satisfying $\mathcal{L}_Y \omega|_U = 0$ on some collar neighbourhood U of the boundary. For cosymplectic manifolds we can ask a similar question, however it does not suffice to impose conditions only at the boundary. We will need the following definition:

Definition 3.5.9. Let $R \in \mathfrak{X}(M)$, we define the complex of R**-basic** differential forms

$$\Omega^\bullet_{R-bas}(M) := \{\alpha \in \Omega^\bullet(M) : \iota_R \alpha = 0, \mathcal{L}_R \alpha = 0\}.$$

The R**-basic cohomology** $H^\bullet_{R-bas}(M)$ is defined as the cohomology of this complex.

Lemma 3.5.10. *Let (C^{2n+1}, η, θ) be a cosymplectic manifold for which $\eta|_{\partial C}$ is nowhere symplectic. Let R be the Reeb vector field of (η, θ) (i.e. $\iota_R\theta = 1, \iota_R \eta = 0$), and let $\iota : \partial C \to U$ denote the inclusion map. Then the following are equivalent:*

- *There exits a nowhere vanishing vector field V in a neighbourhood of the boundary U such that*
  - *V is transverse to the boundary,*
  - *$(\mathcal{L}_V\eta)|_U = 0$,*
  - *$(\iota_V\theta)|_U = 0$.*
- *There exits a nowhere vanishing vector field W in a neighbourhood of the boundary U such that*
  - *W is transverse to the boundary,*
  - *$(\mathcal{L}_W\eta)|_{\partial C} = 0$,*
  - *$(\iota_W\theta)|_{\partial C} = 0$,*
  - *$\iota^* : H^1_{R-bas}(U) \to H^1_{R|_{\partial C}-bas}(\partial C)$ is surjective.*

Proof. We first note that as $\eta|_{\partial C}$ is nowhere symplectic, R is tangent to ∂C and we can consider $R|_{\partial C} \in \mathfrak{X}(\partial C)$.
"$\Leftarrow$": We observe that $[(\iota_V\eta)|_{\partial C}] \in H^1_{R|_{\partial C}-bas}(\partial C)$. Hence there exists $[\gamma'] \in H^1_{R-bas}(U)$ such that $(\iota_V\eta)|_{\partial C} - \gamma'|_{\partial C} = df$ for some $f \in C^\infty(\partial C)$. Using the identification $U \simeq \partial C \times [0,1)$, we define an extension $\widetilde{f} \in C^\infty(U)$ of f by $\widetilde{f}(x,t) = (1-t)f(x)$. We observe that $(d\widetilde{f})(R) = 0$, hence $\gamma(R) = 0$. By non-degeneracy of $\eta|_{\ker\theta}$ we can find $W \in \ker\theta$ such that $\gamma|_{\ker\theta} = \iota_W\eta$. Because $\gamma(R) = 0$ we thus conclude that $\gamma = \iota_W\eta$. By definition of W we have $(\iota_W\theta)|_U = 0$ and as γ is closed we find that $(\mathcal{L}_W\eta)|_U = 0$. We are thus left to show that W is transverse to the boundary. As V is transverse to ∂C we have that $(\iota_V(\eta^n \wedge \theta))|_{\partial C}$ is a volume form. Because $(\iota_V\eta)|_{\partial C} = (\iota_W\eta)|_{\partial C}$ a direct computation shows that $(\iota_V(\eta^n \wedge \theta))|_{\partial C} = (\iota_W(\eta^n \wedge \theta))|_{\partial C}$, which implies that W is transverse to ∂C.
"$\Rightarrow$": On U we can write $R|_U = R|_{\partial C} + f\frac{\partial}{\partial t}$, for some $f \in C^\infty(U)$. Using the local form of η on U as in Proposition 3.2.6 we see that $\iota_{R|_U}\eta|_U = f\beta = 0$. Because $\beta \neq 0$ we conclude that $f = 0$, hence $R|_U = R|_{\partial C}$. Now let $[\alpha] \in H^1_{R|_{\partial C}-bas}(\partial C)$, then $[\mathrm{pr}_1^*\alpha] \in H^1_{R-bas}(U)$ and $\iota^*[\mathrm{pr}_1^*\alpha] = [\alpha]$. This finishes the proof that ι^* is surjective. $\square$

Chapter 4

Open book decompositions

In this chapter we study *open book decompositions*. In the previous chapters we discussed how symplecitic foliations arise from glueing symplectic foliations which are tame near the boundary. For instance in Example 3.4.8 we have constructed the Reeb foliation on S^3 using a decomposition of S^3. Open book decompositions will generalise this decomposition. These open books will be one of our main tools in constructing symplectic foliations in the next chapter.

In Section 4.1 we will recall the basic definitions of open book decomposition. Furthermore, we will describe a procedure which constructs open books from certain complex valued functions. In Section 4.2 we will consider the interplay between open books and contact geometry. In Section 4.3 we will describe one possible way to obtain symplectic foliations from open book decompositions. In Section 4.4 we will describe a method of constructing open book decompositions from complex line bundles which admit sections transverse to the zero-section. After this we will specialize to complex line bundles arising from integral symplectic manifolds. We will call the open books which we obtain in this manner *Donaldson open books*, and these open books will be used in the next chapter to construct symplectic foliations.
Orientations: The convention for orientations induced on the boundary, will be "inward normal first". That is, for an oriented manifold M we say that a frame $\{X_1, \dots, X_n\}$ on ∂M is positively oriented if $\{Y, X_1, \dots, X_n\}$ is a positively oriented frame on M. Here Y denotes the inward pointing normal vector.

4.1 Definitions and constructions of open book decompositions

In this section we recall the definition of an open book decomposition of a manifold. These decompositions will carry a natural foliation outside of a codimension-two submanifold, pointing towards their use in constructing foliations.

4.1.1 Basics of open book decompositions

Definition 4.1.1. An **open book decomposition** of a manifold M consists of:

- A codimension-two embedded submanifold B with trivial normal bundle.
- A submersion $\pi : M \setminus B \to S^1$.

Furthermore, B admits a tubular neighbourhood $\tau(B)$ together with a diffeomorphism $\psi : \tau(B) \to B \times D^2$ such that the following diagram commutes:

$$\begin{array}{ccc} \tau(B) \setminus B & \xrightarrow{\simeq} & B \times (D^2 \setminus \{0\}) \\ & \searrow^{\pi} & \downarrow \mathrm{Ang} \\ & & S^1 \end{array}$$

Here, $\mathrm{Ang} : D^2 \setminus \{0\} \to S^1, z \mapsto \frac{z}{|z|}$ denotes the angular function. We use the following terminology:

- The submanifold B is called the **binding** of the open book.
- $P := \pi^{-1}(1)$ is called the (open) **page** of the open book.
- A tubular neighbourhood satisfying the above properties is called **adapted**. Note that the tubular neighbourhood is not part of the data of an open book decomposition.

We furthermore use the following notation:

- $\tau_\varepsilon(B) = B \times \{x \in D^2 : \|x\| \leq \varepsilon\} \subset \tau(B)$,
- $C_\varepsilon = \overline{M \setminus \tau_\varepsilon(B)}$,
- $\pi_\varepsilon := \pi|_{C_\varepsilon}$,
- The ε**-page**: $P_\varepsilon := \pi_\varepsilon^{-1}(\{1\})$.

The local form of π around the binding gives the following result:

Proposition 4.1.2. *Both $\overline{P}$ and P_ε are manifolds with boundary, with boundary diffeomorphic to B. Furthermore, $\overline{P}$ and P_ε are diffeomorphic.*

Example 4.1.3. The motivating example for the name 'open book' is the following decomposition of $\mathbb{R}^3$. As a binding B we choose the z-axis, and we define $\pi : \mathbb{R}^3 \setminus B \to S^1$ by $\pi(x, y, z) = (x, y)/ |(x, y)|$. Clearly the z-axis has trivial normal bundle and π is a submersion. Let $\tau(B) = \{(x, y, z) : |x|^2 + |y|^2 \leq 1\} = B \times D^2$ which is clearly a tubular neighbourhood of B, on which π satisfies the required local form. This decomposition is visualized in Figure 4.1.

Example 4.1.4. An even simpler example is the open book decomposition of $\mathbb{C}$. As binding we choose the origin, and we let $\pi_{\mathrm{st}} : \mathbb{C}^* \to S^1$ be the angle function. We will call this open book the **standard open book decomposition of** $\mathbb{C}$, and we will show in Section 4.1.2 that any open book decomposition can be realized from this one.

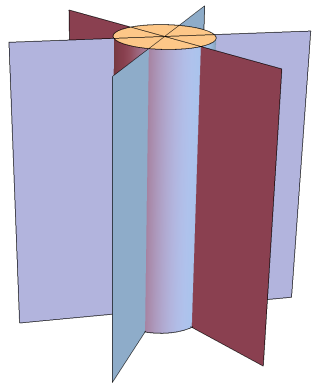


Figure 4.1: A visualisation of the open book decomposition of $\mathbb{R}^3$. The binding is the z-axis, and the pages are the planes of constant cylindrical coordinate.

Foliations from open books

If a manifold admits an open book decomposition under one additional assumption it admits a foliation:

Theorem 4.1.5 ([Law71])**.** *Let $\pi : M \setminus B \to S^1$ be an open book decomposition on a compact manifold. If the binding B admits an S^1-fibration, then M admits a codimension-one foliation.*

Proof. Decompose $M = \tau(B) \cup C$, with $C = \overline{M \setminus \tau(B)}$. We have that C admits a circle fibration, and by the local form of π around B, we see that $\pi|_{\partial C} : \partial C \to S^1$ is a submersion. Hence we can apply Lemma 3.4.5 to obtain a foliation tame near the boundary on C. Because B fibres over S^1 so does $\tau(B) \simeq B \times D^2$. This fibration clearly satisfies the assumption that the restriction to the boundary is a submersion. Hence we can apply Lemma 3.4.5 again, and glue the resulting foliation to the foliation on C. We conclude that M admits a codimension-one foliation. □

Monodromies

An open book decomposition gives the means to decompose a manifold as $M = \tau(B) \cup_{\partial C} C$, where C is the closure of the complement of $\tau(B)$. Now M can be studied by studying both components separately. By assumption $\tau(B) \simeq B \times D^2$, the other component also has a good description in terms of *monodromies*:

Definition 4.1.6. A diffeomorphism $\varphi : P \to P$ is called a **monodromy** of the open book $\pi : M \setminus B \to S^1$ if $P \times_{\mathbb{Z}} \mathbb{R}$, is diffeomorphic to $M \setminus B$. Here the $\mathbb{Z}$-action on $P \times \mathbb{R}$ is defined by

$$n \cdot (p, t) = (\varphi^n(p), t - n).$$

A monodromy is called **adapted** if is the identity on a neighbrouhood of the boundary of the closed page.

Proposition 4.1.7. *Every open book decomposition $\pi : M \setminus B \to S^1$ on a compact manifold admits an adapted monodromy.*

Proof. Let $X \in \mathfrak{X}(M \setminus B)$ be a vector field such that $\pi_* X = \frac{\partial}{\partial\varphi}$, where $\frac{\partial}{\partial\varphi}$ denotes the angular vector field. Because of the local form of π, we can take X such that on an adapted tubular neighbourhood $\tau(B)$ we have $\Phi_*(X) = \frac{\partial}{\partial\varphi}$, where Φ denotes the diffeomorphism $\tau(B) \setminus B \simeq B \times (D^2 \setminus \{0\})$. Hence the time-one-flow, φ_X^1, is the identity on $\tau(B) \setminus B$. The proof that φ_X^1 gives a monodromy goes along the same lines as the proof of Proposition A.2.5. □

We will now give an alternative description of open book decompositions:

Definition 4.1.8. An **abstract open book** is a pair (Q, Φ), where

- Q is a compact manifold with boundary ∂Q,
- $\Phi : Q \to Q$ is a diffeomorphism which is the identity around ∂Q.

Because Φ is the identity near the boundary, we have that the mapping torus of (Q, Φ) has boundary given by

$$\partial(Q \times_{\mathbb{Z}} \mathbb{R}) = \partial Q \times S^1.$$

This allows us to define the following manifold:

$$M(Q, \Phi) := (Q \times_{\mathbb{Z}} \mathbb{R}) \cup_{\partial Q \times S^1} \partial Q \times D^2.$$

The following is proven easily:

Proposition 4.1.9. *Let $B := \partial Q \times \{0\}$, and define $\pi : M(Q, \Phi) \setminus B \to S^1$ by*

$$\pi|_{(Q \times_{\mathbb{Z}} \mathbb{R}) \setminus B}([q, t]) = [t]$$
$$\pi|_{(\partial Q \times D^2) \setminus B}(q, (r, \theta)) = \theta.$$

Then π defines an open book decomposition of $M(Q, \Phi)$.

The following proposition shows that we can also reverse this process.

Proposition 4.1.10. *Let $\pi : M \setminus B \to S^1$ be an open book decomposition on a compact manifold, and let φ be an adapted monodromy. Then $M \simeq M(Q, \Phi)$ with*

$$Q = P_\varepsilon, \quad \Phi = \varphi|_{P_\varepsilon},$$

for $\varepsilon > 0$ small enough.

Proof. By definition $P \times_{\mathbb{Z}} \mathbb{R} \simeq M \setminus B$. Because φ is the identity near the boundary, we see that $P_\varepsilon \times_{\mathbb{Z}} \mathbb{R} \simeq C_\varepsilon$. Hence $M(P_\varepsilon, \varphi|_{P_\varepsilon}) \simeq C_\varepsilon \cup \tau_\varepsilon(B) = M$. □

The last proposition gives us a good description of the outside component of the open book. Combining this with the form of the tubular neighbourhood we obtain the following:

$$M \simeq (B \times D^2) \cup P_\varepsilon \times_{\mathbb{Z}} \mathbb{R}.$$

4.1.2 Pullback open books

In this section we will give an alternative description of open book decompositions, making use of the standard open book decomposition of $\mathbb{C}$ as in Example 4.1.4.

Definition 4.1.11. We say that a function $f : M \to N$ is **transverse to an open book decomposition** $\pi : N \setminus B \to S^1$ if:

- f is surjective,
- f is transverse to B,
- f is transverse to every page.

The next theorem shows that one can obtain open book decompositions using the standard open book decomposition of $\mathbb{C}$.

Theorem 4.1.12. *Let M be a compact manifold and let $f : M \to \mathbb{C}$ be transverse to $\pi_{\mathrm{st}} : \mathbb{C}^* \to S^1$. Define $B := f^{-1}(\{0\})$, and $\pi : M \setminus B \to S^1$ by*

$$\pi(z) = \frac{f(z)}{|f(z)|}.$$

Then $\pi : M \setminus B \to S^1$ is an open book decomposition of M.

Proof. B **submanifold**: Because f is transverse to $\{0\}$ this is immediate.
π **submersion**: Because f is transverse to the page we have for any $\varphi \in S^1$ and all $x \in f^{-1}(\pi_{\mathrm{st}}^{-1}(\{\varphi\}))$ that

$$T_{f(x)}\mathbb{C} = df_x(T_xM) \oplus T_{f(x)}\pi_{\mathrm{st}}^{-1}(\{\varphi\}).$$

As π_{st} is a submersion the above proves that $f^*\pi_{\mathrm{st}}$ is a submersion.
Tubular neighbourhood: We note that the normal bundle to a regular level set always has trivial normal bundle. Let $\mathcal{U}$ be any tubular neighbourhood of B and let $q : \mathcal{U} \to B$ denote the retraction. We define

$$\varphi : \mathcal{U} \to B \times \mathbb{C}, \quad \varphi(z) = (q(z), f(z)).$$

Let $z \in B$, then $(d\varphi)_z$ decomposes as

$$dq_z \oplus df_z : T_zB \oplus T_z\mathcal{N}(B) \to T_zB \oplus T_{f(z)}\mathbb{C}.$$

As z is a regular value of f and $T_zB = \ker df$, we find that $df_z : T_z\mathcal{N}(B) \to T_{f(z)}\mathbb{C}$ is surjective. Because q is the identity on B we conclude that $(d\varphi)_z$ is an isomorphism for all $z \in B$.
By the inverse function theorem, there exists a neighbourhood of every $y \in B$ such that φ

restricted to this neighbourhood is a diffeomorphism. By taking the union of all these neighbourhoods, and taking the intersection with $\mathcal{U}$, we get an open neighbourhood $\widetilde{\mathcal{U}} \subset \mathcal{U}$ such that $\varphi|_{\widetilde{\mathcal{U}}}$ is a diffeomorphism onto its image. Clearly we have that $B \times \{0\}$ is in this image. For all $z \in B$ we have that there is an open U_z and $\varepsilon_z > 0$ such that $U_z \times D_{\varepsilon_z} \subset \varphi(\widetilde{\mathcal{U}})$. This collection forms a cover of B and by compactness of B we can take a finite subcover, $\{U_k \times D_{\varepsilon_k}\}$. Now we have that $U_k \times D_{\min_k \varepsilon_k}$ is a subset of $\cap_k U_k \times D_{\varepsilon_k}$. We thus conclude that $B \times D_{\min_k \varepsilon_k} \subset \varphi(\widetilde{\mathcal{U}})$ and we can define an open neighbourhood $\tau(B) := \varphi^{-1}(B \times D_{\min_k \varepsilon_k}) \cap \mathcal{U}$. Finally we see that $\varphi|_{\tau(B)}$ is a diffeomorphism onto $B \times D_{\min_k \varepsilon_k}$. By the explicit form of φ we directly have that π has the required local form on $\tau(B)$. □

It turns out that any open book decomposition can be obtained by the previous construction.

Theorem 4.1.13. *Let M be a compact manifold. There is a 1:1 correspondence:*

$$\left\{ \begin{array}{c} \textit{Open book decompositions} \\ \pi : M \setminus B \to S^1 \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Functions } f : M \to \mathbb{C} \\ \textit{transverse to } \pi_{st} : \mathbb{C}^* \to S^1 \end{array} \right\} \Big/ \sim$$

where $f \sim g$ if $f^{-1}(\{0\}) = g^{-1}(\{0\})$ and there exists $\rho : M \setminus B \to \mathbb{R}_{>0}$ such that $f = \rho g$.

Proof. Given an open book decomposition $\pi : M \setminus B \to S^1$ we will define $f : M \to \mathbb{C}$ in the following manner. Define $f|_B = 0$ and write

$$f = f_r e^{i f_\theta}, \quad f_r, f_\theta : M \to \mathbb{R}.$$

We define f_θ by $f_\theta|_{\pi^{-1}(\{\varphi\})} = \varphi$. Consider $\rho : D^2 \to \mathbb{R}_{\geq 0}$ such that

$$\rho(\varphi, r) = \begin{cases} r, & \text{if } r \leq \varepsilon, \\ 1, & \text{if } r > 1 - \varepsilon. \end{cases}$$

Now we define

$$f_r(z) = \begin{cases} \rho(r) & \text{on } \tau(B) \text{ with } z = (b, r) \\ 1 & \text{on } M \setminus \tau(B). \end{cases}$$

To see that f defined in this way is smooth, we note that on a smaller tubular neighbourhood $\tau_\varepsilon(B) \subset \tau(B)$, f coincides with $\varphi^* \mathrm{pr}_2$, where $\mathrm{pr}_2 : B \times D^2 \to D^2$ denotes the projection onto the second factor. By construction we have

$$\begin{aligned} \frac{f(z)}{|f(z)|} &= e^{i f_\theta(z)} \\ &= \pi(z). \end{aligned}$$

so f is smooth. We now have to show that f is transverse to $\pi_{\mathrm{st}} : \mathbb{C}^* \to S^1$. To show that 0 is a regular value of f we note that on $\tau_\varepsilon(B)$ the function f coincides with $\varphi^* \mathrm{pr}_2$. As pr_2 is a

submersion this shows that 0 is a regular value of f.
Because π is a submersion and $\pi = f^*\pi_{\mathrm{st}}$ we have that df_z is surjective onto the complement of $\ker(d_z\pi_{\mathrm{st}})$. Because $\ker(d_z\pi_{\mathrm{st}})$ is precisely the page though z, we conclude that f is transverse to π_{st}.
To finish the proof we have to show that given $\pi : M \setminus B \to S^1$ such that $\pi = f/|f| = g/|g|$, there exists $\rho : M \setminus B \to \mathbb{R}_{>0}$ such that $f = \rho g$. One easily checks that f_r/g_r satisfies these properties. □

Remark 4.1.14. The above correspondence gives another interpretation of open book decompositions. Because $\pi_{\mathrm{st}} : \mathbb{C}^* \to S^1$ is precisely the angular function, an open book decomposition can be seen as some sort of angular function on the manifold. Now the role of the origin is played by a codimension-two submanifold, and the rays in $\mathbb{C}^*$ are replaced by the pages of the open book.

4.2 Open books and contact geometry

Open book decompositions are used widely in both foliation theory and contact geometry. For instance the construction of a contact structure on T^{2n+1} [Bou02] made use of open book decompositions. The following definition gives the open books which are most natural to use in contact geometry.

Definition 4.2.1. An open book decomposition $\pi : M \setminus B \to S^1$ is said to **support** a contact form $\alpha \in \Omega^1(M)$ if the following conditions are satisfied:

i. The form $\alpha_B := \alpha|_B$ is a contact form.

ii. The restriction of $d\alpha$ to every open page is symplectic, that is $d\alpha|_{\pi^{-1}(\{\varphi\})}$ is symplectic for all $\varphi \in S^1$.

iii. The following two orientations on B coincide: The orientation induced by α_B, and the orientation induced by viewing B as the boundary of P_ε, where P_ε is oriented by $d\alpha|_{P_\varepsilon}$.

Remark 4.2.2. It might seem more natural to ask in (iii) in the above definition that B is oriented as the boundary of $\overline{P}$. However one must be careful here. Although $\overline{P}$ and P_ε are diffeomorphic, $d\alpha|_{\overline{P}}$ is not symplectic. Indeed as the Reeb vector field of α is tangent to B it is also tangent to $\overline{P}$. Hence $d\alpha|_{\overline{P}}$ has a non-trivial kernel.

Remark 4.2.3. Open book decompositions are useful for constructing foliations because they carry a foliation outside of a codimension-two submanifold. When an open book decompositions supports a contact form, assumption ii) in Definition 4.2.1 ensures that this foliation is in fact symplectic. This motivates that to construct symplectic foliations it is useful to start with open books supporting a contact form.

Remark 4.2.4. In this setting, all pieces of the open book carry canonical orientations. We have that B caries the orientation induced by α_B and P the orientation induced by $(d\alpha)|_P$. When considering tubular neighbourhoods around the binding it is useful to consider those open books which are also adapted to these orientations. That is, tubular neighbourhoods $\tau(B)$ such that the orientation on $\tau(B)$ coincides with the product orientation on $B \times D^2$, where D^2 is endowed with the standard orientation. We will call these tubular neighbourhoods **oriented adapted neighbourhoods**. Note that if an adapted neighbourhood is not oriented one can simply apply an orientation reversing diffeomorphism of D^2 to make it so.

We will now give an alternative description of the second and third property in Definition 4.2.1.

Lemma 4.2.5. *Let $\pi : M \setminus B \to S^1$ be an open book decomposition and $\alpha \in \Omega^1(M)$ a contact form. Then (ii) and (iii) from Definition 4.2.1 are equivalent to the following: There exists a smooth positive function $f_{\pi,\alpha}$ on M such that*

$$(d\pi)(R_\alpha) = f_{\pi,\alpha} \frac{\partial}{\partial \varphi}, \tag{4.1}$$

where R_α denotes the Reeb vector field of α.

Proof. **(ii)**: We will first show that (ii) is equivalent to the existence of a nowhere-vanishing function satisfying (4.1). The Reeb vector field of α spans the kernel of $d\alpha$. Hence we see that for any $\varphi \in S^1$ the restriction $d\alpha|_{\pi^{-1}(\{\varphi\})}$ is symplectic if and only if

$$T_x M = T_x(\pi^{-1}(\{\varphi\})) \oplus \mathbb{R} \cdot (R_\alpha)_x \quad \text{for all } x \in f^{-1}(\{\varphi\}).$$

Or more concisely, by seeing the pages as a foliation $\mathcal{F}$ on $M \setminus B$ we have

$$T(M \setminus B) = T\mathcal{F} \oplus \mathbb{R} \cdot R_\alpha,$$

because $T\mathcal{F}$ is precisely $\ker(d\pi)$. In turn this is equivalent to the fact that

$$(d\pi)(R_\alpha) = f_{\pi,\alpha} \frac{\partial}{\partial \varphi},$$

for some non-vanishing smooth function $f_{\pi,\alpha}$ on M.

(iii): We will now show that $f_{\pi,\alpha}$ is positive is and only if (iii) is satisfied. Let $2n+1 = \dim M$. We work on an adapted oriented tubular neighbourhood. Let $Y_1, \ldots, Y_{2n-1} \in \mathfrak{X}(B_0)$ be a local frame on B, where B is viewed as $\partial \overline{P}$ and we thus denoted it as B_0. Now transport this local frame to a local frame $\widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1} \in \mathfrak{X}(B_\varepsilon)$ on B, but now with B viewed as ∂P_ε and we thus denoted it as B_ε. We can pick a normal vector for $B_0 \subset \overline{P}$, such that its corresponding normal vector for $B_\varepsilon \subset \partial P_\varepsilon$ is precisely ∂_r. Hence the original frame is positively oriented if and only if $\widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}, \partial_r \in \mathfrak{X}(P)$ is positively oriented. Because $\iota_R(\alpha \wedge (d\alpha)^n) = (d\alpha)^n$ we see that this

is the case if and only if $R, \widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}, \partial_r \in \mathfrak{X}(M)$ is positively oriented.
By $(d\pi)(R) = f_{\pi,\alpha}\partial_\varphi$, and the local form of π on the tubular neighbourhood, we have that on $B \times D^2$ the Reeb vector fields takes the form $R = R_B + f_{\pi,\alpha}\partial_\varphi$. Because the tubular neighbourhood is oriented, we see that $\widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}$ is positively oriented with respect to α_B if and only if $\widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}, \partial_r, \partial_\varphi$ is positively oriented on $\tau(B)$. Because $f_{\pi,\alpha} \neq 0$ we see that this is the case if and only if $\widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}, \partial_r, \operatorname{sgn}(f_{\pi,\alpha})R$ is positively oriented on $\tau(B)$. This is the case if and only if $\operatorname{sgn}(f_{\pi,\alpha})R, \widetilde{Y}_1, \ldots, \widetilde{Y}_{2n-1}, \partial_r$ is positively oriented. This proves that the orientations on B induced by P and α_B coincide if and only if $f_{\pi,\alpha} > 0$. □

Remark 4.2.6. Note that the above does not imply that R_α is π-projectible to a vector field on S^1. The lemma shows that property (ii) and (iii) in Definition 4.2.1 can be considered as follows. Property (ii) ensures that the Reeb vector field is transverse to the pages, and property (iii) ensures that the Reeb vector field points in the "counter-clockwise direction".

Starting with a contact structure, we can always find an open book supporting it:

Theorem 4.2.7 ([Gir02])**.** *To any contact structure ξ on a closed manifold M of dimension at least equal to three there exists an open book decomposition (B, π) supporting ξ.*

Conversely if we start with an open book decomposition, under some conditions we can find a contact form which is supported by it:

Theorem 4.2.8 (Thm 7.3.3, [Gei08])**.** *Let (Q, Φ) be an abstract open book with the following properties:*

- *Q admits an exact symplectic form $\omega = d\beta$.*
- *The vector field Y defined by $\iota_Y \omega = \beta$ is transverse to ∂Q, pointing outwards.*
- *The monodromy Φ is a symplectomorphism of (Q, ω).*

Then $M = M(Q, \Phi)$ admits a contact structure supported by the open book decomposition $\pi : M \setminus B \to S^1$.

An open book supporting a contact structure admits natural monodromies:

Proposition 4.2.9. *Let $\pi : M \setminus B \to S^1$ be an open book decomposition on a compact manifold which supports a contact form $\alpha \in \Omega^1(M)$. Then the flow of a suitable scaled version of the Reeb vector field induces a monodromy of π, which preserves the symplectic structure on the page.*

Proof. By Remark 4.2.5 we have that $(d\pi)(R_\alpha) = f_{\pi,\alpha}\frac{\partial}{\partial\varphi}$, with $f_{\pi,\alpha}$ a positive function on M. Now define

$$X = \frac{1}{f_{\pi,\alpha}} R_\alpha.$$

This vector field satisfies $\pi_* X = \frac{\partial}{\partial\varphi}$. As in Proposition 4.1.7 we get that the time-one-flow of X gives a monodromy of the open book.
Note that $d\iota_{R_\alpha} d\alpha = 0$, hence by the Cartan's formula we have $\mathcal{L}_{R_\alpha}(d\alpha) = 0$. This shows that $(\varphi^t_X)^*(d\alpha)$ is constant. Hence we find that $(\varphi^1_X)^*(d\alpha) = d\alpha$, which finishes the proof. □

Remark 4.2.10. We note that the monodromy of the previous proposition is never adapted. Indeed as the binding is a contact submanifold, around the binding the Reeb vector field R_α always has a component in the direction of it.

4.3 Symplectic foliations from open books

In this section we will prove a theorem similar to Theorem 4.2.8, but now for symplectic foliations. Continuing the general motto of obtaining symplectic foliations via decompositions of manifolds, the natural question is what further assumptions we need for a manifold which admits an open book decomposition to admit a symplectic foliation. One answer to this question is the following:

Theorem 4.3.1. *Let M^{2n+1} be a compact manifold. Assume that we have the following:*

- *An open book decomposition $\pi : M \setminus B \to S^1$.*
- *A symplectic form $\omega \in \Omega^2(\overline{P})$ which is of cosymplectic type at the boundary.*
- *An adapted monodromy $\varphi : P \to P$ of the open book which satisfies $\varphi^*\omega = \omega$.*

Then M admits a codimension-one symplectic foliation.

Remark 4.3.2. The disadvantage of this theorem is that it is rather difficult to apply in examples. The main reason for this is the existence of adapted symplectic monodromies. It is always possible to perturb a monodromy to one that is adapted, but by doing so it is hard to keep the monodromy symplectic. Therefore we will not make use of the above theorem in our constructing of a symplectic foliation on S^5 and we will proceed differently. Still this theorem is the natural generalisation of Theorem 4.1.5

Proof of Theorem 4.3.1. Recall that M admits the following decomposition:

$$M \simeq (\overline{P} \times_{\mathbb{Z}} \mathbb{R}) \cup_{\partial\overline{P} \times S^1} (\partial\overline{P} \times D^2),$$

where $\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R}) \simeq \partial\overline{P} \times S^1$ because φ is the identity near the boundary (here we used the diffeomorphism between $\overline{P}$ and P_ε given by Proposition 4.1.2). We will construct symplectic foliations tame near the boundary on $\partial\overline{P} \times D^2$ and $\overline{P} \times_{\mathbb{Z}} \mathbb{R}$ for which the symplectic structures on the boundary leaves coincide. Then by glueing the foliations together we obtain a symplectic foliation on M. Let X be a vector field on $U \subset \overline{P}$ transverse to $\partial\overline{P}$ such that $\mathcal{L}_X \omega_U = 0$.

Step 1: Foliation on $\partial\overline{P} \times D^2$: By Lemma 3.2.4 the pair $(\omega|_{\partial\overline{P}}, (\iota_X\omega)|_{\partial\overline{P}}) =: (\eta, \theta)$ is a cosymplectic structure on $\partial\overline{P}$. We now apply Lemma 3.5.1 to $(\eta, -\theta)$ obtain a symplectic foliation on $\partial\overline{P} \times D^2$, tame near the boundary. The symplectic structure on the boundary leaf is given by

$$\eta + \theta \wedge d\varphi.$$

Step 2: Obtaining a cosymplectic structure on $\overline{P} \times \mathbb{R}$: Consider $\mathrm{pr}_1^*\omega \in \Omega^2(\overline{P} \times \mathbb{R})$. Let $\psi : \overline{P} \times \mathbb{R} \to \overline{P} \times \mathbb{R}$ denote the action defining $\overline{P} \times \mathbb{R}$. Because $\varphi^*\omega = \omega$ we also have that $\psi^*\mathrm{pr}_1^*\omega = \mathrm{pr}_1^*\omega$, hence there exists a form $\Omega \in \Omega^2(P \times_{\mathbb{Z}} \mathbb{R})$ such that $p^*\Omega = \mathrm{pr}_1^*\omega$. Where $p : \overline{P} \times \mathbb{R} \to P \times_{\mathbb{Z}} \mathbb{R}$ denotes the quotient map. A similar argument for pr_2^*dt yields a form $\zeta \in \Omega^1(P \times_{\mathbb{Z}} \mathbb{R})$ such that $p^*\zeta = \mathrm{pr}_2^*dt$. Because p^* is injective and clearly $\omega^n \wedge dt \neq 0$ (we drop the projections from the notation) we see that $p^*(\omega^n \wedge dt) = \Omega^n \wedge \zeta \neq 0$. This shows that (Ω, ζ) is a cosymplectic structure on $\overline{P} \times \mathbb{R}$.
Step 3: $(\overline{P} \times \mathbb{R}, \Omega, \zeta)$ has boundary of s-type: We shrink the neighbourhood U, if necessary, such that $\varphi|_U = \mathrm{id}_U$. Furthermore, we shrink U such that $p(U \times \mathbb{R})$ is a collar neighbourhood of $\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})$. We now extend X trivially to a vector field on $U \times \mathbb{R}$, we then have that $(\varphi|_U)_*X = X$, hence X descends to a vector field V on $p(U \times \mathbb{R})$, which satisfies $V_{p(x)} = (dp)_x(X_x)$ for all $x \in U \times \mathbb{R}$.
We will now show that the vector field V satisfies the assumptions of boundary of s-type. Because $V = p_*X$ it directly follows that $\iota_V\theta = \iota_X dt = 0$. Similarly $\iota_V\Omega = p^*(\iota_X\omega)$, hence $\mathcal{L}_V\Omega = p^*(\mathcal{L}_X\omega) = 0$. Hence V satisfies the assumptions of Proposition 3.2.6.ii), provided that it is transverse to the boundary. To show this we note that $\Omega^n \wedge \zeta$ is a volume form on $P \times_{\mathbb{Z}} \mathbb{R}$, hence it suffices to show that $(\iota_V(\Omega^n \wedge \zeta))|_{\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})}$ is a volume form. We compute

$$\begin{aligned}\iota_V(\Omega^n \wedge \zeta) &= n(\iota_V\Omega) \wedge \Omega^{n-1} \wedge \zeta \\ &= np^*(\iota_X\omega) \wedge p^*\omega^{n-1} \wedge p^*dt \\ &= np^*(\iota_X\omega^n \wedge dt).\end{aligned}$$

Because X is transverse to $\partial\overline{P}$, we see that $(\iota_X\omega^n)|_{\partial\overline{P}} \neq 0$. Using the above computation and the fact that $p(\partial\overline{P} \times \mathbb{R}) = \partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})$, we conclude that $(\iota_X(\Omega^n \wedge \zeta))|_{\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})} \neq 0$, which finishes the proof that $(\overline{P} \times_{\mathbb{Z}} \mathbb{R}, \Omega, \zeta)$ has boundary of s-type.
Step 4: Conclusion: We can now apply Theorem 3.5.7 to obtain a symplectic foliation, with symplectic structure on the boundary leaf given by:

$$\Omega|_{\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})} + \iota_V(\Omega|_{\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})}) \wedge \zeta|_{\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})}.$$

To finish we have to show that $\partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R})$ is symplectomorphic to $\partial(B \times D^2)$. To do as such consider the diffeomorphism

$$\begin{aligned}\rho : \partial\overline{P} \times S^1 &\to \partial(\overline{P} \times_{\mathbb{Z}} \mathbb{R}) \\ (x, [t]) &\mapsto [(x, t)].\end{aligned}$$

We have that the following diagram commutes:

$$\begin{array}{ccc} \partial\overline{P}\times\mathbb{R} & \xrightarrow{\text{id}} & \partial(\overline{P}\times\mathbb{R}) \\ \big\downarrow{\scriptstyle \text{id}\times q} & & \big\downarrow{\scriptstyle p|_{\partial(\overline{P}\times\mathbb{R})}} \\ \partial\overline{P}\times S^1 & \xrightarrow{\rho} & \partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R}) \end{array}$$

Using the commutativity of this diagram we have that

$$\begin{aligned} \rho^*\Omega|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})} &= (\text{id}\times q)_*(p|^*_{\partial(\overline{P}\times\mathbb{R})}\Omega|_{\partial(\overline{P}\times\mathbb{R})}) \\ &= (\text{id}\times q)_*\omega|_{\partial\overline{P}\times\mathbb{R}} \\ &= \omega|_{\partial\overline{P}\times S^1} \\ &= \eta. \end{aligned}$$

Similarly we have

$$\begin{aligned} \rho^*\zeta|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})} &= (\text{id}\times q)_*(p|^*_{\partial(\overline{P}\times\mathbb{R})}\zeta|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})}) \\ &= (\text{id}\times q)_*(dt|_{\partial\overline{P}\times\mathbb{R}}) \\ &= d\varphi. \end{aligned}$$

As before we have that $p^*\iota_V\Omega = \iota_X\omega$, hence

$$\rho^*(\iota_V(\Omega|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})})) = \theta.$$

Combining the above we see that

$$\rho^*(\Omega|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})} + \iota_V(\Omega|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})})\wedge\zeta|_{\partial(\overline{P}\times_{\mathbb{Z}}\mathbb{R})}) = \eta + \theta\wedge d\varphi.$$

We can thus glue this foliation with the foliation obtained in step 1 to obtain a codimension one symplectic foliation on M. □

4.4 Open books from complex line bundles

In this section we associate to any complex line bundle which admits a transverse section, an open book decomposition on the associated principal S^1-bundle. Furthermore, when this complex line bundle arises from a symplectic manifold, we will show that this open book decomposition supports a certain contact form. The constructions in this section are inspired by the Hopf fibration. We will conclude this section by applying our general theory to the Hopf fibration to obtain the open book we will use to prove the existence of a symplectic foliation on S^5.

4.4.1 Principal bundles and vector bundles

We recall some of the basic relations between principal bundles and vector bundles.

Proposition 4.4.1. *There is a 1:1 correspondence between:*

$$\left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{complex line bundles} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Isomorphism classes of} \\ \textit{principal } S^1 - \textit{bundles} \end{array} \right\}$$

Where the map from the right to left is taking the S^1-bundle with respect to some Riemannian metric, and the map from left to right is taking the vector bundle associated to the representation

$$\rho : S^1 \to \mathrm{GL}(\mathbb{C})$$
$$\rho(\lambda)v = \lambda v.$$

We denote by $E(M, \mathbb{C})$ the vector bundle under this correspondence.

Under this correspondence we can relate sections of the line bundle to certain functions on the principal bundle.

Lemma 4.4.2 ([Cra16]). *Let $h : M \to S$ be a principal G-bundle, and let $\rho : G \to GL(V)$ be a representation of G. Then there is a 1:1 correspondence between $\Gamma(E(M, V))$ and G-equivariant functions, $C^\infty(M, V)^G$. Given $f \in C^\infty(M, V)^G$, $\sigma_f \in \Gamma(E(M, V))$ is induced by the map $M \to M \times V, x \mapsto (x, f(x))$.*

Lemma 4.4.3. *Let $f \in C^\infty(M, V)^G$, and let σ_f be the corresponding section of $h' : E(M, V) \to S$. Then 0 is a regular value of f if and only if σ_f is transverse to the zero-section.*

Proof. Choose $U \subset S$ such that $M|_{h^{-1}(U)} \simeq U \times G$ and $E|_{h'^{-1}(U)} \simeq U \times V$. Inspecting the definition of σ_f, we see that locally σ_f takes the following form:

$$\sigma_f : U \to E|_{h'^{-1}(U)} \quad x \mapsto (x, f(x)).$$

Hence locally $d\sigma_f = \mathrm{id} \oplus df$, from which it directly follows that σ_f is transverse to the zero-section if and only if 0 is a regular value of f. □

4.4.2 Open books on principal S^1-bundles associated to complex line bundles

Given a section of a complex line bundle, we can consider the corresponding equivariant function on the S^1-bundle. The idea is to apply Theorem 4.1.12 to obtain an open book decomposition on this S^1-bundle.

Theorem 4.4.4. *Let $L \to S$ be a complex line bundle over a compact manifold, and let $\sigma \in \Gamma(L)$ be a section which is transverse to the zero-section. Let $M = P(L)$ be the principal S^1-bundle* *corresponding to* L*, and let* $f_\sigma \in C^\infty(M,\mathbb{C})^{S^1}$ *be the equivariant function corresponding to* σ*. Define* $N := \sigma^{-1}(\{0\})$*, and* $B := h^{-1}(N)$*. Then*

$$\pi : M \setminus B \to S^1, \quad \pi(z) = \frac{f_\sigma(z)}{|f_\sigma(z)|}$$

defines an open book decomposition of M*. Furthermore, this open book decompositions admits* S^1*-invariant adapted tubular neighbourhoods together with* S^1*-equivariant trivializations.*

Proof. Because σ is transverse to the zero-section we have by Lemma 4.4.3 that 0 is a regular value of f_σ. Because f_σ is equivariant, we have for all $p \in M \setminus B$ that $df(V_p) = (\frac{\partial}{\partial\varphi})_{f(p)}$, where V is the infinitesimal generator of the S^1-action. This shows that f is transverse to $\pi_{\text{st}} : \mathbb{C}^* \to S^1$. We can thus apply Theorem 4.1.12 to obtain an open book decomposition on M. To construct invariant tubular neighbourhoods we can start with an invariant tubular neighbourhood with equivariant retraction, which exists by Theorem A.3.1. The tubular neighbourhood we then obtain will be invariant and as in the proof of Theorem 4.1.12

$$\varphi : \mathcal{U} \to B \times \mathbb{C}, \quad \varphi(z) = (q(z), f(z)).$$

will induce equivariant trivialisations. □

The main advantage of these open books is the following fact:

Lemma 4.4.5. *The monodromy of an open book constructed using Theorem 4.4.4 is trivial.*

Proof. Let $V \in \mathfrak{X}(M \setminus B)$ be the infinitesimal generator of the action. Because the binding is invariant under the action we have that V is tangent to B. Let $\pi : M \setminus B \to S^1$ denote the open book decomposition. We have

$$\begin{aligned}(d\pi)_p(V_p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi(e^{2\pi i t} \cdot p) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{f(e^{2\pi i t} \cdot p)}{|f(p)|} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{2\pi i t}\frac{f(p)}{|f(p)|} \\ &= \left(\frac{\partial}{\partial\varphi}\right)_{f(p)/|f(p)|},\end{aligned}$$

hence the time-one-flow of V generates a monodromy for the open book. Because the time-one-flow of V is the identity we conclude the statement. □

One often encounters sections of complex line bundles which are tensor products of some other line bundle. We can relate the corresponding principal bundles in the following way:

Lemma 4.4.6. *Let* L *be a complex line bundle, and let* $\mathbb{Z}_k$ *act on* $P(L)$ *via the* k*-th roots of unity. Then* $P(L)/\mathbb{Z}_k \simeq P(L^k)$*.*

Proof. Let g be a Riemannian metric on L, and let g^k denote the induced metric on L^k. That is, for all $v_i, w_i \in T_x M$

$$g_x^k(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k) = \frac{1}{k^2} \sum_{i,j=1}^{k} g_x(v_i, v_j).$$

Define $\widetilde{\varphi} : P(L) \to P(L^k)$ fiberwise by

$$\widetilde{\varphi}_x : P(L)_x \to P(L^k)_x, \quad v \mapsto v \otimes \cdots \otimes v.$$

By choice of metric on L^k we see that $\widetilde{\varphi}$ indeed maps into $P(L^k)$. We also see $\widetilde{\varphi}$ is $\mathbb{Z}_k$-invariant and thus induces a map $\varphi : P(L)/\mathbb{Z}_k \to P(L^k)$. Let $v_1 \otimes \cdots \otimes v_k \in P(L^k)_x$, and let $g_2, \ldots, g_k \in S^1$ be such that $g_i v_i = v_1$. Then the map $\psi : P(L^k) \to P(L)/\mathbb{Z}_k$, defined fiberwise by

$$\psi_x : v_1 \otimes \cdots \otimes v_k \mapsto [(g_2 \cdots g_k)^{1/k} v_1],$$

gives an inverse for φ. □

Suppose that we are given a complex line bundle $L \to S$, and a section of L^k which is transverse to the zero-section. By Theorem 4.4.4 we obtain an open book decomposition on $P(L^k)$. The following lemma gives rise to an open book decomposition on $P(L)$.

Lemma 4.4.7. *Let $L \to S$ be a complex line bundle and let $\sigma \in \Gamma(L^k)$ be transverse to the zero-section. Let $\pi : P(L^k) \setminus B \to S^1$ be the open book decomposition from Theorem 4.4.4. Let $\zeta : P(L) \to P(L^k)$ denote the composition of the quotient map and the diffeomorphism of Lemma 4.4.6. Define $\widetilde{B} = \zeta^{-1}(B)$, then*

$$\widetilde{\pi} := \zeta^* \pi : P(L) \setminus \widetilde{B} \to S^1$$

forms an open book decomposition of $P(L)$. Moreover, the monodromy of this open book has order k.

Proof. Let $f_\sigma : M \to \mathbb{C}$ be the equivariant function corresponding to σ. The fact that $\widetilde{\pi}$ defines an open book decomposition follows from Theorem 4.1.12 applied to $\zeta^* f_\sigma$. Let $V, \widetilde{V}$ denote the infinitesimal generators on $P(L^k), P(L)$ respectively. Then $\zeta_* \widetilde{V} = V$ and by naturality of the flow we have

$$\zeta(\varphi^1_{\widetilde{V}}(x)) = \varphi^1_V(\zeta(x)) = \zeta(x).$$

Hence $\varphi^1_{\widetilde{V}}(x) = \omega_i \cdot x$, for some k-th root of unity ω_i. We thus conclude that the monodromy has order k. □

4.4.3 Donaldson open books

We will now endow the open books arising from Theorem 4.4.4 with a contact form which is supported by the open book. First we recall some basic definitions concerning principal bundles.

Basic forms

Let $M \to S$ be a principal G-bundle, the infinitesimal action $a : \mathfrak{g} \to \mathfrak{X}(M)$ is given by:

$$a(v)_p := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} p \cdot \exp(tv)$$

Definition 4.4.8. A **connection one-form** on a principal G-bundle $M \to S$ is a one-form

$$\alpha \in \Omega^1(M, \mathfrak{g})$$

which is G-invariant and satisfies $\alpha(a(X)) = X$ for all $X \in \mathfrak{g}$. A **basic form** $\alpha \in \Omega^k_{\mathrm{bas}}(P, V)$ is a form which is invariant and satisfies $\iota_{a(X)}\alpha = 0$ for all $X \in \mathfrak{g}$.

Remark 4.4.9. Throughout this section we are only considering principal S^1-bundles. Because the Lie algebra of S^1 is trivial, we will not distinguish between elements of $\Omega^1(M, \mathrm{Lie}(S^1))$ and elements of $\Omega^1(M)$.

Recall the following fact:

Theorem 4.4.10 ([Cra16])**.** *Let $h : M \to S$ be a principal G-bundle. And let $\rho : G \to Gl(V)$ be a representation. Let $E = E(S, V)$ be the corresponding vector bundle, then*

$$h^* : \Omega^k(S, E) \to \Omega^k_{bas}(M, V),$$

is an isomorphism.

Recall that a cohomology class $a \in H^n(M)$ is called **integral** if under the De-Rham isomorphism $H^n(M) \simeq H^n(M, \mathbb{R})$, a can be considered as an element of $H^n(M, \mathbb{Z})$. We say that a closed differential form is integral if its cohomology class is integral. This definition is useful because of the following:

Lemma 4.4.11. *Let $a \in H^2(M)$ be integral, then there exists a vector bundle L, unique up to isomorphism, with $c_1(L) = a$.*

Proof. Let $\mathcal{E}$ denote the sheaf of smooth complex-valued functions on M. Consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 0.$$

The first Chern-class can be defined as the connecting morphism in the induced long exact sequence:

$$H^1(X, \mathcal{E}) \longrightarrow H^1(X, \mathcal{E}^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{E}).$$

Because the sheaf $\mathcal{E}$ is fine we see that c_1 is an isomorphism. Because a can be represented as an element in $H^2(X, \mathbb{Z})$ this concludes the proof. □

The following theorem will give the contact form which will be supported by the open books which we will construct.

Theorem 4.4.12 (Boothby-Wang - First Part, [Gei08])**.** *Let (S, ω) be a closed integral symplectic manifold. Let L be the complex line bundle with $c_1(L) = [\omega_S]$, and let $h : M \to S$ be the corresponding principal S^1-bundle. Then there exists connection one-form α on M whose curvature is ω (i.e. $d\alpha = h^*\omega$). Moreover any such α is a contact form whose Reeb vector field, R, coincides with the infinitesimal generator of the S^1-action.*

Proof. We claim the following:

Claim 1. For any connection one-form $\alpha' \in \Omega^1(M)$ on M there exists a unique closed two-form $\omega' \in \Omega^2(S)$ satisfying $d\alpha' = \pi^*\omega'$.

Claim 2. Let $\alpha, \alpha' \in \Omega^1(M)$ be two connection one-forms on M, and let $\omega, \omega' \in \Omega^2(S)$ be such that $d\alpha' = h^*\omega'$, $d\alpha = h^*\omega$. Then $[\omega] = [\omega']$.

We will postpone the proofs of the claims, and first use them to prove the theorem. Let $\alpha' \in \Omega^1(M)$ be any connection one-form and consider $\omega' \in \Omega^2(S)$ such that $d\alpha' = h^*\omega'$. By the second claim we have $[\omega] = [\omega']$. Let β be such that $\omega - \omega' = d\beta$. We have that $\alpha := \alpha' + h^*\beta$ satisfies $d\alpha = h^*\omega$. The form α is a connection one-form satisfying $\alpha \wedge (d\alpha)^n = \alpha \wedge h^*\omega^n$. Hence:

$$\iota_R(\alpha \wedge (d\alpha)^n) = h^*\omega^n \neq 0.$$

Thus α is a contact form. Now as $(\varphi_R^t)^*\alpha = \alpha$, we get that $\mathcal{L}_R(\alpha) = 0$, and using Cartan's formula we see that $\iota_R(d\alpha) = 0$. Because α is a connection one-form $\iota_R\alpha = 1$ and we conclude that R is indeed the Reeb vector field of α. To prove the theorem we are thus left to prove the claims.

Proof of Claim 1. Consider $E(M, \mathbb{R})$, the vector bundle associate to the trivial representation of S^1 in $\mathbb{R}$. Because α' is invariant, we have that $\mathcal{L}_R\alpha' = 0$. By Cartan's formula,

$$\begin{aligned}\mathcal{L}_R\alpha' &= d\iota_R\alpha' + \iota_R d\alpha' \\ &= d(1) + \iota_R(d\alpha'),\end{aligned}$$

hence $\iota_R(d\alpha') = 0$. Furthermore, as α' is S^1-invariant, so is $d\alpha'$. We conclude that $d\alpha'$ is a basic form. Hence there exists a unique $\omega' \in \Omega^2(S)$ such that $d\alpha' = h^*\omega'$. We directly see that ω' is closed. □

Proof of Claim 2.
We note that $(\alpha - \alpha')(R) = 0$, hence $\alpha - \alpha'$ is a basic form. There thus exists $\beta \in \Omega^1(S)$ such that $\alpha - \alpha' = h^*\beta$. We find that $h^*(\omega - \omega') = h^*d\beta$. By injectivity of h^*, we have $\omega - \omega' = d\beta$ which finishes the proof of the claim. □

Now we endow the open book we obtained in Theorem 4.4.12 with a contact form. The following theorem states when this contact form is supported.

Theorem 4.4.13. *Let (S, ω) be a compact integral symplectic manifold, and let $L \to S$ be the complex line bundle with $c_1(L) = [\omega]$. Let $h : M \to S$ denote the principal S^1-bundle corresponding to L. Furthermore, let $\sigma \in \Gamma(L)$ be a section transverse to the zero-section, and endow M with the open book arising from Theorem 4.4.4:*

$$B := f_\sigma^{-1}(\{0\}), \quad \pi : M \setminus B \to S^1, \quad \pi(z) = \frac{f_\sigma(z)}{|f_\sigma(z)|}.$$

Furthermore, endow M with α, the contact form arising from Theorem 4.4.12. Then

i. The binding B is a contact submanifold of M if and only if $N = h(B)$ is a symplectic submanifold of S.

ii. If N is a symplectic submanifold, then the open book decomposition supports α.

Proof. **i)**: Because B is invariant under the S^1-action, and V is the infinitesimal generator of the action, V is tangent to B. Hence we can restrict V to a vector field V_B on B. Because Theorem 4.4.12 ensures that V coincides with the Reeb vector field of α it directly follows that $\alpha_B(V_B) = \alpha(V) = 1$, where $\alpha_B = \alpha|_B$. Completely analogously it follows that $\iota_{V_B}(d\alpha_B) = \iota_V(d\alpha) = 0$. Now we find:

$$\begin{aligned}(\alpha_B \wedge (d\alpha_B)^{n-1})(V_B, \cdot) &= (d\alpha_B)^{n-1} \\ &= h^*(\omega|_N^{n-1}).\end{aligned}$$

We conclude that B is a contact submanifold if and only if N is a symplectic submanifold.
ii): By construction we have that the Reeb vector field of α coincides with the infintisemal generator of the action. In the proof of Lemma 4.4.5 we have shown that $(d\pi)(V) = \frac{\partial}{\partial \varphi}$. Hence after applying Lemma 4.2.5 we conclude that the open book decomposition supports α. □

Definition 4.4.14. We call any open book decomposition arising from Theorem 4.4.13 a **Donaldson open book**. More precisely a Donaldon open book consists of the data $\pi : M \setminus B \to S^1$, $h : (M, \alpha) \to (S, \omega)$.

One of the advantages of Donaldson open books is that the symplectic structure on the page can be related to the symplectic structure on the base manifold:

Lemma 4.4.15. *Let $\pi : M \setminus B \to S^1$, $h : M \to S$ be a Donaldson open book. The inclusion $P \hookrightarrow M \setminus B$ induces a symplectomorphism*

$$\psi : (P, (d\alpha)|_P) \to (S \setminus N, \omega|_{S \setminus N})$$

Proof. To prove that the inclusion induces a diffeomorphism, we show that every S^1-orbit intersects P precisely once. Let $z \in M \setminus B$, and write $f(z) = e^{i\theta} |f(z)|$, for some $\theta \in [0, 2\pi)$. Then $w = e^{-i\theta} \cdot z \in P$. Hence for every S^1-orbit in $M \setminus B$, there is at least one point which is in P. Because $f(\lambda \cdot z) = \lambda f(z)$ we see that this point is unique. The fact that ψ is a symplectomorphism follows directly from $h^*\omega = d\alpha$. □

Hopf-fibration

We will now apply this general procedure to the Hopf-fibration.

Theorem 4.4.16. *Let $\mathbb{Z}_3$ act on S^5 by the third-roots of unity. We have that $S^5/\mathbb{Z}_3$ admits a Donaldson open book. The binding of the open book decomposition is an S^1-bundle over a symplectic torus.*

Proof. Consider $\mathcal{O}(1) \to \mathbb{C}P^2$. It is well-known that $P(\mathcal{O}(1))$ is isomorphic to the Hopf-fibration $\widetilde{h} : S^5 \to \mathbb{C}P^2$. Define $L := \mathcal{O}(3)$. By Remark 4.4.7 we have that $P(L)$ is diffeomorphic to $h : S^5/\mathbb{Z}_3 \to \mathbb{C}P^2$. Let $f \in C^\infty(S^5/\mathbb{Z}_3, \mathbb{C})^{S^1}$ be given by

$$[(z_1, z_2, z_3)] \mapsto z_1^3 + z_2^3 + z_3^3.$$

Define $\sigma_f \in \Gamma(L)$ to be the section corresponding to f. Then as f is a submersion by Lemma 4.4.3 we have that σ_f is transverse to the zero-section. Furthermore, as $c_1(\mathcal{O}(1)) = [\omega_{FS}]$, we have that $c_1(L) = 3[\omega_{FS}]$. Lastly we note that $h(f^{-1}(\{0\}))$ is a complex submanifold of a Kähler manifold, and hence is a symplectic submanifold. We can thus apply Theorem 4.4.13 to $\mathcal{O}(3) \to (\mathbb{C}P^2, 3\omega_{FS})$ and σ_f, to obtain a Donaldson open book decomposition on $S^5/\mathbb{Z}_3$.
We can describe the symplectic submanifold using the degree formula for the genus of a smooth algebraic curve in $\mathbb{C}P^2$ [GJ94]. This formula states that the genus of a complex curve in $\mathbb{C}P^2$ defined as the zero set of an irreducible homogeneous polynomial of degree d is given by

$$g = \frac{(d-1)(d-2)}{2}.$$

In our case $d = 3$, hence $g = 1$. We thus conclude that the symplectic submanifold N is diffeomoprhic to a torus. □

Remark 4.4.17. We use a degree three homogenous polynomial in the above proof because of the following. For different degrees, the corresponding symplectic submanifold will be a surface with genus different from one. It is well-known that the only orientable surface which fibers over S^1 is the torus. Hence if we choose $d = 3$, we can apply Theorem 4.1.5 to obtain a foliation on $S^5/\mathbb{Z}_3$.

Chapter 5

Symplectic foliations from Donaldson open books

In this chapter we will study the existence of codimension-one symplectic foliations on manifolds with an open book decomposition with trivial monodromy. If M admits an open book decomposition with trivial monodromy we can write

$$M \simeq (B \times D^2) \cup_{B \times S^1} P_\varepsilon \times S^1.$$

Now for a Donaldson open book P_ε already admits a symplectic structure, hence the outside component will admits a symplectic foliation. However the symplectic structure on P_ε will not be of cosymplectic type at the boundary, and we will need to adapt it before we can apply the symplectic turbulisation theorem.

In Section 5.1 we will prove that an open book decomposition with trivial monodromy which admits a symplectic structure of cosymplectic type at P_ε gives rise to a symplectic foliation. Then we will study the existence of such a symplectic structure on P_ε. In Section 5.2 we will specialize to Donaldson open books, and summarise our results in Theorem 5.2.4. Using this theorem we will construct a codimension-one symplectic foliation on $S^5/\mathbb{Z}_3$. We then directly obtain a symplectic foliation on S^5, recovering Mitsumatsu's result.

Difference with Osorno Torres' proof

We first comment on the difference between our construction of a symplectic foliation on S^5 and Osorno Torres'. The main difference is that Osorno Torres directly constructs the foliation on S^5, whereas we will proceed via $S^5/\mathbb{Z}_3$. Osorno Torres makes use of an open book decomposition supporting a contact form and the general symplectic turbulisation theorem to foliate the outside component of this open book. We will first construct a symplectic foliation on $S^5/\mathbb{Z}_3$ using an open book decomposition with trivial monodromy. Because the monodromy of the open book is trivial we can make use of the trivial symplectic turbulisation theorem to foliate the outside component. Using the trivial turbulisation instead of the general one makes it much easier to keep track of the symplectic structures on the boundary leaves of the foliation. This has the advantage that all computations simplify considerably.

5.1 Symplectic foliations on open books with trivial monodromy

Using the diffeomorphism between $\overline{P}$ and P_ε the following theorem can be seen as a special case of Theorem 4.3.1. The proof in this setting is however much easier and thus included.

Theorem 5.1.1. *Let M be a compact manifold and $\pi : M \setminus B \to S^1$ be an open book decomposition with trivial monodromy. Suppose that P_ε admits a symplectic structure of cosymplectic type at the boundary. Then M admits a codimension-one symplectic foliation.*

Proof. Let $\omega \in \Omega^2(P_\varepsilon)$ denote the symplectic structure with boundary of cosymplectic type, and (η, θ) a cosymplectic structure induced at the boundary. We decompose M by $M = \tau_\varepsilon(B) \cup C_\varepsilon$, and foliate both parts separately. Because the monodromy of the open book is the trivial $C_\varepsilon \simeq P_\varepsilon \times S^1$. We now apply Theorem 3.4.7 to obtain a symplectic foliation on C_ε which is tame near the boundary, and with symplectic structure on the boundary leaf given by:

$$\eta - \theta \wedge d\varphi.$$

Similarly we apply Lemma 3.5.1 to (B, η, θ) to obtain a symplectic foliation tame near the boundary on $B \times D^2 \simeq \tau_\varepsilon(B)$ for which the symplectic structure at the boundary leaf coincides with that of the foliation on C_ε. Using Theorem 2.2.7 we glue the two foliations along the boundary to obtain a codimension-one symplectic foliation on M. □

Symplectic structure of cosymplectic type at the boundary

We will now study when we can adapt a given symplectic structure on a manifold to one which is of cosymplectic type at the boundary. We will need to extend forms defined on the boundary to globally defined closed forms. The following lemma states that an extension in cohomology gives rise to an extension of forms.

Lemma 5.1.2. *Let $N \subset M$ be an embedded submanifold. If $\eta \in \Omega^k(N)$ is a closed form which cohomology class is in the image of the restriction map $\iota^* : H^k(M) \to H^k(N)$, then there exists a closed form $\kappa \in \Omega^k(M)$ such that $\kappa|_N = \eta$.*

Proof. Let $\kappa' \in \Omega^k(M)$ be such that $\iota^*[\kappa'] = \eta$. Let $\xi \in \Omega^{k-1}(N)$, be such that $\iota^*\kappa' - \eta = d\xi$. Now we extend ξ to a globally defined form. Let $E \to N$ be a tubular neighbourhood of N and let $x \mapsto |x|$ be a metric on E. Let $h : E \to \mathbb{R}$ be a smooth function such that $h(x) = 1$ for $|x| < 1$ and $h(x) = 0$ for $|x| > 2$. Then the form $d(h\xi) \in \Omega^k(E)$ coincides with $d\xi$ on N and can be extend smoothly to a form $\rho \in \Omega^k(M)$ which is zero outside of E. We define $\kappa := \kappa' - \rho$, which satisfies $\iota^*\kappa = \eta$. □

Proposition 5.1.3. *Let (M, ω) be a compact symplectic manifold with boundary, together with a cosymplectic structure (η, θ) on ∂M such that:*

- *$[\eta] \in H^2(\partial M)$ is in the image of the restriction map $H^2(M) \to H^2(\partial M)$,*
- *$\omega|_{\partial M} \wedge \theta = 0$.*

Then there exists a symplectic structure $\omega' \in \Omega^2(M)$ of cosymplectic type at the boundary.

Proof. Because $[\eta]$ is in the image of the restriction map we can use Lemma 5.1.2 to find a closed form $\kappa \in \Omega^2(M)$ for which $\kappa|_{\partial M} = \eta$. Now as M is compact we can chose δ small enough such that $\omega' := \omega + \delta\kappa$ is still symplectic. Because $\omega|_{\partial M} \wedge \theta = 0$, we see that θ is an admissable form for $\omega|_{\partial M} + \delta\eta$, hence ω' is of cosymplectic type at the boundary. □

5.2 Symplectic foliations on Donaldson open books

Normal forms

We wish to describe the symplectic structure of the symplectic manifold around the symplectic submanifold which is part of the data of a Donalson open book. To be complete we will recall precisely what we mean with this. Let $L \to (S, \omega)$ be a complex line bundle over an integral symplectic manifold with $c_1(L) = [\omega]$. Let $\sigma \in \Gamma(L)$ be transverse to the zero-section and assume that $N := \sigma^{-1}(0)$ is a symplectic submanifold. Let $h : M \to S$ be the principal S^1-bundle associated to L and define $B := h^{-1}(N)$. Furthermore, we have a connection one-form $\alpha \in \Omega^1(M)$ such that the following diagram commutes:

$$\begin{array}{ccc} (M, d\alpha) & \xrightarrow{\ h\ } & (S, \omega) \\ \uparrow & & \uparrow \\ (B, d\alpha_B) & \xrightarrow{\ h_B\ } & (N, \omega_N) \end{array}$$

Endow $B \times \mathbb{C}$ with the diagonal S^1-action, and define the form

$$\Omega_\pm = d((1 \mp r^2)\alpha + r^2 d\varphi) \in \Omega^2(B \times \mathbb{C}).$$

Lemma 5.2.1. *Orient $\mathcal{N}(N)$ as the symplectic normal bundle of N, and endow $L|_N$ by the orientation induced by its complex structure. The form $\Omega_\pm$ defined above descends to a form $\underline{\Omega}_\pm \in \Omega^2(B \times_{S^1} \mathbb{C})$. There exists a tubular neighbourhood $\mathcal{U} \subset S$ of N together with a symplectomorphism:*

$$(\mathcal{U}, \omega) \simeq (B \times_{S^1} D^2_\varepsilon, \underline{\Omega}_\pm)$$

With plus-sign, respectively minus-sign if $d^v\sigma : \mathcal{N}(N) \to L|_N$ is orientation preservering respectively orientation reversing.

Proof. We will apply Theorem A.1.5 and use the notation introduced there. Although B is precisely the principal S^1-bundles associated to $L|_N$ one should be careful that the complex structure on $L|_N$ and the complex structure on $\mathcal{N}(N)$ which is compatible with the symplectic structure need not coincide. The two are isomorphic as real vector bundles however, with isomorphism given by the vertical derivative of σ. When $d^v\sigma$ is orientation preserving the vector bundles are isomorphic as complex vector bundles. In this case we are precisely in the setting of Theorem A.1.5 with $B' = B$, $\alpha' = \alpha_B$, $\sigma = \omega_N$ and consequently $\Omega_+ = \Omega$. When $d^v\sigma$ is orientation reversing, then $\overline{L}|_N$ and $\mathcal{N}(N)$ are isomorphic as complex vector bundles. Now the actions on B' and B are precisely conjugate, which we denote by $B' = \overline{B}$. Also $\alpha' = -\alpha_B$ and $\sigma = -\omega_N$. Studying the expression for $\underline{\Omega}$ in Theorem A.1.5 we see that it coincides precisely with $\underline{\Omega}_-$, which finishes the proof. □

We can describe boundary of this tubular neighbourhood as follows:

Lemma 5.2.2. *The map* $\widetilde{\psi} : B \times \partial D^2_\varepsilon \to B, (b, \lambda) \mapsto (\frac{\lambda}{|\lambda|})^{-1} b$ *induces a diffeomorphism*

$$\psi : B \times_{S^1} \partial D^2_\varepsilon \to B.$$

Let $\Phi : \mathcal{U} \to B \times_{S^1} D^2_\varepsilon$ *denote the diffeomorphism from Lemma 5.2.1, then:*

$$\omega|_{\partial\mathcal{U}} = (\psi \circ \Phi|_{\partial M})^*((1 \mp \varepsilon^2) d\alpha_B).$$

With minus-sign, respectively plus-sign if $d^v\sigma : \mathcal{N}(N) \to L|_N$ *is orientation preservering respectively orientation reversing.*

Proof. The fact that ψ is a diffeomorphism is verified easily. Let $q : B \times \partial D^2_\varepsilon \to B \times_{S^1} \partial D^2_\varepsilon$ denote the quotient map. Define $\alpha'_B \in \Omega^1(B \times_{S^1} \partial D^2_\varepsilon)$ by $\alpha'_B := \psi^* \alpha_B$. We have $q^*\alpha_B = \widetilde{\psi}^*\alpha_B = \alpha_B$, because α_B is S^1-invariant. Because

$$\begin{aligned} q^*\underline{\Omega}|_{B\times_{S^1}\partial D^2_\varepsilon} &= \Omega|_{B\times\partial D^2_\varepsilon} \\ &= (1 - \varepsilon^2) d\alpha_B, \end{aligned}$$

and q^* is injective we conclude that

$$\underline{\Omega}_\pm|_{B\times_{S^1}\partial D^2_\varepsilon} = (1 \mp \varepsilon^2) d\alpha'_B.$$

Hence $\omega|_{\partial\mathcal{U}} = \Phi|^*_{\partial\mathcal{U}}((1 \mp \varepsilon^2) d\alpha'_B)$, and thus $\omega|_{\partial\mathcal{U}} = (\psi \circ \Phi|_{\partial M})^*((1 \mp \varepsilon^2) d\alpha_B)$, which finishes the proof. □

Conclusion

Let $\pi : M \setminus B \to S^1, h : M \to S$ be a Donaldson open book. Let $\tau_\varepsilon(B)$ denote a tubular neighbourhood adapted to π. Recall that such a tubular neighbourhood admitted S^1-equivariant trivialisations, so have that $h(\tau_\varepsilon(B)) \simeq B \times_{S^1} D^2_\varepsilon$. In conclusion we can pick $\tau_\varepsilon(B)$ such that $h(\tau_\varepsilon(B))$ coincides with $\mathcal{U}$ from Lemma 5.2.1. We denote $C_\varepsilon = M \setminus \tau_\varepsilon(B)$ and $S_\varepsilon = S \setminus \mathcal{U}$.
The following lemma allows us to use P_ε and S_ε interchangeably:

Lemma 5.2.3. *Let $h : M \to S$ be a Donaldson open book. The inclusion $P_\varepsilon \hookrightarrow C_\varepsilon$ induces a symplectomorphism*

$$(P_\varepsilon, (d\alpha)|_{P_\varepsilon}) \to (S_\varepsilon, \omega|_{S_\varepsilon}).$$

Proof. The argument is completely analogous as for Lemma 4.4.15, using the fact that the tubular neighbourhood $\tau_\varepsilon(B)$ is S^1-invariant. □

We are now ready to prove the main theorem of this thesis.

Theorem 5.2.4. *Let $h : M \to S$ be a Donaldson open book, and let (η, θ) be a cosymplectic structure on the binding B. Using the notation of Lemma 5.2.2, assume that*

- *$[(\psi \circ \Phi|_{\partial\mathcal{U}})^*\eta]$ is in the image of the pull-back of the inclusion $H^2(S_\varepsilon) \to H^2(\partial S_\varepsilon)$,*
- *$d\alpha_B \wedge \theta = 0$.*

Then M admits a codimension-one symplectic foliation.

Proof. Consider the cosymplectic structure on ∂S_ε given by $((\psi\circ\Phi|_{\partial\mathcal{U}})^*\eta, (\psi\circ\Phi|_{\partial\mathcal{U}})^*\theta)$. Because $d\alpha_B \wedge \theta = 0$ Lemma 5.2.1 implies that $\omega|_{\partial\mathcal{U}} \wedge (\psi \circ \Phi|_{\partial M})^*\theta = 0$. Therefore we apply Proposition 5.1.3 to obtain a symplectic structure of cosymplectic type at the boundary on S_ε. Because $S_\varepsilon \simeq P_\varepsilon$ we can now apply Theorem 5.1.1 to conclude the statement. □

Application to S^5

Using the previous theorem it is now relatively easy to obtain a symplectic foliation on S^5, recovering Mitsumatsu's result [Mit11] as well as a symplectic foliation on $S^5/\mathbb{Z}_3$.

Corollary 5.2.5. *The lens space $S^5/\mathbb{Z}_3$ and S^5 admit a codimension-one symplectic foliation.*

Proof. Recall the Donaldson open book constructed for $h : (S^5/\mathbb{Z}_3, d\alpha) \to (\mathbb{C}P^2, \omega)$ constructed in Theorem 4.4.16. We will show that it satisfies the assumptions of Theorem 5.2.4.
Surjectivity of pullback: We consider the Mayer-Vietoris sequence for $S := \mathbb{C}P^2$ using the decomposition $S = \tau(N) \cup (S \setminus N)$, with $\tau(N) = h(\tau_\varepsilon(B))$ where $\tau_\varepsilon(B)$ is as before:

$$\cdots \to H^2(S) \to H^2(S \setminus N) \oplus H^2(\tau(N)) \to H^2(\tau(N) \setminus N) \to H^3(S)$$

Observe that $N \hookrightarrow \tau(N)$, $S_\varepsilon = S \setminus \tau(N) \hookrightarrow S \setminus N$, and $\partial S_\varepsilon \hookrightarrow \tau(N) \setminus N$ all induce homotopy equivalences. Combining this with the fact that $H^3(S) = \{0\}$ we obtain the following exact sequence

$$\cdots \to H^2(S) \to H^2(S_\varepsilon) \oplus H^2(N) \stackrel{\iota^*-j^*}{\to} H^2(\partial S_\varepsilon) \to 0,$$

with $\iota : \partial S_\varepsilon \hookrightarrow S_\varepsilon$. Because $H^2(S)$ is generated by $[\omega]$ we can apply Lemma 5.2.3 to find that $H^2(S) \to H^2(S_\varepsilon)$ is trivial. Because N is a two-dimensional symplectic submanifold we find

that $H^2(S) \to H^2(N)$ is an isomorphism. Because the sequence is exact we find that $j^* = 0$, and thus that ι^* is surjective.
Cosymplectic structure: Because $h(B) \simeq T^2$, we can pick a global closed coframe $\theta_1, \theta_2 \in \Omega^1(h(B))$. Let $h_B := h|_B$, $\alpha_B = \alpha|_B$ and define

$$(\eta, \theta) := (h_B^* \theta_2 \wedge \alpha_B, h_B^* \theta_1).$$

Recall that $d\alpha = h^*(\omega)$, hence $d\alpha_B = h_B^*(\omega|_{h(B)})$. Because both $\theta_1 \wedge \theta_2$ and $\omega|_{h(B)}$ are volume forms we find that

$$d\alpha_B = (h_B^* f) h_B^* \theta_1 \wedge h_B^* \theta_2,$$

for some nowhere vanishing function f on $h(B)$. The fact that α_B is contact now directly implies that (η, θ) is a cosymplectic structure. We also have that $d\alpha_B \wedge h_B^* \theta_1 = 0$. We thus apply Theorem 5.2.4 to obtain a symplectic foliation on $S^5/\mathbb{Z}_3$, and by pulling the foliation back we obtain a symplectic foliation on S^5. □

Remark 5.2.6. In the proof that the pull-back of the inclusion induces a surjective map in cohomology we have only used the facts that the symplectic manifold S is four-dimensional and compact, $H^2(S)$ is one-dimensional, and $H^3(S) = \{0\}$.

Remark 5.2.7. We motivate that the assumptions in Theorem 5.2.4 can be thought of purely as assumptions on the symplectic geometry involved. Firstly, Donaldson open books arises from integral symplectic manifolds with particular symplectic submanifolds. Secondly, the existence of a cosymplectic structure on the binding can be viewed as a requirement on the symplectic normal bundle of the symplectic submanifold. We thus see that constructing symplectic manifolds using Theorem 5.2.4 boils down to finding particular symplectic submanifolds in integral symplectic manifolds.

We can also apply Theorem 5.2.4 to obtain the symplectic foliation on S^3.

Corollary 5.2.8. *The manifold S^3 admits a symplectic foliation.*

Proof. Consider the complex line bundle $\mathcal{O}(1) \to \mathbb{C}P^1$. We have that $P(\mathcal{O}(1))$ is isomorphic to the Hopf-fibration $h : S^3 \to \mathbb{C}P^1$. Consider the function

$$f : S^3 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 + z_2,$$

which is readily seen to be a submersion, and S^1-equivariant if we endow $\mathbb{C}$ with the standard S^1-action. Hence the corresponding section $\sigma_f \in \Gamma(\mathcal{O}(1))$ is transverse to the zero-section. Furthermore, $c_1(\mathcal{O}(1)) = [\omega_{FS}]$, and $h(f^{-1}(\{0\}))$ is trivially a symplectic submanifold. Hence we can apply Theorem 4.4.13 to $\mathcal{O}(1) \to (\mathbb{C}P^1, \omega_{FS})$ to obtain a Donaldson open book decomposition on S^3. The assumptions from Theorem 5.2.4 are trivially satisfied, so we obtain a codimension-one symplectic foliation on S^3. □

Properties of the foliation

We now study some properties of the symplectic foliation constructed in Corollary 5.2.5.

Lemma 5.2.9. *The leaves of the foliation on $S^5/\mathbb{Z}_3$ as constructed in Corollary 5.2.5 are:*

- *Diffeomorphic to $T^2 \times \mathbb{R}^2$ in the interior component.*
- *Diffeomorphic to $\mathbb{C}P^2 \setminus N$ in the outside component.*
- *One compact leaf, diffeomorphic to $B \times S^1$, seperating the two components.*

Proof. The foliation is obtained using the decomposition $S^5/\mathbb{Z}_3 \simeq B \times D^2 \cup P_\varepsilon \times S^1$, hence the leaf seperating the two components is diffeomorphic to $B \times S^1$.
Outside component: The leaves in the outside component are obtained by applying the turublisation procedure to $S^1 \times P_\varepsilon$. Hence by Lemma 3.4.3 we have that the leaves of the foliation are diffeomorphic to Int P_ε, which is diffeomorphic to $\mathbb{C}P^2 \setminus N$.
Inside component: Inspecting the proof we see that the cosymplectic structure on B is of the form $(\eta, f^*d\varphi)$, with $f : B \to S^1$. Now the leaves of the foliation are obtained by an applictaction of Lemma 3.5.1. This lemma can be seen as an application of the general symplectic turbulisation theorem to a cosymplectic structure of the form $(\eta', \mathrm{pr}_1^* f^* d\varphi)$, where $\mathrm{pr}_1^* : B \times D^2 \to B$ is the projection onto the first factor. We thus have by Remark 3.5.8 that the leaves of the turbulised foliation on $B \times \mathrm{Int}\, D^2$ are diffeomorphic to the leaves of the foliation defined by $\mathrm{pr}_1^* f^* d\varphi$. To finish the proof we are left to show that the leaves of this foliation are diffeomorphic to $T^2 \times \mathrm{Int}\, D^2$.
Fibers: For all $\varphi \in S^1$ we have $f^{-1}(\{\varphi\}) = h_B^{-1}(S^1 \times \{\lambda\})$, hence the fibers of f can be seen as two-dimensional manifolds, which fibre over S^1. Because $(\eta, f^*d\varphi)$ is a cosymplectic structure, we have that the fibers of f are symplectic manifolds. Hence, they are orientable sufraces and thus diffeomorphic to tori. We thus see that the foliation defined by $f^*d\varphi$ has leaves diffeomorphic to tori, and thus $\mathrm{pr}_1^* f^* d\varphi$ has leaves diffeomorphic to $T^2 \times \mathrm{Int}\, D^2$, which finishes the proof. □

Lemma 5.2.10. *The foliation on S^5 as constructed in Corollary 5.2.5 coincides with Lawson's foliation. Hence the leaves of the foliation on S^5 are:*

- *Diffeomorphic to $T^2 \times \mathbb{R}^2$ in the interior component.*
- *Diffeomorphic to three-covers of $\mathbb{C}P^2 \setminus N$ in the outside component.*
- *One compact leaf, diffeomorphic to $\widetilde{B} \times S^1$, seperating the two components.*

where $\widetilde{B} = S^5 \cap f^{-1}(\{0\})$, with $f : \mathbb{C}^3 \to \mathbb{C}$ the polynomial $f(z) = z_1^3 + z_2^3 + z_3^3$.

Proof. **Coincide with Lawson**: Lawson's foliation is obtained using the open book decomposition of S^5 and applying the turbulisation procedure to both components of the decomposition. We note that we can choose adapted tubular neighbourhoods $\tau(B), \tau(\widetilde{B})$ such that the following diagrams commute.

$$\begin{array}{ccc} \tau(\widetilde{B}) & \xrightarrow{\zeta} & \tau(B) \\ \downarrow & \swarrow & \\ S^1 & & \end{array} \qquad \begin{array}{ccc} S^5 \setminus \widetilde{B} & \xrightarrow{\zeta} & (S^5/\mathbb{Z}_3) \setminus B \\ \downarrow{\scriptstyle \widetilde{\pi}} & \swarrow{\scriptstyle \pi} & \\ S^1 & & \end{array}$$

Let C denote the complement of a tubular neighbourhood of B in $S^5/\mathbb{Z}_3$, and $\widetilde{C}$ denote the complement of a tubular neighbourhood of $\widetilde{B}$ in S^5. Recall that the foliation tame near the boundary on C is defined by

$$\alpha := \begin{cases} \pi^* d\varphi & \text{on } C \setminus U \\ f(t)\pi^* d\varphi + g(t)dt & \text{on } U. \end{cases}$$

The foliation on $\widetilde{C}$ obtained using the turbulisation procedure is defined by

$$\widetilde{\alpha} := \begin{cases} \widetilde{\pi}^* d\varphi & \text{on } \widetilde{C} \setminus U \\ f(t)\widetilde{\pi}^* d\varphi + g(t)dt & \text{on } U. \end{cases}$$

By commutativity of the diagram we see that $\zeta^*\alpha = \widetilde{\alpha}$, which proves that the foliation on $\widetilde{C}$ obtained by turbulising and the foliation obtained by taking the pull-back of the foliation on C coincide. A similar argument holds for the inside component. Hence we conclude that the foliation constructed in Corollary 5.2.5 coincides with Lawson's foliation. We will finish the proof by describing the leaves of Lawson's foliation.

Outside: The foliation on the outside component is diffeomorphic to the foliation given by the pages of the open book. Hence the leaves in the outside component of the foliation on S^5 are diffeomorphic to three-covers of $\mathbb{C}P^2 \setminus N$.

Inside: The foliation on $\widetilde{B} \times D^2$ is diffeomorphic to the foliation given by the fibres of $\widetilde{B} \times D^2 \to S^1$. Completely analogous to the case of $S^5/\mathbb{Z}_3$ one can prove that these are diffeomorphic to $T^2 \times \mathbb{R}^2$. Hence we conclude that the leaves on $\tau(\widetilde{B})$ are diffeomorphic to $T^2 \times \mathbb{R}^2$. □

Remark 5.2.11 (Relating foliations)**.** The argument in Lemma 5.2.10 can also be used to relate foliations obtained from Theorem 5.1.1 to foliation obtained from Theorem 4.1.5. Let $\pi : M \setminus B \to S^1$ be an open book deocomposition with trivial monodromy for which P_ε admits a symplectic structure of cosymplectic type. Because $\partial P_\varepsilon \simeq B$ inherits a cosymplectic structure, we have by Tischler's theorem that B fibres over S^1. Inspecting the proof of Theorem 5.1.1 one finds that the underlying foliation coincides with the one constructed in Theorem 4.1.5. Hence Theorem 5.1.1 can be seen as an extension of Theorem 4.1.5 to the symplectic setting.

Remark 5.2.12 (S^7)**.** The previous remark also suggests that it is not viable that one can produce a symplectic foliation on S^7 using our methods. Lawson uses the polynomials $p(z) = z_0^2 + \cdots + z_n^2$ to obtain open book decompositions on the spheres S^{2^k+3}. He then constructs foliations on these spheres using Theorem 4.1.5. However the compact leaf of the resulting foliations turns out to be diffeomorphic to $SO(n+1)/SO(n-1)$ (see for instance [Law71]). It can be shown ([Tor15]) that $SO(n+1)/SO(n-1)$ does not admit a symplectic structure. Hence Lawson's foliation on the higher dimensional spheres does not admit a symplectic structure.

Tameness

Recall that we called a foliated differential form $\omega_{\mathcal{F}}$ tame around a leaf L, if there exists a closed extension of ω_L to an open around L. We say that a foliated differential form $\omega_{\mathcal{F}} \in \Omega^2(\mathcal{F})$ is **tame**, if there exists a closed differential form $\widetilde{\omega} \in \Omega^2(M)$ such that $\widetilde{\omega}|_{\mathcal{F}} = \omega_{\mathcal{F}}$.

Lemma 5.2.13. *The symplectic foliations of Corollary 5.2.5 are not tame.*

Proof. The proof is identical for S^5 and $S^5/\mathbb{Z}_3$. Let $\omega_{\mathcal{F}}$ denote the foliated form and assume to the contrary that there exists $\widetilde{\omega} \in \Omega^2(S^5)$ satisfying $\widetilde{\omega}|_{\mathcal{F}} = \omega_{\mathcal{F}}$. Observe that $H^2(S^5) = \{0\}$, hence there exists $\theta \in \Omega^1(S^5)$ such that $\widetilde{\omega} = d\theta$. By the previous remark we have that the foliation has a compact leaf, which we denote by L. We see that $\omega_L = d(\theta|_L)$, hence ω_L is an exact symplectic structure on a compact manifold, which is a contradiction. □

Remark 5.2.14. In the previous lemma we only used that the foliation has a compact leaf and that $H^2(S^5) = \{0\}$. Foliations obtained using Theorem 4.1.5 always come with compact leaves. We thus see that on manifolds with $H^2(M) = \{0\}$, these foliations never admit a tame leafwise symplectic structure. It is still an open question whether S^5 admits a tame symplectic foliation, although it is conjectured that this is not the case. In three-dimensions there exists the *Novikov's compact leaf theorem*, which states that a codimension-one foliation on any three-dimensional manifold for which the universal cover is contractible admits a compact leaf. This in particular shows that S^5 does not admit a tame foliation. Unfortunately, in general Novikov's theorem is false in higher dimensions; there exists foliations on S^5 with only non-compact leaves [Mei12].

Chapter 6

Complex foliations

In this chapter we study the existence of complex foliations. Just as for symplectic foliations this has proven to be a rather difficult problem. In 2002 Meersseman and Verjovsky claimed to have constructed a complex foliation on the five-sphere [MV02]. However, it turned out that they made a mistake, and the complex foliation did not live on the five-sphere but on a different five-dimensional manifold [MV11]. So the question whether S^5 admits a complex foliation is still open and we aim to make some progress towards an answer. In this chapter we outline a possible way to approach the existence of a complex foliation on S^5, motivated by our results in the previous chapters.

Complex foliations are, just like symplectic foliations, most easily described from an infinitesimal point of view. The infinitesimal analogue of a complex foliation is a *CR-structure*. In Section 6.1 we give the definition of CR-structures and complex foliations and describe their relation. Just as presymplectic structures arise from symplectic manifolds with boundary we will see that CR-structures arise from complex manifolds with boundary. In Section 6.2 we will summarize our construction of a symplectic foliation on S^5, and describe a plan to construct a complex foliation on S^5. In the subsequent sections we will carry out some of the described steps in this plan. In Section 6.6 we will use what we have learned in the preceding sections to revise our plan of constructing a complex foliation on S^5.

6.1 CR-structures

Intuitively, a complex foliation is a foliation for which each of the leaves carries the structure of a complex manifold, such that these structures vary smoothly from leaf to leaf. The following definition makes this precise:

Definition 6.1.1. A foliation $\mathcal{F}$ on a smooth manifold M^{2p+q} is said to be a **complex foliation** if there exists a (maximal) smooth foliation atlas $(\varphi_i : U_i \to \mathbb{C}^p \times \mathbb{R}^q)_{i \in I}$, for which the transition functions

$$\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y)), \quad (x,y) \in \mathbb{C}^p \times \mathbb{R}^q$$

are such that $g_{ij}(\cdot, y)$ is holomorphic for every $y \in \mathbb{R}^q$.

Because the transition functions $g_{ij}(\cdot, y)$ are holomorphic, we can endow every leaf of the foliation with the structure of a complex manifold.
Just as complex manifolds can be described using almost complex structures, complex foliations can also be considered from an infinitesimal point of view. Given a complex foliation $\mathcal{F}$ one can define an almost complex structure J on $T\mathcal{F}$ by setting $J|_L$ to be the almost complex structure corresponding to the complex structure on L. Because the charts used to define the complex structure on each leaf L come from a foliation atlas we see that J is smooth. The pair $(T\mathcal{F}, J)$ is an example of a *CR-structure*.

Definition 6.1.2. Let N be a smooth $(2n+1)$-dimensional manifold. An **almost CR-structure**[1] on N is a pair $(\mathcal{H}, J)$, with $\mathcal{H}$ a codimension-one distribution on N, and $J : \mathcal{H} \to \mathcal{H}$ an almost complex structure, i.e. J is a bundle map and $J^2 = -\mathrm{id}$.

We first collect some definitions concerning almost CR-structures and then study their integrability.

Definition 6.1.3. Let $(\mathcal{H}, J)$ and $(\mathcal{H}', J')$ be two CR-structures on manifolds N, N' respectively. A **CR-map** between $(\mathcal{H}, J)$ and $(\mathcal{H}', J')$ is a smooth map $f : N \to N'$ such that $df \circ J = J' \circ df$.

We consider the complex linear extension of J to $\mathcal{H}_{\mathbb{C}} := \mathcal{H} \otimes \mathbb{C}$, which we still denote by J. The complexified bundle admits the decomposition

$$\mathcal{H}_{\mathbb{C}} := \mathcal{H}^{(1,0)} \oplus \mathcal{H}^{(0,1)},$$

where $\mathcal{H}^{(1,0)}, \mathcal{H}^{(0,1)}$ are the $+i$, respectively $-i$-eigenspaces of J.

Definition 6.1.4. We say that an almost CR-structure $(\mathcal{H}, J)$ is **integrable** if

$$[\Gamma(\mathcal{H}^{(0,1)}), \Gamma(\mathcal{H}^{(0,1)})] \subset \Gamma(\mathcal{H}^{(0,1)}).$$

We will also call an integrable almost CR-structure simply a CR-structure.

Using the fact that conjugation induces an isomorphism between $\mathcal{H}^{(1,0)}$ and $\mathcal{H}^{(0,1)}$, we readily see that integrability is equivalent to

$$[\Gamma(\mathcal{H}^{(1,0)}), \Gamma(\mathcal{H}^{(1,0)})] \subset \Gamma(\mathcal{H}^{(1,0)}).$$

[1] CR either stands for Caucy-Riemann or for Complex-Real

Definition 6.1.5. We say that an almost CR-structure $(\mathcal{H}, J)$ is **Levi-flat** if

$$[\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}),$$

i.e. $\mathcal{H}$ is an involutive distribution.

Definition 6.1.6. We say that an almost CR-structure $(\mathcal{H}, J)$ is **Levi-non-degenerate** if the map

$$\Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(TM/\mathcal{H}), \quad (X, Y) \mapsto [X, Y] \bmod \mathcal{H}$$

is non-degenerate, i.e. $\mathcal{H}$ is a contact distribution.

Now we return to the question of integrability of CR-structures. Given a Levi-flat almost CR-structure $(\mathcal{H}, J)$ we have by Frobenius' theorem that $\mathcal{H}$ integrates to a foliation. Moreover, if $(\mathcal{H}, J)$ is integrable, we have by the Newlander-Nirenberg theorem that every leaf admits the structure of a complex manifold. In fact the Newlander-Nirenberg theorem holds parametrically, hence the defined complex charts on the leaves will combine into a foliation atlas. In conclusion we have:

Proposition 6.1.7. *There is a 1:1 correspondence*

$$\{\textit{Complex foliations}\} \overset{1:1}{\longleftrightarrow} \{\textit{Levi-flat, integrable almost CR-structures}\}.$$

Nijenhuis tensor

Recall that for a $(1,1)$-tensor $A : TM \to TM$, we can define its Nijenhuis tensor by

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY].$$

The following is well-known:

Theorem 6.1.8 (Newlander-Nirenberg)**.** *An almost complex structure $J : TM \to TM$ is integrable if and only if $N_J = 0$.*

However, if we want to define a Nijenhuis tensor for an almost CR-structure, a small subtlety arises. Because the almost CR-structure is only defined on the distribution $\mathcal{H}$, the formula for the Nijenhuis tensor is a priori not well-defined. Therefore we have to impose an extra condition before we can define it. So suppose that $(\mathcal{H}, J)$ is an almost CR-structure such that $[JX, Y] + [X, JY] \in \Gamma(\mathcal{H})$ for all $X, Y \in \Gamma(\mathcal{H})$, then we define $N_J : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(TN)$ by the above formula.

Proposition 6.1.9. *An almost CR-structure $(\mathcal{H}, J)$ is integrable if and only if*

$$[JX, Y] + [X, JY] \in \Gamma(\mathcal{H}) \quad \textit{for all } X, Y \in \Gamma(\mathcal{H})$$

and $N_J = 0$.

Proof. "$\Leftarrow$": Let $X, Y \in \Gamma(\mathcal{H}^{(0,1)})$, then $N_J(X,Y) = 2[X,Y] - 2iJ[X,Y]$. Hence $J[X,Y] = -i[X,Y]$, from which we conclude that $(\mathcal{H}, J)$ is integrable.
"$\Rightarrow$": Let $X, Y \in \Gamma(\mathcal{H})$, and write $X = X_1 + X_2, Y = Y_1 + Y_2$, with $X_1, Y_1 \in \Gamma(\mathcal{H}^{(1,0)})$ and $X_2, Y_2 \in \Gamma(\mathcal{H}^{(0,1)})$. We have

$$\begin{aligned}[JX, Y] &= [iX_1 - iX_2, Y] \\ [X, JY] &= [X, iY_1 - iY_2].\end{aligned}$$

Hence

$$\begin{aligned}[JX, Y] + [X, JY] &= i[X_1, Y_1] + i[X_1, Y_2] - i[X_2, Y_1] - [iX_2, Y_2] \\ &\quad + i[X_1, Y_1] - i[X_1, Y_2] + i[X_2, Y_1] - i[X_2, Y_2] \\ &= i[X_1, Y_1] - i[X_2, Y_2].\end{aligned}$$

From which it follows that $[JX, Y] + [X, JY] \in \Gamma(\mathcal{H})$. The proof that the Nijenhuis-tensor vanishes is just as in the complex case. □

We note that the almost complex structure of an almost CR-structure induces an orientation on the hyperplane distribution. Just as with foliations we have that on orientable manifolds this implies that the hyperplane distribution is co-orientable. In conclusion:

Proposition 6.1.10. *Let M be an orientable manifold together with an almost CR-structure $(\mathcal{H}, J)$. Then $\mathcal{H}$ is co-orientable.*

6.1.1 Complex manifolds with boundary

One way to obtain CR-structures is from complex manifolds with boundary. Let (M, J) be an almost complex manifold with boundary and let $N := \partial M$. Then $J|_N$, does not map TN to itself and we thus study $\mathcal{H} := TN \cap J(TN)$.

Lemma 6.1.11. *Let (M, J) be an almost complex manifold with boundary and let $N := \partial M$. Then $(\mathcal{H}, J_{\mathcal{H}}) := (TN \cap J(TN), J|_{\mathcal{H}})$ defines an almost CR-structure.*

Proof. Clearly $\mathcal{H}$ is a distribution on N, and $J_{\mathcal{H}}$ is an almost complex structure. What remains to be shown is that the codimension of $\mathcal{H}$ is one. Because $J(TN)$ cannot be contained in TN, otherwise N would carry an almost complex structure, we have $TN + J(TN) = TM|_N$. Hence as both TN and $J(TN)$ are codimension-one subbundles of $TM|_N$, we have that the intersection $TN \cap J(TN)$ is a codimension-one distribution on N. □

To check the integrability of this CR-structure we need the following:

Lemma 6.1.12. *Let (M, J) be an almost complex manifold with boundary, and let $(\mathcal{H}, J_{\mathcal{H}})$ denote the induced almost CR-structure on the boundary. For any vector field Y defined near ∂M for which both Y and JY are tangent to ∂M, we have $Y|_{\partial M} \in \Gamma(\mathcal{H})$.*

Proof. Because $Y|_{\partial M} = -J_{\mathcal{H}}((JY)|_{\partial M})$, this is immediate. □

Lemma 6.1.13. *Let (M, J) be a complex manifold with boundary and let $N := \partial M$. Then $(\mathcal{H}, J_{\mathcal{H}}) := (TN \cap J(TN), J|_{\mathcal{H}})$ defines an integrable almost CR-structure.*

Proof. Let $X_0, Y_0 \in \Gamma(\mathcal{H})$, and consider extensions $X, Y \in \mathfrak{X}(M)$. Because $N_J = 0$, we have

$$J([JX, Y] + [X, JY])) = [JX, JY] - [X, Y].$$

Because the right-hand side of this equation is tangent to ∂M, so is the left-hand side. Because $[JX, Y] + [X, JY]$ is tangent to ∂M, we can use Lemma 6.1.12 to find $([JX, Y] + [X, JY])|_{\partial M} \in \mathcal{H}$. Because $(JX)|_{\partial M} = J_{\mathcal{H}}X_0$ and similarly for Y we find $[J_{\mathcal{H}}X_0, Y_0] + [X_0, J_{\mathcal{H}}Y_0] \in \mathcal{H}$. Finally we remark that $N_{J_{\mathcal{H}}}(X_0, Y_0) = N_J(X, Y)|_{\partial M} = 0$, which finishes the proof. □

6.2 Strategy of constructing a complex foliation on S^5

6.2.1 Review of symplectic foliation on S^5

We quickly recall the main ingredients of our construction of a symplectic foliation on S^5.
Open books with trivial monodromy: Instead of constructing the symplectic foliation on S^5 we decided to make a detour and construct first a symplectic foliation on $S^5/\mathbb{Z}_3$. We did so because $S^5/\mathbb{Z}_3$ admits an open book decomposition with trivial monodromy, hence we had the decomposition:

$$S^5/\mathbb{Z}_3 = B \times D^2 \cup P_\varepsilon \times S^1.$$

We then proceeded by constructing symplectic foliations on both components and glueing them together.
Glueing symplectic foliations: To glue symplectic foliations we introduced the notion of a symplectic foliation tame near the boundary, and proved in Theorem 2.2.7 that it is possible to glue this.
Symplectic foliation on $M \times S^1$: For a symplectic manifold of cosymplectic type near the boundary we constructed a symplectic foliation tame near the boundary on $M \times S^1$ (Proposition 3.4.7). We did so by making use of the normal form around the boundary of symplectic manifolds with boundary of cosymplectic type.
Symplectic foliation on $B \times D^2$: For a cosymplectic manifold B we constructed a symplectic foliation tame near the boundary on $B \times D^2$ (Lemma 3.5.1). The proof of this fact that only used the local form of the symplectic structure on D^2 around the boundary.
Conclusion: Finally we constructed a symplectic structure of cosymplectic type on P_ε using a Donaldson open book decomposition. Afterwards we used the above to obtain a symplectic foliation on $S^5/\mathbb{Z}_3$ and hence on S^5.

6.2.2 Plan for constructing a complex foliation on S^5

To construct a complex foliation on S^5 one could proceed by using the same decomposition of $S^5/\mathbb{Z}_3$. Now we describe some further steps one could take.
Glueing complex foliations: Just as in the symplectic case we need to be able to glue complex foliations. To this end one should define an appropriate notion of complex foliation tame near the boundary and prove that these can be glued. In Section 6.3 we will do precisely this.
Local form: If we want to proceed using turbulisation we first need to define a local model of complex manifolds around the boundary. In the symplectic setting the local form arose from properties of symplectic manifolds with boundary of cosymplectic type. So one should first find a class of complex manifolds which plays the role of symplectic manifolds with boundary of cosymplectic type. For purpose of exposition call such a class "complex manifolds with nice boundary". The definition of symplectic manifolds with boundary of cosymplectic type was somewhat intrinsic. We can however cheat a bit and simply define a complex manifold with nice boundary to be a complex manifold which satisfies a particular local form around the boundary.
Turbulisation, $M \times S^1$: Using this local form, one could try to mimic the proof of turbulisation. In Section 6.4 we will first define a local model and study some of its properties. Then we will give an attempt on proving a turbulisation theorem for complex manifolds with nice boundary. However our attempt fails, and we will make some comments on why we think it does. We will however be able to prove the existence of a complex foliation tame near the boundary on $D^{2n} \times S^1$. This will provide us with some insight into how one might proceed in a general setting.
Complex foliation on $B \times D^2$: The other ingredient for the proof is the existence of a complex foliation tame near the boundary on manifolds of the form $B \times D^2$. In the symplectic setting we studied the case where B is a cosymplectic manifold. In the complex case, we will thus need to define an analogue of cosymplectic manifolds. These should be CR-manifolds with certain properties. We remark that the proof in the symplectic case only used the local form of the symplectic structure on D^2. We will show that the complex structure on D^2 also admits a local form around the boundary. So if one can find the right class of CR-manifolds, obtaining a complex foliation tame near the boundary on $B \times D^2$ might be doable.

6.3 Glueing complex foliations

Just as for the symplectic foliations we will construct complex foliations by glueing two foliations on manifolds with boundary. The question is again when this is possible. Recall the extended space M_∞ as defined at the beginning of Section 2.1.

Definition 6.3.1. Let $(\mathcal{H}, J)$ be an almost CR-structure on a manifold with boundary such that the almost CR-structure is tangent to the boundary. We extend the almost CR-structure to a hyperplane distribution $(\mathcal{H}_\infty, J_\infty)$ on M_∞, by taking as hyperplanes $\partial M \times \{t\}$, and as almost complex structure the structure induced by ∂M. We call this the **trivial extension** of

$(\mathcal{H}, J)$ to M_∞.

Definition 6.3.2 ([MV02]). Let $(\mathcal{H}, J)$ be an almost CR-structure on a manifold with boundary. We call $(\mathcal{H}, J)$ **tame near the boundary** if

- $(\mathcal{H}, J)$ is tangent to the boundary, i.e. the tangent spaces of the connected components of ∂M are elements of the hyperplane distribution.
- There exists a collar neighbourhood k of ∂M such that the trivial extension $(\mathcal{H}_\infty, J_\infty)$ is smooth.

Lemma 6.3.3 (Lemma 1, [MV02]). *Let N_1, N_2 be two orientable manifolds endowed with almost CR-structures $(\mathcal{H}_1, J_1)$ and $(\mathcal{H}_2, J_2)$ which are tame near the boundary. Assume that there exists a diffeomorphism $\varphi : \partial N_1 \to \partial N_2$ preserving the induced almost complex structures. Then there exists an almost CR-structure on the glued space $M := N_1 \cup_\varphi N_2$, which restricts to the original almost CR-structures on N_1 and N_2. The almost CR-structure is integrable if and only if the CR-structures on N_1 and N_2 are integrable. It is Levi-flat if and only if the CR-structures on N_1 and N_2 are Levi-flat.*

Proof. To glue the hyperplane distributions we note that they are co-orientable by Proposition 6.1.10 and evoke Theorem 2.1.7. We define an operator $J : \mathcal{H}_1 \cup \mathcal{H}_2 \to \mathcal{H}_1 \cup \mathcal{H}_2$, by J_1 on $\mathcal{H}_1$ and J_2 on $\mathcal{H}_2$. A priori this operator is only continuous on M, and smooth on $M \setminus \partial N_1$. We will now show that it is in fact smooth on the entirety of M. Pick $x \in \partial N_1$ and consider local coordinates $\{(x, t)\}$ on U adapted to ∂N_1, that is locally ∂N_1 is given by $t = 0$. Identify J on U with a map

$$\varphi : U \to \mathrm{Gl}(\mathbb{R}^{2n}) \subset \mathbb{R}^N, (x, t) \mapsto J(x, t).$$

Identifying U with an open in $\mathbb{R}^n$ we can talk of the partial derivatives of this map. Because the almost CR-structures on N_1 and N_2 are tame near the boundary we have that $\lim_{t \nearrow 0}(\partial_2^i)\varphi(x, t) = 0$ and $\lim_{t \searrow 0}(\partial_2^i)\varphi(x, t) = 0$. Hence all partial derivatives of J are continuous, and we conclude that J is smooth. This shows that the almost CR-structure is in fact smooth on the entirety of M, hence we conclude the statement. The relation between the integrability and Levi-flatness of the glued CR-structure and the original CR-structures is clear. □

6.4 Towards turbulisation

The symplectic turbulisation theorem was proved using a local form of symplectic manifolds near their boundary. In this section we will construct a local model for complex manifolds near the boundary. Although we are unable to prove a normal form theorem in general, we will show that the complex structure on D^{2n} around its boundary does satisfy this local model. We will then try to prove a turbulisation theorem for manifolds which satisfy this local model. However this will not work and we will give some thoughts on why it does not.

6.4.1 Local model

Let $(\mathcal{H}, J)$ be a co-orientable almost CR-structure on N. Because the CR-structure is co-orientable there exists a globally defined vector field $W_0 \in \mathfrak{X}(N)$ transverse to $\mathcal{H}$. We consider $M := N \times \mathbb{R}$, and define

$$\widetilde{J} : TM \to TM, \quad \begin{cases} X & \mapsto J(X) \quad \text{if } X \in \mathcal{H}, \\ W_0 & \mapsto \frac{\partial}{\partial t}, \\ \frac{\partial}{\partial t} & \mapsto -W_0. \end{cases}$$

This clearly defines an almost complex structure on M, the question is whether it is integrable.

Lemma 6.4.1. *Let $(\mathcal{H}, J)$ be a co-orientable integrable almost CR-structure. The almost complex structure $\widetilde{J}$ as defined above is integrable if and only if $[X, W_0] \in \mathcal{H}$ and $J[X, W_0] = [J(X), W_0]$ for all $X \in \mathcal{H}$.*

Proof. It is easily verified that the vanishing of the Nijenhuis-tensor is equivalent to

$$[X, W_0] + \widetilde{J}([\widetilde{J}X, W_0]) = 0, \quad X \in \Gamma(\mathcal{H}).$$

Which is in turn equivalent to the relation:

$$\widetilde{J}[X, W_0] = [JX, W_0].$$

Because the right-hand side is an element of $\Gamma(TN)$, so must the left-hand side. This results in the condition that $[X, W_0] \in \Gamma(\mathcal{H})$ for all $X \in \Gamma(\mathcal{H})$. In conclusion we have that $\widetilde{J}$ is integrable if and only if $[W_0, X] \in \mathcal{H}$ and $J[X, W_0] = [J(X), W_0]$ for all $X \in \mathcal{H}$. □

Lemma 6.4.2. *The conditions in Lemma 6.4.1 are equivalent to the following:*

$$[W_0, \Gamma(\mathcal{H}^{(1,0)})] \subset \Gamma(\mathcal{H}^{(1,0)}), \quad [W_0, \Gamma(\mathcal{H}^{(0,1)})] \subset \Gamma(\mathcal{H}^{(0,1)}) \tag{6.1}$$

Proof. If the conditions of Lemma 6.4.1 are satisfied then it follows directly that (6.1) is satisfied, so we are left to show the converse. Let $X \in \Gamma(\mathcal{H})$ and write $X = X_1 + X_2$ for some $X_1 \in \Gamma(\mathcal{H}^{(1,0)})$ and $X_2 \in \Gamma(\mathcal{H}^{(0,1)})$. Then $[X, W_0] = [X_1, W_0] + [X_2, W_0]$, and thus $[X, W_0] \in \Gamma(\mathcal{H}_{\mathbb{C}})$. Because $[X, W_0]$ is clearly real we conclude $[X, W_0] \in \Gamma(\mathcal{H})$. Now

$$\begin{aligned} J[X_1 + X_2, W_0] &= J[X_1, W_0] + J[X_2, W_0] \\ &= i[X_1, W_0] - i[X_2, W_0] \\ &= [JX_1, W_0] + [JX_2, W_0] \\ &= [JX, W_0], \end{aligned}$$

which finishes the proof. □

We introduce the notation

$$W_0 \vee \frac{\partial}{\partial t} : TM \to TM, \begin{cases} X & \mapsto 0 \quad \text{if } X \in \mathcal{H}, \\ W_0 & \mapsto \frac{\partial}{\partial t}, \\ \frac{\partial}{\partial t} & \mapsto -W_0. \end{cases}$$

Taking into account the splitting $TM = \mathcal{H} \oplus \mathbb{R} \cdot W_0 \oplus \mathbb{R} \cdot \frac{\partial}{\partial t}$ we write $\widetilde{J} = J + W_0 \vee \frac{\partial}{\partial t}$. The same formula for the almost complex structure from Lemma 6.4.1 can be used to endow $N \times S^1$ with an integrable almost complex structure; with $\frac{\partial}{\partial t}$ replaced by $\frac{\partial}{\partial \varphi}$. Using the above notation this is given by $J + W_0 \vee \frac{\partial}{\partial \varphi}$.

Finding a vector field which satisfies the first condition in Lemma 6.4.1 is not difficult:

Lemma 6.4.3. *Let $(\mathcal{H}, J)$ be a co-orientable integrable almost CR-structure on N. There exists $W_0 \in \mathfrak{X}(N)$ transverse to $\mathcal{H}$ satisfying $[W_0, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H})$ in any of the following cases:*

i. The distribution $\mathcal{H}$ is a unimodular foliation.

ii. The distribution $\mathcal{H}$ is a contact distribution.

Proof. i): Suppose that $\mathcal{H} = \ker \theta$ and chose $W_0 \in \mathfrak{X}(N)$ such that $\theta(W_0) = 1$. We have, for $X \in \Gamma(\mathcal{H})$,

$$\begin{aligned} 0 = d\theta(X, W_0) &= X(\theta(W_0)) - W_0(\theta(X)) - \theta([X, W_0]) \\ &= -\theta([X, W_0]), \end{aligned}$$

hence $[X, W_0] \in \Gamma(\mathcal{H})$.
ii): Suppose that $\mathcal{H} = \ker \alpha$, and let $W_0 \in \mathfrak{X}(N)$ be the Reeb vector field of α. We have, for $X \in \Gamma(\mathcal{H})$,

$$\begin{aligned} 0 = d\alpha(X, W_0) &= X(\alpha(W_0)) - W_0(\alpha(X)) - \alpha([X, W_0]) \\ &= -\alpha([X, W_0]) \end{aligned}$$

hence $[X, W_0] \in \Gamma(\mathcal{H})$. □

Remark 6.4.4. This local model resembles the general local model for symplectic manifolds near their boundary, as given in Proposition 3.2.1. Now the role of the admissible form θ is played by the vector field W_0, this also illustrated by the proof of the previous lemma.

For a symplectic structure of cosymplectic type at the boundary, ω, there exist a vector field X near the boundary, transverse to the boundary such that $\mathcal{L}_X \omega|_U = 0$. For complex manifolds we would like to have something similar. The following lemma gives a condition on a complex manifold with boundary such that there exists enough data to construct a local model around the boundary. It does however not ensure that the complex structure satisfies this local model.

Lemma 6.4.5. *Let* $(M,\widetilde{J})$ *be a complex manifold, and suppose that there exists a vector field* W *near the boundary, tangent to and non-vanishing on the boundary such that*

$$\mathcal{L}_W(\widetilde{J})(X) := [\widetilde{J}X, W] - \widetilde{J}[X,W] = 0$$

for all X *near the boundary. Then* $W_0 := W|_{\partial M}$ *satisfies the assumptions of Lemma 6.4.1 for the CR-structure induced on the boundary.*

Proof. Let $X \in \Gamma(\mathcal{H})$, and let $\widetilde{X}$ be an extension of X. We note that $\widetilde{J}\widetilde{X}|_{\partial M} = JX$, hence $[\widetilde{J}\widetilde{X}, W]$ is tangent to the boundary. Because $[\widetilde{J}\widetilde{X}, W] = \widetilde{J}[\widetilde{X}, W]$, we thus also have that $\widetilde{J}[\widetilde{X}, W]$ is tangent to the boundary. Hence by Lemma 6.1.12 we find that $[\widetilde{X}, W]|_{\partial M} = [X, W_0] \in \Gamma(\mathcal{H})$. Restricting the equality $[\widetilde{J}\widetilde{X}, W] = \widetilde{J}[\widetilde{X}, W]$ to the boundary yields, $[JX, W_0] = J[X, W_0]$. This finishes the proof. □

Remark 6.4.6. For symplectic manifolds of cosymplectic type near the boundary, the vector field transverse to the boundary satisfying $\mathcal{L}_X\omega|_U = 0$ gave rise to a collar neighbourhood on which the symplectic structure takes a particular form. The vector field W of the above lemma is however tangent to the boundary and cannot be used to build a collar neighbourhood. However the vector field $\widetilde{J}W$ is transverse to the boundary, so perhaps this vector field can be used to construct a normal form for the complex structure.

Example 6.4.7 (D^{2n})**.** Consider $S^{2n-1} \subset D^{2n}$, and the integrable CR-structure $(\mathcal{H}, J)$ on S^{2n-1} induced by the complex structure J_0 on D^{2n}. Let $r = \sqrt{\sum_{i=1}^n x_i^2 + y_i^2}$ and define

$$W = \frac{1}{r}\sum_{i=1}^n x^i \frac{\partial}{\partial y_i} - y^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(D^{2n} \setminus \{0\}).$$

A direct computation shows that $\mathcal{L}_W(J_0) = 0$, hence by Lemma 6.4.5 we have that the complex structure defined in Lemma 6.4.1 is integrable. We have that

$$\widetilde{J}W = \frac{1}{r}\sum_{i=1}^n x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}$$

is a normal vector for ∂D^2. Consider the collar neighbourhood $k : \partial M \times [0,1) \to U$ defined by the flow of this vector field. Because $k_*(W_0) = W$, wee see that on U the complex structures $\widetilde{J}$ and J_0 coincide.

6.4.2 Turbulisation, failed attempt

Let M be a complex manifold with boundary, then $M \times S^1$ carries a natural complex foliation. However, this complex foliation is not tame near the boundary. We will assume that M has the local normal form described in Lemma 6.4.1 and try to adapt the foliation on $M \times S^1$ into one which is tame near the boundary.

Let ω be a symplectic structure of cosymplectic type at the boundary, and let (η, θ) be a cosymplectic structure induced on the boundary. In the turbulisation we interpolated between the local form, $\eta + \theta \wedge dt$, and the desired form, $\eta - \theta \wedge d\varphi$. If we want to continue analogously for the complex case we should deform $J - W_0 \vee \frac{\partial}{\partial t}$ into $J - W_0 \vee \frac{\partial}{\partial \varphi}$. Naively one would define:

$$\widetilde{J} = J - aW_0 \vee \frac{\partial}{\partial t} - bW_0 \vee \frac{\partial}{\partial \varphi}.$$

$$\begin{cases} X \mapsto J(X), \quad X \in \mathcal{H} \\ W_0 \mapsto -a\frac{\partial}{\partial t} - b\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial t} \mapsto aW_0 \\ \frac{\partial}{\partial \varphi} \mapsto bW_0 \end{cases}$$

for some good pair of functions (a, b). We compute

$$\widetilde{J^2} : \begin{cases} X \mapsto -X, \quad X \in \mathcal{H} \\ W_0 \mapsto -(a^2 + b^2)W_0 \\ \frac{\partial}{\partial t} \mapsto -a^2\frac{\partial}{\partial t} - ab\frac{\partial}{\partial \varphi} \\ \frac{\partial}{\partial \varphi} \mapsto -ab\frac{\partial}{\partial t} - b^2\frac{\partial}{\partial \varphi} \end{cases}$$

Recall that the turbulised foliation is defined by $\theta = fd\varphi + gdt$, where (f, g) is a good pair of functions. So for $\widetilde{J}$ to restrict to an almost complex structure on the leaves of the foliation we need to ensure $\widetilde{J^2}(X) = -X$ for all $X \in \ker\theta$. This results in the following equations for the functions:

$$\begin{aligned} a^2 + b^2 &= 1 \\ a^2 - ab\frac{f}{g} &= 1 \\ b^2 - ab\frac{g}{f} &= 1 \end{aligned}$$

One easily verifies that this set of equations does not have a solution within the non-negative functions, which shows that $\widetilde{J}$ cannot restrict to an almost complex structure on $\ker\theta$.

Remark 6.4.8. Intuitively it seems more difficult to turbulise complex structures than symplectic structures because of the following. The set of complex structures forms a closed subset of the bundle-automorphisms on a manifold, the set of symplectic structures however forms an open subset of the closed two-forms. Whilst turbulising symplectic structures we had to ensure that the two-form remained non-degenerate along the leaves, i.e. we had to maintain an inequality. However, when we try to turbulise complex structures we have to makes sure that the bundle automorphism squares to minus the identity, i.e. we had to maintain an equality.

6.5 Complex foliation on $D^{2n} \times S^1$

Although we are unable to prove a turbulisation theorem in general we will show the existence of a complex foliation tame near the boundary on the solid torus. Using this we will show that $S^{2n} \times S^1$ admits a complex foliation.

Lemma 6.5.1 (Lemma 2.bis,[MV02])**.** *There exists a complex foliation tame near the boundary on $D^{2n} \times S^1$.*

Proof. Let $d : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism satisfying $d'(t) > 1$ for all $t > 0$ and $d(t) = t$ for all $t \leq 0$. For instance we could pick $d(t) = t + e^{-1/t}$. We define $M = (\mathbb{C}^n \times [0, \infty)) \setminus \{(0, 0)\}$, and consider the $\mathbb{Z}$-action generated by

$$g : (w, t) \mapsto (2w, d(t)).$$

In Lemma 6.5.2 below we will prove that the $\mathbb{Z}$-action is properly discontinuous and $M/\mathbb{Z}$ is a smooth manifold diffeomorphic to $D^{2n} \times S^1$.
Obtaining foliation: Endow M with the product foliation. We note that the action identifies points on the leaf $\mathbb{C}^n \times \{t\}$ with points on the leaf $\mathbb{C}^n \times \{d(t)\}$, but no two points in the same leaf. Thus the images of the leaves under the projection map form a partition of $\text{Int}D^{2n} \times S^1$ into subspaces biholomorphic to $\mathbb{C}^n$. Furthermore, as the projection map is a local diffeomorphism the foliation charts on M induce foliation charts on $\text{Int}D^{2n} \times S^1$. So the interior of the solid torus is endowed with a complex foliation. The action on M preserves the boundary, hence $\partial M = \mathbb{C}^n \setminus \{0\}$ gets mapped diffeomorphically onto $S^{2n-1} \times S^1$ via the quotient map. It is endowed with a complex structure, because it is the discrete quotient of a complex manifold[2]. In conclusion the complex foliation on M descends to a complex foliation on $D^{2n} \times S^1$ which is tangent to the boundary.
Tame near boundary: We will now show that this foliation is in fact tame near the boundary. We remark that the $\mathbb{Z}$-action on M can be extended to a properly discontinuous action on $X := \mathbb{C} \times \mathbb{R} \setminus (\{0\} \times (-\infty, 0])$. The action on $\mathbb{C}^n \setminus \{0\} \times \{t\} \subset X$ for $t \leq 0$, gives as quotient $S^{2n-1} \times S^1$. In conclusion the space $X/\mathbb{Z}$ is given by $(S^{2n-1} \times S^1) \times (-\infty, 0] \cup D^{2n} \times S^1$. We also see that the product foliation on X descends to a complex foliation on $X/\mathbb{Z}$. Now this foliation is precisely the trivial extension of the complex foliation on $(S^{2n-1} \times S^1) \times (-\infty, 0] \cup D^{2n} \times S^1$. Hence we see that the complex foliation on $D^{2n} \times S^1$ is tame near the boundary, which finishes the proof. □

Lemma 6.5.2. *The $\mathbb{Z}$-action defined in the above proof is properly discontinuous, and the manifold $M/\mathbb{Z}$ is diffeomorphic to $D^{2n} \times S^1$.*

Proof. Given any diffeomorphism $d : \mathbb{R} \to \mathbb{R}$ such that $|d'(t)| > 1$ for all $t > 0$, we can consider the $\mathbb{Z}$-action as defined in the above proof. We will prove that this action is properly

[2]For more details on this see [Huy05], this manifold is an example of a *Hopf manifold*.

discontinuous. We note that $|g(z,t)| > |(z,t)|$, hence for small enough ε the set

$$U = \{(z,t) \in M : |(z_0,t_0) - (z,t)| < \varepsilon\}$$

is such that its images under the $\mathbb{Z}$-action are all disjoint. Because the action is clearly smooth we conclude that $M/\mathbb{Z}$ is a smooth manifold.
Compactness: We will show that the set

$$V = \{(z,t) : |(z,t)| \leq 1, \left|g^{-1}(z,t)\right| \leq 1\}$$

gives a fundamental domain for the action. Because $|g(z,t)| > (z,t)$ one can apply g^{-1} often enough such that (z,t) is equivalent to an element in V. By construction of V we see that no two points in its interior are identified. Thus $M/\mathbb{Z}$ is obtained by identifying points in the boundary of V. Because V is compact, we thus conclude that $M/\mathbb{Z}$ is compact.
Independent of choice: Instead of proving directly that $M/\mathbb{Z}$ is diffeomorphic to $D^{2n} \times S^1$, we will proceed via a detour. Now for any diffeomorphism $d : \mathbb{R} \to \mathbb{R}$ such that $|d'(t)| > 1$ for all $t > 0$ we denote $M_d := M/\mathbb{Z}$, where the $\mathbb{Z}$-action is defined using d. We will show that the for different choices of d the manifolds M_d are all diffeomorphic.
Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(s) = 0$ for all $s \leq 0$ and $f(s) \geq 1$ for all $s \geq 1$. For $d, \widetilde{d} : \mathbb{R} \to \mathbb{R}$ such that $|d'(t)| > 1$ for all $t > 0$ we define a family of diffeomorphisms

$$d_s(t) = f(s)d(t) + (1 - f(s))\widetilde{d}(t).$$

We define a $\mathbb{Z}$-action on $M \times \mathbb{R}$ by

$$((x,t),s) \mapsto ((2x, d_s(t)), s)$$

Because $|d_s'(t)| > 1$ we see that the quotient is a manifold, which we denote by $\widetilde{M}$. We also note that $\widetilde{M}$ admits a natural fibration over $\mathbb{R}$, which has as fibres M_{d_s}. Because the fibres are compact, we see that the fibration is proper. We can thus apply Ehresmann's theorem (A.2.2) to see that the fibration is locally trivial. Because $\mathbb{R}$ is connected we thus conclude that all fibres of $\widetilde{M} \to \mathbb{R}$ are diffeomorphic. In particular we have $M_d \simeq M_{\widetilde{d}}$. Now take d as in the proof of the above lemma, and define $\widetilde{d}(t) = 2t$. To finish the proof we are thus left to prove that $M_{\widetilde{d}} \simeq D^{2n} \times S^1$. For the $\mathbb{Z}$-action associated to $\widetilde{d}$ the fundamental domain is given by:

$$V = \{(z,t) \in M : 1 \leq |(z,t)| \leq 2\} = [1,2] \times D^{2n}.$$

Now $M_{\widetilde{d}}$ is obtained from this by glueing $\{1\} \times D^{2n}$ with $\{2\} \times D^{2n}$ via the identity map. In conclusion $M_{\widetilde{d}} \simeq D^{2n} \times S^1$, which finishes the proof. □

Example 6.5.3 ([MV02])**.** We have

$$S^{2n} \times S^1 = D^{2n} \times S^1 \cup_{\mathrm{id}} D^{2n} \times S^1,$$

hence by applying the Lemma 6.5.1 twice, together with Lemma 6.3.3, we obtain a complex foliation on $S^{2n} \times S^1$.

Remark 6.5.4. It is interesting to note that although the above ensures the existence of a complex foliation on $S^{2n} \times S^1$, the existence of a symplectic foliation is still an open problem for $n \geq 3$. For $S^4 \times S^1$ the existence is known ([Tor15]). The underlying foliation of the symplectic foliation on $S^4 \times S^1$ however differs from the above. In fact the foliation constructed in the above example cannot be symplectic, because the compact leaf of the foliation is diffeomorphic to $S^{2n-1} \times S^1$ and $H^2(S^{2n-1} \times S^1) = \{0\}$ for $n \geq 2$.

6.6 Concluding remarks

We finish this chapter by reviewing our plan for constructing a complex foliation on S^5, using what we have learned from our attempts in the previous sections.

Remark 6.6.1 (Turbulisation)**.** One key property of complex manifolds that was used in Lemma 6.5.1, is the fact that quotients of complex manifolds are again complex manifolds (Proposition 2.1.13, [Huy05]). A similar result does not hold in general for symplectic manifolds. Instead of constructing complex foliations on manifolds using some variant of a turbulisation procedure, one could try to construct complex foliations out of quotients of foliated manifolds. Turbulisation used the strength of symplectic structures; namely that small perturbations by closed forms keep the structures symplectic. In the complex case it might thus be better to use the strength of complex structures and proceed via taking quotients.

Remark 6.6.2 (Meersseman and Verjovsky's attempt)**.** Although [MV02] contained a mistake, perhaps there are still things to learn from it. Meersseman and Verjovsky explain in [MV11] the mistake in their proof, which we summarize here. In the construction they decompose S^5 into two components. At first they believed that this decomposition was precisely as in the standard open book decomposition on S^5, but this turned out to be wrong. The inside component was, instead of being diffeomorphic to $B \times D^2$ diffeomorphic to some other manifold. However, the boundary of this manifold was still diffeomorphic to $B \times S^1$, so glueing it to the outside component of the open book of S^5 still results in a smooth manifold. This manifold however has a non-trivial fundamental group, so cannot be diffeomorphic to S^5.
So although the proof in the inside component fails, the proof for the outside component seems correct. Studying this will perhaps provide some more insight into the construction of complex foliations.

Remark 6.6.3 (Lawson's foliation)**.** Also interesting to note is that Meersseman and Verjovsky claimed to have proven that Lawson's foliation does not admit a leafwise complex structure [MV07]. This pre-print has appeared between the first paper with the mistake and the correction to it, so the author is not certain whether the proof is entirely correct. Although the author hasn't been able to take a thorough look on the paper, the proof of the non-existence of a leafwise complex structure on Lawson's foliation does appear to be independent of the previous paper.

Chapter 7

Outlook

In this final chapter we give an outlook on possible further research.

7.1 Other examples of symplectic foliations

So far, Theorem 5.2.4 produced only one example of a symplectic foliation. It would be interesting to find other cases in which this theorem can be applied. We elaborate on two possibilities in the below.

7.1.1 $S^4 \times S^1$

The existence of a symplectic foliation on $S^4 \times S^1$ has been established in different ways. Osorno Torres constructs a symplectic foliation on $S^4 \times S^1$ using an achiral Lefschetz fibration on S^4 [Tor15]. Mori also proves the existence using an open book decomposition [Mor15]. The fact that Mori uses an open book decomposition in his proof leads to the following:

Question 1. *Can one construct a symplectic foliation on $S^4 \times S^1$ using Theorem 5.2.4 or by similar means?*

7.1.2 Other lens spaces

The following idea has been suggested to the author by J. Stienstra. The open book decomposition on S^5 was constructed using a degree three homogeneous polynomial. We used a polynomial of degree three because the genus-degree formula for a curve in $\mathbb{C}P^2$ ensured that the binding of the open book fibres over S^1. This resulted in the fact that $S^5/\mathbb{Z}_3$ admitted a Donaldson open book decomposition, and consequently a symplectic foliation. There exists modifications of projective space in algebraic geometry, called *weighted projective spaces*. For these spaces there also exists genus-degree formulae for curves which differ from the one in $\mathbb{C}P^2$ [Hos16]. Although these weighted projective spaces are not smooth in general, they are in some instances. The actions used to define general lens spaces and weighted projective spaces are quite similar. This leads to the following:

Question 2. *Can certain lens spaces be realized as principal S^1-bundles over (smooth) weighted projective spaces? If so, can one use Theorem 5.2.4 or similar arguments to construct symplectic foliations on these spaces?*

7.2 Contact geometry v.s. symplectic foliations

In this thesis we saw that contact geometry played an important role in constructing symplectic foliations. The main ingredient for constructing a symplectic foliation on S^5 was the existence of an open book decomposition supporting a contact form. This hints at a possible relation between these two structures. In this section we state a conjecture regarding this relation.

7.2.1 A conjecture

Foliations and contact structures can be seen as opposites of each other. Still there are relations between these structures. In 1998, Eliashberg and Thurston found a way to deform contact structures on three-dimensional manifolds into foliations [Eli98]. To do this, they invented *confoliations*, hyperplane distributions which capture both foliations and contact structures. It is not clear whether a relation between foliations and contact structures can be found in higher dimensions. We believe that there exist deformations in dimension three because foliations on three-dimensional manifolds are always symplectic. Therefore, we believe that a relation between *symplectic* foliations and contact structures is more viable. The following conjecture is, in our opinion, the right generalization of the work of Eliashberg and Thurston:

Conjecture 1. *Symplectic foliations and contact structures are, in an appropriate sense, deformations of each other.*

Below, we will motivate why this conjecture could hold and what its applications are. Furthermore, we will go into the statement in more detail, and describe specific methods to approach the conjecture.

Motivation

Another hint that there is a relation between contact structures and symplectic foliations is the fact that they both carry the same topological data, often called the *almost structure*. This is a hyperplane distribution defined by the kernel of a one-form, together with a non-degenerate two-form along the distribution. Recently, Borman, Eliashberg and Murphy established an h-principle for contact structures [BEM15]. An h-principle is, loosely speaking, a manner of deforming an almost structure into a classical one. Given a symplectic foliation, one can forget the differentiable structure and then use the h-principle to deform it into a contact structure.

This is a partial answer to Conjecture 1, but has the problem that the deformation does not take in account the differentiable structure.

Applications

A fruitful use of the conjecture is to transport statements about contact geometry to statements about symplectic foliations and vice versa. Some statements for contact structures tend to be more difficult to prove than the corresponding statement for symplectic foliations, and vice versa. For instance, the existence of a symplectic foliation on T^{2n+1} is trivial, whereas the existence of a contact structure on T^{2n+1} was only proven recently [Bou02]. On the other hand we have seen in this thesis that the existence of a symplectic foliation on S^{2n+1} is a very difficult problem, whereas there exists a canonical contact structure on S^{2n+1}.

7.2.2 The five-sphere

Through the work of Osorno Torres and this thesis we now have a good understanding of the symplectic foliation on S^5. It is thus natural to study Conjecture 1 first in this specific case. Mori has found a way to deform the symplectic foliation on S^5 into the standard contact structure [Mor15]. He gives one possible extension of Eliashberg and Thurston's confoliations to higher dimensions called *$\varepsilon\tau$-confoliations*. He proves that the contact structure on S^5 can be deformed into the symplectic foliation via these $\varepsilon\tau$-confoliations. His methods are however very technical in nature, and we therefore ask the following:

Question 3. *Is there a more geometrical way of deforming the contact structure on S^5 to the symplectic foliation?*

Appendix A

Appendix

A.1 Symplectic normal forms

Here we collect some results concerning local forms of symplectic manifolds around symplectic submanifolds. Let $N \subset M$ be a symplectic submanifold. Then the tangent bundle admits the decomposition $TM|_N = TN \oplus TN^\omega$. We see that TN^ω defines a normal bundle for $N \subset M$. We in fact have that TN^ω inherits the structure of a *symplectic vector bundle*:

Definition A.1.1. Let M^{2n} be a smooth manifold. A **symplectic vector bundle** is a pair (E, ω), where $E \to M$ is a real vector bundle of rank $2n$, and $\omega \in \Gamma(\Lambda^2 E^*)$ is such that ω_x is symplectic for every $x \in M$.
An **isomorphism of symplectic vector bundles** $(E, \omega), (E', \omega')$ is a bundle isomorphism $\Phi : E \to E'$ which satisfies $\Phi^*\omega' = \omega$.

Definition A.1.2. A complex structure J on a symplectic vector bundle (E, ω) is said to be **compatible** with ω if $g \in \Gamma(\mathrm{Sym}^2 T^*M)$ defined by

$$g(u, v) := \omega(u, Jv), \quad \text{for all } u, v \in \Gamma(TM)$$

defines a fibrewise metric on E.

Theorem A.1.3 ([Can05])**.** *For any symplectic vector bundle (E, ω), there exists a complex structure on E which is compatible with ω. For any two such complex structures J, J' the resulting complex vector bundles $(E, J), (E, J')$ are isomorphic as complex vector bundles.*
Two symplectic vector bundles are isomorphic if and only if the corresponding complex vector bundles are isomorphic.

It turns out that the symplectic normal bundle TN^ω fully characterizes the behaviour of the symplectic form around the neighbourhood:

Theorem A.1.4 ([Can05])**.** *Let (M_0, ω_0) and (M_1, ω_1) be symplectic manifolds with diffeomorphic compact symplectic submanifolds X_0 respectively X_1. Suppose that there is an isomorphism*

$\varphi : \mathcal{N}(X_0) \to \mathcal{N}(X_1)$ *of the corresponding symplectic normal bundles covering a symplectomorphism* $\varphi : (X_0, \omega_0|_{X_0}) \to (X_1, \omega_1|_{X_1})$. *Then there exist tubular neighbourhoods* $\mathcal{U}_0 \subset M_0$ *and* $\mathcal{U}_1 \subset M_1$ *of* X_0 *respectively* X_1 *and a symplectomorphism* $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$.

Let $(N, \omega_N) \subset (S, \omega)$ be a codimension-two symplectic submanifold. Consider the normal bundle $\mathcal{N}(N)$ of N in S. Because N is a codimension-two submanifold, the normal bundle is a two-dimensional real symplectic vector bundle. Now we endow $\mathcal{N}(N)$ with a complex structure compatible with the fiberwise symplectic form (see Theorem A.1.3). Now we consider the induced principal S^1-bundle which we denote by $h : B' \to N$. Endow B' with a connection one-form $\alpha' \in \Omega^1(B')$ and let $\sigma \in \Omega^2(N)$ denote the induced curvature (i.e. $h^*\sigma = d\alpha'$). Endow $B' \times \mathbb{C}$ with the diagonal S^1-action, and define the form

$$\Omega = d((1 - r^2)\alpha' + r^2 d\varphi) \in \Omega^2(B' \times \mathbb{C}).$$

Theorem A.1.5. *The form* Ω *as defined above descends to a form* $\underline{\Omega} \in \Omega^2(B' \times_{S^1} \mathbb{C})$. *Furthermore, there exists a tubular neighbourhood* $\mathcal{U}$ *of* $N \subset S$ *which is symplectomorphic to*

$$(\mathcal{U}, \omega) \simeq (B' \times_{S^1} D^2_\varepsilon, \Omega' := \omega_N - \sigma + \underline{\Omega}).$$

Proof. Ω **descends**: To prove that Ω descends to a form on $B' \times_{S^1} \mathbb{C}$, we will first show that $B' \times \mathbb{C} \to B' \times_{S^1} \mathbb{C}$ is a principal bundle. Because the S^1-action on B' is free, we have that the S^1-action on $B' \times \mathbb{C}$ is free as well. Because the S^1-actions on B' and $\mathbb{C}$ are proper, we also have that the action on $B' \times \mathbb{C}$ is proper. We remark that the infinitesimal generator of the S^1-action is given by $X = R_{\alpha'} + \partial_\theta$. Using Cartan's formula we have

$$\begin{aligned} \iota_X \Omega &= -d(\iota_X((1 - r^2)\alpha' + r^2 d\varphi)) + \mathcal{L}_X((1 - r^2)\alpha' + r^2 d\varphi) \\ &= -d(1) \\ &= 0. \end{aligned}$$

We conclude that Ω is a basic form, hence it descends to a form $\underline{\Omega} \in \Omega^2(B' \times_{S^1} \mathbb{C})$.
Symplectomorphism: To prove that there exists a symplectomorphism we will use Theorem A.1.4. We first note that $h(B')$ is a submanifold of $B' \times_{S^1} \mathbb{C}$ diffeomorphic to N, we denote $h(B') = N'$. We have

$$\begin{aligned} h^*((\omega_N - \sigma + \underline{\Omega})|_{N'}) &= (h^*\omega_N - d\alpha' + d((1 - r^2)\alpha' + r^2 d\varphi))|_{B'} \\ &= (h^*\omega_N)|_{B'} \end{aligned}$$

We thus find that $\Omega'|_{N'} = \omega_N$. To finish the proof we are left to show that the symplectic normal bundles $\mathcal{N}(N)$ and $\mathcal{N}(N')$ are isomorphic as symplectic vector bundles. To do as such we recall from Theorem A.1.3 that two symplectic vector bundles are isomorphic if and only if the associated complex vector bundles are isomorphic. Because $d(r^2 d\varphi)$ is compatible with the standard complex structure on $\mathbb{C}$ we observe that the symplectic normal bundle $\mathcal{N}(N')$ is precisely $(B' \times_{S^1} \mathbb{C}, \Omega')$. Recall that B' was constructed as the principal S^1-bundle associated to the complex vector bundle $\mathcal{N}(N)$, hence $B' \times_{S^1} \mathbb{C}$ and $\mathcal{N}(N)$ are isomorphic as complex line bundles. This proves that $\mathcal{N}(N)$ and $\mathcal{N}(N')$ are isomorphic as symplectic vector bundles, which finishes the proof. □

A.2 (Locally trivial) fibrations

We recall some results on the behaviour of (locally trivial) fibrations.

Definition A.2.1. A smooth map $\pi : M \to N$ is called **locally trivial** if for every $x \in M$, there exists an open neighbourhood U of $\pi(x)$ together with a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \pi^{-1}(\{\pi(x)\})$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \pi^{-1}(\{\pi(x)\}) \\ \downarrow \pi & \swarrow \mathrm{pr}_1 & \\ U & & \end{array}$$

Theorem A.2.2 (Ehresmann)**.** *Let M and N be smooth manifolds and let $\pi : M \to N$ be a proper surjective submersion. Then π is a locally trivial fibration.*

There is also a version for manifolds with boundary, but now we need to impose an extra condition on the boundary.

Theorem A.2.3 (Ehresmann with boundary)**.** *Let M and N be smooth manifolds, and assume that M has a boundary. Let $\pi : M \to N$ be a proper surjective submersion, such that $\pi|_{\partial M}$ is also as submersion. Then π is a locally trivial fibration.*

The following lemma will give a normal form for fibrations near the boundary.

Lemma A.2.4. *Let M and N be smooth manifolds, and assume that M has compact boundary. Let $\pi : M \to N$ be a proper surjective submersion, such that $\pi|_{\partial M}$ is also as submersion. Then there exists a collar neighbourhood of the boundary U such that the following diagram commutes:*

$$\begin{array}{ccc} \partial M \times [0,1) & \xrightarrow{\simeq} & U \\ \downarrow {\scriptstyle \pi|_{\partial M} \times \mathrm{id}} & & \downarrow {\scriptstyle \pi} \\ S^1 \times [0,1) & \xrightarrow{pr_1} & S^1 \end{array}$$

Claim 1. The fibers of π are transverse to the boundary.

Proof. Denote $m = \dim M$ and $n = \dim N$. Let $x \in \partial M$, and $p = \pi(x)$. By assumption we have that

$$(d\pi|_{\partial M})_x : T_x\partial M \to T_pN$$

is surjective. Hence the null-space of $(d\pi|_{\partial M})_x$ is at most $(m-1-n)$-dimensional. Now $T_x(\pi^{-1}(p)) = \ker(d\pi)_x$, hence $\dim T_x(\pi^{-1}(b)) \geq m-n$. Because of dimension reasons we thus see that $T_x(\pi^{-1}(p))$ cannot be completely contained in $T_x\partial M$. Hence $T_x\partial M + T_x\pi^{-1}(p) = T_xM$, which shows that the fibres of π are transverse to the boundary. □

Claim 2. There exists a vector field X near the boundary which is tangent to the fibers of π and points inwards.

We will postpone the proof of this claim, and use it to prove the theorem. Because ∂M is compact and X points inwards we have that the flow of X exists for some finite time, which we take equal to 1. We consider the collar neighbourhood $U \simeq \partial M \times [0,1)$ defined by the flow of this vector field. Because X is taken such that $d\pi(X) = 0$, we have that $\pi|_U$ is constant in the second variable. Because the fibers of π are transverse to the boundary, so is X and thus we have for $x \in \partial M$ that $\pi(x,0) = \pi|_{\partial M}(x)$. We conclude that the diagram commutes. To finish the proof we are left to prove the claim.
Proof of Claim 2:
Let $U \simeq \partial M \times [0,1)$ be any collar neighbourhood of the boundary. Consider an extension of the collar neighbourhood $\widetilde{U} = \partial M \times (-\delta, 1)$, and $\widetilde{\pi} : \widetilde{U} \to B$ an extension of π. Clearly $\widetilde{\pi}$ is still submersive at points in the boundary, hence for all $z \in \partial M$ there exists a neighbourhood V of $(z,0)$ such that $\widetilde{\pi}$ is a submersion on V. On V we can thus use local coordinates on which π becomes a projection. Using this local form and the fact that the fibres of π are transverse to ∂M, we see that we can find a vector field X' on V such that $d\widetilde{\pi}(X') = 0$ and $dt(X') > 0$. Using compactness of ∂M and a partition of unity argument we can find a vector field X on an open neighbourhood of the boundary with the required properties.

A.2.1 Mapping Tori

Here we recall how circle fibrations can be viewed as mapping tori.

Theorem A.2.5. *Let M be a compact manifold and let $\pi : M \to S^1$ be a submersion. Let $X \in \mathfrak{X}(M)$ be such that $\pi_*(X) = \frac{\partial}{\partial\varphi}$. Let φ denote the time-one-flow of X and denote $N = \pi^{-1}([0])$. Then there exists a diffeomorphism of N such that $M \simeq N \times_{\mathbb{Z}} \mathbb{R}$. Here the action is generated by $(x,t) \mapsto (\varphi(x), t-1)$.*

Proof. Consider the flow $\varphi_X^t : M \to M$ of this vector field, which as M is compact exists for all $t \in \mathbb{R}$. Consider

$$\varphi_X^t|_{M_{[0]}} : M_{[0]} \to M_{[t]}.$$

Claim 1. $\varphi_X^t|_{M_{[0]}}$ is well-defined and a diffeomorphism.

Proof. By naturality of the flow, we have that

$$\begin{aligned}\pi(\varphi_X^t(x)) &= \varphi_{\pi_* X}^t(\pi(x)) \\ &= \varphi_{\frac{\partial}{\partial\varphi}}^t([0]) \\ &= [t],\end{aligned}$$

which shows that the target is indeed $M_{[t]}$. Now the map is smooth as the fibers of π are submanifolds, and its inverse is given by φ_X^{-t}. □

Now consider the map

$$\begin{aligned}\widetilde{\psi} : N \times \mathbb{R} &\to M \\ (x,t) &\mapsto \varphi_X^t(x).\end{aligned}$$

Clearly this map is constant on the fibers of the action of $\mathbb{Z}$ on $N \times \mathbb{R}$, and thus descends to a smooth map $\psi : N \times_{\mathbb{Z}} \mathbb{R} \to M$. By the previous claim we have that this map is bijective. Also as $\widetilde{\psi}$ is a local diffeomorphism, so is ψ. We conclude that ψ is a diffeomorphism which finishes the proof. □

Note that using a connection on M one can always find a vector field as described in the above lemma.

Theorem A.2.6. *Let M be a compact manifold with boundary and let $\pi : M \to S^1$ be a submersion, such that $\pi|_{\partial M}$ is also a submersion. Denote $N = \pi^{-1}([0])$, then there exists a diffeomorphism of N such that $M \simeq N \times_{\mathbb{Z}} \mathbb{R}$, where the action is generated by the time-one-flow of X.*

Proof. Pick a vector field $X \in \mathfrak{X}(M)$, which satisfies $\pi_* X = \frac{\partial}{\partial\varphi}$ and is tangent to the boundary and points inward. Such a vector can be found using Lemma A.2.4. Now the proof goes exactly the same as for the case without boundary. □

A.3 Invariant tubular neighbourhoods

We recall the existence of invariant tubular neighbourhoods:

Theorem A.3.1 ([Kan07]). *Let G be a Lie group and let M be a principal G-bundle. Let N be a closed G-invariant submanifold of M. Then there exists a G-invariant tubular neighbourhood of N in M together with a G-equivariant retraction.*

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