Why is the Universe Spatially Flat?

An argument from conformal symmetry.

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Abstract

The 't Hooft criterion states that smallness of a physical parameter can be considered *technically natural* if setting it to zero would enhance the symmetry of the system. In this work we propose to extend this criterion to include space-time isometries and curvature. Within this framework, the observed smallness of the curvature of the universe can be understood as technically natural (and protected from quantum fluctuations) if setting curvature to zero increases the number of isometries of space-time.

In General Relativity, setting the curvature of an expanding space-time to zero does not lead to additional isometries. However, such a symmetry enhancement does happen if we consider Conformal Gravity, where we modify gravity to be conformally symmetric.

Studying the behaviour of conformal isometries leads us to formulate the Geometric Isolation conjecture. This asserts that a manifold that decomposes nontrivially into two or more isolated factors admits only a limited number of conformal symmetries. In such a setting, the conformal Killing equation can be greatly simplified and will yield fewer independent solutions than allowed by the dimension of the manifold.

We apply these principles to cosmological space-times that factor as $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ indicates the time-like direction and Σ an expanding, space-like 3-manifold. We will deviate from the usual FLRW space-times where Σ is chosen to be one of S^3, $\mathbb{R}^3$ or $\mathbb{H}^3$ and instead allow Σ to be one of the eight geometries from Thurston's Geometrization Conjecture. While the Conformal Gravity setting introduces extra isometries for any choice of Σ, we will show that only the flat geometry $\Sigma \simeq \mathbb{R}^3$ restores the full conformal group in four dimensions.

As a result, flat space-time represents a point of exceedingly enhanced symmetry, thus satisfying the requirement for our extension of 't Hooft technical naturalness.

Contents

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Notations & Conventions

Throughout this work we will adopt the following conventions.

- Units are such that $\hbar = c = 1$
- We use the 'mostly minus' signature, where time-like directions have a negative entry in the metric $g_{\mu\nu}$. In four dimensions $g_{\mu\nu}$ =diag$(-1, 1, 1, 1)$.
- We adopt the Einstein summation convention where repeated indices are summed over. We will mention explicitly whenever we break this convention or denote indices not summed over with a hat.
- Partial derivatives are denoted by ∂, while covariant derivatives are denoted ∇.
- Tensors are denoted with boldface $\mathbf{T}$, while tensor components carry indices $T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}$.
- We will refer to both $g_{\mu\nu}$ and $ds^2 = g_{\mu\nu} x^\mu x^\nu$ as the metric as these can be used interchangeably.

1 Introduction & Motivation

1.1 The Flatness Problem

As the title of this documents suggests, the universe we live in is (approximately) flat. Indeed, recent integrated results from the Planck satellite [21] put the curvature density of the universe, in units of the critical density, at zero to one part in two-hundred: $\Omega_K = 0.000 \pm 0.005$. This is a surprising fact made even more remarkable by the realization that, due to the expansion of the universe, it must have been even smaller (by a factor of 10^{60}) in the early universe. This was first pointed out by Robert Dicke in 1969 [6] and is known as the **flatness problem**. While inflation solves this problem, along with the monopole problem and the horizon problem, we would like to pursue an alternative approach in this document.

Instead, we ask the question 'Is it natural to expect curvature to be close to zero in our universe?'. We suggest to extend 't Hooft's definition of *technical naturalness*, [16] which states that

> *"A physical parameter is allowed to be small only if setting it to 0 would increase the symmetry of the system."*

to include curvature and space-time isometries.

A more modern reading of this statement is to say that quantum corrections will generate only small modifications to a physical parameter if setting it to zero increases the symmetry of the system. Since primordial curvature fluctuations emerge from quantum fluctuations during inflation, we argue that a small value for the curvature is protected if the zero curvature limit admits more isometries.

From the observational fact that space is approximately homogeneous and isotropic on large scales, we split the metric of space-time $\mathbf{g} = \overline{\mathbf{g}} + \delta\mathbf{g}$ into a background part $\overline{\mathbf{g}}$ that is homogeneous and isotropic and a perturbative part $\delta\mathbf{g}$. The Einstein-Hilbert action then splits similarly as

$$\mathcal{S}_{\mathrm{EH}}[\mathbf{g}] = \mathcal{S}_{bg}[\overline{\mathbf{g}}] + \mathcal{S}_{pert}[\overline{\mathbf{g}}, \delta\mathbf{g}]. \tag{1.1}$$

The isometries of the background metric $\overline{\mathbf{g}}$ will be symmetries of the background action $\mathcal{S}_{bg}$. In general, $\overline{\mathbf{g}}$ will contain a curvature parameter k, which we now interpret as a physical parameter in $\mathcal{S}_{bg}$. We can ask the question if setting it to zero enhances the symmetries of the background action? If so, then we expect the perturbations $\delta\mathbf{g}$ to remain small compared to $\overline{\mathbf{g}}$ and by 't Hooft technical naturalness this should preserve smallness of k.

In ordinary Einstein gravity, a cursory investigation shows that no extra symmetry arises at zero curvature. Recall that dynamics of our universe is well-described against a Friedmann-Lemaître-Robertson-Walker background. In this picture, space-time decomposes as $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ is the direction of time while Σ is one of three maximally-symmetric, three-dimensional manifolds, $\mathbb{R}^3$, $\mathbb{H}^3$ or S^3, that describe the spatial sections. We can describe all three FLRW space-times simultaneously using the metric

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\Big], \qquad d\Omega_2^2 = d\theta^2 + \theta^2\sin^2(\phi). \tag{1.2}$$

In this metric the parameter k represents the curvature, which yields $\Sigma = S^3$ for $k > 0$, $\Sigma = \mathbb{R}^3$ for $k = 0$ and $\Sigma = \mathbb{H}^3$ for $k < 0$. The observational results mentioned impose severe constraints on this parameter k, but are still consistent with a tiny, but nonzero value. As such, data allow for any of the three FLRW geometries to occur, as long as the curvature radius $1/\sqrt{|k|}$ is large compared to the size of the observable universe. Writing out $\mathcal{S}_{bg}$ using the metric from (1.2) gives us

$$\mathcal{S}_{bg}[\bar{\mathbf{g}}] = \frac{M_p^2}{2} \int d^4x \ \left[6r^2 a(t) \sqrt{\frac{\sin^2(\theta)}{1-kr^2}} \Big(k + \dot{a}^2(t) + a(t)\ddot{a}(t)\Big)\right]. \tag{1.3}$$

We now interpret k as a physical parameter in this action. Now note that the presence of the scale factor a manifestly breaks four out of ten symmetries of the Poincaré group, namely time translations P_t and Lorentz boosts L_{ti}. This is no different whether $\Sigma = \mathbb{R}^3$, $\mathbb{H}^3$ or S^3 and accordingly the number of symmetries of (1.3) does not change if we set k to zero. Hence there is no requirement from symmetry for k to remain small by 't Hooft naturalness. Consequently we are forced to conclude that a universe with (close to) zero curvature is unnatural in Einstein gravity.

In the rest of this work we will show that the situation changes drastically in a conformal setting. Indeed, in a theory of conformal gravity, [19] the flat FLRW space-time will represent a unique point of exceedingly enhanced global symmetry. In what follows, all reference to 'the manifold of space-time' will refer to the background metric.

1.2 Conformal Symmetry and Scale-Invariance

As mentioned, we will be working in a conformal extension of gravity, where the symmetry of the theory is enlarged. We will extend the Poincaré group of space-time isometries to include conformal isometries. These are space-time transformations of the type

$$x \to x' \qquad g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x), \tag{1.4}$$

with $\Omega^2(x)$ a non-vanishing smooth function. In establishing this additional symmetry, we will pick up extra gauge symmetry under Weyl transformations. Such a transformation rescales the metric tensor *without* touching the space-time coordinates, i.e.

$$g_{\mu\nu}(x) \to \Lambda^2(x) g_{\mu\nu}(x), \tag{1.5}$$

with again $\Lambda^2(x)$ a non-vanishing smooth function.

Such transformations represent local rescalings that preserve angles and ratios of distances. However, distances are locally scaled by a factor of Ω or Λ depending on the type of transformation. Therefore, any field theory that is conformally invariant should not contain any intrinsic length scales, as these would transform under (1.4) and (1.5). Hence any constants or parameters in the theory should be dimensionless.

1.3 The Case for Scale-Invariance

There is a historic precedent for studying scale-invariant theories. Already in 1909 Bateman [1] noted that Maxwell's equations are conformally invariant in 4 dimensions. In 1984, Belavin, Polyakov, and Zamolodchikov [2] showed that a massless scalar field in 2 dimensions has this symmetry as well, which paved the way for the CFT description of String theory that is still used today. Groundbreaking work was done in by Maldacena [20] halfway through the 90s, when he published the most cited work in High Energy Physics by proposing the ADS/CFT correspondence. This insight finds many uses throughout physics to this day.

Perhaps the most compelling example comes from the 1964 symmetry breaking papers, [7], [14] and [15], that showed how masses can be generated dynamically as a result of spontaneous symmetry breaking. Through this mechanism, most particles in the Standard Model acquire mass through their interaction with the Higgs boson. As a result, the Standard Model is almost conformally invariant, containing only dimensionless coupling constants (in natural units), save for the Higgs mass term.

We argue that it is not unreasonable to think that General Relativity may enjoy similar symmetry at higher energies. There are several considerations from cosmology that support this.

Firstly, the phenomenal success of linear, free-field perturbation theory tells us that much of the dynamics of the universe is already captured by a low-energy description. This indicates that we have some freedom to experiment at higher energies without invalidating the theory.

Secondly, Planck observations tell us that the primordial power spectrum is approximately scale-invariant ($n_s - 1 \approx -0.035$) [21]. Since these primordial perturbations come from the high-energy conditions of the early universe, a scale-invariant UV-completion of gravity may help elucidate this rather peculiar property.

Lastly, some modern approaches to inflation formulate it in terms of spontaneous breaking of time translations. [4] Since conformal symmetry is manifestly broken today, it is quite possible that we may describe the start of the inflationary epoch by breaking not just time translations but also the additional conformal symmetries.

1.4 Conformal Gravity

There are, however, two major obstacles to formulating a conformally invariant theory of gravity. Both of these issues are addressed in [19], which forms the basis of this thesis, in which the authors construct a classically conformal theory of gravity. We will not reproduce the results of this paper in full here, as we will only need the existence of conformal symmetry in most of our own calculations, but mention briefly the steps the authors take.

The first and most important of these is that the geodesic equation, along with many geometric tensors, fails to be conformally invariant even at the classical level. There is no way to resolve this in Einstein gravity, where the Levi-Civita connection is fully specified in terms of the metric tensor. However, the authors argue, this is not the most general metric-affine connection as the requirement that $\Gamma^{\alpha}_{\mu\nu}$ is symmetric in its two lower indices sets torsion to zero. Consequently, the natural language for a conformal extension of GR is Cartan gravity, where the connection is allowed to include torsion,

$$\Gamma^{\lambda}_{\mu\nu} = \mathring{\Gamma}^{\lambda}_{\mu\nu} + \mathcal{T}^{\lambda}{}_{\mu\nu} + \mathcal{T}_{\mu\nu}{}^{\lambda} + \mathcal{T}_{\nu\mu}{}^{\lambda}. \tag{1.6}$$

In this equation Γ indicates the full connection, which is taken to include the Levi-Civita connection $\mathring{\Gamma}$ and torsion terms $\mathcal{T}$. The extra torsion components can be taken to transform precisely in the correct manner to cancel the offending terms in the geodesic equation. This construction will make the theory classically conformally symmetric in the absence of matter fields. It is worth noting that the presence of torsion does not affect the symmetries of the system, a remarkable fact we will observe in section 3.2

A second complication arises directly from the Einstein Field Equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{T_{\mu\nu}}{M_p^2}. \tag{1.7}$$

In the presence of matter fields, or in the quantum regime, the theory will contain a dimensionful quantity in the form of the Planck mass M_p, which is defined via Newton's constant as $\sqrt{\hbar c/8\pi G} \approx 2.435 \times 10^{18}$kg. This will introduce an intrinsic length scale to the theory, namely the Planck length $\ell_p = \sqrt{\hbar G/c^3} \approx 1.61 \times 10^{-35}$m, which violates conformal invariance of the theory. The authors resolve this problem by introducing a dilaton field, Φ, whose expectation value generates the Planck mass locally,

$$\langle \Phi \rangle = M_p, \tag{1.8}$$

in analogue to the BEH mechanism in the Standard Model. This scalar field can be given an appropriate conformal weight so that its action remains conformally invariant. A similar procedure makes the action for matter fields conformally symmetric.

We will adopt this construction implicitly, although we will not use its minutiae for the calculations in this work. In what follows, we will simply presuppose conformal invariance and go from there.

2 Symmetries

Both in physics and in mathematics, there is much to gain from studying the symmetries of an object or theory. These give rise to conserved quantities, allow us to pick a favourable description of the system under consideration and give us insight into the nature of the object itself. We can distinguish between two categories of symmetries, **Dynamical** symmetries and **Gauge** symmetries.

Dynamical symmetries, sometimes also called True or Global symmetries, are fundamental to the theory itself. States related by a dynamical symmetry transformations represent distinct points in the phase-space of the system that are nonetheless considered equivalent by the theory. Roughly speaking, such symmetries will map a solution to the equations of motion onto a different, physically distinct, but nonetheless valid solution. Dynamical symmetries give rise to conserved quantities through Noether's theorem and give us additional insight about the nature of the theory.

A textbook example of this type of symmetry in classical mechanics is the invariance of the action of a massive, point-like particle under translations. We can take any solution to the equation of motion $F = m\ddot{x}$ and shift it by a constant vector to obtain a different legitimate solution. This leads directly to the conservation of (linear) momentum in the absence of external forces, which is the statement of Newton's First Law of Motion.

Gauge symmetries, on the other hand, are an artefact of our description. The configuration space of parameters and variables in the theory is larger than the space of physically distinct states. Gauge symmetries are exactly those transformations of configuration space that stabilize physical phase space. That is to say, the orbits of gauge transformations relate different configurations that represent the same physical state. We may dispose of these spurious degrees of freedom by shrinking the configuration space, i.e. gauge-fixing our theory, as many problems become much easier upon employing the right choice of gauge.

One of the first symmetries of this type that most physicists will encounter comes from classical electrodynamics. We may shift the vector potential $A_\mu(x)$ by the derivative of a scalar $\partial_m \chi(x)$ and obtain two different points in configuration space that correspond to the same electromagnetic fields. There are a number of different gauges we can employ to fix this, such as the Coulomb gauge $\nabla \cdot A = 0$.

2.1 The Automorpishm Group

To formalize the notion of symmetry, we say that two objects A and B are 'the same' if there exists an **isomorphism** that relates them. An isomorphism is a bijective map $\phi : A \to B$ that is *structure-preserving* with respect to a structure of interest on A and B. For instance, if A and B are finite-dimensional vector spaces, then an isomorphism ϕ between the two must respect the vector-space structure of addition and scalar multiplication. Ergo, ϕ must be an invertible, linear transformation, which exists if and only if A and B have the same dimension.

Most relevant to our discussion are isomorphisms between an object A and itself, also called **automorphisms**. Since these preserve A's internal structure, we should think of the set of automorphisms, $\mathrm{Aut}(A)$, as the set of symmetry transformations on A. The key property of $\mathrm{Aut}(A)$ is that it forms a group under composition of maps. Indeed, composing any two automorphisms yields a third automorphism, $\phi_2 \circ \phi_1 = \phi_3 \in \mathrm{Aut}(M)$; the identity map $\mathbb{1}_A$ acts as the group identity, $\phi \circ \mathbb{1}_A = \mathbb{1}_A \circ \phi = \phi$ for any ϕ; and all automorphisms are, by definition, invertible. This property helps us classify the symmetries, as we will demonstrate in later sections.

2.2 Diffeomorphisms

Recall that General Relativity is formulated in terms of a space-time manifold M, equipped with a (dynamical) metric $\mathbf{g}$ and whatever matter fields $\mathbf{T}$ we choose to include.[1] For this reason, any legitimate symmetry of this theory should at least preserve the differentiable structure of the manifold M. The largest symmetry group we find using this condition is therefore $\mathrm{Aut}(M)$, which consists of **diffeomorphisms** from M to itself: smooth, invertible maps whose inverse is likewise smooth. Tensor fields, in particular, are part of the differentiable structure of M via tensor products of the (co)tangent spaces TM and T^*M. Therefore any globally defined, true tensor must also be preserved by any diffeomorphism $\phi \in \mathrm{Aut}(M)$,

$$\mathbf{T}' = \phi^* \mathbf{T} = \mathbf{T}. \tag{2.1}$$

Now suppose we have chosen a chart $(U, \chi : U \to \mathbb{R}^n)$ around some point $p \in U \subset M$ so that we may describe the neighbourhood U of p in local coordinates and let ϕ be a diffeomorphism from M to itself. Then we can find a coordinate chart $(V, \psi : V \to \mathbb{R}^n)$ around $\phi(p)$ by writing $V = \phi(U)$ and $\psi = \chi \circ \phi^{-1}$. In other words, we may view diffeomorphisms as *general coordinate transformations* $x \to x'$ on our space-time manifold M.

This equivalence tells us that we may expand a tensor $\mathbf{T}$ over a coordinate basis in *any* coordinate frame of our choice and obtain an equivalent description,

$$T^I_J(x)\Big(\frac{\partial}{\partial x^I} \otimes dx^J\Big) = \mathbf{T} = \mathbf{T}' = T'^I_J(x')\Big(\frac{\partial}{\partial x'^I} \otimes dx'^J\Big). \tag{2.2}$$

Here we have employed multi-indices $I = \mu_1, \ldots, \mu_p$ and $J = \nu_1, \ldots, \nu_q$ for brevity. This equation tells us that the tensor coordinate functions $T^I_J(x)$ must transform opposite to its basis vectors, $\frac{\partial}{\partial x}$ and dx. From the way $\frac{\partial}{\partial x}$ transforms as a derivative, while dx transforms as an integral measure, we then derive the transformation law for the tensor coordinate functions $T^I_J(x)$,

$$T^{I'}_{J'}(x') = T^I_J(x) \left(\frac{\partial x'^{I'}}{\partial x^I} \cdot \frac{\partial x^J}{\partial x'^{J'}}\right). \tag{2.3}$$

Formally we say that the tensor $\mathbf{T}$ is **invariant** under diffeomorphisms, i.e. it does not change, while its coordinate functions $T^I_J(x)$ are **covariant**, i.e. they change opposite to the coordinates. In physics we tend to conflate the tensor with its coordinate functions and, in doing so, lose this distinction. Since we will be working primarily in local coordinates we will adopt this convention and state that *Tensors are covariant under general coordinate transformations.*

If we can formulate the equations of motion of a theory in terms of true tensors, the solutions must exist at the level of the manifold. Any particular choice for local coordinates in which to express these solutions is simply one of many ways of expressing it. In our language above, the equations of motion and their solutions will be under coordinate transformations. Hence any physical statement we make in some coordinate frame O will map onto an identical statement in some different coordinate frame $\widetilde{O}$, simply expressed differently according to the transformation law (2.3). This puts general coordinate transformations (or, equivalently, diffeomorphisms) solidly in the category of gauge transformations. In particular, they are the gauge transformations of General Relativity.[2]

[1] In our case, we will also specify a nonstandard connection Γ via (1.6), but this will not change the discussion in this section.

[2] Of course, there may be extra gauge transformations coming from the matter fields. These will not take the form of space-time transformations and so are not immediately relevant to our discussion.

2.3 Isometries

A stronger symmetry requirement is to ask that any given coordinate transformation leaves a tensor $T^I_J(x)$ invariant rather than covariant. Such transformations are called the **symmetries** of the tensor $T^I_J(x)$. Of particular interest to us are the symmetries of the space-time metric, also called its **Isometries**, defined by

$$(\phi^* g)_{\mu\nu} = g_{\mu\nu}. \tag{2.4}$$

We can think of isometries as coordinate transformations $x \to x'$ that don't necessitate a change in the local coordinate expression of the metric, i.e.

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x') \qquad \leftrightarrow \qquad g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu\nu}(x')dx'^\mu dx'^\nu. \tag{2.5}$$

Note that the right-hand side contains $g_{\mu\nu}(x')$ rather than $g'_{\mu\nu}(x')$, which would be the case for a general diffeomorphism.

Notions of distances and angles are preserved under isometries so that any valid geometric statement in the frame O gets mapped onto distinct, but valid statement in $\widetilde{O}$. Hence solutions to the geodesic equation or the Einstein equations get mapped onto different, but equally legitimate solutions. Hence isometries are part of the dynamical symmetries of the theory and we may construct conserved quantities out of them. Typically only spatial translations and rotations belong to the isometry group of expanding space-times. The associated conserved quantities will include linear momentum and angular momentum, but will not include energy, which is linked to time translations.

These transformations form the Lie group $\mathrm{Isom}(M, g) < \mathrm{Aut}(M)$, which means they form a mostly continuous group with a handful of discrete transformations relating different components. This will prove to be very useful in finding the generators of the isometry group, of which there are at most $\frac{n(n+1)}{2}$ for a manifold of dimension n. Unfortunately, a general space-time there will have zero isometries as any small deviation from perfect isotropy and homogeneity will break almost every symmetry.

2.4 Conformal Isometries

We now introduce the notion of a **conformal structure** on our manifold. This is a slight modification of the Riemannian structure where, rather than endowing our manifold with a single metric $\mathbf{g}$, we equip it with a class of metrics $[\mathbf{g}]$. Two metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ belong to the same **conformal class** $[\mathbf{g}]$ if we may write $\mathbf{g}_1 = \Omega^2 \mathbf{g}_2$ for some non-vanishing, smooth function Ω, called the **conformal factor**.

This leads us directly to an enhancement of gauge symmetry. After all, if two choices of metric within the same conformal class are considered equivalent, we should be free to change between representatives of this class. Hence we may freely rescale the distances locally by a nonvanishing smooth function Λ as

$$g_{\mu\nu} \to \Lambda^2 g_{\mu\nu} \simeq g_{\mu\nu} \tag{2.6}$$

to obtain an equivalent description; this is called a **Weyl Transformations**. Note that this transformation does not touch the coordinates on the manifold, it is a modification of the metric tensor itself.

As in the Riemannian case, we may consider diffeomorphisms ϕ with the requirement that

$$(\phi^* g)_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{2.7}$$

with $\Omega \neq 0$. Note that $\Omega = 1$ reduces this equation to (2.4) and so diffeomorphisms satisfying (2.7) are conformal generalization of isometries. Appropriately we will call them **Conformal Isometries**. As befor, we can think of conformal isometries as coordinate transformations $x \to x'$ whose effect of the metric is changing the argument and multiplying by a factor of $\Omega(x')^2$:

$$g_{\mu\nu}(x) \to \Omega^2(x')g_{\mu\nu}(x') \qquad \leftrightarrow \qquad g_{\mu\nu}(x)dx^\mu dx^\nu = \Omega^2(x')g_{\mu\nu}(x')dx'^\mu dx'^\nu. \tag{2.8}$$

Again, we write $g_{\mu\nu}(x')$ rather than $g'_{\mu\nu}(x')$ on the right-hand side. Note that Λ in (2.6) and Ω in (2.7) have similar effects on the metric $g_{\mu\nu}$ and are both referred to as the *conformal factor*. However they arise in a very different fashion, respectively from a simple choice of representative and from a change of coordinates. It is important to keep this distinction in mind. To avoid confusion with other sources, whenever we refer to a conformal transformation or conformal isometry we mean transformations of the type (2.7) and we will refer to transformations of the type (2.6) only as Weyl transformations.

As mentioned, the conformal isometries form a more general class of metric symmetries that preserve angles and ratios of distances, but not distances themselves. With the modifications to General Relativity described in section 1.4 the conformal isometries too will map solutions to the geodesic or Einstein equations onto other valid solutions, meaning that conformal symmetry represents an enlargement of the dynamical symmetry group of gravity.

The conformal isometries form the Lie group $\mathrm{Conf}(M, g)$ that indeed fits in between the isomorphisms and automorphisms, $\mathrm{Isom}(M, g) < \mathrm{Conf}(M, g) < \mathrm{Aut}(M)$. We can see this immediately by noting that solutions to (2.4) will also solve (2.7) for $\Omega = 1$. For a manifold of $n \geq 3$ dimensions the conformal group will be at most $\frac{(n+2)(n+1)}{2}$-dimensional, [18] exemplifying the enlargement of symmetry.

3 The (Conformal) Killing Equation

3.1 A Bit of Lie theory

Deriving conformal isometries directly from the equation (2.7) is difficult to do for a general space. We may be able to infer some of the transformations from the manifest symmetries of the metric, but getting all by guesswork (or proving that there are less than the maximum number) is a tall order. Fortunately both isometries and conformal isometries form not just a group under composition, they form the Lie groups $\mathrm{Isom}(M)$ and $\mathrm{Conf}(M)$. This means we may characterize the smooth symmetry transformations ϕ as belonging to a one-parameter subgroup. That is to say, there is a set of smoothly varying transformations ϕ_s, indexed by $s \in \mathbb{R}$, so that $\phi_0 = \mathbb{1}_M$, $\phi_1 = \phi$ and $\phi_s \circ \phi_t = \phi_{s+t}$ for all $s, t \in \mathbb{R}$. We can study the behaviour of this one-subgroup close to the identity at $s = 0$ by considering the *generator* associated to this transformation, ξ_ϕ, which is just the first derivative at zero:

$$\xi_\phi = \frac{\partial}{\partial s}\phi_s\Big|_{s=0}. \tag{3.1}$$

ξ_ϕ can be thought of as representing the infinitesimal transformation $x^\mu \to x^\mu + \xi^\mu$ and we can derive it by solving a linearized version of (2.7),

$$\mathfrak{L}_{\xi_\phi} g_{\mu\nu} = 2\omega g_{\mu\nu}. \tag{3.2}$$

Here $\mathfrak{L}$ is the so-called *Lie derivative* and we have written $\Omega^2 = e^{2\omega}$. The ξ's that solve this equation form a vector space so that any linear combination of them solves (3.2). What's more, we can equip this vector space canonically with a commutator bracket $[\ ,\ ]$ to turn it into the Lie algebra $\mathfrak{isom}(M)$ (for $\omega = 0$) or $\mathfrak{conf}(M)$ (for ω free). We can take a linearly independent set of solutions $\xi^{(i)}$ and reobtain the group-level transformations by solving (3.1) to get[3]

$$\phi_s^{(i)} = \exp(s\xi^{(i)}). \tag{3.3}$$

In practice, however, we will simply recognize the Lie algebra by its commutation relations and immediately state the overlying group.

3.2 Local Conformal Invariance

Since we will work in local coordinates throughout this text it is useful to express (3.2) in local coordinates,

$$-\delta_\xi g_{\mu\nu}(x) = 2\omega(x) g_{\mu\nu}(x), \tag{3.4}$$

where the subscript ξ indicates the infinitesimal coordinate transformation $x^\mu \to x'^\mu(x) = x^\mu + \xi^\mu(x)$. This will change the metric according to the tensor transformation rule (2.3) we derived earlier,

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \to g'_{\mu\nu}(x) = g_{\alpha\beta}(x-\xi)\frac{\partial(x-\xi(x))^\alpha}{\partial x^\mu}\frac{\partial(x-\xi(x))^\beta}{\partial x^\nu}. \tag{3.5}$$

We expand the right-hand side to first order in ξ, omitting explicit dependence on x for brevity,

$$g'_{\mu\nu} = g_{\mu\nu} - \xi^\alpha\partial_\alpha g_{\mu\nu} - g_{\alpha\nu}\partial_\mu\xi^\alpha - g_{\mu\beta}\partial_\nu\xi^\beta + O(\xi^2) \tag{3.6}$$

[3]Technically we are only able to construct $\mathrm{Isom}_e(M)$ and $\mathrm{Conf}_e(M)$ in this manner, the connected component of the Lie group containing the identity. For example, in 4-dimensional Minkowski space-time we would recover $\mathrm{Isom}_e(\mathbb{R}^{1,3}) = \mathrm{SO}^+(1,3) \rtimes \mathbb{R}^4$, which contains only the proper (parity-preserving), orthochronous (preserving the direction of time) Lorentz transformations. This is not a problem for us, however, as the quotient G/G_e contains only discrete transformations.

Now we define the variation of $g_{\mu\nu}$ with respect to ξ as

$$\begin{aligned}
\delta_\xi g_{\mu\nu} :=&\ g'_{\mu\nu} - g_{\mu\nu} + \cancel{O(\xi^2)} && (3.7)\\
=&\ -\xi^\alpha \partial_\alpha g_{\mu\nu} - g_{\alpha\nu}\partial_\mu \xi^\alpha - g_{\mu\beta}\partial_\nu \xi^\beta && (3.8)\\
=&\ -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, && (3.9)
\end{aligned}$$

where the step from the second to the last line follows from the definition of the Levi-Civita connection tensor:

$$\begin{aligned}
\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu =& \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^{\ \beta}_{\mu\nu}\xi_\beta && (3.10)\\
=& \partial_\mu(g_{\alpha\nu}\xi^\alpha) + \partial_\nu(g_{\mu\alpha}\xi^\alpha) - \xi_\alpha g^{\beta\alpha}\Big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\Big) && (3.11)\\
=& \cancel{\partial_\mu(g_{\alpha\nu})\xi^\alpha} + g_{\nu\alpha}\partial_\mu(\xi^\alpha) + \cancel{\partial_\nu(g_{\mu\alpha})\xi^\alpha} + g_{\mu\alpha}\partial_\nu(\xi^\alpha) && (3.12)\\
& -\xi^\alpha\Big(\cancel{\partial_\mu g_{\alpha\nu}} + \cancel{\partial_\nu g_{\mu\alpha}}\Big) + \xi^\alpha \partial_\alpha g_{\mu\nu} && (3.13)\\
=& g_{\nu\alpha}\partial_\mu \xi^\alpha + g_{\mu\alpha}\partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu}. && (3.14)
\end{aligned}$$

Hence we may restate local conformal invariance of the metric, equation (3.4), as

$$\nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) = 2\omega(x) g_{\mu\nu}(x). \tag{3.15}$$

Note that the covariant derivatives ∇ in this equation arise directly from the symmetric part of the connection. Introducing torsion to make the theory conformally invariant will not change (3.15) in the slightest. As alluded to in 1.4 this means the symmetries of the theory are unaffected by the presence of torsion.

3.3 The Killing Equation

If we take $\omega(x) = 0$ ($\Omega(x) = e^{\omega(x)} = 1$) in equation (3.15), we recover a well-known equation called the **Killing equation**:

$$\boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.} \tag{3.16}$$

This equation allows us to find the vector fields ξ that generate isometry transformations, also known as **Killing vectors**. Given some energy-momentum tensor $T_{\mu\nu}$ that is covariantly conserved ($\nabla^\mu T_{\mu\nu} = 0$) we can construct a conserved current

$$J^\xi_\mu = \xi^\nu T_{\mu\nu} \qquad \to \qquad \nabla \cdot J = 0. \tag{3.17}$$

In general, the Killing equation and its solutions are not preserved by either conformal or Weyl transformations. This is to be expected as the Killing vectors are particular to one representative of a conformal class, not to the class itself. However, Killing vectors will be mapped onto their conformal equivalent, which we will discuss next.

3.4 The Conformal Killing Equation

If we allow $\omega(x)$ to be generic, we may eliminate it by taking the trace of (3.15), yielding

$$\nabla \cdot \xi = n\omega(x), \tag{3.18}$$

where n is the dimension of space-time. We may then rewrite (3.15) as

$$\boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2g_{\mu\nu}}{n}\nabla \cdot \xi = 0.} \tag{3.19}$$

This is the conformal generalization of the Killing equation (3.16) and therefore called the **conformal Killing equation**. Since it generalizes the Killing equation, any Killing vector automatically solves it. However, as mentioned earlier, it admits a (potentially) greater set of solutions we will call **conformal Killing vectors**. Any conformal Killing vector that is not also a Killing vector we will refer to as **proper**. The conformal Killing vectors are particular to the conformal class $[g_{\mu\nu}]$ of the metric and are thus preserved under both conformal and Weyl transformations. We will show the latter property explicitly here.

Under a Weyl transformation, the geometric objects that appear in (3.19) transform as

$$\begin{aligned} &g_{\mu\nu} \to \widetilde{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}, \qquad \xi^\mu \to \tilde{\xi}^\mu = \xi^\mu, \qquad \xi_\mu \to \tilde{\xi}_\mu = \Lambda^2 \xi_\mu, \\ &\Gamma^\alpha_{\mu\nu} \to \widetilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \Lambda^{-1}(\partial_\mu \Lambda \delta^\alpha_\nu + \partial_\nu \Lambda \delta^\alpha_\mu - g_{\mu\nu}\partial^\alpha \Lambda). \end{aligned} \tag{3.20}$$

We expand the Covariant Derivatives ∇ in (3.19) to obtain

$$\partial_{(\mu}\tilde{\xi}_{\nu)} - \widetilde{\Gamma}^\alpha_{\mu\nu}\tilde{\xi}_\alpha - \frac{1}{n}\widetilde{g}_{\mu\nu}\left(\partial_a \tilde{\xi}^\alpha + \widetilde{\Gamma}^\alpha_{\alpha\beta}\tilde{\xi}^\beta\right) = 0 \tag{3.21}$$

and consider the transformation of each term under (3.20) separately. We will find that all derivatives acting on Λ cancel exactly,

$$\partial_{(\mu}\tilde{\xi}_{\nu)} = \Lambda^2 \partial_{(\mu}\xi_{\nu)} + \cancel{2\Lambda \xi_{(\nu}\partial_{\mu)}\Lambda}, \tag{3.22}$$

$$-\widetilde{\Gamma}^\alpha_{\mu\nu}\xi_\alpha = -\Lambda^2 \Gamma^\alpha_{\mu\nu}\xi_\alpha - \cancel{2\Lambda \xi_{(\nu}\partial_{\mu)}\Lambda} + \cancel{\Lambda g_{\mu\nu}\xi_\alpha \partial_\alpha \Lambda}, \tag{3.23}$$

$$-\frac{1}{n}\widetilde{g}_{\mu\nu}\partial_a \tilde{\xi}^\alpha = -\frac{\Lambda^2}{n} g_{\mu\nu}\partial_\alpha \xi^\alpha, \tag{3.24}$$

$$\frac{1}{n}\widetilde{g}_{\mu\nu}\widetilde{\Gamma}^\alpha_{\alpha\beta}\tilde{\xi}^\beta = \frac{\Lambda^2}{n} g_{\mu\nu}\Gamma^\alpha_{\alpha\beta}\xi^\beta + \cancel{\Lambda g_{\mu\nu}\xi_\alpha \partial^\alpha \Lambda}, \tag{3.25}$$

and taking the sum on either side of equations (3.22) through (3.25) tells us that

$$\tilde{\nabla}_\mu \tilde{\xi}_\nu + \tilde{\nabla}_\nu \tilde{\xi}_\mu - \frac{2}{n}\widetilde{g}_{\mu\nu}\tilde{\nabla}\cdot\tilde{\xi} = \Lambda^2\left(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{2}{n} g_{\mu\nu}\nabla\cdot\xi\right). \tag{3.26}$$

Indeed, as long as Λ is non-vanishing on M, two representatives $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu} = \Lambda^2 g_{\mu\nu}$ of the same conformal class will admit the same number of conformal Killing vectors. This powerful result will make studying FLRW space-times in section 6 much more manageable, as it means we can eliminate the scale factor $a(t)$ from our calculations entirely.

As dynamical symmetries, we may construct (conformally) conserved currents from conformal Killing vectors. Here we have to be a bit careful as the full covariant derivative $\overline{\nabla}$ picks up additional contributions from the torsion tensor. As a result, if $T_{\mu\nu}$ is torsion-covariantly conserved ($\overline{\nabla}^\mu T_{\mu\nu} = 0$), we can construct a current $J^\xi_\mu = \xi^\nu T_{\mu\nu}$, for which

$$\overline{\nabla}\cdot J^\xi = T^{\mu\nu}\,\overline{\nabla}_\mu\,\xi_\nu = T^{\mu\nu}\,\overline{\nabla}_{(\mu}\,\xi_{\nu)} = T^{\mu\nu}\,\nabla_{(\mu}\,\xi_{\nu)} = T^{\mu\nu} g_{\mu\nu}\frac{\nabla\cdot\xi}{n} = T^\alpha_{\ \alpha}\frac{\nabla\cdot\xi}{n}. \tag{3.27}$$

Here we have used that $\Gamma^\alpha_{\ (\mu\nu)} = \mathring{\Gamma}^\alpha_{\mu\nu}$ so that $\overline{\nabla}$ reduces to the ordinary covariant derivative ∇ when symmetrized. Hence J is conserved only if the energy-momentum tensor is traceless, $T^\alpha_{\ \alpha} = 0$.[4] So we may only construct conserved quantities from proper conformal Killing vectors ($\nabla\cdot\xi \neq 0$) if $T^{\mu\nu}$ is traceless, i.e. if matter is conformal.

[4]This is a familiar requirement for those who have studied conformal field theories, for instance in the context of string theory. Here the vanishing of the trace of the stress-energy tensor is one of the defining (and necessary) features of the classical conformal theory.

4 The Conformal Killing Algebra of Flat Space-Time

To develop some understanding of the conformal Killing equation and its solutions, let us consider the simplest case of a flat space-time $\mathbb{R}^{p,q}$. Take the metric to be

$$\eta_{\mu\nu} = \text{diag}(\underbrace{-1,\ldots,-1}_{p \text{ times}},\underbrace{+1,\ldots,+1}_{q \text{ times}}) \tag{4.1}$$

so that the conformal Killing equation becomes, depending on how we write it,

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\omega\ \eta_{\mu\nu} = 0, \tag{4.2}$$

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{n}\eta_{\mu\nu}\partial\cdot\xi = 0. \tag{4.3}$$

This case will be straightforward enough to solve by hand. We will follow a line of reasoning similar to chapter 5 of [11] in what follows.

4.1 Solving the Conformal Killing Equation

We can contract (4.2) with $\partial^\mu\partial_\rho$ to get

$$\partial^2\partial_\rho\xi_\nu + \partial_\rho\partial_\nu\partial\cdot\xi - 2\partial_\rho\partial_\nu\omega = 0, \tag{4.4}$$

$$\partial^2\partial_\rho\xi_\nu + (n-2)\partial_\rho\partial_\nu\omega = 0, \tag{4.5}$$

where we have used (3.18) to go from the first line to the second. Now symmetrize this equation to obtain

$$\partial^2(\partial_\rho\xi_\nu + \partial_\nu\xi_\rho) + 2(n-2)\partial_\rho\partial_\nu\omega = 0. \tag{4.6}$$

Now take the trace and apply again (3.18) to obtain

$$4(n-1)\partial^2\omega = 0. \tag{4.7}$$

Hence, for $n \neq 1$, $\partial^2\omega = 0$. Rewrite (4.6) using (4.2) as

$$\cancel{2\partial^2\omega} + 2(n-2)\partial_\rho\partial_\nu\omega = 0, \tag{4.8}$$

so that for $n \neq 2$, $\partial_\mu\partial_\nu\omega = 0$ for any μ and ν.[5] It follows directly that from (4.2) that

$$\partial_\mu\partial_\nu\partial_{(\rho}\xi_{\sigma)} = 0 \tag{4.9}$$

and so $\partial_\mu\partial_\nu\partial_\rho\xi_\sigma$ must be antisymmetric in its last two indices. However, partial derivatives commute and so by symmetry in the first three indices we may write

$$\partial_\mu\partial_\nu\partial_\rho\xi_\sigma = \partial_\mu\partial_\rho\partial_\nu\xi_\sigma = -\partial_\mu\partial_\rho\partial_\sigma\xi_\nu \tag{4.10}$$

$$= -\partial_\mu\partial_\sigma\partial_\rho\xi_\nu = \partial_\mu\partial_\sigma\partial_\nu\xi_\rho \tag{4.11}$$

$$= \partial_\mu\partial_\nu\partial_\sigma\xi_\rho. \tag{4.12}$$

Hence $\partial_\mu\partial_\nu\partial_\rho\xi_\sigma$ is also symmetric in its last two indices and it must vanish. We can then make the ansatz

$$\xi^\mu(x) = A^\mu + B^\mu_\alpha x^\alpha + \frac{1}{2}C^\mu_{\alpha\beta}x^\alpha x^\beta. \tag{4.13}$$

Note from this equation that $C^\mu_{\alpha\beta} = C^\mu_{\beta\alpha}$. Here A, B and C are constant tensors that we may derive conditions on by plugging (4.13) back into the conformal Killing equation.

[5] For $n = 2$ we will get the Virasoro algebra, which is infinite-dimensional.

The conformal Killing equation contains only derivatives of ξ and so A is a completely free parameter that we will write as

$$A^\mu = a^\mu. \tag{4.14}$$

To obtain constraints on B we plug the ansatz into (4.3) and equate all constant terms:

$$B_{\mu\nu} + B_{\nu\mu} - \frac{2}{n}\eta_{\mu\nu}B^\alpha_\alpha = 0. \tag{4.15}$$

Hence the antisymmetric part, $B_{[\mu\nu]}$, is free and we shall denote it by $\lambda_{\mu\nu}$, while the symmetric part is constrained by

$$B_{(\mu\nu)} = \frac{1}{n}\eta_{\mu\nu}B^\alpha_\alpha =: d\eta_{\mu\nu}. \tag{4.16}$$

We will write B simply as

$$B^\mu_\alpha = \delta^\mu_\alpha d + \lambda^\mu_\alpha. \tag{4.17}$$

Conditions on C are found in a similar fashion, but this time equating all terms linear in x

$$\Big(C_{\mu\nu\alpha} + C_{\nu\mu\alpha} - \frac{2}{n}\eta_{\mu\nu}C^\beta_{\beta\alpha}\Big)x^\alpha = 0. \tag{4.18}$$

Now shuffle indices around to write

$$\begin{aligned} &\partial_\mu``(\nu\rho)" + \partial_n``(\mu\rho)" - \partial_\rho``(\mu\nu)" \\ &= \cancel{C_{\nu\rho\mu}} + C_{\rho\nu\mu} + \cancel{C_{\mu\rho\nu}} + C_{\rho\mu\nu} - \cancel{C_{\mu\nu\rho}} - \cancel{C_{\nu\mu\rho}} \\ &- \frac{2}{n}\eta\nu\rho C^\beta_{\beta\mu} - \frac{2}{n}\eta\mu\rho C^\beta_{\beta\rho} + \frac{2}{n}\eta\mu\nu C^\beta_{\beta\rho} = 0. \end{aligned} \tag{4.19}$$

We rewrite this to find the expression

$$C_{\rho\mu\nu} = \eta_{\mu\rho}\frac{C^\beta_{\beta\nu}}{n} + \eta_{\nu\rho}\frac{C^\beta_{\beta\mu}}{n} - \eta_{\mu\nu}\frac{C^\beta_{\beta\rho}}{n}. \tag{4.20}$$

From this we deduce that the behaviour of C is fully determined by its trace $C^\beta_{\beta\mu}$. We may define the free parameters $b^\mu := C^\beta_{\beta\mu}/n$ and write C as

$$C^\mu_{\alpha\beta} = \delta^\mu_\alpha b_\beta + \delta^\mu_\beta b_\alpha - \eta_{\mu\nu}b^\mu. \tag{4.21}$$

4.2 The Conformal Group

Putting this all together nets us the final form of the infinitesimal transformation.

$$\boxed{x'^\mu = x^\mu + \underbrace{a^\mu}_{\text{Translations}} + \underbrace{\lambda^\mu_\alpha x^\alpha}_{\text{Lorentz Transformations}} + \underbrace{x^\mu d}_{\text{Dilations}} + \underbrace{2(b\cdot x)x^\mu - x^2 b^\mu.}_{\text{Special Conformal Transformations}}} \tag{4.22}$$

The first two terms should be familiar as they form the Poincaré transformations. The third term simply represents a constant rescaling of all the coordinates while the last term is truly novel. We can understand this term slightly better by considering the global transformations:

$$\begin{aligned} &\textit{Translations} && x'^\mu = x^\mu + A^\mu && (4.23a) \\ &\textit{Lorentz} && x'^\mu = \Lambda^\mu_\nu x^\nu && (4.23b) \\ &\textit{Dilations} && x'^\mu = D\,x^\mu && (4.23c) \\ &\textit{SCTs} && x'^\mu = \frac{x^\mu - B^\mu x^2}{1 - 2B\cdot x + B^2x^2} && (4.23d) \end{aligned}$$

Note that a special conformal transformation for a given B is singular at exactly one point, $x^\mu = B^\mu/B^2$. This is because the true conformal group acts on the compactification of $\mathbb{R}^{p,q}$, which is $S^{p,q}$, so that the singular point is mapped to the point added by this procedure.

It is a short calculation to show that a special conformal transformation is equivalent to the following chain of maps, $x^\mu \to x^\mu/x^2$, $x^\mu \to x^\mu + B^\mu$, $x^\mu \to x^\mu/x^2$, i.e. a translation by B sandwiched in between two inversions.

4.3 Generators and Commutators

We can construct the Lie algebra $\mathfrak{conf}(\mathbb{R}^{p,q})$ from the infinitesimal transformations as follows. Let $f(x)$ represent any tensor coordinate field with suppressed indices. Now we may write for the translations, for instance, that

$$f(x') = f(x + a^\mu) = f(x) + \alpha^\mu \partial_\mu f(x) + O(\xi^2). \tag{4.24}$$

We say that the operator $P_\mu = \partial_\mu$ is the generator corresponding to the killing vector $\xi^\mu = a^\mu$. If we do this for all transformations in (4.22) we get the set of generators below.

Translations	$P_\mu = \partial_\mu$	(4.25a)
Lorentz	$L_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$	(4.25b)
Dilation	$D = x^\mu \partial_\mu$	(4.25c)
SCTs	$K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu$	(4.25d)

These generators satisfy the commutation relations

$$[P_\mu, D] = P_\mu \qquad [K_\mu, D] = -K_\mu \tag{4.26a}$$
$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu} - L_{\mu\nu}) \tag{4.26b}$$
$$[P_\mu, L_{\alpha\beta}] = \eta_{\mu\beta} P_\alpha - \eta_{\mu\alpha} P_\beta \tag{4.26c}$$
$$[K_\mu, L_{\alpha\beta}] = \eta_{\mu\beta} K_\alpha - \eta_{\mu\alpha} K_\beta \tag{4.26d}$$
$$[L_{\mu\nu}, L_{\alpha\beta}] = \eta_{\mu\alpha} L_{\nu\beta} + \eta_{\nu\beta} L_{\mu\alpha} - \eta_{\mu\beta} L_{\nu\alpha} - \eta_{\nu\alpha} L_{\mu\beta}. \tag{4.26e}$$

From these we see that the algebra closes and, thus, is well-defined. Let $n = p + q$ be the dimension of $\mathbb{R}^{p,q}$, then notice that there are n generators coming from translations and $n(n-1)/2$ from the Lorentz transformations, a single one from dilations and n extra from special conformal transformations. They form the $(n+2)(n+1)/2$ generators of the algebra $\mathfrak{conf}(\mathbb{R}^{p,q})$, which contains $\mathfrak{isom}(\mathbb{R}^{p,q})$ with $n(n+1)/2$ isometry generators as a subalgebra. Unsurprisingly perhaps, $\mathbb{R}^{p,q}$ is not only maximally symmetric in the isometric sense, but also in the conformal sense.

4.4 Identifying the Conformal Algebra

Unfortunately the commutation relations given in equation (4.26) are not particularly enlightening when written in this form. Fortunately, on the other hand, the generators form a vector space and we are free to choose a different basis in which to express the algebra. To this end, introduce a new set of antisymmetric generators J_{ab} acting on $\mathbb{R}^{p+1,q+1}$. We label the extra spatial directions $+$ and the extra temporal direction $-$ so that for $a, b \in \mu, +, -$ we define J_{ab} in the following manner:

$$J_{+\mu} = \frac{P_\mu + K_\mu}{2}, \qquad J_{-\mu} = \frac{P_\mu - K_\mu}{2}, \tag{4.27}$$
$$J_{\mu\nu} = L_{\mu\nu}, \qquad J_{-+} = D.$$

Now it is straightforward to check that these generators will satisfy the $\mathfrak{so}(p+1,q+1)$ commutation relations

$$[J_{ab}, J_{cd}] = \eta_{ac}J_{bd} + \eta_{bd}J_{ac} - \eta_{ad}J_{bc} - \eta_{bc}J_{ad}. \tag{4.28}$$

Here η_{ab} =diag$(-1, \eta_{\mu\nu}, +1)$ is the obvious extension of the metric on $\mathbb{R}^{p,q}$ to $\mathbb{R}^{p+1,q+1}$. This shows quite immediately that $\mathfrak{conf}(\mathbb{R}^{p,q}) \simeq \mathfrak{so}(p+1,q+1)$. So, up to discrete transformations such as parity transformations, we derive that $\mathrm{Conf}(\mathbb{R}^{p,q}) \simeq \mathrm{SO}(p+1,q+1)$.

5 General Space-Times

For more general space-times the presence of curvature will make life difficult. Brute-force power series solutions can be attempted, but this will only yield valid solutions if the connection tensors $\Gamma^{\alpha}_{\mu\nu}$ consist of functions that admit finite Laurent series. In particular, this will fail for spaces with positive curvature, whose connection tensors contain trigonometric functions. This approach will work better for hyperbolic spaces, but little insight is gained. Instead, we wish to develop an analytic approach.

5.1 Embedding

Our first approach is to embed M isometrically into a higher-dimensional space M' whose symmetries we know. A natural choice here would be to pick $M' = \mathbb{R}^{p,q}$ as we have just derived the conformal group for this space. As it turns out, we can indeed do this, as proven in 1970 by Greene [13] and independently by Clarke [5] in the same year. Their result can be stated as follows.

Theorem 5.1 (Pseudo-Riemannian Embedding)**.** *An n-dimensional pseudo-Riemannian manifold M with a metric of rank r and signature s can be isometrically embedded into a pseudo-Euclidean space $\mathbb{R}^{p,q}$ for p and q given by*

$$p = n - \frac{1}{2}(r+s) + 1 \tag{5.1}$$

$$q = \begin{cases} \frac{1}{2}n(3n+11), & \textit{if } M \textit{ compact.} \\ \frac{1}{6}n(2n^2+37) + \frac{5}{2}n^2 + 1, & \textit{otherwise.} \end{cases} \tag{5.2}$$

In our case, the manifold M will have $n = 4$, its metric will be nondegenerate so that $r = n = 4$ and we will have only one time-like direction so that $s = n - 2 = 2$. Hence any manifold we consider can be embedded into $\mathbb{R}^{2,q}$ for various values of q. The idea is that, since the embedding is isometric, all symmetries of M must be expressible in terms of symmetries of $\mathbb{R}^{p,q}$ that preserve the embedding equation. For example, we may embed the n-sphere S^n into $\mathbb{R}^{n+1}$ by the equation

$$\vec{X}^2 = L^2, \tag{5.3}$$

where L is the radius of the sphere. The only isometries of $\mathbb{R}^{n+1}$ that preserve this equation are rotations around the origin, which form the group $SO(n+1)$ that we indeed associate as the isometries of the n-sphere.

In the case of conformal isometries, we should relax the requirement by asking that symmetries of $\mathbb{R}^{n+1}$ preserve the embedding equation *up to a conformal factor*. Hence the conformal requirement is that

$$\vec{X'}^2 = \Omega^2(X)\vec{X}^2 = \Omega^2(X)L^2. \tag{5.4}$$

We see immediately that dilations $x \to D\,x$ in $\mathbb{R}^{p,q}$, with conformal factor $\Omega^2(X) = D^2$ satisfy this condition. However, there is no guarantee that there exists a transformation in the coordinates of S^n with this effect. To this end, consider the explicit embedding

$$X^1 = L\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{n-1})\sin(\phi) \tag{5.5a}$$

$$X^2 = L\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{n-2})\cos(\phi) \tag{5.5b}$$

$$X^3 = L\sin(\theta_1)\sin(\theta_2)\ldots\cos(\theta_{n-2}) \tag{5.5c}$$

$$\vdots \tag{5.5d}$$

$$X^{n+1} = L\cos(\theta_1), \tag{5.5e}$$

where $\theta_i \in [0, \pi)$ and $\phi \in [0, 2\pi)$. Note that there is no transformation of the angles θ_i and ϕ that has the effect of multiplying X^μ by an overall factor of D. Instead, the effect of the $\mathbb{R}^{n+1}$-dilation on S^n can only be realised by a Weyl transformation, which is not part of the dynamical symmetries of S^n.

Hence, when using embeddings to find (conformal) isometries, one must be careful. Conformal isometries of $\mathbb{R}^{p,q}$ that preserve the embedding transformation may not correspond to dynamical transformations of the embedded space M. Figuring out whether or not this happens can be a tricky problem to tackle and for that reason we will prefer to use the conformal Killing equation in explicit calculations most of the time.

5.2 Product Manifolds

A second approach is as follows. Suppose M is a product manifold that decomposes into two nontrivial pieces N_1 and N_2 so that we may write

$$M = N_1 \times N_2, \qquad \mathbf{g}_M = \mathbf{g}_{N_1} \oplus \mathbf{g}_{N_2}, \tag{5.6}$$

where $\mathbf{g}_{N_1}$ depends only on the coordinates of N_1 and $\mathbf{g}_{N_2}$ depends only the coordinates of N_2. Suppose that we know about the symmetries of the factors N_1 and N_2. What can we learn about symmetries of M?

Observe that the metric decomposes into block diagonal form, where each block is the metric on the appropriate factor. For isometries it is clear that the following holds

$$\mathrm{Isom}(M) \supseteq \mathrm{Isom}(N_1) \times \mathrm{Isom}(N_2), \tag{5.7}$$

as an isometry from either factor N, when acting on M will preserve both block diagonals in (5.6). Equality holds if N_1 and N_2 are compact, [8] but for more general manifolds the full isometry group may be bigger (as is the case when both N_1 and N_2 are flat).

A similar analysis, unfortunately, does not yield much for the conformal group. A conformal isometry ϕ_i of N_i acting on M will multiply only the $\mathbf{g}_{N_i}$ block diagonal in (5.6) by a conformal factor Ω^2 rather than the entire metric. It cannot, therefore, be an element of $\mathrm{Conf}(M)$ unless $\Omega = 1$, in which case it is a regular isometry.

The only way to find proper elements of $\mathrm{Conf}(M)$ in this way is by looking for $\phi_1 \in \mathrm{Conf}(N_1)$ and $\phi_2 \in \mathrm{Conf}(N_2)$ with identical conformal factors and having them act simultaneously on M. Locally, this amounts to finding $\xi_1 \in \mathfrak{conf}(N_1)$ and $\xi_2 \in \mathfrak{conf}(N_2)$ so that $\nabla \cdot \xi_1 = \nabla \cdot \xi_2 \propto \omega$ and then taking $\xi_M = (\xi_1, \xi_2)$. Note that ξ_1 can only depend on coordinates of N_1 and so ω is independent of coordinates of N_2. But the same argument works the other way around and we are forced to conclude that ω is independent of all coordinates, i.e. it is a constant.

Looking back at (4.22) we see that dilations ($\xi_d^\mu = d\, x^\mu$) satisfy this property, with $\omega_d \propto d$. However, special conformal transformations ($\xi_b^\mu = 2(b \cdot x)x^\mu - x^2 b^\mu$) do not, with $\omega_b \propto b \cdot x$.

5.3 Geometric Isolation

We now wish to argue that for a manifold M with a nontrivial decomposition, special conformal transformations do not induce (nontrivial) conformal isometries on M "most" of the time. The argument, while imprecise, goes like this.

First define the notion of geometric isolation.

Definition 5.2 (Geometric Isolation)**.** Let $x^{\hat{\mu}}$ and $x^{\hat{\nu}}$ be two coordinate directions of a manifold M. Then we say that these two directions are **Geometrically Isolated** if the following three conditions are satisfied.

- $g_{\hat{\mu}\hat{\nu}} = 0$.
- $g_{\hat{\mu}\alpha}$ does not depend on the coordinate $x^{\hat{\nu}}$ for any α.
- $g_{\hat{\nu}\alpha}$ does not depend on the coordinate $x^{\hat{\mu}}$ for any α.

Furthermore, suppose that N_1 and N_2 are factors in a product decomposition of M. Then we say that N_1 is geometrically isolated from N_2 if all coordinate directions of N_1 are isolated from those of N_2 in the above sense.

Suppose that M admits a decomposition as presented in (5.6) into two factors N_1 and N_2. By assumption, the metric $\mathbf{g}_{N_1}$ does not depend on the coordinates of N_2 and vice-versa, and the block diagonal decomposition of $\mathbf{g}_N$ means there are no components $g_{\mu\nu}$ where x^μ is a coordinate of N_1 and x^ν is a coordinate of N_2. That is to say, N_1 and N_2 are factors that are geometrically isolated.

This means that we may embed each of them into a flat manifold $\mathbb{R}^{p_i,q_i}$ $(i = 1, 2)$ so that M gets embedded into $\mathbb{R}^{p,q}$ for $p = p_1 + p_2$ and $q = q_1 + q_2$. As a result, the embedding equation $\vartheta : M \hookrightarrow \mathbb{R}^{p,q}$ factors as $(\vartheta_1, \vartheta_2) : N_1 \times N_2 \hookrightarrow \mathbb{R}^{p_1,q_1} \times \mathbb{R}^{p_2,q_2}$ in such a way that the embedding equation of N_1 is independent of the coordinates of N_2 and vice versa. Any conformal symmetry on $\mathbb{R}^{p,q}$ that induces a symmetry on M should respect this factorization conformally.

For a local special conformal transformation, $x^\mu \to x^\mu + 2(b\cdot x)x^\mu - x^2 b^\mu$, a given coordinate $x^{\hat{\nu}}$ picks up contributions from all other coordinates. This is not a problem as long as we can rewrite these contributions as a global conformal factor. However, contributions from special conformal transformations are *additive*, not multiplicative. So unless the embedding equations ϑ_1 and ϑ_2 are exactly such that this still happens, we will violate the factorization of ϑ. Hence for "most" embeddings, special conformal transformations do not induce conformal isometries on M.

A notable exception to the above is when N_1 and N_2 are both flat, in which case M can be trivially embedded into $\mathbb{R}^{p,q}$ by $X^\mu = x^\mu$. While a full investigation is beyond the scope of this work, we feel justified in formulating the Geometric Isolation Conjecture as follows.

Conjecture 5.3 (Geometric Isolation)**.** *Let M be any pseudo-Riemannian manifold that admits the following decomposition*

$$M = N_1 \times N_2, \qquad \mathbf{g}_M = \mathbf{g}_{N_1} \oplus \mathbf{g}_{N_2}, \tag{5.8}$$

subject to the following extra conditions.

1. *N_1 and N_2 are both indecomposable and not isomorphic to a flat pseudo-Riemannian manifold; or, only one satisfies this property and the other is isomorphic to $\mathbb{R}^{a,b}$ for some a and b.*
2. *N_1 and N_2 are nontrivial, i.e. of at least dimension one.*
3. *N_1 and N_2 are geometrically isolated as defined in definition 5.2.*

Then no (nontrivial) conformal isometry of M is inherited from special conformal transformations upon embedding M isometrically into $\mathbb{R}^{p,q}$.

It is clear that this conjecture can be generalized to an arbitrary product in an obvious manner. Without further proof we will assume that the conjecture holds true for the manifolds we will study in this work. In the following section, we will show that this conjecture simplifies our life immensely when it comes to product manifolds.

5.4 Transverse-Longitudinal Decomposition

The covariant divergence $\nabla \cdot \xi$ in equation (3.19) motivates us to formally decompose ξ as

$$\xi^\mu = \xi^\mu_L + \xi^\mu_T \equiv \nabla^\mu \theta + \chi^\mu, \tag{5.9}$$

where θ is a scalar and χ is transverse to ∇ in the sense that $\nabla \cdot \chi = 0$. Using (3.18) we see that all contributions to the conformal factor comes from the longitudinal part of ξ,

$$\nabla^2 \theta = \nabla \cdot \xi = n\omega(x). \tag{5.10}$$

This shows us that there are essentially three types of solutions to the conformal Killing equation.

1. First we have Killing vector solutions satisfying $\nabla \cdot \xi = 0$.
2. Secondly we have longitudinal, proper, conformal Killing vectors, which we describe as $\xi^\mu = \nabla^\mu \theta$, satisfying $\omega = \frac{\nabla^2 \theta}{n} \neq 0$.
3. Lastly we have proper, conformal Killing vectors with both a longitudinal component $\nabla^\mu \theta$ and a transverse component χ^μ that do not solve equation (3.19) separately, but whose sum does. These, too, satisfy $\omega = \frac{\nabla^2 \theta}{n} \neq 0$.

Referring back to our results in section 4 we see that translations and Lorentz transformations, being isometries, fall into the first category. Dilations ($\xi^\mu = d\, x^\mu$) can be written as $\xi^\mu = d\, \frac{\partial^\mu x^2}{2}$ and fall into the second category. Special conformal transformations ($\xi^\mu = 2(b \cdot x)x^\mu - x^2 b^\mu$) fall into the third category as $\partial^{[\nu}\xi^{\mu]} = 2(b^\nu x^\mu - x^\nu b^\mu) \neq 0$ and so ξ^μ cannot be written in the form $\xi^\mu = \partial^\mu \theta$ for any θ.

5.5 Product Manifolds and Geometric Isolation

Now suppose that we are in the situation outlined in the Geometric Isolation conjecture, where M decomposes into two (or more) geometrically isolated factors N_i. Then, upon embedding M into $\mathbb{R}^{p,q}$, translations, Lorenz transformations and dilations of $\mathbb{R}^{p,q}$ will induce conformal isometries on M, but not special conformal transformations. As a result, we will only need to look for conformal Killing vectors of the first two types: Killing vectors and proper, longitudinal, conformal Killing vectors. Moreover, as $\omega \propto d$ is constant for dilations, we only need to consider CKV with a constant conformal factor. Since covariant derivatives commute when acting on a scalar, we may reduce the conformal Killing equation (3.19) to

$$\boxed{\nabla_\mu \nabla_\nu \theta - g_{\mu\nu}\omega = 0, \qquad \nabla^2 \theta = n\omega,} \tag{5.11}$$

where ω is just a constant.

We can push the analysis a bit further. We can see what happens if we choose $\hat{\mu}$ and $\hat{\nu}$ such that $x^{\hat{\mu}}$ is a coordinate in N_i and $x^{\hat{\nu}}$ is a coordinate in N_j for $i \neq j$. Given that all factors in the decomposition of M are geometrically isolated, it follows that $g_{\hat{\mu}\hat{\nu}} = 0$ and, for all β, $g_{\hat{\mu}\beta}$ is independent of $x^{\hat{\nu}}$ while $g_{\hat{\nu}\beta}$ is independent of $x^{\hat{\mu}}$. Now write out the $\Gamma^{\alpha}_{\hat{\mu}\hat{\nu}}$ components of the Levi-Civita connections,

$$\Gamma^{\alpha}_{\hat{\mu}\hat{\nu}} = \frac{1}{2} g^{\alpha\beta}\Big(\partial_{\hat{\mu}} g_{\hat{\nu}\beta} + \partial_{\hat{\nu}} g_{\hat{\mu}\beta} - \partial_\beta g_{\hat{\mu}\hat{\nu}}\Big), \tag{5.12}$$

and see that these terms vanish for every α. Hence this component of the connection vanishes. We expand the covariant derivatives in the $\hat{\mu}\hat{\nu}$-component of (5.11),

$$0 = \nabla_{\hat{\mu}} \nabla_{\hat{\nu}} \theta - \cancel{g_{\hat{\mu}\hat{\nu}} \omega} = \nabla_{\hat{\mu}} \partial_{\hat{\nu}} \theta = \partial_{\hat{\mu}} \partial_{\hat{\nu}} \theta - \cancel{\Gamma^{\alpha}_{\hat{\mu}\hat{\nu}} \partial_{\alpha} \theta} = \partial_{\hat{\mu}} \partial_{\hat{\nu}} \theta. \tag{5.13}$$

What remains admits the very easy solution $\theta = \theta_1(x^{\hat{\mu}}) + \theta_2(x^{\hat{\nu}})$. We can repeat this analysis for every such pair to conclude that θ must be of the form

$$\boxed{\theta = \theta_1(x_{N_1}) + \theta_2(x_{N_2}) + \ldots + \theta_k(x_{N_k})} \tag{5.14}$$

where the N_i subscript on x indicates that the function θ_i depends only on coordinates of N_i.

Combining this with (5.11) represents a monumental simplification over the general case. We no longer have to solve a large set of coupled partial differential equations where each component of ξ couples directly to every other via $\nabla \cdot \xi$. Instead, different factors of M couple only through the auxiliary Poisson equation $\nabla^2 \theta = n\omega$, which is simplified by the fact that ω is constant. The ansatz (5.14) means that we can solve the equation

$$\boxed{\nabla_{\mu_i} \nabla_{\nu_i} \theta_i - g_{\mu_i \nu_i} \omega = 0,} \tag{5.15}$$

where no summation is implied, separately for θ_i in each factor N_i, with coordinates x^{μ_i}. Once solved, we simply impose that $\nabla^2 \theta = n\omega$, so that the total number of equations we have to solve is drastically reduced.

This simplification procedure, along with the steps that lead up to it, can be considered the central result of this thesis. In what follows we will apply it to space-times of interest.

6 Cosmological Space-Times

While $\mathbb{R}^{1,3}$ with its standard pseudo-Euclidean metric is easy to describe, it is also static. This is contrary to observation, which holds that *space* is more or less homogeneous and isotropic, while space-time as a whole is not. An accurate model for the background manifold of space-time should reflect this, as well as the fact that space appears to be expanding.

A very natural choice for the above is to decompose space-time as $M = \mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents the direction of time and Σ is a 3-manifold that represents the space-like slices of the universe. We equip this manifold with the following metric,

$$ds^2 = -dt^2 + a^2(t)d\sigma^2. \tag{6.1}$$

The dynamics of the expanding universe are captured in the evolution of the scale factor a through the Friedmann equations. Note that if we were to choose Σ maximally symmetric, i.e. $\Sigma = \mathbb{R}^3$, S^3 or $\mathbb{H}^3$, we are in the Friedmann-Lemaître-Robertson-Walker case mentioned in section 1.1 and can rewrite (6.1) as the FLRW metric given in (1.2). Most of modern cosmology uses the flat FLRW case as a background space-time from which small perturbations can deviate.

While these three FLRW geometries are certainly well-known, we argue that there are other valid choices for Σ that are on equal footing. Any 3-manifold characterized by a spatial curvature radius $\sim 1/\sqrt{|k|}$ that is large compared to the observable universe is a potential candidate for the background of the universe; as long as its is homogeneous and isotropic enough to fit current observations. For this reason we broaden our scope to the class of space-time manifolds that decompose as $\mathbb{R} \times \Sigma$, with a metric given by (6.1), but relaxing the strict requirement of homogeneity and isotropy since local observers can never establish these properties globally. This allows for a much larger zoo of 3-manifolds Σ.

6.1 The Eight Thurston Geometries

While there is no complete classification of all possible 3-manifolds, there is a full classification of closed, oriented 3-manifolds by the mathematician William Thurston. He first proposed this famous Geometrization Conjecture in 1982 after many years of work on the topic. Grigori Perelman provided a partial proof of the conjecture in 2003, which has been expanded upon in subsequent years so that the conjecture is considered true. Even though it should therefore be called the Thurston-Perelman Geometrization Theorem, it has retained the 'conjecture' moniker in literature. Many different formulations of the conjecture exist, but we will present the following.

> **Conjecture 6.1** (Thurston Geometrization)**.** *Every closed, oriented three-dimensional Riemannian manifold Σ can be decomposed into pieces which have geometric structures.*

Central to this statement is the notion of a *geometric structure*. We will expand upon this briefly, paraphrasing parts of Thruston's original work [22] combined with a very easily readable overview of the conjecture by Grady [12]. We refer the interested reader to either source for a more detailed overview.

A **Geometry** is a pair $(X, \mathrm{Isom}(X))$ consisting of a simply-connected, complete and homogeneous Riemannian manifold X with its isometry group.

A complete Riemannian manifold Σ has a **Geometric Structure** based on X if it is isometric to the quotient X/Γ. Γ is taken to be a discrete subgroup of $\mathrm{Isom}(X)$ without any fixed points.

With these definitions in mind, we may expand upon the conjecture by Thurston's classification of the eight maximal geometries in three dimensions.

> *Any maximal, simply connected, three-dimensional geometry X that admits a compact quotient is equivalent to one of the eight geometries below.*

- $\mathbb{R}^3$
- $\mathbb{H}^3$
- S^3
- Nil
- $\widetilde{\mathrm{SL}(2,\mathbb{R})}$
- $\mathbb{H}^2 \times \mathbb{R}$
- $S^2 \times \mathbb{R}$
- Solv

If we understand the symmetries of these eight geometries we can describe a much larger class of FLRW-type spacetimes. Not only can we consider all possible closed 3-manifolds this way, each of the eight geometries above can be made a valid candidate for the spatial slices of our universe. Moreover, since these eight geometries are *maximal*, there is a wide range of other open manifolds we can construct from these. For example, the manifold $\mathbb{H}^2 \times S^1$ can be constructed as $\mathbb{H}^2 \times \mathbb{R}/\mathbb{1}_{\mathbb{H}^2} \times \mathbb{Z}$.

There are, however, also several limitations to the conjecture that prevent it from being a full classification of 3-manifolds.

While Thurston's conjecture classifies all possible simply-connected geometries in three dimensions, it says very little about spaces that do not admit a compact quotient. There may be other noncompact 3-manifolds that do not fit in the above classification, so we shouldn't take the conjecture as a full classifications of the possible spatial slices. In particular, any noncompact, *multiply-connected* 3-manifold is not covered in this classification.

The *orientability* condition on Σ precludes us from considering Möbius type geometries. This somewhat limits the scope of our investigation, at least from the point of view of the global geometry.

Completeness of the metric should be interpreted as geodesic completeness. That is to say, we require that every geodesic can be extended indefinitely so that Σ may not contain any punctures or holes. Black Hole solutions to the Einstein Field Equations contain exactly such singularities. Therefore they are not covered under Thurston's Geometrization Conjecture and should be studied separately.

Lastly, recall again that aggregates of cosmological observations put the curvature energy density Ω_K at a value of 0.000 ± 0.005, [21]. So any curved choice for Σ is constrained to have a very large curvature radius $1/\sqrt{|k|}$. Similar constraints will hold for spaces with global anisotropies or inhomogeneities.

Thurston's classification of closed 3-manifolds can be thought of as a generalization of the two-dimensional classification of Riemann Surfaces. This lower-dimensional variant holds that any such surface is conformally equivalent to S^2/Γ, $\mathbb{R}^2/\Gamma$ or $\mathbb{H}^2/\Gamma$, where Γ is again a discrete subgroup without fixed points. The three-dimensional case differs primarily in that Σ does not necessarily consist of a single piece, but may decompose into multiple pieces, each of which is based on a single maximal geometry.

It should also be noted that this work is not the first to study FLRW space-times using Thurston's conjecture. Seminal work was done by Fagundes in 1985 [9] and improved upon in 1992 [10] by the same author. In these works, Fagundes establishes a partial correspondence between the eight Thurston geometries and the classification of spatially homogeneous metrics into Bianchi-type and Kantowski-Sachs types. Our study of Thurston geometries in a conformal setting appears to be novel, however.

6.2 Symmetry Restoration

Since we are indeed in a conformal setting, we have an extra gauge transformation at our disposal. We may use this to restore some of the symmetry lost in foliating space-time into space and time but eliminating the scale factor a from the metric entirely. Recall that we equip cosmological space-times with the metric

$$ds^2 = -dt^2 + a^2(t)d\sigma^2. \tag{6.2}$$

First transform this metric to conformal time by integrating $dt = a(t)d\eta$ to write

$$ds^2 = a^2(t)\Big(- d\eta^2 + d\sigma^2\Big). \tag{6.3}$$

Recall that we showed in section 3.4, starting from equation (3.20), that the conformal Killing equations and its solutions are invariant under Weyl transformations. Explicitly, if ξ is a conformal Killing vector for a metric $\mathbf{g}$, then it is also a conformal Killing vector for the metric $\widetilde{\mathbf{g}} = \Lambda^2\mathbf{g}$.

Now note that (6.3) can be rewritten as $ds^2 = a^2(t)\widetilde{ds^2}$, where $\widetilde{ds^2} = -d\eta^2 + d\sigma^2$. This means that these two metrics belong to the same conformal class and are related by a Weyl transform $g_{\mu\nu} \to \widetilde{g_{\mu\nu}} = \Lambda^2 g_{\mu\nu}$ with $\Lambda = a^{-1}$. From the results of section 3.4 we then conclude that $g_{\mu\nu}$ and $\widetilde{g_{\mu\nu}}$ must admit the *same* conformal Killing vectors. Hence, without loss of generality, we can use the metric

$$ds^2 = -d\eta^2 + d\sigma^2 \tag{6.4}$$

to solve for the conformal Killing vectors of (6.3), which are related to the conformal Killing vectors of (6.2) by a simple coordinate transformation.

From the form of the metric (6.4) we see immediately that (conformal) time translations $\eta \to \eta' = \eta + a^0$ are restored as conformal isometries. Depending on the form of Σ, there may be more, which we will have to find by solving the conformal Killing equation explicitly.

There is one caveat, however, namely that a Weyl transformation with $\Lambda = a^{-1}$ is not well-defined at $a = 0$. If the history of the universe is such that $a = 0$ for some finite t in the past, then any generator involving time will not exponentiate to a global one-parameter subgroup. For instance, in the case of the generator of time translations P_t, the global transformation $\phi_t = \exp(sP_t)$ is not well-defined for negative s.[6] Hence we are restricted to positive s only, which means we find a one-parameter *semigroup*. Nonetheless the local symmetries are worth studying.

[6] It is not well defined for negative s because this would attempt to evolve us to a time 'before' the singularity at $a = 0$. We cannot do this as geodesics are not extendible to a time before this point.

7 The Conformal Algebra of the Thurston Geometries

In this section we will (attempt to) calculate the conformal algebra of Cosmological space-times $\mathbb{R} \times \Sigma$, where Σ is given by one of the eight Thurston Geometries. The starting point for each of these calculations is the metric (6.4), so that we don't have to show the steps in the previous section every time. For the purpose of legibility we will write often t instead of η after the discussion of flat geometry, but conformal time η is implied every time we write t.

For spaces of positive curvature we will prefer to use the embedding method outlined in section 5.1. For spaces of negative curvature we will solve the conformal Killing equations explicitly, solving the Killing equation first to find the Killing vectors and then solving the conformal Killing equation to find longitudinal conformal Killing vectors, in line with the discussion in sections 5.3 and 5.5.

7.1 Flat Geometry $\mathbb{R}^{1,3}$

For the flat Thurston Geometry very little remains to calculate. From the previous section we see that the metric of this geometry may be presented as

$$ds^2 = -d\eta^2 + \delta_{ij} dx^i dx^j. \tag{7.1}$$

This puts us in the situation of section 4 and we immediately recover the full conformal algebra.

$$\boxed{\mathrm{Conf}(\mathbb{R}^{1,3}) = \mathrm{SO}(2,4), \quad \mathfrak{conf}(\mathbb{R}^{1,3}) = \mathfrak{so}(2,4) = \langle P_\mu, L_{\mu\nu}, D, K_\mu \rangle\,.} \tag{7.2}$$

Here the angular brackets $\langle ... \rangle$ indicate that the algebra is spanned by the elements between them. For the sake of completeness we reproduce the explicit form of the generators from section 4 below, where now μ α and β run over the elements η, x, y and z.

$$P_\mu = \partial_\mu \qquad L_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu \qquad D = x^\mu \partial_\mu \qquad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu \tag{7.3}$$

These will satisfy the commutation relations given by (4.26), which are reproduced below. Again μ α and β now run over η, x, y and z

$$[P_\mu, D] = P_\mu \qquad [K_\mu, D] = -K_\mu \tag{7.4a}$$
$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu} - L_{\mu\nu}) \tag{7.4b}$$
$$[P_\mu, L_{\alpha\beta}] = \eta_{\mu\beta} P_\alpha - \eta_{\mu\alpha} P_\beta \tag{7.4c}$$
$$[K_\mu, L_{\alpha\beta}] = \eta_{\mu\beta} K_\alpha - \eta_{\mu\alpha} K_\beta \tag{7.4d}$$
$$[L_{\mu\nu}, L_{\alpha\beta}] = \eta_{\mu\alpha} L_{\nu\beta} + \eta_{\nu\beta} L_{\mu\alpha} - \eta_{\mu\beta} L_{\nu\alpha} - \eta_{\nu\alpha} L_{\mu\beta}. \tag{7.4e}$$

By invariance of the conformal Killing equation under Weyl transformations the algebra generators given in (7.3) also generate the conformal Killing algebra of the original metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \tag{7.5}$$

To rewrite the generators in these coordinates, we simply coordinate transform terms in (7.3) using $dt = a(t) d\eta$. For instance, the generator of conformal time translations, $P_\eta = \partial_\eta$, will simply be $a^{-1}(t)\partial_t$ in the original coordinates.

7.2 Spherical Geometry $\mathbb{R} \times S^3$

Drawing upon the example of section 5.1 we present the following explicit embedding of $\mathbb{R} \times S^3$ into $\mathbb{R}^{1,4}$.

$$X^0 = t \tag{7.6a}$$
$$X^1 = L\sin(\theta_1)\sin(\theta_2)\sin(\phi) \tag{7.6b}$$
$$X^2 = L\sin(\theta_1)\sin(\theta_2)\cos(\phi) \tag{7.6c}$$
$$X^3 = L\sin(\theta_1)\cos(\theta_2) \tag{7.6d}$$
$$X^4 = L\cos(\theta_1), \tag{7.6e}$$

where $t \in \mathbb{R}$, θ_1, $q_2 \in [0, \pi)$ and $\phi \in [0, 2\pi)$. This can be summarized compactly by the statement that $X^0 = t$ and $\vec{X}^2 = L^2$. The question is which of the conformal isometries of $\mathbb{R}^{1,4}$ preserve both equations. To this end, recall the form of these transformations from section 4 on the next page.

To (conformally) preserve $X^0 = t$ we can immediately rule out Lorentz boosts Λ^0_i, but all of the other transformations are permissible, given that we set $B^0 = 0$. We can then rewrite the contribution of a dilation or special conformal transformation to this embedding as multiplication by a factor D or $(1 - 2B \cdot X + B^2X^2)^{-1}$ respectively.

Type	Locally	Globally
Translation	$X'^\mu = X^\mu + a^\mu$	$X'^\mu = X^\mu + A^\mu$
Lorentz	$X'^\mu = X^\mu + \lambda^\mu_\alpha X^\alpha$	$X'^\mu = \Lambda^\mu_\nu X^\nu$
Dilation	$X'^\mu = X^\mu + X^\mu d$	$X'^\mu = D\, X^\mu$
SCT	$X'^\mu = X^\mu + 2(b \cdot X)X^\mu - X^2 b^\mu$	$X'^\mu = \frac{X^\mu - B^\mu X^2}{1 - 2B\cdot X + B^2X^2}$

To (conformally) preserve $\vec{X}^2 = L^2$, we can rule out spatial translations A^i in addition to Lorentz boosts Λ^0_i. Dilations are fine in principle, but we will have to do a bit more work for the special conformal transformations. Given that we have set $B^0 = 0$, write

$$\vec{X'}^2 = \frac{\vec{X}^2 - 2\vec{B}\cdot\vec{X}X^2 + \vec{B}^2X^4}{(1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2)^2} \tag{7.7}$$
$$= \frac{(X^2 + (X^0)^2) - 2\vec{B}\cdot\vec{X}X^2 + \vec{B}^2X^4}{(1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2)^2} \tag{7.8}$$
$$= X^2\frac{1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2}{(1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2)^2} + \frac{(X^0)^2}{(1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2)^2} \tag{7.9}$$
$$= \frac{X^2}{1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2} + \frac{(X^0)^2}{(1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2)^2}. \tag{7.10}$$

If the two denominators agree, this equation will read $\vec{X'}^2 = \vec{X}^2$ and we preserve the embedding of S^3. But requiring that $1 - 2\vec{B}\cdot\vec{X} + \vec{B}^2X^2$ is equal to its square means that it is either zero or one. Zero is not permissible as all SCTs then diverge and one can only be obtained for every X by setting $\vec{B} = 0$. Then we have set $B^\mu = 0$ entirely and no special conformal transformation survives, as suggested by the Geometric Isolation conjecture.

Similar to section 5.1, no transformation of the angles θ_1, θ_2 or ϕ in the embedding (7.6) induces a dilation on the spatial slice of $\mathbb{R}^{1,4}$ and so it is not a conformal isometry for this geometry.

This leaves us with just the X^0-translations and spatial rotations Λ^i_j so that we may write

$$\boxed{\mathrm{Conf}(\mathbb{R}\times S^3)=\mathbb{R}\times\mathrm{SO}(3),\quad \mathfrak{conf}(\mathbb{R}\times S^3)=\mathbb{R}\times\mathfrak{so}(3)=\langle P_t, L_{ij}\rangle\,.} \tag{7.11}$$

Note that, despite the presence of conformal symmetry, we only pick up a single proper conformal Killing vector in the form of a time translation.

7.3 Spherical Geometry $\mathbb{R}^{1,1}\times S^2$

This geometry akin to the previous one. We embed this space-time into $\mathbb{R}^{1,4}$ by $(X^0,X^1)=(t,x)$ and $\overline{X}^2=L^2$, where $\overline{X}$ indicates a 'vector' of the three components (X^2,X^3,X^4).

Repeating the arguments of the previous section we see that the three rotations L_{23}, L_{34}, L_{42} preserve the embedding of S^2, while translations in the X^2, X^3 and X^4 directions violate it. We see separately that the translations P_0, P_1 and the boost L_{01} are fine in the $\mathbb{R}^{1,1}$ sector. Dilations are again appear to be fine, but there is no transformation of the angles on S^2 that induces a dilation on the last three coordinates of $\mathbb{R}^{1,4}$, hence we throw them out. The procedure for the special conformal transformations is almost identical: we argue that $B^0=B^1=0$ and can write the analogue of (7.7) as

$$\overline{X'}^2=\frac{\overline{X}^2-2\overline{B}\cdot\overline{X}X^2+\overline{B}^2X^4}{(1-2\overline{B}\cdot\overline{X}+\overline{B}^2X^2)^2} \tag{7.12}$$

$$=\ldots=\frac{X^2}{1-2\overline{B}\cdot\overline{X}+\overline{B}^2X^2}+\frac{(X^0)^2-(X^1)^2}{(1-2\overline{B}\cdot\overline{X}+\overline{B}^2X^2)^2}. \tag{7.13}$$

Again, we will only obtain a term proportional to $\overline{X}^2$ if the denominators agree, leading to $\overline{B}=0\to B^\mu=0$.

Thus only translations and boosts on $\mathbb{R}^{1,1}$ and rotations on S^2 separately will correspond to conformal isometries on $\mathbb{R}^{1,1}\times S^2$.

$$\boxed{\begin{aligned}&\mathrm{Conf}(\mathbb{R}^{1,1}\times S^2)=\mathbb{R}^2\rtimes\mathrm{SO}(1)\times\mathrm{SO}(2)\\&\mathfrak{conf}(\mathbb{R}^{1,1}\times S^2)=\mathbb{R}^2\rtimes\mathfrak{so}(1)\times\mathfrak{so}(2)=\langle P_t,P_x,L_{tx},L_{23},L_{34},L_{42}\rangle\,.\end{aligned}} \tag{7.14}$$

7.4 Hyperbolic Geometry $\mathbb{R}\times\mathbb{H}^3$

We will tackle the hyperbolic spaces $\mathbb{R}\times\mathbb{H}^3$ and $\mathbb{R}\times\mathbb{R}\times\mathbb{H}^2$ by solving the conformal Killing equation in full. The former will be covered in this section and the latter in the section hereafter.

To describe $\mathbb{H}^3$, we opt for the upper half-space representation as it easier to tackle than the FLRW metric (1.2) with negative k. Taking coordinates $t,y,z\in\mathbb{R}$ and $x>0$, the metric of this space takes the form

$$ds^2_{\mathbb{R}\times\mathbb{H}^3}=-dt^2+\frac{L^2}{x^2}\Big(dx^2+dy^2+dz^2\Big). \tag{7.15}$$

Here $L\propto 1/\sqrt{-k}$ characterizes the curvature. The nontrivial Christoffel symbols are

$$\Gamma^x_{xx}=\Gamma^y_{xy}=\Gamma^z_{xz}=-1/x\qquad\quad \Gamma^x_{yy}=\Gamma^x_{zz}=1/x, \tag{7.16}$$

and the Ricci scalar is $-6/L^2$, indicating that this is indeed a negatively curved space.

As promised, we will first solve the Killing Equation before moving on to the conformal isometries. We may decompose the components of the Killing equation as follows.

Diagonal	**Homogeneous**	**Inhomogeneous**
(1) $\partial_t\xi_t = 0$	(i) $\partial_t\xi_x + \partial_x\xi_t = 0$	(I) $\partial_x\xi_y + \partial_y\xi_x + 2\xi_y/x = 0$
(2) $\partial_x\xi_x + \xi_x/x = 0$	(ii) $\partial_t\xi_y + \partial_y\xi_t = 0$	(II) $\partial_x\xi_z + \partial_z\xi_x + 2\xi_z/x = 0$
(3) $\partial_y\xi_y - \xi_x/x = 0$	(iii) $\partial_t\xi_z + \partial_z\xi_t = 0$	
(4) $\partial_z\xi_z - \xi_x/x = 0$	(iv) $\partial_y\xi_z + \partial_z\xi_y = 0$	

We start by noting that equation (2) is a simple ODE that is solved by $\xi_x = f_x(t,y,z)/x$ for a yet-to-be determined function f_x. Now take the y-derivative of equation (I) to see that, via equation (3),

$$0 = \partial_y^2\xi_x + (\partial_x + 2/x)\partial_y\xi_y = \partial_y^2\xi_x + (\partial_x + 2/x)\xi_x/x = \frac{\partial_y^2 f_x}{x} + \cancelto{0}{(\partial_x + 2/x)\frac{f_x}{x^2}} = \frac{\partial_y^2 f_x}{x}. \quad (7.17)$$

Hence we see that f_x is at most first-order in y. Similarly, we can take the z-derivative of equation (II) and use (4) to conclude that f_x is also at most first-order in z.

Now make the ansatz that $f_x(t,y,z) = 2a_y(t)y + 2a_z(t)z + b(t) + 2d(t)yz$ and use this to find a formal solution to equation (3),

$$\xi_y = f_y(t,x,z) + \frac{1}{x^2}\int dy f_x(t,y,z). \quad (7.18)$$

Plug this solution into (I) to obtain

$$\partial_x f_y - \cancel{\frac{2}{x^3}\int dy f_x(t,y,z)} + \frac{\partial_y f_x}{x} + \frac{2f_y}{x} + \cancel{\frac{2}{x^3}\int dy f_x(t,y,z)} = 0, \quad (7.19)$$

$$\left(\partial_x + \frac{2}{x}\right) f_y = -\frac{\partial_y f_x}{x} \quad \rightarrow \quad f_y(t,x,z) = \frac{c_y(t,z)}{x^2} - \frac{1}{2}\partial_y f_x. \quad (7.20)$$

Again, we can do a similar derivation for z and obtain the following two expressions for ξ_y and ξ_z,

$$\xi_y = \frac{a_y(t)y^2 + b(t)y + c_y(t,z) + d(t)y^2z + 2a_z(t)yz}{x^2} - a_y(t) - d(t)z, \quad (7.21)$$

$$\xi_z = \frac{a_z(t)z^2 + b(t)z + c_z(t,y) + d(t)yz^2 + 2a_y(t)yz}{x^2} - a_z(t) - d(t)y. \quad (7.22)$$

The only y- and z-dependence that we have not made explicit is contained in the functions c_y and c_z. So we plug the above expressions into equation (iv) to determine this dependence,

$$\frac{\partial_y c_z(t,y) + d(t)z^2 + 2a_y(t)z}{x^2} - d(t) + \frac{\partial_z c_y(t,z) + d(t)y^2 + 2a_z(t)y}{x^2} - d(t) = 0. \quad (7.23)$$

We see immediately that $d(t) = 0$ as it is the only function not multiplied by $1/x^2$. We can solve for c_y and c_z by equating powers of y and z,

$$\partial_z c_y(t,z) = -2a_y(t)z + e(t) \quad \rightarrow \quad c_y(t,z) = c_y(t) - a_y(t)z^2 + e(t)z, \quad (7.24)$$

$$\partial_y c_z(t,y) = -2a_z(t)y - e(t) \quad \rightarrow \quad c_z(t,y) = c_z(t) - a_z(t)y^2 - e(t)y. \quad (7.25)$$

Thus we obtain an explicit solution for the spatial Killing Vector components,

$$\xi_x = \frac{2a_y y + 2a_z y + b}{x}, \quad (7.26)$$

$$\xi_y = \frac{a_y(y^2 - z^2) + 2a_z yz + by + c_y + ez}{x^2} - a_y, \quad (7.27)$$

$$\xi_z = \frac{a_z(z^2 - y^2) + 2a_y yz + bz + c_z - ey}{x^2} - a_z, \quad (7.28)$$

in terms of six functions, $a_y(t), a_z(t), b(t), c_y(t), c_z(t), e(t)$. What remains is to check their time-dependence and to find $\xi_t(t,x,y,z)$. Here we see that the two factors of $\mathbb{R}\times\mathbb{H}^3$ don't like to mix even at the level of isometries. We can solve equations (i), (ii) and (iii) formally by writing

$$\xi_t = -\partial_t \int \xi_x \; dx = g(y,z) - \log(x)\Big(2\dot{a_y}y + 2\dot{a_z}y + b\Big), \tag{7.29}$$

$$\xi_t = -\partial_t \int \xi_y \; dy = h(x,z) + \frac{\dot{a_y}(y^2-z^2) + 2\dot{a_z}yz + \dot{b}y + \dot{c_y} + \dot{e}z}{x} - \dot{a}y, \tag{7.30}$$

$$\xi_t = -\partial_t \int \xi_z \; dz = k(x,y) + \frac{\dot{a_z}(z^2-y^2) + 2\dot{a_y}yz + \dot{bz} + \dot{c_z} - \dot{e}y}{x} - \dot{b}z. \tag{7.31}$$

It is immediate from comparing powers of x, y and z that a_y, a_z, b, c_y, c_z, e are all constants with respect to time. Equality of all three expressions means that the integration functions g, h and k are identical and constant. Hence we have found the full Killing algebra consisting of seven independent Killing vectors.

The seven associated symmetry generators are,

$$P_t = \partial_t, \qquad P_y = \partial_y, \qquad P_z = \partial_z, \qquad L_{yz} = z\partial_y - y\partial_z, \qquad \overline{D} = x\partial_x + y\partial_y + z\partial_z,$$
$$F = 2xy\partial_x + (y^2 - z^2 - x^2)\partial_y + 2yz\partial_z, \qquad G = 2xz\partial_x + 2yz\partial_y + (z^2 - y^2 - x^2)\partial_z.$$

The P-generators generate translations, L_{yz} generates rotations in the (y,z)-plane, $\overline{D}$ generates spatial dilations and F and G are generators of the so-called spherical inversions in hyperbolic space. P_t commutes with all other generators while the spatial algebra satisfies the commutation relations on Table 1 below.

	P_y	P_z	L_{yz}	$\overline{D}$	F	G
P_y	0	0	$-P_z$	P_y	$2\overline{D}$	$2L_{yz}$
P_z	0	0	P_y	$-P_z$	$-2L_{yz}$	$2\overline{D}$
L_{yz}	$-P_z$	$-P_y$	0	0	G	$-F$
D	$-P_y$	$-P_z$	0	0	F	G
F	$-2\overline{D}$	$2L_{yz}$	$-G$	$-F$	0	0
G	$-2L_{yz}$	$-2\overline{D}$	F	$-G$	0	0

Table 1: The commutation relations of $\mathfrak{isom}(\mathbb{H}^3)$.

This Lie algebra should be isomorphic to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of the Lie group $\mathrm{PSL}(2,\mathbb{C})$, which is the canonical isometry group of $\mathbb{H}^3$. We may identify the elements we have found above with generators of $\mathfrak{sl}(2,\mathbb{C})$ under the maps,

$$-F \mapsto X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad P_y \mapsto Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad 2\overline{D} \mapsto H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$-G \mapsto iX = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \qquad -P_z \mapsto iY = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \qquad 2L_{yz} \mapsto iH = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is straightforward to check that, under this identification, the commutation relations in Table 1 map to those of $\mathfrak{sl}(2,\mathbb{C})$ coming from the standard $\mathfrak{sl}_2$ triplet $[H,X] = 2X$, $[H,Y] = 2Y$, $[X,Y] = H$.

Moving on to the conformal isometries, we will elect to write out most of the connection tensors and covariant derivatives. Doing this, the components of the Conformal Killing Equation decompose as:

Diagonal		**Homogeneous**		**Inhomogeneous**	
(1)	$\partial_t^2\theta \qquad\qquad + \omega = 0$	(i)	$\partial_t\partial_x\theta = 0$	(I)	$\partial_x\partial_y\theta + \dfrac{2\partial_y\theta}{x} = 0$
(2)	$\partial_x^2\theta + \dfrac{\partial_x\theta}{x} - \dfrac{L^2}{x^2}\omega = 0$	(ii)	$\partial_t\partial_y\theta = 0$	(II)	$\partial_x\partial_z\theta + \dfrac{2\partial_z\theta}{x} = 0$
(3)	$\partial_y^2\theta - \dfrac{\partial_x\theta}{x} - \dfrac{L^2}{x^2}\omega = 0$	(iii)	$\partial_t\partial_z\theta = 0$		
(4)	$\partial_z^2\theta - \dfrac{\partial_x\theta}{x} - \dfrac{L^2}{x^2}\omega = 0$	(iv)	$\partial_y\partial_z\theta = 0$	(Aux)	$\nabla^2\theta = 4\omega$

In accordance with our discussion in section 5.5, the homogeneous equations immediately tell us that θ decomposes as

$$\theta(t, x, y, z) = A(t) + B(x, y) + C(x, z). \tag{7.32}$$

Using this decomposition, equation (I) and (II) read,

$$\partial_x\partial_y B + \partial_y B/x = 0 \qquad \to \qquad B(x, y) = B(y)/x + E_y(x), \tag{7.33}$$

$$\partial_x\partial_z C + \partial_z C/x = 0 \qquad \to \qquad C(x, z) = C(z)/x + E_z(x). \tag{7.34}$$

We combine $E_y(x)$ and $E_z(x)$ into a single function $E(x)$,

$$\theta(t, x, y, z) = A(t) + \frac{B(y) + C(z)}{x} + E(x). \tag{7.35}$$

Now we make use of the simplified form of the diagonal equations. Plug θ first into equation (1),

$$A''(t) + \omega = 0 \qquad \to \qquad A(t) = At - \frac{\omega t^2}{2} \tag{7.36}$$

and then into equation (3),

$$\frac{B''(y)}{x} + \frac{B(y) + C(z)}{x^2} + E'(x) + \frac{\omega L^2}{x^2} = 0. \tag{7.37}$$

From the second term in (7.37) we see that B and C must be constant as the last term does not depend on y or z. We absorb their contributions into $E(x)$, so that (7.37) can be rewritten as

$$E'(x) = \frac{\omega L^2}{x^2} \qquad \to \qquad E(x) = -\frac{\omega L^2}{x}. \tag{7.38}$$

so that

$$\theta(t, x, y, z) = At - \frac{\omega t^2}{2} - \frac{\omega L^2}{x}. \tag{7.39}$$

Finally, plug this into the auxiliary Poisson equation to find

$$4\omega = \nabla^2\theta = -\partial_t^2\theta + \frac{x^2}{L^2}\partial_x^2\theta - \frac{x\partial_x\theta}{L^2} + \text{vanishing terms} = \omega - \frac{2\omega}{x} - \frac{\omega}{x} = \omega - \frac{3\omega}{x}. \tag{7.40}$$

Hence we conclude that $\omega = 0$. So the only conformal killing vector comes from

$$\theta(t, x, y, z) = At \qquad \to \qquad \xi^\mu = (A, 0, 0, 0) \qquad \to \qquad P_t, \tag{7.41}$$

which we already found in the isometries. Hence we find no proper conformal Killing vectors and write

$$\boxed{\begin{aligned}&\mathrm{Conf}(\mathbb{R}\times\mathbb{H}^3) = \mathbb{R}\times \mathrm{PSL}(2,\mathbb{C})\\ &\mathfrak{conf}(\mathbb{R}\times\mathbb{H}^3) = \mathbb{R}\times\mathfrak{sl}(2,\mathbb{C}) = \left\langle P_t, P_y, P_z, L_{yz}, \overline{D}, F, G\right\rangle .\end{aligned}} \tag{7.42}$$

7.5 Hyperbolic Geometry $\mathbb{R}^{1,1}\times\mathbb{H}^2$

Similar to the previous case, we will use the upper half-plane representation for $\mathbb{H}^2$. Pick coordinates $t, y, z \in \mathbb{R}$ and $x > 0$, then the metric of this space can be written as

$$ds^2_{\mathbb{R}^{1,1}\times\mathbb{H}^2} = -dt^2 + \frac{L^2}{x^2}\left(dx^2+dy^2\right) + dz^2. \tag{7.43}$$

Again $L \propto -1/k$ characterizes the curvature. The nontrivial Christoffel symbols are

$$\Gamma^x_{xx} = \Gamma^y_{xy} = -1/x \qquad \Gamma^x_{yy} = 1/x, \tag{7.44}$$

and the Ricci scalar is $-2/L^2$. We decompose the Killing equation as:

Diagonal	Homogeneous	Inhomogeneous
(1) $\partial_t\xi_t = 0$	(i) $\partial_t\xi_x + \partial_x\xi_t = 0$	(I) $\partial_x\xi_y + \partial_y\xi_x + 2\xi_y/x = 0$
(2) $\partial_x\xi_x + \xi_x/x = 0$	(ii) $\partial_t\xi_y + \partial_y\xi_t = 0$	
(3) $\partial_y\xi_y - \xi_x/x = 0$	(iii) $\partial_x\xi_z + \partial_z\xi_x = 0$	
(4) $\partial_z\xi_z = 0$	(iv) $\partial_y\xi_z + \partial_z\xi_y = 0$	
	(v) $\partial_t\xi_z + \partial_z\xi_t = 0$	

Again we see that equation (2) is an ODE that admits the solution $\xi_x = f_x(t,y,z)/x$ for a yet unspecified function f_x. Take the y-derivative of equation (I) to derive, via equation (3), that

$$0 = \partial_y^2\xi_x + (\partial_x + 2/x)\partial_y\xi_y = \partial_y^2\xi_x + (\partial_x+2/x)\xi_x/x = \frac{\partial_y^2 f_x}{x} + \cancelto{0}{(\partial_x+2/x)\frac{f_x}{x^2}} = \frac{\partial_y^2 f_x}{x}. \tag{7.45}$$

Hence we see that f_x is at most first-order in y. We can make the ansatz $f_x(t,y,z) = 2a(t,z)y + b(t,z)$ and plugging this into equation (3) yields

$$\partial_y\xi_y - \frac{f_x}{x^2} = 0 \qquad \to \qquad \xi_y = f_y(t,x,z) + \frac{1}{x^2}\int f_x\,dy = f_y(t,x,z) + \frac{ay^2+by+c}{x^2}. \tag{7.46}$$

Equation (I) now tells us that

$$\cancel{-\frac{2}{x^3}\int f_x\,dy} + \frac{\partial_y f_x}{x} + \frac{2f_y}{x} + \cancel{\frac{2}{x^3}\int f_x\,dy} = 0, \tag{7.47}$$

hence $f_y = -\frac{\partial_y f_x}{2} = -a$ and have completely specified the dependence of ξ_x and ξ_y on x and y:

$$\xi_x = \frac{2a(t,z)y + b(t,z)}{x}, \tag{7.48}$$

$$\xi_y = \frac{a(t,z)y^2 + b(t,z)y + c(t,z)}{x^2} - a(t,z). \tag{7.49}$$

$$\tag{7.50}$$

Next, let's focus on the t and z coordinates of the flat factor $\mathbb{R}^{1,1}$. The diagonal equations (1) and (4) tell us immediately that

$$\xi_t = f_t(x,y,z), \qquad \xi_z = f_z(t,x,y). \tag{7.51}$$

Equation (iiiv) relating these two components can be written as

$$\partial_z f_t(x,y,z) = -\partial_t f_z(t,x,y). \tag{7.52}$$

Now note that the left-hand side of this equation does not depend on t, while the right-hand side is independent of z. As a result, both sides must depend only on x and y so that

$$f_t(x,y,z) = f_t^{(1)}(x,y) + f_t^{(2)}(x,y)z \tag{7.53}$$
$$f_z(t,x,y) = f_z^{(1)}(x,y) + f_z^{(2)}(x,y)t. \tag{7.54}$$

By equation (7.52) it follows that $f_t^{(2)} = -f_z^{(2)}$ so that

$$\xi_t = f(x,y) + h(x,y)z, \tag{7.55}$$
$$\xi_z = g(x,y) - h(x,y)t. \tag{7.56}$$
$$\tag{7.57}$$

What remains is to see what happens when we plug these into equations (i) through (iv). Consider equation (i) and (ii) simultaneously,

$$\partial_x f(x,y) + \partial_x h(x,y)z = -\frac{2\dot{a}(t,z)y + \dot{b}(t,z)}{x}, \tag{7.58}$$
$$\partial_y f(x,y) + \partial_y h(x,y)z = -\frac{\dot{a}(t,z)y^2 + \dot{b}(t,z)y + \dot{c}(t,z)}{x^2} - \dot{a}(t,z). \tag{7.59}$$

Since the right-hand side of equation (7.58) is $O(x^{-1})$, the left hand side of this equation must be too, which tells us that f_t and h can be written as

$$\xi_t = f_1(y) + f_2(y)\log(x) + z(h_1(y) + h_2(y)\log(x)). \tag{7.60}$$

However, the right-hand side of equation (7.59) is $O(x^{-2})$. The only way to make this work with (7.60) is to require f_1, f_2, h_1 and h_2 to be constant with respect to y and a, b and c constant with respect to t. But then the right-hand side of (7.58) vanishes entirely, so that $f_2 = h_2 = 0$. The argument is exactly the same when considering equations (iii) and (iv), so that we may write, for six constants a, b, c, f, g, h, that

$$\xi_t = f + hz, \qquad \xi_z = g - ht, \tag{7.61}$$
$$\xi_x = \frac{2ay+b}{x}, \qquad \xi_y = \frac{ay^2+by+c}{x^2} - a.$$

The associated generators are, in the same order as$a$, b, c, f, g, h,

$$F = (x^2-y^2)\partial_x + 2xy\partial_y, \qquad \overline{D} = x\partial_x + y\partial_y,$$
$$P_y = \partial_y, \quad P_t = \partial_t, \quad P_z = \partial_z, \quad L_{tz} = z\partial_t - t\partial_z.$$

As before, the P-generators generate translations, L_{tz} generates rotations in the (t,z)-plane, $\overline{D}$ generates spatial dilations and F is a generator of so-called spherical inversions. The commutation relations between the set $\{F, \overline{D}, P_y\}$ and the set $\{P_t, P_z, L_{tz}\}$ are trivial so that the algebra factors as $\mathfrak{isom}(\mathbb{R}^{1,1}) \times \mathfrak{isom}(\mathbb{H}^2)$.

The relations among $\{F, \overline{D}, P_y\}$ can be summarized as

$$[\overline{D}, P_y] = -P_y \qquad [\overline{D}, F] = -F \qquad [P_y, F] = 2\overline{D}. \tag{7.62}$$

It is easy to verify that under the identification $D/2 \mapsto H$, $P_y \mapsto X$, $F \mapsto -Y$, we obtain the standard $\mathfrak{sl}_2$ triplet $[H, X] = 2X$, $[H, Y] = 2Y$, $[X, Y] = H$. Hence $\{F, \overline{D}, P_y\}$ form the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, whereas $\{P_t, P_z, L_{tz}\}$ simply form the two-dimensional Poincaré algebra $\mathbb{R}^2 \rtimes \mathfrak{so}(1)$.

Moving on to the conformal isometries, we decompose the conformal Killing equations in a similar fashion.

Diagonal	**Homogeneous**	**Inhomogeneous**
(1) $\partial_t^2\theta \qquad\qquad + \omega = 0$	(i) $\partial_t\partial_x\theta = 0$	(I) $\partial_x\partial_y\theta + \dfrac{2\partial_y\theta}{x} = 0$
(2) $\partial_x^2\theta + \dfrac{\partial_x\theta}{x} - \dfrac{L^2}{x^2}\omega = 0$	(ii) $\partial_t\partial_y\theta = 0$	
(3) $\partial_y^2\theta - \dfrac{\partial_x\theta}{x} - \dfrac{L^2}{x^2}\omega = 0$	(iii) $\partial_t\partial_z\theta = 0$	
(4) $\partial_z^2\theta \qquad\qquad - \omega = 0$	(iv) $\partial_y\partial_z\theta = 0$	
	(v) $\partial_t\partial_z\theta = 0$	(Aux) $\nabla^2\theta = 4\omega$

The homogeneous equations tell us that we may decompose θ as

$$\theta(t, x, y, z) = A(t) + B(x, y) + C(z). \tag{7.63}$$

(I) then reads

$$\partial_x(\partial_y B(x, y)) = -(\partial_y B(x, y))/x, \tag{7.64}$$

so that $B(x, y) = B(y)/x + D(y)$ and

$$\theta(t, x, y, z) = A(t) + B(y)/x + C(z) + D(y). \tag{7.65}$$

(1) and (4) give us a direct expression for A and B as

$$A(t) = -\frac{\omega t^2}{2} + At \qquad C(z) = \frac{\omega z^2}{2} + Cz. \tag{7.66}$$

Plug this into the auxiliary Poisson equation to find that

$$4\omega = \nabla^2\theta = \partial^2\theta = \omega + \omega - \frac{x^2}{L^2}\Big(\frac{B''(y)}{x} + D''(y) + 2\frac{B(y)}{x^3}\Big). \tag{7.67}$$

Hence $\omega = 0$ and we will not find any proper, conformal Killings vectors. We write simply

$$\boxed{\begin{aligned}&\mathrm{Conf}(\mathbb{R}^{1,1} \times \mathbb{H}^2) = \mathbb{R}^2 \rtimes \mathrm{SO}(1) \times \mathrm{PSL}(2, \mathbb{R})\\ &\mathfrak{conf}(\mathbb{R}^{1,1} \times \mathbb{H}^2) = \mathbb{R}^2 \rtimes \mathfrak{so}(1) \times \mathfrak{sl}(2, \mathbb{R}) = \left\langle P_t, P_z, L_{tz}, P_y, \overline{D}, F\right\rangle.\end{aligned}} \tag{7.68}$$

7.6 Hyperbolic Geometry $\mathbb{R} \times \widetilde{\mathbf{SL}(2, \mathbb{R})}$

The $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ Thurston geometry is a bit of a strange beast at first, as there is no easy matrix representation of the space. However, it can be understood in terms of hyperbolic geometry as follows. Any $A \in \mathrm{SL}(2,\mathbb{R})$ can be decomposed by the Iwasawa decomposition, sometimes also referred to as the KAN decomposition. [17] We may find $k > 0$, $n \in \mathbb{R}$ and $\theta \in [0, 2\pi)$ so that we can write A as the following product of three matrices:

$$\begin{aligned} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} k & 0 \\ 0 & 1/k \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} k(\cos\theta + n\sin\theta) & k(n\cos\theta - \sin\theta) \\ \frac{\sin\theta}{k} & \frac{\cos\theta}{k} \end{pmatrix}. \end{aligned} \tag{7.69}$$

The group of orientation-preserving isometries of $\mathbb{H}^2$ is PSL(2,$\mathbb{R}$) and can be identified with the unit tangent bundle of $\mathbb{H}^2$, UT$\mathbb{H}^2$. This tangent bundle inherits a metric from $\mathbb{H}^2$, which we may pull back onto PSL(2,$\mathbb{R}$). $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ then inherits its metric from SL(2,$\mathbb{R}$), which in turn inherits its metric from PSL(2,$\mathbb{R}$). We will go through this procedure explicitly.

Present the hyperbolic plane $\mathbb{H}^2$ as the set $\{x + iy \in \mathbb{C} | x > 0\}$ equipped with the metric

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2\Big) = \frac{dzd\overline{z}}{im(z)^2}. \tag{7.70}$$

The tangent bundle T$\mathbb{H}^2$ is now the set $\{(z, \vec{v}) | z \in \mathbb{H}^2, \vec{v} \in \mathbb{R}^2\}$ endowed with the metric

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2 + dv_x^2 + dv_y^2\Big). \tag{7.71}$$

Now restrict $\vec{v}$ to lie on the unit circle with respect to this metric, i.e. $(v_x^2 + v_y^2)/y^2 = 1$ so that $\vec{v}$ has 'coordinate radius' y, but is nonetheless of unit length. This gives us the Unit Tangent Bundle UT$\mathbb{H}^2 = \{(z, \vec{v}) \in$T$\mathbb{H}^2 | \; \|\vec{v}\| = 1\}$. We may represent any point $(z, \vec{v}) \in$UT$\mathbb{H}^2$ as $(z, im(z)e^{i\phi})$, with $z \in \mathbb{H}^2$ and $\phi \in [0, 2\pi)$. In these coordinates, the metric on UT$\mathbb{H}^2$ is simply

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2\Big) + d\phi^2. \tag{7.72}$$

A matrix $A \in$ PSL(2,$\mathbb{R}$) acts on $\mathbb{H}^2$ by way of a rational linear transformation,

$$A \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \tag{7.73}$$

where $ad - bc = 1$. We now consider the tangent map $A'(z)$ acting on T$_z\mathbb{H}^2$ for some point $z \in \mathbb{H}^2$ by taking the derivative of (7.73).

$$A'(z) = \frac{a}{cz + d} - \frac{azc + bc}{(cz + d)^2} = \frac{acz + ad - azc - bc}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} = \frac{1}{(cz + d)^2}. \tag{7.74}$$

Using the Iwasawa decomposition from (7.69) we obtain the following expressions,

$$A \cdot z = k^2\Big(\frac{\cos\theta(n + z) + \sin\theta(nz - 1)}{\cos\theta + z\sin\theta}\Big), \tag{7.75}$$

$$A'(z) = \frac{k^2}{(\cos^2\theta + z\sin^2\theta)^2} \tag{7.76}$$

Notice that these expressions are unchanged if we send $\theta \to \theta + \pi$, hence $\theta \in [0, \pi)$ and A is truly an element of PSL(2,$\mathbb{R}$) rather than of SL(2,$\mathbb{R}$).

We now consider the action of the pair $(A, A'(z))$ acting on the element $(i, 1) \in$UT$\mathbb{H}^2$,

$$(A, A'(z)) \cdot (i, e^{-i\phi}) = k^2 \Big(\frac{\cos\theta(n+i) + \sin\theta(ni-1)}{\cos\theta + i\sin\theta}, \frac{1}{(\cos^2\theta + i\sin^2\theta)^2} \Big) \tag{7.77}$$

$$= k^2 \Big(\frac{\cancel{\cos\theta + i\sin\theta}}{\cancel{\cos\theta + i\sin\theta}}(n+i), \frac{1}{e^{2i\theta}} \Big) \tag{7.78}$$

$$= k^2 \Big(n + i, e^{-2i\theta} \Big). \tag{7.79}$$

This allows us to map PSL$(2, \mathbb{R})$ bijectively to UT$\mathbb{H}^2$ by identifying

$$x = k^2 n, \qquad y = k^2, \qquad \phi = 2\theta, \tag{7.80}$$

$$k = \sqrt{y}, \qquad n = x/y, \qquad \theta = \phi/2. \tag{7.81}$$

This directly induces the following metric on PSL$(2, \mathbb{R})$,

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2\Big) + d\phi^2 \qquad \leftrightarrow \qquad ds^2 = dn^2 + \frac{4n}{k}dkdn + \frac{4(n^2+1)}{k^2}dk^2 + 4d\theta^2. \tag{7.82}$$

This metric can then be lifted to SL$(2, \mathbb{R})$ and from there on to the universal cover $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. Since SL$(2, \mathbb{R})$ can be viewed as the double cover of a circle bundle over $\mathbb{H}^2$, taking the universal will unwrap this circle to a line. So we may think of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ as matrices in the Iwasawa decomposition (7.69) that are not invariant under the transformation $\theta \to \theta + 2\pi$. Since θ appears only in periodic functions it is easy to see why the universal cover does not admit an easy matrix group representation.

The upshot of this is that we may promote the coordinate $\theta \in [0, 2\pi)$ to a coordinate $z \in \mathbb{R}$ and write the metric for $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ as

$$ds^2 = \frac{1}{y^2}\Big(dx^2 + dy^2\Big) + dz^2. \tag{7.83}$$

This is exactly the form of the metric for the $\Sigma = \mathbb{H}^2 \times \mathbb{R}$ Thurston Geometry and we can simply borrow the algebra results from the previous section. However, the Lie group formed by the spatial transformations will differ. For $\mathbb{H}^2 \times \mathbb{R}$ we will find PSL$(2, \mathbb{R})$, whereas in this case the global transformations form the group $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ itself.

$$\boxed{\begin{aligned} &\mathrm{Conf}(\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}) = \mathbb{R}^2 \rtimes \mathrm{SO}(1) \times \widetilde{\mathrm{SL}(2, \mathbb{R})} \\ &\mathfrak{conf}(\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}) = \mathbb{R}^2 \rtimes \mathfrak{so}(1) \times \widetilde{\mathfrak{sl}(2, \mathbb{R})} = \big\langle P_t, P_z, L_{tz}, P_y, \overline{D}, F \big\rangle\,. \end{aligned}} \tag{7.84}$$

7.7 The Nil-Geometry

We may think of the Nil geometry as $\mathbb{R}^{1,3}$ equipped with the metric

$$ds^2 = -dt^2 + dx^2 + dy^2(1 + \frac{x^2}{L^2}) - \frac{2x}{L}dydz + dz^2. \tag{7.85}$$

It admits quite a zoo of Levi-Civita connection symbols, namely

$$\Gamma^x_{yy} = -\,x/L^2, \qquad \Gamma^y_{xy} = \Gamma^y_{yx} = x/2L^2, \qquad \Gamma^z_{xz} = \Gamma^z_{zx} = -x/2L^2, \tag{7.86}$$

$$\Gamma^x_{yz} = \Gamma^x_{zy} = 1/2L, \qquad \Gamma^z_{xy} = \Gamma^z_{yx} = \frac{x^2 - L^2}{2L^3}, \qquad \Gamma^y_{xz} = \Gamma^y_{zx} = -1/2L.$$

From here we see that solving the Killing equation of this metric is a difficult task. As we have not managed to accomplish this yet, we will instead present only the calculation for the longitudinal conformal Killing vectors. As in the previous cases, we will see that the Nil-geometry picks up little additional conformal symmetry.

Diagonal		**Homogeneous**		**Inhomogeneous**
(1) $\partial_t^2\theta$	$+\,\omega = 0$	(i) $\partial_t\partial_x\theta = 0$	(I)	$\nabla_x\nabla_y\theta = 0$
(2) $\partial_x^2\theta$	$-\,\omega = 0$	(ii) $\partial_t\partial_y\theta = 0$	(II)	$\nabla_y\nabla_z\theta + \frac{x\omega}{L} = 0$
(3) $\partial_y^2\theta + \frac{x\partial_x\theta}{L^2}$	$-\left(1+\frac{x^2}{L^2}\right)\omega = 0$	(iii) $\partial_t\partial_z\theta = 0$	(III)	$\nabla_z\nabla_x\theta = 0$
(4) $\partial_z^2\theta$	$-\,\omega = 0$		(Aux)	$\nabla^2\theta = 4\omega$

As before, we can write θ as

$$\theta = A(t) + B(x,y,z) \tag{7.87}$$

The diagonal equations (1), (2) and (4) now tell us subsequently that

$$A(t) = At - \frac{\omega t^2}{2}, \tag{7.88}$$

$$\partial_x^2 B = \omega \qquad \to \qquad B(x,y,z) = C_1(y,z) + C_2(y,z)x + \frac{\omega x^2}{2} \tag{7.89}$$

$$\partial_z^2 B = \omega \qquad \to \qquad B(x,y,z) = D_1(x,y) + D_1(x,y)z + \frac{\omega z^2}{2}. \tag{7.90}$$

This allows us to refine our ansatz for θ as

$$\theta = A(t) + B(y) + C(y)x + D(y)z + E(y)xz + \frac{\omega}{2}(x^2+z^2-t^2). \tag{7.91}$$

Consider now the auxiliary Poisson equation, which can be expanded as

$$\begin{aligned} 4\omega = \nabla^2\theta &= -\partial_t^2\theta + \partial_x^2\theta + \partial_y^2\theta + \Big(1+\frac{x^2}{L^2}\Big)\partial_z^2\theta + \frac{2x\partial_y\partial_z\theta}{L} \\ &= 2\omega + \Big(B''(y) + C''(y)x + D''(y)z + E''(y)xz\Big) + \omega\Big(1+\frac{x^2}{L^2}\Big) + \frac{2x}{L}\Big(D'(y) + E'(y)x\Big) \\ &= \Big(3\omega + B''(y)\Big) + x\Big(C''(y) + \frac{2D'(y)}{L}\Big) + z\Big(D''(y)\Big) + xz\Big(E''(y)\Big) + \frac{x^2}{L}\Big(\frac{\omega}{L} + 2E'(y)\Big). \end{aligned} \tag{7.92}$$

To satisfy this we must require

$$B(y) = By + \frac{\omega y^2}{2} \qquad D(y) = Dy \qquad C(y) = Cy - \frac{Dy^2}{L} \qquad E(y) = -\frac{\omega y}{2L}, \tag{7.93}$$

so that we may rewrite θ as

$$\theta = At + By + Cxy - \frac{Dxy^2}{L} + Dyz - \omega\frac{xyz}{2L} + \frac{1}{2}\omega\left(-t^2+x^2+y^2+z^2\right). \tag{7.94}$$

From here, consider equation (3), which says that

$$0 = -\frac{2Dx}{L} + \cancel{\omega} + \frac{x}{L^2}\Big(Cy - \frac{Dy^2}{L} - \omega\frac{yz}{2L} + \cancel{\omega x}\Big) - \Big(\cancel{1} + \cancel{\frac{x^2}{L^2}}\Big)\omega \tag{7.95}$$

$$= \frac{x}{L}\Big(-2D + \frac{Cy}{L} - \frac{Dy^2}{L^2} - \omega\frac{yz}{2L^2}\Big). \tag{7.96}$$

Since every term in this equation contains a different power of the coordinates, it is clear that $C = D = \omega = 0$. Plugging what remains into equation (I) yields

$$0 = \partial_x\partial_y\theta - \frac{x}{2L^2}\partial_y\theta - \frac{x^2-L^2}{2L^3}\partial\theta = \frac{xB}{2L^2}. \tag{7.97}$$

As a result, only time translations P_t remain. Hence, regardless of what the isometry group of the Nil geometry is, it will only pick up one extra generator in a conformal context.

7.8 The Solv-Geometry

We will treat the Solv Geometry similar to the Nil geometry in that we will only study the longitudinal conformal isometries. Like the Nil geometry, we may think of Solv geometry as $\mathbb{R}^{1,3}$ equipped with a non-flat metric

$$ds^2 = -dt^2 + dx^2 + dy^2 e^{2x/L} + dz^2 e^{-2x/L}. \tag{7.98}$$

The following components of its connection tensors are nonzero,

$$\Gamma^x_{yy} = -e^{2x/L}/L, \qquad \Gamma^x_{zz} = e^{-2x/L}/L, \qquad \Gamma^y_{xy} = \Gamma^y_{yx} = 1/L, \qquad \Gamma^z_{xz} = \Gamma^z_{zx} = -1/L. \tag{7.99}$$

We may write the conformal Killing equation as

Diagonal			**Homogeneous**		**Inhomogeneous**
(1) $\partial_t^2\theta$	$+\omega = 0$	(i)	$\partial_t\partial_x\theta = 0$	(I)	$\partial_x\partial_y\theta - \dfrac{\partial_y\theta}{L} = 0$
(2) $\partial_x^2\theta$	$-\omega = 0$	(ii)	$\partial_t\partial_y\theta = 0$	(II)	$\partial_x\partial_z\theta + \dfrac{\partial_z\theta}{L} = 0$
(3) $\partial_y^2\theta + e^{2x/L}\dfrac{\partial_x\theta}{L}$	$-e^{2x/L}\omega = 0$	(iii)	$\partial_t\partial_z\theta = 0$		
(4) $\partial_z^2\theta - e^{-2x/L}\dfrac{\partial_x\theta}{L}$	$-e^{-2x/L}\omega = 0$	(iv)	$\partial_y\partial_z\theta = 0$	(Aux)	$\nabla^2\theta = 4\omega$

We can immediately decompose θ as

$$\theta = A(t) + B(x, y) + C(x, z) \tag{7.100}$$

and set $A(t) = At - \frac{\omega t^2}{2}$ via equation (1). Equation (2) then reads

$$\partial_x^2 B(x, y) + \partial_x^2 C(x, z) = \omega \tag{7.101}$$

so that we can write

$$\theta = At + \Big(B_0(y) + C_0(z)\Big) + x\Big(B_1(y) + C_1(z)\Big) + \frac{\omega}{2}\Big(x^2 - t^2\Big). \tag{7.102}$$

Plugging this into equations (I) and (II) gives us, respectively,

$$B_1'(y) - \frac{B_0'(y) + xB_1'(y)}{L} = 0, \tag{7.103}$$

$$C_1'(z) + \frac{C_0'(z) + xC_1'(z)}{L} = 0. \tag{7.104}$$

Since each of the terms of either equation is multiplied by a different powers of the coordinates, they must all vanish simultaneously. B_0 and C_0 can be eliminated from θ entirely as they do not contribute to the conformal Killing vector $\xi^\mu = \nabla^m\theta$. Absorb B_1 and C_1 into a new constant parameter E so that we may write

$$\theta = At + Dx + \frac{\omega}{2}\Big(x^2 - t^2\Big). \tag{7.105}$$

Using this expression the Poisson equation reads

$$4\omega\nabla^2\theta = \partial^2\theta = -\partial_t^2\theta + \partial_x^2\theta + e^{-2x/L}\partial_y^2\theta + e^{2x/L}\partial_z^2\theta = 2\omega \tag{7.106}$$

and we see that $\omega = 0$. Equation (3) then yields

$$\frac{Ee^{2x/L}}{L} = 0, \tag{7.107}$$

so that, the time translation P_t is the only conformal isometry Solv picks up.

8 Conclusion

In conclusion, we propose to extend 't Hooft's criterion for technical naturalness to include space-time isometries and curvature.

Our motivation for this is the observed large-scale homogeneity and isotropy of the universe. This suggests that we may describe the space-time metric of the universe in terms of a background metric $\overline{\mathbf{g}}$ with perturbations $\delta\mathbf{g}$, with a similar decomposition for the action into $\mathcal{S}_{bg}$ and $\mathcal{S}_{pert}$. In this picture, we identify the isometries of $\overline{\mathbf{g}}$ with the symmetries of $\mathcal{S}_{bg}$ and interpret the curvature parameter k of the background as a physical parameter in the theory. With this extension of 't Hooft's criterion, we can understand smallness of curvature in our universe as technically natural if setting k to zero enhances the number of isometries of the background.

If smallness of the curvature is indeed technically natural, we can argue that a small value for the curvature parameter is protected from quantum fluctuations during inflation.

In ordinary General Relativity, the background has at most six isometries for any expanding, cosmological space-time. In particular, the three classical Friedmann-Lemaître-Robertson-Walker space-times have exactly six isometries regardless of the value of the curvature parameter k in the metric (1.2). Hence a flat geometry with $k = 0$ is no more symmetric than any curved geometry with $k \neq 0$ and we cannot appeal to technical naturalness.

We showed in section 6.2 that symmetry restoration occurs for the background of cosmological space-times in a conformal setting. As a result, the group of isometries of flat space-time is enhanced from six to the maximal number of 15 conformal isometries. Our calculations show that curved space-times, however, pick up little extra symmetry due to the Geometric Isolation conjecture proposed in section 5.5. The results of our calculations are presented below.

Geometry	Isometry Group	Conformal Generators	CKVs added	Total No.
$\mathbb{R}^{1,3}$	$\mathbb{R}^3 \rtimes \mathrm{SO}(3)$	P_η, D, L_{ij}, K_μ	+9	15
$\mathbb{R} \times S^3$	SO(4)	P_η	+1	7
$\mathbb{R} \times \mathbb{H}^3$	$\mathrm{PSL}(2, \mathbb{C})$	P_η	+1	7
$\mathbb{R}^{1,1} \times S^2$	$\mathbb{R} \times \mathrm{SO}(3)$	P_η, $L_{\eta z}$	+2	6
$\mathbb{R}^{1,1} \times \mathbb{H}^2$	$\mathbb{R} \times \mathrm{PSL}(2, \mathbb{R})$	P_η, $L_{\eta z}$	+2	6
$\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}$	$\mathbb{R} \times \widetilde{\mathrm{SL}(2, \mathbb{R})}$	P_η, $L_{\eta z}$	+2	6
Nil		P_η	+1	≤ 7
Solv		P_η	+1	≤ 7

Table 2: *A summary of results from section 7, detailing the conformal Killing vectors (CKVs) picked up by various Thurston geometries in Conformal Gravity. The isometry groups of Nil and Solv are not presented as we have not derived them; instead we present an upper bound. As these two manifolds are the least symmetric of the group, it is unlikely they will reach this upper limit.*

From this table we see that the flat geometry $\mathbb{R}^{1,3}$ has more than twice as many conformal generators as any other geometry. In our conformal enhancement of General Relativity, therefore, setting k to zero puts us at a point of exceedingly enhanced symmetry. So we conclude that, in conformal gravity, a universe with small curvature is indeed technically natural.

The 't Hooft criterion, unfortunately, does not tell us anything about which of the Thurston geometries is realised. At best, we can infer a slight preference for the first three geometries on

Table 2 by their higher degree of symmetry.

There are some limitations to and imprecisions in our approach that future effort may seek to iron out.

First of all, in section 5.1 we have tacitly assumed that any that any conformal isometry of a pseudo-Riemannian manifold M will extend to a conformal isometry of $\mathbb{R}^{p,q}$ upon embedding $\vartheta : M \hookrightarrow \mathbb{R}^{p,q}$ isometrically. This is by no means a trivial statement and it should be checked that this holds.

Secondly the Geometric Isolation conjecture in 5.3 is just that, a conjecture with evidence presented in support of it, but without definite proof. To check its general validity, one could embed $\vartheta : M \hookrightarrow \mathbb{R}^{p,q}$ and then pullback the action of a special conformal transformation on $\mathbb{R}^{p,q}$ to M. Express the requirement that it should be a conformal isometry in local coordinates and see what sort of constraints this puts on the form of the embedding map ϑ. Since this requirement boils down to solving a PDE, this should in principle be possible.

Thirdly, as outlined in section 6.1, Thurston's Geometrization conjecture is not a full classification of all possible 3-manifolds, but of the maximal geometries that admit a compact quotient and the closed and oriented manifolds that follow from them. Maximality of these geometries implies that there is no other 3-manifold containing them with a bigger isometry group, hence there will be no other manifold with a compact quotient that is more symmetric than the eight Thurston Geometries. Nonetheless, it is conceivable that there are manifolds Σ without such a compact quotient so that the cosmological space-time $\mathbb{R} \times \Sigma$ has a number of independent conformal Killing vectors closer to the 15 we get for flat space-time.

The conformal group of a manifold forms a Lie group, for which we also have classification schemes. It may be possible to approach this problem via Lie theory rather than directly from geometry, as we have done. It would suffice to find an upper limit on the number of independent conformal Killing vectors for a non-flat geometry.

Lastly, since we have been able to eliminate the scale factor a entirely, one may argue that there is no particular need to decompose M as $\mathbb{R} \times \Sigma$ anymore. As such, we should also consider indecomposable 4-manifolds. Ideally, in fact, we would wish to obtain a classification of 4-manifolds similar to Thurston's work for 3-manifolds. However, such a classification is still an open problem in mathematics today.

However, there are specific indecomposable space-times of interest that future work can investigate, most notably de Sitter space, Anti-de Sitter space and Black Hole space-times such as the Schwarzschild solution.

Similarly, there are ample open questions concerning cosmologies based on conformal gravity.

An interesting question is how inflation relates to conformal symmetry. On large scales, inflationary fluctuations are almost conformal and a spectral tilt n_s different from 1 measures deviation from scale invariance. We can therefore investigate at what energy scales conformal symmetry is broken and if this can be made to coincide with the start of the inflationary epoch? If conformal symmetry persists until some finite time t_c, how does the evolution of the universe change for times $t < t_c$? Can this still be made consistent with today's observations and, if so, does it provide a good fit to the data?

There is also plenty more work to be done on Conformal Gravity as a field theory itself.

The obvious first question to ask is whether or not conformal symmetry gets carried over from the classical regime to the quantum regime. Whether or not this happens may have consequences for any model of inflation as perturbations to primordial spectra are (partially) generated by quantum fluctuations during inflation. Additionally, we may hope that any conformal symmetry that survives quantization helps to renormalize the theory. It may grant us some insight into the conditions of the early universe and into what a UV-completion of gravity may look like.

Furthermore, it is well-known that string theory contains classical General Relativity. Given that conformal symmetry is an important component of string theory, we may hope to investigate a similar connection to conformal gravity where conformal symmetry is inherited directly from string theory. Such a description could help us learn more about the nature of conformal gravity, especially in studying the quantum regime.

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