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Bribing in Second Price Auctions:
The Effect on Efficiency and Payoffs

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Abstract

We analyse the effect of two simple forms of collusion on the efficiency and payoffs of a two-bidder, second-price auction for a single good, where both bidders have private and independent valuations. In the first form, the auction is preceded by one round of bribing, i.e. one bidder can try to bribe the other to stay out of the auction. In the unique and continuous equilibria of this model, bidders with a type below a certain threshold, reveal themselves as the amount they offer depends uniquely on their type. All bidders with a type above the threshold, offer the same amount. This can result in an inefficient auction. Due to the bribing, the payoffs of the bidders increase, while the income of the seller decreases, compared to the regular second-price auction payoffs. In the two-rounds bribing model, both bidders get the opportunity to bribe the other in exchange for her commitment to leave the auction. These bribes are offered in turn and the order is determined exogenously. There exists an efficient equilibrium, in which the first bidder offers her valuation minus her surplus. The second bidder either accepts, or rejects and counteroffers the surplus of the first bidder. Due to full revelation of the first bidders type, the auction remains efficient. In this model, there is always one bidder who excludes herself from the auction. Hence, the payoff of the seller is zero. Furthermore, due to information asymmetry, the first bidder's payoff remains the same as in a regular second-price auction, while the second bidder's payoff increases.

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Nomenclature

β	a behavioural strategy
Δ_m	the strategy space
Γ_i	the income of bidder i
$\mathbb{R}_+$	the real positive numbers
$\mathbb{1}_X$	the indicator function for event X
μ	the belief system
$\pi_i(\theta_i)$	the surplus of bidder i given her valuation
θ_i	the valuation bidder i has of an object
$\vec{p^*}$	the best reply
$\vec{p}$	the probability distribution of a strategy
$A(I_i)$	the set of player i's actions possible at a certain information set
$A(x)$	the highest type of bidder 2 that accepts the bribe x of bidder 1
b	the bribing function of bidder 1
b^*	the highest bid offered
$b_2(\cdot\vert x)$	the bribing function of bidder 2
b_i	the amount bidder i offers for an object
h	a history is a possible sequence of actions in an extensive game
H_i	the set of histories after which player i moves
I_i	an information set known to player i
k_i	the highest bid offered by all bidders except for bidder i
p^*	the selling price of an object
p^r	the reserve price of an object
R	the income of the seller
r	the amount bidder 2 counteroffers in the two-rounds bribing model
W_i	the expected payoff of bidder i
W_s	the expected income of the seller
x	the amount bidder 1 offers to bidder 2

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1. Introduction

A serious problem in auctions is collusion among several participants. A simple form of collusion involves a group of cartel members, who meet up before the auction to decide who will participate in the auction. All members report their valuation simultanuously, on which is then based which member will submit what amount on behalf on the whole cartel. However, in reality, these collusive agreements are determined sequentially; the different parties negotiate with each other until an agreement is reached. During the negotiation process, the parties can signal their private valuations to each other. This signalling leads to the question whether it is optimal to fully reveal private information or not. ([6], p. 191)

In this thesis, we examine the effect bribing has on the efficiency and the payoffs of a second-price auction. In a second-price auction, the participant with the highest bid wins the object, but only pays the second-highest amount offered. Whether an auction is efficient or not, depends on who obtains the object. If the participant with the highest valuation of the object always wins the item, the auction is called efficient. If not, the auction is called inefficient. We will examine two different bribing models. The first model is from an article of Eső and Schummer ([1]), and the second model is from an article of Rachmilevitch ([6]). In both models, the setting is a second-price auction for a single object with two participants.

The model of Eső and Schummer is a one-round bribing model. In here, only one of the participants, exogenously chosen, say bidder 1, can offer the other participant, bidder 2, a bribe in exchange for her commitment to leave the auction. As a result, bidder 1 becomes the sole bidder in the auction. Eső and Schummer concluded that bribing leads to an inefficient auction. The model of Rachmilevicth is an extension of this one-round bribing model. Bidder 1 has again the opportunity to offer bidder 2 a bribe to stay out of the auction. Next, bidder 2 cannot only decide whether to accept or reject this offer, but she can also make a counteroffer. Now bidder 1 must choose between accepting or rejecting the offer. If both players reject the bribes, the auction will proceed as a regular second-price auction. In contract to Eső and Schummer, Rachmilevitch found that in a two-rounds bribing model, the auction remains efficient.

We will compare the analysis of both articles and examine the differences between the one-round bribing model where only one participant has bargaining power, and the two-rounds bribing model where both participants can initiate a collusive agreement.

2. Game Theory

Game theory analyses situations in which competition and cooperation occur between several parties. By studying this mathematical discipline, we gain insights in the decision-making of the involved parties, the interaction between them, and the outcome of different strategies. Game theoretic models are applicable in almost all study fields, but particularly in political, economic and social disciplines; examples of these applications are the voting for political candidates, the problem of a fair allocation of taxes and competitive animal behaviour. ([5], p. 1; [4], p. 1)

In game theory, *players* are the parties that make the decisions. Every player has a set of feasible *actions* and is not only affected by her own action, but also by the actions of others. Furthermore, all players have certain *preferences* concerning their action *profile*, the list of a player's possible actions. These preferences depend on the *payoff* these actions yield. ([4], p. 13). A main assumption in game theory is that all players act *rational*: given a set of actions, they choose the best action in line with their preferences. ([4], p. 5) Taking her actions and preferences into account, a player determines a *strategy* to play a game. This strategy states for each decision that must be made, which action she will take. ([5], p. 54)

Henceforth, we assume there are only two players participating in a game and that both are rational. The payoffs of the game can be presented in *normal form*, which is a $m \times n$-bimatrix (A, B), where $m, n \in \mathbb{N}_{>0}$. This implies that (row-)player 1 has m different strategies with payoffs A, and that (column-)player 2 has n different strategies with payoffs B. It follows that both players can choose between different strategies. The set of all possible strategies of a player, is called the *strategy space*. Denoting the probability distribution of the strategy as $\vec{p}$, the strategy space is defined as follows:

$$\Delta_m := \left\{ \vec{p} = (p_1, \dots, p_m) \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^{m} p_i = 1,\ p_i \leq 0 \quad \forall i = 1, \dots, m \right\}.$$

If $\exists k \in [1, m]$ such that $p_k = 1$, strategy $\vec{p}$ of player 1 is called a *pure strategy*; if $\forall i\ p_i \neq 1$, strategy $\vec{p}$ is called a *mixed strategy*. ([5], p. 25-26) Suppose now that player 1 plays $\vec{p}$ and player 2 plays $\vec{q}$. The expected payoff of player 1 is equal to $\vec{p}A\vec{q}$, and the expected payoff of player 2 is $\vec{p}B\vec{q}$. ([5], p. 38)

In general, the optimal strategy of player 1 depends on the strategy of player 2. Therefore, when determining her optimal strategy, a player must consider the actions the other player can choose. She must form a *belief* about the other player's strategy. ([4], p. 21)

Definition 2.1 (Best Reply ([5], p. 38)). *A strategy $\vec{p^*}$ of player 1 is a best reply to a strategy $\vec{q}$ of player 2 in an $m \times n$-bimatrix game (A, B) if*

$$\vec{p^*}A\vec{q} \;\geq\; \vec{p}A\vec{q} \qquad \forall \vec{p} \in \Delta^m.$$

Similarly, $\vec{q^*}$ of player 2 is a best reply to $\vec{p}$ of player 1 if

$$\vec{p^*}B\vec{q} \;\geq\; \vec{p}B\vec{q} \qquad \forall \vec{q} \in \Delta^n.$$

Definition 2.2 (Nash Equilibrium ([5], p. 39)). *A pair of strategies ($\vec{p^*}$, $\vec{q^*}$) in a bimatrix game (A, B) is a Nash equilibrium if $\vec{p^*}$ is a best reply of player 1 to $\vec{q^*}$, and $\vec{q^*}$ is a best reply of player 2 to $\vec{p^*}$. A Nash equilibrium ($\vec{p^*}$, $\vec{q^*}$) is called pure if both $\vec{p^*}$ and $\vec{q^*}$ are pure strategies.*

Furthermore, if $\vec{p^*}A\vec{q} > \vec{p}A\vec{q}$ for all $\vec{p} \in \Delta^m$ and $\vec{p^*}B\vec{q} > \vec{p}B\vec{q}$ for all $\vec{q} \in \Delta^n$ then ($\vec{p^*}$, $\vec{q^*}$) is a strict Nash equilibrium.

To illustrate the theory mentioned above, we will discuss the well-known *Prisoner's Dilemma*. In this particular game, we have two prisoners who committed a major crime together; they are interrogated in separate rooms. There is enough evidence to convict them of a minor crime, but not sufficient to declare them guilty of the major crime, unless one of the suspects witnesses against the other. Thus, both prisoners can either cooperate (C) and stay quiet, of defect (D) and betray their partner. If they cooperate, they will only be convicted of the minor crime and go to prison for one year. If either of them defects, she will be freed to witness, and the other suspect will spend ten years in jail. If both prisoners defect, the punishment is nine years of prison. We can represent this in normal form, which gives us figure 1.

The figure makes clear that for both prisoners it is better to play D. It yields a higher payoff no matter what the other prisoner does; in other words, C is strictly dominated by D. This indicates that strategy D is the best reply for both suspects. Consequently, (D, D) is a Nash equilibrium with payoffs $(-9, -9)$.

Note that both prisoners could obtain a higher payoff, if they would cooperate. Therefore, the Nash equilibrium (D, D) is not *Pareto optimal.* ([5], p. 6; [4], p. 14-15)

Prisoner A \ Prisoner B	Cooperate	Defect
Cooperate	-1 (A), -1 (B)	-10 (A), 0 (B)
Defect	0 (A), -10 (B)	-9 (A), -9 (B)

Figure 1: Prisoner's Dilemma in Normal Form

2.1. Signalling Games

In a two-player signalling game, only player 1 is informed about a certain variable that is relevant to both players; information is *asymmetric*. In general, player 1 will take an action, which will be observed by player 2, who then sequentially takes an action that will affect both players. The asymmetric information can be modelled with a chance node called *Nature*, that assigns a *type* to player 1. Player 2 is unaware of the type of player 1, and only responds to player 1's action. In some cases, the action of player 1 can *signal* her private information. This happens if each type of player 1 takes a different action. In that case player 2 can determine the type of player 1 by observing the latter's action. Equilibria in which this occurs, are called *separating equilibria.* If all types of player 1 choose the same action, the equilibrium is called a *pooling equilibrium.* ([4], p. 331-334)

To clarify, we will discuss an adjusted version of the *Entry Deterrence* game. Suppose a market is monopolized by a single firm, and a new firm wants to enter. The entrant is either *strong* or *weak*, both with probability $p = \frac{1}{2}$. Only the entrant knows her type and she will enter the market independent of whether she is strong of weak. After observing her type, the entrant may decide to remain *unready* (U) or to get *ready* (R) to fight with the incumbent by advertising her products on TV. The incumbent observes the entrant's action and responds by either colluding (C), or fighting (F) by dropping her prices significantly. ([4], p. 332) We can represent this extensive form game[1] as in figure 2, where *Nature* determines the type of the entrant. Note that both the incumbent and the entrant have four strategies, respectively (FF, FC, CF, CC) and (UU, UR, RU, RR). For each strategy of the incumbent, the left action is played after observing ready, and the right action after observing unready. The entrant plays the left action of her strategy after observing weak, and the right action after observing strong.

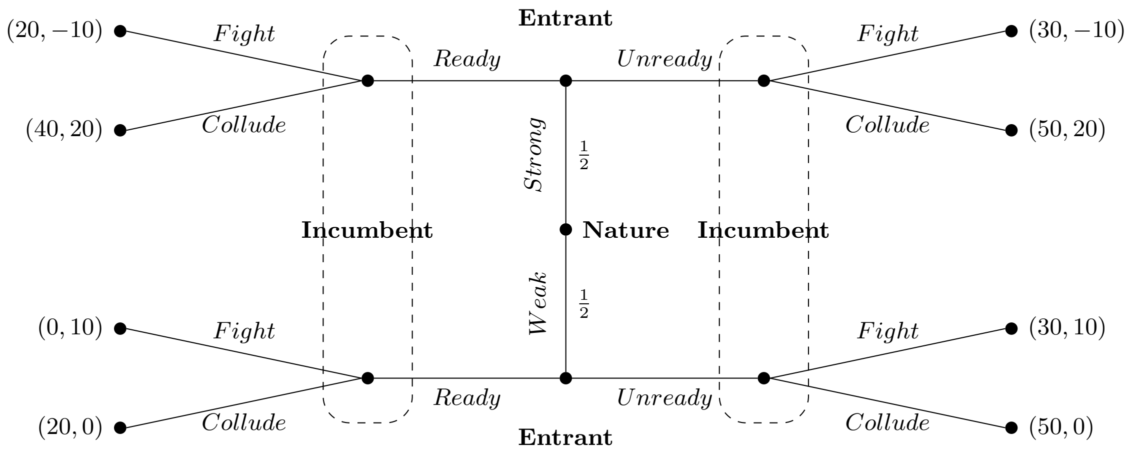

Figure 2: Entry Deterrence with Imperfect Information

We can show that this game has a separating equilibrium, i.e. (UR, CF). Here, a weak entrant plays U, a strong entrant plays R, and the incumbent plays C after observing ready and F after observing unready. If the entrant is weak, she will always prefer to remain unready as her payoff is higher regardless of the action of the incumbent. The incumbent will respond with fighting as this yields a higher payoff. If the entrant chooses ready, she signals the incumbent that her type is strong; if her type would be weak, she would have chosen unready. Knowing this, the incumbent will choose to collude as this

[1]The extensive form is a way of presenting games using a game tree consisting of nodes and edges. Since each node corresponds to a different point in time, this graphical representation is often used to model sequential games, in which players move in turn.

gives her a payoff of 20 rather than -10. Consequently, we have a separating equilibrium (UR, CF), where the entrant signals her type. If the entrant is weak, she chooses unready and the incumbent will choose to fight; if the entrant is strong, she chooses ready and the incumbent will choose to collude.

The other possible equilibrium in the game is (UU, CC). If a strong entrant chooses to remain unready as well, the incumbent will belief that the entrant is strong with probability $\frac{1}{2}$ and weak with probability $\frac{1}{2}$. Her expected payoff of colluding is $(20 + 0) \cdot \frac{1}{2} = 10$ and the expected payoff of fighting is $(-10 + 10) \cdot \frac{1}{2} = 0$. Hence, colluding is the best reply for the incumbent, and (UU, CC) is a pooling equilibrium. ([4], p. 333)

2.2. Weak Sequential Equilibria

A refinement of Nash equilibria, are weak sequential equilibria. To define the latter, we start by introducing other terminology.

Firstly, in an extensive game a *history* is a possible sequence of actions. ([4], p. 153) The set of histories after which player i moves, is called H_i. In case of perfect information, player i knows exactly which history $h \in H_i$ has occurred. However, in case of imperfect information, player i is informed about a partition of H_i. Suppose there is a history set $H_i = \{A, B, C\}$, where A, B and C are possible sequences. Furthermore, the information partition of player i contains two information sets $I_i = \{B\}$ and $I_i = \{A, C\}$. This implies that if she moves after B, she is informed that B has occurred, but if she moves after A or C, she is only informed that B has not occurred. ([4], p. 313)

As mentioned before, players form a belief about the history that has happened. In case of perfect information, an information set only consists of one history. Player i then beliefs that this history has happened with probability 1. However, if an information set contains multiple histories, player i assigns probabilities to each history. We call this set of probabilities the *belief system* μ of the particular information set. ([4], p. 323)

Given that $A(I_i)$ is the set of player i's actions possible at a certain information set ([4], p. 318), we can define her *behavioural strategy* β. This β is a function that assigns a probability distribution to the set of actions in $A(I_i)$, for all of player i's information sets I_i. A combination of a behavioural strategy and a belief system (μ, β) is called an *assessment*. ([4], p. 324-325) We can now define a weak sequential equilibrium as follows:

Definition 2.3 (Weak Sequential Equilibrium ([5], p. 261))**.** *A weak sequential equilibrium in an extensive form game is an assessment such that the following two conditions are satisfied:*

(i) sequential rationality: the players choose optimally given their beliefs;

(ii) consistency: beliefs are consistent with the history at every information set reached.

The first requirement implies that all players must maximize their expected payoffs given their beliefs, and the second requirement implies that for each player at every information set she reaches playing her strategy, her beliefs about the history are correct and updated. ([4], p. 329)

To illustrate, we will discuss another version of the Entry Deterrence game, see figure 3. Only taking into account pure strategies, this game has two Nash equilibria, (U, C) and (O, F). Consider the first equilibrium. Choosing to collude is always the optimal strategy for the incumbent, so the first condition is satisfied. To satisfy the second requirement, the incumbent must belief that the history is U. Hence, $\beta = (U, C)$ combined with the belief μ that U is the history, is a weak sequential equilibrium. Now consider the second equilibrium. As colluding always yields a higher payoff than fighting, fighting is not the optimal choice regardless of the incumbent's belief. Ergo, (O, F) does not satisfy the first condition, meaning there is no assessment with $\beta = (O, F)$, that is a weak sequential equilibrium. ([4], p. 330)

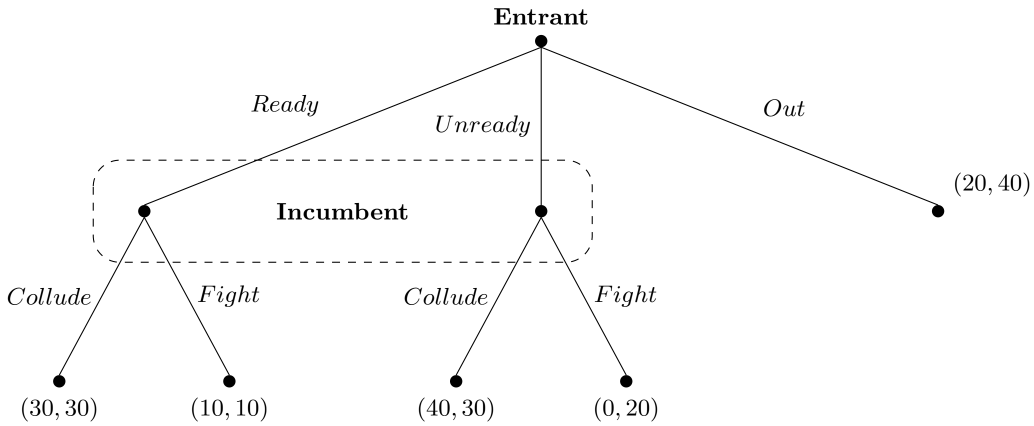

Figure 3: Weak Sequential Equilibria in the Entry Deterrence Game

3. Second-Price Auction

Auctions are market mechanisms that determine to whom and at what price items will be sold. They are particularly helpful when it is hard to set a market price, for instance when selling a painting of Van Gogh. ([2], p. 1) There are several types of auctions; for this thesis however, we only examine the *second-price auction*. First, we will introduce the necessary terminology.

To comprehend the behaviour of the bidders, it is important to understand the difference between a *value*, a *bid*, the *selling price* and the *reserve price*. The value θ_i bidder i has of an object is the maximum amount she is willing to pay. The bid b_i is what she offers for the object. There are three strategies she can follow: *sincere bidding*, where she offers $b_i = \theta_i$, *overbidding*, where she offers $b_i > \theta_i$, and *underbidding*, where she offers $b_i < \theta_i$. The selling price p^* is the final price the highest bidder will actually pay; this depends on the type of auction. In a second-price auction, this is the amount the second-highest bidder has offered. At last, the reserve price p^r is the minimum price for which the seller is willing to sell her object. Consequently, if $p^* < p^r$, the seller may decide to break off the sale. ([2], p. 2-5)

Henceforward, the highest bid will be denoted by b^*. Now suppose bidder i has offered the most, i.e. $b_i = b^*$. Her *income* is now equal to her valuation of the object, thus $\Gamma_i = \theta_i$. Her *surplus* corresponds with the difference between her income and costs, ergo $\pi_i(\theta_i) = \Gamma_i - p^*$. If bidder i has not made the highest bid, both her income and surplus are equal to zero. The *seller's income* is logically equal to the selling price, i.e. $R = p^*$. ([2], p. 6-7)

In the next section, the setting is a particular type of second-price auctions, where the bidders all have *private* and *independent* values. This means that the valuation of an arbitrary bidder i is only known to her, and is independent of the values that the other bidders assign to the object. The latter implies that the values of all bidders are independently distributed. Furthermore, the bidders are *risk neutral*, which means that they play the strategy that maximizes their expected surplus. ([2], p. 17-18) In a second-price auction, the expected surplus of bidder i is equal to her chance of winning the object times the surplus she receives from winning. Thus, given that bidder i wins and bidder j offered the second highest bid, $E[\pi_i(\theta_i)] = P[winning](\theta_i - b_j)$. Moreover, the auction is a *sealed-bid* auction, meaning that the bidders can only simultaneously submit one bid. Ergo, the bidders are unware of the amount the other participants have offered. This gives bidders the incentive to bid their true value. ([2], p. 7, 11)

Theorem 3.1. *In a second-price sealed-bid auction with private, independent values, it is a dominant strategy for every bidder i to bid her true value, $b_i = \theta_i$.*

Proof. Let $k_i = max_{j \neq i}\ b_j$ be the highest bid offered by all bidders except for bidder i. If bidder i considers overbidding, $b_i > \theta_i$, the following situations can happen:

- If $b_i > \theta_i > k_i$, bidder i wins the object with a surplus of $\theta_i - k_i > 0$. However, if she had bid sincerely, she would have obtained the exact same surplus.

- If $b_i > k_i > \theta_i$, bidder i wins the object with the negative surplus of $\theta_i - k_i < 0$. She would have been better off bidding sincerely and receiving a surplus of zero by not winning the object.
- If $k_i > b_i > \theta_i$, bidder i does not win the object. However, by bidding sincerely she would not have obtained the object either, ergo, her surplus would have been the same.

Hence, sincere bidding yields a surplus that is at least as high as the surplus of overbidding.

Now suppose that bidder i considers underbidding, $b_i < \theta_i$. There are again three situations:

- If $k_i < b_i < \theta_i$, bidder i wins the object with a surplus of $\theta_i - k_i > 0$. However, bidding sincerely would have given her the same surplus.
- If $b_i < k_i < \theta_i$, bidder i does not obtain the object and receives a surplus of zero. However, if she had bid her true value, she would have won the object and received a surplus of $\theta_i - k_i > 0$.
- If $b_i < \theta_i < k_i$, bidder i does not win the object. As bidding sincerely would not have assigned the object to her either, her surplus would have been the same.

Hence, sincere bidding yields a surplus that is at least as high as the surplus of underbidding.

We may conclude that sincere bidding never yields a lower surplus than overbidding or underbidding. Thus, bidder i has no incentive to deviate from bidding her true value. Sincere bidding is therefore a dominant strategy. ([3], p. 245; [2], p. 22-23) □

An auction is called *efficient* if the objects are assigned to the bidders whose valuations of the objects are the highest. Given that all bidders submit their true value in a second-price sealed-bid auction, the bidder with the highest valuation will win the object. Hence, this type of auction is efficient. ([2], p. 31)

4. One-Round Bribing Model

Suppose we have a second-price sealed-bid auction for a single object, with two risk-neutral bidders who both have private, independent values. Eső and Schummer ([1]) examined whether this type of auction is immune to a particular form of collusion. The form where one bidder can bribe the other to stay out of the auction.

In the model presented by Eső and Schummer, the private valuations of both bidders, $\theta_1, \theta_2 \in [0, 1]$ are drawn independently according to the same differentiable cumulative distribution function F. They assume that $0 < F'(t) < \infty \ \forall t \in [0, 1]$. Apart from the private valuations, everything is commonly known. After the bidders are informed about their valuations, but before the start of the auction, bidder 1 is given the opportunity to offer bidder 2 a bribe $x \in [0, 1]$. If bidder 2 accepts the offer, she commits to leaving the auction, which is modelled as her bidding 0. If she rejects the offer, the auction will continue as a regular second-price sealed-bid auction.

As proven in section 3, bidders bid their true valuation in a second-price auction. The exception here is if bidder 2 accepts the offer of bidder 1 and must bid 0. In that case, the payoffs for 1 and 2 are respectively $\theta_1 - x$ and x. If bidder 2 rejects the bribe, or bidder 1 initially decided not to bribe, i.e. offer 0, the surpluses are $\max(\theta_1, \theta_2) - \min(\theta_1, \theta_2)$ for the bidder with the highest valuation, and 0 for the other. Note that, for the simplification of the formulas, the reserve price p^r is set at zero. Nonetheless, if $p^r > 0$, the results of Eső and Schummer continue to hold.

As defined in section 2.2, a weak sequential equilibrium is a combination of strategies and beliefs, that satisfies the conditions of sequential rationality and consistency. Eső and Schummer have defined a strategy for bidder 1 as a function that maps types into offers, $b : [0, 1] \rightarrow \mathbb{R}_+$. Since bidder 1 herself can determine how much she offers, she can signal information to bidder 2 about her type. A strategy for bidder 2 specifies the set of types that accept the offer $x \in \mathbb{R}_+$, $A(x) \subseteq [0, 1]$. In particular, her equilibrium strategies are sets of the form $\mathbb{A}(x) = [0, A(x)\rangle$[2], such that any offer $x \in \mathbb{R}_+$ is accepted by a set. Thus, the weak sequential equilibrium of this model is a combination of a pair of strategies $\beta = (x, \mathbb{A}(x))$ and a posterior belief distribution μ. Here, the beliefs are those formed by bidder 2 about the type of bidder 1, after she received the offer $x \in \mathbb{R}_+$.

4.1. Results

Eső and Schummer proved that there is an equilibrium in which the strategy of bidder 1 is continuous. In this equilibrium, the bribing function b is strictly increasing on a certain interval $[0, \bar{\theta}_1)$; on the interval $[\bar{\theta}_1, 1]$ the function is constant. This threshold exists, as for bidders with a high enough type, it suffices to signal that they are *sufficiently strong* to make sure that their bribes are accepted. Furthermore, if bidder 1 offers a bribe $b(\theta_1)$ that

[2] In certain games, there are equilibria that are completely similar, apart from the behaviour of one single type. For example, if all types $\theta_2 < \frac{1}{2}$ accept a certain offer, and all types $\theta_2 > \frac{1}{2}$ reject it, we have the borderline type $\theta_2 = \frac{1}{2}$ who is indifferent between both actions. Therefore, $\forall\ 0 \leq a \leq 1$, the notation $[0, a\rangle$ is introduced; this means either $[0, a)$ or $[0, a]$. Hence, we can have unique equilibria, where only certain types on the interval boundaries might behave differently.

is below $b(\bar{\theta}_1)$, her type θ_1 is revealed. Bidder 2 will only accept if this offer is higher than her payoff in the auction would be, i.e. $\theta_2 - \theta_1 < b(\theta_1)$, which is equivalent to $\theta_2 < \theta_1 + b(\theta_1)$. To make sure that bidder 1 reveals her type and not slightly overbids, the bribing function has to be characterized as follows:

Theorem 4.1. *([1], p. 309)*
Suppose F is log-concave.[3] *In any weak sequential equilibrium in which bribing occurs, if bidder 1's bribing strategy function b is continuous, then it is the unique solution to the following equation satisfying $b(0) = 0$:*

$$b'(\theta_1) = \begin{cases} \dfrac{F'(\theta_1 + b(\theta_1))(\theta_1 - b(\theta_1)}{F(\theta_1 + b(\theta_1)) - F'(\theta_1 + b(\theta_1))(\theta_1 - b(\theta_1)} & \text{if} \quad \theta_1 + b(\theta_1) < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Conversely, there exists a weak sequential equilibrium in which 1's continuous strategy b is described by equation 1, with $b(0) = 0$.

Note that equation 1 tells us that b strictly increases up to a certain $\bar{\theta}_1$, after which it is constant with $\bar{\theta}_1 + b(\bar{\theta}_1) = 1$. Hence, if the valuation of bidder 1 is below $\bar{\theta}_1$, she signals her type to bidder 2 by offering a bribe.

Building further on theorem 4.1, the main result found by Eső and Schummer is that in equilibrium, the bribing function of bidder 1 has to be characterized as in following theorem.

Theorem 4.2. *([1], p. 311)*
Suppose F is log-concave. We have a weak sequential equilibrium if and only if the bribing function b is such that

(i) for some $\hat{\theta}_1 \leq \bar{\theta}_1$,

$$b(\theta_1) = \begin{cases} b^*(\theta_1) & \text{if} \quad \theta_1 < \hat{\theta}_1 \\ \hat{b} \equiv \hat{\theta}_1 - F(\hat{\theta}_1 + b^*(\hat{\theta}_1))(\hat{\theta}_1 + b^*(\hat{\theta}_1)) & \text{if} \quad \theta_1 > \hat{\theta}_1 \end{cases} \tag{2}$$

and $b(\hat{\theta}_1) \in \{b^*(\hat{\theta}_1), \hat{b}\}$, where b^* is the continuous bribing function described in theorem 4.1, and

(ii) $\hat{b} \geq 1 - E\big[\theta_1 \mid \theta_1 \geq \hat{\theta}_1\big]$.

Figure 4 shows us that the functions b and b^* are the same if $\hat{\theta}_1 = \bar{\theta}_1$.

Now given that $b' = \lim_{\theta_1 \uparrow \hat{\theta}_1} b(\theta_1)$ and $b" = \{b(\theta_1) : \theta_1 > \hat{\theta}_1\}$, we can denote bidder 2's strategy as

$$A(b(\theta_1)) = \begin{cases} b^{-1}(b(\theta_1)) + b(\theta_1) = \theta_1 + b(\theta_1) & \text{if} \quad b(\theta_1) < b' \\ 1 & \text{if} \quad b(\theta_1) = b". \end{cases} \tag{3}$$

It follows that inefficiency occurs. Whenever $\theta_2 > \theta_1$ and $b(\theta_1) = b"$, bidder 2 will ac-

[3] F is log-concave if $\frac{d[F/F']}{d\theta} \geq 0$.

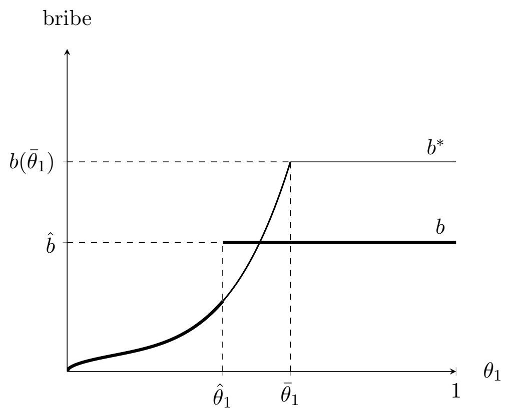

Figure 4: Bribing Function of the One-Round Bribing Model

cept the bribe, even though she would have won the object in a regular second-price auction.

Conclusively, we have found a unique and continuous bribing equilibrium in which inefficiency can occur. The equilibrium is mostly-separating, since every bidder with a type below the threshold $\hat{\theta}_1$ perfectly reveals herself by offering a bribe. All bidders with a type above the threshold offer the same amount. Every bribe is accepted with a positive probability and the highest bribe is always accepted.

4.2. Concrete Effect

To concretize the effect, we will determine the effect on the efficiency and the payoffs, given that θ_1 and θ_2 are independently and identically distributed from $U(0,1)$.

4.2.1. Effect on Efficiency

To decide whether the bribing round has effect on the efficiency of the auction, we will determine the best reply of bidder 2 given the strategy of bidder 1. Given the uniform distribution, we can present the strategy of bidder 1 as follows:

$$b(\theta_1) = \begin{cases} \frac{1}{2}\theta_1, & \text{if} \quad \theta_1 \in \left[0, \frac{2}{3}\right] \\ \frac{1}{3}, & \text{if} \quad \theta_1 \in \left[\frac{2}{3}, 1\right]. \end{cases} \tag{4}$$

Remark. Note that the strategy of bidder 1 is mostly-separating. If $\theta_1 < \frac{2}{3}$, bidder 1 signals her type to bidder 2 by offering her a bribe.

The strategy of bidder 2 is of the form $[0, A(x)\rangle$, i.e. for a given bribe $b(\theta_1)$, she accepts if $\theta_2 < A(x)$. Suppose that $b(\theta_1) < \frac{1}{3}$, that is $\theta_1 = 2x$. bidder 2 only accepts the bribe if her possible payoff of the game is below $b(\theta_1)$. Her payoff $W_2(\theta_1, \theta_2)$ is equal to

$$W_2(\theta_1, \theta_2) = \begin{cases} 0, & \text{if} \quad \theta_2 \leq \theta_1 \\ \theta_2 - \theta_1, & \text{if} \quad \theta_2 \geq \theta_1. \end{cases} \tag{5}$$

Hence, bidder 2 accepts the bribe if $\theta_2 \leq \theta_1 = 2x$, that is $\frac{1}{2}\theta_2 \leq x$. However, she also accepts if $\theta_2 \geq \theta_1 = 2x$ and $\theta_2 - 2x \leq x$; i.e. if $2x \leq \theta_2 \leq 3x$. Thus, if $\theta_2 \geq 3x$ she rejects the offer and if $\theta_2 \leq 3x$ she accepts. Ergo, we have found that $A(x) = 3x$, which implies that $[0, A(x)\rangle = [0, 3x\rangle$. If $b(\theta_1) = \frac{1}{3}$, it follows immediately that $A(x) = 3 \cdot \frac{1}{3} = 1$.

This leads to the conclusion that inefficient outcomes occur; for instance if $\theta_1 = \frac{2}{3}$ and $\theta_2 = 1$, bidder 2 will accept the offer, even though she could have won the object.

4.2.2. Probability of Winning the Auction

Next, we will calculate the probability that bidder 1(2) wins.

We know that bidder 1 wins if bidder 2 accepts her bribe. Bidder 2 can obtain the object if she does not accept the bribe and wins the auction. We will calculate the probability that bidder 2 wins the object and use it to determine the probability that bidder 1 wins.

Bidder 2 does not accept the bribe and wins the auction if $\theta_1 < \theta_2 < \frac{2}{3}$. We can represent this in a graph, see figure 5. If the valuations give a coordinate in area I, bidder 2 wins. This happens with probability $\frac{1}{2} - (\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}) = \frac{4}{9}$. It follows that bidder 1 wins the object with probability $\frac{5}{9}$. Hence, a one-round bribe increases the chances of obtaining the object for the bidder who offers the bribe.

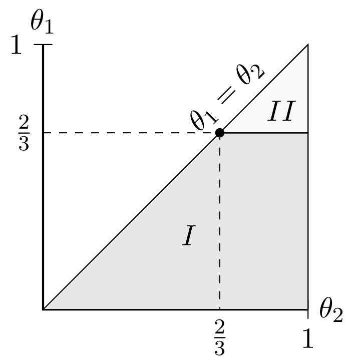

Figure 5: Winning Graph

Remark. Area II shows all the combinations of θ_2 and θ_1, where bidder 1 wins even though the valuation of bidder 2 is higher. We can call this the *inefficiency area.*

4.2.3. Surplus without a Bribing Round

To be able to compare the expected payoffs of the bidders and the expected income of the seller with the situation without a bribing round, we will first compute the surpluses in the normal situation without a bribe.

Due to symmetry, the expected surpluses of both bidders are equal. Given that $\mathbb{1}_X$ is the indicator function for event X, we can calculate the surpluses W_1 and W_2 as follows:

$$
\begin{aligned}
W_1 = W_2 &= \int_0^1 \int_0^1 (\ \theta_2 - \theta_1\ )\ \mathbb{1}(\ \theta_2 \geq \theta_1\ )\, d\theta_2\, d\theta_1 \\
&= \int_0^1 \int_{\theta_1}^1 (\ \theta_2 - \theta_1\ )\, d\theta_2\, d\theta_1 \\
&= \int_0^1 [\ \tfrac{1}{2}\theta_2^2 - \theta_2 \cdot \theta_1\ ]_{\theta_2=\theta_1}^{\theta_2=1}\, d\theta_1 \\
&= \int_0^1 (\ \tfrac{1}{2} - \theta_1 - \tfrac{1}{2}\theta_1^2 + \theta_1^2\ )\, d\theta_1 \\
&= \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6}\ = \tfrac{1}{6}.
\end{aligned}
$$

The expected income of the seller W_s is equivalent to

$$
\begin{aligned}
W_s &= \int_0^1 \int_0^1 \theta_1\ \mathbb{1}(\ \theta_2 \geq \theta_1\ )\, d\theta_2\, d\theta_1 + \int_0^1 \int_0^1 \theta_2\ \mathbb{1}(\ \theta_1 \geq \theta_2\ )\, d\theta_2\, d\theta_1 \\
&= 2 \cdot \int_0^1 \int_{\theta_1}^1 \theta_1\, d\theta_2\, d\theta_1 \\
&= 2 \cdot \int_0^1 [\ \theta_1 \cdot \theta_2\ ]_{\theta_2=\theta_1}^{\theta_2=1}\, d\theta_1 \\
&= 2 \cdot \int_0^1 (\ \theta_1 - \theta_1^2\ )\, d\theta_1 \\
&= 2 \cdot (\ \tfrac{1}{2} - \tfrac{1}{3}\ ) = \tfrac{1}{3}.
\end{aligned}
$$

Hence, we have found that in the case of no bribing, $W_1 = W_2 = \frac{1}{6}$ and $W_s = \frac{1}{3}$.

4.2.4. Effect on Payoff Bidders

Now, suppose that bidder 1 can offer a bribe to bidder 2, as in the model of Eső and Schummer. Bidder 1 only has a positive payoff if her bribe is accepted. This happens if $\theta_1 > \frac{2}{3}$, and if $\theta_2 < \theta_1 < \frac{2}{3}$. In figure 6 this corresponds with area's I and II respectively.

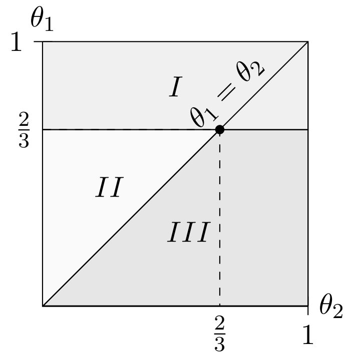

Figure 6: Payoff Graph

Hence, the payoff of bidder 1 is:

$$
\begin{aligned}
W_1 &= \iint_{I} \left(\theta_1 - \tfrac{1}{3} \right) d\theta_2\, d\theta_1 + \iint_{II} \left(\theta_1 - \tfrac{1}{2}\theta_1 \right) d\theta_2\, d\theta_1 \\
&= \int_{\frac{2}{3}}^{1} \int_{0}^{1} \left(\theta_1 - \tfrac{1}{3} \right) d\theta_2\, d\theta_1 + \int_{0}^{\frac{2}{3}} \int_{0}^{\theta_1} \left(\tfrac{1}{2}\theta_1 \right) d\theta_2\, d\theta_1 \\
&= \int_{\frac{2}{3}}^{1} \left(\theta_1 - \tfrac{1}{3} \right) d\theta_1 + \int_{0}^{\frac{2}{3}} \left(\tfrac{1}{2}\theta_1^2 \right) d\theta_1 \\
&= \left[\tfrac{1}{2}\theta_1^2 - \tfrac{1}{3}\theta_1 \right]_{\theta_2=\frac{2}{3}}^{\theta_2=1} + \left[\tfrac{1}{6}\theta_1^3 \right]_{\theta_2=0}^{\theta_2=\frac{2}{3}} \\
&= \left(\tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{2} \cdot \tfrac{4}{9} + \tfrac{1}{3} \cdot \tfrac{4}{9} \right) + \left(\tfrac{1}{6} \cdot \tfrac{8}{27} \right) = \tfrac{35}{162}.
\end{aligned}
$$

Ergo, the expected payoff received by bidder 1 is $W_1 = \frac{35}{162}$. For bidder 2 we proceed in the same manner. There are two cases in which she has a positive payoff. She can either accept the bribe, or reject the bribe but win the auction. Bidder 2 accepts the bribe if $\theta_1 > \frac{2}{3}$ or $\theta_2 < \theta_1 < \frac{2}{3}$. In figure 6 these restrictions correspond respectively with the area's I and II. The expected payoff of this situation is:

$$
\begin{aligned}
W_2^1 &= \iint_{I} \tfrac{1}{3}\, d\theta_2\, d\theta_1 + \iint_{II} \left(\tfrac{1}{2}\theta_1 \right) d\theta_2\, d\theta_1 \\
&= \tfrac{1}{9} + \tfrac{1}{2} \int_{0}^{\frac{2}{3}} \int_{0}^{\theta_1} \theta_1\, d\theta_2\, d\theta_1 \\
&= \tfrac{1}{9} + \tfrac{1}{2} \int_{0}^{\frac{2}{3}} \left[\theta_1 \cdot \theta_2 \right]_{\theta_2=0}^{\theta_2=\theta_1} d\theta_1 \\
&= \tfrac{1}{9} + \tfrac{1}{2} \int_{0}^{\frac{2}{3}} \theta_1^2\, d\theta_1 = \tfrac{13}{81}.
\end{aligned}
$$

The other case is if she rejects the auction, but wins the object, that is if $\theta_1 < \theta_2 < \frac{2}{3}$. This corresponds with area III. The payoff she receives can be calculated as follows:

$$
\begin{aligned}
W_2^2 &= \iint_{III} \left(\theta_2 - \theta_1 \right) d\theta_2\, d\theta_1 \\
&= \int_{0}^{\frac{2}{3}} \int_{\theta_1}^{1} \left(\theta_2 - \theta_1 \right) d\theta_2\, d\theta_1 \\
&= \int_{0}^{\frac{2}{3}} \left[\tfrac{1}{2}\theta_2^2 - \theta_2 \cdot \theta_1 \right]_{\theta_2=\theta_1}^{\theta_2=1} d\theta_1 \\
&= \int_{0}^{\frac{2}{3}} \left(\tfrac{1}{2} - \theta_1 + \tfrac{1}{2}\theta_1^2 \right) d\theta_1 \\
&= \left[\tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_1^2 + \tfrac{1}{6}\theta_1^3 \right]_{\theta_2=0}^{\theta_2=\frac{2}{3}} \\
&= \tfrac{1}{3} - \tfrac{2}{9} + \tfrac{1}{6} \cdot \tfrac{8}{27} = \tfrac{13}{81}.
\end{aligned}
$$

The total expected payoff received by bidder 2 is then $W_2 = W_2^1 + W_2^2 = \frac{26}{81}$.

Note that the expected payoff of bidder 2 increases much more as a result of the bribing round. Without the bribing round, bidder 2 only receives a positive payoff if she wins the object. However, with the introduction of the bribing round, she can also receive a positive payoff by accepting the bribe of bidder 1. Suppose $\theta_1 > \theta_2$ and bidder 2 accepts the bribe, the payoff of bidder 2 now has increased compared to the situation without a bribing round. Bidder 1 still only receives a positive payoff if she wins the auction. Nonetheless, due to the inefficiency area, the expected payoff of bidder 1 still increases.

4.2.5. Effect on Income Seller

As determined in the previous subsection, both bidder's expected payoffs increase if bribing is possible. Consequently, we expect that the income of the seller will decrease. To confirm whether this is true, we will calculate her expected income.

She only profits if the bribe is rejected. This happens if $\theta_1 < \theta_2 < \frac{2}{3}$, i.e. area III in figure 6. We obtain

$$\begin{aligned} W_s &= \int\int_{III} \theta_1 \, d\theta_2 \, d\theta_1 \\ &= \int_0^{\frac{2}{3}} \int_{\theta_1}^{1} \theta_1 \, d\theta_2 \, d\theta_1 \\ &= \int_0^{\frac{2}{3}} \left[\ \theta_1 - \theta_1^2 \ \right]_{\theta_2=\theta_1}^{\theta_2=1} d\theta_1 \\ &= \left[\ \tfrac{1}{2}\theta_1^2 - \tfrac{1}{3}\theta_1^3 \ \right]_{\theta_2=\theta_1}^{\theta_2=1} d\theta_1 \\ &= \tfrac{1}{2} \cdot \tfrac{4}{9} - \tfrac{1}{3} \cdot \tfrac{8}{27} = \tfrac{10}{81}. \end{aligned}$$

Thus the expected income of the seller is $W_s = \frac{10}{81}$, which is much less than the $\frac{1}{3}$ she receives if bribing is not possible. We can conclude that a one-round bribe leads to a larger profit for the bidders, but a loss for the seller.

5. Two-Rounds Bribing Model

Similar to Eső and Schummer, Rachmilevitch ([6]) examined whether a second-price sealed-bid auction is immune to a certain type of collusion. Rachmilevitch extended the game of Eső and Schummer by analysing a two-rounds bribing model.

The setting of Rachmilevitch's model is the same. We have a second-price sealed-bid auction for a single object, and two risk-neutral bidders with private, independent values. These values $\theta_i \in [0,1]$ with $i = 1,2$ are drawn independently according to a log-concave distribution function F_i, of which f_i is a strictly positive and differentiable density function. The reserve price is again set at zero for simplicity, but one can generalize the model and use a reserve price $p^r \in (0,1)$ without changing the results. Furthermore, everything is commonly known, except the private valuations.

Before the auction starts, the two bidders can, in turn, offer each other a bribe; by accepting a bribe a bidder commits herself to staying out of the auction, which is again modelled as bidding 0. More specifically, we have two rounds of bribing. In the first round, bidder 1 can offer bidder 2 a bribe, which 2 can either accept or reject. If she accepts, she leaves the auction, implying that bidder 1 becomes the sole participant and therefore obtains the object. If she rejects, she can make a counteroffer, which bidder 1 must accept or reject. Accepting leads again to an auction with only one participant. Rejecting leads to a regular second-price sealed-bid auction, in which they bid truthfully and the bidder with the highest valuation has a payoff of $\max(\theta_1, \theta_2) - \min(\theta_1, \theta_2)$, and the other a payoff of 0. It is also possible for both bidders to not offer a bribe, this is modelled as offering 0.

A strategy of bidder 1 is defined as the bribing function $b : [0,1] \to \mathbb{R}_+$ and a family of functions $\{a^{\theta_1}\}_{\theta_1} \in [0,1]$, where $a^{\theta_1} : \mathbb{R}_+ \to \{\text{accept, reject}\}$ gives the response of type θ_1 to all possible counteroffers made by bidder 2. A strategy for bidder 2 is defined as a family of functions $\{b_2(\cdot|x)\}_{x \in \mathbb{R}_+}$, where $b_2(\cdot|x) : [0,1] \to \mathbb{R}_+ \cup \{\text{accept}\}$ gives the behaviour of bidder 2 after she is offered the amount x. Here, $r \in \mathbb{R}_+$ means that 2 rejects the bribe and counteroffers the amount r. Henceforth, unless otherwise specified, the term *bribing function* always refers to the bribing function of bidder 1; not to the counter-bribing function of bidder 2.

A weak sequential equilibrium is, as defined earlier, a strategy-belief pair (μ, β), that satisfies the conditions of sequential rationality and consistency. Here, given the beliefs μ_1, β_1 is the best reply of bidder 1 against β_2 in each of 1's information sets; μ_1 are bidder 1's beliefs about the type of bidder 2. Analogously, given μ_2, β_2 is the best reply against β_1 in each of 2's information sets, with μ_2 bidder 2's beliefs about the type of bidder 1. In section 5.1, an equilibrium is a weak sequential equilibrium that satisfies four extra requirements. Given that b is the equilibrium bribing function, for every bribe x offered, $b^{-1}(x)$ is the set of bidder types who make this offer. The four requirements are:

1. If bidder 1 receives an unexpected offer $x \leq 1$ from bidder 2, any belief she might form, gives probability 1 to the event $\{\theta_2 \geq x\}$;
2. The offer 0 is never accepted by a bidder with a strictly positive type;

3. $\forall x$ in the range of b, $b^{-1}(x)$ has a minimum;

4. $\forall x$ in the range of b, $b^{-1}(x)$ is convex.

As discussed in section 4.1, Eső and Schummer proved that in a one-round bribing model inefficiency occurs with a positive probability, since there is no incentive for bidder 1 to fully reveal her type. To clarify, suppose we have an equilibrium in which the bribing function of bidder 1 is strictly increasing, and thus fully revealing. Now assume that a bidder with type $\theta_1 = 1$ pretends to be $1 - \epsilon$, for a small $\epsilon > 0$. Bidder 2 will now still accept the lowered bribe, as she cannot hope to receive more than ϵ in the regular second-price auction. Hence, all types above a certain threshold will offer the same amount. In the two-rounds bribing model of Rachmilevitch, this no longer holds. In the second round, bidder 2 can reject an offer without triggering the regular auction. Ergo, it no longer suffices for bidder 1 to signal that her type is sufficiently strong to make sure her bribe is accepted.

Remark. In the model of Eső and Schummer, bidder 1 would never offer an amount higher than her valuation. However, in the two-rounds model low types of bidder 1 might do so, hoping that their offer will be rejected and lead to a higher counteroffer. This leads to pooling equilibria in the first round, where bidders with a low type all offer the same amount, even though it might lead to a negative payoff. This is because the probability that bidder 2 has a higher type is large enough, implying that 2 will reject the bribe and will counteroffer.

5.1. Results

We start by introducing the lemmas Rachmilevitch used to describe the structure of the efficient equilibria of his model. The first lemma says that if bidder 2 with type θ_2 accepts a certain bribe of bidder 1, then so do all bidders with a type below θ_2; this can be proved using contradiction. The second lemma implies that the bribing function b is strictly increasing and can be proved using lemma 5.1. Lemma 5.3 and 5.4 define $b(\theta_1) \equiv \theta_1 - \pi_1(\theta_1)$ as the bribe offered by bidder 1 with type θ_1, if her bribe is strictly positive. The proofs are based on the previous lemmas and the model of Eső and Schummer.[4]

Lemma 5.1. *(Monotone acceptance). Consider an arbitrary equilibrium and let b be its bribing function. If $b(\theta_1) > 0$ for some θ_1, then bidder 2's equilibrium-response to the offer $b(\theta_1)$ is monotonic: if θ_2 accepts $b(\theta_1)$, then so do all $\theta_2' < \theta_2$.*

Lemma 5.2. *(Revelation by strictly positive bribes). Let b be the bribing function of an efficient equilibrium. Then if θ_1 and θ_1' are two types of bidder 1 such that $b(t) > 0$ for both $t \in \{\theta_1, \theta_1'\}$, then $b(\theta_1) \neq b(\theta_1')$.*

Lemma 5.3. *Let b be a bribing function of an efficient equilibrium. Then there is a*

[4] For the interested reader, the full proofs of all lemmas are gathered in Appendix A. The proofs of propositions 5.1, 5.2 and 5.3, and theorem 5.5 can be found on page 196-197 and in the Appendix of the article of Rachmilevitch [6].

constant $C \geq 0$ such that the following holds for each θ_1:

$$b(\theta_1) > 0 \Rightarrow b(\theta_1) = \theta_1 - \pi_1(\theta_1) - C.$$

Lemma 5.4. *(Characterization of strictly positive bribes). The constant described in lemma 5.3, C, is zero. In particular, in any efficient equilibrium, if type θ_1 offers a strictly positive bribe then her offer is given by:*

$$b(\theta_1) \equiv \theta_1 - \pi_1(\theta_1).$$

In the proof of lemma 5.3, Rachmilevitch denotes the counteroffer of bidder 2 as $r(\theta_1)$. He then states there is a cut-off type $\theta_2 = \theta_1$, who is indifferent between offering this, and accepting the offer of bidder 1: $b(\theta_1) = \theta_1 - r(\theta_1)$. It follows from this equation and lemma 5.4 that in an efficient equilibrium with bribing function b, bidder 2 counteroffers $\pi_1(\theta_1)$ if she rejects the offer of bidder 1. Furthermore, lemma 5.3 and 5.4 imply that in an efficient equilibrium, a bidder with type $\theta_1 = 0$, offers a bribe of zero. At last, the lemmas result in the following corollaries.

Corollary 5.4.1. *Let b^* be the bribing function of an efficient equilibrium. Suppose that b^* is continuous. Then either $b^* = b$ or $b^* = 0$.*

Corollary 5.4.2. *Let b^* be the bribing function of an efficient equilibrium. Suppose that b^* is non-decreasing. Then there is a $t \geq 0$ such that $b^*(\theta_1) = b(\theta_1)$ for $\theta_1 \geq t$ and $b^*(\theta_1) = 0$ otherwise.*

The following proposition is obtained using the corollaries.

Proposition 5.1. *Suppose that $f_1'(0) > 0$. Let b^* be the bribing function of an efficient equilibrium. Suppose that b^* is either continuous or non-decreasing (or both). Then $b^* = b$.*

The idea behind proposition 5.1, is that it eliminates the *no-bribing* equilibria in the second round. If a small offer ϵ is made in the second round, all $\theta_1 \in [0, \epsilon]$ will accept it. Since, $f_1'(0) > 0$, relatively-strong types are more likely in the neighbourhood of zero than relatively-weak ones. Ergo, by counteroffering a relatively low amount, bidder 2 can eliminate opponents who are relatively strong. Hence, bribing will occur in the second round. Due to efficiency, this leads to the fact that bribing must also occur in the first round. Using the lemmas, it follows that b, defined as in lemma 5.4, is the bribing function.

The proposition only applies to equilibria whose bribing function is *minimally well-behaved*, i.e. either continuous or non-decreasing. However in this bribing game there exist equilibria which are not that well-behaved. In fact, there even exist efficient equilibria like that. Consider for instance the bribing function $\tilde{b}$ of equation 6, which is certainly neither continuous, nor non-decreasing. Rachmilevitch proved that there is an efficient equilibrium with this bribing function.

Proposition 5.2. *Suppose that $E[\theta_2] \geq \frac{1}{2}$. Then there exists an efficient equilibrium* *whose bribing function is* $\tilde{b}$*, with* $\tilde{b}$ *defined as follows:*

$$\tilde{b}(\theta_1) = \begin{cases} 0, & if \quad \theta_1 \in \mathbb{Q} \\ b(\theta_1), & if \quad \theta_1 \notin \mathbb{Q}. \end{cases} \tag{6}$$

Furthermore, if the condition $f_1'(0) > 0$ does not hold, the bribing function of a well-behaved equilibrium is not necessarily b.

Proposition 5.3. *Suppose that* F_i *is uniform for each* $i = 1, 2$*. Then a no-bribing equilibrium exists. In it, bidder* 1*'s bribing function is identically zero, bidder* 2 *reciprocates zero with the counter offer zero, and the auction is played non-cooperatively with probability one.*

The following theorem embodies the main result of Rachmilevitch. It states the necessary and sufficient condition for an efficient equilibrium in which the bribing function of bidder 1 is equal to b.

Theorem 5.5. *There exists an efficient equilibrium whose bribing function is* b *if and only if* $E[\theta_2] \geq \frac{1}{2}$.

The condition $E[\theta_2] \geq \frac{1}{2}$ originates from the following lemma:

Lemma 5.6. $E[\theta_2] \geq \frac{1}{2}$*, if and only if* $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$ $\forall \theta_1 \in [0, 1]$.

We will show the necessity of this condition. Suppose that for some θ_1, $\pi_1(\theta_1) > \frac{\theta_1}{2}$. Furthermore, assume this type θ_1 was truthfully revealed in the first round and that $\theta_2 > \theta_1$. Now, bidder 2 has an incentive to deviate from the equilibrium by offering $\pi_1(\theta_1) - \epsilon$ for a small $\epsilon > 0$. Bidder 1 will believe with probability 1 that $\theta_2 \geq \pi_1(\theta_1) - \epsilon$. Hence, she will accept the lower offer if $\pi_1(\theta_1) - \epsilon \geq \theta_1 - (\pi_1(\theta_1) - \epsilon)$, which is true for small $\epsilon > 0$. However, if $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$, the previous condition no longer holds, and bidder 2 has no incentive to deviate from the equilibrium.

The efficient equilibrium is now defined as follows. In the first round, bidder 1 offers $b = \theta_1 - \pi_1(\theta_1)$, and therefore fully reveals her type. In the second round, bidder 2 accepts the offer if $\theta_2 \leq \theta_1$. If she rejects the bribe, she makes the counteroffer $\pi_1(\theta_1)$, which is always accepted by bidder 1. Hence, the equilibrium payoff of bidder 1 is equal to her payoff in a regular second-price auction, independent of the type of bidder 2. The payoff of bidder 2, however, increases independently of whether she chooses to accept or reject. She accepts the offer of bidder 1 if $\theta_2 \leq \theta_1$. Her payoff in a regular auction would have been zero, but now she obtains $b = \theta_1 - \pi_1(\theta_1) > 0$. If she rejects the offer, i.e. if $\theta_2 \geq \theta_1$, her payoff would have been $\theta_2 - \theta_1$. Now however, she obtains $\theta_2 - \pi_1(\theta_1)$, and since $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$, her payoff has again increased.

The difference in the effect on the payoff, is due to information asymmetry. Since bidder 2 knows bidder 1's type before having to make her move, she is able to exclude bidder 1 from the auction at the lowest price possible.

Remark. One might think that after bidder 2 rejects the offer of bidder 1, which in an efficient equilibrium implies that $\theta_2 > \theta_1$, she counteroffers an arbitrarily small amount. Bidder 1 would have already observed that bidder 2 is a stronger opponent since she rejected the offer, and therefore accept any offer. However, if this were the case, low types of bidder 2 would pretend to have a higher type, hoping that bidder 1 will accept a small offer. Ergo, this is not part of an equilibrium.

Conclusively, there exists a continuous bribing equilibrium, of which the first round is fully separating. All types of bidder 1 perfectly reveal themselves by offering a bribe. Due to this full revelation of bidder 1's type, the equilibrium is efficient.

5.2. Concrete Effect

We will again concretize the effect on the efficiency and the payoffs, given that θ_1 and θ_2 are independently and identically distributed from $U(0,1)$.

5.2.1. Effect on Efficiency

As explained earlier, in the model of Rachmilevitch, there is no threshold. Consequently, the strategy of bidder 1 is simply to bribe $b(\theta_1) = \theta_1 - \pi_1(\theta_1)$. The best reply of bidder 2 now is to accept if $\theta_2 \leq \theta_1$ and to reject if $\theta_2 \geq \theta_1$. Thus, we have $[0, A(x)\rangle = [0, \theta_1\rangle$. Hence, the bidder with the highest valuations always obtains the object, which makes the auction efficient despite the bribing. Note that as the auction is efficient, both bidders win the auction with probability $\frac{1}{2}$.

For the uniform distribution, the specific expected payoff $\pi_1(\theta_1)$ and bribing function $b(\theta_1)$ are

$$\begin{aligned}\pi_1(\theta_1) &= \int_0^{\theta_1} \left(\theta_1 - \theta_2 \right) d\theta_2 \\ &= \left[\theta_1\theta_2 - \tfrac{1}{2}\theta_2^2 \right]_{\theta_2=0}^{\theta_2=\theta_1} = \tfrac{1}{2}\theta_1^2\end{aligned}$$

and thus

$$b(\theta_1) = \theta_1 - \pi_1(\theta_1) = \theta_1 - \tfrac{1}{2}\theta_1^2.$$

5.2.2. Effect on Payoff Bidders and Income Seller

In section 4.2.3 we calculated the expected income of the seller and the expected surplus of both bidders in a situation without a bribing round. Recall that we found the values $W_s = \frac{1}{3}$, $W_1 = \frac{1}{6}$ and $W_2 = \frac{1}{6}$.

Now consider the two-rounds bribing model, where the amount offered by bidder 1 is equal to $b(\theta_1) = \theta_1 - \frac{1}{2}\theta_1^2$. Bidder 1 can make a profit in two distinct ways. Firstly, if bidder 2 accepts her bribe, she gains $\theta_1 - b(\theta_1) = \frac{1}{2}\theta_1^2$. Secondly, if bidder 2 rejects her bribe and makes a counteroffer, she obtains $r = \pi_1(\theta_1) = \frac{1}{2}\theta_1^2$. As she receives the same payoff in both cases, we can calculate it the following way:

$$\begin{aligned}
W_1 &= \int_0^1 \int_0^1 \pi_1(\theta_1)\, d\theta_1\, d\theta_2 \\
&= \int_0^1 \int_0^1 \left(\tfrac{1}{2}\theta_1^2 \right) d\theta_1\, d\theta_2 \\
&= \int_0^1 \left[\tfrac{1}{6}\theta_1^3 \right]_{\theta_1=0}^{\theta_1=1} d\theta_2 \\
&= \left[\tfrac{1}{6}\theta_2 \right]_{\theta_2=0}^{\theta_2=1} = \tfrac{1}{6}.
\end{aligned}$$

For bidder 2 we also have two possible situations in which she can get a positive payoff. If she accepts the bribe, she gains $b(\theta_1) = \theta_1 - \frac{1}{2}\theta_1^2$. If she rejects the bribe, she gains $\theta_2 - r = \theta_2 - \frac{1}{2}\theta_1^2$. We will start by calculating her expected payoff if she accepts the offer; this is equal to

$$\begin{aligned}
W_2^1 &= \int_0^1 \int_0^1 b(\theta_1)\, \mathbb{1}\left(\theta_1 \geq \theta_2 \right) d\theta_1\, d\theta_2 \\
&= \int_0^1 \int_{\theta_2}^1 \left(\theta_1 - \tfrac{1}{2}\theta_1^2 \right) d\theta_1\, d\theta_2 \\
&= \int_0^1 \left[\tfrac{1}{2}\theta_1^2 - \tfrac{1}{6}\theta_1^3 \right]_{\theta_1=\theta_2}^{\theta_1=1} d\theta_2 \\
&= \int_0^1 \left(\tfrac{1}{3} - \tfrac{1}{2}\theta_2^2 + \tfrac{1}{6}\theta_2^3 \right) d\theta_2 \\
&= \left[\tfrac{1}{3}\theta_2 - \tfrac{1}{6}\theta_2^3 + \tfrac{1}{24}\theta_2^4 \right]_{\theta_2=0}^{\theta_2=1} \\
&= \tfrac{1}{3} - \tfrac{1}{6} + \tfrac{1}{24} = \tfrac{5}{24}.
\end{aligned}$$

Next, we calculate her expected payoff if she rejects and makes the counteroffer r; we now obtain

$$\begin{aligned}
W_2^2 &= \int_0^1 \int_0^1 \left(\theta_2 - \pi_1(\theta_1) \right) \mathbb{1}\left(\theta_2 \geq \theta_1 \right) d\theta_1\, d\theta_2 \\
&= \int_0^1 \int_0^{\theta_2} \left(\theta_2 - \tfrac{1}{2}\theta_1^2 \right) d\theta_1\, d\theta_2 \\
&= \int_0^1 \left[\theta_1\theta_2 - \tfrac{1}{6}\theta_1^3 \right]_{\theta_1=0}^{\theta_1=\theta_2} d\theta_2 \\
&= \int_0^1 \left(\theta_2^2 - \tfrac{1}{6}\theta_2^3 \right) d\theta_2 \\
&= \left[\tfrac{1}{3}\theta_2^3 - \tfrac{1}{24}\theta_2^4 \right]_{\theta_2=0}^{\theta_2=1} = \tfrac{7}{24}.
\end{aligned}$$

Hence the total expected payoff of bidder 2 is $W_2 = W_2^1 + W_2^2 = \frac{5}{24} + \frac{7}{24} = \frac{1}{2}$.

At last, we will discuss the expected income for the seller. As a result of the two-rounds bribing, there is always one bidder that accepts a bribe from the other bidder, and therefore commits to staying out of the auction. Even though, the auction is still efficient, the seller will never receive a positive income, which means unfortunately that $W_s = 0$. Note that her income is now zero as $p^r = 0$; if $p^r > 0$, her income is logically $W_s = p^r > 0$.

6. Conclusion

In this thesis, we have examined the effect of two simple forms of collusion on the efficiency and payoffs of a second-price auction. The specific setting was a second-price sealed-bid auction for a single object, with two risk-neutral bidders. Both bidders had private and independent valuations $\theta_1, \theta_2 \in [0, 1]$. Recall that in a regular second-price auction bidders bid their true value. Hence, the payoff of the bidder with the highest valuation is $\max(\theta_1, \theta_2) - \min(\theta_1, \theta_2)$, while the payoff of the other bidder is zero.

The form of collusion Eső and Schummer studied, is a one-round bribing model. Here, only bidder 1 gets the opportunity to exclude bidder 2 from the auction by offering her a bribe. If she accepts, she commits to leaving the auction, i.e. submitting a bid of zero. If she rejects, the auction continues as a regular second-price auction, in which the bidders bid their true valuation. The main result was that there exists a unique and continuous bribing equilibrium in which inefficiency occurs with a positive probability. All bidders with a type below a certain threshold $\hat{\theta}_1$ offer a unique amount as a function of their type. Hence, since these bidders reveal their type, the equilibrium is mostly-separating. The bidders with a type above $\hat{\theta}_1$ all offer the same bribe. This threshold exists since it is enough for bidder 1 to signal that she is sufficiently strong to make sure her bribe is accepted. Hence, every bribe is accepted with a positive probability and the highest bribe is always accepted, which leads to the positive probability of inefficiency.

The collusion type studied by Rachmilevitch is an extension of the model of Eső and Schummer. The main result was that there exists an efficient equilibrium whose bribing function is b if and only if $E[\theta_2] \geq \frac{1}{2}$. The equilibrium is now as follows. In the first round, given that bidder 1 offers a strictly positive bribe, she offers $b(\theta_1) = \theta_1 - \pi_1(\theta_1)$. In the second round, if $\theta_2 < \theta_1$, bidder 2 accepts this offer. The payoffs are now respectively $\pi_1(\theta_1)$ and $b(\theta_1)$. If $\theta_2 > \theta_1$, bidder 2 rejects and counteroffers $\pi_1(\theta_1)$, which bidder 1 always accepts. Now the payoffs for bidder 1 and 2 are respectively $\pi_1(\theta_1)$ and $\theta_2 - \pi_1(\theta_1)$. Given that $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$, the payoff of bidder 2 always increases compared to the regular second-price auction, while the payoff of bidder 1 remains the same. The two-rounds bribing model is more beneficial for bidder 2 than for bidder 1 due to information asymmetry. Since b is strictly increasing, the first round is fully separating. Since bidder 2 is aware of the type of bidder 1, she can bribe 1 to stay out of the auction by offering the minimum price.

Conclusively, if we have a second-price auction with private and independent values, it matters whether the bribing model consists of one or two rounds. In a one-round model, efficiency is impossible, since bidder 1 has no incentive to fully reveal her type. There is pooling among types of bidder 1 above the threshold, and thus inefficiency will occur with a positive probability. In a two-rounds model, it is no longer enough for bidder 1 to signal that her type is sufficiently strong. Ergo, there is no threshold. Bidder 2 will only accept the bribe is her type is actually lower than θ_1. Hence, we have full revelation of the type of bidder 1, and thus efficiency.

At last, in sections 4.2 and 5.2 we found the payoffs given a uniform distribution. In the case of no bribing, the total amount circulating in the auction is $W_s + W_1 + W_2 = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3} = \frac{108}{162}$. In the one-round bribing model it was $\frac{10}{81} + \frac{35}{162} + \frac{26}{81} = \frac{107}{162}$, and in the two-rounds bribing model we found $0 + \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$. Hence, for the inefficient model, the total amount decreased slightly, while in the efficient model, the amount remained the same. This could be a coincidence, but might be an interesting angle to examine further.

7. References

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Appendices

A. Proofs Lemmas Two-Rounds Bribing Model

The proofs are adopted from Rachmilevitch ([6]) using a slightly changed notation. They are shown here, so that the interested reader does not have to look up the original article.

Lemma A.1. *(Monotone acceptance). Consider an arbitrary equilibrium and let b be its bribing function. If $b(\theta_1) > 0$ for some θ_1, then bidder 2's equilibrium-response to the offer $b(\theta_1)$ is monotonic: if θ_2 accepts $b(\theta_1)$, then so do all $\theta_2' < \theta_2$.*

Proof. Make the aforementioned assumptions and assume by contradiction that θ_2 and θ_2' are two types of bidder 2 such that (in the equilibrium under consideration) the former type accepts $b(\theta_1)$, the latter rejects it and counters it with the offer r, and $\theta_2' < \theta_2$. Let p denote the probability that bidder 1 accepts r. Note that $p < 1$, because otherwise we would have $\theta_2' - r \geq b(\theta_1) \geq \theta_2 - r$, in contradiction to $\theta_2' < \theta_2$. The fact that $p < 1$ implies that the auction that follows the rejection of r happens on the equilibrium path. Let ϕ be bidder 2's payoff function in this auction. The incentive constraints for θ_2 and θ_2' are therefore:

$$p(\theta_2' - r) + (1-p)\phi(\theta_2') \geq b(\theta_1) \geq p(\theta_2 - r) + (1-p)\phi(\theta_2) \quad (7)$$

which implies

$$p(\theta_2' - \theta_2) \geq (1-p)(\phi(\theta_2) - \phi(\theta_2')) \geq 0. \quad (8)$$

Inequality 8 implies $p = 0$, and substituting $p = 0$ into 7 gives $\phi(\theta_2') \geq \phi(\theta_2)$, and therefore $\phi(\theta_2') = \phi(\theta_2)$. This means that the winning probability of θ_2' in the auction that follows the rejection of r is zero, because in that auction whenever θ_2' wins then surely θ_2 wins, and θ_2's payoff from winning is greater than θ_2''s by the constant $\Delta \equiv \theta_2 - \theta_2' > 0$. The fact that θ_2''s winning probability in the aforementioned auction is zero implies that rejecting $b(\theta_1) > 0$ and countering it with r is strictly sub-optimal – in contradiction to equilibrium. □

Lemma A.2. *(Revelation by strictly positive bribes). Let b be the bribing function of an efficient equilibrium. Then if θ_1 and θ_1' are two types of bidder 1 such that $b(t) > 0$ for both $t \in \{\theta_1, \theta_1'\}$, then $b(\theta_1) \neq b(\theta_1')$.*

Proof. Let b be the bribing function of an efficient equilibrium and suppose that there is a $b(\theta_1) > 0$ for some θ_1. Let θ_1 be the minimal type who offers $b(\theta_1)$. By lemma A.1, all $\theta_2 \leq k$ accept $b(\theta_1)$ for some cut-off $k = k(b(\theta_1))$.[5] By efficiency, $k \leq \theta_1$; otherwise, efficiency would fail for any type-realization where bidder 1's type is θ_1 and bidder 2's type is in (θ_1, k). I argue that $k = \theta_1$. To see this, assume by contradiction that $k < \theta_1$ and let r denote the counter-bribing function employed by bidder 2 in this equilibrium after she rejects $b(\theta_1)$. Efficiency dictates that θ_1 rejects all counters $r(t)$, $t \in (k, \theta_1)$. This implies that on the equilibrium path any such counter $r(t)$ (for $t \in (k, \theta_1)$) is rejected by bidder 1. The reason is that θ_1 prefers to reject a counter-offer and compete in the auction, then so does any other $\theta_1' > \theta_1$,[6] and by assumption θ_1 is the minimal type who offers $b(\theta_1)$. This means that for types $\theta_2 \in (k, \theta_1)$, the continuation payoff that

[5] It is easy to see that $k > 0$, because all $\theta_2 < b(\theta_1)$ accept $b(\theta_1)$.

[6] A proof of this fact is analogous to that of lemma A.1, and is therefore omitted.

follows the rejection of $b(\theta_1) > 0$ is zerowhich is impossible in equilibrium. Therefore $k = \theta_1$.

Now suppose that there is a $\theta_1' > \theta_1$ such that $b(\theta_1') = b(\theta_1)$. Therefore, by requirement 4 (see section 5), there is an interval of types who offer $b(\theta_1)$. Look at some $\theta_2 \in (k, \theta_1') = (\theta_1, \theta_1')$. This type rejects $b(\theta_1)$ and she therefore assigns probability at least $\eta \equiv \frac{b(\theta_1)}{\theta_1'} > 0$ to the event {bidder 1's type $< \theta_2$}.[7] This is true for every such θ_2, no matter how close to k. Therefore, $P(\theta_1 = k \mid b(\theta_1)) \geq \eta$, which is impossible, because there is a continuum of briber-types who offer $b(\theta_1)$.

□

Lemma A.3. *Let b be a bribing function of an efficient equilibrium. Then there is a constant $C \geq 0$ such that the following holds for each θ_1:*

$$b(\theta_1) > 0 \Rightarrow b(\theta_1) = \theta_1 - \pi_1(\theta_1) - C.$$

Proof. Let b be such a function and consider a θ_1 such that $b(\theta_1) > 0$. It follows from the combination of lemma A.2 and the equilibrium's efficiency that bidder 2 accepts this offer if $\theta_2 < \theta_1$ and rejects if $\theta_2 > \theta_1$. It follows that bidder 2's rejection of the revealing offer $b(\theta_1)$ implies that $\theta_2 \geq \theta_1$and therefore implies that bidder 1 can only obtain a positive payoff through the acceptance of a counter bribe, so she indeed accepts the counter, if it is strictly positive.

Now consider $\theta_2 \in (\theta_1, \theta_1 + b(\theta_1))$. This θ_2 rejects the revealing offer $b(\theta_1) > 0$, thereby forgoing the sure payoff $b(\theta_1)$. Since $\theta_2 \in (\theta_1, \theta_1 + (\theta_1))$ she will not counter with zero, because this will trigger the auction with certainty, in which case her net payoff will be smaller than $b(\theta_1)$. Therefore θ_2's counter is strictly positive. We noted above that bidder 1 can only obtain a positive payoff through the acceptance of a counter bribe, so she accepts the strictly positive counter. Moreover, it is easy to see that no two such offers can be made in equilibrium: if two strictly positive counters are proposed by bidder 2 and accepted by the self-revealing θ_1, then the more generous one cannot be a part of the equilibrium. Therefore, a single offer is made, call it $r(\theta_1)$, and the cut-off type $\theta_2 = \theta_1$ is indifferent between offering it and accepting bidder 1's offer:

$$\theta_1 - r(\theta_1) = b(\theta_1). \tag{9}$$

Therefore, θ_1's equilibrium payoff is $F_2(\theta_1)(\theta_1 - b(\theta_1)) + (1 - F_2(\theta_1))r(\theta_1) = \theta_1 - b(\theta_1)$. On the other hand, since our bribing game can be thought of as a revelation mechanism, it follows that θ_1's expected equilibrium payoff is $\pi_1(\theta_1) + C$ where C is the equilibrium payoff of type $\theta_1 = 0$, which is obviously non-negative in our game (the reason is that the efficient equilibrium implements the same allocation as the dominant-strategy equilibrium). Therefore $\theta_1 - b(\theta_1) = \pi_1(\theta_1) + C$, or $b(\theta_1) = \theta_1 - \pi_1(\theta_1) - C$, as argued.

□

Lemma A.4. *(Characterization of strictly positive bribes). The constant described in lemma A.3, C, is zero. In particular, in any efficient equilibrium, if type θ_1 offers a strictly*

[7] Once such θ_2 rejects $b(\theta_1)$, she knows, because of the equilibrium's efficiency, that the only way for her to end up with a positive payoff is in the event $\{\theta_1 \leq \theta_2\}$. Letting p be the probability of {bidder 1's type $< \theta_2$} the following obviously must be satisfied $p\theta_2 \geq b(\theta_1) \Rightarrow p \geq \eta$

positive bribe then her offer is given by:

$$b(\theta_1) = \theta_1 - \pi_1(\theta_1).$$

Proof. Assume by contradiction that $C > 0$. Note that $\theta_1 = C$ offers a zero bribe; otherwise, by lemma A.3, her offer would be $C - \pi_1(C) - C = -\pi_1(C) < 0$, which is impossible. Therefore, the expected equilibrium payoff of type $\theta_1 = C$ is obtained by triggering a counteroffer of bidder 2. Hence, this payoff is bounded from above by:

$$\int_0^1 \max\{r(t), \max\{C - t, 0\}\} f_2(t)\, dt,$$

where r is the function describing the counter-offers of bidder 2 that follows the rejection of the bribe offer zero. The reason is that once arrived in this second round, bidder 1's payoff cannot decrease if, in addition to bidder 2's offer $r(\theta_2)$ bidder 1 is provided with the information regarding bidder 2's type (the function r need not be invertible); conditional on this information, bidder 1 can optimally decide whether to accept the counteroffer or compete in the auction.

Now notice that $C = \int_0^1 r(t) f_2(t)\, dt$, since C is the equilibrium payoff of $\theta_1 = 0$ and it hence must be equal to the expected counter-bribe. Since $C > 0$ it follows that r is strictly positive on a set of positive measure; it therefore follows from Eső and Schummer that r is weakly increasing and strictly positive on an interval of the form $[x, 1]$, which implies:

$$\begin{aligned}\int_0^1 \max\{r(t), \max\{C - t, 0\}\} f_2(t)\, dt &< \int_0^1 \left[\ r(t) + \{\max\{C - t, 0\}\ \right] f_2(t)\, dt \\ &= \int_0^1 r(t) f_2(t)\, dt + \int_0^1 \max\{C - t, 0\} f_2(t)\, dt = C + \pi_1(C).\end{aligned}$$

However, recalling that the equilibrium payoff of type $\theta_1 = C$ is $C + \pi_1(C)$, we obtain $C + \pi_1(C) < C + \pi_1(C)$ – a contradiction.

□

Lemma A.5. *$E[\theta_2] \geq \frac{1}{2}$, if and only if $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$ $\forall \theta_1 \in [0, 1]$.*

Proof. Suppose that $\pi_1(\theta_1) \leq \frac{\theta_1}{2}$ $\forall \theta_1 \in [0, 1]$. Taking $\theta_1 = 1$ gives $1 - E[\theta_2] \leq \frac{1}{2}$, or $E[\theta_2] \geq \frac{1}{2}$. Conversely, suppose that $E[\theta_2] \geq \frac{1}{2}$. We need to prove that $\gamma(\theta_1) \geq 0$, where $\gamma(\theta_1) \equiv \frac{\theta_1}{2} - \pi_1(\theta_1)$. Note that $\gamma'(\theta_1) = \frac{1}{2} - F_2(\theta_1)$, hence $\gamma''(\theta_1) = -f_2(\theta_1) \leq 0$, so γ is concave. Also, $\gamma(0) = 0$ and $\gamma(1) = \frac{1}{2} - (1 - E[\theta_2])) = E[\theta_2] - \frac{1}{2} \geq 0$, where the inequality is by assumption. Therefore, $\gamma(\theta_1) \geq \theta_1\gamma(1) + (1 - \theta_1)\gamma(0) = \theta_1\gamma(1) \geq 0$.

□