

Finding Better Tools for Mathematics Education:
GeoGebra as a Possible Replacement for the Graphing Calculator

Wesley van Woudenberg

University of Utrecht

Author Note

Wesley van Woudenberg, Master student Science Education and Communication, University of Utrecht, 3761959, July 2017, Master Thesis, 30 ECTS

This research paper is supervised by Paul Drijvers, and Michiel Doorman as a second examiner, Department of Mathematics Education, University of Utrecht and supported by and designed at CITO, Arnhem

This paper is intended for the Dutch mathematics education, and addresses ICT use in class.

Correspondence concerning this research proposal should be addressed to Wesley van Woudenberg, Barneveld, Lange voren 70.
Contact: w.j.j.vanwoudenberg@students.uu.nl

Abstract

The graphing calculator (GC) is a proper device that brings ICT in the mathematics classroom, but due to several drawbacks, it is wise that we try to find better tools. This study focusses on GeoGebra (GGB) as a possible replacement of it, and therefore an experiment was set up in order to find out what the transition to GeoGebra on graphical-numerical assessments does to three aspects: test scores, opinions and formation of conceptual processes (instrumental genesis). In six classes, an intervention of three lesson took place where students learned to work with GeoGebra and made tests with one of the two tools. An analysis on differences between the tools on the three different aspects turned out that students' test scores were at least as good in the case that students made the test with GeoGebra. Furthermore, students liked the graphing calculator better because they were more used to it, but it turned out that they became more positive about GeoGebra when they used the program more often. Also, zooming functions, all answers in one screen, and the greater amount of freedom in the program, appeared to be substantial advantages of GeoGebra. Therefore, we can say that GeoGebra has a lot of potential and that the use of it in classrooms is recommended.

Keywords: graphing calculator, GeoGebra, graphical-numerical assessment, instrumental genesis

Finding Better Tools for Mathematics:

GeoGebra as a possible Replacement for the Graphing Calculator

Since the beginning of mankind, humans created tools to facilitate their work. From hunting gear to taming horses and from abacuses to computers, mankind was innovative to invent tools that eased his job. Also in mathematics education, different kinds of tools are used. Pen, paper and drawing gear are common artefacts, but nowadays a technical tool called a calculator is an important aid in the learning processes of the young mathematicians. Calculators made it possible to get ICT into classrooms. More sophisticated versions, as the graphing calculator, were developed and became part of the standard equipment in different countries because national policies enforced teachers and students to work with graphing calculators during national examinations (Drijvers, 2009). After an initial phase of struggles, the use of this handheld technology was not only perceived optimistically by students and teachers but also by researchers and educators, who valued the use of handheld technology because of the permanent availability and the sense of ownership that the students experienced, which was due to the freedom to use their own devices whenever they want and the freedom to make as many mistakes as they want (Trouche & Drijvers, 2010).

In the Netherlands, the graphing calculator is part of the examination since 1998. Currently some serious drawbacks are noticed. The graphing calculator is technologically outdated compared to smartphones and tablets, is an expensive investment especially in the case students already possess a laptop or tablet and it is hard to determine what kind of resources the students actually possess because of the different models and because it is easily expandable with all kinds of downloadable applications. These hindrances are described by the Dutch College of Testing and Examination (CvTE, 2015) and therefore the CvTE is looking for new, cheaper and safer possibilities. Indeed, the question rises whether it could be possible to abolish the graphing calculator in the distant future and make use of virtual graphing calculators on smartphones or use other digital software programs on tablets or laptops (CvTE, 2015).

This study will focus on dynamic geometry software called GeoGebra, as a possible replacement for the graphing calculator. According to a lot people GeoGebra has good potential in mathematics education because it is easy to use, because of its proficient zooming functions, because it could expand the mathematical thinking and creativity processes and because it enhanced students understanding and learning abilities if used in a computer assisted instruction (Saha, Ayub, & Tarmizi, 2010). This might mean that in a future where every student has his own laptop or tablet with GeoGebra, a graphing calculator is redundant. Before this is possible, a lot of questions should be answered. Questions such as ‘To what extent is it possible to adapt tests and exams for which the graphing calculator is required in such a way that GeoGebra can be used as a resource?’ or ‘How well do students perform with GeoGebra during school and national examinations and what are their experiences?’ or ‘What are the experiences of teachers with mathematics tests using GeoGebra and what pressure does this exert on the organization of testing in schools?’ are all valid and important questions to be answered before policies can be changed.

Finding good answers to all of these questions, reaches beyond the scope of one single study. In order to find out whether GeoGebra is at least as good as the graphing calculator, it must be clear whether students in general can use GeoGebra in a similar way they use their

graphing calculator. Therefore, the aim of this study is to examine whether students are able to use GeoGebra for the exact same operations as they know them with the graphing calculator and this study also tries to trace possible problems and advantages when working with GeoGebra. The focus will be on five main operations which are characteristic for all graphical functions because of resemblance. This study will focus not only on the performances of students working with GeoGebra conducting certain techniques they are already familiar with when working with the graphing calculator, it also measures how students learn to use GeoGebra and what they think of the program. The research question of this study is ‘What effects does the use of GeoGebra have on tenth-grade pre-university students in performing graphical-numerical assessments, compared to the use of a graphing calculator?’ These effects will be measures in terms of usability, test results and opinions in three different sub-questions, which will be described in the next chapter.

Theoretical Framework

A lot of scientists agree that technology in the mathematics classroom has a lot of potential (Hodanbosi, 2001; Jupri, Drijvers, Van den Heuvel-Panhuizen, 2015; Saha et al., 2010; Tarmizi, Ayub, Bakar, & Yunus, 2010). In using technology in the classroom, an important difference between working *with* and working *through* technology has been made by Stacey and Wiliam (2013). Assessment *with* technology can be seen as traditional pen-and-paper tests where students are allowed to use technology which supports them in their work, whereas assessment *through* technology means that students are questioned and administered with technological means. This study focusses on the first kind of assessment. Working with technology relieves students of laborious computations and precise drawing operations, and this reduced focus on basic skills enables a larger focus on higher order thinking skills such as modelling, problem solving and conceptual understanding (Drijvers et al., 2016).

A lot of countries acknowledge this potential and have therefore integrated the use of graphing calculators in their national policies. Although questions arose about whether task-items are designed properly, a lot of countries have managed to find a workable situation where students are allowed, recommended or required to use technology at national mathematics examinations (Drijvers, 2009). An early study on this commonly used graphing calculator made clear that besides all the advantages, still 75% of participating students identified difficult or counter-intuitive features (Hennessy, Fung, & Scanlon, 2001). It is not easy to decide upon which kind of tool to use in mathematics education, because it is not easy nor straightforward to foresee the effects on the learning process (Bokhove & Drijvers, 2010), but because of the different kinds of mentioned limitations of the graphing calculator, GeoGebra is by some people seen a good alternative to it. Experiences with coordinate geometry which increased learning outcomes (Saha et al., 2010), indicate the potential of this software program. Although this subject differs from graphical-numerical calculations it seems plausible that GeoGebra is a good tool for this as well because of its intuitive use. It is known that students appreciate to overcome knowledge gaps themselves (Drijvers, 2016), and this is where GeoGebra is advantageous. Buttons can be found on more intuitive places and are easily recognizable by certain icons.

This different kind of usage, calls for an explorative study which compares the use of GeoGebra with that of the graphing calculator. The quality of working with GeoGebra will be tested with the *Theory of Instrumental Genesis*. This theory implies the creation of different

kinds of techniques when working with tools and is best described in view of four dualities (Drijvers, Godino, Font, & Trouche, 2013). The first duality is the difference between an *artefact* and an *instrument*. An artefact is a tool that can be used, for example an axe. When a person becomes familiar with an artefact and uses it for a specific goal, for example to cut trees, it becomes an instrument. Both the graphing calculator and GeoGebra are artefacts, but will only be instruments when used properly by students to perform calculations. The ongoing and time-consuming process for a user to turn an artefact into an instrument is called *instrumental genesis*. A second duality is the one between *instrumentalisation* and *instrumentation*. The relationship between a user and an artefact is bilateral. Instrumentalisation refers to the usage of an artefact by its user while instrumentation implies that the tool guides the user wherein new conceptions emerge. This dual nature within instrumental genesis shows that the student's thinking is shaped by the artefact, but also shaped the artefact. The third duality is the most important one and comprises of the difference between *schemes* and *techniques*. A scheme is a network of concepts in the brain of a user which allow him to think how to deal with a certain problem, whereas a technique refers to the actions a user performs when trying to overcome the problem. Therefore, techniques are the visible counterparts of a scheme which is only present in the mind of the student. In this study, techniques are traced in order to shed light on the schemes. At last the term *instrumental genesis* itself is complemented by *instrumental orchestration*. The first focuses on the learner, while the latter refers to the intentional and systematic organization and use of artefacts by the teacher (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2014; Trouche, 2004). In order to properly trace the genesis of new schemes and mastering of new techniques, a proper instrumental orchestration is needed. At first the teacher needs to master both the functionalities of the artefact as well as the mathematical and didactical backgrounds of the mathematical subjects. Only in this way the teacher is able to pose the right questions at the right time and is he capable of formulating and structuring tasks and assessments the right way. Another important aspect is that students should be enabled to discover things themselves. Learning for peers makes sure that the information is better understood (Trouche, 2004) and because the knowledge is created with peers it feels more like their own knowledge which is more long-lasting (Mayer, 2014).

In learning the right techniques, it is eminent that students practice with assessments in order to fully master them. Assessment can be *summative* or *formative*. In summative assessment, the goal is to evaluate the knowledge at the end of the instruction period, whereas in formative assessment the goal is to “form” the knowledge in the process (EACEA/Eurydice, 2011). The use of technological tools affects both kinds of assessment and therefore both kinds must be looked upon. According to Stacey and Wiliam (2013) the use of technology impacts the way students operate and reason. Because students no longer spend all their time on computation and drawing tasks, other types of skills are needed, new kinds of tasks with different goals emerge which affect the validity and reliability of the assessment. Mathematical literacy abilities, conceptual understanding, strategies, modelling and problem-solving skills can be assessed more easily (Stacey & Wiliam, 2013), which is a welcome benefit.

Three sub-questions will structure the research. Summative assessment is measured in the first question and formative assessment also in the third. The second question will shed an important different light on the research.

1. To what extent does the use of GeoGebra influence test results, compared to a graphing calculator?
2. To what extent do students prefer to use GeoGebra instead of a graphing calculator?
3. How does the instrumental genesis evolve using GeoGebra?

Methodology

Overview of the Study

This study tries to find differences for the graphing calculator and GeoGebra regarding students' schemes. The focus of this experiment will be on five numerical-graphical schemes of window settings (1), images (2), zeroes (3), extreme values (4) and intersections (5). In order to find these differences an experiment which consisted of three lessons was set up, see table 1 for an overview. In the first lesson these five schemes were recapitulated with the graphing calculator, in the second lesson students learned to use these five schemes with GeoGebra and in the third lesson students made a test with one of the artefacts and filled in a questionnaire. All classes will be divided into two groups, which will be clarified in the next section.

Two research paths can be distinguished in this research. At first there is a quantitative path with test results (sub-question 1) and questionnaire outcomes (sub-question 2). These are the written documents by students. It is hypothesized that working with GGB leads to higher results and that students will favor GeoGebra over the graphing calculator. The second path is a qualitative one (sub-question 3). The hypothesis is that students can use GeoGebra fast and more fluently.

Table 1

Design of the study

Lesson	Activity	Sub-question number (path 1)	Student group	Role of researcher	Sub-question number (path 2)
1	Recapitulation of schemes with GC	(1, 2)	A and B	-	
2	Learn and practice schemes with GGB	(1, 2)	A and B	Observes couple with observation instrument	3
3	Test with GC	1	A	Record video	3
	Test with GGB	1	B	Record video with speech	3
	Survey	2	A and B	-	

Methods

Test Results. For the first sub-question, a quasi-experiment was set up. Each class was divided into two randomly selected groups. Both groups receive the same instructions during the first two lessons. In the third lesson one of the groups made a test with the graphing calculator and the other group made a similar test with GeoGebra. The former group will be

referred to as GC-students and the latter as GGB-students. Students got 30 minutes to finish the test.

The test consisted of four questions. In each question students were asked to find output solutions and draw sketches. In the first question students needed to draw a sketch of a third-degree graph (1a) and find all zeroes (1b). In the second question the order of drawing and writing down solutions is reversed, which implied that they needed to find extreme values of a fourth-degree function (2a) and draw a sketch of its graph (2b). The reversed order was chosen deliberately in order to find differences in completeness of sketches and output solutions. In the third question intersection points of a third-degree function h with the line $y = x - 1$ should be found (3a), a sketch with all intersections needed to be drawn (3b) and $h(x) = k(x)$ needed to be solved, where k was also a third-degree function (3c). The level of this last sub-question is somewhat higher, because this question did not straightforwardly say that intersections should be found. The last question was of the highest level, because of the context-rich environment. At first, students needed to sketch a difficult fifth-degree graph and give their chosen domain and range (4a). This function gave the amount of internet visitors at a certain site for different times at one day, so therefore the best possible domain was $[0, 24]$, because of the twenty-four hours in a day. At second, images should be found for 6.00 a.m. (4b), 4.30 p.m. (4c) and 10.45 p.m. (4d). Then, an extreme value (4e) and intersection points for 6000 visitors should be found (4f), and at last students were asked to tell how much extreme values this function has (4g). Both the answers *two* and *four* are considered correct, because the function had two extreme values within the domain $[0, 24]$ and four in total.

All functions are designed in such a way that at least one of the output solutions cannot be found within the *standard window*. Students need to zoom out at least one time to find all output solutions. See figures 1 and 2 for examples. For the graphing calculator, the standard window settings are $X_{min} = -10$, $X_{max} = 10$, $Y_{min} = -10$ and $Y_{max} = 10$ and GeoGebra starts with $X_{min} = -5$, $X_{max} = 25$, $Y_{min} = -13$ and $Y_{max} = 7$, which is, although the window in GeoGebra is very flexible, considered as the standard window for GeoGebra.

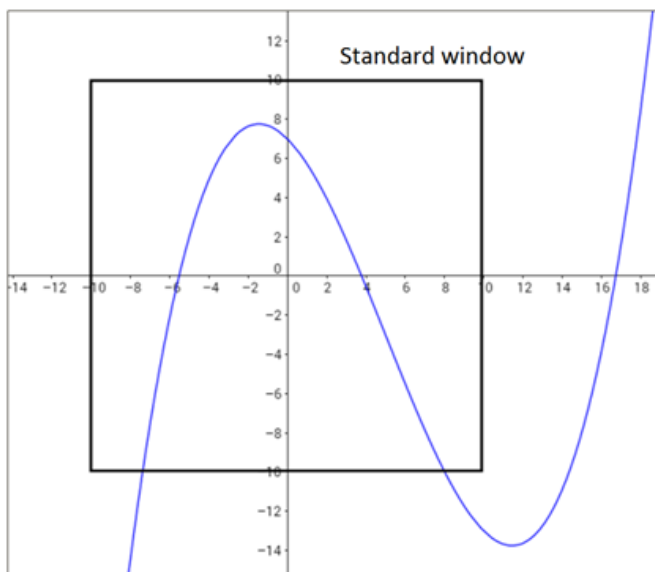


Figure 1. Graph of test question 1 with GC standard window

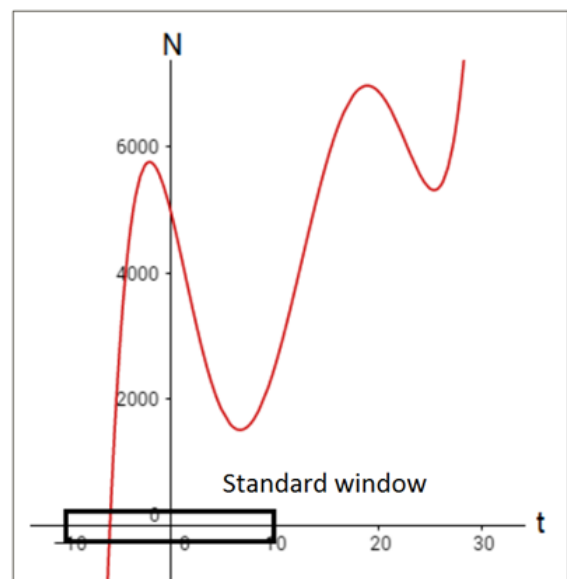


Figure 2. Graph of test question 4 with GC standard window

Opinions and Preferences. For the second sub-question a survey was conducted by means of a questionnaire. Students filled in the questionnaire directly after the test. This was done to make sure that students that made the test, could not be absent when filling in the questionnaire. Between the end of the test and filling in the questionnaire, students were not allowed to talk to ensure that their opinions were not influenced by their peers.

The questionnaire consisted of different items. At first students were asked to give some basic characteristics like *name, class, school, gender, math-subjects, artefact used with test* (this last one was item 1). Item 2 asked about prior experiences with GeoGebra, here students could circle n (never used before), s (sometimes used before) or r (regularly used before). Items 3 to 16 can be answered on a 1 to 5 Likert scale. Item 17 asked what artefact the student prefers to use. After this the students were asked to give one or two reasons to explain their choice.

Instrumental Genesis. For the third sub-question observations were conducted. Before the observation methods are explained, it is important to take a closer look at the five main schemes where this study focusses on.

1. Window settings
2. Images
3. Zeroes
4. Extreme values
5. Intersections

The first scheme comprises of the understanding that functions will only be visible within a window when the domain and range are chosen properly. For the graphing calculator, the window is quite static, as for GeoGebra, the techniques for this seem more intuitive. This is because users can zoom in and out with the mouse wheel, the window can be moved by dragging and domains and ranges can be easily altered by stretching or compressing the x -respectively y -axis. For the graphing calculator, students need to make up at least their own interval for the domain, where after they can use *ZoomFit*, which automatically adapts the range to the chosen domain for the current graphs. It is possible to zoom in and out with the graphing calculator but it takes longer and seems to work less intuitive.

The other four schemes are calculation options, that are available in both the graphing calculator and GeoGebra. An important difference between the two artefacts is that for the last three schemes, GeoGebra gives all output solutions at once in the algebra screen. Finding those solutions with the graphing calculator requires a student not only to select every point one at a time, but also a proper domain should be thought of, because points outside the chosen domain cannot be found. Because of the dynamic interface of GeoGebra it is quite likely that students will perform better with GeoGebra and like GeoGebra better than the graphing calculator.

During the second lesson, where students learn to work with GeoGebra, the researcher sat down with a pair of students and made notes of their conversation with an observation instrument. This second lesson is divided into four questions, dealing with the five main schemes. The observation instrument is based on a similar instrument used by Jupri et al. (2015) and is set up in such a manner that for each question a division into different sub-schemes is made. These sub-schemes are split up into the different techniques that can be

visible. The researcher can mark different techniques he sees and write notes behind it. Afterwards all observation instruments will be compared in order to find important differences or similarities.

During the test in the third lesson observations were carried out by filming students. For each class, one GC-student was filmed in the classroom together with all other students that made the test, and one student GGB-student was filmed in a different room. This was done in order to enable these students to speak all his thoughts aloud. This made the instrumental genesis during the test clearer for the researcher. It was made sure that these students were the two persons being observed in the second lesson, to measure differences more closely. Afterwards all videos were coded in a time and action matrix for a part of it. For each of the five main schemes it was determined when and how the action took place. For how well the action was, a distinction between poor (wrong actions), moderate (good actions but not directly helpful) and good actions was made.

Procedure

In the different phases of the experiment the teacher, the students and the researcher had different roles. Teachers were experts in using GeoGebra, because they volunteered to participate in this experiment. Before the first phase, the researcher instructed the teacher how to act. This was done by means of instruction forms for each lesson, see Appendices A, B and C. Also, with the help of the teacher, the researcher decided which two students could be chosen as the subjects for observations. These students were talkative persons which made it easier to distinguish different techniques and schemes.

In the first lesson the teacher instructed the students to recapitulate the five schemes of the graphing calculator by giving them one large assessment, see Appendix D, in which every scheme is repeated in different sub-questions. Students worked in pairs and the teacher walked through the classroom in order to support couples that needed help. By repeating the prior knowledge for every student, it was made sure that all students of different schools had the same entry level. Some school used another model of the graphing calculator. Most schools used a TI model, but one of the schools worked with the Casio model. A special Casio version of the assessment was developed to minimize differences, see Appendix E. In this lesson, there is one thing that students learned for the first time. Students learned that an n th-degree function has at most $n - 1$ extreme values. They needed to learn this to be aware of the fact that a function could have more extreme values or other outcome solutions than visible within their window.

In the second lesson students worked with GeoGebra on laptops or computers, in couples, in order increase the effect of learning from peers (Mayer, 2014). The teacher explained the basics of GeoGebra and instructed the class to make the second assessment, see Appendix F for the TI version, with questions similar to the first assessment. Here, the couples tried to master the GeoGebra functionalities on their own. In both lessons, the guidance of the teacher was minimal, because that could influence the instrumental genesis (Jupri et al., 2015). In the first two lessons, it was important that the teacher paid specific attention on how the students needed to write their answers because proper answering is fundamental to this study. Another method to ensure this, was that all the student answers were written down on special answer sheets, which were collected by the researcher afterwards, see Appendix G for an example of lesson 1.

In the third lesson students made pen-and-paper tests, whereby they are allowed to use one instrument, depending on the group they are randomly placed in. See Appendices H and I for respectively the test for the GC- and GGB-students. After 30 minutes the test was finished and students were asked to fill in the questionnaire.

Data Collection

In total 131 students in tenth-grade pre-university education (mathematics B) took part in this study. Out of these students, 122 made the test, of which 59 were male and 63 female. In order to collect reliable data, it was important to have as little variables as possible, so therefore no eleventh-grade, mathematics A or pre-higher education classes took part in the experiment. Different schools throughout the Netherlands, using different kinds of mathematics sourcebooks and different models of graphing calculators, were selected to increase representativeness. In total five teachers from four different schools participated with six classes. The lessons lasted 45 to 50 minutes and took place in the period February-May. All participating students were familiar with the graphing calculator, whom most of them learned to know at the beginning of the schoolyear.

Each experiment consisted of three lessons. Each first lesson was carried out by only the teacher, the second lessons were observed by the researcher and during the third lessons colleagues of the researcher recorded videos of two students per class. Before the start of the first experiment, concept lessons were piloted in a secondary school in Bemmelen, where some students were already used to work with GeoGebra. Adaptations were made to optimize the assessments and the test. This implied that redundant information was left out of the assessments and the length of the test was reduced.

Data Analysis

In this research, both formative and summative assessment were measured and analyzed. Summative assessment consists of checking the students' tests, which was analyzed for sub-question 1. Formative assessment consists of learning to use an artefact, which was not only present in the first two lessons, but also during the test. Observations of both lesson 2 and 3 were analyzed for sub-question 3.

Test Results. For the first sub-question students' answers to the test were analyzed using a coding scheme. For each part of the assignment students could score points. At first, a distinction was made between five different categories. For each category, a description is given. When students meet these requirements, a point for that letter or category was earned, see figure 3 for a student example of question 2 with a complete coding scheme.

I	Input	Students wrote down the formula they typed in their artefacts.
F	Function	Students wrote down what function, option, button or command they used to find their answer.
O	Output	Students wrote down their output solutions. For questions 1-3 these were subcategorized in O1 and O2.
	O1	O1 output solutions were situated within the standard window.
	O2	O2 output solutions were situated outside of the standard window
C	Conclusion	Students wrote down a corresponding conclusion or answer to the question.
S	Sketch	Students drew a proper sketch.

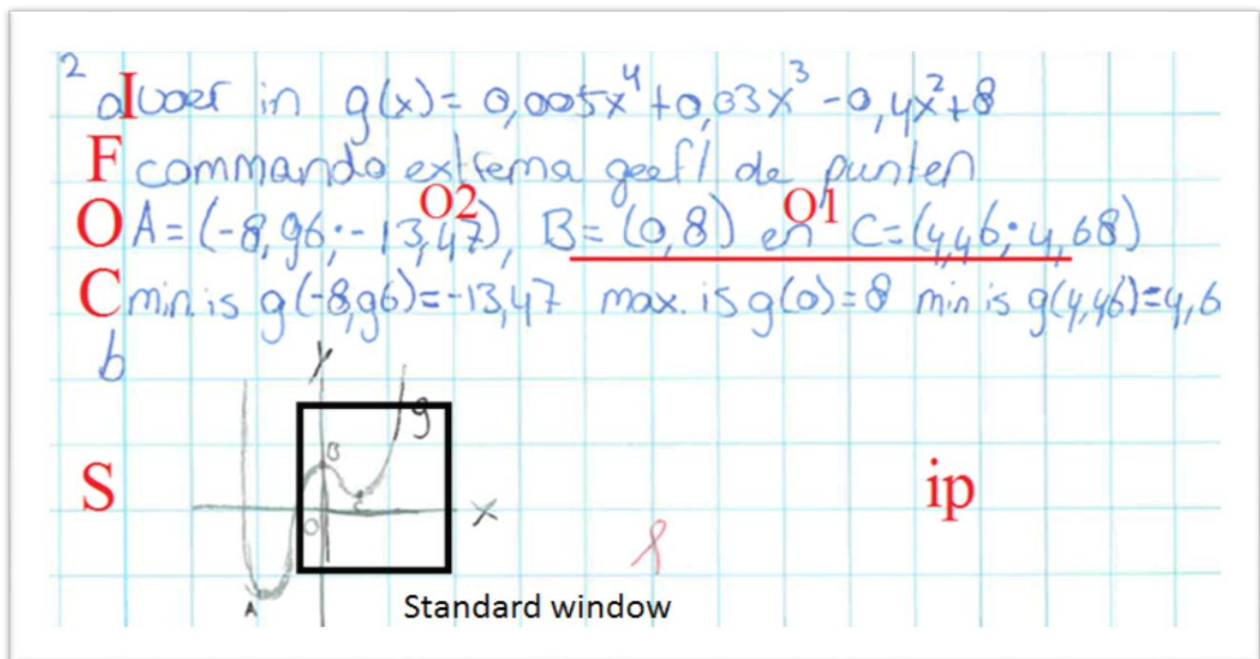


Figure 3. An example of a perfect GGB-student answer for question 2, with corresponding coding scheme.

Because output solutions and sketches determine whether a student properly executed the task, only these two categories counted as *regular points*. Only if students entered their input wrongly, one regular point was subtracted from the total amount. Corresponding output solutions or sketches could then still earn the student points. O-points were granted when students wrote down the right answer. Answers that were rounded off wrongly were approved because this did not harm the understanding of the task. S-points were granted differently per assignment. A distinction between *poor*, *good* and *moderate* sketches was made. In the former case sketches were wrong or missing, in the latter case domains were chosen too small or too broad but ranges suited with that domain. For question 1, where students started with drawing a sketch, good and moderate sketches earned the student a regular point, for question 2 and 3 only good sketches earned a regular point, because the assignment told to sketch *all* output solutions. For question 4 moderate sketches granted one regular point, unless the student did not choose a range according to the function in their chosen domain, and good sketches granted two regular points. Two points could be earned for the sketch of question 4 because it was a hard fifth-degree function in a context question. Two special points remain in question 4a, where students had to write down a proper domain and range for their sketch. Students earned the X-point when they chose the domain $[0, 24]$. The Y-point was earned when students chose a range that made the whole graph visible within their chosen domain.

Besides regular points, students could also score *insight points*. In general, it could be said that students scored an insight point (ip) if they have seen all important characteristics of the function *within sight*. For each insight point a short description for earning it is given. In the final analysis of the results, differences between GC- and GGB-students' regular (maximum is 23 points) and insight scores (maximum is 6) were compared.

1. Two extreme values and three zeroes of the third-degree function must be visible in the sketch. GC-students also got this insight point when they found their O2 zero, GGB-students then scored only a half insight point because it cannot be said with certainty whether they zoomed out or not, and just copying an answer from the algebra screen would not attest to insight.
2. All three extreme values of the fourth-degree functions must be visible in the sketch or written down as output solutions. If only the latter is the case, then GGB-students score only a half insight point.
- 3ab Similar to question 2 for both extreme values of a third-degree function with all three intersection points with $y = x - 1$.
- 3c Only all three intersecting points of two third-degree functions needed to be written down.
- 4 context This insight point was earned when a student understands that within the context only the interval $[0, 24]$ is relevant because of the twenty-four hours in a day.
- 4 graph This insight point was earned when a student drew a *good* sketch.

This means that most insight points were earned when students saw the total global progress of the graph. Only one insight point (4 context) could be earned for the understanding of the context.

Opinions and Preferences For the second sub-question, the questionnaire was analyzed. For four different kinds of items a distinction between three characteristics was made. Numbers of the corresponding questionnaire items were noted between brackets, see appendix J for the entire questionnaire.

- a. The students made the test with the graphing calculator or with GeoGebra (1)
- b. Regular points
- c. Insight points

The four categories of items are the following.

1. Students' opinion on the artefacts (11, 13, 14, 16, 17)
2. Students' previous experience with GeoGebra (2)
3. Students' estimation of their own skills (4, 6, 8, 10, 12, 15)
4. Students' motivation for components of the experiment (3, 5, 7, 9)

Statistic tests determined significant differences between characteristics for different categories of items. For Likert scale categories 1, 3 and 4 parametric tests were used in order to maintain robustness (Whadgave & Khairnar, 2016). For the second category (item 2) the answers *sometimes* and *never* are combined as one answer because the difference between the two appeared to little, because some students who said they never used GeoGebra before, did use it before, according to the teacher, without realizing it.

At last, the open question was analyzed. The students' reasons for their preferred artefact were sorted in positive, neutral and negative categories for both the graphing calculator as GeoGebra. Similar answers were added up, to distinguish commonly given answers. Afterwards the different types of answers were analyzed.

Instrumental Genesis. For the final sub-question observation instruments and video codings were analyzed, see Appendix K. In both matrixes, a comparison was made to find important differences and similarities. Important differences between the GC and GGB usage were noted. The particular parts of the videos where these events took place, were looked at by four researchers, in order to increase the reliability and validity of the results. Important results were discussed and analyzed. These analyses were made for each of the five main schemes.

The observation instruments served as an inspirational tool to find differences in the videos. Some of the observations of pairs in lesson 2 were also filmed because of practical reasons. These videos served as a measurement tool for comparing differences as well.

Results

Test Results

Overview. In this section, we give an overview of the scored points. Review of the Kolmogorov-Smirnov test for normality suggested that scores were not normally distributed for both the regular points ($KS = .095, df = 122, p = .009$) and insight points ($KS = .213, df = 122, p < .001$). Therefore, non-parametric Mann-Whitney tests were executed in order to find significant differences between GC- and GGB-students. This indicated that regular point scores for GGB-students ($Mdn = 15$) was not significantly higher than GC-students ($Mdn = 14$), $U = 1568, p = .135$, but for insight points GGB-students ($Mdn = 4$) scored significantly higher than GC-students ($Mdn = 3$), $U = 1020, p < .001, r = .40$. The following sections will elaborate on these scores.

Figure 4 shows an overview of scores for each category of test questions. For each category, we can see the difference between scores of GC- and GGB-students.

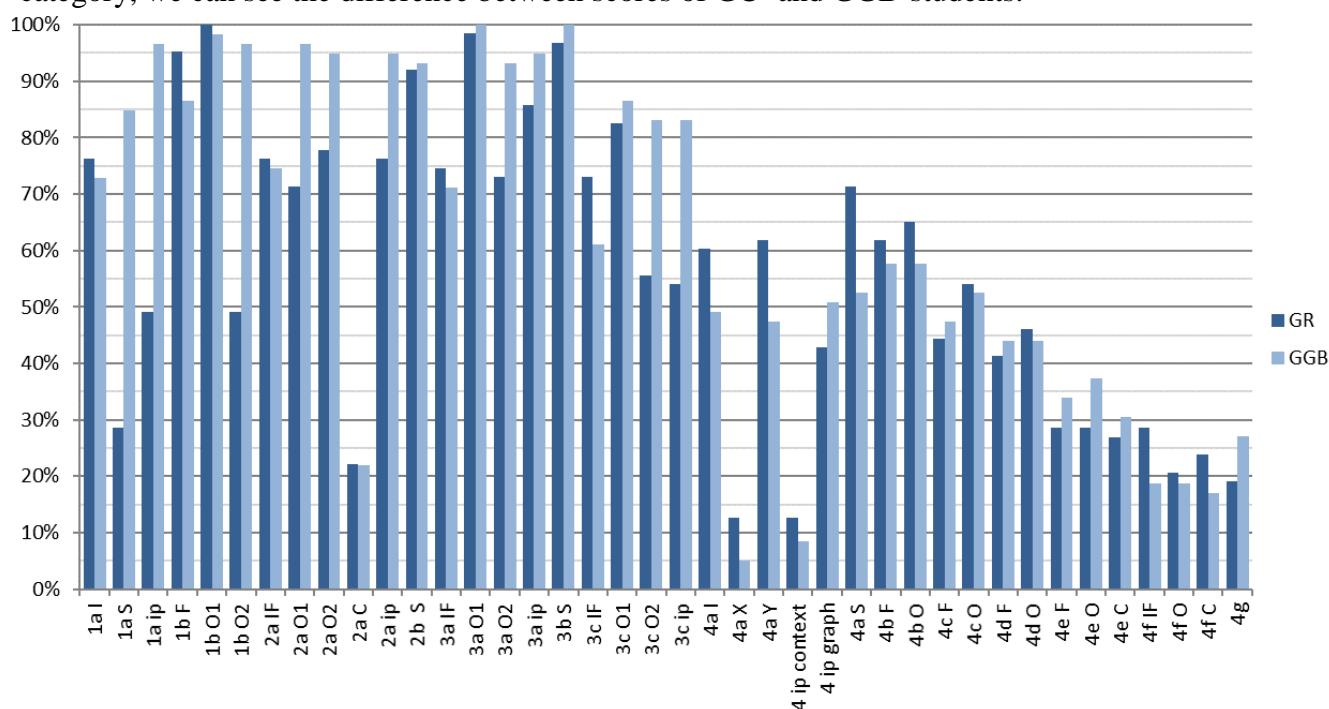


Figure 4. Percentages of correct student answers per category

Regular points. For the regular points, we did not find a significant difference between GC- and GGB-students in general. But when we examined the results more closely, we noted some important differences. We will focus on differences between sketches and output solutions.

Sketches. Table 2 shows the amounts of poor, moderate and good sketches for all questions. Only students who could make the corresponding questions were included. With these numbers, Chi-square tests were computed and noted in table 3. In most cases, too many expected values were lower than 5, in which case a Fisher's exact test was used.

We found significant differences for questions 1, 2 and 4. Many GC-students drew a sketch in which not all output solutions were visible. They significantly more often made moderate sketches, which was due to the *ZoomFit* function for the graphing calculator. The higher p value for question 3 could be due to the O2 intersecting point being very close to the standard window ($x = 12.2$), which made it more obvious to GC-students that another output solution could be found outside the standard window.

Table 2

Amounts of sketches drawn poorly (p), moderately (m) or good (g)

artefact	question 1a			question 2b			question 3b			question 4a		
	p	m	g	p	m	g	p	m	g	p	m	g
GC	2	43	18	5	9	49	2	9	52	17	17	21
GGB	2	7	50	4	0	55	0	4	55	19	4	24

Table 3

Chi-square test of independence, or Fisher's exact test

Question	df	N	χ^2	p	Cramer's V
1a	2	122	-	<.001	-
2b	2	122	-	.009	-
3b	2	122	-	.144	-
4a	2	102	7.78	.020	.276

Output solutions. Table 4 shows the amounts of correct and incorrect answers for questions 1-3 and table 5 shows the same for question 4. With these numbers, Chi-square or Fisher's exact tests were computed and noted in table 6.

Except for question 2 we found no significant differences for O1 output solutions. For question 2 this was due to the coordinate (0, 8). Very often a graphing calculator approximates the $x = 0$ in the scientific notation $p \cdot 10^q$, with q negative, in which case GC-students sometimes only wrote down p as the x -coordinate.

We found significant differences for O2 output solutions for questions 1, 2 and 3. This implied that GC-students very often forgot to zoom out, in order to find output solutions outside the standard window. For question 4 we did not find any significant differences in output solutions. This could be due to the length of the test because, as can be seen in table 6, not every student was able to finish the entire test. Another reason for this could have been the difficulty of this fifth-degree function. Not only all four extreme values must be found outside

the standard window, also tenth-grade pre-university students had too little experience with these kind of context questions. For a lot of students this was probably the very first question in which they needed to think about window settings themselves, because basically nothing could be seen in the standard screen. This kind of assignment was practiced in questions 5-9 of the assessment form of the second lesson with GeoGebra, but 43 out of 103 students were not able to start with question 5 because of lack of time. A lot of students who could make (some of) these last questions forgot to zoom out in which case they still did not practice this kind of assessment.

Table 4

Amounts of correct (1) and incorrect (0) output solutions for questions 1-3 within (O1) and outside of (O2) the standard window

artefact	q.1b O1		q.1b O2		q.2a O1		q.2a O2		q.3a O1		q.3a O2		q.3c O1		q.3c O2	
	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
GC	0	63	32	31	18	45	14	49	1	62	17	46	6	52	23	35
GGB	1	58	2	57	2	57	3	56	0	59	5	55	3	51	5	49

Table 5

Amounts of correct (1) and incorrect (0) output solutions for question 4

artefact	q.4a domain		q.4a range		q.4b		q.4c		q.4d		q.4e		q.4f		q.4g	
	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
GC	47	8	16	39	3	41	3	34	5	29	8	18	7	13	6	12
GGB	45	3	20	28	8	33	3	30	5	26	5	22	7	11	1	15

Table 6

Chi-square test of independence, or Fisher's exact test

Question	<i>df</i>	<i>N</i>	χ^2	<i>p</i>	Cramer's <i>V</i>	Odds ratio
1b O1	1	122	-	.299	-	-
1b O2	1	122	34.06	< .001	.528	29.42
2a O1	1	122	14.10	< .001	.340	11.40
2a O2	1	122	7.46	.006	.247	5.33
3a O1	1	122	-	.331	-	-
3a O2	1	122	8.73	.003	.267	5.08
3c O1	1	112	-	.352	-	-
3c O2	1	112	13.78	< .001	.351	6.44
4a domain	1	103	-	.174	-	-
4a range	1	103	-	.182	-	-
4b	1	85	-	.081	-	-
4c	1	70	-	.883	-	-
4d	1	65	-	.874	-	-
4e	1	53	-	.300	-	-
4f	1	38	-	.804	-	-
4g	1	34	-	.051	-	-

Insight points. For the insight points we found a significant difference between GC- and GGB-students in general. When we examined differences for each of the six insight points, similar results as for output solutions were found. Table 7 shows for each question the numbers of students who scored the insight point (1) and who did not (0). GGB-students that scored only half a point are counted as 0. Table 8 shows calculations of Chi-square tests of independence, comparing the scored insights points for GC- and GGB-students per question. We found significant differences for questions 1, 2 and 3c. For questions 3b (table 3) and 4 (table 6) we already found similar results.

Table 7

Scored insight points per question

artefact	question 1		question 2		question 3ab		question 3c		q. 4 context		q. 4 graph	
	0	1	0	1	0	1	0	1	0	1	0	1
GC	32	31	15	48	9	54	24	34	47	8	28	27
GGB	6	53	4	55	3	56	5	49	43	5	18	30

Table 8

Chi-square test of independence

Question	<i>df</i>	<i>N</i>	χ^2	<i>p</i>	Cramer's <i>V</i>	Odds ratio
1	1	122	23.45	< .001	.438	9.12
2	1	122	6.72	.010	.235	4.30
3b	1	122	2.91	.088		
3c	1	112	15.04	< .001	.366	6.92
4 context	1	103	.40	.529		
4 graph	4	103	1.86	.172		

Opinions and Preferences

Students' opinions on the artefacts. In this section five items on students' opinions on the artefacts are considered. The first four can be found in table 9.

Table 9

Average scores on opinion items

number	item	<i>N</i>	<i>M</i>	<i>SD</i>
11	The GC is pleasant to work with.	122	4.19	.71
13	The place where I can find functions/buttons in the GC is chosen intuitively.	122	3.73	.99
14	GGB is pleasant to work with.	111	3.14	1.17
16	The place where I can find functions/buttons in GGB is chosen intuitively.	109	3.34	1.14

Item 17. For the last item, 93 out of 122 answered that they preferred the graphing calculator over GeoGebra. This means that only 23.8% of the students liked GeoGebra better. Table 10 gives an overview of arguments for the answers to this item. Here we find that the most common reason for preferring the graphing calculator is just that they are more used to it ($n = 35$). Another common reason for this was the space on the table that an artefact consumes ($n = 33$), as appears from number of times students referred to the compactness of graphing calculators and the place-consuming characteristic of laptops. Fortunately, also arguments that

related more to the content were given. A lot of students said that typing the input is easy for a graphing calculator ($n = 14$) and less easy for GeoGebra ($n = 14$). This could be due to the larger keyboard, the using of dots instead of commas and the further distance between keyboard and screen for GeoGebra. Furthermore, students gave more positive content-related arguments for GeoGebra. They argued for example that GeoGebra is more clear because all the contents (and output solutions) are situated in one large screen. Some students oppose this and say that they figuratively drown in all the possible options that GeoGebra offers in the screen, the toolbar and on the keyboard. At last some minor GeoGebra bugs are reported.

Table 10

Average scores on opinion items

Opinion GC	<i>n</i>	Opinion GGB	<i>n</i>
The GC is handy and compact	20	GGB is more clear	12
Typing input is easy on the GC	12	GGB works faster	7
With the GC I work faster	7	Zooming with GGB is more convenient	7
The GC works more easily	7	In GGB I can find the right buttons more easily	7
On the GC buttons are more clear	6	GGB works conveniently	5
The GC has calculator options	5	GGB works more easily	5
The GC is more clear	5	With GGB you have everything in one screen	4
On the GC, I can more easily find buttons	4	GGB also gives output outside the current screen	4
The GC works more pleasantly	3	GGB has more possibilities to find solutions	4
The GC has more options	1	GGB already plots a graph when typing input	3
All calculators resemble each other so therefore the GC works more easily	1	GGB gives clear answers	2
		GGB plots graphs more conveniently	2
		With GGB has one large screen	2
		Becoming acquainted with GGB goes faster	1
		With GGB you can use commands	1
		With GGB you can use a mouse instead of buttons	1
		Typing input works more easily on the GC	1
Subtotal (positive arguments)	71		68
I am used to the GC	35	GGB is too unknown to me	8
		I never use GGB before	4
		Becoming acquainted with GGB requires more time	1
Subtotal (neutral arguments)	35		13
Zooming with the GC works clumsy	2	Typing input works less easily	14
The GC plots graphs very slowly	2	With GGB a laptop takes much place on my table	13
		I prefer not using a laptop with mathematics	11
		GGB suffers from a lot of small problems	5
		With GGB my computer needs to startup first	5
		I do not understand GGB	3
		With GGB you find buttons everywhere	3
		GGB works clumsy	3
		GGB sometimes jams	2
		With GGB a keyboard has too many options	2
		Sitting behind a computer gives RSI complaints	1
		GGB without a mouse does not work convenient	1
		With GGB, a computer can crash	1
		With GGB, I sometimes move my graphs	1
		For GGB a lot of prior knowledge is needed	1
		The window in GGB is too flexible	1
Subtotal (negative arguments)	4		67
Total	91		148

When we take a closer look at which students gave which answer we find an important difference in table 11. We calculated a Chi-square square test of independence comparing the frequency of preferences for GC- and GGB-students. We found that students who made the test with GeoGebra were much more likely to develop a preference for GeoGebra (42%) than students who made the test with the graphing calculator (7%), $\chi^2(1, N = 121) = 21.47$, $p < .001$, $V = .42$, odds ratio = 10.66.

Table 11

Artefact usage plotted against artefact preference

artefact	Preference for the GC - percentage		Preference for GGB - percentage		Total
GC	58	93.5%	4	6.5%	62
GGB	34	57.6%	25	42.4%	59
Total	92	76.0%	29	24.0%	121

Coherence opinion items. An internal coherence for items 11, 13, 14, 16 and 17 was found. The Cronbach's alpha for these five opinion items was .72 when scales for the GC-items 11 and 13 were reversed. This implies that these five items can be seen as a coherent set of questions, representing a scale for preference.

We found a same kind of coherence when we compared differences between the two groups of students for these five items. In table 11 we already found this for item 17, but the same is true for the other four items, as can be found in table 12. We found that the artefact that students used to make the test was defining for their opinion. GC-students were significantly more positive about the graphing calculator and significantly less positive about GeoGebra and vice versa. These students not only had 30 more minutes experience with GeoGebra, but also the test confirmed that they were adequately capable to work with it. In this case, students became more positive about the program.

Table 12

Independent t tests

Item	sort description	artefact	<i>N</i>	<i>M</i>	<i>SD</i>	<i>df</i>	<i>t</i>	<i>p</i>	effect size
11	<i>GC pleasant</i>	<i>GC</i>	63	4.38	.55	119	3.26	.001	.59
		<i>GGB</i>	58	3.97	.81				
13	<i>GC intuitive</i>	<i>GC</i>	63	3.92	.99	119	2.27	.025	.41
		<i>GGB</i>	58	3.52	.96				
14	<i>GGB pleasant</i>	<i>GC</i>	51	2.76	1.03	108	3.41	.001	.66
		<i>GGB</i>	59	3.49	1.18				
16	<i>GGB intuitive</i>	<i>GC</i>	50	3.04	1.09	106	2.54	.013	.49
		<i>GGB</i>	58	3.59	1.14				

Students' previous experience with GeoGebra. Not only the artefact used for the test influenced students' preferences, also prior experience with GeoGebra influenced it, appeared from the questionnaire. In table 13 we see that students who used GeoGebra regularly before thought significantly more positive about it. This also appeared from a Fisher's exact test with item 17. Students how regularly used GeoGebra before were more likely to prefer GeoGebra (67%) that students how used it sometimes or never before (21,7%), $p = .029$, odds ratio = 7.20.

Table 13

Independent t tests, differentiating opinions of students who never (n), sometimes (s) or regularly (r) used GeoGebra before.

Item	sort description	used before	<i>N</i>	<i>M</i>	<i>SD</i>	<i>df</i>	<i>t</i>	<i>p</i>	effect size
14	GGB pleasant	n/s	104	3.10	1.15	108	2.22	.028	0.93
		r	6	4.17	1.17				
15	GGB own-skills	n/s	106	3.18	1.31	110	2.44	.016	1.20
		r	6	4.50	.84				
16	GGB intuitive	n/s	102	3.25	1.11	106	3.05	.003	1.45
		r	6	4.67	.82				

Students' estimation of their own skills. In this section six items on students own-skills are considered. Means and standard deviations for these items can be found in table 14.

Table 14

Average scores on own-skills items

number	item	<i>N</i>	<i>M</i>	<i>SD</i>
4	In my opinion, the first lesson was easy.	106	4.25	.87
6	In my opinion, the second lesson was easy.	110	3.05	1.12
8	In my opinion, the test was easy.	120	3.87	.84
10	My understanding of the GC/GGB was sufficient to make the test.	123	3.96	1.04
12	I know where to find the right functions/buttons on the GC.	122	4.55	.60
15	I know where to find the right functions/buttons in GGB.	113	3.26	1.32

In the previous section, we already found that students that regularly used GeoGebra before, ranked their own-skills higher than students who used it less. For item 15 we also found another interesting difference between the two groups of students. An independent-samples *t* test pointed out that students who made the test with GeoGebra ranked their own-skills significantly higher ($M = 3.86$, $SD = 1.15$) than students who made the test with the graphing calculator ($M = 2.57$, $SD = 1.15$), $t(110) = 5.96$, $p < .001$, effect size 1.20. Also here, we find a more positive attitude towards GeoGebra, when having 30 minutes of experience with the program and confirmation from the test in addition.

Students' motivation for components of the experiment. In this section, we regard four items on students' motivation, see table 15. For this category, we did not find anything important. This might primarily be due to a bad internal consistency. The Cronbach's alpha for these four items was .58, which implies that these questions cannot be seen as one scale for motivation. It would have been interesting to weight test results with a motivation rank in order to be able to reduce the importance of results of badly motivated students. We did try this with only item 9, but that did not work because according to Spearman's rank tests there were correlations between this item and the amount of regular points ($p = .026$, $r_s = .20$) and the amount of insight points ($p = .011$, $r_s = .23$). As could be expected of tenth-graders, students were able to evaluate their performances well. The higher students ranked their effort for the test, the higher their regular and insight scores turned out to be.

Table 15

Average scores on motivation items

number	item	<i>N</i>	<i>M</i>	<i>SD</i>
3	In my opinion, the three lessons were useful.	110	3.42	.86
5	In my opinion, the first lesson was fun.	106	3.12	.83
7	In my opinion, the second lesson was fun.	110	3.16	.97
9	I made the test as good as I possibly could.	122	4.44	.70

Instrumental Genesis

It is complicated to describe how the instrumental genesis with GeoGebra evolved. This was primarily due to the prior knowledge students had with graphical-numerical assessments with the graphing calculator. The instrumental genesis was not unique but derived from what they already knew from the graphing calculator. Therefore, in the following description of the analyses on the observation instruments and video codings, a comparison between the two artefacts was made for the five main schemes.

Observation instruments Most of the observations of the second lessons gave an unambiguous view of the learning process. At first, we saw that students executed the first four assignments without much trouble. Buttons were found quickly and students mentioned they liked the fact that GeoGebra gives all output solutions at once. At the same time, we saw that students needed time to play around, to learn to know the program, and to write down their answers neatly. Furthermore, we saw that students were struggling with the zooming functions, more we than expected. This was especially the case when students worked on laptops without a mouse. The lack of a mouse wheel but also of a numeric keypad on smaller laptops was unfortunate.

For assignments 5-9 we saw that it was crucial for the understanding of the students that they could choose proper window settings and that they were able to work fluently with it. And still then, many students did not realize that there were more intersecting points to be found, outside the standard window. GeoGebra is inconsistent here because it can find all intersections of polynomial functions at once, but this is not possible for exponential or trigonometric functions. Besides this, many students were not able to finish all these questions. Students simply did not have enough time to figure out the deeper meaning of these last assignments and the inconsistency of GeoGebra. An important finding here was, that only one lesson of GeoGebra was too short to understand and execute stretching and compressing axes.

Video codings. In the videos, some similar patterns were found in comparing GC- and GGB-students. A first observation was that, for lower level questions, schemes are the same for both artefacts. Only the techniques differ a little, but in the videos we saw that students were able to use these functionalities with GeoGebra fluently.

Higher level questions, on the contrary, showed something different. How well the question was answered, when it was asked in a different way than just ‘find the extreme values’ or ‘find zeroes’, depended on the understanding of the students, on how well they participated in the second lesson, on how fluently they can change window settings and on

how intuitive the program works. For the latter condition we will make two comparisons of the two artefacts with examples from the videos. The first one deals with images and the last one with intersections and window settings.

A student who we will call John made the test with GeoGebra but was also filmed during the second lesson because of practical reasons. In the second lesson he wanted to find the image $f(-2) = 9$ and at the test $N(6) \approx 1528$. In the lesson, John removed the x between brackets in the entire formula $f(x) = 0,2x^4 - 1,8x^3 - 0,1x^2 + 3,1x - 2$, in which case GeoGebra calculates $f(-2) = 9$ but presents it together with the -2 as 11 at the left side of an equation. Therefore, John failed to learn how to calculate images in the second lesson, even after checking his answers, from which he learned that his answers were wrong. At the test, John remembered that this was not a good approach. In GeoGebra he had the liberty to type in $x = 6$. This gave him a vertical line, after which he was able to find an answer by intersecting the graph with this line. This is a nice example of how GeoGebra has multiple ways of finding correct answers and works intuitively.

Another example where the greater amount of freedom in GeoGebra plays an advantageous role, was with a student who we will call Peter. Peter needed to solve $h(x) = k(x)$, in which h and k are both third-degree functions. Peter was puzzled because first he thought he had to find an answer to this equation algebraically, but he soon found out that this was too hard. Peter did not realize he had to find intersections. Peter did not reach the part of the second lesson where this was explained, so he had basically no prior knowledge. But in GeoGebra, Peter had the liberty to place a *new point*, which would not have been possible with a graphing calculator. When he selected this option and then clicked at one of the three points where the graphs intersected, an intersection point was drawn. In the algebra screen, Peter read that it was an intersection point, and at that moment he realized that intersecting points were places where two formulas are equal. Another important finding in this example was that Peter was very skilled in adapting window settings. Peter used his personal laptop, both in the second lesson as at the test and got personal help from the researcher in the second lesson, so zooming was very easy for him.

This was different for his classmate Josh, with whom Peter worked together in the second lesson. Josh made the test with the graphing calculator and struggled to find intersection points of the fifth-degree function of question 4 with $y = 6000$. Josh could not get this graph within his window properly, it was squeezed to the y -axis because he has chosen the domain interval $[-1000, 1000]$. Josh had very little time left and as a last attempt to see *more* of the function than just a vertical line near the y -axis, Josh changed his interval to $[-10\ 000, 10\ 000]$, which worked counterproductive. Because the time ran out, Josh could write down an answer anymore. This is a characteristic example of the difference between the graphing calculator and GeoGebra.

Discussion

Conclusions

The purpose of this research was to evaluate the differences between the graphing calculator and GeoGebra. For each sub-question a summary of the most important findings is given.

To what extent does the use of GeoGebra influence test results, compared to a graphing calculator? For the first sub-question, we found that after one lesson of working with GeoGebra, students were able to perform just as well on graphical-numerical assessments when working with GeoGebra. From this we can conclude that using GeoGebra instead of a graphing calculator will not be harmful. In fact, in the case that output solutions outside the standard window exist, students perform even better with GeoGebra. Another finding was that we could not draw that conclusion for context questions for which the graph had all its extreme values outside the standard window. Therefore, we could say that students are significantly more skillful in adapting window settings. This was due to the lack of experience with this level of assessment, and to the lack of time for test question 4. Because of these reasons, weaker students were not able to show their true potential. At last we found that the GC-option *ZoomFit*, helped students do draw moderate sketches.

To what extent do students prefer to use GeoGebra instead of a graphing calculator? An answer to the second sub-question was that students, who are already acquainted with the graphing calculator and then learn to work with GeoGebra during one lesson, favor the graphing calculator over GeoGebra. The most common arguments for this choice were, that students were just more used to the graphing calculator, that students do not like the larger size of the device on their tables and that students found it more bothersome to type their input in GeoGebra. Another important finding was that if students worked with GeoGebra more often, they were more positive about it. This appeared from students' previous experience with the program and from the artefact students used with the test. It was found that working 30 minutes longer with GeoGebra and having the experience of making a test that confirms your capabilities with it, significantly increases your attitude toward it.

How does the instrumental genesis evolve using GeoGebra? The last sub-question was a difficult one to answer. It was complicated to describe how the instrumental genesis evolved. Therefore, a comparison between the two artefacts was made for each of the five main schemes.

For the schemes zeroes, extreme values and intersections, we found that the techniques for finding the output solutions was very similar. Students liked it that GeoGebra gives all output solutions at once and were much less tended to overlook solutions that were not visible in their standard or current window, but the most important finding was that these schemes were actually the same.

For images, we found that techniques were quite different. Although students were able to type the image and press enter, we saw that students were not yet accustomed to typing. Students thought of more intuitive ways of finding it, for example by entering the original for x as a vertical line, where after they searched for the intersecting point. It seemed that the technique for finding images was less intuitive with GeoGebra, when students are acquainted with the graphing calculator

For window settings, we found that students clearly needed proper instruction. We saw that it was not only important to show how axes can be stretched or compressed, also students need to try to do it themselves, because otherwise they will not remember what button to press before being able to do so. Because a lot of students were not capable of stretching and compressing axes, many of them did not fully master the zooming techniques of GeoGebra. When students mastered the zooming techniques, we learned that these techniques worked intuitively and faster, because the techniques for finding proper window settings with the graphing calculator was much more difficult because students then need to make up the interval for at least the domain themselves.

Discussion of Results

In this study, we compared two artefacts on usability and insight. But does insight in certain functions lead to a better understanding of functions in general? Insight point were granted when students had the most important characteristics within the sight of their window, and we saw that insight increased when using GeoGebra. Whether this also increases the understanding of functions in general goes beyond the scope of this research. For the graphing calculator, we can say that it forces a student to think about the interval for the domain. GeoGebra could make a student lazy at this point. Because zooming and dragging axes is so easy, a student is less likely to skip this and think of general patterns for certain functions, although this argument could be refuted by saying that typing a function also costs a lot of time in which case GGB-students are also stimulated to search for general pattern in functions.

We found that GeoGebra finds all output solutions for polynomial functions at once, and that students appreciate that. But for more difficult functions, not all output solutions can be found at once. This inconsistency and students' often blindly copying of answers from the algebra screen, could lead to worse results. But when students become aware of this inconsistency they are more likely tended to check their answers. Therefore, we might say that GeoGebra invites students to search for more output solutions. Whether students really learn from the greater amount of insight, or that they just copy the answers without thinking, is hard to say, and there probably is a difference between weak and strong students here.

We found that the greater amount of freedom in GeoGebra could support understanding, but also that, for weaker students, this could feel like a confusing raft of options and buttons. Although these redundant buttons could lead to a cognitive overload (Mayer, 2014), strong students can profit here. The stronger students, who are often the ones who proceed in a beta oriented study, are likely to achieve a greater level of understanding, whereas weaker students will always struggle with that. For the latter group, only the formative process seems harder.

At last it is important that an artefact should always be used with the awareness of the classroom context in which it is going to be used (Bokhove & Drijvers, 2010). In a world where computers become increasingly important, it is peculiar to use an external calculator with limited options. Besides the mentioned advantages, GeoGebra has a lot more options like parameter sliders and drawing possibilities for geometry. ICT tools can take over a lot from students and this should work fluently and intuitively. Here GeoGebra seems to have a substantial advantage.

Limitations

Because this the experiment took place within daily practices of schools, teachers and students, some unforeseen minor limitations were noted. At first not all students were present at each of the three lessons. Some students had other obligations or went on school trips for other courses, so therefore some of them were missing during one or more lessons. When students missed the first or the third lesson, this was not a real problem, but when the second lesson where students learned to work with GeoGebra was missed, students were automatically assigned to the GC-test. At second, one of the teachers planned her lessons wrongly. Because of this the researcher was not able to observe the second lesson in person, so therefore this lesson needed to be recorded on video. Afterwards, it seemed that the teacher did not use the instruction form, what implied that she gave very little instruction. Students were thrown in at the deep end, which could also have had a slightly negative impact on GeoGebra results. These were minor advantages for the graphing calculator, because these students were far more likely to favor the graphing calculator and preform worse on GGB-tests, but they were opposed to a minor advantage for GeoGebra. One of the teachers did not carry out the plan of randomly assigning artefacts for tests as planned. Each two students of this class that worked together in the second lesson, could choose who of them would make the test with the graphing calculator and who would do it with GeoGebra, what could have influenced the results. Fortunately, these advantages for one of the artefacts canceled each other out.

Students already knew the five schemes for the GC, so they knew what to look for. For a better comparison between the two artefacts it would have been better to pay the same amount of attention to both. Another option was to compare test results of two groups of students who only learned to work with one of the two artefacts, which might have reduced the effects of prior knowledge of the other artefact.

Another limitation was that the main researcher was not able to record the videos of the tests himself because of practical reasons. Colleagues who were asked to do this, caused some irregularities like forgetting to ask the GGB-students to think aloud, or like giving the GGB-students minor hints. Whereas the first problem only caused the coding to be more difficult, the latter could have improved GGB-scores. This was not regarded as a major problem because the students already had an idea of what they wanted to do, and they had only forgotten where to find the corresponding button. This was due to the short time span of the experiment. When it would have been practically possible to take more time for students to learn GeoGebra more thoroughly, this would have been less an issue.

Another problem was the duration of the test. Due to practical issues, it was not possible to make the test last longer than 30 minutes. Even after reducing the size of the test, many weaker students could not finish it. If also weaker students would have been able to finish it, it was more likely that significant differences for question 4 would be found.

Further Research

Suggestions for further research are numerous. Here students were already acquainted with the graphing calculator, so it would be interesting to research how well students perform when they only know GeoGebra. Or what will the outcome of this study be if it would be

completely reversed? A study that first learns students to work with GeoGebra and then introduces the graphing calculator for one lesson, may have an interesting different outcome.

Also, it would be interesting to examine effects in the long run. Results in this study show that it matters how much time students spend with GeoGebra before formulating how you think of it. This might also influence results.

Other suggestions could be a more in-depth research on understanding. An example could be a question that asks students to finish a graph of a fifth-degree function of which only the start and three extreme values are drawn. Does the student understand that an odd-degree function will increase or decrease just as it started, and that therefore one extra extremum must be drawn? This would work best if students have learned to work with only one of the two artefacts and make this test without it, in order to see differences in general understanding.

In summary, we can say that GeoGebra offers a lot of potential. It would be most interesting to develop pilot programs to test whether these results still hold for student groups without prior knowledge of the graphing calculator. We hope that policy makers, schools and teachers are challenged to take this exciting step into the future.

Acknowledgements. I thank P. H. M. Drijvers for supervising my research project and CITO for supporting this study in their project 'GeoGebra als vervanging van de Grafische rekenmachine bij wiskundetoetsen in de tweede fase van havo en vwo'. I thank J. Remijn for supervising my internship at CITO, along with I. Claus for their assistance in visiting schools and supportive feedback. I thank all the teachers and students for their participation.

References

- Bokhove, C., & Drijvers, P. H. M. (2010). Digital tools for algebra education: Criteria and evaluation. *International Journal of Computers for Mathematics Learning*, *15*(1), 45-62. doi: 10.1007/s10758-010-9162-x
- CvTE. (2015, March 6). *De toekomst van de grafische rekenmachine*. Retrieved from https://www.hetcvte.nl/nieuws/20150306/de_toekomst_van_de_grafische.
- Drijvers, P. H. M. (2009). Tools and tests: Technology in national final mathematics examinations. In C. Winslow (Ed.), *Nordic Research on Mathematics Education, Proceedings from NORMA08* (pp. 225-236). Rotterdam: Sense.
- Drijvers, P. H. M., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey: 2016*. Cham, Switzerland: Springer International Publishing AG.
- Drijvers, P. H. M., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2014). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213–234.
- Drijvers, P. H. M., Godino, J.D., Font, V., & Trouche, L. (2013). One episode, two lenses: A reflective analysis of student learning with computer algebra from instrumental and onto-semiotic perspectives. *Educational Studies in Mathematics*, *82*(1), 23-49.
- EACEA/Eurydice. (2011). *Mathematics education in Europe: Common challenges and national policies*. Brussels: Eurydice.
- Hennessy, S., Fung, P., & Scanlon, E. (2001). The role of the graphic calculator in mediating graphing activity, *International Journal of Mathematical Education in Science and Technology*, *32*(2), 267-290, doi: 10.1080/00207390010022176
- Hodanbosi, C. L. (2001). *A comparison of the effects of using a dynamic geometry software program and construction tools on learner achievement*. United States, OH: Kent State University Kent.
- Jupri, A., Drijvers, P. H. M., & Van den Heuvel-Panhuizen, M. (2015). An instrument theory view on students' use of an applet for algebraic substitution. *International Journal of Technology in Mathematics Education*, *123*(2), 63-80. doi: 10.1564/tme_v23.2.02
- Mayer, R. E. (2014). Cognitive theory of multimedia learning. In R. E. Mayer (Ed.), *Cambridge Handbook of Multimedia Learning second edition* (pp. 43-71). Cambridge: University Press.
- Saha, R. A., Ayub, A. F. M., & Tarmizi, R. A. (2010). The effects of GeoGebra on mathematics achievement: Enlightening coordinate geometry learning. *Procedia - Social and Behavioral Sciences*, *8*, 686–693. doi:10.1016/j.sbspro.2010.12.095
- Stacey, K., & Wiliam, D. (2013). Technology and assessment in mathematics. In M.A. (Ken) Clements et al. (Eds.), *Third International Handbook of Mathematics Education* (pp. 721-751). New York: Springer. doi: 10.1007/978-1-4614-4684-2_23

- Tarmizi, R. A., Ayub, A. F. M., Abu Bakar, K., & Yunus, A. S. (2010). Effects of technology enhanced teaching on performance and cognitive load in calculus. *International Journal of Education and Information Technologies*, 4(2), 109–120.
- Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281–307.
- Trouche, L., & Drijvers, P. H. M. (2010). Handheld technology: Flashback into the future. *ZDM, the International Journal on Mathematics Education*, 42(7), 667-681.
- Wadgave, U., & Khairnar, M. (2016). Parametric tests for Likert scale: For and against. *Asian Journal of Psychiatry*, 24, pp.67-68. doi: 10.1016/j.ajp.2016.08.016

Instructieformulier Les 1

Herhalingsles GR



Universiteit Utrecht



Doel van de les

De eerste les bestaat uit het herhalen van hoe een vijftal opdrachten grafisch-numeriek opgelost kan worden met de GR. Het gaat om het berekenen van functiewaarden, nulpunten, minima, maxima en snijpunten. Daarnaast moet een leerling behendig zijn in het instellingen van juiste window-instellingen.

Benodigheden

- Opgaveblad GR
- Invulblad GR
- Antwoordenblad GR

Rol van de docent

Onderzoek inleiden - 5 minuten

Voordat de leerlingen aan het werk gaan is het uiteraard belangrijk dat er iets verteld wordt over het onderzoek. Leidt het dus in door dingen te vertellen als

- dat het doel van het onderzoek is om het gebruik van GeoGebra te testen als eventuele vervanger van de grafische rekenmachine
- hoeveel lessen het in beslag gaat nemen
- dat er (een) onderzoeker(s) langs komt
- dat leerlingen een GR- of GGB-toetsje gaan maken
- dat er van de toets van een paar leerlingen die dit geen probleem vinden filmbeelden opgenomen worden
- dat ze een vragenlijst mogen invullen over hun voorkeur.

Opgavenblad bespreken - 5 minuten

Na de inleiding van het onderzoek bespreek je het doel van de eerste les. Wijs de leerlingen hierbij op het voorblad en merk op

- dat de nadruk van de les ligt op het instellen van het window en de vijf functies van het CALC (TI) of G-Solv (Casio) menu
- dat leerlingen zelfstandig of in tweetallen door het opgaveblad mogen werken
- dat het goed is dat leerlingen zichzelf aanwennen om de invoer, gebruikte functie, oplossing en (eventueel) conclusie (zie schema) op te schrijven (hiermee worden de uiteindelijke toetsresultaten ook inzichtelijker)
- dat ze werken op een invulblad die na afloop worden ingenomen (naam erop)
- dat leerlingen een antwoordenblad voorin de klas mogen pakken als ze klaar zijn
- dat vraag 13 de laatste vraag is, dat ze hierna kunnen nakijken en daarna de extra opgave mogen proberen.

Ondersteunen - 25 tot 30 minuten

Loop door de klas en bied ondersteuning waar nodig.

Volgende les introduceren - 10 minuten

Licht tien minuten voor het einde van het lesuur toe wat er de volgende les staat te gebeuren. Zodra je hiermee klaar bent kunnen de leerlingen (eventueel) nog doorwerken tot de bel.

-Vertel de leerlingen dat het doel van de volgende les is om de vijf soorten grafische berekeningen met GeoGebra uit te voeren.

-Laat zien hoe de leerling GeoGebra opent en laat ze alvast een zien hoe je de nulpunten van bijvoorbeeld $y = -x^2 + 3$ kunt bepalen.

-Laat zien wanneer de balk van de examenversie van GGB rood wordt en instrueer de leerlingen om geen andere dingen te doen, zodat de balk blauw blijft.

-Vertel waar de volgende les plaatsvindt (computerlokaal?) .

-Laat de leerlingen alvast tweetallen vormen zodat ze elkaar de volgende les direct kunnen opzoeken.

-Vraag twee leerlingen die samenwerken en zich er niet voor schamen hun mening te geven toestemming voor nader onderzoek. Dit houdt in dat de onderzoeker de volgende les samen zit met dit tweetal en hen nauwkeurig observeert, en dat ze tijdens het maken van de toets gefilmd zullen worden.

Rol van de leerling

De leerlingen werken deze les zelfstandig of in tweetallen aan het opdrachtenblad. Ze schrijven hun antwoorden op het invulblad dat ze na afloop inleveren.

Na vraag 13 zijn leerlingen klaar. Ze kunnen dan een antwoordenblad voorin de klas pakken om de opdrachten na te kijken. Indien ze voortijdig klaar zijn kunnen ze de extra opgave proberen. Indien leerlingen het niet afkrijgen leveren ze het invulblad toch in.

Rol van de onderzoeker(s)

Tijdens de eerste les is de onderzoeker om praktische redenen niet aanwezig.

Instructieformulier Les 2

Introductieles GeoGebra



Universiteit Utrecht



Doel van de les

De tweede les bestaat uit het kennis maken met GeoGebra(GGB). Ook met GGB leren de leerlingen een vijftal opdrachten grafisch-numeriek op te lossen. Het gaat om het berekenen van functiewaarden, nulpunten, minima, maxima en snijpunten. Daarnaast moet een leerling behendig zijn in het instellingen van juiste window-instellingen. Dit kan door scrollen, het beeld slepen en de assen slepen.

Benodigheden

- Opgaveblad GGB
- Invulblad GGB
- Antwoordenblad GGB

Rol van de docent


Introductie - 10 minuten

- Laat bij binnenkomst de tweetallen elkaar opzoeken en hun computer of laptop en GeoGebra opstarten.
- Introduceer de onderzoeker en kondig aan dat hij zichzelf voorstelt.
- Deel terwijl de onderzoeker zich voorstelt de opgavebladen uit.
- Bespreek het voorblad. Leg nadruk op
- Tweeledigheid: Juiste window instellen en de grafisch-numerieke berekeningen
- Meerdere manieren van oplossen: onderscheid tussen knoppen op de werkbalk en commando's in de invoerbalk.
- Opschrijven van antwoorden: duidelijk en volledig, zie tabel (knop of commando)
- Leerlingen schrijven ook deze les hun uitwerkingen op een invulblad wat nadien wordt ingeleverd. Leerlingen werken in tweetallen samen, maar beide schrijven antwoorden op.
- Dat je tussendoor één keer de klas stillegt om een aantal dingen te bespreken en dat je hierbij leerlingen naar voren vraagt om hun methodes te demonstreren.
- Dat als leerlingen denken dat ze alle mogelijkheden geprobeerd en onder de knie hebben, dat ze door kunnen met een volgend onderdeel.

Zelfstandig werken - 40 tot 45 minuten

Laat de leerlingen zelfstandig (in tweetallen) werken. Ze mogen in hun eigen tempo aan de opgaven werken.

Leg na een minuut of tien de hele klas stil om een aantal dingen te bespreken. Laat de input van de leerlingen komen. Vraag ze naar voren om het ze te laten demonstreren. Bespreek in ieder geval het volgende

- Hoe de as-verhouding aangepast kan worden (slepen assen met  of rechtermuisknop, $xAs : yAs$ verhouding aanpassen)
- Wat het verschil is tussen een commando en het gebruiken van een knop. Neem bijvoorbeeld opgave 2 om dit mee te demonstreren (voer de functie zelf van tevoren al even in).
- Hoe een leerling het aantal decimalen kan aanpassen (menuknop rechtsboven, opties, afronden)

Nadien kunnen de leerlingen zelfstandig verder werken. Vertel het volgende

- Indien leerlingen klaar zijn kunnen ze eerst een antwoordenblad voorin de klas pakken om opgave 1 t/m 4 na te kijken. Daarna kunnen ze werken aan de bonusopgaven 5 t/m 9 (als leerlingen niet aan deze opgave toekomen, of als ze hem juist heel goed maken is dat ook interessant voor het onderzoek, vertel dit).

Afronding

Vertel de leerlingen waar de volgende les plaatsvindt en dat ze daar ter plekke horen of ze de toets gaan maken met de GR of met GGB.

Rol van de leerling

Leerlingen werken in tweetallen aan de hand van het opdrachtenblad. Ze schrijven hun antwoorden op een invulblad dat ze na afloop inleveren. Na opgave 4 kunnen leerlingen nakijken. Ze kunnen dan een antwoordenblad voorin de klas pakken om de opgaven na te kijken. Indien ze voortijdig klaar zijn kunnen ze de extra opgaven proberen. Indien leerlingen het niet afkrijgen leveren ze het blad toch in.

Rol van de onderzoeker(s)

De onderzoeker stelt zich aan het begin van de les voor en vertelt kort iets over het onderzoek. Hierna schuift hij direct aan bij het aangewezen tweetal. Hij zal de acties van deze leerlingen nauwgezet observeren en registreren. Na afloop neemt de onderzoeker de les door met de docent en ook blikken ze samen vooruit op het afnemen van de toets. Hierbij delen ze onder andere samen de groep (willekeurig) in tweeën.

Instructieformulier Les 3

Toets en vragenlijst



Universiteit Utrecht



Doel van de les

In de derde les worden de toets en een vragenlijst afgenomen.

Benodigheden

- Toets GR
- Toets GGB
- Vragenlijst leerlingen
- Toetspapier van de school

Rol van de docent

Introductie - 5 tot 10 minuten

Voorafgaand aan de les is de groep in tweeën gedeeld.

- Zorg ervoor dat de leerlingen vlot weten of ze een computer moeten gebruiken, of dat ze de toets juist gaan maken met de GR.
- Laat de leerlingen hun plek opzoeken en eventueel de computers opstarten.
- Vraag en stimuleer leerlingen om zo goed mogelijk hun best te doen, in belang van het onderzoek.
- Vraag leerlingen hun antwoorden uitgebreid op te schrijven, in belang van het onderzoek en wijs leerlingen erop dat dit boven aan het toetsblad staat weergeven.
- Wijs leerlingen erop dat het belangrijk is dat het stil blijft tot na het invullen van de vragenlijst. Hierdoor beïnvloeden ze elkaars mening niet.

Toets - 30 minuten

Leerlingen werken zelfstandig in stilte aan de toets.

- Benadruk dat het belangrijk is dat de naam van de leerling bovenaan het uitwerkingsblaadje staat.
- Als ze klaar zijn leggen ze de toets op de hoek van de tafel, uitwerkingsblaadje onder.
- Vijf minuten voor het einde van dit halfuur kunnen de blaadjes van leerlingen die klaar zijn opgehaald worden, deze mogen vast beginnen aan de vragenlijst.

Vragenlijst – 5 tot 10 minuten

Haal de toetsblaadjes op en deel direct de vragenlijsten uit. Zorg ervoor dat het stil blijft tot na het invullen van de vragenlijst.

Rol van de leerling

Leerlingen maken de toets en vullen daarna de vragenlijst in. Eén leerling wordt gefilmd in het lokaal (GR) en één leerling wordt gefilmd in een ander lokaal (GGB), zie volgende alinea.

Rol van de onderzoeker(s)

De onderzoeker stemt met de docent af waar de leerlingen zullen gaan zitten. Samen installeren ze een camera, die de onderzoeker heeft meegenomen in het lokaal, gericht op een tafel. Eén leerling van het tweetal die door de onderzoeker nauwgezet is geobserveerd in les 2 maakt hier de toets met de GR. Hij wordt gevraagd, steeds voordat hij op *graph* klikt, zijn invoer nadrukkelijk aan de camera te tonen. Daarnaast wordt een camera geïnstalleerd in aparte ruimte waar de andere leerling van het specifieke tweetal zal zitten om zijn toets met GeoGebra te maken. De onderzoeker zal hierbij zitten en steeds aan de leerling vragen zijn gedachten hardop uit te spreken.

Na afloop overhandigt de docent al het materiaal aan de onderzoeker. Afsluitend ondervraagt de onderzoeker de docent in een mini-interview naar de bevindingen van de docent.

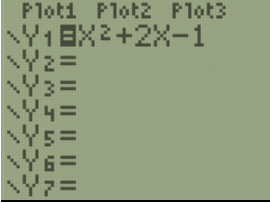
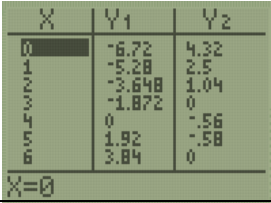
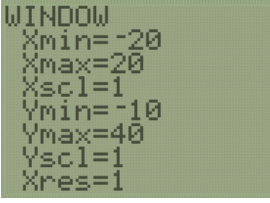
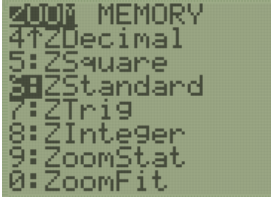
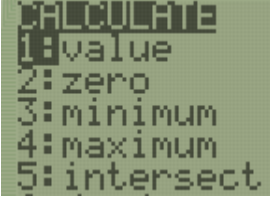
Opgaveblad onderzoek GeoGebra



Universiteit Utrecht

Les 1 – De grafische rekenmachine - Ti

Dit schooljaar heb je kennis gemaakt met de grafische rekenmachine (GR). Tijdens deze les fris je je geheugen hiervan op. We behandelen het instellen van het venster (*window*) en vijf opdrachten, die je kunt uitvoeren met eerste vijf functies uit het *CALC*-menu. Grijs gemarkeerde woorden zijn knoppen op de GR en **dikgedrukte woorden** zijn de verschillende opdrachten die we behandelen.

Invoer en tabel	<code>y =</code>		<code>table</code>
	Formules invoeren		Tabel inzien - Startwaarde en stapgrootte aanpassen met <code>tblset</code>
Scherminstellingen	<code>window</code>		<code>zoom</code>
	Handmatig window aanpassen		6 ZStandard – standaard scherminstellingen 0 ZoomFit – y-as automatisch aanpassen op X_{min} en X_{max} .
Berekenen van...	<code>calc</code>		
	<ol style="list-style-type: none"> 1 ...functiewaarden – Kan ook met <code>trace</code> 2 ...nulpunten – Linkergrens / rechtergrens / gokje 3 ...minimum – Linkergrens / rechtergrens / gokje 4 ...maximum – Linkergrens / rechtergrens / gokje 5 ...snijpunten – Grafiek 1 selecteren / grafiek 2 selecteren / gokje 		

BELANGRIJK:

Bij de opdracht 'Bereken ...' mag je de GR gebruiken. Het is hierbij wel van belang dat je niet zomaar het antwoord opschrijft, maar goed aangeeft hoe je hebt gewerkt. Noteer altijd in ieder geval de volgende vier zaken.

De voorbeeldvraag is: *Bereken het minimum van $y = x^2 + 1$.*

I	Invoer	Noteer welke formules je intypt	Voer in $y_1 = x^2 + 1$
F	Functie	Noteer de functie uit het <i>calc</i> -menu	Optie minimum geeft
O	Oplossing	Noteer de gevonden oplossing	$x = 0$ en $y = 1$
C	Conclusie	Geef antwoord op de vraag	Dus het minimum is 1

Lees goed en beantwoord de genummerde vragen met een nette uitwerking in je schrift.

Veel succes!

Juiste *window* instellen / functiewaarden berekenen

Gegeven is de functie f met het functievoorschrift
 $f(x) = x^2 - 3x - 2$.

Je kunt functiewaarden op verschillende manieren berekenen. Door 1 in te vullen op de plek van x bereken je de functiewaarde bij $f(1)$. Als je lastigere formules gebruikt of veel verschillende functiewaarden van dezelfde functie wilt berekenen is het handiger om de GR te gebruiken. In het *CALC*-menu vind je de optie *value* die de y -waarde berekent bij de x -waarde die jij invoert.

1. Bereken $f(-1)$ algebraïsch en controleer je antwoord met de optie *value*. Schrijf je berekeningen op het invulblad.
2. Bereken $f(5\frac{1}{2})$ met de functie *value*.
3. Bereken $f(-84)$ met de functie *value*.

Om met *value* een y -waarde te kunnen berekenen, moet de betreffende x -waarde wel in beeld zijn. In het *WINDOW*-menu kun je de grootste en kleinste waarden van je assenstelsel veranderen. Pas indien nodig dus X_{min} of X_{max} aan.

Waarschijnlijk is het je opgevallen dat oplossingen buiten Y_{min} en Y_{max} wel worden gegeven, ook al vallen ze buiten beeld. Ondanks dat is het vaak handig dat je de grafiek mooi "in beeld" krijgt. De grafische rekenmachine heeft een handige optie om de waarden voor Y_{min} en Y_{max} automatisch aan te passen aan de X_{min} en X_{max} van de ingevoerde formules. In het *ZOOM*-menu vind je 0 : *ZoomFit*.

4. Neem $X_{min} = -100$ en $X_{max} = 100$. Gebruik *ZoomFit* en wacht tot de grafiek getekend is. Geef Y_{min} en Y_{max} .
In het *ZOOM*-menu kun je met 6 : *ZStandard* snel je *window*-instellingen terug zetten naar $X_{min} = -10$, $X_{max} = 10$, $Y_{min} = -10$ en $Y_{max} = 10$.

➤ Gebruik *ZStandard* om beide assen weer van -10 tot 10 in te stellen.

Nulpunten berekenen

Om de snijpunten met de x -as (nulpunten) te bepalen wordt de tweede functie, genaamd *zero*, van het *CALC*-menu gebruikt. Deze functie geeft de x - en y -coördinaat van dit nulpunt. Uiteraard is de y -waarde hierbij altijd gelijk aan 0 , omdat de hoogte altijd 0 is.

Om de functie *zero* goed te gebruiken, moet je de GR een beetje helpen met zoeken. Geef eerst een linker- en rechtergrens (*left/right bound*) door (twee keer) op *enter* te drukken. Binnen deze grenzen zoekt de GR naar een snijpunt met de x -as. Als laatste geef je een gokje (*guess*) door je cursor naar het dichtstbijzijnde punt te verplaatsen en op *enter* te drukken. Dit gokje is alleen belangrijk als er binnen je twee grenzen meerdere nulpunten te vinden zijn. Je GR geeft de oplossing die het dichtst bij je gokje in de buurt zit.

5. Bereken in twee decimalen nauwkeurig de nulpunten van f .

Extreme waarden berekenen

Gegeven is de functie p met het functievoorschrift

$$p(x) = -0,2x^3 + 0,6x^2 + 2,6x - 3.$$

Bewaar deze functie voor opgaven 10 t/m 13.

De GR kan ook de coördinaten van plaatselijk hoogste en laagste punten berekenen. Deze worden ook wel extreme waarden, minima/maxima of toppen genoemd.

Om de waarden van toppen te berekenen kunnen de functies *minimum* en *maximum* uit het *CALC*-menu gebruikt worden. Deze gebruik je net als *zero*, door eerst een linker- en rechtergrens te bepalen waarbinnen de GR naar een top zoekt en door de cursor zo dicht mogelijk naar de betreffende top te brengen en als op enter te drukken (het gokje).

6. Bereken de extreme waarden van p . Rond af op twee decimalen.
Noteer je eindantwoord als volgt: min. is $p(\dots) = \dots$ en max. is $p(\dots) = \dots$

Niet altijd zijn alle toppen direct op je scherm zichtbaar.

Gegeven is de functie q met het functievoorschrift

$$q(x) = 0,01x^4 + 0,1x^3 - 0,8x^2 - x + 2.$$

Bewaar ook deze functie.

7. Schets de grafiek. Gebruik $X_{min} = -10$, $X_{max} = 10$, $Y_{min} = -10$ en $Y_{max} = 10$.
Je hebt een grafiek met twee extreme waarden geschetst. Toch heeft deze grafiek nog een derde top.
8. Bereken alle extreme waarden van q in twee decimalen nauwkeurig. Neem $X_{min} = -16$, $X_{max} = 10$ en gebruik *ZoomFit*. Zorg ervoor dat je functie p niet mee laat plotten door het $=$ -teken, dat voor de invoer van deze functie in het 'y =' menu staat, uit te schakelen.
Hier is sprake van een **regelmaat**. In de formule van p is 3 het grootste getal als exponent bij grondtal x . We noemen p een derdegraadsfunctie en deze functie heeft 2 extreme waarden. De functie q is een vierdegraadsfunctie en deze heeft 3 extreme waarden. Aan de graad (hoogste exponent) van een functie kun je direct aflezen hoeveel extreme waarden die functie maximaal heeft.
9. Hoeveel extreme waarden heeft de functie $k(x) = x^{99} - x^{19} + 0,6x^7 + 1$ maximaal?
Indien je dus bijvoorbeeld maar drie extreme waarden in beeld ziet bij een vijfdegraadsfunctie, realiseer je dan dat je waarschijnlijk nog een vierde kunt vinden als je de window-instellingen aanpast.

Snijpunten berekenen / vergelijkingen oplossen

Gegeven zijn nogmaals de functies p en q met de functievoorschriften

$$p(x) = -0,2x^3 + 0,6x^2 + 2,6x - 3 \text{ en}$$

$$q(x) = 0,01x^4 + 0,1x^3 - 0,8x^2 - x + 2$$

Met de functie *intersect* kan de GR snijpunten van grafieken bepalen. Een aantal soorten vragen kunnen zich voordoen. Steeds probeer je eerst zelf een antwoord op de vraag te vinden. Daarna kun je de uitleg lezen die erna komt.

10. Bepaal de coördinaten van de snijpunten van p en q . Rond af op één decimaal.

In het geval dat je snijpunten moet bepalen van twee grafieken plot je allereerst beide grafieken. Om de coördinaten van een snijpunt te vinden gebruik je *intersect* door eerst de twee betreffende grafieken te selecteren en vervolgens een weer een gokje (guess) te doen. In het geval dat je meerdere snijpunten hebt, zoals bij vraag 10, gebruik je *intersect* dus meerdere keren en zorg je ervoor dat je gokje steeds in de buurt van een ander snijpunt komt te liggen.

11. Los op $-0,2x^3 + 0,6x^2 + 2,6x - 3 = x + 1$. Rond zo nodig af op één decimaal.

In het geval dat je een vergelijking moet oplossen ben je geïnteresseerd naar de x waarvoor deze vergelijking voldoet. Je wilt dus weten voor welke x -waarde de formule aan de linkerkant van het =-teken gelijk is aan de rechterkant. De linker- en rechterkant zijn dus eigenlijk gewoon formules die je kunt invoeren op je GR. Door *intersect* te gebruiken vind je grafische rekenmachine direct voor welke x (en y) de formules elkaar snijden en dus gelijk zijn.

12. Bereken in één decimaal nauwkeurig de x -waarden waarvoor de hoogte van p gelijk is aan 3.

Ook bij vragen van deze soort gebruik je *intersect*. De grafiek waarvan je de snijpunten met p wilt bepalen is dan een lijn met hoogte 3. Deze lijn kun je plotten door $y = 3$ in te voeren.

13. De functies p en q hebben vier snijpunten. Bepaal de coördinaten van het meest linker snijpunt, indien je bij vraag 10 geen vier snijpunten gevonden hebt. Rond af op één decimaal.

In sommige gevallen liggen niet alle snijpunten in het standaard scherm van de grafische rekenmachine. Omdat x^4 sneller groeit dan x^3 als x steeds negatiever wordt, valt er aan de linkerkant nog een snijpunt van p en q buiten beeld. Pas hierom eerst X_{min} aan en gebruik eventueel *ZoomFit*. De GR vindt snijpunten dus alleen als ze binnen X_{min} en X_{max} liggen.

Extra. Plot $y_1 = 2^x$ en $y_2 = x^4$ in het standaardscherm. Je ziet twee snijpunten. Toch is er een derde snijpunt. Probeer dit snijpunt op je scherm te krijgen en geef de coördinaten. Welke functie groeit “op den duur” het snelst?

Opgaveblad onderzoek GeoGebra



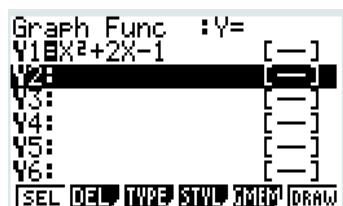
Universiteit Utrecht



Les 1 – De grafische rekenmachine - Casio

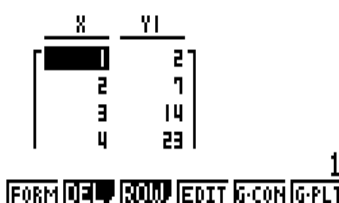
Dit schooljaar heb je kennis gemaakt met de grafische rekenmachine (GR). Tijdens deze les fris je je geheugen hiervan op. We behandelen het instellen van het venster (*window*) en vijf opdrachten, die je kunt uitvoeren in *G-Solv* menu. **Grijs gemarkeerde woorden** zijn knoppen op de GR en **dikgedrukte woorden** zijn de verschillende opdrachten die we behandelen.

Invoer en tabel



MENU – 5

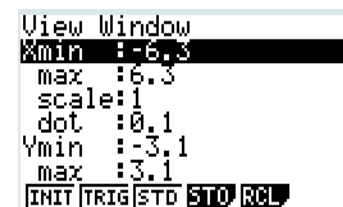
Formules invoeren



MENU – 7

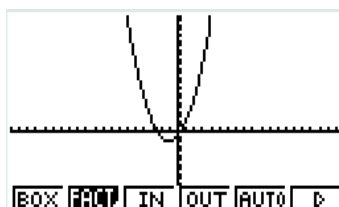
Tabel inzien - Startwaarde en stapgrootte aanpassen met **F1 FORM – F5 SET**

Scherminstellingen



V-Window

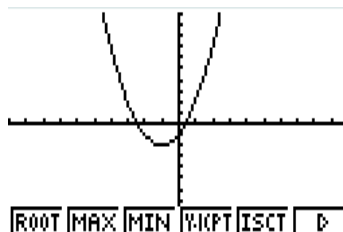
Window aanpassen **F3 STD -** standaard instellingen



Zoom

F5 AUTO - *y*-as automatisch aanpassen op X_{min} en X_{max} .

Berekenen van...



G-Solv

F6-F1 Y-CAL ...**functiewaarden** – Kan ook met **Trace**
F2 ROOT ...**nulpunten** – Kan ook met **F6-F2 X-CAL** ($y = 0$)
F3 MIN ...**minimum**
F2 MAX ...**maximum**
F5 ISCT ...**snijpunten** – selecteer eerst de juiste grafieken

BELANGRIJK:

Bij de opdracht 'Bereken ...' mag je de GR gebruiken. Het is hierbij wel van belang dat je niet zomaar het antwoord opschrijft, maar goed aangeeft hoe je hebt gewerkt. Noteer altijd in ieder geval de volgende vier zaken.

De voorbeeldvraag is: *Bereken het minimum van $y = x^2 + 1$.*

I	Invoer	Noteer welke formules je intypt	Voer in $y_1 = x^2 + 1$
F	Functie	Noteer de functie uit het <i>G-Solv</i> menu	Optie MIN geeft
O	Oplossing	Noteer de gevonden oplossing	$x = 0$ en $y = 1$
C	Conclusie	Geef antwoord op de vraag	Dus het minimum is 1

Lees goed en schrijf je antwoorden met nette uitwerkingen op het invulblad.

Veel succes!

Opgaveblad onderzoek GeoGebra



Universiteit Utrecht

Les 2 – GeoGebra

Dit schooljaar heb je kennis gemaakt met de grafische rekenmachine (GR). Je hebt in het *CALC* menu vijf verschillende functies leren gebruiken. Dit zijn *value*, *zero*, *minimum*, *maximum* en *intersect*. Vandaag ga je werken met GeoGebra. Misschien heb je dit programma al wel eens gebruikt om vlakke figuren mee te tekenen of andere meetkundige opdrachten mee uit te voeren. Met GeoGebra kun je ook functievoorschriften invoeren en daarmee kan je dezelfde opdrachten uitvoeren als op de GR. In deze les ga je samen met je docent en je klasgenoten kennismaken met deze functies van GeoGebra (GGB). Net als met de GR heb je met GGB soms meerdere mogelijkheden om dezelfde opdracht uit te voeren. Deze mogelijkheden staan tussen haakjes. Je mag dan kiezen welke methode je fijner vindt.


Juiste *window* instellen

Zoomen en verplaatsen

-

Scrollen en slepen

Verhouding assen aanpassen-

Gebruik  en sleep een as**Functiewaarden** berekenen

Commando

-

bijv. $f(-1)$ (of *Functie onderzoeker*Tabblad *Punten*)**Nulpunten** berekenen*Nulpunten* $nulpunten[f]$)

(of Commando

Extreme waarden berekenen*Extrema* $extrema[f]$)

(of Commando

Snijpunten berekenen*Snijpunten* $snijpunten[f]$)

(of Commando

BELANGRIJK:

Ook bij het gebruik van GGB is het belangrijk dat je aangeeft hoe je te werk bent gegaan. Bij de opdracht 'Bereken ...' mag je de GGB gebruiken. Noteer altijd in ieder geval de volgende vier zaken. De voorbeeldvraag is: *Bereken het minimum van $y = x^2 + 1$.*

I	Invoer	Noteer welke formules je intypt	Voer in $f(x) = x^2 + 1$
F	Functie	Noteer de knop uit de werkbalk of het commando van de invoerbalk	Optie extrema geeft of commando extrema[f] geeft
O	Oplossing	Noteer de gevonden oplossing	$x = 0$ en $y = 1$
C	Conclusie	Geef antwoord op de vraag	Dus het minimum is 1

Lees goed en beantwoord de genummerde vragen met een nette uitwerking in je schrift.

Veel succes!

Juiste *window* instellen / functiewaarden berekenen

Gegeven is de functie f met het functievoorschrift.

$$f(x) = 0,2x^4 - 1,8x^3 - 0,1x^2 + 3,1x - 2$$

Let erop dat je bij het invoeren van je functie in GeoGebra een punt gebruikt i.p.v. een komma.

1. Bereken met behulp van GeoGebra $f(-2)$, $f(0)$, $f(1)$, $f(2)$, $f(5)$ en $f(8)$.
Probeer het ook eens met de **functieonderzoeker**.

Nulpunten berekenen

Gegeven is de functie g met het functievoorschrift

$$g(x) = 0,01x^3 + 0,42x^2 - 2,6x + 2,2$$

2. Bereken met behulp van GeoGebra de nulpunten van g . Rond af op drie decimalen.

Extreme waarden berekenen

Gegeven is de functie h met het functievoorschrift

$$h(x) = 0,01x^4 - 0,2x^3 - 4x^2 - 3,5x + 8$$

3. Bereken met behulp van GeoGebra de extreme waarden van h . Rond af op twee decimalen.

Snijpunten berekenen

Gegeven zijn de voorgaande functies g en h met de functievoorschriften

$$g(x) = 0,01x^3 + 0,42x^2 - 2,6x + 2,2 \text{ en}$$

$$h(x) = 0,01x^4 - 0,2x^3 - 4x^2 - 3,5x + 8.$$

4. Vind met behulp van GeoGebra de waarden van x waarvoor geldt dat $g(x) < h(x)$. Rond af op één decimaal.

Let op dat GeoGebra niet altijd alle oplossingen vindt. Bij veeltermfuncties zoals f , g en h , waarbij de x steeds als een grondtal van een macht voorkomt vindt GeoGebra wel alle oplossingen, maar voor ander soort functies zoals die van p in de volgende vraag geldt dit niet. Vind dan de snijpunten door ze in beeld te brengen en zelf een snijpunt te creëren.

Gegeven zijn de functies p en q met $p(x) = 2^x$ en $q(x) = x^b$ met b een geheel getal groter dan 1. Voor $b = 2$ geldt bijvoorbeeld $q = x^2$.

Rond bij het maken van de volgende opgaven zo nodig af op twee decimalen.

5. Vind met behulp van GeoGebra de coördinaten van alle snijpunten van p en q met $b = 2$.
6. Vind met behulp van GeoGebra de coördinaten van alle snijpunten van p en q met $b = 3$.
7. Voor $b = 7$ geldt $q(x) = x^7$. Hoeveel snijpunten vermoed je dat p en q hebben in dit geval hebben?
8. Stel een vermoeden op over het aantal snijpunten van p en q voor verschillende b .
9. Geef de grootste waarde van b waarvoor je alle snijpunten van p en q in GeoGebra redelijkerwijs in beeld kunt brengen.

Invulblad Les 1 – GR



Universiteit Utrecht



Naam: _____

Juiste window instellen / functiewaarden berekenen

1.

2.

3.

4.

Nulpunten berekenen

5.

Appendix H - Test GC

VWO 4 Wiskunde B	Toets 30 minuten	GR - Ti Versie A	4 opgaven
-----------------------------------	-----------------------------------	-----------------------------------	------------------

Voor deze toets mag je als hulpmiddel de grafische rekenmachine gebruiken. Andere hulpmiddelen zijn niet toegestaan.

Schijf bij elke vraag nauwkeurig op hoe je aan je antwoord bent gekomen. Noteer in ieder geval altijd:

- invoer
 - functie/knop
 - oplossing(en)
 - conclusie
- } Indien oplossingen en conclusie hetzelfde zijn, schijf het slechts één keer op.

Start met $X_{min} = -10$, $X_{max} = 10$, $Y_{min} = -10$ en $Y_{max} = 10$ of gebruik *ZStandard*.

1. Gegeven is de functie f waarvoor geldt $f(x) = 0,02x^3 - 0,3x^2 - x + 7$
 - a Schets de grafiek
 - b Vind alle nulpunten van f . Rond af op twee decimalen.

2. Gegeven is de functie g waarvoor geldt $g(x) = 0,005x^4 + 0,03x^3 - 0,4x^2 + 8$
 - a Bereken de extreme waarden van g in twee decimalen nauwkeurig.
 - b Schets de grafiek van g waarin alle toppen te zien zijn.

3. Gegeven is de functie h waarvoor geldt $h(x) = -0,008x^3 + 0,1x^2 + 1,3x - 5$
 - a Vind de coördinaten van de snijpunten van de grafiek van h met de lijn met de vergelijking $y = x - 1$. Rond af op twee decimalen.
 - b Schets de grafiek van h en de lijn $y = x - 1$ waarin alle snijpunten te zien zijn.

Gegeven in de functie k waarvoor geldt $k(x) = 0,01x^3 + 0,2x^2 - 2x + 4$

 - c Los op $h(x) = k(x)$. Rond af op twee decimalen.

4. Het aantal bezoekers van een internetsite waarop allerlei goederen te koop worden aangeboden is voor een doordeweekse dag te benaderen door het model $N = 0,02t^5 - 1,225t^4 + 22,5t^3 - 84t^2 - 646t + 5000$. Hierin is N het aantal bezoekers en t de tijd in uren met $t = 0$ om 0:00 uur.
 - a Schets de grafiek van N . Welke X_{min} , X_{max} , Y_{min} en Y_{max} heb je genomen?
 - b Hoeveel bezoekers zijn er 's ochtends om zes uur?
 - c Hoeveel bezoekers zijn er 's middags om half vijf?
 - d Hoeveel bezoekers zijn er 's avonds om kwart voor elf?
 - e Hoe laat is het aantal bezoekers het grootst? Hoeveel bezoekers zijn er dan? Rond af op gehelen.
 - f Er zijn 6000 bezoekers. Hoe laat kan het zijn? Rond af op hele of halve uren.
 - g Hoeveel extreme waarden heeft de grafiek van N ? Geef als je twijfelt over je antwoord aan waarom je twijfelt.

Noteer als laatste je **eindtijd** in de linkerbovenhoek van je antwoordenblad.

Appendix I - Test GGB

VWO 4 Wiskunde B	Toets 30 minuten	GGB Versie A	4 opgaven
-----------------------------------	-----------------------------------	-------------------------------	------------------

Voor deze toets mag je als hulpmiddel GeoGebra gebruiken. Andere hulpmiddelen zijn niet toegestaan.

Schrijf bij elke vraag nauwkeurig op hoe je aan je antwoord bent gekomen. Noteer in ieder geval altijd:

-invoer

-functie/knop

-oplossing(en)

-conclusie.

} Indien oplossingen en conclusie hetzelfde zijn, schrijf het slechts één keer op.

1. Gegeven is de functie f waarvoor geldt $f(x) = 0,01x^3 - 2x + 1$
 - a Schets de grafiek
 - b Vind alle nulpunten van f . Rond af op twee decimalen.

2. Gegeven is de functie g waarvoor geldt $g(x) = 0,005x^4 + 0,03x^3 - 0,4x^2 + 8$
 - a Bereken de extreme waarden van g in twee decimalen nauwkeurig.
 - b Schets de grafiek van g waarin alle toppen te zien zijn.

3. Gegeven is de functie h waarvoor geldt $h(x) = -0,008x^3 - 0,3x^2 + 2x + 13$
 - a Vind de coördinaten van de snijpunten van de grafiek van h met de lijn met de vergelijking $y = x - 1$. Rond af op twee decimalen.
 - b Schets de grafiek van h en de lijn $y = x - 1$ waarin alle snijpunten te zien zijn.

Gegeven is de functie waarvoor geldt $k(x) = 0,01x^3 + 0,5x^2 - 5x + 9$

 - c Los op $h(x) = k(x)$. Rond af op twee decimalen.

4. Het aantal bezoekers van een internetsite waarop allerlei goederen te koop worden aangeboden is voor een doordeweekse dag te benaderen door het model $N = 0,02t^5 - 1,225t^4 + 22,5t^3 - 84t^2 - 646t + 5000$. Hierin is N het aantal bezoekers en t de tijd in uren met $t = 0$ om 0:00 uur.
 - a Schets de grafiek van N . Welke X_{min} , X_{max} , Y_{min} en Y_{max} heb je genomen?
 - b Hoeveel bezoekers zijn er 's ochtends om zes uur?
 - c Hoeveel bezoekers zijn er 's middags om half vijf?
 - d Hoeveel bezoekers zijn er 's avonds om kwart voor elf?
 - e Hoe laat is het aantal bezoekers het grootst? Hoeveel bezoekers zijn er dan? Rond af op gehelen.
 - f Er zijn 6000 bezoekers. Hoe laat kan het zijn? Rond af op hele of halve uren.
 - g Hoeveel extreme waarden heeft de grafiek van N in totaal? Geef als je twijfelt over je antwoord aan waarom je twijfelt.

Noteer als laatste je **eindtijd** in de linkerbovenhoek van je antwoordenblad.

Appendix J - Questionnaire

GeoGebra-GR vragenlijst



Universiteit Utrecht



Naam: _____ Klas: _____ School: Geslacht M / V Wiskunde A / B / C / D	Helemaal mee oneens	Mee oneens	Neutraal	Mee eens	Helemaal mee eens
Bedankt voor je deelname aan dit onderzoek! Vul deze vragenlijst in door kruisjes te zetten in het vakje van jouw mening. Beantwoord de vragen zo eerlijk mogelijk. Je docent gebruikt de antwoorden niet voor cijfers of iets dergelijks.					
Omcirkel wat voor jou van toepassing is					
1. Ik heb de toets gemaakt met	GR / GeoGebra				
2. Hiervoor gebruikte ik GeoGebra <i>nooit</i> (n) / <i>soms</i> (s) / <i>regelmatig</i> (r).	n / s / r				
Mening over de drie lessen van het onderzoek					
3. Ik vond de drie lessen nuttig.					
4. Ik vond de eerste les waarin de GR kennis herhaald werd makkelijk.					
5. Ik vond de eerste les leuk.					
6. Ik vond de tweede les waarin we met GeoGebra leerden omgaan makkelijk.					
7. Ik vond de tweede les leuk.					
8. Ik vond de toets makkelijk.					
9. Ik heb mijn best gedaan om de toets zo goed mogelijk te maken.					
10. Ik had voldoende begrip van de grafische rekenmachine of GeoGebra om de toets goed te kunnen maken.					
Mening over grafische rekenmachine					
11. Ik vind de grafische rekenmachine prettig om mee te werken.					
12. Ik weet waar ik de juiste functies (zoals nulpunten/minimum/snijpunten) op de grafische rekenmachine kan vinden.					
13. De plek waar ik deze functies/knoppen van de grafische rekenmachine kan vinden vind ik handig gekozen.					
Mening over GeoGebra					
14. Ik vind GeoGebra prettig om mee te werken.					
15. Ik weet waar ik de juiste functies (zoals nulpunten/minimum/snijpunten) in GeoGebra kan vinden.					
16. De plek waar ik deze functies/knoppen in GeoGebra kan vinden vind ik handig gekozen.					
Omcirkel wat voor jou van toepassing is					
17. Ik werk het liefst met	GR / GeoGebra				

Probeer één of twee redenen op te schrijven voor je antwoord op vraag 17.

Observation instrument



Universiteit Utrecht

Datum: 19-05-2017 Observator: WesleySchool: JFC Geobserveerde leerling: _____

Vraag 1

Conceptual aspect (scheme)	Related action (technique)	+/-	Opmerkingen / toelichting
Substitutie: Functiewaarde vind je door getal te substitueren voor x	Exploreert door -2 in te vullen en (algebraïsch) door te rekenen		
Invoeren: Functievoorschrift moet worden ingevoerd om grafiek te kunnen tekenen	Voert de functie in in de invoerbalk		
	Voert functie foutief in		
	Herstelt fout tijdig		
Kijkvenster instellen: De instelling van het kijkvenster bepaalt welke eigenschappen van een functie we zien	Zoomt uit		
	Versleept venster		
	Duwt assen samen (interval aanpassen)		
Functiewaarden berekenen: Functiewaarde berekenen door getal als argument van f in te voeren.	Schat uitkomst		
	Voert $f(-2)$ in in de invoerbalk		
	Schat uitkomst		
	Voert $f(\dots)$ in in de invoerbalk		
Samenhang formule – grafiek – tabel: Inzien dat functiewaarden kunnen worden afgelezen in een tabel	Zoekt de functieonderzoeker		
	Selecteert f		
	Exploreert functieonderzoeker		
	Voert waarde voor x in.		
	Breidt tabel uit		
	Schrijft y -waarde over uit algebravenster		