An Introduction to Lie Groups, Lie Algebras and their Representation Theory

Bachelor Thesis

Written by

Robert Slob, Utrecht University

Supervised by Dr. J.W. van de Leur

June 13, 2017

Abstract

In this thesis, we give an extensive introduction to Lie groups and Lie algebras. We conclude the thesis by providing the basic concept of the finite representation theory of a semisimple lie Algebra. The reader is expected to have some general knowledge of group theory, linear algebra, representation theory and topology. We do not demand the reader to have much prior knowledge of topology, hence we will also devote a great part of this thesis to discussing the preliminary knowledge necessary for Lie theory (e.g. manifolds, tangent spaces, vector fields, and integral curves).

Contents

1 Introduction

A Lie group represents the best-developed theory of continuous symmetry of mathematical structures, which causes it to play a major role in modern geometry. Alike, it plays a renowned role in modern theoretical physics and it formed a foundation in the theory of algebraic groups.

A Lie group is a group that is a smooth manifold as well, with the additional property that its group operations are smooth. This gives rise to some interesting extra properties in comparison to normal groups. Furthermore, this allows one to make use of topology on Lie groups to deduce more properties.

Finally, every Lie group gives rise to a Lie algebra. This interesting relation between Lie groups and Lie algebras allows one to study Lie groups in terms of their algebras, hence we have a relation between geometric and linear objects.

We believe that the Lecture notes on Lie groups of van den Ban (see [2]) contain a great structure for introducing Lie groups and Lie algebras. For this reason, we used these notes as a guidance for our structure in the chapters 3, 5 and 6.

2 Introduction to manifolds

We will give the formal definition of a Lie group in chapter 3. As Lie groups are smooth manifolds, we will start on this subject.

2.1 Manifolds

We start off by noting some general properties of *topological spaces*. As we shall see later on, we require a manifold to possess some of these properties.

Definition 2.1. Let X be some topological space, then

(a) A *neighbourhood* of a point p in X is any open subset U of X which contains the point p.

(b) We say that X is *second countable* if it has a countable base, i.e. there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of X such that any open subset of X can be written as the union of elements of $\mathcal{U}$.

(c) An *open cover* of X is a collection of open subsets $\{U_\alpha\}_{\alpha \in A}$ of X, such that their union equals X.

(d) The space X is said to be *Hausdorff* if any two distinct points $x, y \in X$ have disjunct neighbourhoods, i.e. there exist opens U, V of X with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.

Lastly we note that subsets of a topological space (with the induced topology) will inherit both the properties second countability and Hausdorffness.

We will take a look at $\mathbb{R}$ endowed with the Euclidean topology in order to illustrate these properties.

Example 2.2. We let $\mathbb{R}$ be endowed with the Euclidean topology in this example. Let p be a point in $\mathbb{R}$, then $(p-a,\ p+a)$ is a neigbourhood of p for all real $a > 0$. Let q be another distinct point in $\mathbb{R}$, we will show that $\mathbb{R}$ is Hausdorff.

We let $\epsilon := |p-q| > 0$, then $(p - \frac{1}{2}\epsilon,\ p + \frac{1}{2}\epsilon)$ and $(q - \frac{1}{2}\epsilon,\ q + \frac{1}{2}\epsilon)$ are two disjunct opens containing respectively p and q. As p and q were chosen randomly, $\mathbb{R}$ endowed with the Euclidean topology is Hausdorff.

We show that $\mathbb{R}$ is second countable as well. We let $\mathcal{B}$ denote a countable base of $\mathbb{R}$ defined as $\mathcal{B} := \{\ (a, b) \mid a, b \in \mathbb{Q},\ a < b\}$, i.e. $\mathcal{B}$ is the set of open intervals in $\mathbb{R}$ with rational endpoints. Now, as $\mathbb{Q}$ is countable, we see that $\mathcal{B}$ forms a countable basis.

The following property is distinctive of manifolds. Afterwards we will give the definition of a manifold.

Definition 2.3. A topological space X is *locally Euclidean of dimension* n if every point $p \in X$ has a neighbourhood U such that a homeomorphism φ from U onto an open subset of $\mathbb{R}^n$ exists. Such a pair (U, φ) is called a *chart*. A chart (U, φ) is said to be centered at a point $p \in X$ if φ maps p onto 0, i.e. $\varphi(p) = 0$.

Definition 2.4. A *manifold* is a topological space X that is Hausdorff, second countable and moreover, locally Euclidean. The manifold is then said to be of dimension n if it is locally Euclidean of dimension n.

As Lie groups are smooth manifolds, the next subsection will address smoothness of manifolds. However, before we do this, we conclude this subsection by showing an interesting property of the image of the intersection of two charts, which serves an introductory purpose for the next subsection.

Remark 2.5. Let X be a manifold that has the two charts (U, φ) and (V, ψ). As $U \cap V$ is open in U and the homeomorphism $\varphi : U \to \mathbb{R}^n$ maps onto an open subset of $\mathbb{R}^n$, the image $\varphi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Alike, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

2.2 Smoothness

As one may know, a function is said to be *smooth* or C^∞ if its partial derivatives to any order exist. For smooth manifolds however, we require that the homeomorphisms of its charts are more than just smooth; the smoothness needs to be "compatible" with the other charts as well. The next definition will make this precise.

Definition 2.6. Two charts (U, φ) and (V, ψ) of a manifold are said to be C^∞*-compatible* if both the maps

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V) \quad \text{and} \quad \psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

are smooth.

Remark 2.7. Note that if the intersection of U and V in above definition is empty, the charts (U, φ) and (V, ψ) are automatically C^∞-compatible.

In this thesis, we will only be talking about C^∞-compatible charts. Hence in order to make everything more readable, we will omit the "C^∞" part and just speak of compatible charts. We will now define an atlas, a collection of compatible charts, after which the definition of a smooth manifold will follow shortly.

Definition 2.8. An *atlas* on a local Euclidean space X is a collection $\{(U_\alpha, \phi_{U_\alpha})\}$ of pairwise compatible charts which covers all of X, i.e. such that $\bigcup_\alpha U_\alpha = X$.

An atlas can be *maximal*, which means that the atlas is not contained in any other larger atlas. In order to be precise, let $\mathcal{A}$ be a maximal atlas on the local Euclidean space X. If $\mathcal{B}$ is another atlas on X which contains $\mathcal{A}$, then $\mathcal{A} = \mathcal{B}$.

Finally, we give our definition of a smooth manifold.

Definition 2.9. A *smooth manifold* is a manifold X together with a maximal atlas.

It can be shown that any atlas on a locally Euclidean space is contained in some *unique* maximal atlas. This isn't hard to prove, but we find that it falls outside the scope of this thesis, hence we will refer to [1] for a proof. We will use the result however, as it implies that finding any atlas for a manifold is sufficient for it to be smooth.

Lemma 2.10. Any atlas on a locally Euclidian space is contained in a *unique* maximal atlas.

Proof. See [1, Proposition 5.10] for a proof. □

Corollary 2.11. Let X be a manifold and $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on X. Then X is a *smooth manifold.*

Proof. This is a direct consequence of Lemma 2.10. □

We conclude this subsection by providing a useful lemma concerning open subsets of smooth manifolds.

Lemma 2.12. Let X be a smooth manifold and Y be an open subset of X. Then Y is a smooth manifold as well.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be the atlas of X, then we let $\phi_\alpha|_{U_\alpha \cap Y}$ denote the restriction of ϕ_α to the set $U_\alpha \cap Y$. Next we note that $\{U_\alpha \cap Y, \phi_\alpha|_{U_\alpha \cap Y}\}$ then forms an atlas for Y, hence Y is locally Euclidean. Since Y is a subset of X, it inherits its Hausdorffness and second countability as we mentioned earlier. We conclude that Y is a manifold and by Corollary 2.11, we get our result. □

We end this section with some examples of smooth manifolds.

Example 2.13. We start off with some easier examples of smooth manifolds, after which we will take a look at the harder case of the n-sphere.

(a) We look at $\mathbb{R}^n$ even though this is quite the trivial example. It can easily be seen by Example 2.2 that this space is Hausdorff and has a countable basis, hence it is second countable. Since the single chart $\{(\mathbb{R}^n, 1_{\mathbb{R}^n})\}$ is an atlas on $\mathbb{R}^n$, where $1_{\mathbb{R}^n}$ denotes the identity map, we conclude that it is a smooth manifold.

(b) Take a look at the unit circle S^1 defined by $x^2 + y^2 = 1$. This is a subspace of $\mathbb{R}^2$, hence by the preceding example of $\mathbb{R}^n$, we see that it is Hausdorff and second countable. Take a look at the following charts

$$\varphi_{left}(x, y) = y, \ \varphi_{right}(x, y) = y$$
$$\varphi_{top}(x, y) = x, \ \varphi_{bottom}(x, y) = x$$

which each cover half of the circle, leaving out the endpoints. Our goal is to show that these charts form an atlas for S^1. We start by proving the compatibility of the top and right chart, whose domain overlap in the quarter where x and y are positive. Both the transition maps $\varphi_{right} \circ \varphi_{top}^{-1}$ and $\varphi_{top} \circ \varphi_{right}^{-1}$ are given by

$$a \mapsto \sqrt{1 - a^2}$$

which we see is a map $(0, 1) \to (0, 1)$. It follows that this map is smooth. The rest of the charts are pairwise compatible in the same manner; we conclude that we have found an atlas for S^1, hence it is a smooth manifold.

(c) In this last example, we generalize the previous result of the circle S^1 to the n-sphere S^n with radius 1. As this sphere is a subspace of $\mathbb{R}^{n+1}$, the Hausdorffness and second countability follow. For the atlas, we make use of the *stereographic projection* as our homeomorphism on $U_1 = S^n \setminus \{0, 0, \dots, 0, 1\}$ and $U_2 = S^n \setminus \{0, 0, \dots, 0, -1\}$ onto $\mathbb{R}^n$. We start by making the homeomorphisms explicit.

$$\varphi_1(x_1, x_2, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n)$$
$$\varphi_2(x_1, x_2, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots, x_n)$$

from which we can calculate that the inverse of these homeomorphisms is given by

$$\varphi_1^{-1}(y_1, y_2, \dots, y_n) = \frac{1}{1 + ||y||^2}(2y_1, 2y_2, \dots, ||y||^2 - 1)$$
$$\varphi_2^{-1}(y_1, y_2, \dots, y_n) = \frac{1}{1 + ||y||^2}(2y_1, 2y_2, \dots, 1 - ||y||^2)$$

where $||y|| := \sum_{i=1}^n y_i^2$. We see that these homeomorphisms are smooth. We will now show that the charts are compatible as well. We take a look at the map $\varphi_1 \circ \varphi_2^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, which is made explicit by

$$\varphi_1 \circ \varphi_2^{-1}(y_1, y_2, \dots, y_n) = \left(\frac{1}{1 - \frac{1 - ||y||^2}{1 + ||y||^2}}\right)\left(\frac{1}{1 + ||y||^2}\right)(2y_1, \dots, 2y_n) = \frac{1}{||y||^2}(y_1, \dots, y_n)$$

Then, since the zero vector does not belong to the domain of this map, we see that it is smooth. The other charts are pairwise compatible in the same manner, hence we have found an atlas. We conclude that the n-sphere is a smooth manifold.

3 Lie groups

As the necessary theory concerning smooth manifolds is covered, we can now give the formal definition of a Lie group. After that, we will discuss subgroups and morphisms on Lie groups. We will be using the standard naming conventions of groups in this chapter.

3.1 Definition of a Lie group

Definition 3.1. A *Lie group* is a smooth manifold G equipped with a group structure such that the maps $\mu : G \times G \to G,\ (x, y) \mapsto xy$ and $\imath : G \to G,\ x \mapsto x^{-1}$ are smooth.

Remark 3.2. Some Lie groups may be commutative. The group operation may be denoted additively in this case, i.e. $(x, y) \mapsto\ x + y$. The neutral element will be denoted by 0 in this case.

Example 3.3. We will now give two examples of Lie groups.

(a) We look back at $\mathbb{R}^n$ again, which we already saw to be a smooth manifold in 2.13(a). We let $G = \mathbb{R}^n$ and it is well known that G isn't a group when equipped with standard multiplication because of the 0-element in G. However, when we equip it with addition + and choose 0 as its neutral element, G is a group. Now, as both addition and the inverse map $\imath : a \mapsto -a$ are smooth, G is a Lie group.

(b) We look at $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$. As this is an open subset of $\mathbb{R}$, it is a smooth manifold by Lemma 2.12. When we equip $\mathbb{R}^*$ with scalar multiplication and choose 1 as its neutral element, $\mathbb{R}^*$ becomes a Lie group.

Next, we'll talk about the *product manifold*. Let G_1, G_2 be Lie groups. We can give the product manifold $G := G_1 \times G_2$ the *product group structure*, i.e. let $(x_1, x_2)(y_1, y_2) := (x_1y_1, x_2y_2)$ where $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$. The neutral element of this product manifold is given by $e_G = (e_{G_1}, e_{G_2})$. We will show in the next lemma that this turns the product manifold into a Lie group.

Lemma 3.4. The product manifold $G := G_1 \times G_2$, where G_1 and G_2 are Lie groups, as defined above is a Lie Group.

Proof. We define the multiplication map $\mu : G \times G \to G$ as

$$\mu((x_1, x_2), (y_1, y_2)) := (\mu_1(x_1, y_1), \mu_2(x_2, y_2))$$

from which we see that we can write μ as $[\mu_1 \times \mu_2]((x_1, y_1), (x_2, y_2))$, which in turn gives natural rise to the function composition $\mu = (\mu_1 \times \mu_2) \circ (I_{G_1} \times S \times I_{G_2})$. Here I_{G_1} and I_{G_2} are the identity functions of respectively G_1 and G_2, and S is the switch map, i.e. $S(x, y) = (y, x)$. From this it follows that μ is a composition of smooth functions, hence μ is smooth itself.

Next, the inversion map $\imath$ is given by $\imath := (\imath_1, \imath_2)$, where $\imath_1, \imath_2$ denote the inversion map of respectively G_1 and G_2. Now since both $\imath_1$ and $\imath_2$ are smooth maps, the map $\imath$ itself is seen to be smooth.

Lastly, since G_1 and G_2 were smooth manifolds, the product manifold G is a smooth manifold as well, from which we conclude that G is a Lie group. □

3.2 Morphisms and subgroups

One might have guessed that since morphisms and subgroups play a key role in group theory, they have to be important in Lie theory as well, which is indeed true. We will start off by discussing morphisms.

Morphisms can be made compatible with Lie groups. For this, some small yet intuitive adjustments need to be made, which we will now show.

Definition 3.5. Let G and H be Lie groups, then

(a) A *Lie group homomorphism* is a group homomorphism $\varphi : G \to H$, where φ is a smooth map.

(b) A *Lie group isomorphism* from G to H is a bijective Lie group homomorphism $\varphi : G \to H$, whose inverse φ^{-1} is a Lie group homomorphism as well.

(c) A *Lie group automorphism* on G is a Lie group isomorphism from G onto itself.

Remark 3.6. A Lie group isomorphism is a smooth bijective map between manifolds, whose inverse is smooth as well. From this we see that a Lie group isomorphism is a *diffeomorphism* as well.

We'll now move on to subgroups. The next important lemma shows us that a subgroup of a Lie group only needs to be a smooth manifold to be a Lie group itself.

Lemma 3.7. Let G be a Lie group and let $H \subset G$ be both a subgroup and a smooth manifold, then H is a Lie group.

Proof. Let $\mu_G : G \times G \to G$ be the multiplication map of G. We can restrict this map to H, i.e. we define the multiplication map $\mu_H := \mu|_{H\times H}$. Then, since μ is smooth and $H \times H$ is a smooth submanifold of $G \times G$, so is μ_H as map $H \times H \to G$. Then, since H is a subgroup of G, μ_H maps into the smooth submanifold H of the manifold G, from which it follows that μ_H is smooth as map $H \times H \to H$ as well.

We can define the inverse map $\imath_H := \imath|_H$ in the same manner, from which we obtain our result. □

Corollary 3.8. Let G be a Lie group and H be an open subset of G. If H is a group, then H is a Lie group.

Proof. This follows by combining Lemma 2.12 and Lemma 3.7. □

In order to provide a more clear understanding of the given definitions, we will take a look at the real matrix group.

Example 3.9. Let n be some positive integer. We let $\mathrm{M}(n,\mathbb{R})$ denote the set of $n \times n$-matrices with real values. We can equip $\mathrm{M}(n,\mathbb{R})$ with addition, which makes it easy to see that this set can be identified with $\mathbb{R}^{n^2}$. Then $\mathrm{M}(n,\mathbb{R})$ follows to be a smooth manifold by making use of Example 2.13(a).

Next, we let $A \in \mathrm{M}(n,\mathbb{R})$, then we denote with A_{ij} the value of A in the i-th row and the j-th column, where i, j are positive integers no greater than n. Using this, we can define the coordinate function ρ_{ij} on $\mathrm{M}(n,\mathbb{R})$ as the map $A \mapsto A_{ij}$. Using this, we define the well known *determinant* on $\mathrm{M}(n,\mathbb{R})$

$$\det A = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\rho_{1\sigma(1)}(A)\dots\rho_{n\sigma(n)}(A)$$

where S_n is the permutation group on the set $\{1, 2, \dots, n\}$ and $\mathrm{sgn}(\sigma)$ denotes the sign of a permutation $\sigma \in S_n$. By this definition, we see that the determinant is a polynomial of coordinate functions, hence it is a smooth map.

It is a well-known result that if a matrix $A \in \mathrm{M}(n, \mathbb{R})$ has a nonzero determinant, its inverse A^{-1} exists in $\mathrm{M}(n, \mathbb{R})$. We will denote the set of these matrices by

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{M}(n, \mathbb{R}) \mid \det A \neq 0\}$$

Now when we equip this set with matrix multiplication, we shall show that this makes it a Lie group, which is known as the *general linear group*.

First off, note that $\mathrm{GL}(n, \mathbb{R})$ is the pre-image of the open subset $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ of $\mathbb{R}$ under the det function. Now, as det was continuous, it follows that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathrm{M}(n, \mathbb{R})$. Using this result, it follows that $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold by Lemma 2.12.

Then, for $\mathrm{GL}(n, \mathbb{R})$ to be a Lie group, we need only the multiplication map μ and the inverse map $\imath$ to be smooth. As we said, $\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ was given by standard matrix multiplication, which in terms of the coordinate function becomes

$$\rho_{ij}(\mu(A, B)) = \sum_{k=1}^{n} \rho_{ik}(A)\rho_{kj}(B)$$

from which we see that μ is smooth.

Next, we move on to the inverse map $\imath : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$, for which we need to define the *comatrix* first. We start by defining the *minor* M_{ij}. The action of the minor M_{ij} on A is given by the deletion of the i-th row and j-th column of A, followed by the calculation of the determinant over the remainder of the matrix, see Example A.1.

The *cofactor* C_{ij} of a matrix is obtained by multiplying the minor with $(-1)^{i+j}$, i.e. $C_{ij} := (-1)^{i+j} M_{ij}$. Note that since det is a smooth map, both the minor and the cofactor are smooth. The *comatrix* C of A is then defined as

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

We need the transpose of this comatrix for our inverse map, which is better known as the *adjugate matrix* $Adj(A) := C^T$. Now, Cramer's rule says that the inverse of matrices with a nonzero determinant is given by

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{\det A} Adj(A)$$

Next we define $\imath$ in terms of the coordinate functions

$$\rho_{ij}(\imath(A)) = \frac{1}{\det A} \rho_{ij}(Adj(A)) = \frac{1}{\det A} C_{ji}$$

from which it follows that the inverse map $\imath$ is smooth and we can finally conclude that $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

3.3 Matrices on real vector spaces

Although working with $\mathbb{R}^n$ (or $n \times n$ real matrices) is comfortable, it limits us as well. Luckily, it turns out that we do not need to make a lot of extra effort in order to work with real vector spaces instead. To illustrate this, we will extend the above example to a real vector space.

Example 3.10. Let V be a real vector space of dimension n and let $\mathbf{v} = (v_1, \dots, v_n)$ be an ordered basis of V. With e_i we denote the i-th standard basis vector of $\mathbb{R}^n$. We know that there exists a unique linear isomorphism $\varphi_\mathbf{v}$ from $\mathbb{R}^n$ onto V, which maps the j-th standard basis vector e_j onto v_j.

Now let $\mathbf{w}$ be some second basis of V, we can then define $\varphi_\mathbf{w}$ in the same manner as $\varphi_\mathbf{v}$. Next, we define the linear isomorphism $L := \varphi_\mathbf{v}^{-1}\varphi_\mathbf{w}$ from $\mathbb{R}^n$ onto itself and we note that this is a diffeomorphism as well. It then follows that V must be a smooth manifold and hence we have that $\varphi_\mathbf{v}$ is a diffeomorphism, which is independent of our choice of basis of V.

With $\mathrm{End}(V)$ we denote the set of linear endomorphisms of V, i.e. linear maps from V into itself. We can equip the space $\mathrm{End}(V)$ with multiplication, addition and scalar multiplication, which turns it into a vector space. Let A be some element of $\mathrm{End}(V)$, then with $\mathrm{mat}(A) = \mathrm{mat}_\mathbf{v}(A)$ we will denote the matrix A with respect to the earlier mentioned basis $\mathbf{v}$ of V. This may cause some confusion as $\mathrm{End}(V)$ was a set of linear maps, however as we know from linear algebra, all these linear maps are uniquely represented by some matrix, thus this notation makes sense. An entry A_{ij} of this matrix can then be calculated using the correspondence $Av_j = \sum_{i=1}^n A_{ij} v_i$, for all $1 \leq j \leq n$.

We can equip $\mathrm{M}(n, \mathbb{R})$ from the previous example with entry-wise addition and scalar multiplication, which turns it into a vector space. From this, we can readily verify that mat is actually a linear isomorphism from $\mathrm{End}(V)$ onto $\mathrm{M}(n, \mathbb{R})$. An interesting property of this map is that function composition in $\mathrm{End}(V)$ corresponds to matrix multiplication in $\mathrm{M}(n, \mathbb{R})$, more precisely

$$\mathrm{mat}(A \circ B) = \mathrm{mat}(A)\ \mathrm{mat}(B)$$

for all $A, B \in \mathrm{End}(V)$.

We end this example by showing that the determinant and trace function are independent of our choice of basis for V. For this, we let $\mathbf{w} = (w_1, \dots, w_n)$ again be a second ordered basis of V. Now note that we could write the matrix of $\mathrm{mat}_\mathbf{v}(A)$ as $e_v^{-1} \circ A \circ e_v$ with respect to the standard basis of $\mathbb{R}^n$. Let T denote the matrix with respect to the standard basis of $\mathbb{R}^n$ which corresponds to the linear endomorphism $L = e_v^{-1} \circ e_w \in \mathrm{End}(\mathbb{R}^n)$. Now note the following relation

$$e_v^{-1} A\ e_v = L \circ (e_w^{-1} A\ e_w) \circ L^{-1}$$

from which we see that

$$\mathrm{mat}_\mathbf{v}(A) = T\ \mathrm{mat}_\mathbf{w}(A)\ T^{-1}$$

Then, since both the trace and determinant are invariant under conjugation, we can conclude that

$$\det(\mathrm{mat}_\mathbf{v}(A)) = \det(\mathrm{mat}_\mathbf{w}(A)) \qquad \text{and} \qquad \mathrm{tr}(\mathrm{mat}_\mathbf{v}(A)) = \mathrm{tr}(\mathrm{mat}_\mathbf{w}(A))$$

for all $A \in \mathrm{End}(V)$. The trace and determinant are thus independent of our choice of basis of V, hence there exist unique maps det, tr: $\mathrm{End}(V) \to \mathbb{R}$ such that $\det(A) = \det(\mathrm{mat}A)$ and $\mathrm{tr}(A) = \mathrm{tr}(\mathrm{mat}A)$ for any basis of V.

3.4 Comparing Lie groups at different points

We end this chapter by discussing the comparison of different points on a Lie group, which has an important consequence when looking at subgroups. The following will illustrate this idea of homogeneity.

Let G be a Lie group and x be an arbitrary point of G. We define *left translation* by x as the map $L_x : G \to G$, given by $y \mapsto \mu(x, y) = xy$. This map is bijective as its inverse is given by $L_{x^{-1}}$. Then, since both these maps are smooth, left transition follows to be a diffeomorphism. We define *right translation* $R_x : G \to G$ by $y \mapsto \mu(y, x) = yx$, which is a diffeomorphism in the same manner.

Let a, b be two points belonging to G, we then see that the translation maps $L_{ba^{-1}}$ and $R_{a^{-1}b}$ are diffeomorphisms on G that map a onto b. We can use this in order to compare properties of G at different points. The next lemma illustrates this.

Lemma 3.11. Let G be a Lie group and H a subgroup of G. Let $h \in H$ be a point (in most applications, h will be the neutral element). The following statements are equivalent:

(a) H is a submanifold of G at the point h.

(b) H is a submanifold of G.

Before we provide the proof, we would like to elaborate (a). The statement says that there exists a chart (U, φ) centered at h with U open in H.

Proof. It is obvious that (b) implies (a). Hence we assume (a) and prove that (b) holds as well. Let n denote the dimension of G and m the dimension of H at h. We know that $m \leq n$ holds and moreover, we know that there exists some open U in G and a diffeomorphism φ of U onto $\mathbb{R}^n$ such that $\varphi(h) = 0$. Then, as φ is injective, the following holds

$$\varphi(U \cap H) = \varphi(U) \cap \varphi|_U(H) = \varphi(U) \cap T$$

where T is some m-dimensional space.

Now let k be an arbitrary point in H and define $a := kh^{-1} \in G$. The left translation map L_a is a diffeomorphism on G which maps k onto h. Our goal is to show that H is a submanifold of dimension m at k as well. For this, note that as $a \in H$, the translation L_a maps the subset H bijectively onto itself. Hence we can define $U_k := L_a(U)$, which is an open neighbourhood of k in G. Next, we define the map $\varphi_k := \varphi \circ L_a^{-1}$, which is a diffeomorphism from U_k onto $\varphi(U)$. We then have that the following holds

$$\varphi_k(U_k \cap H) = \varphi_k(L_a U \cap L_a H) = \varphi_k \circ L_a(U \cap H) = \varphi(U \cap H) = \varphi(U) \cap T$$

from which we can finally conclude that H is a submanifold of dimension m at k as well. As k was chosen randomly in H, (b) follows. □

4 Tangent space and vector fields

We would like to discuss *vector fields* on manifolds. Some may know that in $\mathbb{R}^n$, a vector field is represented by a real-valued function which assigns a vector to any point of $\mathbb{R}^n$. However, in order be able to define a vector field on a manifold we first need to define a tangent space, which we will do in the next subsection.

4.1 Tangent space

Roughly speaking, a tangent space of a point p is a vector space that contains the possible "directions" at which one can tangentially pass through p. The elements of this space are then called the tangent vectors. We will now formalize this.

Definition 4.1. Let M be a smooth manifold, then we say that X is a *tangent vector* at a point $p \in M$ if there exists some smooth path $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = X$, where I is some real open interval containing 0.

Using the tangent vector, the definition of the tangent space comes intuitively.

Definition 4.2. When we equip the tangent vectors at a point p in a manifold M with addition and scalar multiplication, it turns into a vector space, which is known as the *tangent space*. This tangent space is then written as $T_p(M)$ or shortly T_pM.

We can take the disjoint union over all the tangent spaces of the points in our manifold M, which is known as the *tangent bundle* $TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM$. It can be shown that TM is a smooth manifold itself (see [5, lemma 4.1] for a proof).

We will end this subsection by defining the differential in the context of tangent spaces.

Definition 4.3. Let $\varphi : M \to N$ be a smooth map of manifolds. Let $x \in M$ be some point, then we define the *differential* of φ at x as the linear map $\mathrm{d}\varphi_x : T_xM \to T_{\varphi(x)}N$. In order to define what the actual map does to a tangent vector, let $X \in T_xM$ be a tangent vector and $f : N \to \mathbb{R}$ be a smooth map, then

$$\mathrm{d}\varphi_x(X)(f) = X(f \circ \varphi)$$

which is why the differential of φ at x is also known as the *pushforward*.

4.2 Vector fields

Since we have dealt with the prerequisite knowledge, we can now give our definition of a vector field on a manifold.

Definition 4.4. A vector field on a manifold M is a map $v : M \to TM$ which assigns a tangent vector to any point in M.

As we are working with smooth manifolds, we will give a notion of smoothness for vector fields as well. For this, let $(U, (x^i))$ be a coordinate chart of a smooth manifold M and let $p \in U$. Let v be a vector field on M, then $v(p) = \sum v^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$, where $\left.\frac{\partial}{\partial x^i}\right|_p$ is a point-derivation at p.

If M has dimension n, this then gives us n maps $v^i : U \to \mathbb{R}$, which are known as the *component functions*. We say that v is smooth on M iff all the component functions v^i are smooth on M. The collection of all smooth vector fields on a manifold M is denoted by $\mathfrak{X}(M)$.

In the rest of this thesis, by G we denote some arbitrary Lie group.

Definition 4.5. Let $v \in \mathfrak{X}(G)$ be a vector field and $f \in C_p^\infty$, we say that v is *left invariant* if

$$\mathrm{d}(L_g)_h \; v(h)(f) = v(gh)(f) \tag{1}$$

for all $g, h \in G$.

Remark 4.6. It becomes apparent from the above definition that the requirement on the differential we saw in Definition 4.3, namely that the differential is to be taken at some point $x \in G$, is redundant when working on a left invariant vector field.

There are some issues with the definition of the pushforward in the context of vector fields as well. To illustrate this, let $\varphi : M \to N$ be a smooth map on manifolds and let v_M be a vector field on M. We would like to identify the pushforward of φ by v_M with some vector field v_N on N. However, if φ is not surjective, there would be no natural way to define the pushforward outside of the image $\varphi(v_M)$. Moreover, if φ is not injective, we may have multiple choices for the pushforward at a certain point.

We are thus required to redefine the differential in the context of vector fields in order to deal with this impreciseness.

Definition 4.7. Let $\varphi : M \to N$ be a smooth map of manifolds. Let v_M be a vector field on M. We can apply the differential of φ element-wise on all of v_M as follows

$$\mathrm{d}(\varphi)v_M = \varphi_* v_M = \{\, \mathrm{d}(\varphi)_v v_M \mid v \in v_M \,\}$$

which we will call the *differential of φ on a vector field*. We can think of this differential $\mathrm{d}(\varphi)v_M$ as a "subset" of the tangent bundle TN.

Using this new notation and omitting f from (1), we see that a vector field v is equivalently left invariant if

$$\mathrm{d}(L_g)\ v(h) = (L_g)_* v(h) = v(gh) \tag{2}$$

for all $g, h \in G$.

We denote the collection of smooth left invariant vector fields by $\mathfrak{X}_L(G)$, and by choosing $h = e$ in the above definition, we see that a left invariant vector field $v \in \mathfrak{X}_L(G)$ is completely determined by its value at the identity, i.e. by $v(e) \in T_eG$. Hence the map $v \mapsto v(e)$ from $\mathfrak{X}_L(G)$ onto T_eG must be injective.

We will end this chapter by proving that this map is surjective as well.

Lemma 4.8. The map $X \mapsto v_X$ is a linear isomorphism from T_eG onto $\mathfrak{X}_L(G)$, its inverse being given by the map $v \mapsto v(e)$.

Proof. Let $X \in T_eG$ and define the vector field v_X on G as $v_X(g) := \mathrm{d}(L_g)_e X$, with $g \in G$. We start by showing that v_X is a smooth map. For this, note that $(g, h) \mapsto L_g(h)$ is smooth as map $G \times G \to G$. Now when we differentiate this map with respect to h at $h = e$ in the direction of $X \in T_eG$, it follows that $\mathrm{d}(L_g)_e X = v_X(g)$ is smooth as a map $G \to TG$. Hence v_X is a smooth vector field on G. By this, it follows that $X \mapsto v_X$ defines a linear map $T_eG \to \mathfrak{X}(G)$.

Our goal is to show that it actually maps into $\mathfrak{X}_L(G)$. For this, fix $X \in T_eG$. We are going to differentiate L_{gh} at e, but before we do this, we note that $L_{gh} = L_g \circ L_h$. We can thus make use of the chain rule

$$\mathrm{d}(L_{gh})_e X = \mathrm{d}(L_g \circ L_h)_e X = \mathrm{d}(L_g)_h \mathrm{d}(L_h)_e X = \mathrm{d}(L_g) v_X(h)$$

Now look back at the definition of v_X with $(gh) \in G$ as input and by combining this with the above, we get

$$v_X(gh) = \mathrm{d}(L_{gh})_e X = \mathrm{d}(L_g) v_X(h)$$

and we see that v_X is left invariant, hence $X \mapsto v_X$ indeed maps into $\mathfrak{X}_L(G)$.

Finally, note that $v_X(e) = X$ and with this we conclude that the map $v \mapsto v(e)$ is not only injective, but also surjective as map from $\mathfrak{X}_L(G)$ onto T_eG. We conclude that this map is a linear isomorphism, its inverse given by $X \mapsto v_X$, hence our result follows. □

5 Integral curves and the exponential map

In this chapter, we will work towards defining the exponential map. This is an important map in Lie theory as we can use it to connect Lie algebras and their corresponding Lie groups. However, before we can define this map, we need to define so-called integral curves.

5.1 Integral curves

In order to get an understanding for integral curves, we will start by defining an integral curve for Cartesian coordinates. After this, we will move on to the case of smooth manifolds.

Definition 5.1. Let v be a n-dimensional vector field with Cartesian coordinates, i.e. $v = (v_1, v_2, \ldots, v_n)$, and let $x(t)$ be a parametric curve with Cartesian coordinates. Then $x(t)$ is an *integral curve* of v if $\dfrac{dx_i}{dt} = v_i(x_1, x_2, \ldots, x_n)$ for all $i \in \{1, \ldots, n\}$. A more convenient way to write this is as $\dfrac{d}{dt}x(t) = x'(t) = v(x(t))$.

Next, we will see that the definition for smooth manifolds comes intuitively from here.

Definition 5.2. Let M be some smooth manifold, v a smooth vector field on M and p some point belonging to M. Let α be some smooth map $\alpha : J \to M$, where J is an open interval of $\mathbb{R}$ that contains some t_0. Then α is an *integral curve* through p if:

(a) $\alpha(t_0) = p$

(b) $\alpha'(t) = v(\alpha(t))$ for all $t \in J$

We say that p is the *initial point* of the integral curve α.

Remark 5.3. In above definition, it would suffice to require M to be a Banach manifold of class C^r with $r \geq 2$. In the same manner, v and α do not to need be smooth, but could be of class C^{r-1}. However, as we only work with smooth manifolds in this thesis, we chose not to bother.

Just like an atlas, an integral curve can be *maximal*. If M is a smooth manifold, the integral curve $\alpha : J \to M$ is said to be maximal if the domain of any other integral curve with the same initial point is a subset of J. We will now provide an example of an integral curve.

Example 5.4. Let $M = \mathbb{R}^3$ be our smooth manifold and let $v(x) := (-x_1, -2x_2, 4x_3)$ be a vector field on M. We would like to find a maximal integral curve α with initial point $p = (1, -1, 2)$. We will show that $\alpha(t) = (e^{-t}, -e^{-2t}, 2e^{4t})$ satisfies this requirement.

First off, note that $\alpha(0) = p$. Next, we obtain by direct calculation that

$$\alpha'(t) = (-e^{-t}, 2e^{-2t}, 8e^{4t}) = v(e^{-t}, -e^{-2t}, 2e^{4t}) = v(\alpha(t))$$

and from this it follows that $\alpha(t)$ is an integral curve with initial point p. Finally, since α has all of $\mathbb{R}$ as domain, we see that it is a maximal integral curve as well.

Remark 5.5. If we had chosen the vector field $v(x) = (-x_1, -2x_2, 4x_3^2)$ in the preceding example, we would have had to find an integral curve that complies with this new third coordinate $v_3(x)$ of the vector field. This coincides with finding a solution to the differential equation

$$y'(t) = 4(y(t))^2, \; y(0) = 2$$

From this, we find that $y(t) := \frac{2}{1-8t}$. However, note that as y is not defined on the whole of $\mathbb{R}$, we would have to pick a smaller open interval as the domain of our integral curve α.

We will now shift our attention towards the tangent space at the identity.

Remark 5.6. Before we continue, we recall v_X from Lemma 4.8, the vector field corresponding to some tangent vector X of the tangent space T_eG. This vector field will have a major role in the rest of this work.

Next we will define a maximal integral curve for a vector field v_X, after which we prove some properties concerning the domain of this curve.

Definition 5.7. Let $X \in T_eG$ and v_X be its corresponding vector field. We define α_X as the *maximal integral curve* of v_X that has e as its initial point.

Lemma 5.8. Let $X \in T_eG$, then the maximal integral curve α_X has the whole of $\mathbb{R}$ as domain. Furthermore, the following condition holds for all $s, t \in \mathbb{R}$:

$$\alpha_X(s+t) = \alpha_X(s)\alpha_X(t)$$

Proof. We let α be some other integral curve of v_X. Now let $g \in G$ and define $\alpha_1(t) := g\alpha(t)$ which, as we will see in a bit, is an integral curve as well. We differentiate α_1 with respect to t, from which we obtain the following

$$\frac{d}{dt}\alpha_1(t) = \frac{d}{dt}(L_g \circ \alpha)(t) = \mathrm{d}(L_g)\frac{d}{dt}\alpha(t) = \mathrm{d}(L_g)v_X(\alpha(t)) = v_X(\alpha_1(t))$$

where we used that v_X is left invariant in the last step. From this, it follows that α_1 is an integral curve of v_X as well.

Next, we let I denote the domain of α_X, fix some point $t_1 \in I$ and put $x_1 = \alpha_X(t_1)$. We then redefine α_1 as follows (equivalent with fixing $g = x_1$ in our initial definition)

$$\alpha_1(t) := x_1\alpha_X(t)$$

which turns α_1 into an integral curve of v_X with initial point x_1 and domain I.

The maximal integral curve of v_X through x_1 is however given by $\alpha_2\colon t \mapsto \alpha_X(t+t_1)$, which has domain $I - t_1$. Then, since α_1 has I as its domain, it follows that $I \subset I - t_1$. This however implies that, as I is not empty, $s + t_1 \in I$ for all $s, t_1 \in I$ and we may finally conclude that $I = \mathbb{R}$.

Fortunately, we just did most of the work necessary to prove the second part of this lemma. First off, we fix some $s \in \mathbb{R}$. Then, as we just saw, the maximum integral curve through s is given by $\alpha_{x_1}\colon t \mapsto \alpha_X(t+s)$. However, we saw that another maximal integral curve through s was given by $\alpha_{x_2} : t \mapsto \alpha_X(s)\alpha_X(t)$ as well. As maximal integrals are unique, it follows that α_{x_1} and α_{x_2} must be the same integral curve, hence our assertion follows. □

We will prove one more property concerning the smoothness of the integral curve α_x as a map, after which we will define the exponential map.

Lemma 5.9. The map $(t, X) \mapsto \alpha_X(t)$ is smooth as map $\mathbb{R} \times T_eG \to G$.

Proof. We use a well known fact of local flows to prove our statement. For this, let Φ_X denote the flow of v_X, then the following local result can be proven: The map $(X, t, x) \mapsto \Phi_X(t, x)$ is smooth (see [4, Lang] for a proof and more on the subject). Now, as $\Phi_X(t, e) = \alpha_X(t)$, it follows that in particular the map $(X, t) \mapsto \Phi_X(t, e) = \alpha_X(t)$ is smooth, as required. □

5.2 Exponential map

Definition 5.10. The *exponential map* $\exp = \exp_G : T_eG \to G$ is defined by

$$\exp(X) := \alpha_X(1)$$

From above definition, we see that the integral curve α_X and the exponential map are closely related. From this, it should not come as too much of a surprise that the next corollary uses the results we obtained in the previous subsection in order to prove some properties of the exponential map.

Corollary 5.11. The exponential map is smooth as map $T_eG \to G$. Furthermore, the following holds for all the $s, t \in \mathbb{R}$ and $X \in T_eG$

(a) $\exp(tX) = \alpha_X(t)$

(b) $\exp(s + t)X = \exp(sX) \exp(tX)$

Proof. The first assertion (the exponential map being smooth) follows directly by using Lemma 5.9 with $t = 1$.

Now for the other two assertions:

(a) Let s be some real number and take a look at the curve $c(t) := \alpha_X(st)$. As $c(0) = \alpha_X(0) = e$, it is an integral curve through e. We can take its derivative with respect to t, from which we obtain

$$\frac{d}{dt}c(t) = \frac{d}{dt}\alpha_X(st) = s\alpha'_X(st) = sv_X(\alpha_X(st)) = v_{sX}(\alpha_X(st)) = v_{sX}(c(t))$$

We see that $c(t)$ is the maximal integral curve of v_{sX} through the point e, and we can thus write $c(t) = \alpha_{sX}(t)$. We put $t = 1$ and by looking back to our initial definition of $c(t)$, we see that

$$\alpha_X(s) = c(1) = \alpha_{sX}(1) = \exp(sX)$$

and as s was chosen randomly, our result follows.

(b) Let s, t be real numbers. We combine (a) and Lemma 5.8 to see that

$$\exp(s + t)X = \alpha_X(s + t) = \alpha_X(s)\alpha_X(t) = \exp(sX) \exp(tX)$$

□

We will look at the Lie group $\mathrm{GL}(V)$ for an example of the exponential map.

Example 5.12. Let V be a finite-dimensional real linear space of dimension n. It was shown in Example 3.10 that $\mathrm{GL}(V)$ is a Lie group. Furthermore, we know that the neutral element is given by $I = I_V$. Using this, we fix some $X \in \mathrm{End}(V)$ and take a look at the map $\gamma : t \mapsto I + tX$. It was shown in Example 3.10 that $\mathrm{GL}(V)$ is open in $\mathrm{End}(V)$, from this it follows that there exists some real open interval J such that the smooth path γ maps into $\mathrm{GL}(V)$ on this interval J. Note that

$$\gamma(0) = I \quad \text{and} \quad \gamma'(0) = X$$

hence X is a tangent vector at $e = I$ of $\mathrm{GL}(V)$ and as X was chosen randomly in $\mathrm{End}(V)$, we conclude that

$$\mathrm{End}(V) \subseteq T_e\mathrm{GL}(V) \tag{3}$$

Moreover, since the dimension of $\mathrm{End}(V)$ is n^2 and the dimension of a tangent space is at most the dimension of its smooth manifold, we see that the sets in (3) must be equal, hence we conclude that

$$T_e\mathrm{GL}(V) = \mathrm{End}(V) \tag{4}$$

Our next goal is to make the left invariant vector field of $\mathrm{GL}(V)$ explicit. For this, let $x \in \mathrm{GL}(V)$ and note that $l_x : \mathrm{GL}(V) \to \mathrm{GL}(V)$ is the restriction of the map $L_x : \mathrm{End}(V) \to \mathrm{End}(V)$ given by $A \mapsto xA$ to $\mathrm{GL}(V)$. Using this in combination with (4), it follows that $\mathrm{d}(l_x)_e = L_x$. Hence we can conclude that for all $X \in \mathrm{End}(V)$, its corresponding vector field v_X is left invariant if $v_X(x) = xX$ holds.

When we look back at our definition of the maximal integral curve α_X, it follows from the preceding that the curve must satisfy the following condition

$$\frac{d}{dt}\alpha_X(t) = v_X(\alpha_X(t)) = \alpha_X(t)X$$

We finish this example by looking at a solution of this equation. We see that the map $t \mapsto e^{tX}$ is a solution which agrees on the initial value of I_V. Since the maximal integral curve is unique, we have that $\alpha_X(t) = e^{tX}$.

We conclude that in the case of $\mathrm{GL}(V)$, the exponential map is given by Euler's exponential function $X \mapsto e^X$, $\mathrm{End}(V) \to \mathrm{End}(V)$.

Next, we define the one-parameter subgroup and after that, we will prove an interesting lemma which provides us with a connection between the tangent space T_eG, a one-parameter subgroup and the exponential map.

Definition 5.13. Let α be a smooth group homomorphism from the real line as additive group to G, i.e. α is a smooth map $(\mathbb{R}, +) \to G$ for which $\alpha(s+t) = \alpha(s)\alpha(t)$ holds for all $s, t \in \mathbb{R}$. Then α is said to be a *one-parameter subgroup* of G.

The naming comes from the fact that the image of α forms a subgroup of G.

Lemma 5.14. For any tangent vector $X \in T_eG$ there exists a *unique* one-parameter subgroup α that has X as its tangent vector at the point $t = 0$. This map is given by $\alpha(t) := \exp tX$.

Our goal is to show that any one-parameter subgroup α with $\alpha'(0) = X$ must be given by $\alpha(t) = \exp tX$, but first we need to show that the map $t \mapsto \exp tX$ is a one-parameter subgroup itself.

Proof. Let $X \in T_eG$. By looking back at Corollary 5.11(b), it follows that $t \mapsto \exp tX$ is a group homomorphism from $(\mathbb{R}, +)$ to G. In Corollary 5.9, it was shown that the map $(t, X) \mapsto \alpha_X(t)$ is smooth. Using this in combination with Corollary 5.11(a), our map $t \mapsto \exp tX$ is smooth and we conclude that it is a one-parameter subgroup.

We let α be some one-parameter subgroup for which $\alpha'(0) = X$. Note that as α is a group homomorphism, it must be that $\alpha(0) = e$. Then, by taking the derivative of $\alpha(t)$ with respect to t, we obtain

$$\frac{d}{dt}\alpha(t) = \frac{d}{ds}\alpha(t+s)\big|_{s=0} = \frac{d}{ds}\alpha(t)\alpha(s)\big|_{s=0} = \mathrm{d}(l_{\alpha(t)})\alpha'(0) = v_X(\alpha(t))$$

from which we see that α is the maximal integral curve of v_X with initial point e, i.e. we have that $\alpha = \alpha_X$. In Corollary 5.11(a) it was shown that $\exp tX = \alpha_X(t)$, from where it finally follows that $\alpha(t) = \exp tX$. $\square$

We end this section by providing a lemma which turns out to be quite important for our next section: the Lie algebra.

Lemma 5.15. Let G, H be Lie groups and $\varphi : G \to H$ be a Lie group homomorphism. Then the following is a *commutative diagram*:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ T_eG & \xrightarrow[\mathrm{d}\varphi_e]{} & T_eH \end{array}$$

Proof. Let $X \in T_eG$. Define $\alpha(t) := \varphi(\exp_G(tX))$, then it can be readily verified that this is a unique one-parameter subgroup of H. If we differentiate α at $t = 0$, we obtain

$$\alpha'(0) = \mathrm{d}\varphi_e(\mathrm{d}(\exp_G)_0)X = d\varphi_e X$$

from which it follows by the previous lemma, that α must also equal $\alpha(t) = \exp_H(t(d\varphi_e X))$. By fixing $t = 1$, we obtain our result. $\square$

6 The adjoint representation and Lie algebra

We could have chosen to give the formal definition of a Lie Algebra and work from there. However, as the title suggests, we have chosen to work towards the definition by means of the adjoint representation. This allows us to use some of the results of the previous section, which is an extra asset as it might help to clarify that chapter.

In this section, we again denote by G a Lie group.

6.1 The adjoint representation

Let $x \in G$, we define the conjugation map $C_x : G \to G$ as $C_x = L_x \circ R_{x^{-1}}$. Since both left and right translation by any element of G are smooth diffeomorphisms on G, so is conjugation. Note that C_x fixes the identity, hence the differential of C_x at e is a linear automorphism of the tangent space T_eG, i.e. $\mathrm{d}(C_x)_e \in \mathrm{GL}(T_eG)$.

Definition 6.1. Let $x \in G$, we then define the map $\mathrm{Ad} : G \to \mathrm{GL}(T_eG)$, by $\mathrm{Ad}(x) := \mathrm{d}(C_x)_e$. The map Ad is said to be the *adjoint representation* of G in T_eG.

In order to get an understanding of the adjoint representation, we return to the example of $\mathrm{GL}(V)$.

Example 6.2. Let $G = \mathrm{GL}(V)$, where V is a real finite-dimensional vector space. As was mentioned in Example 3.10, $\mathrm{End}(V)$ may be identified with $\mathrm{Mat}(V)$. Combining this with the discussion in Example 3.9, it follows that $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$. The tangent space of $\mathrm{GL}(V)$ at $e = I_V$ may thus be identified with $\mathrm{End}(V)$.

Let $x \in G$, we then see that the conjugation map C_x is a restriction of the map $A \mapsto xAx^{-1}$ with $\mathrm{End}(V)$ as its domain (and image), we will denote this map as $\mathcal{C}_x$. We may thus conclude that the adjoint representation is in this case simply given by $\mathrm{Ad}(x) = \mathrm{d}(C_x)_e = \mathcal{C}_x$.

This suggests that the adjoint representation is nothing more than conjugation by an element x on the tangent space of the identity element. Making this precise is a little harder however, but luckily we can make use of the exponential map.

Lemma 6.3. Let $X \in T_eG$ and $x \in G$, then

$$x \exp X \, x^{-1} = \exp(\ \mathrm{Ad}(x)X\ )$$

Proof. We start by noting that conjugation is a Lie group homomorphism as map $G \to G$. We can thus use Lemma 5.15 with $\varphi = C_x$ and $H = G$, from which we see that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{C_x} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ T_eG & \xrightarrow[\mathrm{d}(C_x)_e]{} & T_eG \end{array}$$

By using that $\mathrm{d}(C_x)_e = \mathrm{Ad}$, the result follows by this diagram. □

As Lemma 5.15 requires the map to be a Lie group homomorphism, the following lemma will prove to be useful later on.

Lemma 6.4. The adjoint representation is a Lie group homomorphism as map $G \to \mathrm{GL}(T_eG)$.

Proof. We start by showing that it is a group homomorphism. For this, note that $\mathrm{Ad}(e) = I_{T_eG}$. Next, let $x, y \in G$, then we use that conjugation by xy is the same as conjugating by y first and by x after. Then by the chain rule we have that

$$\mathrm{Ad}(xy) = \mathrm{d}(C_{xy})_e = \mathrm{d}(C_x \circ C_y)_e = \mathrm{d}(C_x)_{C_y(e)} \circ \mathrm{d}(C_y)_e = \mathrm{Ad}(x)\mathrm{Ad}(y)$$

where we used that the identity element is invariant under conjugation.

We need to show that it is a smooth map as well. For this, we use that $(x, y) \mapsto xyx^{-1}$ is smooth as map $G \times G \to G$. If we differentiate this map with respect to y at $y = e$, we see that $x \mapsto \mathrm{Ad}(x)$ is smooth as map $G \to \mathrm{End}(T_eG)$. Then, since $\mathrm{GL}(T_eG)$ is open in $\mathrm{End}(T_eG)$, it follows that the adjoint representation is smooth as map $G \to \mathrm{GL}(T_eG)$ as well. □

Let $I := I_{T_eG}$, then we just saw that $\mathrm{Ad}(e) = I$. Next, as $T_I(\mathrm{Gl}(T_eG)) = \mathrm{End}(T_eG)$, the differential of Ad at e must be a linear map $T_eG \to \mathrm{End}(T_eG)$. Motivated by this, we define the following map.

Definition 6.5. We define *ad* as the linear map $T_eG \to \mathrm{End}(T_eG)$ given by $\mathrm{ad} := \mathrm{d}(\mathrm{Ad})_e$.

The following is a useful consequence of applying the chain rule on the map we just defined. Let $X \in T_eG$, then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp(tX)) = \mathrm{ad}(X)$$

As we mentioned, we did the preceding work in order to be able to use Lemma 5.15 with Ad as linear map.

Lemma 6.6. For all $X \in T_eG$ we have that the following holds:

$$\mathrm{Ad}(\exp X) = e^{\mathrm{ad}\ X}$$

Proof. Using Lemma 6.4, we are able to choose $\varphi = \mathrm{Ad}$ as our Lie group homomorphism in Lemma 5.15 with $H = \mathrm{GL}(T_eG)$. Note that $\mathrm{d}(Ad)_e = \mathrm{ad}$. Then, since $T_eH = T_I(\mathrm{GL}(T_eG)) = \mathrm{End}(T_eG)$ and as $\exp_H$ is simply given by the map $X \mapsto e^X$, we see that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(T_eG) \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle e^{(\cdot)}} \\ T_eG & \xrightarrow[\mathrm{ad}]{} & \mathrm{End}(T_eG) \end{array}$$

from which our lemma follows. □

In our next example, we will start approaching the Lie algebra. Some of the readers may know that the lie bracket of an associative algebra is given by the commutator, i.e. by $[x, y] = xy - yx$. As $\mathrm{GL}(V)$ is an associative algebra (where V is a finite-dimensional real linear space), the next example hints towards this result.

Example 6.7. We denote by V again a finite-dimensional real linear space and we fix some $x \in \mathrm{GL}(V)$. Then, as we saw earlier, the map $\mathrm{Ad}(x) : \mathrm{End}(V) \to \mathrm{End}(V)$ is given by $Y \mapsto xYx^{-1}$. We can substitute x with e^{tX} and we differentiate our expression with respect to t at $t = 0$ to obtain

$$\left.\frac{d}{dt}\right|_{t=0} \left(e^{tX} Y e^{-tX}\right) = XY - YX$$

where the equality followed by the product rule. However, by using the chain rule, we see that

$$(\mathrm{ad}\ X)\ Y = \left.\frac{d}{dt}\right|_{t=0} \left(e^{tX} Y e^{-tX}\right)$$

from which we can conclude that $(\mathrm{ad}\ X)\ Y$ is the so-called *commutator* of X and Y.

6.2 Lie brackets

It is now time to define a Lie bracket on our linear space T_eG.

Definition 6.8. Let $X, Y \in T_eG$, we define the *Lie bracket* $[X, Y] \in T_eG$ as

$$[X, Y] := (\text{ad } X)\ Y$$

When we're discussing multiple groups, we will use an underscore to denote their respective Lie brackets, i.e. the Lie bracket of the Lie group H is then given by $[\cdot, \cdot]_H$.

We start by proving with an easily seen property of the Lie bracket, which is necessary in order to prove the lemma after it.

Corollary 6.9. The map $(X, Y) \mapsto [X, Y]$ is *bilinear* as a map $T_eG \times T_eG \to T_eG$.

Proof. Recall that ad is linear as map $T_eG \to \text{End}(T_eG)$. Using this, the bilinear property of our map follows evidently. □

Lemma 6.10. The Lie bracket is *alternating*, i.e. $[X, X] = 0$. Moreover, this implies that it is *anticommutative* as well, i.e. $[X, Y] = -[Y, X]$.

Proof. Let $Z \in T_eG$, we then have the following for all $s, t \in \mathbb{R}$:

$$\exp(tZ) = \exp((s + t - s)Z) = \exp(sZ)\exp(tZ)\exp(-sZ)$$

where we made use of Lemma 5.11 (b) in the last step. Now, since $\exp(sZ)$ is an element of G, we can apply Lemma 6.3 with $x = \exp(sZ)$. Then

$$\exp(tZ) = \exp(\ \text{Ad}(\exp(sZ))(tZ))$$

We take the derivative of this equality with respect to t at $t = 0$ to obtain

$$Z = \frac{d}{dt}\bigg|_{t=0} \exp(t\ \text{Ad}(\exp(sZ))Z)) = \text{Ad}(\exp(sZ))Z$$

We then take the derivative of this equality with respect to s at $s = 0$, which gives us the following

$$0 = \frac{d}{ds}\bigg|_{s=0} \text{Ad}(\exp(sZ))Z = \text{ad}(Z)\ Z = [Z, Z]$$

hence we have the alternating property.

For the anticommutativity, we let $Z = X + Y$ and use the bilinear property of the Lie bracket to find

$$0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] \tag{5}$$

in which we use that $[X, X] = 0 = [Y, Y]$ and the anticommutativity follows. □

We will show that there exists an interesting connection between the Lie bracket and the differential at the identity of a Lie group homomorphism. This shouldn't come as too much of a surprise however, considering our definition of the Lie bracket.

Lemma 6.11. Let $\varphi : G \to H$ be a homomorphism of Lie groups and $X, Y \in T_eG$, then

$$\mathrm{d}\varphi_e([X,Y]_G) = [\mathrm{d}\varphi_e X, \mathrm{d}\varphi_e Y]_H$$

Proof. Let $x \in G$, we then denote by C_x^G the conjugation by x as action on G. Now note that, since φ is a homomorphism, the following equality holds: $\varphi \circ C_x^G = C_{\varphi(x)}^H \circ \varphi$. We can then take the differential at e of both sides of this equation to obtain the following commuting diagram:

$$\begin{array}{ccc} T_eG & \xrightarrow{\mathrm{d}(\varphi)_e} & T_eH \\ {\scriptstyle \mathrm{Ad}_G(x)}\uparrow & & \uparrow{\scriptstyle \mathrm{Ad}_H(\varphi(x))} \\ T_eG & \xrightarrow[\mathrm{d}(\varphi)_e]{} & T_eH \end{array}$$

We differentiate the result in the direction of $X \in T_eG$ at $x = e$, obtaining the next commuting diagram:

$$\begin{array}{ccc} T_eG & \xrightarrow{\mathrm{d}(\varphi)_e} & T_eH \\ {\scriptstyle \mathrm{ad}_G(X)}\uparrow & & \uparrow{\scriptstyle \mathrm{ad}_H(\mathrm{d}(\varphi)_e X)} \\ T_eG & \xrightarrow[\mathrm{d}(\varphi)_e]{} & T_eH \end{array}$$

Lastly, we fix some $Y \in T_eG$. It then follows by the above diagram that

$$\mathrm{d}(\varphi)_e[X,Y]_G = \mathrm{d}(\varphi)_e((\mathrm{ad}_G\ X)\ Y) = (\mathrm{ad}_H\ (\mathrm{d}(\varphi)_e X))\ \mathrm{d}(\varphi)_e Y = [\mathrm{d}(\varphi)_e X, \mathrm{d}(\varphi)_e Y]_H$$

as we required. □

We prove one final lemma before we give the definition of a Lie algebra. We shall prove, as we will soon see, the final property of a Lie algebra: the *Jacobi identity*.

Lemma 6.12. For all $X, Y, Z \in T_eG$, we have:

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] \tag{6}$$

Proof. We will be making use of Lemma 6.11; we let $\varphi = \mathrm{Ad}$ and $H = \mathrm{GL}(T_eG)$ in this lemma. Then, as we saw earlier, $e_H = I$ and moreover, $T_IH = \mathrm{End}(T_eG)$. Now note that by this choice of H, the following holds for all $A, B \in \mathrm{End}(T_eG)$:

$$[A,B]_H = AB - BA$$

Next, we note that $\mathrm{d}(\varphi)_e = \mathrm{ad}$ and put $[\cdot,\cdot]_G = [\cdot,\cdot]$, then we obtain from the previous lemma that

$$\mathrm{ad}\big((\mathrm{ad}X)\ Y\big) = \mathrm{ad}\big([X,Y]\big) = [\mathrm{ad}X, \mathrm{ad}Y]_H = \mathrm{ad}X\ \mathrm{ad}Y - \mathrm{ad}Y\ \mathrm{ad}X$$

for all $X, Y \in T_eG$. When we apply above relation to $Z \in T_eG$, we get our result. □

6.3 Lie algebra

We will now give the formal definition of a Lie algebra and we will see that our previous definition of the Lie bracket agrees on the properties of a Lie algebra.

Definition 6.13. A *Lie algebra* is a vector space $\mathfrak{g}$ over a field F together with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, which satisfies the following conditions for all $X, Y, Z \in \mathfrak{g}$:

(a) The Lie bracket is *alternating*:

$$[X, X] = 0$$

(b) The Lie bracket satisfies the *Jacobi identity*:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Note that in the general case we do not assume that a product $X \cdot Y$ is defined on our vector space $\mathfrak{g}$, in particular no associative product. We can only assume that there is the Lie bracket, hence don't have that $[X, Y] = XY - YX$ in general.

Remark 6.14. As we did in (5), we can use the bilinear and alternating property together to show that a Lie Algebra is anticommutative as well. Furthermore, if the characteristic of the field F does not equal 2, it can be shown that anticommutativity implies alternativity.

Next, by making use of the anticommutative property, it can be shown that (6) is equivalent to condition (b) in above definition. In the same manner, another equivalent notation is given by

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

where X, Y, Z are elements of the Lie algebra. This particular notation shows that $\mathrm{ad}X$ acts as a derivation.

As we mentioned earlier, we were already working towards an example of a Lie algebra.

Corollary 6.15. The linear space T_eG equipped with the map $(X, Y) \mapsto [X, Y] := (\mathrm{ad}X)\ Y$ as was defined in Definition 6.8, forms a Lie algebra.

Proof. It was shown in Corollary 6.9 that the Lie bracket is bilinear. In Lemma 6.10, we proved the anticommutative property. Lastly, the Jacobi identity follows from Lemma 6.12. □

Example 6.16. We will show that the vector space $\mathfrak{g} = \mathbb{R}^3$ with the cross product $\times$ as its Lie bracket is a Lie algebra. Let $a, b \in \mathbb{R}^3$, then

$$a \times b := \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

from which it is not hard to see that the cross product is a bilinear operation and in the same manner we see that it is alternating.

The Jacobi identity can be obtained by direct calculation. However, we will prove this identity by making use of Lagrange his formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$ with $a, b, c \in \mathbb{R}^3$ and where $(\cdot)$ denotes the inner product. Using this, we see that

$$\begin{aligned} a \times (b \times c) + c \times (a \times b) + b \times (c \times a) = &(a \cdot c)b - (a \cdot b)c \\ &+(c \cdot b)a - (c \cdot a)b \\ &+(b \cdot a)c - (b \cdot c)a \end{aligned}$$

and by using that the inner product is commutative, we obtain the Jacobi identity from this.

We'll end this chapter by giving a notion of homomorphisms on Lie algebras and work out some of the definitions given earlier for the Lie group $\mathbb{R}^n$, which should aid in understanding these definitions.

In order to be as conventional as possible, we'll be using Roman capitals to denote Lie groups and use Gothic lower case letters for Lie algebras.

Definition 6.17. Let $\mathfrak{g}$,$\mathfrak{h}$ be Lie algebras. A *Lie algebra homomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ that is compatible with the Lie bracket, i.e. it satisfies the condition

$$\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{h}}$$

for all $X, Y \in \mathfrak{g}$.

As it will often be the case that we talk about a Lie group and its associated Lie algebra, we'll denote its algebra with the same letter as the group, i.e. a Lie group G his associated Lie algebra is denoted by $\mathfrak{g}$. Now, unless it is stated otherwise, with the Lie algebra of a Lie group G we will denote the linear space T_eG with the "adjoint Lie bracket", as was stated in Corollary 6.15.

Our final chapter will be concerned with the finite representation of a Lie algebra; however, the definition of a representation of a Lie algebra differs slightly from the definition of a representation of a group. Luckily, it follows directly from our previous definition.

Definition 6.18. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

from $\mathfrak{g}$ to the Lie algebra of endomorphisms on some vector space V equipped with the commutator as its Lie bracket.

The next corollary will provide us with a clear connection between Lie groups, Lie homomorphisms and their associated Lie algebras.

Corollary 6.19. Let $\varphi : G \to H$ be a Lie group homomorphism. The associated tangent map $d(\varphi)_e : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras. Furthermore, the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow[\mathrm{d}(\varphi)_e]{} & \mathfrak{h} \end{array}$$

Proof. The first statement was proven in Lemma 6.11 and the second in Lemma 5.15. □

Example 6.20. We consider $G = \mathbb{R}^n$ as our Lie group. Note that its Lie algebra $\mathfrak{g} = T_0\mathbb{R}^n$ can be associated with $\mathbb{R}^n$ itself. We know that G is a commutative group, hence it is invariant under conjugation. This implies that the adjoint representation $\mathrm{Ad}(x)$ of G in $\mathfrak{g}$ must equal the identity $I_\mathfrak{g}$, for all the $x \in G$. This in turn implies that $\mathrm{ad}(X) = 0$ for all the $X \in \mathfrak{g}$, hence the Lie bracket $[X, Y]$ equals zero as well, independent of our choice of $X, Y \in \mathfrak{g}$.

Now for the one-parameter subgroup and exponential. Let $X \in \mathfrak{g}$, the one-parameter subgroup $\alpha_X : \mathbb{R} \to G$ must be given by $\alpha_X(t) := tX$. From this it follows that its exponential map is given by $\exp(X) = X$, for all $X \in \mathfrak{g}$.

The next part of this example is more substantial, however it involves some topological knowledge. We consider the following Lie group homomorphism $\varphi = (\varphi_1, \ldots, \varphi_n) : \mathbb{R}^n \to \mathbb{T}^n$ given by $\varphi_j(x) = e^{2\pi i x_j}$. Note that φ is a local diffeomorphism and that the kernel of φ is given by $\mathbb{Z}^n$. It then follows from the isomorphism theorem of groups that there exists some group isomorphism $\hat{\varphi} : \mathbb{T}^n \to \mathbb{R}^n/\mathbb{Z}^n$. We can use this isomorphism to transfer the manifold structure of $\mathbb{T}^n$ onto $\mathbb{R}^n/\mathbb{Z}^n$, which turns $H := \mathbb{R}^n/\mathbb{Z}^n$ into a Lie group.

Note that this manifold structure on H must be the unique manifold structure for which the canonical mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is a local diffeomorphism. Hence the map π is a Lie group homomorphism itself. We end this example by examining the associated Lie algebra homomorphism $d(\varphi)_e : \mathfrak{g} \to \mathfrak{h}$.

First off, note that the Lie algebra homomorphism must be bijective, as π was a diffeomorphism. This implies that $d(\varphi)_e$ is actually an isomorphism of Lie algebras, hence we can identify $\mathfrak{h}$ with $\mathfrak{g} = \mathbb{R}^n$ through $d(\varphi)_e$. Using Corollary 6.19, it then follows that the exponential map $\exp_H : \mathbb{R}^n \to H$ of H is given by $\exp_H(X) = \pi(X) = X + \mathbb{Z}^n$, for all the $X \in \mathbb{R}^n$.

7 Special linear Lie algebra and its representation

In the first subsection of this section, we drop the abstraction and study the *special linear Lie algebra* $\mathfrak{sl}_\mathfrak{n}$. After that, we look at the representation of the special linear Lie algebra for $n = 2$, which plays a major role in the study of the representation of Lie algebras as it is often used as a model to study the representation of other Lie algebras. Furthermore, the algebra $\mathfrak{sl}_2(\mathbb{C})$ plays an important role in the study of relativity in physics.

Afterwards, we will generalize the representation we found for $\mathfrak{sl}_2$ to the dimension n. We then end the chapter by discussing the representation of semisimple Lie algebras in general.

7.1 The Lie algebra $\mathfrak{sl}_\mathfrak{n}$

We start this subsection by determining the Lie algebra $\mathfrak{sl}_\mathfrak{n}$ using the results of the previous section. We will first define the special linear group.

Definition 7.1. The *special linear group* $\mathrm{SL}(n, \mathbb{F})$ is the set of $n \times n$-matrices over the field $\mathbb{F}$ with determinant 1 (see Example 3.9 for the definition of the determinant).

If $\mathbb{F}$ equals $\mathbb{R}$ or $\mathbb{C}$ in the preceding definition, it can be readily verified that $\mathrm{SL}(n, \mathbb{F})$ forms a Lie group. Now, as we are only interested in this case, $\mathbb{F}$ will be either $\mathbb{R}$ or $\mathbb{C}$ in the rest of this section. Next, we will take a look at the tangent space at the identity of the special linear group.

Example 7.2. Let $G = \mathrm{SL}(n, \mathbb{F})$. Recall from Definition 4.1 that X is a tangent vector at the identity I_G of the special linear group if there exists some smooth path $\gamma : I \to G$, where I is some real open interval, such that $\gamma(0) = I_G$ and $\gamma'(0) = X$. Now note that

$$\frac{d}{dt}\Big|_{t=0} \det\gamma(t) = \frac{d}{dt}(1) = 0$$

as $\gamma(t)$ maps into G. However, by using Jacobi's formula to elaborate the left side of this equation, we see that

$$\frac{d}{dt}\Big|_{t=0} \det\gamma(t) = \mathrm{tr}\big(\mathrm{Adj}(\gamma(t))\ \gamma'(t)\big)\Big|_{t=0} = \mathrm{tr}\big(\mathrm{Adj}(I)\ X\big) = \mathrm{tr}(X)$$

from which we conclude that the tangent space $T_e\mathrm{SL}(n, \mathbb{F})$ is the space of $n \times n$-matrices that have a zero trace.

Corollary 7.3. The Lie algebra $\mathfrak{sl}_\mathfrak{n}$ is given by the $n \times n$-matrices with a vanishing trace. Its Lie bracket is the commutator.

Proof. The first claim follows directly by the preceding example. We saw in Example 6.7 (combined with Definition 6.8) that the Lie bracket of the Lie algebra of $\mathrm{GL}(V)$ was given by the commutator. Now, since this result holds for subgroups of $\mathrm{GL}(V)$ as well and since $\mathrm{SL}(n, \mathbb{F})$ is a subgroup, our second claim follows. □

7.2 Finite representations of $\mathfrak{sl}_2(\mathbb{C})$

In the rest of this section, we choose the complex numbers $\mathbb{C}$ for our field $\mathbb{F}$. We start this subsection by determining a basis for $\mathfrak{sl}_2(\mathbb{C})$, which we will then use to obtain its finite representation.

Remark 7.4. We could be more general and only require our field $\mathbb{F}$ to be algebraically closed and have characteristic zero. However, since just $\mathfrak{sl}_2(\mathbb{C})$ is used in applications, this does not add much value and, moreover, working in $\mathbb{C}$ makes the rest of this text more comprehensive.

Example 7.5. We look for the 2×2-matrices over $\mathbb{C}$ with a vanishing trace. One can readily verify that these are all given by

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad \text{where } a, b, c \in \mathbb{C}$$

hence we see that the following matrices provide us with a basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

By calculating the Lie bracket of these basis elements, we obtain the following important relations

$$[E, F] = H,\ [H, F] = -2F \text{ and } [H, E] = 2E \tag{7}$$

The above-mentioned relations will be used in the following theory of representations. However, before we can continue, we first need to assume one Lemma concerning such a representation.

Lemma 7.6. Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2$, then H acts diagonalizably on V.

Proof. This is a consequence of $\mathfrak{sl}_2$ having a semisimple-nilpotent decomposition. For a proof, see [6, p. 12]. $\square$

We will be talking a lot about finite-dimensional irreducible representations, hence in order to make everything more readable, we let V denote a finite-dimensional irreducible representation of $\mathfrak{sl}_2$ in the rest of this subsection.

Corollary 7.7. Any irreducible finite-dimensional representation V can be written as a decomposition of eigenspaces

$$V = \bigoplus V_\alpha, \quad \text{where } V_\alpha = \{v \in V \mid Hv = \alpha v\}$$

Proof. This is a direct consequence of Lemma 7.6. $\square$

Definition 7.8. Let $V = \bigoplus V_\alpha$ be a decomposition of V into eigenspaces. We call the eigenspace V_α a *weight space* with *weight* α.

We will now determine how the matrices E and F act on such a weight space.

Lemma 7.9. Let v be an element of a weight space V_α as was described above, then

$$Ev \in V_{\alpha+2} \text{ and } Fv \in V_{\alpha-2}$$

Proof. Using the relations mentioned in (7), we see that

$$H(Ev) = E(Hv) + [H, E]v = E\alpha v + 2Ev = (\alpha + 2)Ev$$

and by looking back at our definition of V_α, we see that Ev is an element of $V_{\alpha+2}$. Through similar calculations for Fv, we see that $Fv \in V_{\alpha-2}$, from which we obtain our result. $\square$

By the preceding lemma, we see that the actions of E and F on a weight space cause the weight of the space to respectively decrease/increase by 2. Later on, we will see that this plays a fundamental role in the representation of Lie algebras in general as well.

The following lemma shows us an important consequence of this result.

Lemma 7.10. The eigenvalues of V form an unbroken string that differs by 2.

Proof. Let α be the weight of some weight space of V. Note that for each $v \in V$, both Ev and Fv have to be elements of V as well. Then by the preceding lemma, we see that the following holds:

$$W = \bigoplus_{n \in \mathbb{Z}} V_{\alpha+2n} \subseteq V$$

Then, since W is a subrepresentation of V, it follows from the irreducibility of V that W and V must be equal. Hence we see that the weights of the weight spaces of V are given by $\{\dots, \alpha - 2,\ \alpha,\ \alpha + 2, \dots\}$ and as these weights represent the eigenvalues of V, we obtain our result. $\square$

Corollary 7.11. Let α be the weight of some weight space of V. Then there exist smallest nonnegative integers N, N' such that the following holds for all the integers $k \geq 0$

$$V_{\alpha+2(N+k)} = 0 = V_{\alpha-2(N'+k)}$$

Proof. In the preceding lemma, we saw that V can be written as the following decomposition of eigenspaces:

$$V = \bigoplus_{n\in\mathbb{Z}} V_{\alpha+2n} \tag{8}$$

Then, since V is finite-dimensional, we can find some smallest nonnegative integer N such that $V_{\alpha+2N} = 0$. Next, we note that for all the nonnegative integers k

$$E^k(V_{\alpha+2N}) = 0 = V_{\alpha+2(N+k)}$$

from which we obtain our result for N. In the same manner, we find our result for N' by replacing the action of E with that of F. $\square$

The following lemma will show us that there exists a basis of V produced solely by using one specific element of V in combination with F.

Lemma 7.12. There exists some $v \in V$ and some non negative integer k such that $\{v, Fv, F^2v, \dots, F^kv\}$ is a basis of V.

Proof. We will start by specifying the element v. Let V_α be the weight space with the largest weight. We let v be some element of V_α and we note that in particular $Ev = 0$.

Let $W := \langle v, Fv, F^2v, \dots \rangle$ be the span created by v and the action of F, we will show that this makes W a subrepresentation of V. We start by noting that if F^jv isn't zero, each of $\{v, Fv, F^2v, \dots\}$ are elements of different weight spaces, i.e. of $\{V_\alpha, V_{\alpha-2}, V_{\alpha-4}, \dots\}$ respectively. In combination with (8), this makes these elements linearly independent. We can thus create a basis of W given by the nonzero F^iv.

Next, we look at the actions of $E,\ F$ and H on W. It is obvious by the relations mentioned in (7) that the actions of F and H will just transfer W into itself. However, the action of E remains unclear, hence we will compute this now.

First, we know that $Ev = 0$; then secondly

$$E(Fv) = F(Ev) + [E, F]v = Hv = \alpha v$$

and continuing for F^2v, we get

$$E(F^2v) = F(EFv) + [E, F]Fv = \alpha Fv + HFv$$

and we can then use Lemma 7.9 on the last term to obtain

$$E(F^2v) = \alpha Fv + (\alpha - 2)Fv = (2\alpha - 2)Fv$$

By continuing in this manner, it can be shown by induction that $E(F^i)v$ is a multiple of $F^{i-1}v$ for all positive integers i. Even though this proof doesn't provide much difficulty, we chose to omit it.

With this, we have shown that W is a subrepresentation of V, which makes them equal by the irreducibility of V. Moreover, by Corollary 7.11 it follows that there exists some smallest nonnegative integer N' such that $V_{\alpha-2N'} = 0$. From this, we see that $F^{N'}v$ must be zero. We then let $k = N' - 1$, and see that $\{v, Fv, F^2v, \ldots, F^kv\}$ is a basis for $W = V$, hence we have our result. □

Although this provides us with an idea of what an irreducible representation V looks like, we can say even more: it turns out that α must be an integer and that k from our preceding lemma equals α. We will show this now.

Theorem 7.13. Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, then there exists some nonnegative integer α such that V can be written as the following decomposition of eigenspaces

$$V = V_{-\alpha} \bigoplus V_{-\alpha+2} \bigoplus \cdots \bigoplus V_{\alpha-2} \bigoplus V_{\alpha} \tag{9}$$

in which all the V_i have dimension one.

Proof. First off, we note that since H has trace zero, its representation in V must have a vanishing trace as well. Next, we note that the eigenvalues of the representation of H were given by the weights of V. From linear algebra, we know that the trace of a matrix equals the sum of its eigenvalues, hence the sum of the weights of the weight spaces of V must equal zero. It then follows by combining Lemma 7.10 and Corollary 7.11 that the α corresponding to the weight space with the highest weight V_α must be an integer and that the eigenvalues have to be given by $\{-\alpha, -\alpha+2, \ldots, \alpha-2, \alpha\}$, from which (9) follows.

Our next goal is to prove that the dimensions of these V_i equal 1. We saw in Lemma 7.12 that there exists a basis of V of dimension $k+1$ for some nonnegative integer k. When we compare that basis to our decomposition in (9), we see that if v belongs to V_α, then $Fv \in V_{\alpha-2}$ and so on. Hence we find that this k equals α, from which it follows that we have a basis of V with dimension $\alpha + 1$. Since our decomposition in (9) contains $\alpha + 1$ eigenspaces and none of these eigenspaces are zero, our result follows. □

We can thus conclude that for every integer $n \geq 1$, there exists an up to isomorphism unique, irreducible representation of dimension n as is defined in (9). We end this subsection by working out some of these representations when we fix the α in (9).

Example 7.14. As was mentioned above, we look at the representations of (9) for fixed α over the field $\mathbb{C}$:

- For $\alpha = 0$, this gives us the trivial representation on $\mathbb{C}$.
- For $\alpha = 1$, we get the standard representation on $\mathbb{C}^2$, which should be clear by looking at Corollary 7.7 with -1 and 1 as eigenvalues.
- The case of $\alpha = 2$ is more interesting: this instance is given by the adjoint representation of H, since ad H takes H to 0, E to $2E$ and F to $-2F$.

We see that the standard representation is given by the case of $\alpha = 2$.

7.3 Finite representations of $\mathfrak{sl}_n(\mathbb{C})$

We will reuse the theory of $\mathfrak{sl}_2$ in order to find irreducible representations for the special linear Lie algebra with dimension n. We start this chapter by defining the set of homogeneous polynomials.

Definition 7.15. Let $V = \mathbb{C}[x_1, x_2, \dots, x_n]$ be a polynomial ring in n variables. Let d be some nonnegative integer. The monomials of degree d are given by the set

$$M_d = \{cx_1^{i_1}x_2^{i_2}\dots x_n^{i_n} \mid i_1 + i_2 + \dots + i_n = d,\ c \in \mathbb{C}^*\}$$

This set forms a basis for the *set of homogeneous polynomials of degree* d, denoted by V_d, which is then given by the span of M_d:

$$V_d = \mathrm{Span}(M_d) = \mathrm{Span}\{x_1^{i_1}x_2^{i_2}\dots x_n^{i_n} \mid i_1 + i_2 + \dots + i_n = d\}$$

Lemma 7.16. The dimension of V_d is given by $\binom{d+n-1}{d}$

Proof. We know that the set $\{x_1^{i_1}x_2^{i_2}\dots x_n^{i_n} \mid i_1+i_2+\dots+i_n = d\}$ is a basis of V_d, hence it suffices to count the elements in this set. We see that we can use the so-called "stars and bars" method from combinatorics, where we have d stars representing the i_j and $n-1$ bars representing the x_i. For example, let $n = 5$, then the monomial $x_1x_2^2x_4^3$ belonging to M_6 would be represented by

$$x_1|x_2x_2||x_4x_4x_4| \ \simeq\ \star|\star\star||\star\star\star|$$

We see that there are $d + n - 1$ objects that need to be arranged, and if we fix the position of the d stars, we have $n-1$ places left for $n-1$ bars. We conclude that the position of the stars determines the entire arrangement, hence we see that we have $\binom{d+n-1}{d}$ possibilities for our arrangement. Then, since each arrangement corresponds to a monomial in our basis, we obtain our result. □

Remark 7.17. By the definition of the binomial, it follows that another way to write the dimension of V_d is given by $\binom{d+n-1}{n-1}$. Another interesting way to see this result is by first fixing the position of the bars instead of the stars in the preceding proof.

By using the preceding, we can create an explicit representation corresponding to the standard representation of $\mathfrak{sl}_2$ which we will afterwards intuitively adapt to the case of dimension n.

Example 7.18. An explicit representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension 3 (hence $\alpha = 2$) is given by the span $V_2 = \langle x_1^2, x_1x_2, x_2^2\rangle$ over $\mathbb{C}$ with

$$E = x_1\frac{\partial}{\partial x_2}$$
$$F = x_2\frac{\partial}{\partial x_1}$$
$$H = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}$$

We can intuitively generalize this representation to the case of dimension n. We let our vector space V be given by the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ and let V_d denote the set of homogeneous polynomials of degree d. Our next goal is to find similar elements like E, F and H. We generalize E and F to dimension n as follows

$$E_{ij} = x_i \frac{\partial}{\partial x_j}, \text{ for } i < j \tag{10}$$

$$E_{ji} = x_j \frac{\partial}{\partial x_i}, \text{ for } i < j \tag{11}$$

where F would resemble the case (11). Note that i, j are elements of $\{1, 2, \ldots, n\}$; then for H we choose

$$H_i = x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}, \text{ for } 1 \leq i < n \tag{12}$$

We are only left with defining H_n. We know by Corollary 7.3 that the matrices in $\mathfrak{sl}_\mathfrak{n}$ have a vanishing trace; using this, we define

$$H_n = -\sum_{i=1}^{n-1} H_i = x_n \frac{\partial}{\partial x_n} - x_1 \frac{\partial}{\partial x_1} \tag{13}$$

Note that the E_{ij} in (10) and (11), and H_i in (12) form a basis of $\mathfrak{sl}_\mathfrak{n}$, just like we saw for $\mathfrak{sl}_2$.

Definition 7.19. We define the representation V_d of $\mathfrak{sl}_\mathfrak{n}$ by E_{ij} and H_i as in (10), (11), (12) and (13). Note that this representation is faithful.

We will now show that this representation is indeed a Lie algebra representation.

Lemma 7.20. The set of homogeneous polynomials of degree d over $\mathbb{C}[x_1, x_2, \ldots, x_n]$ form a representation for $\mathfrak{sl}_\mathfrak{n}$.

Proof. Note that the action of this representation on polynomials of V_d either maps the polynomial to 0, or to another polynomial of degree d. Then, since the Lie bracket on both spaces is given by the commutator, it is not hard to see that the representation is compatible with the Lie bracket by its linearity, from which we get our result. □

Our next goal is to show that the representation we just found is irreducible. However, we first need to define the lexicographical ordering on monomials for this.

Definition 7.21. Let $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ and $x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ be two n-dimensional monomials of degree k, i.e. $i_1 + i_2 + \cdots + i_n = k = j_1 + j_2 + \cdots + j_n$. We denote that one monomial has a higher *lexicographical ordering* by $>^{lex}$ and respectively lower by $<^{lex}$. Then

$$x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} >^{lex} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$$

implies that either $i_1 > j_1$ or

$$i_1 = j_1 \text{ and } x_2^{i_2} \ldots x_n^{i_n} >^{lex} x_2^{j_2} \ldots x_n^{j_n}$$

Lemma 7.22. The representation V_d given in Definition 7.19 is irreducible.

Proof. Let W be some nonzero subrepresentation of V_d. Let $w \in W$, then w is some polynomial of degree d. We can take the monomial with the highest lexicographical ordering of w, which we denote by μ, then

$$\mu = m_0 x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$$

Now note that

$$(E_{n1})^{m_1}(\mu) = m_0(m_1!) x_2^{m_2} \dots x_n^{m_n+m_1}$$

and by continuing for E_{n2} and so on until $E_{n,(n-1)}$, we obtain the monomial

$$m_0(m_1!)\dots(m_{n-1}!)x_n^d$$

Since μ had the highest lexicographical order, we see that by repeating the actions of $(E_{n1})^{m_1}$, $(E_{n2})^{m_2}, \dots$ on w, the other terms of this polynomial must become 0, hence we see that

$$m_0(m_1!)\dots(m_{n-1}!)x_n^d \in W$$

Using this element, we will show that we can generate every monomial of V_d through actions of E_{ij}. Let $\alpha = m_0(m_1!)\dots(m_{n-1}!)$ and $w_d = w_0 x_1^{w_1} x_2^{w_2} \dots x_n^{w_n}$ be some other monomial of degree d, then

$$(E_{1n})^{w_1}(E_{2n})^{w_2}\dots(E_{n-1,n})^{w_{n-1}}(\alpha x_n^d) = \beta w_d \in W \tag{14}$$

for some $\beta \in \mathbb{C}$.

We conclude that we can create all monomials in V_d, from which we see that $W = V_d$, hence we have found our result. □

7.4 Representations of semisimple Lie algebras

We end this thesis by elaborating on the finite-dimensional representations of semisimple Lie algebras in general. A semisimple Lie algebra is the direct sum of simple Lie algebras. A simple Lie algebra is defined as a non-abelian Lie algebra $\mathfrak{g}$ whose only ideals are $\{0\}$ and itself.

Remark 7.23. We require our Lie algebras to be semisimple as the representation of these algebras is completely reducible. Moreover, the Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ acts diagonally on $\mathfrak{g}$.

As we mentioned at the start of this section, $\mathfrak{sl}_2(\mathbb{C})$ is often used as an example to study other representations of Lie algebras, which we will do here as well.

Let $\mathfrak{g}$ be some complex semisimple Lie algebra that we would like to find the irreducible representations of. First off, we need to find something similar to the role of the element H in $\mathfrak{sl}_2$. However, in general, no single element will serve this function of H. Instead, we need to find a maximal abelian subalgebra which acts diagonally, known as the *Cartan subalgebra*. We will denote this subalgebra by $\mathfrak{h}$. For instance, in the case of $\mathfrak{sl}_\mathfrak{n}$, this Cartan subalgebra is given by the span $\mathfrak{h} = \langle H_1, H_2, \dots, H_{n-1} \rangle$.

After we have found the Cartan subalgebra, we can decompose $\mathfrak{g}$ by the action of $\mathfrak{h}$, which we will do as follows: Let $\alpha \in \mathfrak{h}^*$, i.e. a linear function on $\mathfrak{h}$, then we let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x,\ h \in \mathfrak{h}\} \tag{15}$$

and using this, we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \bigoplus \left(\bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \right)$$

where $\mathfrak{h}$ is often denoted by $\mathfrak{g}_0$. Then, since $\mathfrak{g}$ is finite-dimensional, we have a finite collection of these α, which are known as the *roots*; correspondingly, $\mathfrak{g}_\alpha$ is known as a *root space*. We let R be the set of roots of these root spaces, i.e. $R = \{\alpha \mid \mathfrak{g}_\alpha \neq \emptyset\}$.

If we look at the irreducible representations of $\mathfrak{g}$, it should be clear that we can decompose a representation V in the same manner, i.e. we can write $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, where $V_\alpha = \{v \in V \mid hv = \alpha(h)v,\ h \in \mathfrak{h}\}$. With $P(V)$ we denote the set of weights of a representation V. We will now look at the action of a root space on a weight space.

Lemma 7.24. Let $\mathfrak{g}_\alpha$ and V_β be defined as above, then

$$\mathfrak{g}_\alpha V_\beta = V_{\alpha+\beta}$$

Proof. Let $h \in \mathfrak{h}$, $v \in V_\beta$ and x be some element of $\mathfrak{g}_\alpha$, we then have the following equality:

$$h(xv) = [h,x]v + x(hv) = \alpha(h)(xv) + \beta(h)(xv)$$

from which our result follows. □

Note that the preceding is very similar to what we saw earlier in $\mathfrak{sl}_2$. However, in the case of $\mathfrak{sl}_2$, the weights of the weight spaces differed by 2, but it should be clear that in the general case, we don't have such a value. In order to deal with this, we will look at a basis for the root system R. We introduce this idea by taking a closer look at the root system of $\mathfrak{sl}_\mathfrak{n}$.

Example 7.25. Let $\mathfrak{g} = \mathfrak{sl}_\mathfrak{n}$. As we mentioned earlier, a basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is given by $\{H_1, H_2, \dots, H_{n-1}\}$.

We will show that we can use the explicit representation of homogeneous polynomials as was given in Definition 7.19 in order to make the root system of $\mathfrak{g}$ explicit. By looking back at the definition of the root system we saw in (15), it follows that it suffices to calculate the Lie bracket of H_i and E_{jk} with $1 \leq i < n$, $j \neq k$ and $1 \leq j,k \leq n$, then

$$\begin{aligned}
[H_i, E_{jk}] &= [x_i \frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}}, x_j \frac{\partial}{\partial x_k}] && (16)\\
&= \delta_{ij} x_i \frac{\partial}{\partial x_k} - \delta_{i+1,j} x_{i+1} \frac{\partial}{\partial x_k} - \delta_{ik} x_j \frac{\partial}{\partial x_i} + \delta_{i+1,k} x_j \frac{\partial}{\partial x_{i+1}} && (17)\\
&= \delta_{ij} E_{ik} - \delta_{i+1,j} E_{i+1,k} - \delta_{ik} E_{ji} + \delta_{i+1,k} E_{j,i+1} && (18)\\
&= (\delta_{ij} - \delta_{i+1,j} - \delta_{ik} + \delta_{i+1,k}) E_{jk} && (19)
\end{aligned}$$

The last equality is obtained since the Kronecker delta δ becomes zero if its variables aren't equal. Then, by using that $j \neq k$, we obtain the following result

$$[H_i, E_{jk}] = E_{jk} \begin{cases} 2 & \text{for } i = j = k-1 \\ 1 & \text{for } i = j \neq k-1 \quad \text{or} \quad k = i+1 \neq j+1 \\ -1 & \text{for } i = j-1 \neq k \quad \text{or} \quad k = i \neq j-1 \\ -2 & \text{for } i = j-1 = k \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

Next, we note that $E_{jk} = [E_{j,j+1}, [E_{j+1,j+2}, [\dots, [E_{k-2,k-1}, E_{k-1,k}] \dots]$ for $j < k$. In a similar manner we have that $E_{jk} = [E_{j,j-1}, [E_{j-1,j-2}, [\dots, [E_{k+2,k+1}, E_{k+1,k}] \dots]$ for $j > k$. From this, it is straightforward to start by calculating the roots of $E_{j,j+1}$ and $E_{j,j-1}$. However, by making use of the previous calculation we saw in (19), it follows that

$$\begin{aligned} [H_i, E_{j,j+1}] &= [x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}, x_j \frac{\partial}{\partial x_{j+1}}] \\ &= (2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}) E_{j,j+1} \end{aligned}$$

and since

$$\begin{aligned} [H_i, E_{j+1,j}] &= [x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}, x_{j+1} \frac{\partial}{\partial x_j}] \\ &= -(2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}) E_{j+1,j} \end{aligned}$$

we see that the root of $E_{j,j+1}$ is the negation of the root of $E_{j+1,j}$. From this, we see that all the roots are symmetric around 0.

Now, let α_i be the root of $E_{i,i+1}$, then by (20) we have that

$$\alpha_i(H_j) = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

From this, it follows that the root space $\mathfrak{g}_{\alpha_i}$ must consist solely of the element $\{E_{i,i+1}\}$. We will now show that we can form a basis $\triangle = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ of the roots where α_i is defined as above. Let $j < k$, it then follows by the equality $E_{jk} = [E_{j,j+1}, [E_{j+1,j+2}, [\dots, E_{k-1,k}] \dots]$ that the root of E_{jk} is given by $\alpha_j + \alpha_{j+1} + \dots + \alpha_{k-1}$. The root space of E_{jk} for $j > k$ is similarly given by $-(\alpha_{j-1} + \alpha_{j-2} + \dots + \alpha_k) = -(\alpha_k + \alpha_{k+1} + \dots + \alpha_{j-1})$. From this, it follows that a root is either positive or negative with respect to our basis $\triangle$, hence we can speak of positive and negative roots.

Using this, we can determine the weights of our representation V_d. We start by calculating the actions of H_i and E_{ij} on a monomial $\mu := x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$ of degree d. Note that this suffices since the monomials form a basis of V_d, then

$$\begin{aligned} H_i \mu &= (m_i - m_{i+1}) \mu \\ E_{ij} \mu &= m_j x_1^{m_1} \dots x_i^{m_i+1} \dots x_j^{m_j-1} \dots x_n^{m_n} \end{aligned}$$

Next, we note that the monomial x_1^d is maximal since

$$E_{ij}x_1^d = 0 \text{ for all } j > i$$

Then, as the action of any positive root on x_1^d maps it to 0, we see that the weight λ of the weight space containing x_1^d must be the *highest weight*, which is similar to what we saw in $\mathfrak{sl}_2$. Moreover, since any other monomial can be formed solely by actions of E_{jk} with $j > k$ on x_1^d, we see that the weight of every monomial in V_d must be given by $\lambda - \sum_{n_i \in \mathbb{Z}_{\geq 0}} n_i\alpha_i$. Using this, we obtain the following important relation for the weights of V_d

$$P(V_d) \subseteq \Big\{\lambda - \sum_{n_i \in \mathbb{Z}_{\geq 0}} n_i\alpha_i\Big\}$$

The monomial x_1^d from the preceding example is known as the *highest-weight vector.*

It turns out that in the general case for a complex semisimple Lie algebra $\mathfrak{g}$, it is always possible to fix a basis $\triangle = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ of roots of $\mathfrak{g}$ such that any root α can be written as the sum of these roots, i.e.

$$\alpha = \sum_{n_i \in \mathbb{Z}} n_i\alpha_i \text{ for all } \alpha \in R$$

Moreover, either all the $n_i \geq 0$ or all $n_i \leq 0$, hence we have positive and negative roots, and these roots are symmetric around 0 as well.

Alike, a highest weight vector like x_1^d always exists in finite-dimensional irreducible representations V. Similar to what we saw in the preceding example and for $\mathfrak{sl}_2$ as well, we can use this highest weight vector to generate the representation, i.e. if λ denotes the weight of this highest weight vector, we have that $P(V) \subseteq \{\lambda - \sum\limits_{k_i \in \mathbb{Z}_{\geq 0}} k_i\alpha_i\}$.

We end this section by stating the main result of the classification of these representations.

Theorem 7.26 (Theorem of the highest weight). Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra with basis $\{\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_l\}$. Then

1. Every finite-dimensional irreducible representation of $\mathfrak{g}$ has a unique highest weight and moreover, two representations with the same weight are isomorphic.

2. If μ_0 is the highest weight of an irreducible representation, then $\mu_0(\mathfrak{h}_i)^\dagger$ is a nonnegative integer for all $i = 1, 2, \dots, l$.

3. Conversely, if $\mu_0 \in \mathfrak{h}^*$ such that $\mu_0(\mathfrak{h}_i)$ is a nonnegative integer for all $i = 1, 2, \dots, l$, then there exists an irreducible representation of $\mathfrak{g}$ with μ_0 as its highest weight.

The latter weights are known as *dominant integral weights.*

By this theorem, we conclude that there exists a one-to-one correspondence between finite-dimensional irreducible representations of semisimple complex Lie algebras and dominant integral weights. See [7, chapter 9] for more on the subject, including a proof of the Theorem of the highest weight.

†Note that μ_0 is a sum of roots, hence we can talk about the action of μ_0 on $\mathfrak{h}_i$.

8 References

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A Minor

Example A.1. Say that we would like to calculate the minor M_{32} of some 3×3 matrix

$$A = \begin{pmatrix} 3 & 0 & 2 \\ -9 & 7 & -4 \\ 1 & 0 & 6 \end{pmatrix}, \text{ then } M_{32}(A) = \det \begin{pmatrix} 3 & \square & 2 \\ -9 & \square & -4 \\ \square & \square & \square \end{pmatrix} = \det \begin{pmatrix} 3 & 2 \\ -9 & -4 \end{pmatrix} = 6$$