Axiomatic sheaf cohomology theories

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Introduction

The notion of cohomology is one that arises in many subjects related to algebraic topology. One of its elementary uses is to distinguish between spaces. Cohomology has the property that two homeomorphic spaces will have isomorphic cohomology groups so if two spaces have different cohomology groups they cannot be homeomorphic. Two of these notions of cohomology are the de Rham cohomology, which is defined only for differentiable manifolds, and singular cohomology, which is defined for general topological spaces.

One difference between these two notions of cohomology is that singular cohomology can be defined using different coefficients while the de Rham cohomology cannot. This makes singular cohomology an even finer tool to distinguish between spaces, for if two spaces have the same cohomology in one set of coefficients their cohomology could be different in another. One example would be the cohomology of the real projective plane and a point. The singular cohomology spaces $H^p(M, \mathbb{R})$ with coefficients in $\mathbb{R}$ will for both cases be a copy of $\mathbb{R}$ for $p = 0$ and 0 otherwise. The singular cohomology spaces $H^p(M, \mathbb{Z})$ will however be different, a copy of $\mathbb{Z}$ for $p = 0$ for both spaces, but the first cohomology space for the projective plane will be equal to $\mathbb{Z}/2\mathbb{Z}$ while it is zero for the point.

The notion of cohomology with coefficients can be generalized by introducing the notion of a sheaf. Besides being a natural generalization of classical cohomology, sheaf cohomology also has many other uses besides its use to differentiate between spaces. One can for instance use the first cohomology space with coefficients in the sheaf of non-zero real functions $H^1(M, C^\infty(M; \mathbb{R}^*))$ to classify isomorphism classes of real line bundles and by taking functions with coefficients in $\mathbb{C}^*$ classify isomorphism classes of complex line bundles.

The notions of cohomology as given above have many properties, and there are many different definitions of cohomology, therefore the natural question to ask is whether these definitions are unique. This question will give rise to the notion of axiomatic sheaf cohomology theory which will give us cohomology for any given sheaf and will carry enough axioms to be unique.

The notion of axiomatic sheaf cohomology theory will turn out to be quite strong, in fact so strong that many classical notions of cohomology will not give rise to it. Then the question remains whether these notions of cohomology still relate to the notion of an axiomatic sheaf cohomology theory.

In this thesis we will first introduce sheaves as our main object of interest. Sheaves will turn out to be easy to work with but hard to define and thus we will introduce the concept of a presheaf. These presheaves will turn out to be easier to define and in turn will give rise to sheaves. To determine the uniqueness of the notion of cohomology we will define the notion of a sheaf cohomology theory in an axiomatic way, of which the axioms arise naturally from the known notions of cohomology. Using sheaf theoretical arguments we will then show that this axiomatic definition defines cohomology uniquely. To show existence of such a theory we will explicitly consider the notion of Čech cohomology and show that it gives rise to a cohomology theory. To answer the question how the notions of cohomology which will not give rise to a cohomology theory relate to this we will introduce and study the notion of sheaf resolutions. These objects will define cohomology for a certain class of sheaves for which the cohomology modules will be isomorphic to the corresponding sheaf cohomology modules.

Structure of this thesis

This thesis is organised as follows. We will mainly be following [3] but with some adaptations on the order of chapters. We will first show uniqueness of cohomology theories and then show existence using Čech cohomology. In Section 1 we will introduce the notion of a sheaf, which will be a topological space providing the coefficients for our cohomology. In Section 2 we will recall and extend the notion and properties of cochain complexes which are sequences from which cohomology arises naturally. In Section 3 we will give the definition of an axiomatic cohomology theory and show that it is unique. In Section 4 we will construct the Čech cohomology for arbitrary sheaves and show that it satisfies the properties of an axiomatic sheaf cohomology theory, thus showing the existence of such an axiomatic theory. In Section 5 we will define the notion of a sheaf resolution and show that the cohomology that it gives rise to is isomorphic to the corresponding sheaf cohomology. In Sections 6 and 7 we will define a notion of the de Rham and singular cohomology via these resolutions and also show that these definitions coincide with the more classical definitions.
We assume that the reader has a thorough understanding of point-set topology. Although some knowledge of differential geometry could be useful, this is only essential for Section 6 which can be omitted without loss of continuation.

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1 Sheaves

In this section we will define the notion of a sheaf and a presheaf of K-modules and show some of their basic properties.

1.1 Preliminaries

Let V be a real vector space. By definition we can multiply elements of V by real numbers, more formally, we have an operation $\mathbb{R} \times V \to V$ which satisfies the defining properties of vector spaces. Similarly if V is a complex vector space we have an operation $\mathbb{C} \times V \to V$. The notion of a K-module is a generalization of this concept by replacing $\mathbb{R}$ or $\mathbb{C}$ by an arbitrary ring.

Definition 1.1. Let K be a ring with $\mathbf{1}_K$ as identity. A *K-module* is an abelian group with an operation $(G, +)$ and an operation $K \times G \to G$ such that for all $f, g \in G$ and $k, l \in K$ the following holds:

1. $k(f + g) = kf + kg$
2. $(k + l)f = kf + lf$
3. $(kl)f = k(lf)$
4. $\mathbf{1}_K f = f$ ♢

Example 1.2. Let $K = \mathbb{Z}$, we have that the notion of a $\mathbb{Z}$-module coincides with that of an abelian group. For let G be any abelian group. We define an operation $\mathbb{Z} \times G \to G$ by $(n, f) \mapsto n \cdot f$, where $n \cdot f$ is defined by

$$n \cdot f = \begin{cases} \underbrace{f + \ldots + f}_{n \text{ times}} & \text{for } n \geq 1 \\ 0 & \text{for } n = 0 \\ \underbrace{f^{-1} + \ldots + f^{-1}}_{n \text{ times}} & \text{for } n \leq -1 \end{cases}$$

It is easy to show that this gives an operation $K \times G \to G$ which satisfies all the properties of a K-module. △

Definition 1.3. A map $\varphi : G \to H$ is a *K-module homomorphism* if

$$\varphi(kf + g) = k\varphi(f) + \varphi(g)$$

for all $f, g \in G$ and $k \in K$. A *K-module isomorphism* is a bijective K-module homomorphism. ♢

Equivalent to the notion of a quotient group or quotient vector space is the notion of a quotient module.

Definition 1.4. Let H be a submodule of a module G over K (a submodule is a subset of G which is itself a module over K). We define the *quotient module* G/H by the following equivalence relation: $a, b \in G$ are equivalent if and only if $b - a \in H$. It is easy to show that the quotient module is a module over K. ♢

Another notion we will need in the definition of a sheaf is that of a local homeomorphism.

Definition 1.5. A map $f : X \to Y$ is a *local homeomorphism* if for any point $x \in X$ there exists an open neighbourhood U of x such that $f(U)$ is open and $f|_U$ is a homomorphism. ♢

Note that all local homeomorphisms are necessarily open maps. Indeed let V be any open in X. Let $\{U_i\}$ be an open cover of V such that the restriction of f to each U_i is invertible. Then $f(V) = \cup_i f|_{U_i} (U_i \cap V)$ which is clearly open.

In this thesis we will assume that K is a ring and M a paracompact Hausdorff space. Recall that a paracompact space is a space M for which every open cover M admits a locally finite refinement.

1.2 Sheaves and presheaves

We first define the notion of a sheaf.

Definition 1.6. A *sheaf* $\mathcal{S}$ *of* K*-modules over* M consists of a topological space $\mathcal{S}$ together with a surjective map $\pi : \mathcal{S} \to M$ satisfying:

1. π is a local homeomorphism of $\mathcal{S}$ onto M.
2. $\pi^{-1}(m)$ is a K-module for each $m \in M$.
3. subtraction and multiplication of scalars are continuous.

We call the map π the *projection* and the K-module $\mathcal{S}_m = \pi^{-1}(m)$ the *stalk over* m. ♢

Denote the set of pairs (s_1, s_2) such that s_1 and s_2 are in the same stalk by $S \circ S$. With (3) we mean that the map from $S \circ S$ to S given by $(s_1, s_2) \mapsto s_1 - s_2$ is continuous as well as the map $s \mapsto ks$ where $k \in K$.

Example 1.7. A trivial, but not unimportant, example of a sheaf is the *constant sheaf* $\mathcal{G} = M \times G$, where G is a K-module with the discrete topology and $\mathcal{G}$ has the product topology. The projection π is defined by $\pi(m, g) = g$. With this definition it is clear that the stalk $\mathcal{S}_m$ is just the K-module G for every m, so 1.6.2 is satisfied. As G has the discrete topology it is clear that 1.6.1 and 1.6.3 are satisfied as well and thus we have that $\mathcal{G}$ is indeed a sheaf. △

For another example of a sheaf, we will need the notion of a *germ* of a function.

Definition 1.8. Let $m \in M$ and let f and g be smooth functions defined on open sets containing m. Then f and g are said to have the same *germ* at m if they agree on some neighbourhood of m. This induces an equivalence relation where two functions are said to be equivalent if they have the same germ at m. We denote the set of germs of smooth functions at m by F_m, and the germ of a function f at m by $\mathbf{f}_m$. ♢

Example 1.9. Let

$$\mathscr{C}^\infty(M) = \bigcup_{m \in M} F_m.$$

Together with the map $\pi : \mathscr{C}^\infty(M) \to M$, which sends $\mathbf{f}_m \in F_m$ to $m \in M$, this will become the *sheaf of germs of smooth functions on* M. We put a topology on $\mathscr{C}^\infty(M)$ by associating with every smooth function f, and every open U in M, the set

$$\bigcup_{m \in U} \mathbf{f}_m.$$

These sets form a basis for a topology on $\mathscr{C}^\infty$. We could show that this is a sheaf, but we will give a more general proof later in this section. △

Definition 1.10. A *local section of* $\mathcal{S}$ *over an open* U is a continuous map $f : U \to \mathcal{S}$ such that $\pi \circ f = \mathrm{id}$. With the 0-section we mean the map that assigns to every $m \in U$ the zero element of $\mathcal{S}_m$. We denote with $\Gamma(U, \mathcal{S})$ the set of sections of $\mathcal{S}$ over U and with $\Gamma(\mathcal{S})$ the *global sections* of $\mathcal{S}$, the sections which are defined on all of M. ♢

If we define addition and multiplication on $\Gamma(U, \mathcal{S})$ by

$$(f+g)(m) := f(m) + g(m)$$
$$kf(m) := k(f(m)), k \in K$$

we see that $\Gamma(U, \mathcal{S})$ becomes a K-module.
We will need the following properties of sections

Lemma 1.11. *(a) If two (local) sections agree on a point $p \in M$ then they agree on a neighbourhood of p.*
(b) Every element of $\mathcal{S}$ is the value of some local section of $\mathcal{S}$.

Proof. Let f and g be two sections defined on $V \subset M$ of $\mathcal{S}$ which agree on $p \in M$. Since π is a local homeomorphism there is a neighbourhood U of $f(p) = g(p)$ on which π has a local inverse. Since both f and g are this local inverse of π we see that f and g agree on $\pi(U) \cap V$ which is open as π is an open map.
Because π is a local homeomorphism at every point $p \in \mathcal{S}$, there exists a neighbourhood U of p, such that π restricted to U is invertible. This inverse is in fact a section of π, and $\pi^{-1}(\pi(p)) = p$, which proves the lemma. □

We will study maps between sheaves which respect the sheaf structure.

Definition 1.12. Let $\mathcal{S}$ and $\mathcal{S}'$ be sheaves on M with projections π and π' respectively. A continuous map $\varphi : \mathcal{S} \to \mathcal{S}'$ such that $\pi' \circ \varphi = \pi$ is called a *sheaf mapping*. Note that a sheaf map maps stalks onto stalks. A sheaf mapping which is a homomorphism of K-modules on each stalk is called a *sheaf homomorphism*. A *sheaf isomorphism* is a sheaf homomorphism with an inverse which is also a sheaf homomorphism. $\diamondsuit$

Lemma 1.13. *Sheaf mappings are local homeomorphisms and thus open maps.*

Proof. Locally $\varphi = \pi'^{-1} \circ (\pi' \circ \varphi) = \pi'^{-1} \circ \pi$, which is a local homeomorphism as it is the composition of local homeomorphisms. $\square$

We will describe how to define quotients of sheaves, to do this we first have to define the notion of a subsheaf.

Definition 1.14. A *subsheaf* of a sheaf $\mathcal{S}$ is an open set $\mathcal{R}$ together with the restricted projection map $\pi|_{\mathcal{R}} : \mathcal{R} \to \mathcal{S}$ such that the stalk $\mathcal{R}_m = \pi|_{\mathcal{R}}^{-1}(m)$ is equal to $\mathcal{R} \cap \mathcal{S}_m$ and $\mathcal{R}_m$ a submodule of $\mathcal{S}_m$ for each $m \in M$. $\diamondsuit$

It is easy to show that a subsheaf with the subspace topology inherited from the sheaf $\mathcal{S}$ is again a sheaf.

Definition 1.15. Let $\mathcal{R}$ be a subsheaf of $\mathcal{S}$. For each $m \in M$, let $\mathcal{F}_m$ denote the quotient module $\mathcal{S}_m/\mathcal{R}_m$, and

$$\mathcal{F} = \bigcup_{m \in M} \mathcal{F}_m.$$

Let $\tau : \mathcal{S} \to \mathcal{F}$ be the quotient map, and give $\mathcal{F}$ the quotient topology. We set the projection $\widetilde{\pi}$ to be the map that sends every element of $\mathcal{F}_m$ to m. Then $\mathcal{F}$ is the *quotient sheaf* of $\mathcal{S}$ modulo $\mathcal{R}$. $\diamondsuit$

Lemma 1.16. *The quotient sheaf $\mathcal{F}$ is a sheaf.*

Proof. It is clear that $\widetilde{\pi}^{-1}(m)$ is a K-module. To show that $\widetilde{\pi}$ is a local homeomorphism let U be a neighbourhood of a point $p \in M$ such that $\pi|_U$ is invertible. Then we define an inverse of $\widetilde{\pi}|_U$ by $\tau \circ \pi|_U^{-1}$. It is easy to show that this is indeed an inverse of $\widetilde{\pi}|_U$. Since τ is continuous because $\mathcal{F}$ has the quotient topology, we see that $\tau \circ \pi|_U^{-1}$ is continuous as well. To show that subtraction is continuous let $s_1, s_2 \in \mathcal{F}_m$, and let $s_1', s_2' \in \mathcal{S}_m$ be representatives of s_1 and s_2 respectively. Then $s_1 - s_2 = \tau(s_1' - s_2')$ is continuous because it is the composition of τ and the subtraction map on $\mathcal{S}_m$. Similarly we have that multiplication with $k \in K$ is continuous. Hence we conclude that $\mathcal{F}$ is a sheaf. $\square$

Example 1.17. Let $\varphi : \mathcal{S} \to \mathcal{F}$ be a sheaf homomorphism. We will construct a subsheaf called the *kernel* of φ. The stalk $(\ker\varphi)_m$ consists of all $p \in \mathcal{S}_m$ such that $\varphi|_{\mathcal{S}_m}(p) = 0 \in \mathcal{F}_m$. This set is open

because it is the inverse image of the zero-sheaf $M \times \{0\}$ under φ, and $M \times \{0\} \cong M = \pi(\mathcal{S})$ which is open because π is an open map.
Because $\varphi|_m$ is a K-module homomorphism, the stalk $(\ker\varphi)_m = \ker(\varphi|_m)$ is a submodule of $\mathcal{F}_m$. It is now clear that $(\ker\varphi)_m = \ker\varphi \cap \mathcal{F}_m$, and we see that $\ker\varphi$ is a subsheaf of $\mathcal{F}$.
Similarly we have that the *image* $\varphi(\mathcal{S})$ of φ, is a subsheaf of $\mathcal{F}$. It is open by Lemma 1.13, and from the definition of sheaf homomorphisms it follows directly that $(\operatorname{im}\varphi)_m = \operatorname{im}(\varphi|_m) = \operatorname{im}\varphi \cap \mathcal{F}_m$. △

The following theorem is a generalization of the first isomorphism theorem for modules.

Theorem 1.18 (First Isomorphism)**.** *Let $\varphi : \mathcal{S} \to \mathcal{F}$ be a sheaf homomorphism. For $s \in \mathcal{S}_m$ the map $\psi : \mathcal{S}/\ker\varphi \to \operatorname{im}\varphi$ given by $(s + (\ker\varphi|\mathcal{S}_m)) \mapsto \varphi(s)$ is a sheaf isomorphism.*

Proof. As φ is a sheaf map, ψ is a sheaf map as well. From the definition of ψ we see that $\psi|_{\mathcal{S}_m} : (\mathcal{S}/\ker\varphi)_m \to (\operatorname{im}\varphi)_m$ is a K-module homomorphism. By the definition of the quotient sheaf and the previous example we have that $(\mathcal{S}/\ker\varphi)_m = \mathcal{S}_m/\ker(\varphi|_{\mathcal{S}_m})$ and $(\operatorname{im}\varphi)_m = \operatorname{im}(\varphi|_{\mathcal{S}_m})$. Using the first isomorphism theorem for modules we see that $\psi|_{\mathcal{S}_m}$ is a K-module isomorphism. Hence we conclude that ψ is a sheaf isomorphism. □

A collection of sheaves and sheaf homomorphisms that is of great importance are the exact sequences.

Definition 1.19. A sequence of sheaves $\{\mathcal{S}_i\}$ and homomorphisms

$$\cdots \to \mathcal{S}_i \to \mathcal{S}_{i+1} \to \mathcal{S}_{i+2} \to \cdots$$

is called *exact* if at each stage the image of a given homomorphism is the kernel of the next. Exact sequences of K-modules are defined in the same way. By definition of a sheaf homomorphism, a sequence in sheaves is exact if and only if the corresponding sequence in stalks

$$\cdots \to (\mathcal{S}_i)_m \to (\mathcal{S}_{i+1})_m \to (\mathcal{S}_{i+2})_m \to \cdots$$

is exact. ♢

Definition 1.20. An exact sequence of sheaves of the form

$$0 \to \mathcal{S} \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{T} \to 0$$

is called a *short exact sequence.* ♢

In this case, since the map φ is injective, we can identify $\mathcal{S}$ with its image $\varphi(\mathcal{S})$ in $\mathcal{F}$. Because the kernel of ψ is $\varphi(\mathcal{S})$, we conclude that $\mathcal{T}$ is isomorphic to $\mathcal{F}/\mathcal{S}$.
The notion of a sheaf is a global one. A natural way in which sheaves arise is via presheaves, which are much more local objects. This makes presheaves much easier to work with. At the end of this section we will show that presheaves give rise to sheaves.

Definition 1.21. A *presheaf* $P = \{S_U : \rho_{U,V}\}$ is a collection of K-modules on M consisting of a K-module S_U for every U open in M, and a homomorphism $\rho_{U,V} : S_V \to S_U$ for each inclusion $U \subset V$ of opens in M such that $\rho_{U,U} = \text{id}$, and such that $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$ whenever $U \subset V \subset W$. ♢

Example 1.22. Assign to every open $U \subset M$ the K-module $C^\infty(U)$, and let $\rho_{U,V}$ be the map restricting functions $f : V \to \mathbb{R}$ to functions $f|_U : U \to \mathbb{R}$. This defines a presheaf $\{C^\infty(U); \rho_{U,V}\}$. Instead of smooth functions on U one could also take continuous functions or local sections of a sheaf over U, these too will give rise to a presheaf. △

Definition 1.23. Let $P = \{S_U; \rho_{U,V}\}$ and $P' = \{S'_U; \rho'_{U,V}\}$ be presheaves on M. A *presheaf homomorphism* of P to P' is a collection of homomorphisms $\varphi_U : S_U \to S'_U$ such that

$$\rho'_{U,V} \circ \varphi_V = \varphi_U \circ \rho_{U,V}$$

whenever $U \subset V$, that is the following diagram commutes:

$$\begin{tikzcd} S_V \arrow[r, "\varphi_V"] \arrow[d, "\rho_{U,V}"'] & S'_V \arrow[d, "\rho'_{U,V}"] \\ S_U \arrow[r, "\varphi_U"] & S'_U \end{tikzcd}$$

A *presheaf isomorphism* is a presheaf homomoprhism where all the φ_U are isomorphisms of K-modules. ♢

1.3 The relation between sheaves and presheaves

In this section we will construct presheaves from sheaves and vice versa and show that for some classes of sheaves and presheaves these constructions are inverses of each other.

Definition 1.24. Each sheaf $\mathcal{S}$ gives rise to a presheaf $\{\Gamma(U, \mathcal{S}); \rho_{U,V}\}$, where $\Gamma(U, \mathcal{S})$ are the sections of $\mathcal{S}$ over U, and $\rho_{U,V}$ is the restriction of a section. Because the maps $\rho_{U,V}$ are restrictions of functions it is clear that they satisfy the properties of a presheaf. We shall denote the map of sheaves to presheaves by α. We call $\alpha(\mathcal{S})$ the *presheaf of sections of* $\mathcal{S}$. ♢

We will construct a sheaf from a presheaf by considering a construction somewhat similar to germs of functions. The following definition will give rise to the stalks of this sheaf.

Definition 1.25. Let $P = \{S_U; \rho_{U,V}\}$ be a presheaf of K-modules on M. Let $m \in M$. Let

$$\mathcal{S}_m = \bigsqcup_{U \ni m} S_U / \sim$$

be the disjoint union modulo an equivalence relation. Where $f \in S_U$ and $g \in S_V$ are equivalent if and only if there is a neighbourhood W of m with $W \subset U \cap V$ such that $\rho_{W,U} f = \rho_{W,V} g$, i.e. f is equivalent to g if they have the same germ at m. ♢

The set $\mathcal{S}_m$ will become the stalk of the sheaf associated to the presheaf P, hence we give it the structure of a K-module.

Lemma 1.26. *Let $m \in U$, and denote by $\rho_{m,U} : S_U \to \mathcal{S}_m$ the projection which assigns to each element of S_U its equivalence class. Let $f \in S_U$ and $g \in S_V$ be representatives of classes s_1 and s_2 in $\mathcal{S}_m$ respectively. Let W be a neighbourhood of m such that $W \subset U \cap V$. Defining addition on $\mathcal{S}_m$ by*

$$s_1 + s_2 = \rho_{m,W}(\rho_{W,U} f + \rho_{W,V} g)$$

and multiplication by $k \in K$ by

$$ks_1 = \rho_{m,U}(kf).$$

These operations give $\mathcal{S}_m$ the structure of a K-module.

Proof. We will check that these maps are well-defined. Assume that $f \in S_U$ and $\widetilde{f} \in S_{\widetilde{U}}$ represent the same class s_1 in $\mathcal{S}_m$, i.e. there exists a $Z \subset U \cap \widetilde{U}$ such that $\rho_{Z,U} f = \rho_{Z,\widetilde{U}} \widetilde{f}$. Let $s_2 = \rho_{p,V} g$ where $g \in S_V$ and let W be a neighbourhood of m such that $W \subset \widetilde{U} \cap V$ and $W \subset Z$, then

$$\rho_{m,W}(\rho_{W,\widetilde{U}} \widetilde{f} + \rho_{W,V} g) = \rho_{m,W}(\rho_{W,Z} \circ \rho_{Z,\widetilde{U}} \widetilde{f} + \rho_{W,V} g) = \rho_{m,W}(\rho_{W,U} f + \rho_{W,V} g),$$

which shows that addition is well-defined, similarly one can show that multiplication is well-defined. It is easy to show that these operations give $\mathcal{S}_m$ the structure of a K-module for which the $\rho_{m,U}$ are all homomorphisms. □

Definition 1.27. The *associated sheaf* to a presheaf P is the sheaf

$$\beta(P) = \bigcup_{m \in M} \beta(P)_m,$$

where $\beta(P)_m = \mathcal{S}_m$ is the K-module defined in Definition 1.25. Let $\pi : \beta(P) \to M$ be the projection such that $\pi(\beta(P)_m) = m$. We set a topology on $\mathcal{S}$ by taking as a basis for a topology sets of the form

$$O_f = \{\rho_{p,U} f : p \in U\}$$

for all $f \in S_U$ and U open in M. ♢

In Example 1.9 we claimed that the given set $\mathscr{C}^\infty(M)$ was a sheaf. Now the construction there corresponds precisely to the set $\beta(P)$ where P is the presheaf of smooth functions as described in Example 1.22. Because the stalk $\mathcal{S}_m$ corresponds to the germs of smooth functions at m, and the topology and projection map agree in both cases. So if we show that $\beta(P)$ is indeed a sheaf for any presheaf P, then $\mathscr{C}^\infty(M)$ is a sheaf as claimed.

Proposition 1.28. *The associated sheaf $\beta(P)$, as defined in Definition 1.27, is a sheaf.*

Proof. We first check that the sets O_f indeed form a basis for a topology on $\mathcal{S}$. It is clear that these sets cover $\mathcal{S}$. Now take $s \in O_f \cap O_g$, say $s = \rho_{p,U} f = \rho_{p,V} g$. Then there exists a neighbourhood $W \subset U \cap V$ of p, such that $s = \rho_{p,W} \circ \rho_{W,U} f = \rho_{p,V} \circ \rho_{W,V} g$, hence $s \in O_{\rho_{W,U} f} \subset O_f \cap O_g$. We conclude that the opens O_f indeed form a basis for a topology on $\mathcal{S}$.
We will now show that π is a local homeomorphism. The map π restricted to a set O_f, takes the form $\pi(\rho_{p,U} f) = p$, which has a well-defined inverse $p \mapsto \rho_{p,U} f$. Both π^{-1} and π are continuous. Indeed let $f \in S_U$, then $\pi(O_f) = U$ and if $V \subset M$ open then $\pi^{-1}(V) = \bigcup_{g \in S_V} O_g$.
From the definition it is clear that $\pi^{-1}(m)$ is a K-module for all $m \in M$ so we are left to show that substraction and multiplication by scalars are continuous on $\mathcal{S}$. Let $f \in S_U$, we will show that the inverse image of $O_f = \{\rho_{p,U} : p \in U\}$ under taking differences is open. Let s_1, s_2 be elements of $\mathcal{S}_m$ such that $\rho_{p,U} f = s_1 - s_2$. Let $g \in S_V$ and $h \in S_W$ be representatives of s_1 and s_2 respectively. Then there exists a neighbourhood $Q \subset U \cap V \cap W$ such that

$$\rho_{Q,U} f = \rho_{Q,V} g - \rho_{Q,W} h.$$

Hence $\rho_{p,Q} f = \rho_{p,Q} g - \rho_{p,Q} h$ for all $p \in Q$, which shows that the open set

$$O_{\rho_{Q,V} g} \times O_{\rho_{Q,W} h} \cap \mathcal{S} \circ \mathcal{S}$$

gets mapped into O_f. If we let $\{(s_1^i, s_2^j)\}$ denote the set of all pairs whose difference is f and let g_i, h_i be there representatives we conclude that the open

$$\bigcup_i (O_{\rho_{Q,V} g_i} \times O_{\rho_{Q,W} h_i}) \cap \mathcal{S} \circ \mathcal{S}$$

is the inverse image of O_f. Thus we conclude that $\mathcal{S} \circ \mathcal{S} \to \mathcal{S} : (s_1, s_2) \mapsto s_1 - s_2$ is continuous.
Now let $f \in S_U$. We will show that the inverse image of O_f under multiplication by $k \in U$ is open. Let $s \in s_m$ such that $ks = \rho_{p,U} f$, and let $h \in S_V$ such that $\rho_{p,V} h = s$. There exists a neighbourhood $W \subset U \cap V$ of p such that $\rho_{W,V} ks = \rho_{W,U} f$. It is clear that $O_{\rho_{W,V} s}$ gets mapped into O_f by multiplication by k. From the above we also see that this $O_{\rho_{W,V} s}$ is the complete pre-image of O_f, which completes the proof that multiplication by k is continuous, and hence we conclude that $\beta(\mathcal{S})$ is a sheaf. $\square$

Each sheaf homomorphism $\varphi : \mathcal{S} \to \mathcal{F}$ gives rise to a presheaf homomorphism between $\alpha(\mathcal{S})$ and $\alpha(\mathcal{F})$ by composing elements of $\Gamma(\mathcal{S}, U)$ with φ. A presheaf homomorphism induces a sheaf homomorphism by the following lemma.

Lemma 1.29. *Let $\{\varphi_U\} : P \to P'$ be a presheaf homomorphism. Then the induced map $\varphi : \beta(P) \to \beta(P')$ defined by:*

$$\varphi : x \mapsto (\rho'_{p,U} \circ \varphi_U)(f), \tag{1.1}$$

where $f \in S_U$ is a representative of $x \in \beta(P)_p$, is a sheaf homomorphism.

The map φ is depicted in the following diagram:

$$\begin{tikzcd} \beta(P)_p \arrow[r, "\varphi"] & \beta(P')_p \\ S_U \arrow[u, "\rho_{p,U}"] \arrow[r, "\varphi_U"] & S'_U \arrow[u, "\rho'_{p,U}"'] \end{tikzcd}$$

Proof. This map is well-defined. Indeed let $g \in S_V$ be another representative of x and let $W \subset U \cap V$ be the neighbourhood on which f and g agree. Then

$$\varphi(x) = \rho'_{p,V}(\varphi_V(g)) = \rho'_{p,W}(\varphi_W(\rho_{W,V}(g))) = \rho'_{p,W}(\varphi_W(\rho_{W,U}(f))) = \rho'_{p,U}(\varphi_U(f)).$$

We will now show that this map is a sheaf homomorphism. Using the fact that both $\rho'_{p,U}$ and φ_U are K-module homomorphisms we see that the map $x \mapsto (\rho'_{p,U} \circ \varphi_U)(f)$ also is a K-module homomorphism. Hence $\varphi|_{\beta(P)_p}$ is a K-module homomorphism for all p. Let π, π' be the projection maps on $\beta(P), \beta(P')$ respectively. From the definition of π' we have that $\pi' \circ \rho'_{p,U} = p$. Hence $(\pi' \circ \varphi)(x) = p$ for all $x \in \beta(P)_p$ and because $\pi(x) = p$ we conclude that $\pi' \circ \varphi = \pi$. Thus we conclude that φ is a sheaf homomorphism. $\square$

We will now consider if the maps β and α are inverses of each other. We have the following proposition:

Proposition 1.30. *The sheaves $\mathcal{S}$ and $\beta(\alpha(\mathcal{S}))$ are canonically isomorphic.*

Proof. Let $\xi \in \beta(\alpha(\mathcal{S}))$ be the germ at p of some section f over U, i.e. $\xi = \rho_{p,U} f$. We will show that

$$\varphi : \xi \mapsto f(p)$$

is a sheaf isomorphism. This map is clearly well-defined, because another representative of ξ will have the same value at p as f.

It is a sheaf mapping because if we denote by π the projection on $\mathcal{S}$ and by π' the projection on $\beta(\alpha(\mathcal{S}))$, $\pi \circ \varphi(\xi) = \pi(f(p)) = p$ and furthermore $\pi'(\xi) = \pi'(\rho_{p,U} f) = p$ by definition of π'.

To show that φ is a sheaf homomorphism let $\xi = \rho_{p,U} f$ and $\eta = \rho_{p,V} g$ be elements of $\beta(\alpha(\mathcal{S}))_p$ and let k be an element of K. Then

$$\varphi(\xi + \eta) = (f + g)(p) = f(p) + g(p) = \varphi(\xi) + \varphi(\eta)$$
$$\varphi(k\xi) = (kf)(p) = k(f(p)) = k\varphi(\xi)$$

and thus it follows that φ is a homomorphism on each stalk and thus a sheaf homomorphism.

By Lemma 1.11 we have that every element of $\mathcal{S}$ is the value of some section of $\mathcal{S}$, i.e. for all a in $\mathcal{S}$ there exists a local section f and a point p in M such that $f(p) = a$. We will show that the map

$$\psi : a \mapsto \rho_{p,U} f$$

is an inverse of φ. This map is well-defined because if f and $\widetilde{f}$ agree on p they must agree on some neighbourhood of p, hence f and $\widetilde{f}$ have the same germ at p. This map is also a sheaf map because $\pi' \circ \psi(f(p)) = \pi'(\rho_{p,U} f) = p$ and $\pi(f(p)) = p$.
We now show that ψ is a sheaf homomorphism. Let $a, b \in \mathcal{S}$, let f be a section such that $f(p) = a + b$ and let f_a and f_b be sections such that $f_a(p) = a$ and $f_b(p) = b$. Then

$$\psi(a+b) = \rho_{p,U} f$$
$$\psi(a) + \psi(b) = \rho_{p,U_a} f_a + \rho_{p,U_b} f_b = \rho_{p,U_a \cap U_b}(f_a + f_b).$$

Because f and $f_a + f_b$ agree on p there must thus exists a neighbourhood of p on which they agree, hence they represent the same equivalence class in $\beta(\alpha(\mathcal{S}))$, and we conclude that $\psi(a + b) = \psi(a) + \psi(b)$. Similarly, let $\widetilde{f}$ be a section such that $ka = \widetilde{f}(p)$ and let f be a section such that $f(p) = a$. Then kf and $\widetilde{f}$ agree on p, and thus on a neighbourhood of p. This shows that kf and $\widetilde{f}$ represent the same equivalence class from which we conclude that ψ is a sheaf homomorphism. This finishes the proof that φ is a sheaf isomorphism as claimed. □

However it is in general not true that $\alpha(\beta(P))$ is isomorphic to P. Take for instance the presheaf $P = \{S_U; \rho_{U,V}\}$, where $S_U = K$ for any U, and the restrictions $\rho_{U,V}$ are zero if $U \neq V$. Because the restrictions are zero we have that in $\beta(P)_m$ every element of S_U is equivalent to $0 \in S_U$. So every stalk $\beta(P)_m$ will just be the zero-module. Then if we take the presheaf of sections we will get the zero module for every $U \subset M$. For a presheaf P to be isomorphic to $\alpha(\beta(P))$ we need it to be complete.

Definition 1.31. A presheaf $\{S_U; \rho_{U,V}\}$ on M is said to be *complete* if whenever an open set U is expressed as a (not necessarily countable) union $\bigcup_i U_i$ of open sets in M, the following two conditions are satisfied:

1. (Locality) Whenever f and g in S_U are such that $\rho_{U_i,U} f = \rho_{U_i,U} g$ for all i, then $f = g$.
2. (Glueability) Whenever there is an element $f_i \in S_{U_i}$ for each i such that $\rho_{U_i \cap U_j, U_i} f_i = \rho_{U_i \cap U_j, U_j} f_j$ for all i and j, then there exists $f \in S_U$ such that $f_i = \rho_{U_i,U} f$ for each i. ♢

Proposition 1.32. *If P is a complete presheaf, then $\alpha(\beta(P))$ is canonically isomorphic to P.*

Proof. Let $P = \{S_U; \rho_{U,V}\}$ be a complete presheaf. We define a presheaf homomorphism from P to $\alpha(\beta(P)) = \{\Gamma(\beta(P), U), \rho_{U,V}\}$ as follows. For each open U in M we define a map

$$\varphi_U : S_U \to \Gamma(\beta(P), U) : f \mapsto (p \mapsto \rho_{p,U} f). \tag{1.2}$$

Note carefully that the restriction maps $\rho_{U,V}$ are the same in both P and $\alpha(\beta(P))$. The collection $\{\varphi_U\}$ forms a presheaf homomorphism. Indeed if $f \in S_V$ then

$$(\rho_{U,V} \circ \varphi_V)(f) = \rho_{U,V}(p \mapsto \rho_{p,V} f) = (p \mapsto \rho_{p,U} f)$$
$$(\varphi_U \circ \rho_{U,V})(f) = p \mapsto \rho_{p,U} \rho_{U,V} f,$$

which clearly agree and if $f, g \in S_U$ then $\varphi_U(kf+g) = k\varphi_U(f)+\varphi_U(g)$. To show that $\{\varphi_U\}$ is a presheaf isomorphism we will first prove the injectivity of the homomorphisms φ_U. Assume that f in S_U gets mapped to the zero section in S_U. Then for every point $p \in U$ there exists a neighbourhood $U_p \subset U$ such that $\rho_{p,U_p} f = 0$. Note that the set $\{U_p\}$ covers U. By the locality axiom of a complete presheaf $\rho_{p,U_p} f = \rho_{p,U_p} 0$ for all p. Hence $f = 0 \in S_U$ and we conclude that φ_U is injective. To show that φ_U is surjective let $c : U \to \beta(P)$ be a section. Then for any point $p \in U$ we have that there exists an $f_p \in S_{U_p}$ such that this is a representative of the equivalence class of $c(q)$, i.e.: for all $q \in U_p$ we can find an element $f_p \in S_{U_p}$ such that

$$\rho_{q,U_p} f_p = c(q). \tag{1.3}$$

Then for any $pinU_p$ we get an $f_p \in S_{U_p}$ which form a collection $\{f_p\}$, and a cover $\{U_p\}$ of U. Now take $p, q \in U$, we will show that

$$\rho_{U_p \cap U_q, U_p} f_p = \rho_{U_p \cap U_q, U_q} f_q.$$

By equation (1.3) we have for all $r \in U_p \cap U_q$ that

$$\rho_{r,U_p \cap U_q} \circ \rho_{U_p \cap U_q, U_p} f_p = c(r)$$
$$\rho_{r,U_p \cap U_q} \circ \rho_{U_p \cap U_q, U_q} f_q = c(r).$$

By definition of $\rho_{r,U_p \cap U_q}$ this implies that there exists a neighbourhood $W_r \subset U_p \cap U_q$ of r such that

$$\rho_{W_r,U_p \cap U_q} \circ \rho_{U_p \cap U_q, U_p} f_p = \rho_{W_r,U_p \cap U_q} \circ \rho_{U_p \cap U_q, U_q} f_q.$$

If we construct W_r for all $r \in U_p \cap U_q$ we get a collection $\{W_r\}$ which covers $U_p \cap U_q$. Using the locally axiom of a complete presheaf we conclude that

$$\rho_{U_p \cap U_q, U_p} f_p = \rho_{U_p \cap U_q, U_q} f_q$$

for all p and q in U. Hence by the the gluability axiom of a complete presheaf there exists an $f \in S_U$ such that

$$f_p = \rho_{U_p,U} f \tag{1.4}$$

for all $p \in U$. Now we will show that $\varphi_U(f) = c$. Call the section where f gets mapped to s, then $\rho_{U_p,U} s$ will be the section $q \mapsto \rho_{q,U_p} \rho_{U_p,U} f$ which by equation (1.3) and (1.4) is equal to $c(q)$ for all $q \in U_p$. This concludes the proof that f gets mapped to c and thus the proof of the proposition. □

Remark 1.33. For complete presheaves we indeed have that the maps α and β are inverses of each other. Very often instead of the definition of a sheaf as given in Definition 1.6, one defines a sheaf to be a complete presheaf. It often turns out to be easier to show that something is a complete presheaf, as we do not have to consider the topology on the sheaf. ♢

We will finish this section with some examples which combine most of the theory discussed up to now.

Example 1.34 (Presheaf of discontinuous functions)**.** The collection

$$C(M) = \{f : U \to \mathbb{R}; \rho_{U,V}\},$$

where the maps $\rho_{U,V}$ are the restriction of functions is a complete presheaf. It is clear that this is a sheaf. To show it is a complete presheaf let $U \subset M$ be open and $U = \bigcup_i U_i$.
(Locality): Assume $f, g \in S_U$ and that $\rho_{U_i,U} f = \rho_{U_i,U} g$ for all i, then for all i and for all $x \in U_i$ we have that $f(x) = g(x)$, so it is clear that $f = g$.
(Glueability): Assume that $f, g \in S_{U_i}$ and that $\rho_{U_i \cap U_j, U_i} f_i = \rho_{U_i \cap U_j, U_j} f_j$. We define $f \in S_U$ by $f(x) = f_i(x)$ for all $x \in U_i$. We have that f is well-defined because $x \in U_i \cap U_j$ and we have by assumption that $f_i(x) = f_j(x)$. Furthermore f is still continuous. Indeed if V is any open in Y then $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$ is open by the fact that all the f_i are continuous. △

Example 1.35. An example of a presheaf that is not complete is the presheaf of bounded functions on $\mathbb{R}$. Let U, V be opens such that $U \subset V \subset \mathbb{R}$ and let S_U be the set of bounded functions on U. Together with the restriction maps $\rho_{U,V}$ of functions on V to functions on U this forms a presheaf $\{S_U; \rho_{U,V}\}$. This sheaf is however not complete as it fails the glueability axiom. Let $U_i = (-i, i)$, let $\mathbb{R} = \bigcup_i U_i$ and let $f_i \in S_{U_i}$ be the identity on U_i. Then $\rho_{U_i \cap U_j, U_i} f_i = \rho_{U_i \cap U_j, U_j} f_j$ for all i and j, but there exists no bounded function on the whole of $\mathbb{R}$ that restricts to the identity on every open U_i. Hence we conclude that the presheaf of bounded functions does not satisfy the glueability axiom. △

Another class of sheaves that will be of great importance are the so-called *fine sheaves*.

Definition 1.36. A sheaf $\mathcal{S}$ over M is said to be *fine* if for each locally finite open cover $\{U_i\}$ of M there exist homomorphisms $\{l_i\}$ from $\mathcal{S}$ to itself such that $\operatorname{supp}(l_i) \subset U_i$ and $\sum l_i = 1$. With $\operatorname{supp}(l_i)$ we mean the closure of the set $\{m \in M : l_i|_{\mathcal{S}_m} \neq 0\}$. We call the set $\{l_i\}$ a *partition of unity subordinate to the cover* $\{U_i\}$ *of* M. ♢

Example 1.37. The sheaf of germs of smooth functions, as defined in Example 1.9, is a fine sheaf. Let $\{U_i\}$ be a locally finite open cover of M, then there exists a partition of unity subordinated to $\{U_i\}$, see for instance [1, p. 91], which we denote by $\{\varphi_i\}$. We obtain homomorphisms on the presheaf $\{C^\infty(M), \rho_{U,V}\}$ to itself by defining for $f \in C^\infty(U)$

$$l_{i,U}(f) = \varphi_i|_U \cdot f.$$

The associated sheaf homomorphism l_i from $\mathscr{C}^\infty(M)$ to itself will form a partition of unity subordinated to the cover $\{U_i\}$ of M. Let $x \in \mathscr{C}(M)$ and let $x = \rho_{p,U} s$, then

$$\sum l_i(x) = \sum_i \rho_{p,U} l_{i,U}(s) = \rho_{p,U} \sum_i l_{i,U}(s) = \rho_{p,U} s = x.$$

Now let $p \notin \operatorname{supp}(\varphi_i)$, and let U be an neighbourhood of p which has empty intersection with $\operatorname{supp}(\varphi_i)$. The existence of such a neighbourhood is the result of an elementary topological proposition that states that a paracompact Hausdorff space is normal, see for instance [1, p. 91]. Then $\rho_{p,U} l_{i,U}(f) = 0$ for all f, which implies that $p \notin \operatorname{supp}(l_i)$, from which we conclude that $\operatorname{supp}(l_i) \subset \operatorname{supp}(\varphi_i) \subset U_i$ and hence conclude that the sheaf $\mathscr{C}^\infty(M)$ is indeed fine. △

The notion of smoothness is not a requirement in the previous example. Instead of smooth functions we could have taken continuous functions, or even discontinuous functions and we would still obtain a fine sheaf.

Example 1.38. Let $C^\infty(U, M)(x_0)$ be the space of smooth functions defined on $U \subset M$ vanishing at $x_0 \in M$ and let

$$\{C^\infty(U, M)(x_0); \rho_{U,V}\}$$

be a presheaf with associated sheaf $\mathscr{C}^\infty(M)(x_0)$. It is clear that we have an injection $\mathscr{C}^\infty(M)(x_0) \to \mathscr{C}^\infty(M)$ as this is an inclusion on the presheaf level. We then get a short exact sequence

$$0 \to \mathscr{C}^\infty(M)(x_0) \to \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0) \to 0.$$

We will now study this quotient sheaf. Let $f \in C^\infty(U, M)$ be such that $x_0 \notin U$, it then is clear that $(\mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0))_m = 0$ for $m \in U$. As we have assumed that our space is paracompact Hausdorff and thus normal, we have that every point $m \in M$, $m \neq x_0$, has a neighbourhood which does not contain x_0, hence $(\mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0))_m = 0$ for all $m \neq x_0$. We will show that the stalk at x_0 is equal to $\mathbb{R}$. Let $f \in C^\infty(U, M)$ be a function such that $f(x_0) \neq 0$. Consider the difference of f and the function g with constant value $f(x_0)$ on U, then clearly $g - f$ is a function which vanishes at x_0. Hence f is equivalent to $f + (g - f) = g$ in $(\mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0))_{x_0}$. We thus conclude that $f \in (\mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0))_{x_0}$ is completely determined by its value at x_0 and see that $(\mathscr{C}^\infty(M)/\mathscr{C}^\infty(M)(x_0))_{x_0} \cong \mathbb{R}$. △

The above example is the idea that gives rise to the following definition.

Definition 1.39. Let G be a K-module. The *skyscraper sheaf* $\operatorname{Skysc}(x_0)$ is defined for $x_0 \in M$ to be

$$\operatorname{Skysc}(x_0)_m = \begin{cases} 0 & \text{if } m \neq x_0 \\ G & \text{if } m = x_0 \end{cases}$$ ♢

The following example gives the means to turn a vector bundle into a sheaf.

Example 1.40. Let E be a vector bundle over M, and let $\Gamma(U, E)$ be the space of sections of E over U. Let $\rho_{U,V}$ be the restrictions of sections. Then the collection $\{\Gamma(U, E); \rho_{U,V}\}$ is a presheaf, from which we construct a corresponding sheaf. △

We have by the above that every vector bundle gives rise to a sheaf, it is however not true that every sheaf gives rise to a vector bundle There is a certain class of sheaves that do so which are called the locally free sheaves, but we will not cover these here.

2 Cochain complexes

Cohomology arises naturally from cochain complexes. In this section we will discuss cochain complexes of K-modules and discuss the K-modules which arise from them. We will call these modules cohomology modules associated to the cochain complex. These modules will often coincide with the cohomology modules of a cohomology theory as we will define in the next section.

Definition 2.1. A *cochain complex* U^* is a sequence of K-modules and homomorphisms

$$\cdots \to U^{q-1} \to U^q \to U^{q+1} \to \cdots$$

defined for all integers q such that at each stage the image of a given homomorphism is contained in the kernel of the next. We will refer to the homomorphisms $U^q \to U^{q+1}$ as d^q, or d if this does not cause any confusion. The fact that the image of a homomorphism is contained in the kernel of the next can be written as $d^{q+1} \circ d^q = 0$, or more concisely $d^2 = 0$. We call d^q the q-th *coboundary operator*. We will denote the kernel of d^q by $Z^q(U^*)$ and call its elements the q-th degree *cocycles* of the cochain complex U^*. We will denote the image of d^{q-1} by $B^q(U^*)$ and call its elements the q-th degree *coboundaries*. $\diamondsuit$

Using the fact that $d^2 = 0$, $B^q(U^*)$ is a subset of $Z^q(U^*)$ for all q.

Definition 2.2. The q-th *cohomology module* $H^q(U^*)$ associated to the cochain complex U^* is defined as the quotient module

$$H^q(U^*) = \frac{Z^q(U^*)}{B^q(U^*)}$$

$\diamondsuit$

Example 2.3. Let $\Omega^q(M)$ be the vector space of forms on a differentiable manifold M, and let d denote the exterior derivative operator. It is well known that $d^2 = 0$, hence the sequence

$$\cdots \to \Omega^{q-1}(M) \to \Omega^q(M) \to \Omega^{q+1}(M) \to \cdots$$

is a cochain complex. The cohomology modules associated to this cochain complex, $H^q(\Omega^*(M))$, are called the *de Rham cohomology modules* which we will study in more detail in Section 6. $\triangle$

We would like that a map between cochain complexes induces maps in the corresponding cohomology modules. The following class of maps will have this property.

Definition 2.4. A *cochain map* $U^* \to V^*$ is a collection of homomorphisms $\varphi_q : U^q \to V^q$ such that for each q the following diagram commutes:

$$\begin{tikzcd} U^{q+1} \arrow[r, "\varphi_{q+1}"] & V^{q+1} \\ U^q \arrow[u, "d_U"] \arrow[r, "\varphi_q"] & V^q \arrow[u, "d_V"'] \end{tikzcd} \tag{2.1}$$

$\diamondsuit$

Proposition 2.5. *A cochain map naturally induces homomorphisms in the corresponding cohomology modules.*

Proof. Let $\{\varphi_q\}$ denote such a collection of homomorphisms. Let f be a q-coboundary of U^*. By the commutativity of diagram (2.1), $d_V(\varphi_q f) = \varphi_{q+1}(d_U f) = 0$, hence $\varphi_q f$ is q-coboundary of V^*. Similarly let f be a $(q+1)$-cocycle of U^*, that is $f = d_U g$ for some $g \in U^q$. Then $\varphi_{q+1} f = \varphi_{q+1}(d_U g) = d_V(\varphi_q g)$, hence $\varphi_{q+1} f$ is a $(q+1)$-cocycle of V^*. As all φ_q map cocycles into cocycles and coboundaries into coboundaries we have that the map $H^q(U^*) \to H^q(V^*)$ defined by $[f] \mapsto [\varphi_q(f)]$ is a well-defined homomorphism. We will denote this map by φ^*. □

Definition 2.6. A cochain map for which all the homomorphisms $U^q \to V^q$ are isomorphisms is called an *isomorphism of cochain complexes.* ♢

We will use the following fact quite often so we state it as a lemma.

Lemma 2.7. *Let two cochain maps, $U^* \to V^*$ and $V^* \to W^*$, be given. Then the composition of the induced maps $H^q(U^*) \to H^q(V^*) \to H^q(W^*)$ agrees with the induced map of the composition $H^q(U^*) \to H^q(W^*)$.*

Proof. Denote the cochain maps by $\varphi : U^* \to V^*$ and $\psi : V^* \to W^*$. Then for $[w] \in H^q(U^*)$ we have that $\psi^*(\varphi^*[w]) = [(\psi \circ \varphi)(w)]$ and $(\psi \circ \varphi)^*[w] = [(\psi \circ \varphi)(w)]$ which proves the lemma. □

Definition 2.8. A sequence of cochain maps

$$0 \to U^* \to V^* \to W^* \to 0 \tag{2.2}$$

is called an *exact sequence* if for every q the sequence

$$0 \to U^q \to V^q \to W^q \to 0$$

is a short exact sequence of K-modules. ♢

Definition 2.9. A *homomorphism between short exact sequences* $0 \to U^* \to V^* \to W^* \to 0$ and $0 \to \overline{U}^* \to \overline{V}^* \to \overline{W}^* \to 0$ of cochain complexes consists of cochain maps $U^* \to \overline{U}^*$,$V^* \to \overline{V}^*$ and $W^* \to \overline{W}^*$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & U^* & \to & V^* & \to & W^* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{U}^* & \to & \overline{V}^* & \to & \overline{W}^* & \to & 0 \end{array} \tag{2.3}$$

This is equivalent to saying that the following diagram commutes for all q:

$$\begin{array}{ccccccccc} 0 & \to & U^q & \to & V^q & \to & W^q & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{U}^q & \to & \overline{V}^q & \to & \overline{W}^q & \to & 0 \end{array} \tag{2.4}$$

♢

One concept that is often seen in algebraic topology is the fact that a short exact sequence in some spaces induces a long exact one in some collection of K-modules. We will show that a short exact sequence of cochain complexes induces a long exact sequence in the corresponding cohomology modules.

Proposition 2.10. *Given a short exact sequence of cochain maps*

$$0 \to U^* \to V^* \to W^* \to 0,$$

consisting of short exact sequences

$$0 \longrightarrow U^q \xrightarrow{f^q} V^q \xrightarrow{g^q} W^q \longrightarrow 0,$$

there are homomorphisms

$$H^q(W^*) \xrightarrow{\delta} H^{q+1}(U^*)$$

for each q such that the following sequence is exact.

$$\cdots \longrightarrow H^{q-1}(W^*) \xrightarrow{\delta} H^q(U^*) \xrightarrow{f^*} H^q(V^*) \xrightarrow{g^*} H^q(W^*) \xrightarrow{\delta} H^{q+1}(U^*) \to \dots \tag{2.5}$$

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & U^{q+1} \arrow[u] \arrow[r, "f^{q+1}"] & V^{q+1} \arrow[u] \arrow[r] & W^{q+1} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & U^q \arrow[u] \arrow[r, "f^q"] & V^q \arrow[u] \arrow[r, "g^q"] & W^q \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & U^{q-1} \arrow[u] \arrow[r] & V^{q-1} \arrow[u] \arrow[r, "g^{q-1}"] & W^{q-1} \arrow[u] \arrow[r] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd} \tag{2.6}$$

Proof. We will define $\delta : H^q(W^*) \to H^{q+1}(U^*)$ by chasing over the above diagram. Let $w \in W^q$ be a cocycle. Since g is surjective, let $v \in V^q$ be such that $g(v) = w$. Then $0 = \partial_W w = d_W g(v) = g(d_V v)$, that is, $d_V v \in \ker(g) = \operatorname{im}(f)$. Hence there is a $u \in U^{q+1}$ such that $f(u) = d_V v$. Because $f(u) = d_V v$ we see that $d_V \circ f(u) = d_V^2 v = 0$. Using the fact that f is a cochain map we conclude that $f(d_U u) = 0$. Because f is injective $d_U u = 0$, hence u represents an equivalence class in $H^{q+1}(U^*)$ and we define $\delta[w] = [u]$.
We will show that δ is well-defined. Take $w, w' \in W^q$ such that $[w] = [w']$. Now with the same reasoning as before choose $v, v' \in V^q$ such that $g(v) = w$ and $g(v') = w'$. Now consider $d_V v, d_V v' \in V^{k+1}$. We have $g(d_V v) = d_W g(v) = 0$ and the same for v', thus both are in the kernel of g and thus in the image of f. Hence we can find $u, u' \in U^{k+1}$ such that $f(u) = d_V v$ and $f(u') = d_V v'$. We conclude that

$\delta[w-w'] = [u-u']$. To show that δ is well-defined we are left to show that $u-u'$ is a coboundary. Now consider $w-w' = g(v)-g(v')$. By assumption this in the image of d_W and thus there is a $\underline{w} \in W^{q-1}$ such that $d_W\underline{w} = g(v)-g(v')$. By surjectivity of g there is a $\underline{v} \in V^{q-1}$ such that $g(\underline{v}) = \underline{w}$. Now we consider $v - v' - d_V\underline{v}$, then

$$g(v - v' - d_V\underline{v}) = g(v) - g(v') - g(v) + g(v') = 0.$$

Therefore $v - v' - d_V\underline{v}$ is in the kernel of g hence also in the image of f. Therefore there exists a $\widetilde{u} \in U^k$ such that $f(\widetilde{u}) = v - v' - d_V\underline{v}$. Now

$$d_V f(\widetilde{u}) = d_V(f(v) - (v') - d_V f(\underline{v})) = f(u-u').$$

Therefore $f(d_U\widetilde{u}) = f(u-u')$, and since f is injective we have that $d_U\widetilde{u} = u-u'$. Hence $u-u' \in \mathrm{im} d_U$ consequently $[u-u'] = 0$. We therefore conclude that δ is well-defined.

We will now show that with this definition of δ the sequence (2.5) is indeed exact. We denote by f^* and g^* maps in cohomology classes induced form f and g.

Proof of $\ker(g^*) = \mathrm{im}(f^*)$: Let $[h] \in \mathrm{Im}(f^*)$ hence there is a $[\widetilde{h}]$ such that $f^*[\widetilde{h}] = h$. Now $g^*[h] = g^* \circ f^*[\widetilde{h}] = [(g \circ f)h] = [0]$, hence we have that $[h] \in \ker(g^*)$.

Let $[h] \in \ker(g^*)$, thus $[g(h)] = [0]$. This implies that $g(h) \in \mathrm{im}(d_W)$, which implies that there exists a $w \in W^{q-1}$ such that $d_W w = g(h)$. Because g is surjective there exists a $v \in V^{k-1}$ such that $g(v) = w$. Then $d_V v$ is again an element in V^q which is in $\mathrm{im}(d_V)$ hence we have that $[h - d_V v] = [h]$. By applying g on $h - d_V v$ we get that

$$\begin{aligned} g(h - d_V v) &= g(h) - g(d_V v) \\ &= g(h) - d_W g(v) = g(h) - d_W w = g(h) - g(h) = 0. \end{aligned}$$

We thus conclude that $h - d_V v$ is in the kernel of g, from which we in turn conclude that $h - d_V v$ is in the image of f. Hence there exists a $u \in U^q$ such that $f(u) = h - d_V v$. Then $f^*[u] = [f(u)] = [h - d_V v]$. But as $d_V v$ is in the image of d_V it follows that $f^*[u] = [h]$. We thus conclude that $[h] \in \mathrm{im}(f)$.

Proof of $\mathrm{im}(\delta) = \ker(f^*)$: Let $[h] \in \ker(f^*)$ since f is injective it has trivial kernel and it follows that $f(h) \in \mathrm{im}(d_V)$. Therefore there is a $v \in V^{q-1}$ such that $d_V v = f(h)$. Now we consider $g(v) \in W^{q-1}$. Following the definition of δ we have that $g(d_V v) = d_W g(v) = g(f(h)) = 0$, and thus that $d_V v \in \ker(g)$ and hence we conclude that $d_V v \in \mathrm{im}(f)$. By construction $f(h) = d_V v$, hence $[h] = \delta[g(v)]$ and thus we conclude that $[h] \in \mathrm{im}(\delta)$.

Let $[h] \in \mathrm{im}(\delta)$. Then there exists a $w \in W^{q-1}$, such that $\delta[w] = [h]$. Again we will follow the definition of δ. Because g is surjective there exists a $v \in V^{q-1}$ such that $g(v) = w$. Now $d_V v$ is in the kernel of g, which implies that $g(d_V v) = d_W g(v) = d_W w = 0$. Hence there is an h in U^q such that $f(h) = d_V v$. It follows directly that $f(h)$ is in the image of d_V which than implies that $[h]$ is in the kernel of f^*.

Proof of $\mathrm{im}(g^*) = \ker(\delta)$:

Let $[w] \in \operatorname{im}(g^*)$, hence there is a $[v] \in H^q(V)$ such that $g^*[v] = [w]$. Because v is a cocycle we have that $d_V v = 0$. Then as before we find u such that $f(u) = d_V v$. But as $d_V v = 0$ and f is injective it follows that $u = 0$ and thus that $\delta[g(v)] = [0]$. Which implies that $[w] \in \ker(\delta)$.
Let $[w] \in \ker(\delta)$, hence $\delta[w] = [u] = [0]$, and thus there exists a $\widetilde{u} \in U^k$ such that $d_U \widetilde{u} = u$. As before $f(u) = d_V v$, with $g(v) = w$.
For $x = v - f(\widetilde{u})$,

$$d_V x = d_V v - d_V f(\widetilde{u}) = f(u) - f(u) = 0.$$

Therefore x is a cocycle and thus represents an equivalence class in $H^q(V)$. Now $g^*[x] = [g(v) - g(f(\widetilde{u}))] = [w]$. Hence $[w] \in \operatorname{Im} g^*$, which concludes the proof that sequence (2.5) is exact. ☐

This definition of δ also gives rise to another commutative diagram which will be key to showing the existence of a cohomology theory.

Proposition 2.11. *The homomorphisms (2.3) of short exact sequences of cochain complexes together with the map δ give rise to a commutative diagram:*

$$\begin{tikzcd}
H^q(W^*) \arrow[r, "\delta"] \arrow[d] & H^{q+1}(U^*) \arrow[d] \\
H^q(\overline{W}^*) \arrow[r, "\overline{\delta}"] & H^q(\overline{U}^*)
\end{tikzcd} \tag{2.7}$$

Proof. Denote by $\overline{f}^q$ the cochain map $U^q \to V^q$ and by $\overline{g}^q$ the cochain map $V^q \to W^q$. To show that diagram (2.7) commutes we first construct a commutative lattice, that is, a three-dimensional commutative diagram for which all the sides commute:

$$\begin{tikzcd}
& U^{q+1} \arrow[rr] \arrow[dl] & & V^{q+1} \arrow[rr] \arrow[dl] & & W^{q+1} \arrow[dl] \\
\overline{U}^{q+1} \arrow[rr] & & \overline{V}^{q+1} \arrow[rr] & & \overline{W}^{q+1} & \\
& U^q \arrow[uu, dotted] \arrow[rr, dotted] \arrow[dl, dotted] & & V^q \arrow[uu, dotted] \arrow[rr, dotted] \arrow[dl, dotted] & & W^q \arrow[uu] \arrow[dl] \\
\overline{U}^q \arrow[uu] \arrow[rr] & & \overline{V}^q \arrow[uu] \arrow[rr] & & \overline{W}^q \arrow[uu] &
\end{tikzcd}$$

The horizontal segments of this diagram commute by the fact that we have a homomorphism between short exact sequences, and the vertical segments by the fact that the maps $U^* \to \overline{U}^*$,$V^* \to \overline{V}^*$ and $W^* \to \overline{W}^*$ are cochain maps.
We first describe the composition $H^q(W^*) \to H^{q+1}(U^*) \to H^{q+1}(\overline{U}^*)$ which sends $w \in W^q$ to some $\overline{u}$ in $\overline{U}^{q+1}$. Let $w \in W^q$. By surjectivity of g^q there exists $v \in V^q$ such that $g(v) = w$, we then lift v to $d_V v$.

As we have shown in the definition of δ, there exists a $u \in U^{q+1}$ such that $g(u) = d_V v$, hence $\delta(w) = u$, and finally we map u to $\overline{u} \in \overline{U}^{q+1}$.

We will now describe the composition $H^q(W^*) \to H^q(\overline{W}^*) \to H^{q+1}(\overline{U}^*)$ which sends $w \in W^q$ to some $\overline{u}'$ in $\overline{U}^{q+1}$. Let $\overline{w}'$ be the image of w under $W^q \to \overline{W}^q$. Then by surjectivity of $\overline{g}^q$ there exists a $\overline{v}' \in V^q$ such that $\overline{g}(\overline{v}') = \overline{w}'$. We then lift $\overline{v}'$ to $d_{\overline{V}}\overline{v}'$, and as we have shown in the definition of $\overline{\delta}$ there exists $\overline{u}' \in U^{q+1}$ such that $\overline{g}(\overline{u}') = d_{\overline{V}}\overline{v}'$, and we had that $\delta(\overline{w}) = \overline{u}'$.

To show that diagram (2.7) commutes we are left to show that $\overline{u}' - \overline{u}$ is a coboundary. If we map v to $\overline{v} \in \overline{V}^q$ then by commutativity of the diagram we see that $\overline{v}$ and $\overline{v}'$ both get mapped into $\overline{w}'$, i.e. $\overline{g}(\overline{v} - \overline{v}') = 0$. Thus there exists a $\overline{\overline{u}} \in \overline{U}^q$ such that $\overline{f}(\overline{\overline{u}}) = \overline{v} - \overline{v}'$. We will show that $d_U\overline{\overline{u}} = \overline{u} - \overline{u}'$. To do this we need that $d_{\overline{V}}\overline{v} = \overline{f}(\overline{u})$ which follows directly form the definition of $\overline{\delta}$. We then have that

$$\overline{f}(d_{\overline{U}}\overline{\overline{u}}) = d_{\overline{V}}\overline{f}(\overline{\overline{u}}) = d_{\overline{V}}\overline{v} - d_{\overline{V}}\overline{v}' = \overline{f}(\overline{u}) - \overline{f}(\overline{u}') = \overline{f}(\overline{u} - \overline{u}'),$$

and thus by injectivity of $\overline{f}$ we conclude that $d_U\overline{\overline{u}} = \overline{u} - \overline{u}'$. This completes the proof that diagram (2.7) commutes. $\square$

3 Cohomology theories

In this section we will give the definition of a cohomology theory and show its uniqueness.

Definition 3.1. A *sheaf cohomology theory* $\mathcal{H}$ for M with coefficients in sheaves of K-modules over M consists of

- a K-module $H^q(M, \mathcal{S})$ for each sheaf $\mathcal{S}$ and for each integer q and
- a homomorphism $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}')$ for each homomorphism $\mathcal{S} \to \mathcal{S}'$

such that the following six properties hold:

(a) For $q \leq 0$, $H^q(M, \mathcal{S}) = 0$ and there is an isomorphism $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$ such that for each homomorphism $\mathcal{S} \to \mathcal{S}'$ the following diagram commutes:

$$\begin{tikzcd} H^0(M, \mathcal{S}) \arrow[r] \arrow[d] & \Gamma(\mathcal{S}) \arrow[d] \\ H^0(M, \mathcal{S}') \arrow[r] & \Gamma(\mathcal{S}') \end{tikzcd}$$

(b) If $\mathcal{S}$ is a fine sheaf then $H^q(M, \mathcal{S}) = 0$ for all $q \geq 0$.

(c) If $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ is exact, then there exists a homomorphism $\delta : H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$, called the *connecting homomorphism*, such that the following sequence is long exact:

$$\cdots \to H^q(M, \mathcal{S}') \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}'') \xrightarrow{\delta} H^{q+1}(M, \mathcal{S}') \to \cdots$$

(d) The identity homomorphism $\mathrm{id} : \mathcal{S} \to \mathcal{S}$ induces the identity homomorphism $\mathrm{id} : H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S})$

(e) If the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}' \arrow[dl] \\ \mathcal{S}'' & \end{tikzcd}$$

commutes, then for each q the following diagram commutes:

$$\begin{tikzcd} H^q(M, \mathcal{S}) \arrow[r] \arrow[d] & H^q(M, \mathcal{S}') \arrow[dl] \\ H^q(M, \mathcal{S}'') & \end{tikzcd}$$

(f) For each homomorphism of short exact sequences of sheaves (defined in the same way as a homomorphism of short exact sequences of cochain complexes)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0 \end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd} H^q(M,\mathcal{S}'') \arrow[r] \arrow[d] & H^{q+1}(M,\mathcal{S}') \arrow[d] \\ H^q(M,\mathcal{F}'') \arrow[r] & H^{q+1}(M,\mathcal{F}') \end{tikzcd}$$

◊

As we eventually want to show that all cohomology theories are equivalent, we will introduce the notion of a homomorphism between cohomology theories. Besides defining these to be homomorphisms between the cohomology modules we also ask some commuting properties which will allow us to show the uniqueness of such a homomorphism in the main theorem of this section.

Definition 3.2. Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be cohomology theories on M with coefficients in sheaves of K-modules over M. A *homomorphism of the cohomology theory $\mathcal{H}$ to the theory $\widetilde{\mathcal{H}}$* consists of a homomorphism $H^q(M,\mathcal{S}) \to \widetilde{H}^q(M,\mathcal{S})$ for each q, such that the following conditions hold:

(H_1) For $q = 0$, the following diagram commutes:

$$\begin{tikzcd} H^0(M,\mathcal{S}) \arrow[r, "\cong"] \arrow[d] & \Gamma(\mathcal{S}) \arrow[d] \\ \widetilde{H}^0(M,\mathcal{S}) \arrow[r, "\cong"] & \Gamma(\mathcal{S}) \end{tikzcd}$$

(H_2) For each homomorphism $\mathcal{S} \to \mathcal{F}$ and each integer q the following diagram commutes:

$$\begin{tikzcd} H^q(M,\mathcal{S}) \arrow[r] \arrow[d] & H^q(M,\mathcal{F}) \arrow[d] \\ \widetilde{H}^q(M,\mathcal{S}) \arrow[r] & \widetilde{H}^q(M,\mathcal{S}) \end{tikzcd}$$

(H_3) For each short exact sequence of sheaves

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{F} \to 0$$

and for each integer q the following diagram commutes:

$$\begin{tikzcd} H^q(M,\mathcal{F}) \arrow[r, "\delta"] \arrow[d] & H^{q+1}(M,\mathcal{R}) \arrow[d] \\ \widetilde{H}^q(M,\mathcal{F}) \arrow[r, "\delta"] & \widetilde{H}^{q+1}(M,\mathcal{F}) \end{tikzcd}$$

An *isomorphism of the cohomology theory $\mathcal{H}$ to the theory $\widetilde{\mathcal{H}}$* is a homomorphism where all the homomorphisms $H^q(M,\mathcal{S}) \to \widetilde{H}^q(M,\mathcal{S})$ are isomorphisms. $\diamondsuit$

The following construction will be the main ingredient for proving the uniqueness of axiomatic cohomology theories.

Definition 3.3. Let $\mathcal{S}$ be a a sheaf over M. By a *discontinuous section of $\mathcal{S}$* we mean any (possibly discontinuous) map $f : U \to \mathcal{S}$ such that $\pi \circ f = \mathrm{id}$. By associating the module of discontinuous sections of $\mathcal{S}$ to every open $U \subset M$ we get a presheaf $P\mathcal{S}_0$ whose associated sheaf, $\mathcal{S}_0$, is called the *sheaf of germs of discontinuous sections of $\mathcal{S}$*. $\diamondsuit$

From the sheaf of discontinuous sections we will construct a short exact sheaf sequence which will give us long exact sequences in cohomology modules.

Lemma 3.4. *Denote by $\overline{\mathcal{S}}$ the quotient sheaf $\mathcal{S}_0/\mathcal{S}$ then there exists maps i, τ such that the following sequence is exact:*

$$0 \to \mathcal{S} \xrightarrow{i} \mathcal{S}_0 \xrightarrow{\tau} \overline{\mathcal{S}} \to 0 \tag{3.1}$$

Proof. It is clear that there is an injection from $\alpha(\mathcal{S})$ into $P\mathcal{S}_0$ as this is just the inclusion of continuous sections into discontinuous ones. This injection induces an injection, $\beta(\alpha(\mathcal{S})) \to P\mathcal{S}_0$, on the corresponding sheaves, and since $\beta(\alpha(\mathcal{S}))$ is canonically isomorphic to $\mathcal{S}$ we get an injection $i : \mathcal{S} \to \mathcal{S}_0$. Hence we conclude that (3.1) is exact at $\mathcal{S}$. The map τ sends x to its equivalence class $x + (\mathcal{S})$. Therefore it is clear that the image, $i(\mathcal{S})$ which is isomorphic to $\mathcal{S}$, is the kernel of τ. Hence we conclude that (3.1) is exact at $\mathcal{S}_0$. Finally it is clear that τ is surjective and hence that (3.1) is exact at $\overline{\mathcal{S}}$. We conclude that (3.1) is a short exact sequence. $\square$

We will prove that the sheaf $\mathcal{S}_0$ is fine. This is important because then the cohomology modules $H^q(M,\mathcal{S}_0)$ all become trivial. From the long exact sequence induced by (3.1) we will get important information. To show that $\mathcal{S}_0$ is fine we need the following topology lemma, for a proof see [1, p. 91].

Lemma 3.5. *Let $\{U_i\}$ be a locally finite cover of M. Then there exists a refinement $\{V_i\}$ of $\{U_i\}$ such that $\overline{V}_i \subset U_i$ for each i.*

Lemma 3.6. *The sheaf $\mathcal{S}_0$ is fine for any sheaf $\mathcal{S}$.*

Proof. Let $\{U_i\}$ be a locally finite cover of M and let $\{V_i\}$ be a refinement as in the above lemma. We define a collection of (discontinuous) functions $\{\varphi_i\}$ as follows. Let $x \in M$, then clearly $x \in V_i$ for some (not necessarily unique) i. Then define $\varphi_j(x) = 1$ for $j = i$ and 0 otherwise. In this way it becomes clear that $\sum_i \varphi_i = 1$ and also that $\mathrm{supp}(\varphi_i) \subset U_i$. Hence the set $\{\varphi_i\}$ forms a partition of unity of M subordinated to $\{U_i\}$ of functions which only take the values 1 and 0. We define homomorphisms $\{l_{i,U}\}$ from the presheaf of discontinuous sections of $\mathcal{S}$ to itself by

$$l_{i,U}(s)(m) = \varphi_i(m)s(m)$$

for each discontinuous section s of $\mathcal{S}$ over $U \subset M$ and $m \in U$. These maps are well-defined because the $\varphi_i(m)$ are either 1 or zero so $l_{i,U}(s)(m)$ is a discontinuous section. As in Example 1.37 we get that the associated sheaf homomorphisms $\{l_i\}$ form a partition of unity for $\mathcal{S}_0$. □

We will show that these sheaves of germs of discontinuous sections behave naturally under sheaf homomorphisms:

Lemma 3.7. *Let $\mathcal{S} \to \mathcal{F}$ be a sheaf homomorphism. Then there exists maps between the corresponding sheaves of discontinuous sections $\mathcal{S}_0 \to \mathcal{F}_0$ and the quotients sheaves $\overline{\mathcal{S}} \to \overline{\mathcal{F}}$ such that the following diagram commutes:*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}_0 \arrow[r] \arrow[d] & \overline{\mathcal{S}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}_0 \arrow[r] & \overline{\mathcal{F}} \arrow[r] & 0 \end{tikzcd} \tag{3.2}$$

Proof. We will construct the homomorphism $\varphi : \mathcal{S}_0 \to \mathcal{F}_0$ on the presheaf level. The homomorphism φ induces a homomorphism on presheaves $\{\varphi_U\} : \alpha(\mathcal{S}) \to \alpha(\mathcal{F})$ by sending a section s to $\varphi \circ s$. Clearly as this map is defined point-wise we also have a map from discontinuous sections of $\mathcal{S}$ to discontinuous sections of $\mathcal{F}$. These maps form a presheaf homomorphism $\{\varphi_U\} : P\mathcal{S}_0 \to P\mathcal{F}_0$. Then $\varphi : \mathcal{S}_0 \to \mathcal{F}_0$ is the associated sheaf homomorphism.
From this construction it is clear that if we identify $\mathcal{S}$ with its image in $\mathcal{S}_0$ that $\varphi|_{\mathcal{S}}$ agrees with the given sheaf homomorphisms $\mathcal{S} \to \mathcal{F}$. Hence we can conclude that the first square in diagram (3.2) commutes. Now we define a map $\mathcal{S}_0/\mathcal{S} \to \mathcal{F}_0/\mathcal{F}$ by $[x] \mapsto [\varphi(x)]$. This map is well-defined for if y is another representative of $[x]$, then $x - y \in \mathcal{S}$ hence $\varphi(x - y) \in \mathcal{F}$. Hence we conclude that diagram (3.2) commutes. □

We will now show that there is a canonical homomorphism between two cohomology theories. The fact that it is canonical will allow us to show that a homomorphism between two cohomology theories is in fact an isomorphism.

Theorem 3.8. *Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be cohomology theories on M with coefficients in sheaves of K-modules over M. Then there exists a canonical homomorphism from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.*

Remark 3.9. With canonical we mean that the homomorphism depends only on inherit properties of the cohomology theories. In fact we will show that it depends only on connecting homomorphisms and the isomorphisms $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$. This will be enough to show that a homomorphism between cohomology theories is in fact an isomorphism. ♢

Proof. We will first prove that a cohomology theory homomorphism is necessarily canonical and then we will show its existence. Let $\{\varphi_q\}$ be a homomorphism from $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, and let $\mathcal{S}$ be any sheaf. We will proceed by considering the cases $q \leq 0$, $q = 1$ and $q \geq 2$. For $q < 0$ the statement is trivial. For $q = 0$

we have that both $H^0(M,\mathcal{S})$ and $\widetilde{H}^0(M,\mathcal{S})$ are isomorphic to $\Gamma(\mathcal{S})$ from which we clearly see that φ_0 is determined only by inherit properties of the cohomology theories. For $q = 1$ we construct the following diagram from the sequence in (3.1):

$$\begin{tikzcd}
H^0(M,\mathcal{S}_0) \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & H^0(M,\overline{\mathcal{S}}) \arrow[r, "\delta"] \arrow[d] \arrow[dr, phantom, "2"] & H^1(M,\mathcal{S}) \arrow[r] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "3"] & H^1(M,\mathcal{S}_0) \arrow[d] \\
\widetilde{H}^0(M,\mathcal{S}_0) \arrow[r] & \widetilde{H}^0(M,\overline{\mathcal{S}}) \arrow[r, "\widetilde{\delta}"] & \widetilde{H}^1(M,\mathcal{S}) \arrow[r] & \widetilde{H}^1(M,\mathcal{S}_0)
\end{tikzcd}$$

The squares 1 and 3 commute by Definition 3.2.H_2 applied to sequence (3.1), and square 2 commutes by Definition 3.2.H_3 applied to sequence (3.1). If we also use the fact that $H^0(M,\mathcal{S}) \cong \Gamma(\mathcal{S})$ and the fact that $\mathcal{S}_0$ is fine we get the commutative diagram:

$$\begin{tikzcd}
\Gamma(\mathcal{S}_0) \arrow[r] \arrow[d] & \Gamma(\overline{\mathcal{S}}) \arrow[r, "\delta"] \arrow[d] & H^1(M,\mathcal{S}) \arrow[r] \arrow[d, "\varphi_1"] & 0 \\
\Gamma(\mathcal{S}_0) \arrow[r] & \Gamma(\overline{\mathcal{S}}) \arrow[r, "\widetilde{\delta}"] & \widetilde{H}^1(M,\mathcal{S}) \arrow[r] & 0
\end{tikzcd} \tag{3.3}$$

We have by Definition 3.1.(c) that the rows of Diagram (3.3) are exact. If $a \in H^1(M,\mathcal{S})$ then by the fact that δ is surjective there exists a $b \in \Gamma(\overline{\mathcal{S}})$ such that $\delta(b) = a$. Because the diagram commutes we have that $\varphi_1(a) = \varphi_1(\delta(b)) = \widetilde{\delta}(b)$. This is independent of the choice of b because if we take another $\widetilde{b}$ such that $\delta(\widetilde{b}) = a$, then by the fact that the diagram commutes we have that $\varphi_1(\delta(b)) = \varphi_1(\delta(\widetilde{b}))$. Therefore we conclude that φ_1 is determined only by inherit properties of the cohomology theories. For $q \geq 2$ we construct a diagram likewise but now with the difference that $H^{q-1}(M,\mathcal{S}_0) = 0$ as $\mathcal{S}_0$ is a fine sheaf. We thus get that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & H^{q-1}(M,\overline{\mathcal{S}}) \arrow[r, "\delta_q"] \arrow[d, "\varphi_{q-1}"] & H^q(M,\mathcal{S}) \arrow[r] \arrow[d, "\varphi_q"] & 0 \\
0 \arrow[r] & \widetilde{H}^{q-1}(M,\overline{\mathcal{S}}) \arrow[r, "\widetilde{\delta}_q"] & \widetilde{H}^q(M,\mathcal{S}) \arrow[r] & 0
\end{tikzcd} \tag{3.4}$$

Now we proceed inductively. Assume that $\varphi_n = \widetilde{\delta}_n \circ \varphi_{n-1} \circ \delta_n^{-1}$ for $1 < n \leq q-1$. By the exactness of the rows the maps $\delta_q, \widetilde{\delta}_q$ are isomorphisms. Hence by the commutativity of the diagram we have that $\varphi_q = \widetilde{\delta}_q \circ \varphi_{q-1} \circ \delta_q^{-1}$ from which we conclude that φ_q is determined only by the coboundary operators of both cohomology theories.

We will now show the existence of the homomorphisms. We define the homomorphism for $q = 1$ as the unique map $H^1(M,\mathcal{S}) \to \widetilde{H}^1(M,\mathcal{S})$ for which diagram (3.3) commutes and for $q \geq 2$ we inductively define it as the map $H^q(M,\mathcal{S}) \to \widetilde{H}^q(M,\mathcal{S})$ such that diagram (3.4) commutes.

We have that 3.2.H_3 follows from the fact that we have constructed the homomorphism such that diagram (3.3) commutes.

To show 3.2.H_2 we consider the cases $q \leq 0$, $q = 1$ and $q > 1$. For $q \leq 0$ 3.2.H_2 follows trivially from the fact that $H^q(M,\mathcal{S}) = 0$ for $q < 0$ and $\Gamma(\mathcal{S})$ for $q = 0$. To show 3.2.H_2 for $q = 1$ we will consider the following lattice constructed from two copies of diagram (3.3) and using 3.1.6:

$$
\begin{tikzcd}
& \Gamma(\overline{\mathcal{S}}) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & H^1(M,\mathcal{S}) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & 0 \arrow[dl] \arrow[dd] \\
\Gamma(\overline{\mathcal{F}}) \arrow[rr] \arrow[dd] & & H^1(M,\mathcal{F}) \arrow[rr] \arrow[dd] \arrow[r, phantom, "1"] & & 0 \arrow[dd] & \\
& \Gamma(\overline{\mathcal{S}}) \arrow[rr, dotted] \arrow[dl, dotted] & & \widetilde{H}^1(M,\mathcal{S}) \arrow[rr, dotted] \arrow[dl, dotted] & & 0 \arrow[dl] \\
\Gamma(\overline{\mathcal{F}}) \arrow[rr] & & \widetilde{H}^1(M,\mathcal{F}) \arrow[rr] & & 0 &
\end{tikzcd}
$$

The horizontal sides of this lattice are precisely the diagrams (3.3) for $\mathcal{S}$ and $\mathcal{F}$. By Lemma 3.7 we have a map $\overline{\mathcal{S}} \to \overline{\mathcal{F}}$ thus we have that the left side commutes as well. We now apply property 6 of a cohomology theory to diagram (3.2) to get the following commutative diagram:

$$
\begin{tikzcd}
H^0(M,\overline{\mathcal{S}}) \arrow[r] \arrow[d] & H^1(M,\mathcal{S}) \arrow[d] \\
H^0(M,\overline{\mathcal{F}}) \arrow[r] & H^1(M,\mathcal{F})
\end{tikzcd}
$$

This diagram will be the top side of the lattice. Similarly we get a diagram for the $\widetilde{\mathcal{H}}$ theory, which is the bottom diagram.

We now have that all the sides of the lattice except one side, which is denoted in the diagram by a 1 in the middle, commute. Square 1 is precisely the diagram of 3.2.H_2 for $q = 1$. The idea is to use the commutativity of the other parts of the lattice to write the maps in square 1 as compositions of maps in the other squares of the diagram. Because these squares commutes we then have that square 1 will commutes as well.

We begin by considering the union of the top and front side projected onto the plane:

$$
\begin{tikzcd}
\Gamma(\overline{\mathcal{S}}) \arrow[r, "\delta_{\mathcal{S}}"] \arrow[d, "f_0"'] & H^1(M,\mathcal{S}) \arrow[r] \arrow[d, "f_1"'] & 0 \\
\Gamma(\overline{\mathcal{F}}) \arrow[r, "\delta_{\mathcal{F}}"] \arrow[d, "\mathrm{id}"'] & H^1(M,\mathcal{F}) \arrow[r] \arrow[d, "\varphi^1_{\mathcal{F}}"'] & 0 \\
\Gamma(\overline{\mathcal{F}}) \arrow[r, "\widetilde{\delta}_{\mathcal{F}}"] & \widetilde{H}^1(M,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}
$$

As this is the union of two commutative diagrams it is commutative as well. Let $x \in H^1(M,\mathcal{S})$. By the fact that $\delta_{\mathcal{S}} : \Gamma(\overline{\mathcal{S}}) \to H^1(M,\mathcal{S})$ is surjective we can pick $y \in \Gamma(\overline{\mathcal{S}})$ such that $\delta_{\mathcal{S}}(y) = x$. Hence, by the commutativity of the diagram,

$$(\varphi^1_{\mathcal{F}} \circ f_1 \circ \delta_{\mathcal{S}})(y) = (\widetilde{\delta_{\mathcal{F}}} \circ \mathrm{id} \circ f_0)(y)$$

from which we conclude that

$$(\varphi^1_{\mathcal{F}} \circ f_1)(x) = (\widetilde{\delta_{\mathcal{F}}} \circ f_0)(y). \tag{3.5}$$

Now consider the back and bottom diagram projected onto the plane:

$$\begin{tikzcd}
\Gamma(\overline{\mathcal{S}}) \arrow[r, "\delta_{\mathcal{S}}"] \arrow[d, "\mathrm{id}"'] & H^1(M,\mathcal{S}) \arrow[r] \arrow[d, "\varphi^1_{\mathcal{S}}"'] & 0 \\
\Gamma(\overline{\mathcal{S}}) \arrow[r, "\widetilde{\delta}_{\mathcal{S}}"] \arrow[d, "f_0"'] & \widetilde{H}^1(M,\mathcal{S}) \arrow[r] \arrow[d, "\widetilde{f}_1"'] & 0 \\
\Gamma(\overline{\mathcal{F}}) \arrow[r, "\widetilde{\delta}_{\mathcal{F}}"] & \widetilde{H}^1(M,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

With the same x and y as before we have by the commutativity of this diagram that

$$(\widetilde{f}_1 \circ \varphi^1_{\mathcal{S}} \circ \delta_{\mathcal{S}})(y) = (\widetilde{\delta}_{\mathcal{F}} \circ f_0 \circ \mathrm{id})(y)$$

from which we conclude that

$$(\widetilde{f}_1 \circ \varphi^1_{\mathcal{S}})(x) = (\widetilde{\delta}_{\mathcal{F}} \circ f_0)(y).$$

If we combine the above with (3.5) we get

$$(\widetilde{f}_1 \circ \varphi^1_{\mathcal{S}})(x) = (\varphi^1_{\mathcal{F}} \circ f_1)(x)$$

from which we conclude that square 1 commutes.

For $q > 1$ we construct an analogous lattice:

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & H^{q-1}(M,\overline{\mathcal{S}}) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & H^q(M,\mathcal{S}) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & 0 \arrow[dl] \arrow[dd] \\
0 \arrow[rr] \arrow[dd] & & H^{q-1}(M,\overline{\mathcal{F}}) \arrow[rr] \arrow[dd] & & H^q(M,\mathcal{F}) \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[rr, dotted] \arrow[dl, dotted] & & \widetilde{H}^{q-1}(M,\overline{\mathcal{S}}) \arrow[rr, dotted] \arrow[dl, dotted] & & \widetilde{H}^q(M,\mathcal{S}) \arrow[rr, dotted] \arrow[dl, dotted] & & 0 \arrow[dl] \\
0 \arrow[rr] & & \widetilde{H}^{q-1}(M,\overline{\mathcal{F}}) \arrow[rr] & & \widetilde{H}^q(M,\mathcal{F}) \arrow[rr] & & 0 &
\end{tikzcd}$$

By induction and a similar argument as for the case $q = 1$ one can show that 3.2.H_2 holds for $q \geq 2$.
To show 3.2.H_3, suppose that we have a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{F} \to 0.$$

We want to show that the diagram

$$\begin{tikzcd}
H^q(M,\mathcal{F}) \arrow[r] \arrow[d] & H^{q+1}(M,\mathcal{R}) \arrow[d] \\
\widetilde{H}^q(M,\mathcal{F}) \arrow[r] & \widetilde{H}^{q+1}(M,\mathcal{R})
\end{tikzcd}$$

commutes. To do this we will write the maps $H^q(M,\mathcal{F}) \to H^{q+1}(M,\mathcal{R})$ and $\widetilde{H}^q(M,\mathcal{F}) \to \widetilde{H}^{q+1}(M,\mathcal{R})$ as the composition of some other maps. We will first construct a diagram on the level of sheaves and then consider the induced diagram in cohomology. These diagrams will commute and thus allow us to write the maps $H^q(M,\mathcal{F}) \to H^{q+1}(M,\mathcal{R})$ and $\widetilde{H}^q(M,\mathcal{F}) \to \widetilde{H}^{q+1}(M,\mathcal{R})$ as the composition of some other maps.

Lemma 3.10. *From a short exact sheaf sequence*

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{F} \to 0$$

we can construct maps such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "1"] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{R} \arrow[r] & \mathcal{S}_0 \arrow[r] & \mathcal{S}_0/\mathcal{R} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[u] \arrow[r, phantom, shift left=4ex, "2"] & \mathcal{R}_0 \arrow[r, "g"] \arrow[u, "h"] \arrow[r, phantom, shift left=4ex, "3"] & \overline{\mathcal{R}} \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{3.6}$$

Proof. The map $\mathcal{R} \to \mathcal{S}_0$ is defined to be the composition $\mathcal{R} \to \mathcal{S} \to \mathcal{S}_0$. Because this is the composition of injective maps it is injective as well. This map is a sheaf homomorphisms because it is the composition of sheaf homomorphisms. From this definition it is clear that square 1 commutes. By Lemma 3.7 we have a sheaf homomorphism $\varphi : \mathcal{R}_0 \to \mathcal{S}_0$. Identify $\mathcal{R}$ with its image in $\mathcal{R}_0$. Using the proof or Lemma 3.7 the map $\varphi|_{\mathcal{R}}$ agrees with the map $\mathcal{R} \to \mathcal{S}$. Hence we conclude that square 2 commutes.
Now we will define the maps $\overline{\mathcal{R}} \to \mathcal{S}_0/\mathcal{R}$ and $\mathcal{F} \to \mathcal{S}_0/\mathcal{R}$. We will use the fact that the the maps $g : \mathcal{R}_0 \to \overline{\mathcal{R}}$ and $\mathcal{S} \to \mathcal{F}$ are surjective. Define the map $f : \overline{\mathcal{R}} \to \mathcal{S}_0/\mathcal{R}$ as follows. Let $x \in \overline{\mathcal{R}}$. By the fact that g is surjective, there exists an $a \in \mathcal{R}_0$ such that $g(a) = x$. By mapping a to $\mathcal{S}_0/\mathcal{R}$ via $\mathcal{S}_0$ we get the required map f. If we can show that this map is well-defined we can conclude that square 3 commutes. Let $y \in \overline{\mathcal{R}}$ such that $x - y \in \mathcal{R}$. The map $h : \mathcal{R}_0 \to \mathcal{S}_0$ is constructed in such a way that $h|_{\mathcal{R}}$ agrees with the map $\mathcal{R} \to \mathcal{S}$. Thus $h(x - y) \in \mathcal{R}$ for $x - y \in \mathcal{R} \subset \mathcal{R}_0$. Hence the image of $h(x - y)$ in $\mathcal{S}_0/\mathcal{R}$ is zero and we conclude that $f : \overline{\mathcal{R}} \to \mathcal{S}_0/\mathcal{R}$ is well-defined. If we define the map $\mathcal{F} \to \mathcal{S}_0/\mathcal{R}$ in a similar same way we conclude that diagram (3.6) commutes. $\square$

Now we will consider the diagram in $\mathcal{H}$-cochomology induced by diagram (3.6) (the construction for the theory $\widetilde{\mathcal{H}}$ is completely analogous). Using the fact that a short exact sequence induces a long exact sequence in cohomology we get a long exact sequence for every row in diagram (3.6). Hence we obtain a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \Gamma(\mathcal{S}) \arrow[r] \arrow[d] \arrow[dr, phantom, "1"] & \Gamma(\mathcal{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "3"] & H^1(M,\mathcal{R}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma(\mathcal{S}_0) \arrow[r] & \Gamma(\mathcal{S}_0/\mathcal{R}) \arrow[r] & H^1(M,\mathcal{R}) \arrow[r] & 0 \\
\cdots \arrow[r] & \Gamma(\mathcal{R}_0) \arrow[r] \arrow[u] \arrow[ur, phantom, "2"] & \Gamma(\overline{\mathcal{R}}) \arrow[r] \arrow[u] \arrow[ur, phantom, "4"] & H^1(M,\mathcal{R}) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{3.7}$$

The squares 1 and 2 commute because the maps are the induced maps in sections of the squares 1 and 2 in diagram (3.6). The squares 3 and 4 commute by applying 3.1.(f) to the sequences in diagram (3.6). Finally because $\mathcal{S}_0$ is fine $H^1(M,\mathcal{S}_0)$ and $H^1(M,\mathcal{R}_0)$ are trivial. One can construct a similar diagram for $q \geq 1$:

$$\begin{tikzcd}
\cdots \arrow[r] & H^q(M,\mathcal{S}) \arrow[r] & H^q(M,\mathcal{F}) \arrow[r] \arrow[d] & H^{q+1}(M,\mathcal{R}) \arrow[r] \arrow[d] & \cdots . \\
& 0 \arrow[r] & H^q(M,\mathcal{S}_0/\mathcal{R}) \arrow[r] & H^{q+1}(M,\mathcal{R}) \arrow[r] & 0 \\
& 0 \arrow[r] & H^q(M,\overline{\mathcal{R}}) \arrow[r] \arrow[u] & H^{q+1}(M,\mathcal{R}) \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{3.8}$$

The argument that this diagram commutes is analogous.

We will use these diagrams to write the maps $H^q(M,\mathcal{F}) \to H^{q+1}(M,\mathcal{R})$ as compositions. We will first do this for $q = 0$ and then for $q \geq 1$.

Because $H^0(M,\mathcal{F}) \cong \Gamma(\mathcal{F})$ the map $H^0(M,\mathcal{F}) \to H^1(M,\mathcal{R})$ is in the upper row of (3.7). We will however chase the diagram in such a way that our composition ends at $H^1(M,\mathcal{R})$ in the lower row. We will also consider the maps to quotients spaces instead of the normal spaces, this is done such that the resulting compositing will consist of some isomorphisms.

First we map from $\Gamma(\mathcal{F})$ to $\Gamma(\mathcal{S}_0/\mathcal{R})/\operatorname{im}\Gamma(\mathcal{S}_0)$. Now we will construct a map from $\Gamma(\mathcal{S}_0/\mathcal{R})$ into $\Gamma(\overline{\mathcal{R}})$, to do this we note the following. By the exactness of the middle row of diagram (3.7) the map $\Gamma(\mathcal{S}_0/\mathcal{R})/\operatorname{im}\Gamma(\mathcal{S}_0) \to H^1(M,\mathcal{R})$ is an isomorphism. Similarly, it follows from the exactness of the lower row of diagram (3.7) that the map $\Gamma(\overline{\mathcal{R}})/\operatorname{im}\Gamma(\mathcal{R}_0) \to H^1(M,\mathcal{R})$ is an isomorphism. Using these isomorphisms we we define an isomorphism $\Gamma(\mathcal{S}_0/\mathcal{R})/\operatorname{im}\Gamma(\mathcal{S}_0) \to \Gamma(\overline{\mathcal{R}})/\operatorname{im}\Gamma(\mathcal{R}_0)$ by the composition:

$$\Gamma(\mathcal{S}_0/\mathcal{R})/\operatorname{im}\Gamma(\mathcal{S}_0) \overset{\cong}{\to} H^1(M,\mathcal{R}) = H^1(M,\mathcal{R}) \overset{\cong}{\leftarrow} \Gamma(\overline{\mathcal{R}})/\operatorname{im}\Gamma(\mathcal{R}_0).$$

If we combine the above and the fact that diagram (3.7) commutes we can write the homomorphism $H^0(M,\mathcal{F}) \to H^1(M,\mathcal{R})$ as the composition

$$H^0(M,\mathcal{F}) \overset{\cong}{\to} \Gamma(\mathcal{F}) \to \Gamma(\mathcal{S}_0/\mathcal{R})/\mathrm{im}\Gamma(\mathcal{S}_0) \overset{\cong}{\leftarrow} \Gamma(\overline{\mathcal{R}})/\mathrm{im}\Gamma(\mathcal{R}_0) \overset{\cong}{\to} H^1(M,\mathcal{R}). \tag{3.9}$$

For $q \geq 1$ we proceed analogous. However from (3.7) if follows that $H^q(M, \mathcal{S}_0/\mathcal{R}) \to H^{q+1}(M, \mathcal{R})$ and $H^q(M, \overline{\mathcal{R}}) \to H^{q+1}(M, \mathcal{R})$ are isomorphisms and thus by the fact that the diagram commutes we have that the composition, $H^q(M, \overline{R}) \to H^q(M, \mathcal{S}_0/\mathcal{R})$, is an isomorphism as well. It follows, by the commutativity of diagram (3.8), that the homomorphism $H^q(M, \mathcal{F}) \to H^{q+1}(M, \mathcal{R})$ is the composition

$$H^q(M, \mathcal{F}) \to H^q(M, \mathcal{S}_0/\mathcal{R}) \overset{\cong}{\leftarrow} H^q(M, \overline{\mathcal{R}}) \overset{\cong}{\to} H^{q+1}(M, \mathcal{R}). \tag{3.10}$$

We will now use these compositions to construct commutative diagrams. Again we first consider the case $q = 0$. From sequence (3.9) and the corresponding sequence for the $\widetilde{\mathcal{H}}$ theory we obtain the diagram:

$$\begin{tikzcd}
H^0(M, \mathcal{F}) \arrow[r, "\cong"] \arrow[d] & \Gamma(\mathcal{F}) \arrow[r] \arrow[d] & \Gamma(\mathcal{S}_0/\mathcal{R})/\mathrm{im}\Gamma(\mathcal{S}_0) \arrow[d] & \Gamma(\overline{\mathcal{R}})/\mathrm{im}\Gamma(\mathcal{R}_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & H^1(M, \mathcal{R}) \arrow[d] \\
\widetilde{H}^0(M, \mathcal{F}) \arrow[r, "\cong"] & \Gamma(\mathcal{F}) \arrow[r] & \Gamma(\mathcal{S}_0/\mathcal{R})/\mathrm{im}\Gamma(\mathcal{S}_0) & \Gamma(\overline{\mathcal{R}})/\mathrm{im}\Gamma(\mathcal{R}_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \widetilde{H}^1(M, \mathcal{R})
\end{tikzcd} \tag{3.11}$$

In this diagram the first square commutes by construction of the homomorphism $H^0(M, \mathcal{F}) \to \widetilde{H}^0(M, \mathcal{F})$. The middle squares commute trivially and the last square commutes by definition of the homomorphism $H^1(M, \mathcal{R}) \to \widetilde{H}^1(M, \mathcal{R})$, this follows from diagram (3.3) with $\mathcal{S}$ replaced by $\mathcal{R}$. Hence we conclude that diagram (3.11) commutes. By construction, the composition of the maps in a row agrees with the map $H^0(M, \mathcal{F}) \to H^1(M, \mathcal{R})$. Hence we conclude that the diagram

$$\begin{tikzcd}
H^0(M, \mathcal{F}) \arrow[r] \arrow[d] & H^1(M, \mathcal{R}) \arrow[d] \\
\widetilde{H}^0(M, \mathcal{F}) \arrow[r] & \widetilde{H}^1(M, \mathcal{R})
\end{tikzcd}$$

commutes. This shows $3.2.H_3$ in the case that $q = 0$.

The proof for the case that $q \geq 1$ is analogous, so we will be brief. From sequence (3.10) and the corresponding sequence in the $\widetilde{H}$ theory we obtain the diagram:

$$\begin{tikzcd}
H^q(M, \mathcal{F}) \arrow[r] \arrow[d] & H^q(M, \mathcal{S}_0/\mathcal{R}) \arrow[d] & H^q(M, \overline{\mathcal{R}}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & H^{q+1}(M, \mathcal{R}) \arrow[d] \\
\widetilde{H}^q(M, \mathcal{F}) \arrow[r] & \widetilde{H}^q(M, \mathcal{S}_0/\mathcal{R}) & \widetilde{H}^q(M, \overline{\mathcal{R}}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \widetilde{H}^1(M, \mathcal{R})
\end{tikzcd} \tag{3.12}$$

In which the last square commutes by definition of the homomorphism $H^{q+1}(M, \mathcal{R}) \to \widetilde{H}^{q+1}(M, \mathcal{R})$, and the first two squares commute by $3.2.H_2$. As in the case where $q = 0$ we conclude that the diagram

$$\begin{tikzcd}
H^q(M, \mathcal{F}) \arrow[r] \arrow[d] & H^{q+1}(M, \mathcal{R}) \arrow[d] \\
\widetilde{H}^q(M, \mathcal{F}) \arrow[r] & \widetilde{H}^{q+1}(M, \mathcal{R})
\end{tikzcd}$$

commutes. This shows that $3.2.H_3$ holds for $q \geq 1$ and finishes the proof of the existence of a homomorphism between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. $\square$

Corollary 3.11. *A homomorphism of cohomology theories is automatically an isomorphism. Consequently, any two cohomology theories on M with coefficients in sheaves of K-modules over M are uniquely isomorphic.*

Proof. Assume that we are given a homomorphism $\{\varphi_q\} : \mathcal{H} \to \widetilde{\mathcal{H}}$. Then by the previous theorem we have that there is also a homomorphism $\{\widetilde{\varphi}_q\} : \widetilde{\mathcal{H}} \to \mathcal{H}$. We will show that these maps are inverses of each other and hence isomorphisms. Denote by $\delta : H^q(\overline{\mathcal{S}}) \to H^{q+1}(\mathcal{S})$ and $\widetilde{\delta} : \widetilde{H}^q(\overline{\mathcal{S}}) \to \widetilde{H}^{q+1}(\mathcal{S})$ the connecting homomorphisms of $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ respectively induced by the short exact sequence $0 \to \mathcal{S} \to \mathcal{S}_0 \to \overline{\mathcal{S}} \to 0$. For $q < 0$ both φ_q and $\widetilde{\varphi}_q$ are the trivial map. For $q = 0$ both isomorphism φ_q and $\widetilde{\varphi}_q$ are induced by the isomorphisms $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$ and thus it is clear that they are inverses. For $q = 1$ we have that both φ_1 and $\widetilde{\varphi}_1$ are the the unique map that make the following diagram commutes:

$$\begin{tikzcd}
\Gamma(\mathcal{S}_0) \arrow[r] \arrow[d] & \Gamma(\overline{\mathcal{S}}) \arrow[r, "\delta"] \arrow[d] & H^1(M, \mathcal{S}) \arrow[r] \arrow[d, shift left, "\varphi_1"] & 0 \\
\Gamma(\mathcal{S}_0) \arrow[r] & \Gamma(\overline{\mathcal{S}}) \arrow[r, "\widetilde{\delta}"] & \widetilde{H}^1(M, \mathcal{S}) \arrow[r] \arrow[u, shift left, "\widetilde{\varphi}_1"] & 0
\end{tikzcd}$$

Hence $\varphi_1 \circ \widetilde{\varphi}_1 = \mathrm{id}$. For $q \geq 2$ we proceed inductively. Assume that $\varphi_{q-1} \circ \widetilde{\varphi}_{q-1} = \mathrm{id}$. By the proof of the previous theorem we have that $\varphi_q = \widetilde{\delta} \circ \varphi_{q-1} \circ \delta^{-1}$ and similarly $\widetilde{\varphi}_q = \delta \circ \widetilde{\varphi}_{q-1} \circ \widetilde{\delta}^{-1}$. Then

$$\begin{aligned}
\varphi_q \circ \widetilde{\varphi}_q &= \widetilde{\delta} \circ \varphi_{q-1} \circ \delta^{-1} \circ \delta \circ \widetilde{\varphi}_{q-1} \circ \widetilde{\delta}^{-1} = \widetilde{\delta} \circ \varphi_{q-1} \circ \widetilde{\varphi}_{q-1} \circ \widetilde{\delta}^{-1} \\
&= \widetilde{\delta} \circ \widetilde{\delta}^{-1} = \mathrm{id},
\end{aligned}$$

from which we conclude that all φ_q are isomorphisms and thus that $\{\varphi_q\}$ is a cohomology theory isomorphism. $\square$

4 Čech cohomology

In the previous section we have shown the uniqueness of cohomology theories. In this section we will show that such a cohomology theory exits by constructing the Čech cohomology.

4.1 Definition of Čech cohomology

A first attempt to define cohomology modules will be dependent of the choice of a cover of the topological space M. This definition will use the spaces of Čech cochains which will form a cochain complex, which in turn will give rise to cohomology modules.

Definition 4.1. Let $\mathcal{U} = \{U_i\}$ be an open cover of M. A *q-simplex* σ of $\mathcal{U}$ is a collection $(U_0, \ldots, U_q)$ such that $U_0 \cap \ldots \cap U_q \neq \emptyset$. The set $|\sigma| = U_0 \cap \ldots \cap U_q$ is called the *support* of σ. The $(q-1)$-simplex $(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_q)$ is called the i-th *face* of σ and is denoted by σ^i. The K-module $C^q(\mathcal{U}, \mathcal{S})$ consisting of functions which assign to each q-simplex σ an element of $\Gamma(\mathcal{S}, |\sigma|)$ is called the space of *Čech q-cochains.* ♢

Addition and multiplication with elements of K is defined by

$$(f+g)(\sigma)(m) = f(\sigma)(m) + g(\sigma)(m),$$
$$(k\sigma)(m) = k(\sigma(m)),$$

for all $f, g \in C^q(\mathcal{U}, \mathcal{S})$, σ a q-simplex, and $m \in M$. It is clear that this definition gives $C^q(\mathcal{U}, \mathcal{S})$ the structure of a K-module. We will now define a homomorphism which, together with the spaces of q-cochains will form a cochain complex.

Definition 4.2. The *coboundary operator* is the homomorphism $d^q : C^q(M, \mathcal{S}) \to C^{q+1}(M, \mathcal{S})$,

$$d^q f(\sigma) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|,|\sigma^i|} f(\sigma^i)$$

for all $f \in C^q(\mathcal{U}, \mathcal{S})$ and σ a $(q+1)$-simplex. The map $\rho_{|\sigma|,|\sigma^i|}$ is the restriction of a section defined on $\left|\sigma^i\right|$ to $|\sigma|$. ♢

We will show that the spaces of Čech cochains together with the coboundary operator indeed form a cochain complex which we will denote by $C^*(\mathcal{U}, \mathcal{S})$.

Lemma 4.3. *The sequence*

$$\cdots \to 0 \to C^0(M, \mathcal{S}) \xrightarrow{d^0} C^1(M, \mathcal{S}) \xrightarrow{d^1} C^2(M, \mathcal{S}) \to \cdots$$

is a cochain complex.

To show that this is a cochain complex it suffices to show that $d^{q+1} \circ d^q = 0$. We first need to make the following observation. Let $(\sigma^i)^j = \sigma^{i,j}$ denote the j-th face of the i-th face of σ. Then

$$\sigma^{i,j} = \begin{cases} (U_0, \ldots, \hat{U}_i, \ldots, \hat{U}_{j+1}, \ldots, U_q) & \text{if } j \geq i \\ (U_0, \ldots, \hat{U}_j, \ldots, \hat{U}_i, \ldots, U_q) & \text{if } j < i \end{cases}$$

where the hat denotes that the element is omitted. One can easily check that if $j \geq i$ $\sigma^{i,j} = \sigma^{j+1,i}$ and if $j < i$, $\sigma^{i,j} = \sigma^{j,i-1}$.

Lemma 4.4. *The Čech coboundary operator satisfies $d^{q+1} \circ d^q = 0$.*

Proof. Let σ be a $(q+2)$-simplex and f a q-cochain, in the following we will omit some of the restriction homomorphisms for clarity reasons.

$$\begin{aligned}
(d^{q+1}(d^q(f))(\sigma) &= \sum_{i=0}^{q+2} (-1)^i \rho_{|\sigma|,|\sigma^i|} \sum_{j=0}^{q+1} (-1)^j \rho_{|\sigma^i|,|\sigma^{i,j}|} f(\sigma^{i,j}) && (1) \\
&= \sum_{i=0}^{q+2} \sum_{j=0}^{q+1} (-1)^{i+j} \rho_{|\sigma|,|\sigma^{i,j}|} f(\sigma^{i,j}) && (2) \\
&= \sum_{i=0}^{q+2} \sum_{j \geq i} (-1)^{i+j} f(\sigma^{i,j}) + \sum_{i=0}^{q+2} \sum_{j<i} (-1)^{i+j} f(\sigma^{i,j}) && (3) \\
&= \sum_{i=0}^{q+2} \sum_{j \geq i} (-1)^{i+j} f(\sigma^{j+1,i}) + \sum_{i=0}^{q+2} \sum_{j<i} (-1)^{i+j} f(\sigma^{i,j}) && (4) \\
&= \sum_{i=0}^{q+2} \sum_{j > i} (-1)^{i+j-1} f(\sigma^{j,i}) + \sum_{i=0}^{q+2} \sum_{j<i} (-1)^{i+j} f(\sigma^{i,j}) && (5) \\
&= \sum_{j=0}^{q+2} \sum_{i > j} (-1)^{i+j-1} f(\sigma^{i,j}) + \sum_{i=0}^{q+2} \sum_{j<i} (-1)^{i+j} f(\sigma^{i,j}) && (6) \\
&= 0
\end{aligned}$$

In the fifth line we replaced j by $j-1$ and in the sixth line we interchanged i and j in the first sum. This finishes the proof that $C^*(\mathcal{U}, \mathcal{S})$ is a cochain complex. □

We will now drop the q from the notation d^q as long as it does not result in any confusion.

Definition 4.5. The *q-th Čech cohomology module* of M with respect to the cover $\mathcal{U}$ with coefficients in $\mathcal{S}$ is the module $H^q(C^*(\mathcal{U}, \mathcal{S}))$, which we will denote by $\check{H}^q(\mathcal{U}, \mathcal{S})$. ♢

This definition is clearly dependent of the choice of the cover $\mathcal{U}$ of M. To define the cohomology in a cover independent way we will first consider the result of refining the cover $\mathcal{U}$. The refining of a cover will give rise to a refining map, which in turn will give rise to a cochain map. This cochain map will induce homomorphisms in cohomology modules.

Definition 4.6. Let $\mathcal{B}$ be a refinement of the cover $\mathcal{U}$. As every $V \in \mathcal{B}$ is a subset of some $U \in \mathcal{U}$, there exists a map $\mu : \mathcal{B} \to \mathcal{U}$ such that $V \subseteq \mu(V)$ (we will call such a map a *refining map*). If $\sigma = (V_0, \ldots, V_q)$ is a q-simplex of $\mathcal{B}$, then we denote by $\mu(\sigma)$ the q-simplex $(\mu(V_0), \ldots, \mu(V_q))$ of $\mathcal{U}$. ♢

Define a map $\mu^* : C^*(\mathcal{U}, \mathcal{S}) \to C^*(\mathcal{B}, \mathcal{S})$ by

$$\mu_q(f)(\sigma) = \rho_{|\sigma|,|\mu(\sigma)|} f(\mu(\sigma))$$

for $f \in C^q(\mathcal{U}, \mathcal{S})$, and σ a q-simplex of $\mathcal{B}$.

Lemma 4.7. *The map $\mu^* : C^*(\mathcal{U}, \mathcal{S}) \to C^*(\mathcal{B}, \mathcal{S})$, as defined above, is a cochain map.*

Proof. Let $f \in C^q(\mathcal{U}, \mathcal{S})$ and let σ be a q-simplex of the cover $\mathcal{B}$. Then for all q:

$$\begin{aligned}(d \circ \mu_q)(f)(\sigma) &= \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|,|\sigma^i|} \circ \rho_{|\sigma^i|,|\mu(\sigma^i)|} f(\mu(\sigma^i)) \\ &= \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|,|\mu(\sigma^i)|} f(\mu(\sigma^i))\end{aligned}$$

and also;

$$\begin{aligned}\mu_{q+1}(df)(\sigma) &= \rho_{|\sigma|,|\mu(\sigma)|} \sum_{i=0}^{q+1} (-1)^i \rho_{|\mu(\sigma)|,|\mu(\sigma^i)|} f(\mu(\sigma^i)) \\ &= \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|,|\mu(\sigma)|} \circ \rho_{|\mu(\sigma)|,|\mu(\sigma^i)|} f(\mu(\sigma^i))\end{aligned}$$

which thus shows that μ^* is a cochain map. □

By the fact that μ^* is a cochain map we get induced homomorphisms in cohomology modules:

$$\mu_q^* : \check{H}^q(\mathcal{U}, \mathcal{S}) \to \check{H}^q(\mathcal{B}, \mathcal{S})$$

We will show that if μ and τ are two refining maps from $\mathcal{B}$ to $\mathcal{U}$ that the induced maps in cohomology are equal. To do this we will find homomorphisms $h_q : C^q(\mathcal{U}, \mathcal{S}) \to C^{q-1}(\mathcal{B}, \mathcal{S})$ such that

$$h_{q+1} \circ d^q + d^{q-1} \circ h_q = \tau_q - \mu_q. \tag{4.1}$$

Remark 4.8. This is a situation which we will encounter often. Suppose we have two cochain maps f and g from (U^*, d_U) to (V^*, d_V). If we can find maps $h_q : U^q \to V^{q-1}$ such that

$$h_{q+1} \circ d_U + d_V \circ h_q = f_q - g_q,$$

then f and g induce the same maps in cohomology modules. Indeed, let σ be a q-cocycle of U^*. Then

$$f_q(\sigma) - g_q(\sigma) = h_{q+1}(d_U(\sigma)) + d_V(h_q(\sigma)) = d_V(h_q(\sigma))$$

hence $f_q(\sigma)$ and $g_q(\sigma)$ differ only by a coboundary and thus induce the same maps in cohomology modules. We will call the maps h_q *homotopy operators* between f and g. ♢

Let $\sigma = (V_0, \ldots, V_{q-1})$ be a $(q-1)$-simplex of $\mathcal{U}$. Define q-simplices of $\mathcal{B}$ by

$$\sigma_k = (\mu(V_0), \ldots, \mu(V_k), \tau(V_k), \ldots, \tau(V_{q-1})).$$

We define homomorphisms $h_q : C^q(\mathcal{U}, \mathcal{S}) \to C^{q-1}(\mathcal{B}, \mathcal{S})$ by

$$h_q(f)(\sigma) = \sum_{j=0}^{q-1} (-1)^j \rho_{|\sigma|,|\sigma_k|} f(\sigma_k).$$

We will now show that the homomorphisms h_q satisfy equation (4.1). To do this we first need to make the following observation. Let σ be a $(q-1)$-simplex of the cover $\mathcal{B}$, then:

$$(\sigma^j)_k = \begin{cases} (\mu(V_0), \ldots, \widehat{\mu(V_j)}, \ldots, \mu(V_{k+1}), \tau(V_{k+1}), \ldots, \tau(V_{q-1})) & \text{if } j > k \\ (\mu(V_0), \ldots, \mu(V_k), \tau(V_k), \ldots, \widehat{\tau(V_j)}, \ldots, \tau(V_{q-1})) & \text{if } k < j \end{cases}$$

also;

$$(\sigma_k)^j = \begin{cases} (\mu(V_0), \ldots, \mu(V_k), \tau(V_k), \ldots, \widehat{\mu(V_{j-1})}, \ldots, \tau(V_{q-1})) & \text{if } j > k \\ (\mu(V_0), \ldots, \widehat{\tau(V_j)}, \ldots, \mu(V_k), \tau(V_k), \ldots, \tau(V_{q-1})) & \text{if } k > j \end{cases}$$

For $k > j$, $(\sigma_k)^j = (\sigma^j)_{k-1}$ and for $k < j$, $(\sigma_k)^{j+1} = (\sigma^j)_k$. Also $(\sigma_0)^0 = \tau(\sigma)$ and $(\sigma_{q-1})^q = \mu(\sigma)$.

Lemma 4.9. *The homomorphisms $h_q : C^q(\mathcal{U}, \mathcal{S}) \to C^{q-1}(\mathcal{B}, \mathcal{S})$, as defined in (4.1), are homotopy operators between μ^* and τ^*.*

Proof. In what follows we will omit the restriction homomorphisms for clarity reasons. Let $f \in C^q(\mathcal{U}, \mathcal{S})$ an let σ be a $(q-1)$-simplex of $\mathcal{B}$. Then:

$$\begin{aligned} d^{q-1}(h_q(f)(\sigma)) &= \sum_{j=0}^{q} (-1)^j h_q(f)(\sigma^j) & (1) \\ &= \sum_{j=0}^{q} \sum_{k=0}^{q-1} (-1)^{j+k} f((\sigma^j)_k) & (2) \\ &= \sum_{j=0}^{q} \sum_{k<j} (-1)^{j+k} f((\sigma^j)_k) + \sum_{j=0}^{q} \sum_{k\geq j} (-1)^{j+k} f((\sigma^j)_k) & (3) \\ &= \sum_{j=0}^{q} \sum_{k<j} (-1)^{j+k} f((\sigma^j)_k) + \sum_{j=0}^{q} \sum_{k>j} (-1)^{j+k-1} f((\sigma^j)_{k-1}) & (4) \\ &= \sum_{j=0}^{q} \sum_{k<j} (-1)^{j+k} f((\sigma_k)^{j+1}) + \sum_{j=0}^{q} \sum_{k>j} (-1)^{j+k-1} f((\sigma_k)^j). & (5) \end{aligned}$$

In the forth line we replaced j by $j+1$ and in the fifth line changed the order of the operations on σ. On the other hand we have,

$$\begin{aligned}
h_{q+1}(d^q(f))(\sigma) &= \sum_{k=0}^{q}(-1)^k d(f)(\sigma_k) \\
&= \sum_{k=0}^{q}\sum_{j=0}^{q+1}(-1)^{k+j} f((\sigma_k)^j) \\
&= \sum_{k=0}^{q}\sum_{j\le k}(-1)^{k+j} f((\sigma_k)^j) + \sum_{k=0}^{q}\sum_{j>k}(-1)^{k+j} f((\sigma_k)^j) \\
&= \sum_{k=0}^{q}\sum_{j\le k}(-1)^{k+j} f((\sigma_k)^j) + \sum_{k=0}^{q}\sum_{j\ge k}(-1)^{k+j+1} f((\sigma_k)^{j+1})
\end{aligned}$$

By adding these up all terms cancel except for those where $k = j$. That is:

$$\begin{aligned}
h_{q+1}(d^q(f))(\sigma) + d^{q-1}(h_q(f)(\sigma)) &= \sum_{k=0}^{q}(-1)^{2k} f((\sigma_k)^k) - \sum_{k=0}^{q}(-1)^{2k} f(((\sigma)_k)^{k+1}) \\
&= f((\sigma_0)^0) - f((\sigma_{q-1})^q) = f(\tau(\sigma)) - f(\mu(\sigma)) = \tau_q(f)(\sigma) - \mu_q(f)(\sigma).
\end{aligned}$$

We thus conclude that $\tau_q^* = \mu_q^*$. □

From the above we get that if $\mathcal{B}$ is a refinement of $\mathcal{U}$, then a refining map from $\mathcal{B}$ to $\mathcal{U}$ canonically induces homomorphisms $\check{H}^q(\mathcal{U},\mathcal{S}) \to \check{H}^q(\mathcal{B},\mathcal{S})$. We will use these homomorphisms to define cohomology modules which are independent of the choice of cover.

Definition 4.10. The *q-th Čech cohomology module for M with coeffients in the sheaf of K-modules of* $\mathcal{S}$ is the module

$$\check{H}^q(M,\mathcal{S}) = \bigsqcup_i \check{H}^q(\mathcal{U}_i,\mathcal{S})/\sim .$$

If $\mathcal{U}_i$ is a refinement of $\mathcal{U}_j$ we denote the corresponding refining map by μ_{ji}. We declare that $x_i \in \check{H}^q(\mathcal{U}_i,\mathcal{S})$ and $x_j \in \check{H}^q(\mathcal{U}_j,\mathcal{S})$ are equivalent if and only if there exists a common refinement, $\mathcal{U}_k$ of $\mathcal{U}_j$ and $\mathcal{U}_i$ such that the homomorphisms induced by the refining maps μ_{ik}^*, μ_{jk}^* satisfy $\mu_{ik}^q(x_i) = \mu_{jk}^q(x_j)$. ◇

We will show that the relation given above is indeed an equivalence relation. To do this we first need the following observations about the refining maps. In what follows q will stay fixed and thus omitted from the notation in μ_{ij}^q.

Lemma 4.11. *Let $\mathcal{U}_i$ be a cover of M and let $\mathcal{U}_j$ be a refinement of $\mathcal{U}_i$ which has $\mathcal{U}_k$ as a refinement. The induced refining maps satisfy:*
(1): $\mu_{ii}^* = \mathrm{id}$,
(2): $\mu_{ik}^* = \mu_{jk}^* \circ \mu_{ij}^*$.

Proof. (1): The identity map is a refining map from $\mathcal{U}_i$ to itself just as μ_{ii}. By our discussion after Lemma 4.9 we have that two refining maps between the same covers induce the same maps in cohomology. Hence $\mu_{ii}^* = \mathrm{id}$.
(2): The composition of the refining maps, $\mu_{jk} \circ \mu_{ij}$, is also a refinement map from $\mathcal{U}_i$ to $\mathcal{U}_k$. Hence by the same reasoning we have that μ_{ik} and $\mu_{jk} \circ \mu_{ij}$ both induce the same maps in cohomology. □

Lemma 4.12. *Two covers always admit a common refinement.*

Proof. We will show that this common refinement is given by $\mathcal{U} \cap \mathcal{B} = \{U_i \cap B_j\}$. This is clearly a subcover of both $\mathcal{U}$ and $\mathcal{B}$, also $\cup_j U_i \cap B_j = U_i$ hence the union $\cup_{i,j} U_{i,j}$ is equal to M. □

Lemma 4.13. *The relation as described in Definition 4.10 is an equivalence relation.*

Proof. Let $x_i \in \check{H}^q(\mathcal{U}_i, \mathcal{S})$. By Lemma 4.11 $\mu_{ii}^* = \mathrm{id}$, hence $\mu_{ii}^*(x_i) = x_i$ from which we conclude that the equivalence relation is reflexive. Let $x_j \in \check{H}^q(\mathcal{U}_j, \mathcal{S})$ such that $x_i \sim x_j$. Thus $\mu_{ik}^*(x_i) = \mu_{jk}^*(x_j)$ from which we directly conclude that the relation is symmetric.
We are left to show that the relation is transitive. Since in what follows only the cover changes in the cohomology modules, we will denote $\check{H}^q(M,\mathcal{U}_i)$ simply by $\mathcal{U}_i$. Let x_i, x_j, x_k be elements of the cohomology associated with $\mathcal{U}_i$,$\mathcal{U}_j$,$\mathcal{U}_k$ respectively, such that $x_i \sim x_j$ and $x_j \sim x_k$. Hence there exists a cover $\mathcal{U}_l$ such that $\mu_{il}(x_i) = \mu_{jl}(x_j)$, and a cover $\mathcal{U}_n$ such that $\mu_{jn}(x_j) = \mu_{kn}(x_k)$.
By Lemma 4.12 there exists a common refinement of $\mathcal{U}_l$ and $\mathcal{U}_n$ which we will denote by $\mathcal{U}_m$. Together with the induced refining maps we get the following diagram:

$$\begin{tikzcd}
\mathcal{U}_i \arrow[dr, "\mu_{il}"] \arrow[ddrr, bend right, "\mu_{im}"'] & & \mathcal{U}_j \arrow[dl, "\mu_{jl}"'] \arrow[dr, "\mu_{jn}"] \arrow[dd, "\mu_{jm}"] & & \mathcal{U}_k \arrow[dl, "\mu_{kn}"'] \arrow[ddll, bend left, "\mu_{km}"] \\
& \mathcal{U}_l \arrow[dr, "\mu_{lm}"] & & \mathcal{U}_n \arrow[dl, "\mu_{nm}"'] & \\
& & \mathcal{U}_m & &
\end{tikzcd}$$

By Lemma 4.12 $\mu_{im}(x_i) = \mu_{lm}(\mu_{il}(x_i))$. Because $x_i \sim x_j$, $\mu_{lm}(\mu_{il}(x_i)) = \mu_{lm}(\mu_{jl}(x_j))$. Again by Lemma 4.12 $\mu_{lm}(\mu_{jl}(x_j)) = \mu_{jm}(x_j)$ hence we conclude that $\mu_{im}(x_i) = \mu_{jm}(x_j)$.
Completely analogous we have that as $x_k \sim x_j$ that $\mu_{km}(x_k) = \mu_{jm}(x_j)$. Hence we conclude that $\mu_{im}(x_i) = \mu_{km}(x_k)$ which shows that $x_i \sim x_k$. Hence the relation described in Definition 4.10 is an equivalence relation. □

Remark 4.14. The procedure carried out above is called the *direct limit of a directed system.* A *directed set* is a collection $I = \{i\}$ together with a preorder such that every element i has an upper bound. In this case I is the collection of all covers of M and the preorder is the refinement. Every element has as upper bound since every cover is a refinement of the cover consisting only of M.
A *directed system* is a collection $\{A_i, i \in I\}$ together with homomorphisms $f_{ij} : A_i \to A_j$ for which the

properties of Lemma 4.11 hold. In this case the sets A_i are the cohomology modules $\check{H}^q(\mathcal{U}_i, \mathcal{S})$ and the maps f_{ij} are the maps μ_{ij}.
The *direct limit* is defined as $\bigsqcup_i A_i/\sim$. Where $x_i \in A_i$ is related to $x_j \in A_j$ if and only if there exists A_k such that $f_{ik}(x_i) = f_{jk}(x_j)$. The direct limit is often denoted by $\varinjlim A_i$.
Another example of a direct limit was the construction of the stalks in the discussion on the sheaf associated to a presheaf. Here the direct set was the collection of opens U such that $m \in U$, the direct system consisted of the sets S_U and the restriction homomorphism $\rho_{U,V}$. ♢

4.2 Proof of existence of axiomatic sheaf cohomology

We will now show that the definition of Čech cohomology as given in Definition 4.10 gives rise to an axiomatic sheaf cohomology theory. To do this we will first construct homomorphisms in cohomology modules induced by sheaf homomorphisms.

Lemma 4.15. *A sheaf homomorphism $\varphi : \mathcal{S} \to \mathcal{S}'$ induces a homomorphism $\check{H}^q(M, \mathcal{S}) \to \check{H}^q(M, \mathcal{S}')$.*

Proof. It is easy to show that the map $C^*(\mathcal{U}, \mathcal{S}) \to C^*(\mathcal{U}, \mathcal{S}')$ defined by composing φ with elements of $C^q(\mathcal{U}, \mathcal{S})$ is a cochain map. Hence there are induced homomorphisms $\check{H}^q(\mathcal{U}, \mathcal{S}) \to \check{H}^q(\mathcal{U}, \mathcal{S}')$ which we will denote by φ as well. Because φ commutes with the restriction homomorphisms and thus by

$$\begin{aligned}
\mu_q(\varphi(f))(\sigma) &= \rho_{|\sigma|,|\mu(\sigma)|}\varphi(f(\sigma)) \\
\varphi(\mu_q(f))(\sigma) &= \varphi \circ \rho_{|\sigma|,|\mu(\sigma)|}(f(\sigma)) \\
&= \rho_{|\sigma|,|\mu(\sigma)|}\varphi(f(\sigma))
\end{aligned}$$

we have that it commutes with the refinement homomorphisms. Let $x_i \in C^q(\mathcal{U}_i, \mathcal{S})$ be a representative of an equivalence class in $\check{H}^q(M, \mathcal{S})$. We define the homomorphisms

$$\varphi^* : \check{H}^q(M, \mathcal{S}) \to \check{H}^q(M, \mathcal{S}') \tag{4.2}$$

by $\varphi^*([x_i]) = [\varphi(x_i)]$. We check that this map is well-defined. Let $x_j \in C^q(\mathcal{U}_j, \mathcal{S})$ represent the same equivalence class as x_i. Thus there exists a common refinement $\mathcal{U}_k$ of $\mathcal{U}_i$ and $\mathcal{U}_j$ such that $\mu^q_{ik}(x_i) = \mu^q_{jk}(x_j)$. Hence

$$\varphi^*([x_j]) = [\varphi(x_j)] = [\mu^q_{jk}(\varphi(x_j))] = [\varphi(\mu^q_{jk}(x_j))] = [\varphi(\mu^q_{ik}(x_i))] = [\mu^q_{ik}(\varphi(x_i))] = [\varphi(x_i)] = \varphi^*([x_i])$$

and we conclude that the homomorphisms are well-defined. □

Theorem 4.16. *The Čech cohomology theory, as defined in Definition 4.10, is an axiomatic sheaf cohomology theory.*

Proof. By the above lemma we have all the objects we need for a cohomology theory as given in Definition 3.1, we are left to show that this theory satisfies all the given axioms.

We will first show axiom (a), that is for $q < 0$, $\check{H}^q(M, \mathcal{S}) = 0$ and $\check{H}^0(M, \mathcal{S})$ is isomorphic to $\Gamma(\mathcal{S})$ and this isomorphism commutes with sheaf homomorphisms. Since $\check{H}^q(M, \mathcal{S})$ is defined to be zero for $q < 0$ we will show that $\check{H}^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$. We first show $\check{H}^0(\mathcal{U}, \mathcal{S}) \cong \Gamma(\mathcal{S})$ for any cover $\mathcal{U}$ of M, then it is clear that $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$. Let f be a 0-cochain, which assigns to each 0-simplex (i.e. any open $U_i \in \mathcal{U}$), a section of $\mathcal{S}$ over $\mathcal{U}$. If f is a 0-cocycle we have for every 1-simplex $\sigma = (U_i, U_j)$ that

$$df(\sigma) = \rho_{U_i \cap U_j, U_i} f(U_i) - \rho_{U_i \cap U_j, U_i} f(U_j) = 0,$$

hence $f(U_i)|_{U_i \cap U_j} = f(U_j)|_{U_i \cap U_j}$. Let $m \in M$ and choose U_i such that $m \in U_i$. We define the isomorphism by $\varphi : f \mapsto (m \mapsto f(U_i)(m))$. Then it is clear that the following diagram commutes:

$$\begin{array}{ccc} \check{H}^0(\mathcal{U}, \mathcal{S}) & \longrightarrow & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ \check{H}^0(\mathcal{U}, \mathcal{S}') & \longrightarrow & \Gamma(\mathcal{S}') \end{array} \tag{4.3}$$

Now let $[x_i] \in \check{H}^q(M, \mathcal{S})$, we define $\varphi([x_i]) = [\varphi(x_i)]$. It is clear that if this map is well-defined we have show axiom (a). So let $x_j \in \check{H}^q(\mathcal{U}_j, \mathcal{S})$ be another representative of $[x_i]$. By definition of $\check{H}^q(M, \mathcal{S})$ we have $\mu^q_{ik}(x_i) = \mu^q_{jk}(x_j)$ and thus

$$\varphi(x_j) = m \mapsto x_j(U_j)(m) = m \mapsto \rho_{U_i, \mu^q_{jk}(U_i)} x_i(\mu(U_i))(m),$$

which clearly agrees with $\varphi(x_i)$. Hence we conclude that the following diagram commutes, and thus that we have shown axiom (a):

$$\begin{array}{ccc} \check{H}^0(M, \mathcal{S}) & \longrightarrow & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ \check{H}^0(M, \mathcal{S}') & \longrightarrow & \Gamma(\mathcal{S}') \end{array} \tag{4.4}$$

We will now prove axiom (b), that is $\check{H}^q(M, \mathcal{S}) = 0$ for $q > 0$ and $\mathcal{S}$ a fine sheaf. We will show that $\check{H}^q(\mathcal{U}, \mathcal{S}) = 0$ for $q > 0$ for all locally finite covers $\mathcal{U}$. By the fact that M is paracompact every cover admits a locally finite refinement, so every element of $\check{H}^q(M, \mathcal{S})$ has a representative in $\check{H}^q(\mathcal{U}, \mathcal{S})$ where $\mathcal{U}$ is a locally finite cover of M. Therefore it is sufficient to show that $\check{H}^q(\mathcal{U}, \mathcal{S}) = 0$ for all locally finite $\mathcal{U}$.

Since the sheaf $\mathcal{S}$ is fine, we have a partition of unity $\{l_\alpha\}$ subordinated to the locally finite cover $\mathcal{U}$ of M. Let $f \in C^q(\mathcal{U}, \mathcal{S})$ be a q-cochain. We will construct homotopy operators $h_q : C^p(M, \mathcal{S}) \to C^{p-1}(M, \mathcal{S})$ between d and the identity:

$$d \circ h_q + h_{q+1} \circ d = \text{id} \text{ for all } q \geq 1. \tag{4.5}$$

Let $\sigma = (U_0, \ldots, U_{q-1})$ be a $(q-1)$-simplex, we define a q-simplex by $\sigma_\alpha = (U_\alpha, U_0, \ldots, U_{q-1})$. Then $l_\alpha \circ f(\sigma_\alpha)$ is a section over $|\sigma_\alpha|$. Because the support of l_α is contained in U_α and $|\sigma_\alpha| \subset U_\alpha$ we can extend $l_\alpha \circ f(\sigma_\alpha)$ to the whole of $|\sigma|$ by declaring it to be zero on $|\sigma| \,/\, |\sigma_\alpha|$. Henceforth we see $l_\alpha \circ f(\sigma_\alpha)$ as a section over $|\sigma|$. We now define h_q by

$$h_q(f)(\sigma) = \sum_\alpha l_\alpha \circ f(\sigma_\alpha).$$

Before we show (4.5) we first make the following observation. Let σ be a $(q-1)$-simplex, then

$$\begin{aligned}(\sigma_\alpha)^i &= (U_\alpha, \dots, \hat{U}_{i-1}, \dots, U_q)\\ (\sigma^i)_\alpha &= (U_\alpha, \dots, \hat{U}_i, \dots, U_q)\end{aligned}$$

form which we see that $(\sigma^i)_\alpha = (\sigma_\alpha)^{i+1}$ and $(\sigma_\alpha)^0 = \sigma$.
Let σ be a $(q-1)$ simplex and f a q-cochain, then:

$$\begin{aligned}d(h_q f)(\sigma) &= \sum_{i=0}^{q}(-1)^i \rho_{|\sigma|,|\sigma^i|} h_q f(\sigma^i)\\ &= \sum_{i=0}^{q}(-1)^i \rho_{|\sigma|,|\sigma^i|} \sum_\alpha l_\alpha \circ f((\sigma^i)_\alpha)\\ &= \sum_{i=0}^{q}\sum_\alpha (-1)^i \rho_{|\sigma|,|\sigma^i|} l_\alpha \circ f((\sigma^i)_\alpha)\\ &= \sum_{i=0}^{q}\sum_\alpha (-1)^i l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma^i)_\alpha|} f((\sigma^i)_\alpha)\\ &= \sum_{i=0}^{q}\sum_\alpha (-1)^i l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^{i+1}|} f((\sigma_\alpha)^{i+1})\\ &= \sum_{i=1}^{q+1}\sum_\alpha (-1)^{i-1} l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^{i}|} f((\sigma_\alpha)^{i})\end{aligned}$$

Where we have used the fact that

$$\rho_{|\sigma|,|\sigma^i|} l_\alpha \circ f((\sigma^i)_\alpha) = l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma^i)_\alpha|} f((\sigma^i)_\alpha).$$

On the other hand we have that:

$$\begin{aligned}h_{q+1}(df)(\sigma) &= \sum_\alpha l_\alpha \circ \sum_{i=0}^{q+1}(-1)^i \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^i|} f((\sigma_\alpha)^i)\\ &= \sum_\alpha \sum_{i=0}^{q+1}(-1)^i l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^i|} f((\sigma_\alpha)^i)\\ &= \sum_{i=0}^{q+1}\sum_\alpha (-1)^i l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^i|} f((\sigma_\alpha)^i).\end{aligned}$$

Where in the last line we used that the cover $\mathcal{U}$ is locally finite and thus the sum over α contains only finitely many terms, hence we could change the order of summation. If we combine the above we get that:

$$d(h_q f)(\sigma) + h_{q+1}(df)(\sigma) = \sum_\alpha l_\alpha \circ \rho_{|\sigma_\alpha|,|(\sigma_\alpha)^0|} f((\sigma_\alpha)^0) = \sum_\alpha l_\alpha \circ \rho_{|\sigma_\alpha|,|\sigma|} f(\sigma) = f(\sigma).$$

Hence we conclude that equation (4.5) holds.
Now let a cocycle $f \in C^q(\mathcal{U}, \mathcal{S})$ be given. Then $d(h_q(f)) + h_{q+1}df = f$. Hence we see that $d(h_q(f)) = f$ and thus conclude that $\check{H}^q(\mathcal{U}, \mathcal{S}) = 0$ for $q > 0$. Hence we conclude that $\check{H}^q(M, \mathcal{S}) = 0$ for $q > 0$.
We will now prove axiom (c), that is, a short exact sheaf sequence induces a long exact sequence in cohomology. Let

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{4.6}$$

be a short exact sheaf sequence. This sequence induces a sequence

$$0 \to C^q(\mathcal{U}, \mathcal{S}') \to C^q(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S}'') \tag{4.7}$$

by composing cochains by the homomorphisms in the original sequence. It is easy to show that this sequence is also exact, but we do not have a priori that the map $C^q(\mathcal{U}, \mathcal{S}) \to C^q(\mathcal{U}, \mathcal{S}'')$ is surjective. Thus we will consider a slightly different sequence. Let $\overline{C}^q(\mathcal{U}, \mathcal{S}')$ denote the image of $C^q(\mathcal{U}, \mathcal{S})$ in $C^q(\mathcal{U}, \mathcal{S}'')$. Then the sequence

$$0 \to C^q(\mathcal{U}, \mathcal{S}') \to C^q(\mathcal{U}, \mathcal{S}) \to \overline{C}^q(\mathcal{U}, \mathcal{S}'') \to 0 \tag{4.8}$$

is exact for all q. Because the homomorphisms are cochain maps for the corresponding cochain complexes we get a short exact sequence of cochain complexes

$$0 \to C^*(\mathcal{U}, \mathcal{S}') \to C^*(\mathcal{U}, \mathcal{S}) \to \overline{C}^*(\mathcal{U}, \mathcal{S}'') \to 0. \tag{4.9}$$

This implies, using Proposition 2.11 and 2.10, that the following diagram commutes and the rows are exact:

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{H}^{q-1}(\mathcal{U}, \mathcal{S}'') \arrow[r] \arrow[d, "\mu^*_{q-1}"'] & \check{H}^q(\mathcal{U}, \mathcal{S}') \arrow[r] \arrow[d, "\mu^*_q"'] & \check{H}^q(\mathcal{U}, \mathcal{S}) \arrow[r] \arrow[d, "\mu^*_q"'] & \overline{H}^q(\mathcal{U}, \mathcal{S}'') \arrow[r] \arrow[d, "\mu^*_q"'] & \check{H}^{q+1}(\mathcal{U}, \mathcal{S}') \arrow[r] \arrow[d, "\mu^*_{q+1}"'] & \cdots \\
\cdots \arrow[r] & \overline{H}^{q-1}(\mathcal{B}, \mathcal{S}'') \arrow[r] & \check{H}^q(\mathcal{B}, \mathcal{S}') \arrow[r] & \check{H}^q(\mathcal{B}, \mathcal{S}) \arrow[r] & \overline{H}^q(\mathcal{B}, \mathcal{S}'') \arrow[r] & \check{H}^{q+1}(\mathcal{B}, \mathcal{S}') \arrow[r] & \cdots
\end{tikzcd} \tag{4.10}$$

Because this diagram commutes we have that the homomorphisms in the rows induce homomorphisms in the direct limit of the cohomology modules. This construction is completely analogous to that of the constructing of homomorphisms in cohomolgy from sheaf homomorphisms as given in (4.15). With these induced homomorphisms we get a long exact sequence:

$$\cdots \to \overline{H}^{q-1}(M, \mathcal{S}'') \to \check{H}^q(M, \mathcal{S}') \to \check{H}^q(M, \mathcal{S}) \to \overline{H}^q(M, \mathcal{S}'') \to \check{H}^{q+1}(M, \mathcal{S}') \to \cdots \tag{4.11}$$

We will show that the inclusion map $\overline{C}^*(\mathcal{U}, \mathcal{S}'') \to C^*(\mathcal{U}, \mathcal{S}'')$ induces isomorphisms

$$\overline{H}^q(M, \mathcal{S}'') \to \check{H}^q(M, \mathcal{S}''). \tag{4.12}$$

These isomorphisms, combined with the sequence (4.11), then give rise to a long exact sequence

$$\cdots \to \check{H}^{q-1}(M,\mathcal{S}'') \to \check{H}^{q}(M,\mathcal{S}') \to \check{H}^{q}(M,\mathcal{S}) \to \check{H}^{q}(M,\mathcal{S}'') \to \check{H}^{q+1}(M,\mathcal{S}') \to \cdots \tag{4.13}$$

which proves axiom (c). To show that the inclusion induces isomorphisms in cohomology we define quotient modules

$$\widetilde{C}^q(\mathcal{U},\mathcal{S}'') = C^q(\mathcal{U},\mathcal{S}'')/\overline{C}^q(\mathcal{U},\mathcal{S}'').$$

These, together with the coboundary operator induced by $C^q(\mathcal{U},\mathcal{S}'')$, form a cochain complex $\widetilde{C}^*(\mathcal{U},\mathcal{S}'')$. We will show that the direct limit of the modules $H^q(\widetilde{C}^*(\mathcal{U},\mathcal{S}''))$ is zero for all q. Because the sequence of cochain complexes

$$0 \to \overline{C}^*(\mathcal{U},\mathcal{S}'') \to C^*(\mathcal{U},\mathcal{S}'') \to \widetilde{C}^*(\mathcal{U},\mathcal{S}'') \to 0$$

is exact we then get, as be an induced long exact sequence in cohomology modules:

$$\cdots \to \check{H}^{q-1}(\widetilde{C}^*(\mathcal{U},\mathcal{S}'')) \to \check{H}^{q}(\overline{C}^*(\mathcal{U},\mathcal{S}'')) \to \check{H}^{q}(C^*(\mathcal{U},\mathcal{S}'')) \to \check{H}^{q}(\widetilde{C}^*(\mathcal{U},\mathcal{S}'')) \to \cdots$$

and as before a corresponding sequence in the direct limit modules:

$$\cdots \to \widetilde{H}^{q-1}(M,\mathcal{S}'') \to \overline{H}^q(M,\mathcal{S}'') \to \check{H}^q(M,\mathcal{S}'') \to \widetilde{H}^q(M,\mathcal{S}'') \to \cdots$$

where $\widetilde{H}^q(M,\mathcal{S}'')$ denotes the direct limit of the modules $\check{H}^q(\widetilde{C}^*(\mathcal{U},\mathcal{S}''))$. If we show that $\widetilde{H}^q(M,\mathcal{S}'') = 0$ for all q we get isomorphisms:

$$\overline{H}^q(M,\mathcal{S}'') \to \check{H}^q(M,\mathcal{S}'')$$

for all q and we conclude that (4.13) is exact.

Let $f \in C^q(\mathcal{U},\mathcal{S}'')$ be arbitrary, we construct a refinement $\mu : \mathcal{B} \to \mathcal{U}$ such that $\mu_q(f) \in \overline{C}^q(\mathcal{B},\mathcal{S}'')$. Then it is clear that f is equivalent to zero in $\widetilde{H}^q(M,\mathcal{S}'')$. This refinement will be given by the following lemma.

Lemma 4.17. *Let $\{U_i\}$ be a locally finite cover of M, and let $\{V_i\}$ be a refinement such that $\overline{V_i} \subset U_i$ (this exists by [1, p.91]). Then for each $p \in M$ there exists a neighbourhood W_p such that the following properties hold:*

(a) $W_p \subset V_i$ for some i,

(b) If $W_p \cap V_i \neq \emptyset$, then $W_p \subset V_i$,

(c) W_p lies in the intersection of the U_i containing p,

(d) If σ is a q-simplex of the cover $\mathcal{U}$, and $p \in |\sigma|$ (which by (a) implies that $W_p \subset |\sigma|$), then $\rho_{W_p,|\sigma|} f(\sigma)$ is the image of a section of $\mathcal{S}$ over W_p.

Proof. By the locally finiteness of the cover $\{U_i\}$ there exists a neighbourhood X_p of p such that it intersects only finitely many U_i's. Now take a V_i such that $p \in V_i$ and define $Y_p = X_p \cap V_i$. We see that Y_p satisfies property (a).
Now let $Z_p = Y_p \bigcap_{U_i \ni p} U_i$. This set is open by the choice of Y_p and it clearly satisfies property (c).
Let I be the collection of indices for which $Z_p \cap \overline{V_i} \neq \emptyset$ and $p \in \overline{V_i}$. Because $\overline{V_i} \subset U_i$ and Z_p intersects only finitely many U_i's we have that I is finite. Therefore $A_p = Z_p \bigcap (\bigcap_{i \in I} \overline{V_i})^c$ is an open neighbourhood of p. Hence we conclude that A_p satisfies property (b).

Let σ be a q-simplex of $\mathcal{U}$ such that $p \in |\sigma|$. By property (b) $A_p \subset |\sigma|$. Now consider $f(\sigma)(p)$ and denote $g : \mathcal{S} \to \mathcal{S}''$. By the exactness of (4.6) we have that g is surjective and thus there exists an $a \in \mathcal{S}$ such that $g(a) = f(\sigma)(p)$. Now let s be a local section of $\mathcal{S}$ which takes the value a at $p \in M$. Then $(g \circ s)(p) = f(\sigma)(p)$ and hence there exists a neighbourhood W_p of p, which we take to be a subset of A_p, such that $g \circ s|_{W_p} = f(\sigma)|_{W_p}$. We conclude that W_p satisfies property (d). □

Now let $\mathcal{B}$ be the cover $\{W_p\}$. By property (a) we can choose $V_p \subset U_p$ such that $W_p \subset V_p$. This implies that there exists a refinement map $\mu : \mathcal{B} \to \mathcal{U}$. Let $\sigma = (W_{p_0}, \cdots, W_{p_q})$ be a q-simplex of $\mathcal{B}$ and let $\mu(\sigma) = (U_{p_0}, \ldots, U_{p_q})$ be the corresponding $\mathcal{U}$-simplex. Using the fact that σ is a simplex $W_{p_0} \cap V_{p_i} \neq \emptyset$ for all $0 \leq i \leq q$ and by property (b) $W_{p_0} \subset V_{p_i}$ for all $0 \leq i \leq q$, which implies that $W_{p_0} \subset |\mu(\sigma)|$. Hence for all $f \in C^q(\mathcal{B}, \mathcal{S}'')$:

$$\begin{aligned}\mu_q(f)(\sigma) &= \rho_{|\sigma|,|\mu(\sigma)|} f(\mu(\sigma)) \\ &= \rho_{|\sigma|,W_{p_0}} \circ \rho_{W_{p_0},|\mu(\sigma)|} f(\mu(\sigma))\end{aligned}$$

which by property (d) implies that $\mu_q(f) \in \overline{C}^q(\mathcal{B}, \mathcal{S}'')$. Therefore we have shown that the direct limit of the modules $\check{H}^q(\widetilde{C}^*(\mathcal{U}, \mathcal{S}''))$ is zero. Hence we have the isomorphisms in (4.12) which give rive to the long exact sequence in (4.13). We conclude that axiom (c) holds.
Axiom (d) (the identity map in sheaves induces the identity map in cohomology) is apparent form the definition of the induced homomorphisms as in Lemma 4.9.
Axiom (e) (the composition of two sheaf maps induces the same map in cohomology as the compostion of the induced maps) follows from the fact that the composition of two cochain maps induces the same map on cohomology as the composition of the induced maps. If $\mathcal{S} \to \mathcal{S}''$ is the composition $\mathcal{S} \to \mathcal{S}' \to \mathcal{S}''$, the induced cochain maps $C^*(\mathcal{U}, \mathcal{S}) \to C^*(\mathcal{U}, \mathcal{S}'')$ and $C^*(\mathcal{U}, \mathcal{S}) \to C^*(\mathcal{U}, \mathcal{S}') \to C^*(\mathcal{U}, \mathcal{S}'')$ clearly agree. Hence the maps $\check{H}^q(\mathcal{U}, \mathcal{S}) \to \check{H}^q(\mathcal{U}, \mathcal{S}'')$ and $\check{H}^q(\mathcal{U}, \mathcal{S}) \to \check{H}^q(\mathcal{U}, \mathcal{S}') \to \check{H}^q(\mathcal{U}, \mathcal{S}'')$ agree. And as all these maps commute with the refinement homomorphisms, the induced maps on Čech cohomology modules also agree.

Axiom (f) follows in the same manner as axiom (c). Let a homomorphism of short exact sheaf sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}' \arrow[r] \arrow[d] & \mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'' \arrow[r] & 0
\end{tikzcd}$$

be given. We then get a homomorphism between the short exact sequence (4.9) and a corresponding one for the sheaf $\mathcal{F}$. This gives, using Proposition 2.10, a commutative diagram

$$\begin{tikzcd}
\overline{H}^q(M, \mathcal{S}'') \arrow[r] \arrow[d] & \check{H}^{q+1}(M, \mathcal{S}') \arrow[d] \\
\overline{H}^q(M, \mathcal{F}'') \arrow[r] & \check{H}^{q+1}(M, \mathcal{F}')
\end{tikzcd}$$

which together with the isomorphisms (4.12) shows axiom (f). Therefore we can conclude that Čech cohomology as defined in Definition 4.10 gives rise to an axiomatic sheaf cohomology theory. Hence we conclude the existence of such a theory. $\square$

5 Sheaf Resolutions

In this section we will introduce a way to define more notions of cohomology besides the notion of an axiomatic theory. To do this we will make use of *sheaf-resolutions*. These resolutions give rise to cohomology modules which will be isomorphic to the sheaf cohomology modules. We will give two concrete examples of such resolutions, which will result in the de Rham, and singular cohomology modules. This will however not result in a full cohomology theory as both de Rham and singular only yield resolutions for a certain class of sheaves. There is however a canonical resolution which is defined for all sheaves and gives rise to a sheaf cohomology theory, see for instance [5, p.56].

Definition 5.1. An exact sequence of sheaves of K-modules

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \tag{5.1}$$

is called a *resolution* of the sheaf $\mathcal{S}$. The resolution is called *fine* if each of the sheaves $\mathcal{C}_i$ is fine. ♢

This resolution gives rise to the following cochain complex:

$$\cdots \to 0 \to \Gamma(\mathcal{C}_0) \to \Gamma(\mathcal{C}_1) \to \cdots \tag{5.2}$$

which we will denote by $\Gamma(\mathcal{C}^*)$. The homomorphisms in sequence (5.2) are the composition of the homomorphisms in (5.1) with global sections. It is easy to show that (5.2) is indeed a cochain complex. Take note that the sheaf $\mathcal{S}$ is not a part of sequence (5.2). This will ensure that $H^0(\Gamma(\mathcal{C}^*))$ will be isomorphic to $\Gamma(\mathcal{S})$ as will be shown in the next theorem.
The following lemma will be required for the proof that the cohomology modules induced by the cochain complex (5.2) are isomorphic to the sheaf cohomology modules.

Lemma 5.2. *Let*

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

be a short exact sequence of sheaves, then the following sequence is short exact

$$0 \to \Gamma(\mathcal{S}') \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}''). \tag{5.3}$$

If the sheaf $\mathcal{S}'$ is fine the full sequence is exact

$$0 \to \Gamma(\mathcal{S}') \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}'') \to 0. \tag{5.4}$$

Proof. By axioms (a) and (c) of a cohomology theory we have that the following sequence is exact

$$0 \to \Gamma(\mathcal{S}') \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}'') \to H^1(M, \mathcal{S}') \to \cdots$$

which directly shows that sequence (5.3) is exact. If the sheaf $\mathcal{S}'$ is fine then $H^1(M, \mathcal{S}') = 0$, and we conclude that (5.4) is exact. □

Theorem 5.3. *Let $\mathcal{H}$ be a cohomology theory for M with coefficients in sheaves of K-modules over M. Let*

$$0 \to \mathcal{S} \to \mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \tag{5.5}$$

be a fine resolution of the sheaf $\mathcal{S}$. Then there exists isomorphisms

$$H^q(M, \mathcal{S}) \cong H^q(\Gamma(\mathcal{C}^*)) \text{ for all } q.$$

Proof. By Lemma 5.2 the sequence $0 \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{C}_0) \to \Gamma(\mathcal{C}_1)$ is exact. Hence $H^0(\Gamma(\mathcal{C}^*)) = \ker(\Gamma(\mathcal{C}_0) \to \Gamma(\mathcal{C}_1)) \cong \Gamma(\mathcal{S})$, and because $\mathcal{H}$ is a cohomology theory we conclude that $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S}) \cong H^0(\Gamma(\mathcal{C}^*))$. Now for $q \geq 1$ let $\mathcal{K}_q$ be the kernel of the map $\mathcal{C}_q \to \mathcal{C}_{q+1}$. Using the fact that (5.5) is exact, the sequence

$$0 \to \mathcal{S} \to \mathcal{C}_0 \to \mathcal{K}_1 \to 0 \tag{5.6}$$

is exact as well. Similarly, for $q \geq 1$ we have an exact sequence:

$$0 \to \mathcal{K}_q \to \mathcal{C}_q \to \mathcal{K}_{q+1} \to 0 \tag{5.7}$$

For $q = 1$ we consider the long exact sequence induced by sequence (5.6):

$$\cdots \to \Gamma(\mathcal{C}_0) \to \Gamma(\mathcal{K}_1) \to H^1(M, \mathcal{S}) \to \underbrace{H^1(M, \mathcal{C}_0)}_{=0 \text{ as } \mathcal{C}_0 \text{ is fine}} \to \cdots \tag{5.8}$$

Using the fact that $\Gamma(\mathcal{K}_q) = \ker(\Gamma(\mathcal{C}_q) \to \Gamma(\mathcal{C}_{q+1}))$ and the exactness of sequence (5.8) we get isomorphisms

$$H^1(M, \mathcal{S}) \cong \Gamma(\mathcal{K}_1)/\mathrm{Im}\ \Gamma(\mathcal{C}_0).$$

By definition, $\Gamma(\mathcal{K}_1)/\mathrm{Im}\ \Gamma(\mathcal{C}_0) = H^q(\Gamma(\mathcal{C}^*))$ and thus we conclude that $H^1(M, \mathcal{S}) \cong H^1(\Gamma(\mathcal{C}^*))$.
For $q > 1$ we consider the long exact sequence induced by sequence (5.8):

$$\cdots \to H^{q-1}(M, \mathcal{C}_0) \to H^{q-1}(M, \mathcal{K}_1) \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{C}_0) \to \cdots$$

Because $\mathcal{C}_0$ is a fine sheaf, $H^{q-1}(M, \mathcal{C}_0)$ and $H^q(M, \mathcal{C}_0)$ are zero and thus we get from the above sequence isomorphisms:

$$H^{q-1}(M, \mathcal{K}_1) \cong H^q(M, \mathcal{S}). \tag{5.9}$$

We consider the long exact sequence in cohomology induced by theory $\mathcal{H}$ of sequence (5.6):

$$\cdots \to H^{q-i}(M, \mathcal{C}_i) \to H^{q-i}(M, \mathcal{K}_i) \to H^{q-i+1}(M, \mathcal{K}_{i+1}) \to H^{q-i+1}(M, \mathcal{C}_{i+1}) \to \cdots$$

Because $H^{q-i}(M, \mathcal{C}_i) = 0$ as $\mathcal{C}_i$ is fine we get isomorphisms

$$H^{q-i}(M, \mathcal{K}_i) \cong H^{q-i+1}(M, \mathcal{K}_{i+1})$$

for all $1 < i \le q-1$. We can combine these isomorphisms and conclude that

$$H^q(M,\mathcal{S}) \cong H^{q-1}(M,\mathcal{K}_1) \cong H^{q-2}(M,\mathcal{K}_2) \cong \cdots \cong H^1(M,\mathcal{K}_{q-1}). \tag{5.10}$$

From the long exact sequence induced by sequence (5.7) we get the following exact sequence:

$$0 \to \Gamma(\mathcal{K}_{q-1}) \to \Gamma(\mathcal{C}_{q-1}) \to \Gamma(\mathcal{K}_q) \to H^1(M,\mathcal{K}_{q-1}) \to H^1(M,C_{q-1}).$$

Hence

$$H^1(M,\mathcal{K}_{q-1}) \cong \Gamma(\mathcal{K}_q)/\mathrm{Im}\ \Gamma(\mathcal{C}_{q-1}) \cong H^q(\Gamma(\mathcal{C}^*))$$

which combined with the isomorphisms (5.10) completes the proof. $\square$

6 De Rham Cohomology

In this section we will take the ring K to be the real numbers and we let M be a differentiable manifold. We will use differential forms to define presheaves and corresponding sheaves from which we will construct a fine sheaf resolution. The cohomology modules defined in this way will be the sheaf theoretical versions of the de Rham cohomology modules. At the end of this section we will show that this definition coincides with a more classical definition of the de Rham cohomology modules.

Let $U \subset M$ be an open subset of a differential manifold, which is itself a differential manifold as well. The set of differential q-forms on U is a real vector space, which we denote by $\Omega^q(U)$. Let $\rho_{U,V}$ be the restrictions of forms.

Lemma 6.1. *The set $\{\Omega^q(U); \rho_{U,V}\}$ is a complete presheaf.*

Proof. It is clear that this is indeed a presheaf. The fact that it satisfies the locality axiom of a complete presheaf is apparent. So we are left to show the glueability axiom. Let $U = \bigcup_i U_i$ and let $f_i \in \Omega^q(U_i)$ such that $\rho_{U_i \cup U_j, U_i} f_i = \rho_{U_i \cup U_j, U_j} f_j$. We have to show that there exists an $f \in \Omega^q(U)$ which restricts to f_i on U_i. We define f to be $f_i(x)$ for $x \in U_i$. It is clear that this is well-defined, so we are only left to show that $f \in \Omega^q(U)$. Recall that $\Omega^q(U)$ was defined to be the collection $\Gamma(\Lambda^q T^* M)$. If we let π denote the projection of $\Lambda^q T^* M$, we have that $\pi \circ f_i = \text{id}$ for all i. Hence it directly follows that $\pi \circ f = \text{id}$, and thus that $f \in \Omega^q(U)$. From its definition it is clear that f restricts to f_i on U_i so we have shown the glueability axiom and conclude that $\{\Omega^q(U); \rho_{U,V}\}$ is a complete presheaf. □

Let d denote the exterior derivative operator $\Omega^q(U) \to \Omega^{q+1}(U)$. The exterior derivative operator commutes with the restriction homomorphisms and thus it induces a presheaf homomorphism:

$$d : \{\Omega^q(U); \rho_{U,V}\} \to \{\Omega^{q+1}(U); \rho_{U,V}\}.$$

Let $\mathcal{E}^q(M)$ denote the associated sheaf of germs and let d' denote the sheaf homomorphism induced by d.

Theorem 6.2. *Let $\mathcal{R} = M \times \mathbb{R}$ denote the constant sheaf. For $\mathcal{E}^q(M)$ and d' as defined above, the sequence*

$$0 \to \mathcal{R} \to \mathcal{E}^0(M) \xrightarrow{d'} \mathcal{E}^1(M) \xrightarrow{d'} \cdots \tag{6.1}$$

is a fine resolution of the constant sheaf.

This theorem will, in view of Theorem 5.3, give rise to cohomology modules $H^q(\Gamma(\mathcal{E}^*(M)))$. These modules will be isomorphic to the sheaf cohomology modules $H^q(M, \mathcal{R})$.

The sheaves $\mathcal{E}^q(M)$ are all fine by an argument similar to that for the sheaf of germs of smooth functions as in Example 1.37. So we are left to show that the sequence is exact. We first consider the exactness at $\mathcal{R}$ and $\mathcal{E}^0(M)$.

Lemma 6.3. *The sequence* (6.1) *is exact at* $\mathcal{R}$ *and* $\mathcal{E}^0(M)$.

Proof. It is well known that $\Omega^0(U) = C^\infty(U)$, the smooth functions on U. Hence the sheaf $\mathcal{E}^0(M)$ is the sheaf of germs of smooth functions as constructed in Example 1.37. Then the homorphism $\mathcal{R} \to \mathcal{E}^0(M)$ can be defined by sending an element $a \in \mathcal{R}_m$ to the germ of a function with constant value a. This map is clearly injective, hence sequence (6.1) is exact at $\mathcal{R}$.
It is a well known fact that if $f \in \Omega^0(U) = C^\infty(U)$ that $df = 0$ if and only if f is locally constant. Then $d'\rho_{m,U} f = 0$ if and only if f is locally constant. It is clear that the image of $\mathcal{R}$ in $\mathcal{E}^0(M)$ can be identified with the germs of locally constant functions. Thus we the image of $\mathcal{R}$ is equal to the kernel of d. Hence we conclude that the sequence (6.1) is exact at $\mathcal{E}^0(M)$. $\square$

To show the exactness of (6.1) at all other sheaves we first note that $d'^2 = 0$. This is because the exterior derivative operator satisfies $d^2 = 0$. Hence the image of each homomorphism in (6.1) is contained in the kernel of the next. To show the reverse we will use the following corollary of the Poincaré lemma. For a proof see [3, p. 155].

Lemma 6.4. *Let* U *be the open unit ball in* $\mathbb{R}^n$. *For each* $q \geq 1$ *there exists a linear transformation* $h_q : \Omega^q(U) \to \Omega^{q-1}(U)$ *such that*

$$h_{q+1} \circ d + d \circ h_q = \mathrm{id}\,.$$

We now finish the proof of Theorem 6.2 by showing that the sequence (6.1) is exact at $\mathcal{E}^q$ for $q \geq 1$.

Proof. Let $f \in \mathcal{E}^q(M)_m$ be an element of the kernel of d'. By definition of $\mathcal{E}^q(M)_m$ there exists an $f' \in \Omega^q(U)$ such that $\rho_{m,U} f' = f$. Where $\rho_{m,U}$ denotes the projection of an element in $\Omega^q(U)$ to its equivalence class in $\mathcal{E}^q(M)_m$. We choose a coordinate chart $(U, \widetilde{\varphi}_m)$ with $m \in U$. Then U is diffeomorphic to an open unit ball V in $\mathbb{R}^n$. Let φ_m denote this diffeomorphism and $(\varphi_m)^*$ its pullback. Then $(\varphi_m^{-1})^* f'$ is an element of $\Omega^q(V)$ and we apply Lemma 6.4:

$$h_{q+1}(d(\varphi_m^{-1})^* f')) + d(h_q(\varphi_m^{-1})^* f') = (\varphi_m^{-1})^* f'.$$

Because f is in the kernel of d', we have that f' is in the kernel of d. Thus it follows from the above that:

$$d(h_q(\varphi_m^{-1})^* f') = (\varphi_m^{-1})^* f'.$$

Now we apply φ_m^* on both sides which implies, by the fact that the exterior derivative and pullback commute, that:

$$d(\varphi_m^* h_{q+1}(\varphi_m^{-1})^* f') = f'.$$

By applying $\rho_{m,U}$ we get that:

$$f = \rho_{m,U} d(\varphi_m^* h_{q+1} \varphi_m^{-1})^* f') = d'(\rho_{m,U} \varphi_m^* h_{q+1}(\varphi_m^{-1})^* f').$$

Where in the last step we used the definition of d'. We thus conclude that f is in the image of d' and thus that sequence (6.1) is exact. This concludes the proof that the sequence (6.1) is a fine resolution of the constant sheaf $\mathcal{R}$. $\square$

We now define the de Rham Cohomology groups for M:

Definition 6.5. The *de Rham cohomology groups* of a differentiable manifold M are the modules.

$$H^q_{\text{de Rham}}(M) = H^q(\Gamma(\mathcal{E}^*(M))). \tag{6.2}$$

$\diamondsuit$

The de Rham cohomology groups are in view of Theorem 5.3 isomorphic to the corresponding sheaf cohomology modules.

6.1 Classical de Rham cohomology

We will now show that the definition of the de Rham cohomology as given in Example 2.3 is equivalent to the definition as given in Example 2.3. In Example 2.3 we defined the de Rham cohomology modules $H^q_{\text{Classical}}(M)$ as the cohomology modules associated to the cochain complex:

$$\cdots \to \Omega^{q-1}(M) \to \Omega^q(M) \to \Omega^{q+1}(M) \to \cdots$$

Theorem 6.6. *There exists isomorphisms $H^q_{Classical}(M) \cong H^q_{de\ Rham}(M)$.*

Proof. We will construct these isomorphisms on the cochain level. That is, we will show that there exist isomorphisms between the cochain complex $\Omega^*(M)$ and the cochain complex $\Gamma(\mathcal{E}^*(M))$. These isomorphism will then induce isomorphisms in cohomology modules.

The isomorphisms on the cochain level will arise from the fact that the presheaf $\{\Omega^q(U); \rho_{U,V}\}$ is complete. In Lemma 1.32 we had shown that for a complete presheaf $P = \{S_U; \rho_{U,V}\}$, the presheaf $\alpha(\beta(P))$ was isomorphic to P. We did this by constructing a presheaf isomorphism $\{\varphi_U\}$ (recall that a presheaf isomorphism is a collection of K-module isomorphisms for any open $U \subset M$) given by:

$$\varphi_U : S_U \to \Gamma(\beta(P), U) : f \mapsto (\rho_{m,U} f).$$

In the case of the presheaf $\{\Omega^p(U); \rho_{U,V}\}$ we have that these become isomorphisms

$$\varphi^q_U : \Omega^q(U) \to \Gamma(\mathcal{E}^q, U) : f \mapsto (\rho_{m,U} f).$$

The map φ^q_M then is an isomorphism from $\Omega^q(M)$ to $\Gamma(\mathcal{E}^q)$ so to show it is an isomorphism between cochain complexes we only have to show that $\{\varphi^q_M\}$ is a cochain map.

Let $f \in \Omega^q(M)$ then

$$\begin{aligned}\varphi_M^{q+1}(df) &= p \mapsto \rho_{m,M} df\\ d'(\varphi_M^q f) &= d'(p \mapsto \rho_{m,M} f) = p \mapsto \rho_{m,M} df\end{aligned}$$

where we used that d' is the sheaf homomorphism induced by d. Hence $\{\varphi_M^q\}$ is a cochain map $\Omega^*(M) \to \Gamma(\mathcal{E}^*(M))$ which is an isomorphism of cochain complexes. This shows the existence of isomorphisms

$$H^q(\Omega^*(M)) \to H^q(\Gamma(\mathcal{E}^*(M)))$$

which completes the proof that $H^q_{\text{Classical}}(M)$ and $H^q_{\text{de Rham}}(M, \mathcal{R})$ are isomorphic. $\square$

7 Singular cohomology

In this section we let K be an arbitrary ring and M a paracompact, locally euclidean Hausdorff space. We also assume that there is a metric d given on M. This is no further restriction due to the Urysohn metrizability theorem: [1, p.106]. We will need this metric for some technicalities in the last part of this section. None of these will be dependent on the metric.

We will define the notion of singular cohomology in a sheaf theoretical sense by constructing sheaf resolutions of the constant sheaf $M \times G$ where G is a K-module. These sheaf resolutions will consist of the sheaves of germs of singular cochains which we will define by first constructing presheaves. We will also give a more classical definition of these cohomology modules and show that they are isomorphic to the sheaf theoretical versions.

There is also the notion of differentiable singular cohomology which we will not cover here. Most of this section will carry over word for word to differentiable singular cohomology. See [3] for details. The use of differentiable singular cohomology is mostly theoretical. One of its uses is to proof the de Rham theorem, which gives an explicit isomorphism between differentiable singular and de Rham cohomology. For a good introduction on both differentiable singular cohomology as a proof of the de Rham theorem see [2].

The singular cochains will be defined as maps from the spaces of chains of singular p-simplices to a K-module G. The singular p-simplices will be generalizations of the two dimensional triangle in $\mathbb{R}^2$, or the tetrahedron in $\mathbb{R}^3$.

Definition 7.1. For $p \geq 1$ the set

$$\Delta^p = \{(a_1, \ldots, a_p) \in \mathbb{R}^p : \sum_{i=0}^{p} a_i \leq 1 \text{ and each } a_i \geq 0\}$$

is called the *standard p-simplex* in $\mathbb{R}^p$. For $q = 0$, Δ^q is the space containing only the origin, $\{0\}$. $\diamondsuit$

Definition 7.2. A continuous map $\sigma : \Delta^p \to U$, where U is open in M, is called a *singular p-simplex* in U. $\diamondsuit$

Definition 7.3. We define $S_p(U)$ to be the free abelian group generated by singular p-simplices in U, thus elements of $S_p(U)$ are finite formal linear combinations $\sum_i n_i\sigma_i$ with $n_i \in \mathbb{Z}$. We will call elements of $S_p(U)$ *singular p-chains (with integer coefficients).* $\diamondsuit$

The following definition will give the spaces required to define the singular presheaves.

Definition 7.4. Let $S^p(U, G)$ be the K-module consisting of homomorphisms defined on generators σ of $S_p(U)$, which assign an element of G to σ. We call $S^p(U, G)$ the set of *singular p-cochains* on U. Scalar multiplication and addition on $S^p(U, G)$ defined by

$$(f + g)(\sigma) = f(\sigma) + g(\sigma)$$
$$(kf)(\sigma) = k(f(\sigma))$$

for $f, g \in S^p(U,G)$, $k \in K$ and $\sigma \in S_p(U)$, and give $S^p(U,G)$ the structure of a K-module. ♢

Remark 7.5. By linear extension each cochain in $S^p(U,G)$ defines a homomorphism from $S_p(U)$ to G. That is, each cochain is an element of the dual space $S_p(U)^*$ of $S_p(U)$. ♢

Let $V \subset U$ be open. We define homomorphisms

$$\rho_{V,U} : S^p(U,G) \to S^p(V,G)$$

to be the restriction of homomorphisms of $S_p(U)$ to G to homomorphisms of $S_p(V)$ to G.

Definition 7.6. The *presheaf of singular p-cochains* is the collection

$$\{S^p(U,G); \rho_{U,V}\}.$$

♢

The fact that is indeed a presheaf is clear, but this presheaf is in general not complete as it fails to satisfy the locality axiom. Let f be a cochain that restricts to the zero map on simplices contained in U and V. Then this will give us no information on the behaviour of f on $U \cup V$. Thus we can not ensure that it vanishes on $U \cup V$. We do however have that this presheaf satisfies the glueability axiom which will be of use in the discussion of classical singular cohomology.

Lemma 7.7. *The presheaf of singular p-cochain satisfies the glueability axiom of a complete presheaf.*

Proof. Let $U = \bigcup_i U_i$, and let $f_i \in S^p(U_i, G)$ be such that

$$\rho_{U_i \cap U_j, U_i} f_i = \rho_{U_i \cap U_j, U_j} f_j.$$

Now define $f \in S^p(U,G)$ by $f(\sigma) = f_i(\sigma)$ if $\sigma \in S^p(U_i, G)$, and to be 0 if σ is not in any of the $S^p(U_i, G)$. This clearly gives a well-defined element of $S^p(U,G)$ and we thus conclude that the presheaf of singular p-cochain satisfies the glueability axiom. □

The sheaf associated to the presheaf of singular p-cochains will be denoted by $\mathscr{S}^p(M,G)$.
We want to use these sheaves to define singular cohomology in a sheaf theoretical way. To do this we will construct a sheaf resolution of the constant sheaf $\mathcal{G} = M \times G$. This resolution will consist of the sheaves $\mathscr{S}^p(M,G)$. This sheaf resolution will then give rise to cohomology modules $H^p_{\text{sing}}(M,\mathcal{G})$ which will be isomorphic to the sheaf cohomology modules $H^p(M,\mathcal{G})$. To create this resolution we will define homomorphisms between the sheaves of sinqular cochains. To do this we will first define a coboundary operator $S^p(U,G) \to S^{p+1}(U,G)$ as the dual map of a boundary operator between $S_{p+1}(U)$ and $S_p(U)$.

We will define the boundary of a simplex in such a way that it coincides with the intuitive notion of a boundary. Denote by $v_i = \sigma(e_i)$ (where e_i is the standard basis in $\mathbb{R}^n$ and $e_0 = 0$), we then denote by $[v_0, \ldots, \hat{v}_i, \ldots, v_p]$ the space which remains after removing one vertex in the image of σ. Then intuitively

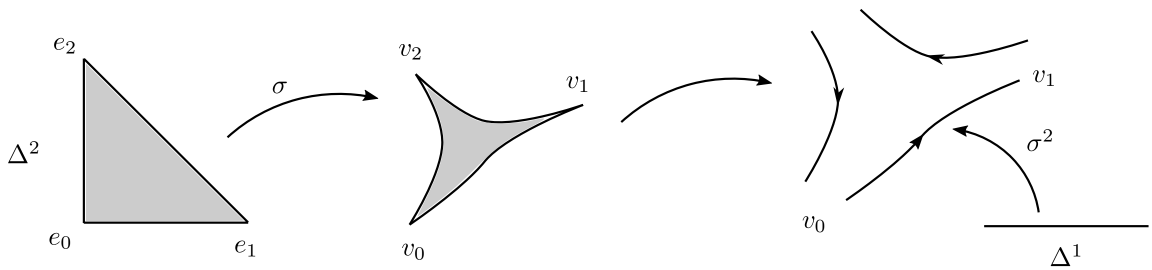

Figure 1: A visualization of the face of a simplex.

$[v_0, \ldots, \hat{v}_i, \ldots, v_p]$ is a simplex of one dimension less then σ, which we will denote by $\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_p]}$, and call the *i-th face* of sigma, σ^i.

We will make this idea more precise. We will define σ^i to be a map from Δ^{p-1} into M, therefore define $k_i^p : \Delta^p \to \Delta^{p+1}$,

$$k_i^p(a_1, \ldots, a_p) := \begin{cases} (1 - \sum_{i=1}^p a_i, a_1, \ldots, a_p) & \text{if } i = 0 \\ (a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_p) & \text{if } 1 \leq i \leq p+1 \end{cases} \tag{7.1}$$

for $p \geq 1$ and $k_0^0(0) = 1, k_1^0(0) = 0$.

We then define the face by $\sigma^i = \sigma \circ k_i^{p-1}$. The boundary of σ will be the signed sum of all its faces. The signs are chosen in such a way that the boundary of the simplex is oriented counter-clockwise.

Definition 7.8. The *boundary* of a singular p-simplex σ is the singular $(p-1)$ chain

$$\partial\sigma = \sum_{i=0}^{p} (-1)^p \sigma^i$$

where σ^i, the $i-$th face, is the $(p-1)$ simplex

$$\sigma^i = \sigma \circ k_i^{p-1}.$$

The maps k_i^p are as in (7.1). ♢

We extend ∂ linearly to $S_p(U)$ to get a homomorphism $S_p(U) \to S_{p-1}(U)$. That is, for a chain $\sigma = \sum_i n_i \sigma_i$ in $S_p(U)$ we define $\partial(\sigma) = \sum_i n_i \partial(\sigma_i)$.

Remark 7.9. The construction of singular cohomology is dual to the notion of singular *homology*. The sequence

$$\cdots \xrightarrow{\partial_{p+2}} S_{p+1}(U) \xrightarrow{\partial_{p+1}} S_p(U) \xrightarrow{\partial_p} S_{p-1}(U) \xrightarrow{\partial_{p-1}} \cdots$$

is called a *chain complex* if the image of ∂_{p-1} is contained in the kernel of ∂_p. The *homology modules* are defined as the quotient modules

$$H_p(M, G) = \ker \partial_p / \operatorname{im} \partial_{p-1}.$$

A classical construction of singular cohomology, which will be given at the end of this section, would then be defined by dualizing the chain complex over G. We will not go into detail on homology, for an excellent introduction see [4]. ♢

We will often first construct an object on then chain level. Then by dualizing that object over G we will get a corresponding object at the cochain level. Therefore it is useful to introduce the dual notions of cochain maps and homotopy operators.

Definition 7.10. A *chain map* $U_* \to V_*$ is a collection of homomorphisms $\varphi_p : U_p \to V_p$ such that for each p the following diagram commutes:

$$\begin{tikzcd} U_{p-1} \arrow[r, "\varphi_{p-1}"] & V_{q-1} \\ U^p \arrow[u, "\partial_U"] \arrow[r, "\varphi_p"] & V^q \arrow[u, "\partial_V"'] \end{tikzcd} \tag{7.2}$$

♢

The following two lemmas show that the notion of chain and cochain complexes are really dual to each other. Both proves are a result of elementary properties of the dual map and thus omitted.

Lemma 7.11. *Let $\varphi_p : U_p \to V_p$ be a collection of homomorphisms which form a chain map. Then the the collection $\varphi_p^* : \mathrm{Hom}(V_p, G) \to \mathrm{Hom}(U_p, G)$ of dual maps forms a cochain map.*

Lemma 7.12. *Let f, g be two chain maps from U_* to V_* which are chain homotopic, that is, there exists maps $h_p : V_p \to U_{p+1}$ such that*

$$\partial \circ h_{p+1} + h_p \circ \partial = f - g,$$

then the dual cochain maps f^*, g^* are cochain homotopic through the dual maps $h_p^* : Hom(U_{p+1}, G) \to Hom(V_p, G)$, that is

$$h_{p+1}^* \circ d + d \circ h_p^* = f^* - g^*.$$

Definition 7.13. The *coboundary operator* $d : S^p(U, G) \to S^{p+1}(U, G)$ is given by

$$df(\sigma) = f(\partial\sigma) \tag{7.3}$$

for all $\sigma \in S_{p+1}(U)$. ♢

The coboundary homomorphism clearly commutes with the restrictions $\rho_{U,V}$ and hence induces a presheaf homomorphism

$$d : \{S^p(U, G), \rho_{U,V}\} \to \{S^{p+1}(U, G), \rho_{U,V}\}.$$

This presheaf homomorphism in turn induces an associated homomorphism between the associated sheaves:

$$d' : \mathscr{S}^p(M, G) \to \mathscr{S}^{p+1}(M, G).$$

Now these definitions of the sheaves $\mathscr{S}^p(M, G)$ and the operators d' will allow us to construct a sheaf resolution of the constant sheaf $\mathcal{G}$.

Claim 7.14. The sequence

$$0 \to \mathcal{G} \to \mathscr{S}^0(M,G) \overset{d'}{\to} \mathscr{S}^1(U,G) \overset{d'}{\to} \dots \tag{7.4}$$

is a fine resolution of the constant sheaf $\mathcal{G}$. $\diamondsuit$

Lemma 7.15. *The sheaves $\mathscr{S}^p(M,G)$ are fine for all p.*

Proof. Let a locally finite cover $\{U_i\}$ of M be given. Take, as in Lemma 3.6, a (discontinuous) partition of unity, $\{\varphi_i\}$, subordinated to $\{U_i\}$ consisting of functions which only take the values 1 and 0. We define a homomorphism of $\{S^p(U,G);\rho_{U,V}\}$ to itself by

$$l_{i,U}(f)(\sigma) = \varphi_i(\sigma(0))f(\sigma)$$

for all $f \in S^p(U,G)$ and $\sigma \in S^p(U)$. Here 0 denotes the origin in Δ^p. It is clear that every $l_{i,U}$ is a homomorphism. Note that the fact that the φ_i are discontinuous is no problem as cochains are not necessarily continuous maps. If $V \subset U$ then $\rho_{V,U} \circ l_{i,U} = l_{i,V} \circ \rho_{V,U}$, hence $\{l_{i,U}\}$ is a presheaf homomorphism of $\{S^p(U,G);\rho_{U,V}\}$ to itself.
We will now show that sheaf homomorphisms l_i, associated to the presheaf homomorphisms $l_{i,U}$ form a partition of unity. For all $f \in S^p(U,G)$ and $\sigma \in S^p(U)$ we have that

$$\sum_i l_i \circ \rho_{m,U} f(\sigma) = \sum_i \rho_{m,U} l_{i,U}(f)(\sigma) = \sum_i \rho_{m,U}\varphi_i(\sigma(0))f(\sigma) = \rho_{m,U} f(\sigma)\sum_i \varphi_i(\sigma(0)) = \rho_{m,U} f(\sigma).$$

Hence $\sum_i l_i = 1$. Now let $m \in M$ such that $m \notin U_i$. We will show that $l_i|_{\mathscr{S}^p(M,G)_m} = 0$ which will finish the proof that $\{l_i\}$ forms a partition of unity. As in Example 1.37 we choose a neighbourhood U of m such that $U \cap \operatorname{supp}(\varphi_i) = \emptyset$. Because $\sigma(0) \in U$ we have that $\sigma(0) \notin \operatorname{supp}(\varphi_i)$. Therefore

$$(l_i \circ \rho_{m,U})f(\sigma) = \rho_{m,U}\varphi_i(\sigma(0))f(\sigma) = 0.$$

We thus conclude that $\{l_i\}$ is a partition of unity for $\mathscr{S}^p(M,G)$ hence the sheaves $\mathscr{S}^p(M,G)$ are fine for all p. $\square$

To show that this sequence is exact at $\mathcal{G}$ we need to show that $\mathcal{G}$ can be injected into $\mathscr{S}^0(M,G)$, for that we will first show the following:

Lemma 7.16. *The sheaf of germs of discontinuous functions on M with values in G is isomorphic to $\mathscr{S}^0(M,G)$.*

Proof. Every element of $S_0(U)$ is a point in U, as it is a map from $\Delta^0 = \{0\}$ to U. Therefore $S_0(U) \cong U$, hence an element of $S^0(U,G)$ is a homomorphism from U to G. Hence the sheaf corresponding to the presheaf $\{S^0(U,G);\rho_{U,V}\}$ is the sheaf of germs of discontinuous functions on M with values in G. $\square$

Now we inject $\mathcal{G}$ into $\mathscr{S}^0(M,G)$ by sending an element $a \in \mathcal{G}_m$ to the germ of the constant function with value a in m. It then becomes clear that the sequence (7.4) is exact at $\mathcal{G}$.
We will now show that the sequence (7.4) is exact. Before we have already shown that the sequence is exact at $\mathcal{G}$. We now show exactness at $\mathscr{S}^0(M,G)$.

Lemma 7.17. *The sequence* (7.4) *is exact at* $\mathscr{S}^0(M,G)$.

Proof. We first show that the image of $\mathcal{G}$ is contained in the kernel of d'. Recall that sheaf homomorphisms are defined stalkwise and we thus only need to check this on the level of stalks. Let $f \in \mathscr{S}^0(M,G)_m$ be an element of the image of $\mathcal{G}$ and let $f' \in S^0(U,G)$ be such that $\rho_{m,U} f' = f$. We have for every singular 1-simplex σ in U that:

$$df'(\sigma) = f'(\partial\sigma) = f'(\sigma^0 - \sigma^1) = f'(\sigma^0) - f'(\sigma^1). \tag{7.5}$$

As in Lemma 7.16 we can identify the image of $\mathcal{G}$ in $\mathscr{S}^0(M,G)$ as the germs of constant functions. Hence f is the germ of a constant function and thus f' is a constant function. As σ^0 and σ^1 are 0-simplices we can identify them with points in U and thus conclude that $df'(\sigma) = 0$. Hence $d'f(\sigma) = 0$ and we conclude that f is in the kernel of d'.
Similarly, let $f \in (\mathscr{S}^0(M,G))_m$ be in the kernel of d' and let $f' \in S^0(U,G)$ be such that $\rho_{m,U} f' = f$. Then $df' = 0$ and by (7.5) $f'(\sigma^0) = f'(\sigma^1)$ for all 1-simplices σ. As before we can identify σ^0 and σ^1 with points in U, and conclude that f' is a locally constant function. This shows that f' is in the image of $\mathcal{G}$ and hence we conclude that the sequence (7.4) is exact at $\mathscr{S}^0(M,G)$. $\square$

To show that sequence (7.4) is exact at the sheaves $\mathscr{S}^q(M,G)$ for all $q \geq 1$ we first show that $d'^2 = 0$. To do this we will show that $\partial^2 = 0$ from which it follows that $d^2 = 0$ and thus also that $d'^2 = 0$.
To show that $\partial^2 = 0$, we first note that the j-th face of the $i-$th face satisfies $(\sigma^i)^j = (\sigma^{j+1})^i$ for $j \geq i$, and $(\sigma^i)^j = (\sigma^j)^{i-1}$ for $j < i$. This is precisely the same as what we noticed in the proof that the Čech cobounary operator squared to zero. The proof that $\partial^2 = 0$ caries over word for word from Lemma 4.4.
Because $d^2 = 0$, and thus also $d'^2 = 0$, we conclude that at every stage in sequence (7.4) the image of d'^p is contained in the kernel of d'^{p+1}. To show that sequence (7.4) is exact we are left to show that the kernel of d'^p is contained in the image of d'^{p+1}.

Lemma 7.18. *The kernel of* d'^p *is contained in the image of* d'^{p+1}.

Proof. Let $f \in (\mathscr{S}^p(M,G))_m$ with $p \geq 1$ such that $d'f = 0$ and let $f' \in S^p(U,G)$ be such that $\rho_{m,U} f' = f$. Hence $df' = 0$ as well. If we can find $g' \in S^{p-1}(U,G)$ such that $dg' = f$, we conclude that $d(\rho_{m,U} g') = f$. It will turn out that we need to take U sufficiently small such that it is homeomorphic to an open unit ball in $\mathbb{R}^{\dim M}$ which is possible due to the locally euclidean structure of M. To find such a g' we will construct cochain homotopy operators between the identity and the trivial map, i.e. we will construct homomorphisms

$$h_p : S^p(U,G) \to S^{p-1}(U,G)$$

such that
$$d \circ h_p + h_{p+1} \circ d = \mathrm{id}\,. \tag{7.6}$$
Assume for the moment that we have such cochain homotopy operators, then
$$f' = d(h_p f') + h_{p+1}(df') = d(h_p f')$$
from which we conclude that f' is in the image of d and thus that f is in the image of d'.
To find the homotopy operators h_p, we first define chain homotopy operators $\widetilde{h}_p$ between the identity and the trivial map, i.e. homomorphisms such that
$$\mathrm{id} = \partial \circ \widetilde{h}_{p+1} + \widetilde{h}_p \circ \partial.$$

Lemma 7.19. *The maps*
$$\widetilde{h}_p : S_{p-1}(U) \to S_p(U) \tag{7.7}$$
defined by
$$\widetilde{h}_p(\sigma)(a_1, \dots, a_p) = \left(\sum_{i=1}^{p} a_i \right) \cdot \sigma \left(\frac{a_2}{\sum_{i=1}^{p} a_i}, \dots, \frac{a_p}{\sum_{i=1}^{p} a_i} \right)$$
for any p-simplex σ and $(a_1, \dots, a_p) \neq 0$, and 0 for $(a_1, \dots, a_p) = 0$, are chain homotopies between the identity and the trivial map, that is
$$\mathrm{id} = \partial \circ \widetilde{h}_{p+1} + \widetilde{h}_p \circ \partial. \tag{7.8}$$

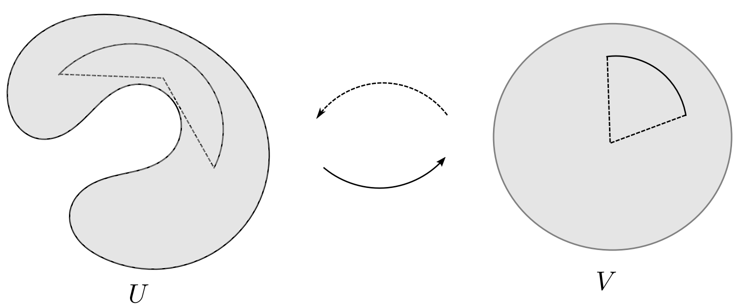

Figure 2: A visualization of the homotopy operators $\widetilde{h}_p$. The 1-simplex in U denotes the simplex $\widetilde{h}_p(\sigma)$.

Remark 7.20. We need to make some remarks on the definition of $\widetilde{h}_p$. Geometrically $\widetilde{h}_p(\sigma)$ will be the cone on σ as depicted in Figure 2. In the definition of $\widetilde{h}_p$ we multiply σ with a real number. To do this we have made an implicit identification between U and an open unit ball in $\mathbb{R}^n$. We have assumed that U was chosen such that it was homeomorphic to an open unit ball in $\mathbb{R}^n$, which we will call V. In the definition of $\widetilde{h}_p$ we therefore first identify U with V. Then σ will have values V and we thus see that multiplication with real numbers is well-defined.

Another note that must be made is that $\widetilde{h}_p(\sigma)$ still has to have values inside U. Again we first identify U with V, and then by the the convexity of the V we have that the cone on σ will still lie in V. If we then identify V back to U we can conclude that $\widetilde{h}_p(\sigma)$ will have values in U. $\diamondsuit$

Lemma 7.21. *The map $\widetilde{h}_p$ is well-defined.*

Proof. We have for all points where $(a_1, \ldots, a_p) \neq 0$ that $a_2/\sum_{i=1}^p a_i + \ldots + a_p/\sum_{i=1}^p a_i \leq 1$. Hence $\sigma(a_2/\sum_{i=1}^p a_i, \ldots, a_p/\sum_{i=1}^p a_i)$ is well-defined. We thus are left to check is that $\widetilde{h}_p(\sigma)$ is continuous at 0. This is due to the fact that $\sigma(a_2/\sum_{i=1}^p a_i, \ldots, a_p/\sum_{i=1}^p a_i)$ is bounded and we thus have that $\lim_{a\to 0} = \widetilde{h}_p(\sigma)(a) = 0$. We clearly see that as $\widetilde{h}_p$ is linear, it can be extended to a homomorphism $S_{p-1}(U) \to S_p(U)$. $\square$

Now we have checked that the maps $\widetilde{h}_p$ are in fact well-defined we can prove the lemma.

Proof of Lemma 7.19. Let $c = \sum_{i=1}^p a_i$. We have that,

$$\begin{aligned}
(\widetilde{h}_p \circ \partial(\sigma))(a_1, \ldots, a_p) &= c \cdot \partial\sigma(a_2/c, \ldots, a_p/c) \\
&= c \cdot \sum_{i=0}^{p} (-1)^i (\sigma \circ k_i^{p-1})(a_2/c, \ldots, a_p/c) \\
&= c\sigma \circ k_0^{p-1}(a_2/c, \ldots, a_p/c) + c \cdot \sum_{i=1}^{p} (-1)^i \sigma(a_2/c, \ldots, a_i/c, 0, a_{i+1}/c, \ldots, a_p/c) \\
&= c\sigma \circ k_0^{p-1}(a_2/c, \ldots, a_p/c) + c \cdot \sum_{i=2}^{p+1} (-1)^{i-1} \sigma(a_2/c, \ldots, a_{i-1}/c, 0, a_i/c, \ldots, a_p/c)
\end{aligned}$$

on the other hand,

$$\begin{aligned}
(\partial \circ \widetilde{h}_{p+1}(\sigma))(a_1, \ldots, a_p) &= \sum_{i=0}^{p+1} (-1)^i \widetilde{h}_{p+1}(\sigma)(k_i^p)(a_1, \ldots, a_p) \\
&= \widetilde{h}_{p+1}(\sigma)(a - c, a_1, \ldots, a_p) + \sum_{i=1}^{p+1} (-1)^i \widetilde{h}_{p+1}(\sigma)(a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_p) \\
&= \widetilde{h}_{p+1}(\sigma)(a - c, a_1, \ldots, a_p) - \widetilde{h}_{p+1}(\sigma)(0, a_1, \ldots, a_p) \\
&\quad + c \cdot \sum_{i=2}^{p+1} (-1)^i \sigma(a_2/c, \ldots, a_{i-1}/c, 0, a_i/c, \ldots, a_p/c).
\end{aligned}$$

If we add these up the sums cancel out and we are left with three terms which we work out separately.

$$\begin{aligned}
&c\sigma \circ k_0^{p-1}(a_2/c, \ldots, a_p/c) = c\sigma\left(1 - \frac{\sum_{i=2}^p a_i}{\sum_{i=1}^p a_i}, a_2/c, \ldots, a_p/c\right) = c\sigma(a_1/c, \ldots, a_p/c) \\
&- c\widetilde{h}_{p+1}(\sigma)(0, a_1, \ldots, a_p) = -c\sigma(a_1/c, \ldots, a_p/c) \\
&\widetilde{h}_{p+1}(\sigma)(a - c, a_1, \ldots, a_p) - \widetilde{h}_{p+1}(\sigma)(0, a_1, \ldots, a_p) = (a - c + c)(\sigma)(a_1, \ldots, a_p) = \sigma(a_1, \ldots, a_p).
\end{aligned}$$

The sum of these terms equals $\sigma(a_1, \ldots, a_p)$, thus we conclude $\partial \circ \widetilde{h}_{p+1} + \widetilde{h}_p \circ \partial = \mathrm{id}$. $\square$

We will now use these chain homotopy operators to define cochain homotopy operators h_p.

Lemma 7.22. *The maps $h_p : S^p(U,G) \to S^{p-1}(U,G)$ defined by*

$$h_p(f)(\sigma) = f(\widetilde{h}_p(\sigma)) \tag{7.9}$$

for any p-simplex σ satisfy

$$d \circ h_p + h_{p+1} \circ d = \mathrm{id}\,.$$

Proof. This is a direct application of Lemma 7.12 with $f = \mathrm{id}$ and g the trivial map. □

Let $f \in (\mathscr{S}^p(M,G))_m$ with $p \geq 1$ be such that $d'f = 0$. Let $f' \in S^p(U,G)$ be a representative of f for some open neighbourhood U which is homeomorphic to an open unit ball in V. Then $df' = 0$ and by the above lemma $f' = d(h_p f') + h_{p+1}(df') = d(h_p f')$. Hence $f = \rho_{m,U} f' = \rho_{m,U} d(h_p f') = d'(\rho_{m,U} h_p f')$ and we conclude that f is in the image of d'. □

This finishes the proof that the sequence (7.4) is a fine resolution of the constant sheaf $\mathcal{G}$. Hence we can define singular cohomology modules in a sheaf theoretical way which in view of Theorem 5.3 will be isomorphic to the corresponding sheaf cohomology modules.

Definition 7.23. The *singular cohomology modules* are the cohomology modules associated to the cochain complex $\Gamma(\mathscr{S}^*(M,G))$:

$$H^p_{\mathrm{sing}}(M,\mathcal{G}) = H^p(\Gamma(\mathscr{S}^*(M,G)))$$

♢

7.1 Classical singular cohomology

In this section we will show that the classical way to define singular cohomology is equivalent to the definition as given in Definition 7.23.

Definition 7.24. The *classical singular cohomology modules of M* with coefficients in a K-module G are defined by:

$$H^q(M;G) = H^q(S^*(M,G))$$

Where $S^*(M,G)$ is the cochain complex:

$$\cdots \to S^{q-1}(M,G) \to S^q(M,G) \to S^{q+1}(M,G) \to \cdots$$

♢

We will show that there exist isomorphisms between $H^q_{\mathrm{sing}}(M,\mathcal{G})$ and $H^q(M;G)$.In the discussion on classical de Rham cohomology these isomorphisms were constructed on the cochain level. For classical singular cohomology we will do the same, but there is a problem. In the discussion on classical de Rham, the isomorphisms of $\Omega^q(M)$ to $\Gamma(\mathcal{E}^q)$ arose form the fact that the presheaf $\{\Omega^q(U); \rho_{U,V}\}$ was complete. We now however have that the presheaf $\{S^q(U,G); \rho_{U,V}\}$ fails to be complete. Thus we will not get isomorphisms from $S^q(U,G)$ to $\Gamma(\mathscr{S}^q(M,G))$ in the same way. To continue we first introduce a space which will measure how much a presheaf fails to be complete. Let $\{S_U; \rho_{U,V}\}$ be a presheaf on M, define a submodule

$$(S_M)_0 = \{s \in S_M : \rho_{m,M}(s) = 0 \text{ for all } m \in M\}$$

of S_M. The following proposition will be the beginning of defining the isomorphisms $S^p(U,G) \to \Gamma(\mathscr{S}^p(M,G))$:

Proposition 7.25. *Let $\{S_U; \rho_{U,V}\}$ be a presheaf on M satisfying the glueability axiom of a complete presheaf, and let $\mathcal{S}$ be the associated sheaf. Then the sequence*

$$0 \to (S_M)_0 \xrightarrow{i} S_M \xrightarrow{\gamma} \Gamma(\mathcal{S}) \to 0 \tag{7.10}$$

is exact. Where γ is the homomorphism which sends $s \in S_M$ to the global section $m \to \rho_{m,M}(s)$ of $\mathcal{S}$.

Remark 7.26. We will show that the submodule $(S_M)_0$ measures how much a presheaf fails to satisfy the locality axiom of a complete presheaf. Assume that we have a complete presheaf and let $s \in S_M$ be such that $\rho_{m,M}(s) = 0$ for all $m \in M$. From the definition of $\rho_{m,M}$ we have for every $m \in M$ that there exists a neighbourhood U_m of m such that $\rho_{U_m,M} s = 0$. By the locally axiom we have $s = 0$ and we conclude that $(S_M)_0 = 0$. So when the presheaf in the proposition is complete we have that the sequence (7.10) induces an isomorphism $\gamma : S_M \to \Gamma(\mathcal{S})$. This is precisely the role γ played in Lemma 1.32 (P is isomorphic to $\beta(\alpha(P))$)) and thus we can see this proposition as a generalization of that lemma. $\diamondsuit$

Proof. Because the map from $(S_M)_0$ to S_M is an injection it is clear that sequence (7.10) is exact at $(S_M)_0$. We identify the image of $(S_M)_0$ in S_M with itself. Then it is clear that an element in the image of i gets mapped to the zero-section by γ. Hence the sequence (7.10) is exact at S_M. So it remains to show that the sequence is exact at $\Gamma(\mathcal{S})$, that is, γ is surjective. Let $t \in \Gamma(\mathcal{S})$. Because $\mathcal{S}$ is the sheaf associated to the presheaf $\{S_U; \rho_{U,V}\}$ we have for any $a \in \mathcal{S}$ that there exists an $s_m \in S_{V_m}$ such that $\rho_{m,V_m} s_m = a$. Now choose, for any $m \in M$, $s_m \in S_{V_m}$ such that $\gamma(s_m) = \rho_{m,U_m} s_m = t(m)$. Then the sections $p \mapsto \rho_{p,V_m} s_m$ and t have the same value at m. Hence by Lemma 1.11, there exists a neighbourhood Y_m of m such that $t|_{Y_m} = \gamma(s_m)|_{Y_m}$. Clearly the set $\{Y_m\}$ covers M. Choose a locally finite refinement $\{U_i\}$ of both $\{Y_m\}$ and $\{V_i\}$. Because each U_i is a subset of some V_m we can choose a V_m for any U_i and define $s_i = \rho_{U_i,V_m} s_m$. By the fact that $\rho_{U_i,V_m}\gamma(s_m) = \gamma(\rho_{U_i,V_m} s_m) = \gamma(s_i)$ we can conclude that:

$$t|_{U_i} = \gamma(s_m)|_{U_i} = \gamma(s_i).$$

We now want to define an $s \in S_M$ such that $\gamma(s) = t$. To do this we will use the fact that the presheaf satisfies the glueability axiom of a complete presheaf. We will construct a cover $\{W_m\}$ of M such that $W_m \subset U_i$ for all m. Then we construct corresponding $s_m \in S_{W_m}$ such that $\rho_{W_m \cap W_n, W_m} s_m =$

$\rho_{W_m \cap W_n, W_n} s_n$. Finally we use the glueability of the presheaf to show there exist a globally defined $s \in S_M$ such that $\gamma(s) = t$. We will first construct the cover $\{W_m\}$.

Lemma 7.27. *Let $\{U_i\}$ be a locally finite cover of M, and let $\{V_i\}$ be a refinement such that $\overline{V_i} \subset U_i$ (this exists by [1, p.91]). Let I_m be the set of indices i for which $m \in \overline{V_i}$. The set I_m is finite by the locally finiteness of $\{U_i\}$. For any $m \in M$ there exists a neighbourhood W_m of m such that the following properties hold:*

1. $W_m \cap \overline{V}_j = \emptyset \quad$ *if* $\quad j \notin I_m$,
2. $W_m \subset \bigcap_{i \in I_m} U_i$,
3. *For s_i and s_j as above* $\rho_{W_m, U_i}(s_i) = \rho_{W_m, U_j}(s_j) \quad$ *if* $\quad i, j \in I_m$.

Proof. By the locally finiteness of $\{U_i\}$, there exists a neighbourhood W'_m of m that intersects only finitely many elements of U_i. Hence the set of indices $J_m = \{j : j \notin I_m, W'_m \cap \overline{V_j} \neq \emptyset\}$ is finite. Then the intersection $(\bigcup_{j \in J} \overline{V_j})^c = \bigcap_{j \in J} \overline{V_j}^c$ is open as it is the complement of a finite union of closed sets. We shrink the neighbourhood W'_m to a neighbourhood $\widetilde{W}_m \subset \bigcap_{j \in J} \overline{V_j}^c$. We conclude that $\widetilde{W}_m$ is a neighbourhood which satisfies property (1).
The set $\bigcap_{i \in I_m} U_i$ is open because I_m is finite. Hence we can shrink $\widetilde{W}_m$ to a neighbourhood $\widetilde{W}'_m \subset \bigcap_{i \in I_m} U_i$ and we conclude that this neighbourhood satisfies property (2).
Let $s_i \in S_{U_i}$ and $s_j \in S_{U_j}$ such that $m \in \overline{V_i} \cap \overline{V_j}$ and $\gamma(s_i) = t|_{U_i}$ and $\gamma(s_j) = t|_{U_j}$. Then $i, j \in I_m$. We have $\gamma(s_i)|_{U_i \cap U_j} = \gamma(s_j)|_{U_i \cap U_j}$ which implies that

$$\rho_{m, U_i \cap U_j}(s_i) = \rho_{m, U_i \cap U_j}(s_j),$$

which by definition of $\rho_{m, U_i \cap U_j}$ implies that there exists a neighbourhood $W_m \subset U_i \cap U_j$ of m which we shrink to a subset of $\widetilde{W}'_m$ such that

$$\rho_{W_m, U_i} s_i = \rho_{W_m, U_j} s_j.$$

We conclude that W_m is a neighbourhood of m which satisfies the properties of the lemma. □

Let $m \in M$. By property (2) of the above lemma, $W_m \subset \bigcap_{k \in I_m} U_k$. We choose an $i \in I_m$ (note that $I_m \neq \emptyset$ for all m) and define $s_m = \rho_{W_m, U_i}(s_i)$. We first show that this is independent of the choice of i. Let $\widetilde{s}_m = \rho_{W_m, U_j}(s_j)$. Then by the third property of the above lemma, $\rho_{W_m, U_i}(s_i) = \rho_{W_m, U_j}(s_j)$. Hence $\widetilde{s}_m = s_m$ and we conclude that the definition of s_m is independent of the choice of i. We will show that with this definition of s_m we have that for all m and n in M:

$$\rho_{W_m \cap W_n, W_m}(s_m) = \rho_{W_m \cap W_n, W_n}(s_n). \tag{7.11}$$

Let $p \in W_m \cap W_n$. Let $i \in I_p$, that is, $p \in \overline{V}_i$. Then because $p \in W_m$, $\overline{V}_i \cap W_m \neq \emptyset$. Hence by property (1), $i \in I_m$. Similarly, $i \in I_n$ and by property (2) we have that $W_m \cap W_n \subset U_i$. Then:

$$\rho_{W_m \cap W_n, W_m} s_m = \rho_{W_m \cap W_n, W_m}(\rho_{W_m, U_i}(s_i)) = \rho_{W_m \cap W_n, U_i}(s_i),$$
$$\rho_{W_m \cap W_n, W_n} s_n = \rho_{W_m \cap W_n, W_n}(\rho_{W_n, U_i}(s_i)) = \rho_{W_m \cap W_n, U_i}(s_i),$$

which shows that $\rho_{W_m \cap W_n, W_m}(s_m) = \rho_{W_m \cap W_n, W_n}(s_n)$. Because the presheaf satisfies the glueability axiom there exists an $s \in S_M$ such that

$$\rho_{W_m, M}(s) = s_m.$$

Then for any $m \in M$ we have:

$$\begin{aligned} \gamma(s)|_{W_m} &= m \mapsto \rho_{m, W_m}(s_m) = m \mapsto \rho_{m, W_m} \circ \rho_{W_m, U_i}(s_i) \\ &= m \mapsto \rho_{m, U_i}(s_i) = \gamma(s_i)|_{W_m} = t|_{W_m}. \end{aligned}$$

Hence we conclude that $\gamma(s) = t$, which completes the proof that sequence (7.10) is exact. □

We apply this proposition to get an exact sequence of cochain complexes.

Lemma 7.28. *The sequence of cochain complexes*

$$0 \to (S^*(M, G))_0 \to S^*(M, G) \xrightarrow{\gamma} \Gamma(\mathscr{S}^*(M, G)) \to 0, \tag{7.12}$$

where the maps are induced from sequence (7.10)*, is exact.*

Proof. By Proposition 7.25 we have that each of the sequences

$$0 \to S_0^q(M, G) \to S^q(M, G) \xrightarrow{\gamma} \Gamma(\mathscr{S}^*(M, G)) \to 0$$

is exact. So to show that these sequences form a sequence of cochain complexes we are left to show that all homomorphisms are cochain maps. Note that the coboundary operator is d in both $S^q(M, G)$ and $S_0^q(M, G)$ and that it is d' in $\Gamma(\mathscr{S}^*(M, G))$. The map $S_0^q(M, G) \to S^q(M, G)$ clearly induces a cochain map as it is an inclusion. To show that $S^q(M, G) \xrightarrow{\gamma} \Gamma(\mathscr{S}^q(M, G))$ induces a cochain map let $s \in S^q(M, G)$. Then

$$\gamma(ds) = m \mapsto \rho_{m, M}(ds)$$
$$d'\gamma(s) = d'(m \mapsto \rho_{m, M} s) = m \mapsto \rho_{m, M} ds,$$

since d' is a sheaf homomorphism induced by d. This completes the prove that $S^q(M, G) \xrightarrow{\gamma} \Gamma(\mathscr{S}^q(M, G))$ induces a cochain map and thus that the sequence (7.12) is a short exact sequence of cochain complexes. □

The following claim will be key to defining the isomorphisms $S^*(M,G) \to \Gamma(\mathscr{S}^*(M,G))$, we will prove it later in this section.

Claim 7.29. The cohomology of $S_0^*(M,G)$ is trivial, that is, $H^q(S_0^*(M,G)) = 0$ for all q. $\diamondsuit$

Now with this claim we can show the existence of the isomorphism between the classical definition of singular cohomology and the sheaf theoretical definition:

Theorem 7.30. *For all q, $H^q(S^*(M,G)) \cong H^q(\Gamma(\mathscr{S}^*(M,G))$.*

Proof. The short exact sequence (7.12) induces a long exact sequence in cohomology:

$$\cdots \xrightarrow{\delta} H^q(S_0^*(M,G)) \longrightarrow H^q(S^*(M,G)) \longrightarrow H^q(\Gamma(\mathscr{S}^*(M,G)) \xrightarrow{\delta} H^{q+1}(S_0^*(M,G)) \to \dots$$

By the above claim $H^q(S_0^*(M,G)) = 0$ for all q, hence $H^q(S^*(M,G)) \cong H^q(\Gamma(\mathscr{S}^*(M,G))$. $\square$

We now return to the proof of the claim.

Lemma 7.31. *The modules $H^q(S_0^*(M,G))$ are trivial for all $q \leq 0$.*

Proof. Because the modules $S_0^q(M,G)$ are all zero for $q < 0$ we clearly see that this holds for $q < 0$. For $q = 0$ we have by the proof of Lemma 7.16 that the presheaf $\{S^0(U,G); \rho_{U,V}\}$ is isomorphic to the sheaf of discontinuous functions. By Example 1.34 this presheaf is complete and by Remark 7.26 we have that $S_0^0(M,G) = 0$. Hence $H^0(S_0^*(M,G)) = 0$. $\square$

To show the claim for $q \geq 1$, let f be a q-cocyle of $S_0^*(M,G)$. By definition there exists an open cover $\mathcal{U} = \{U_i\}$ of M such that for every point m there is a U_i such that $\rho_{U_i,M} f = 0$. If all simplices would be contained in some element of $\mathcal{U}$, we would have that $f = 0$. This is the motivation for the following definition.

Definition 7.32. The cochain complex of *$\mathcal{U}$-small singular cochains*, $S_{\mathcal{U}}^*$, consists of the modules $S_{\mathcal{U}}^p(M,G)$, which have as elements singular cochains that are defined only on $\mathcal{U}$-small simplices, that is, simplices with values contained in elements of the cover $\mathcal{U}$. $\diamondsuit$

It is clear that if let $\mathcal{U}$ in this definition be the cover of M such that for every point m, there is a U_i such that $\rho_{U_i,M} f = 0$, that the restriction of $f \in S_0^q(M,G)$ to $S_{\mathcal{U}}^p(M,G)$ is zero. Thus we would like to be able to work only with $\mathcal{U}$-small singular cochains. This is the idea behind the following claim.

Claim 7.33. The restriction homomorphisms $j_{\mathcal{U}}^p : S^p(M,G) \to S_{\mathcal{U}}^p(M,G)$ which restrict a cochain defined on all simplices to $\mathcal{U}$-small simplices induces a cochain map

$$j_{\mathcal{U}} : S^*(M,G) \to S_{\mathcal{U}}^*(M,G)$$

which induces isomorphisms in cohomology modules. $\diamondsuit$

It is clear that $j_{\mathcal{U}}$ is indeed a cochain map, so we are left to show that it induces isomorphisms in cohomology. We will come back to the proof of this claim in a moment, but will first use it to proof Claim 7.29.

Proof of Claim 7.29. Denote by $K^q_{\mathcal{U}}$ the kernels of the maps $j^p_{\mathcal{U}}$. If we restrict the coboundary operator of $S^*(M,G)$ onto $K^*_{\mathcal{U}}$ we get a cochain complex $K^*_{\mathcal{U}}$. Then we get the following short exact sequence of cochain complexes:

$$0 \to K^*_{\mathcal{U}} \to S^*(M,G) \overset{j_{\mathcal{U}}}{\to} S^*_{\mathcal{U}}(M,G) \to 0.$$

Now consider the long exact sequence in cohomology induced by this sequence:

$$\cdots \to H^q(K^*_{\mathcal{U}}) \to H^q(S^*(M,G)) \xrightarrow{j^*_{\mathcal{U}}} H^q(S^*_{\mathcal{U}}(M,G)) \to H^{q+1}(K^*_{\mathcal{U}}) \to \cdots$$

If we can show that $j_{\mathcal{U}}$ induces isomorphisms in cohomology modules we get from this sequence that:

$$H^q(K^*_{\mathcal{U}}) = 0 \quad \text{for all } q. \tag{7.13}$$

Now we can proof Claim 7.29. Let $f \in S^q_0(M,G)$ such that $df = 0$. Fix the cover $\mathcal{U} = \{U_i\}$ to be such that for every point $m \in M$ there is a U_i such that $\rho_{U_i,M} f = 0$. This cover exists by the definition of $S^q_0(M,G)$. Hence the restriction $j^q_{\mathcal{U}} f = 0$ and $f \in K^q_{\mathcal{U}}$. By the fact that $j_{\mathcal{U}}$ is a cochain map we have that $dj^q_{\mathcal{U}} f = j^{p+1}_{\mathcal{U}} df = 0$. Then by (7.13) there exists a $g \in K^{q-1}_{\mathcal{U}}$ such that $dg = f$. Because it is clear that $K^{q-1}_{\mathcal{U}} \subset S^q_0(M,G)$ this concludes the prove of Claim 7.29. □

We will now return to the prove that $j_{\mathcal{U}}$ induces surjections in cohomology. Since our cover $\mathcal{U}$ is fixed we will omit it from the notation when possible. To prove that j induces surjections in cohomology, we construct a cochain map

$$k : S^*_{\mathcal{U}}(M,G) \to S^*(M,G)$$

such that

$$j \circ k = \mathrm{id}\,.$$

Then it is clear that j must induce surjections in comhomlogy modules.
To prove that j induces injections on cohomology modules we will find homotopy operators $h_p : S^p(M,G) \to S^{p-1}(M,G)$ for all p such that

$$h_{p+1} \circ d + d \circ h_p = \mathrm{id} - k \circ j. \tag{7.14}$$

Hence $k \circ j$ induces the identity map on cohomology modules. Thus we conclude that j induces injection in cohomology and we conclude that j induces isomorphisms in cohomology.
Let $f \in S^p_{\mathcal{U}}(M,G)$. Note that f is only defined on $\mathcal{U}$-small simplices. We want to define $k(f)$ so that it is defined on all singular p-simplices. To do this we will break large p-simplices into smaller ones by a process called *barycentric subdivision* which will be covered in the next section. We then define $k(f)$ on the large p-simplices to be f on the subdivided ones.

7.1.1 Barycentric Subdivision

The process of barycentric subdivision is very technical, but intuitive. The barycentre of a simplex is an abstract version of the barycentre of a triangle. We will will create a chain of small simplices from a larger simplex by using the barycentre as extra vertice. This process is depicted Figure 3.
This process is done in such a way that the boundary of the chain is the same as the boundary of the simplex before subdivision. This makes it intuitive that barycentric subdivision will induce the identity in cohomology. The main part of this section will be to show that this is indeed the case. Then we will use this process to construct the maps from the previous section.

We will begin with the process of barycentric subdivision by first dividing the so called linear p-simplices, and then general singular p-chains.

Definition 7.34. *A linear p-simplex* in Δ^q, i.e a map $\sigma : \Delta^p \to \Delta^q$, is a singular p-simplex of the form

$$(a_1, \ldots, a_q) \mapsto (1 - \sum_{i=1}^{p} a_i)v_0 + a_1 v_1 + \ldots a_p v_p$$

determined by the ordered sequence $(v_0, \ldots, v_p)$ of points in Δ^q, for $q = 0$ this is the map $0 \mapsto v_0$. ♢

Note that the sum of the coefficients of the v_i in this definition is equal to 1. This ensures that the image of a linear p-simplex lies in Δ^q. This is due to an equivalent definition of the standard simplex, namely that it is the smallest set which contains all linear combinations of the form $\sum_i a_i e_i$ with $\sum_i a_i = 1$.

Example 7.35. The identity map on Δ^q is a linear q-simplex ($v_0 = 0$ and v_i is the standard basis of $\mathbb{R}^q$) which we will denote by Δ^q. This linear simplex will be of main interest to us. △

We first define our subdivision operator on linear simplices. Since the identity simplex is a linear simplex, we define the operator on general simplices by composing a singular q-simplex with the subdivided identity simplex.

Definition 7.36. The *barycenter* of a linear p-simplex $\sigma = (v_0, \ldots, v_p)$ in Δ^q is the point

$$b_\sigma = \frac{1}{p+1} v_0 + \cdots + \frac{1}{p+1} v_p.$$

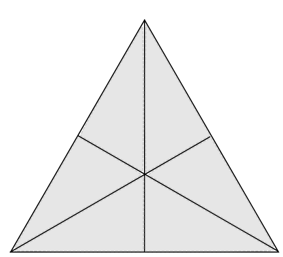

Figure 3: A visualisation of the process of barycentric subdivision. The large simplex is divided into a chain of six smaller ones.

$\diamondsuit$

Let $L_p(\Delta^q)$ denote the free abelian group generated by linear $p-$chains (thus its elements are formal linear combinations of linear p-chains with integer coefficients).

Definition 7.37. Given a linear p-simplex $\sigma = (v_0, \ldots, v_p)$ and a point v in Δ^p we define the *join*, $v\sigma$, of v and σ to be the $(p+1)$-simplex $(v, v_0, \ldots, v_p)$. We extend the join operation to be a linear map form $L_p(\Delta^q)$ to $L_{p+1}(\Delta^q)$. $\diamondsuit$

The following lemma will be of great use later on.

Lemma 7.38. *For any $v \in \Delta^p$ and σ a linear p-simplex,*

$$\partial(v\sigma) = \sigma - v(\partial\sigma). \tag{7.15}$$

Proof. In this proof we will use the notation used in the intuitive introduction of the boundary operator.

$$\begin{aligned}\partial(v\sigma) &= \sigma|_{[v_0,\ldots,v_p]} + \sum_{i=1}^{p+1}(-1)^i\,\sigma|_{[v,\ldots,\hat{v}_{i-1},\ldots,v_p]} = \sigma - \sum_{i=0}^{p}(-1)^i\,\sigma|_{[v,\ldots,\hat{v}_i,\ldots,v_p]}\\ &= \sigma - v\left(\sum_{i=0}^{p}(-1)^i\,\sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_p]}\right) = \sigma - v(\partial\sigma)\end{aligned}$$

$\square$

Definition 7.39. The subdevision operator, Sd_p, is the identity for $p = 0$ and

$$\mathrm{Sd}_p(\sigma) = b_\sigma\,\mathrm{Sd}_{p-1}(\partial\sigma)$$

for $p \geq 1$, where σ is a linear p-simplex. This operator is linearly extended to an operator $\mathrm{Sd}_p : L_p(\Delta^q) \to L_p(\Delta^q)$. $\diamondsuit$

Lemma 7.40. *The barycentric subdivision operator satisfies:*

$$\partial \circ \mathrm{Sd}_{p+1} = \mathrm{Sd}_p \circ \partial. \tag{7.16}$$

That is, Sd *is a chain map of the chain complex $L_*(\Delta^q)$.*

Proof. We proceed by induction. For $p = 0$, by (7.15):

$$(\partial \circ \mathrm{Sd}_1)(\sigma) = \partial(b_\sigma\partial\sigma) = \partial\sigma - b_\sigma\partial^2\sigma = \partial\sigma = (\mathrm{Sd}_0 \circ \partial)(\sigma)$$

which shows (7.16) for $p = 0$. Now assume (7.16) holds for $p = n - 1$. Using equation (7.15) we have that

$$\begin{aligned}\partial(\mathrm{Sd}_{n+1}(\sigma)) &= \partial(b_\sigma\,\mathrm{Sd}_n(\partial\sigma)) = \mathrm{Sd}_n\,\partial\sigma - b_\sigma\partial(\mathrm{Sd}_n(\partial\sigma))\\ &= \mathrm{Sd}_n\,\partial\sigma - b_\sigma(\mathrm{Sd}_{n-1}(\partial\partial\sigma)) = \mathrm{Sd}_n\,\partial\sigma\end{aligned}$$

which shows equation (7.16) for the case that $p = n$. $\square$

To show that the barycentric subdivision operator induces the identity in cohomology we will find cochain homotopy operators by first constructing chain homotopy operators.

Lemma 7.41. *Let R_p be a homomorphism defined by $R_0 = 0$ and for $p \geq 1$ by*

$$R_p(\sigma) = b_\sigma(\sigma - \mathrm{Sd}_p(\sigma) - R_{p-1}(\partial\sigma))$$

for all linear p-simplices in Δ^q. The linear extention of this map to a homomorphism $L_p(\Delta^q) \to L_{p+1}(\Delta^q)$ satisfies

$$\partial \circ R_{p+1} + R_p \circ \partial = \mathrm{id} - \mathrm{Sd}_{p+1}\,. \tag{7.17}$$

That is, R_p is a chain homotopy operator between the subdivision operator and the identity.

Proof. We proceed by induction. Let σ be a 1-simplex in Δ^q. Then:

$$\begin{aligned}(\partial \circ R_1)(\sigma) + (R_0 \circ \partial)(\sigma) &= \partial(b_\sigma\sigma - b_\sigma \,\mathrm{Sd}_1(\sigma)) + 0\\ &= \sigma - b_\sigma(\partial\sigma) - \mathrm{Sd}_1(\sigma) + b_\sigma(\partial \circ \mathrm{Sd}_1\,\sigma)\\ &= \sigma - \mathrm{Sd}_1(\sigma) - b_\sigma(\partial\sigma) + b_\sigma(\mathrm{Sd}_0 \circ \partial\sigma)\\ &= \sigma - \mathrm{Sd}_1(\sigma)\end{aligned}$$

which proves (7.17) for the case that $p = 0$.
Let σ be a linear n-simplex and assume that equation (7.17) holds for $p = n-1$ we have

$$\begin{aligned}\partial R_{n+1}(\sigma) &= \partial(b_\sigma\sigma) - \partial(b_\sigma \,\mathrm{Sd}_{n+1}(\sigma)) - \partial(b_\sigma R_n(\partial\sigma))\\ &= \sigma - b_\sigma(\partial\sigma) - \mathrm{Sd}_{n+1}(\sigma) + b_\sigma(\partial\,\mathrm{Sd}_{n+1}(\sigma)) - R_n(\partial\sigma) + b_\sigma(\partial R_{n-1}(\partial\sigma)\end{aligned}$$

where we applied equation (7.15). Hence

$$\partial R_{n+1}(\sigma) + R_n(\partial\sigma) = \sigma - \mathrm{Sd}_{n+1}(\sigma) + b_\sigma(-\partial\sigma + \partial\,\mathrm{Sd}_{n+1}(\sigma) + \partial R_n(\partial\sigma)).$$

We will now show, using the fact that equation (7.17) holds for $p = n-1$, that $-\partial\sigma + \partial\,\mathrm{Sd}_{n+1}(\sigma) + \partial R_n(\partial\sigma) = 0$,

$$-\partial\sigma + \partial\,\mathrm{Sd}_{n+1}(\sigma) + \partial R_n(\partial\sigma) = -\partial\sigma + \partial\,\mathrm{Sd}_{n+1}(\sigma) + \partial\sigma - \mathrm{Sd}_{n-1}(\partial\sigma) - R_{n-1}(\partial^2\sigma) = 0.$$

Thus we conclude that equation (7.17) holds for $p = n+1$. □

From now on we will drop the subscripts in our notation for Sd and R as it will not result in any confusion. We define the subdivision operator on general singular p-chains. As we have noted before we have that Δ^p is a linear p-simplex.

Definition 7.42. The subdivision operator Sd : $S_p(U) \to S_p(U)$, and the homotopy operators R : $S_p(U) \to S_{p+1}(U)$ are the compositions

$$\mathrm{Sd}(\sigma) = \sigma \circ \mathrm{Sd}(\Delta^p)$$
$$R(\sigma) = \sigma \circ R(\Delta^p)$$

defined on any singular p-simplex and then linearly extended to $S_p(U)$. ♢

One can easily show with this definition of Sd and R that:

$$\mathrm{Sd} \circ \partial = \partial \circ \mathrm{Sd}$$
$$\partial \circ R + R \circ \partial = \mathrm{id} - \mathrm{Sd},$$

that is, Sd is a chain map for the chain complex $S_*(U)$, and R is again a chain homotopy operator between Sd and the identity.
The *diameter* of a simplex is defined to be the largest distance between any two of its points. We have, by the following lemma, that the diameter of the subdivision of a simplex will be smaller then the diameter of the original simplex. For a proof see [4, p.120]

Lemma 7.43. *The diameter of each simplex of* $\mathrm{Sd}(\sigma)$ *will be at most* $p/(p+1)$ *times the diameter of* σ

We will now define a notion which will make the notion of a $\mathcal{U}$-small simplex more precise:

Definition 7.44. The Lesbeque number of an open cover $\mathcal{U}$ of a compact metric space is the number $\delta \in \mathbb{R}_{\geq 0}$ for which all sets of diameter less then δ lie in some element of the open cover. ♢

Lemma 7.45. *The Lesbeque number exists.*

Proof. Since we assumed the metric space to be compact, we can extract a finite subcover $\mathcal{B}$ of $\mathcal{U}$. Let $z_i = \min_{x,y \in B_i}(|x-y|)$. Because B_i is an open in a metric space $z_i > 0$ for all i. Hence $\delta = \min_i z_i$ which exists by the finiteness of the open over $\mathcal{B}$. □

Let σ be a p-simplex in M and let $\mathcal{U}$ be a cover of M. We consider the inverse image cover $\sigma^{-1}(\mathcal{U}) = \{\sigma^{-1}(U_i)\}$. We have that $\sigma^{-1}(\mathcal{U})$ is a cover of the standard p-simplex Δ^p. Hence we have that this cover has a Lesbeque number δ. We now apply the s-fold composition of the subdivision operator, Sd^s, on Δ^p. We have by Lemma 7.43 that the diameter of every simplex in $\mathrm{Sd}^s(\Delta^p)$ tends to zero as s tends to infinity. Hence we can choose s in such a way that the diameter of every simplex in $\mathrm{Sd}^s(\Delta^p)$ is smaller then δ. Now if we take the composition of σ and $\mathrm{Sd}^s(\Delta^p)$ we get that every simplex in $\sigma(\mathrm{Sd}^s(\Delta^p))$ is contained in an element of $\mathcal{U}$. Hence we can conclude that $\sigma \circ \mathrm{Sd}^s$ is an element of the $\mathcal{U}$-small singular simplexes. Now let $s(\sigma)$ denote the minimal number of subdivisions required to ensure that each simplex in $\mathrm{Sd}^{s(\sigma)}(\sigma)$ is contained in an element of $\mathcal{U}$. With this definition it becomes clear that $\mathrm{Sd}^{s(\cdot)}$ maps general simplexes to $\mathcal{U}$-small simplexes, hence we can conclude:

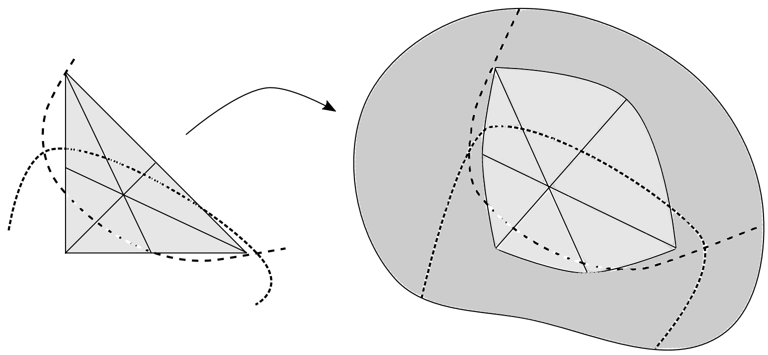

Figure 4: A visualization of the barycentric subdivision of a simplex σ. The striped lines present elements of a cover. We pull the cover back to the standard simplex and by using barycentric subdivision we see that every simplex in the subdivision becomes contained in an element of the cover. Then we apply σ to this chain and get that every element of the subdivision of σ is contained in some element of the cover.

Lemma 7.46. *For any singular p-simplex in M we have*

$$\mathrm{Sd}^{s(\cdot)} : S_p(U) \to (S_{\mathcal{U}}(U))_p$$

where $(S_{\mathcal{U}}(U))_p$ is the space of singular p-chains with values in elements of the cover $\mathcal{U}$.

We will now show that this map is also chain homotopic to the identity. We have already shown this for the case that $s(\cdot) = 1$.

Lemma 7.47. *The maps $T_m : S_p(U) \to S_{p+1}(U)$ defined by*

$$T_m = \sum_{i=0}^{m-1} R(\mathrm{Sd})^i$$

satisfy

$$\partial \circ T_m + T_m \circ \partial = \mathrm{id} - \mathrm{Sd}^m .$$

That is, we have that Sd^m is chain homotopic to the identity.

Proof.

$$
\begin{aligned}
\partial \circ T_m + T_m \circ \partial &= \sum_{i=0}^{m-1} \partial R(\mathrm{Sd})^i + R(\mathrm{Sd})^i \partial \\
&= \sum_{i=0}^{m-1} \partial R(\mathrm{Sd})^i + R\partial(\mathrm{Sd})^i \\
&= \sum_{i=0}^{m-1} (\partial R + R\partial)(\mathrm{Sd})^i \\
&= \sum_{i=0}^{m-1} (\mathrm{id} - \mathrm{Sd})(\mathrm{Sd})^i \\
&= \mathrm{id} - \mathrm{Sd}^m .
\end{aligned}
$$

□

Now we have constructed the subdivision operator we come back to the proof that the restriction map $j_{\mathcal{U}} : S^*(M,G) \to S^*_{\mathcal{U}}(M,G)$ induces isomorphisms in cohomology modules.

We first proof that $j_{\mathcal{U}}$ induces surjections in cohomology, do do this we define a cochain map $k : S^*_{\mathcal{U}}(M,G) \to S^*(M,G)$ such that $j \circ k = \mathrm{id}$. We define the homomorphisms k_p as the dual maps of a composition of barycentric subdivisions.

Definition 7.48. The homomorphisms $k_p : S^p_{\mathcal{U}}(M,G) \to S^p(M,G)$ are the maps for which $k_p = \mathrm{id}$ for $p \leq 0$, and for $p \geq 1$

$$
k_p(f)(\sigma) = f\left(\mathrm{Sd}^{s(\sigma)}(\sigma)\right). \tag{7.18}
$$

♢

Lemma 7.49. *The map k is a cochain map and $j \circ k = \mathrm{id}$. Hence j induces surjections in cohomology.*

Proof. As k is the dual map of a chain map, we have that it is a cochain map. Let $f \in S^p_{\mathcal{U}}(M,G)$ and $\sigma \in (S_{\mathcal{U}})_p$. As the subdivision operator is defined to be the identity on $\mathcal{U}$-small simplices we have that $\mathrm{Sd}^{s(\sigma)}(\sigma) = \sigma$. Hence $k_p(f)(\sigma) = f(\sigma)$ and we conclude that $j \circ k_p = \mathrm{id}$. Thus we conclude that j induces surjections in cohomology. □

Now we will show that j induces injections in cohomology.

Lemma 7.50. *Let $f \in S^p(M,G)$ and $\sigma \in S_{p-1}(M,G)$. The homomorphisms $h_p : S^p(M,G) \to S^{p-1}(M,G)$ defined by $h_p = 0$ for $p \leq 0$ and*

$$
h_p(f)(\sigma) = f\left(R\left(\sum_{0 \leq i \leq s(\sigma)-1} (\mathrm{Sd})^i(\sigma)\right)\right)
$$

for $p \geq 1$ are chain homotopy operators between k and j. That is,

$$
h_{p+1} \circ d + d \circ h_p = \mathrm{id} - k_p \circ j.
$$

Hence j induces injections in cohomology

Proof. By Lemma 7.47 $\mathrm{Sd}^{s(\cdot)}$ is chain homotopic to the identity. Hence by Lemma 7.12 $k_p \circ j$ is cochain homotopic to the identity though the maps h_p. Thus we conclude that $k_p \circ j$ induces the identity on cohomology and we conclude that j induces injections in cohomology. $\square$

By the above lemmas, $j_{\mathcal{U}} : S^*(M, G) \to S^*_{\mathcal{U}}(M, G)$ induces isomorphisms in cohomology which proves Claim 7.33. We had shown that this claim proved Claim 7.29 which in turn proved that there exist isomorphisms

$$H^q(S^*(M, G)) \cong H^q(\Gamma(\mathscr{S}^*(M, G))$$

between the classical and sheaf theoretical definitions of singular cohomology.

Conclusion

In this thesis we have shown the existence and uniqueness of an axiomatic sheaf cohomology theory. We have shown existence by constructing the Čech cohomology, which gave rise to an axiomatic sheaf cohomology theory. We also introduced the notion of a sheaf resolution and the cohomology associated with it. We then showed that under mild conditions this cohomology was isomorphic to the corresponding sheaf cohomology. We have shown that both singular and de Rham cohomology gave rise to sheaf resolutions of which the associated cohomology was isomorphic to the classical cohomology. We can thus conclude that for any K-module G:

$$\check{H}^q(M, \mathcal{G}) \cong H^q_{\text{sing}}(M; G)$$

and if we take $G = \mathbb{R}$:

$$\check{H}^q(M, \mathcal{G}) \cong H^q_{\text{de Rham}}(M) \cong H^q_{\text{sing}}(M; \mathbb{R}).$$

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