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Whittaker vectors on $\mathrm{SL}(2, \mathbb{R})$

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Introduction

The main objects of study in this thesis are the so called *Whittaker vectors* of the principal series representation and two transformation operators called the *Whittaker–Fourier transformation* and the *Harish-Chandra transformation* which are intimately related to these Whittaker vectors. Whittaker vectors were first introduced by Jacquet in [Jac67] and have been studied extensively. The two transformations both find there use in the proof of the so called Whittaker–Plancherel theorem. The proof of this result was originally obtained by Harish-Chandra. Unfortunately the result has only been communicated by Harish-Chandra in private correspondence and the proof has never been published. An independent proof of this theorem is given by Wallach in his book [Wal92]. However the proof presented there is not quite complete. It was pointed out by Van den Ban and Kuit in [vdBK] that a certain lemma ([Wal92, Section 15.3.2]), pertaining to the Harish-Chandra transformation, could not be true in the generality stated in the book, as this would lead to contradictory results. For a more precise account of the statement of the lemma we refer to Section 3.9.

It was the question whether the result stated in this lemma does hold if we impose additional assumptions that led to the subject of this thesis. In this text we give a partial answer to this question in the special case of $\mathrm{SL}(2,\mathbb{R})$. For a precise statement of what we prove we refer to Proposition 3.11. The author would like to point the reader's attention to a recent preprint by Wallach ([Wal]) in which a full proof of the Whittaker–Plancherel Theorem, circumventing the faulty lemma, is given (and it is shown that the lemma holds true when extra conditions are imposed). It should be noted that this preprint was published only recently and after the writing of this thesis had already begun.

We give a quick summary of the structure of this thesis. The text is divided into three chapters. The first of these is dedicated to introducing several concepts and notational conventions that will be used throughout the rest of the text. In this chapter we only give a very brief account of the facts we will need and most proofs will be omitted.

In the second chapter we introduce the notion of *Whittaker vectors* and we spend the first part of this chapter studying the space of these Whittaker vectors. The main result of this section will be Theorem 2.10 which gives a full description of this space of Whittaker vectors. In the second part of this chapter we introduce, using our results of the first part, the concept of the *Whittaker coefficient*. We spend the rest of the chapter on studying this Whittaker coefficient.

In final chapter we study the aforementioned *Whittaker–Fourier transformation* and *Harish-Chandra transformation*. Our analysis of the Whittaker–Fourier transformation will rely on the result on the Whittaker coefficient obtained in the previous chapter. The main focus of the third chapter will be to prove Proposition 3.11 which will provide a partial answer to the question asked in the above introduction.

All proofs presented in this text are by the author unless otherwise stated. The author would like to note however that many proofs presented here, some more than others, as based on the countless suggestions communicated to him by E.P. van den Ban.

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Chapter 1

Preliminaries

In this chapter we introduce several concepts that will be used in later chapters. We also fix several pieces of notation that will be used throughout the text. Most of the topics in this chapter are part of the structure theory and the representation theory of semisimple Lie groups. The discussion of the topics will be quite brief and most proofs are omitted. The reader is assumed to be familiar with the theory of Lie groups and Lie algebras and with the basics of geometric analysis. Most of this chapter is based on the book 'Lie Groups Beyond an Introduction' by A.W. Knapp ([Kna96]) and lecture notes on Harmonic Analysis written by E.P. van den Ban ([vdB]).

1.1 Cartan decomposition

Let $\mathfrak{g}$ be a real semisimple Lie algebra and denote by $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ its Killing form.

Definition 1.1. An involution of $\mathfrak{g}$ is a Lie algebra automorphism θ of $\mathfrak{g}$ that satisfies $\theta^2 = \mathrm{id}$.

If $\mathfrak{g}$ is equipped with such an involution it decomposes as a direct sum of the $+1$-eigenspace and -1-eigenspace of θ which we will denote by $\mathfrak{g}_+$ and $\mathfrak{g}_-$ respectively. Because θ is an automorphism of $\mathfrak{g}$ we have that $B(\theta X, \theta Y) = B(X, Y)$ holds for all $X, Y \in \mathfrak{g}$. It follows that $\mathfrak{g}_+ \perp \mathfrak{g}_-$ with respect to B. Using that θ preserves the Lie brackets it is easy to check that

$$[\mathfrak{g}_+, \mathfrak{g}_+] \subset \mathfrak{g}_+, \quad [\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_- \quad \text{and} \quad [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

In particular we see that $\mathfrak{g}_+$ is a subalgebra of $\mathfrak{g}$.

Definition 1.2. A Cartan involution of $\mathfrak{g}$ is an involution θ of $\mathfrak{g}$ such that the Killing form is negative definite on $\mathfrak{g}_+$ and positive definite on $\mathfrak{g}_-$.

An equivalent definition is to require θ to be such that $\langle X, Y\rangle := -B(X, \theta Y)$ defines a positive definite innner product on $\mathfrak{g}$ (see [vdB, Lemma 15.5]). This inner product is called the *Cartan inner product* on $\mathfrak{g}$. If θ is a Cartan involution we denote $\mathfrak{k} := \mathfrak{g}_+$ and $\mathfrak{p} := \mathfrak{g}_-$. It is easy to check that for $X \in \mathfrak{g}$ we have $\mathrm{ad}(X)^\top = -\mathrm{ad}(\theta X)$ with respect to the Cartan inner product. In particular we have that $\mathrm{ad}(X)$ is anti-symmetric for $X \in \mathfrak{k}$ and is symmetric for $X \in \mathfrak{p}$. We will make use of the following fact.

Proposition 1.3. Every real semisimple Lie algebra can be equipped with a Cartan involution.

For a proof see [Kna96, Corollary 6.18].

We now assume that $\mathfrak{g}$ is the Lie algebra of a connected semisimple Lie group G. For our discussion we fix a Cartan involution θ on $\mathfrak{g}$. We denote by K the analytic subgroup $K := \langle \exp \mathfrak{k} \rangle$. It turns out that as a manifold G can be decomposed into a product of this subgroup K and the vector space $\mathfrak{p}$.

Proposition 1.4. The map $K \times \mathfrak{p} \to G \colon (k, X) \mapsto k \exp(X)$ is a diffeomorphism.

For a proof see [Kna96, Theorem 6.31]. Using this decomposition we can define a lift of θ to G, also denoted by θ, by setting $\theta(k \exp(X)) = k \exp(-X)$.

Proposition 1.5. The map $\theta \colon G \to G$ as defined above is the unique involution of G such that $d\theta(e) = \theta \colon \mathfrak{g} \to \mathfrak{g}$.

For a proof see [Kna96, Theorem 6.31]. It is readily verified that K equals G^θ, the set of fixed points of θ. As a consequence we see that K is a closed subgroup of G.

Proposition 1.6. The subgroup K is compact if and only if the center of G is finite. If this is the case then K is a maximal compact subgroup of G.

For a proof see [Kna96, Theorem 6.31].

1.2 Restricted root system

We let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Such a subalgebra exists by finite dimensionality. It is a fact that any two such maximal abelian subalgebras have the same dimension (see [Kna96, Theorem 6.51]). So the value of $\dim \mathfrak{a}$ is independent of the precise choice of $\mathfrak{a}$. Keeping this in mind we can give the following definition.

Definition 1.7. The split rank of a semisimple Lie group is defined as

$$\operatorname{split\,rank} G := \dim \mathfrak{a}.$$

Here $\mathfrak{a}$ is any choice of maximal abelian subalgebra of $\mathfrak{p}$.

For any $H \in \mathfrak{a}$ we have that $\operatorname{ad}(H)$ is symmetric with respect to the Cartan inner product hence $\operatorname{ad}(H)$ is diagonalizable with real eigenvalues. Since $\mathfrak{a}$ is abelian all maps $\{\operatorname{ad}(H) \mid H \in \mathfrak{a}\}$ are simultaneously diagonalizable. As a result $\mathfrak{g}$ decomposes into a direct sum of simultaneous eigenspaces of these maps. For $\lambda \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid \operatorname{ad}(H)X = \lambda(H)X \quad \text{for all } H \in \mathfrak{a}\}.$$

Definition 1.8. An element $\alpha \in \mathfrak{a}^*$ is called a (restricted) root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. In this case $\mathfrak{g}_\lambda$ is called a (restricted) root space. We denote the set of roots by $\Sigma = \Sigma(\mathfrak{g}; \mathfrak{a})$.

It is easy to check that $\theta \mathfrak{g}_\lambda = \mathfrak{g}_{-\lambda}$ and $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ for $\lambda, \mu \in \mathfrak{a}^*$. The decomposition of $\mathfrak{g}$ as a direct sum of these simultaneous eigenspaces is called the restricted root space decomposition.

Proposition 1.9. As a vector space $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

This sum is orthogonal with respect to the Cartan inner product.

For a proof see [Kna96, Proposition 6.40]. Since $\mathfrak{a}$ is abelian we have $\mathfrak{a} \subset \mathfrak{g}_0$. The maximality of $\mathfrak{a}$ in $\mathfrak{p}$ implies that $\mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$. We define $\mathfrak{m} \subset \mathfrak{k}$ to be $\mathfrak{m} := \mathfrak{g}_0 \cap \mathfrak{k} = Z_{\mathfrak{k}}(\mathfrak{a})$. By using that $\mathfrak{g}_0$ is θ-stable we see that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$

Definition 1.10. A choice of positive roots in Σ is a subset $\Sigma^+ \subset \Sigma$ such that:

1. $\Sigma = \Sigma^+ \cup (-\Sigma^+)$
2. Σ^+ is contained in an open half space in $\mathfrak{a}^*$, i.e. $\Sigma^+ \subset \{\lambda \in \mathfrak{a}^* \mid \lambda(H) > 0\}$ for suitable $H \in \mathfrak{a}$.

For any $\alpha \in \Sigma$ the set $\ker \alpha$ is a hyperplane in $\mathfrak{a}$. Hence the set $\mathfrak{a}^{\mathrm{reg}} := \mathfrak{a} \setminus \cup_{\alpha\in\Sigma} \ker \alpha$ consists of a disjoint union of open convex sets. These connected components of $\mathfrak{a}^{\mathrm{reg}}$ are called the *Weyl chambers* in $\mathfrak{a}$. Let $C \subset \mathfrak{a}^{\mathrm{reg}}$ be such a Weyl chamber. If we define $\Sigma^+ := \{\alpha \in \Sigma \mid \alpha(H) > 0 \text{ for all } H \in C\}$ then this set is a choice of positive roots in Σ. This construction yields a one-to-one correspondence between Weyl chambers and choices of positive roots. For a certain choice of positive roots Σ^+ we will denote the corresponding Weyl chamber by $\mathfrak{a}^+$ (this set is given by $\mathfrak{a}^+ = \{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \Sigma^+\}$).

We fix a choice of positive roots $\Sigma^+ \subset \Sigma$ and define the subspaces

$$\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \overline{\mathfrak{n}} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{-\alpha}.$$

We observe that $\theta\mathfrak{n} = \overline{\mathfrak{n}}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $\alpha + \beta \in \Sigma^+$ for any $\alpha, \beta \in \Sigma^+$ it follows that both $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ are nilpotent subalgebras of $\mathfrak{g}$. We see that $\mathfrak{g}$ decomposes as the following direct sum of subalgebras

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Definition 1.11. A positive root is called simple if it can not be written as the sum of two positive roots. We denote the set of simple roots by $\Delta = \Delta(\Sigma^+)$.

The set Δ is a basis of $\mathfrak{a}^*$ and and we have $\Sigma^+ = \mathbb{N}\Delta \cap \Sigma$.

1.3 Iwasawa decomposition

The following result is known as the *infinitesimal Iwasawa decomposition* of the Lie algebra $\mathfrak{g}$.

Proposition 1.12. The space $\mathfrak{g}$ decomposes as the following direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

For a proof see [Kna96, Proposition 6.43].

We define the following two subgroups of G

$$A := \langle \exp \mathfrak{a} \rangle \quad \text{and} \quad N := \langle \exp \mathfrak{n} \rangle.$$

The subgroup A is abelian and the subgroup N is nilpotent. The infinitesimal Iwasawa decomposition of $\mathfrak{g}$ has the following counterpart on G.

Proposition 1.13. The multiplication map $K \times A \times N \to G\colon (k, a, n) \mapsto kan$ is a diffeomorphism.

For a proof see [Kna96, Proposition 6.46].

In general $\exp\colon \mathfrak{g} \to G$ is only a local diffeomorphism but for both A and N we have the following.

Proposition 1.14. Both A and N are closed subgroups of G. The restrictions of the exponential map $\exp|_{\mathfrak{a}} \colon \mathfrak{a} \to A$ and $\exp|_{\mathfrak{n}} \colon \mathfrak{n} \to N$ are both diffeomorphisms. Furthermore, if we view the vector space $\mathfrak{a}$ as a Lie group $(\mathfrak{a}, +, 0)$ then $\exp \mathfrak{a} \to A$ is an isomorphism of Lie groups.

For a proof of the first part see [vdB, Lemma 17.5 and Lemma 17.13]. For the fact that $\exp\colon \mathfrak{a} \to A$ is actually an isomorphism we remember $\mathfrak{a}$ is abelian and that $\exp(X+Y) = \exp(X)\exp(Y)$ for all $X, Y \in \mathfrak{g}$ with $[X,Y] = 0$.

For the inverse of the map $\exp\colon \mathfrak{a} \to A$ we will use the notation $\log\colon A \to \mathfrak{a}$. Now we define for $\lambda \in \mathfrak{a}^*$ the notation

$$a^\lambda := e^{\lambda(\log a)} \quad \text{for } a \in A.$$

Proposition 1.15. Let $a \in A$ then each root space $\mathfrak{g}_\alpha$ is stable under the action of $\mathrm{Ad}(a)$. Furthermore, for $\alpha \in \Sigma \cup \{0\}$ we have that $\mathrm{Ad}(a)$ acts on $\mathfrak{g}_\alpha$ as $\mathrm{Ad}(a)|_{\mathfrak{g}_\alpha} = a^\alpha \cdot I$.

For a proof see [vdB, Lemma 17.5].

We will denote by $\overline{N}$ the analytic subgroup $\overline{N} := \langle \exp \overline{n} \rangle$. We observe that $\theta N = \overline{N}$. Furthermore, we define $M := Z_K(\mathfrak{a})$ which is a closed subgroup of G. Its Lie algebra equals $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. Since M centralizes A we have that MA is a subgroup of G. It is not difficult to see that MA in turn normalizes N hence the set $P := MAN$ is a subgroup of G. This is a closed subgroup of G since $M \times A \times N$ is closed in $K \times A \times N \cong G$. This subgroup P is called a *minimal parabolic subgroup* of G.

We define the *Iwasawa projection maps* $k\colon G \to K, a\colon G \to A, n\colon G \to N$ to be the composition of the diffeomorphism $G \to K \times A \times N$ and the projections onto K, A and N respectively. For any $g \in G$ we have $g = k(g)a(g)n(g)$. We also introduce the map $H\colon G \to \mathfrak{a}$ which is defined as $H := \log \circ a$.

For $\alpha \in \Sigma$ we denote by $H_\alpha \in \mathfrak{a}$ the unique element that satisfies $B(\cdot, H_\alpha) = \alpha$ on $\mathfrak{a}$.

Proposition 1.16. The map $H\colon G \to \mathfrak{a}$ maps $\overline{N}$ into the space $\sum_{\alpha \in \Sigma^+} \mathbb{R}_{\geq 0} \cdot H_\alpha$. The restricted map $H|_{\overline{N}}\colon \overline{N} \to \sum_{\alpha \in \Sigma^+} \mathbb{R}_{\geq 0} \cdot H_\alpha$ is proper and surjective.

For a proof of this see either [HC58, Lemma 43] or [vdB86, Theorem A.I.]

We consider the inclusion map $K \hookrightarrow G$ which induces a smooth map $K/M \to G/P$. It is easily seen that $G/P \to K/M\colon gP \mapsto k(g)M$ is an inverse to this map hence we obtain the following.

Proposition 1.17. The inclusion map $K \hookrightarrow G$ induces a diffeomorphism $K/M \to G/P$. This diffeomorphism is equivariant under the left actions of K on K/M and G/P.

1.4 $\mathrm{SL}(n, \mathbb{R})$

The Lie group $\mathrm{SL}(2, \mathbb{R})$ will be of special interest throughout this text. In this section we explicitly calculate the subgroups and Lie subalgebras introduced in the previous section in the case of $G = \mathrm{SL}(n, \mathbb{R})$.

The Lie group $\mathrm{SL}(n, \mathbb{R})$ is defined as the subgroup of matrices in $\mathrm{GL}(n, \mathbb{R})$ of determinant one, i.e.

$$\mathrm{SL}(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det A = 1\}.$$

The Lie algebra of this group is $\mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid \operatorname{tr} X = 0\}$. The map $\theta \in \mathrm{Aut}(\mathfrak{sl}(n, \mathbb{R}))$ defined by $\theta X = -X^\top$ provides a Cartan involution on this Lie algebra. The map $\theta \in \mathrm{Aut}(\mathrm{SL}(n, \mathbb{R}))$ defined by $\theta A = (A^{-1})^\top$ is the unique lift of this Cartan involution to $\mathrm{SL}(n, \mathbb{R})$.

For convenience we denote in this section $\mathfrak{g} := \mathfrak{sl}(n, \mathbb{R})$. We immediately see that

$$K = \mathrm{SL}(n, \mathbb{R})^\theta = \mathrm{SO}(n) = \{A \in GL(n, R) \mid AA^\top = I\}$$

and

$$\mathfrak{k} = \mathfrak{g}_+ = \mathfrak{so}(n) = \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X = -X^\top\}.$$

The algebra $\mathfrak{p} = \mathfrak{g}_-$ is given by

$$\mathfrak{p} = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X = X^\top\}.$$

It is easily checked that the subalgebra of diagonal matrices is a maximal abelian subalgeba of $\mathfrak{p}$. We will fix this as our choice for $\mathfrak{a}$, i.e.

$$\mathfrak{a} := \{H \in \mathfrak{sl}(n,\mathbb{R}) \mid H \text{ is diagonal}\} \subset \mathfrak{p}.$$

We denote by E_{ij} an $n \times n$ matrix with the (i,j) entry equal to one and all other entries equal to zero. For $i \neq j$ we have $E_{ij} \in \mathfrak{sl}(2,\mathbb{R})$. Then we see for $H \in \mathfrak{a}$ that $[H, E_{ij}] = (H_{ii} - H_{jj})E_{ij}$. So if we set $\alpha_{ij} \in \mathfrak{a}^*$ to be $\alpha_{ij}(H) = H_{ii} - H_{jj}$ then we see that $\mathfrak{g}_{\alpha_{ij}} = \mathbb{R}E_{ij}$. We have that

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{a} \oplus \bigoplus_{i\neq j} \mathfrak{g}_{\alpha_{ij}}$$

and $\Sigma = \{\alpha_{ij} = e_i - e_j \mid i \neq j\}$. We see that $\mathfrak{g}_0 = \mathfrak{a}$ hence we must have $\mathfrak{m} = 0$. For $i \neq j$ we see that $\ker \alpha_{ij}$ consists of elements $H \in \mathfrak{a}$ with $H_{ii} = H_{jj}$ hence

$$\mathfrak{a}^{\text{reg}} = \{H \in \mathfrak{a} \mid H_{ii} \neq H_{jj} \quad \text{for all} \quad i \neq j\}.$$

We fix the following choice of positive Weyl chamber

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} \mid H_{nn} > \cdots > H_{22} > H_{11}\}.$$

The positive roots are now given by $\Sigma^+ = \{\alpha_{ij} \mid i < j\}$. We see that $\mathfrak{n} = \oplus_{i<j}\mathfrak{g}_{\alpha_{ij}} = \oplus_{i<j}\mathbb{R} \cdot E_{ij}$ consists of the strictly upper triangular matrices and $\overline{\mathfrak{n}}$ of the strictly lower triangular matrices. In conclusion we find

$$\begin{aligned}
\mathfrak{k} &= \mathfrak{so}(n)\\
\mathfrak{a} &= \{H \in \mathfrak{sl}(n,\mathbb{R}) \mid H \text{ is diagonal}\}\\
\mathfrak{m} &= 0\\
\mathfrak{n} &= \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X \text{ is strictly upper triangular}\}\\
\overline{\mathfrak{n}} &= \{X \in \mathfrak{sl}(n,\mathbb{R}) \mid X \text{ is strictly lower triangular}\}.
\end{aligned}$$

The analytic subgroups in G corresponding to these Lie algebras are give by

$$\begin{aligned}
K &= \mathrm{SO}(n)\\
M &= \{m \in SO(n) \mid m \text{ is diagonal with entries } \pm 1\}\\
A &= \{a \in \mathrm{SL}(n,\mathbb{R}) \mid a \text{ is diagonal with positive entries}\}\\
N &= \{n \in \mathrm{SL}(n,\mathbb{R}) \mid n \text{ is upper triangular with 1's along the diagonal}\}\\
\overline{N} &= \{\overline{n} \in \mathrm{SL}(n,\mathbb{R}) \mid \overline{n} \text{ is lower triangular with 1's along the diagonal}\}\\
P = MAN &= \{p \in \mathrm{SL}(n,\mathbb{R}) \mid p \text{ is upper triangular}\}.
\end{aligned}$$

1.4.1 $\mathrm{SL}(2,\mathbb{R})$

In the case of $\mathrm{SL}(2,\mathbb{R})$ we have the standard $\mathfrak{sl}(2,\mathbb{R})$-triple H, X, Y defined by

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These elements satisfy the commutation relations $[H,X] = 2X,\ [H,Y] = -2Y$ and $[X,Y] = H$. In terms of this basis we have $\mathfrak{a} = \mathbb{R}H$, $\mathfrak{a}^+ = \mathbb{R}_{>0}H$, $\mathfrak{n} = \mathbb{R}X$ and $\overline{\mathfrak{n}} = \mathbb{R}Y$.

For $x \in \mathbb{R}$ and $\lambda > 0$ we define

$$a_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad n_x := \exp(x \cdot X) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \overline{n}_x := \exp(x \cdot Y) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

We can now describe the subgroups A, N and $\overline{N}$ as

$$A := \{a_\lambda \mid \lambda > 0\}, \quad N := \{n_x \mid x \in \mathbb{R}\} \quad \text{and} \quad \overline{N} = \{\overline{n}_x \mid x \in \mathbb{R}\}.$$

We note that for $x, y \in \mathbb{R}$ we have $n_x \cdot n_y = n_{x+y}$ hence the map $\exp\colon \mathfrak{n} \to N$ (and similarly $\exp\colon \overline{\mathfrak{n}} \to \overline{N}$) is in fact a group isomorphism. This is a special feature of the group $\mathrm{SL}(2,\mathbb{R})$. Finally we have that $K = \mathrm{SO}(1) \cong S^1$.

The space of roots is given by $\Sigma = \{-\alpha, \alpha\}$ with $\alpha \in \mathfrak{a}^*$ determined by $\alpha(H) = 2$. Our choice of positive roots is such that α is positive so $\Sigma^+ = \{\alpha\}$.

The Iwasawa projection $H\colon G \to \mathfrak{a}$ is such that $g = k(g)\exp H(g) n(g)$ for every $g \in \mathrm{SL}(2,\mathbb{R})$. For $\mathrm{SL}(2,\mathbb{R})$ we can find the following explicit formula for H.

Proposition 1.18. Assume $G = \mathrm{SL}(2,\mathbb{R})$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ then

$$H(g) = \frac{1}{2}\log(a^2 + b^2) \cdot H.$$

Proof. Let $k \in K, \lambda > 0$ and $x \in \mathbb{R}$ such that $g = k a_\lambda n_x$. Then $(g n_{-x})^\top (g n_{-x}) = a_\lambda^2$. An easy calculation yields

$$(g n_{-x})^\top (g n_{-x}) = \begin{pmatrix} a^2 + b^2 & ab + cd - x(a^2 + c^2) \\ ab + cd - x(a^2 + c^2) & (b - xa)^2 + (c - xd)^2 \end{pmatrix} = a_\lambda^2 = \begin{pmatrix} \lambda^2 & 0 \\ 0 & 1/\lambda^2 \end{pmatrix}.$$

This implies that we must have $\lambda = \sqrt{a^2 + b^2}$ hence $H(g) = \frac{1}{2}\log(a^2 + b^2) \cdot H$. $\square$

1.5 Bruhat decomposition

For an element $\alpha \in \mathfrak{a}^*$ a reflection in α is a map $s \in \mathrm{GL}(\mathfrak{a}^*)$ such that $s(\alpha) = -\alpha$ and $\mathfrak{a}^* = \mathbb{R}\alpha \oplus \ker(I - s)$. For every root $\alpha \in \Sigma$ there exists a unique reflection s_α such that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\Sigma) = \Sigma$. We denote by $W(\mathfrak{g};\mathfrak{a}) = \langle s_\alpha \mid \alpha \in \Sigma \rangle$ the *Weyl group* which is the subgroup of $\mathrm{GL}(\mathfrak{a}^*)$ generated by the reflections s_α. To give an alternative description of the Weyl group we note that the group $N_K(\mathfrak{a})$ acts on $\mathfrak{a}^*$ via the co-adjoint action $\mathrm{Ad}^\vee$. The kernel of the map $\mathrm{Ad}^\vee\colon N_K(\mathfrak{a}) \to \mathrm{GL}(\mathfrak{a}^*)$ is $Z_K(\mathfrak{a})$.

Proposition 1.19. The map $\mathrm{Ad}^\vee\colon N_K(\mathfrak{a}) \to \mathrm{GL}(\mathfrak{a}^*)$ maps into $W(\mathfrak{g};\mathfrak{a})$ and induces an isomorphism $N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \cong W(\mathfrak{g};\mathfrak{a})$.

For a proof see [Kna96, Theorem 6.57].

The following result is known as the Bruhat decomposition of G.

Proposition 1.20. Let G be a connected semisimple Lie group with finite center. The map $N_K(\mathfrak{a}) \to P \setminus G/P \colon w \mapsto PwP$ induces a bijection between the Weyl group $W(\mathfrak{g};\mathfrak{a}) \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ and the double coset space $P \setminus G/P$.

A consequence of this result is that we can decompose G as the following disjoint union

$$G = \bigsqcup_{w \in W(\mathfrak{g};\mathfrak{a})} P\widetilde{w}P = \bigsqcup_{w \in W(\mathfrak{g};\mathfrak{a})} N\widetilde{w}P,$$

were $\widetilde{w} \in N_K(\mathfrak{a})$ is a representative of $w \in W(\mathfrak{g};\mathfrak{a})$ under the identification $W(\mathfrak{g};\mathfrak{a}) \cong N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. The sets $B_w := N\widetilde{w}P$ are called the *Bruhat cells* of G. These cells are orbits of the Lie group action $L \times R$ of $N \times P$ on G. It is readily verified that the subgroup $(N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}) \times P$ of $N \times P$ acts transitively on B_w. Furthermore, in light of Lemma 1.21 we see that the action of this subgroup on B_w is free. It follows that

$$\phi_w \colon (N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}) \times P \to B_w \colon (n,p) \mapsto n\widetilde{w}p^{-1}$$

is a bijective map. This map is of constant rank because it is equivariant for the action of $(N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}) \times P$. Since a map that is both injective and of constant rank is an immersion we conclude that B_w is an immersed submanifold of G that is diffeomorphic to $(N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}) \times P$. We see that the dimension of B_w equals $\dim(N \cap \widetilde{w}\overline{N}\widetilde{w}^{-1}) + \dim P$.

Lemma 1.21. We have $\overline{N} \cap P = \{e\}$.

For a proof see [Kna96, Lemma 7.64]. The above discussion has the following consequence.

Proposition 1.22. The set $\overline{N}P$ is an open submanifold of G. Its complement in G consists of submanifolds of codimension at least one.

Proof. Let $\widetilde{w} \in N_K(\mathfrak{a})$ be a representative of $w \in W(\mathfrak{g};\mathfrak{a})$ the longest element in the Weyl-group. This is the element in $W(\mathfrak{g};\mathfrak{a})$ that is uniquely characterized by $w(\Sigma^+) = -\Sigma^+$. Then $\widetilde{w}\overline{N}\widetilde{w}^{-1} = N$ hence $\dim B_w = \dim(N) + \dim P = \dim G$. We conclude that B_w is a codimension 0 immersed submanifold of G hence is an open submanifold of G. For every $w' \in W(\mathfrak{g};\mathfrak{a})$ with $w' \neq w$ we have $\dim B_{w'} < \dim G$. From the Bruhat decomposition of G as $G = B_w \sqcup \sqcup_{w' \neq w} B_{w'}$ we see that the complement of B_w does indeed consists of lower dimensional manifolds. The statements in the lemma for the set $\overline{N}P$ follow from the fact that $\widetilde{w}^{-1}B_w = \widetilde{w}^{-1}N\widetilde{w}P = \overline{N}P$. □

1.6 Induced representations

Let G be a Lie group and H be a closed subgroup of G. Let (σ, V_σ) be a representation of H on a finite-dimensional Hilbert space V_σ. In this section we define the induced representation $\mathrm{ind}_H^G(\sigma)$ which is a representation of G constructed from the representation (σ, V_σ) of H. The discussion in this section follows [vdB, Section 19].

If M is a manifold and E is a vector bundle then we denote by $\Gamma^0(M;E)$ the space of continuous sections of E over M. We equip this space with the topology of uniform convergence on compact sets so that $\Gamma^0(M;E)$ becomes a Fréchet space. By $\Gamma^\infty(M;E)$ we denote the space of smooth sections of E over M. This space we equip with the topology of uniform convergence of all derivatives on compact sets. The space $\Gamma^\infty(M;E)$ is also a Fréchet space. If E is trivial, i.e. $E = M \times V$ for some finite-dimensional vector space V, then we denote these spaces by $C^0(M;V)$ and $C^\infty(M;V)$ respectively. Furthermore, the bundle of densities on M is denoted by $\mathcal{D}_M$.

Definition 1.23. The representation space of the induced representation $\mathrm{ind}_P^G(\sigma)$ is defined as

$$C(G:H:\sigma) := \{f \in C^0(G;V_\sigma) \mid f(gh) = \sigma(h)^{-1}f(g) \quad \text{for all } g \in G, h \in H\}.$$

We let G act on this space via the representation $\pi_\sigma = \mathrm{ind}_H^G(\sigma)$ which is defined to be the left regular representation, i.e. $[\pi_\sigma(g)f](x) = f(g^{-1}x)$.

The space $C(G:H:\sigma)$ is a closed subspace of the Fréchet space $C(G;V_\sigma)$ hence it is itself a Fréchet space. The action of G on $C(G:H:\sigma)$ is continuous with respect to this topology. The space of smooth vectors of this representation is denoted by $C^\infty(G:H:\sigma) := C(G:H:\sigma) \cap C^\infty(G;V_\sigma)$. This space is closed in $C^\infty(G;V_\sigma)$ hence is itself a Fréchet space. Furthermore, we denote by $C_c(G:H:\sigma)$ the subspace of functions $\phi \in C(G:H:\sigma)$ such that the image of $\operatorname{supp}\phi$ under $G \to G/H$ is compact in G/H.

1.6.1 Normalized induced representation

The induced representation as defined above does not come with a natural inner product. So if σ is unitary the induced representation does not inherit this property. We introduce the normalized induced representation to remedy this.

Let $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebras of G and H respectively. For $h \in H$ the map $\mathrm{Ad}(h)$ descends to a map $\overline{\mathrm{Ad}}(h)\colon \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. We define the character $\Delta\colon H \to \mathbb{R}$ as

$$\Delta(h) := \left|\det_{\mathfrak{g}/\mathfrak{h}} \overline{\mathrm{Ad}}(h)\right|^{-1} = \frac{|\det_{\mathfrak{h}} \mathrm{Ad}(h)|_{\mathfrak{h}}|}{|\det_{\mathfrak{g}} \mathrm{Ad}(h)|}.$$

We denote the natural left action of G on G/H by l, i.e. $l_g\colon G/H \to G/H\colon xH \mapsto gxH$. The character Δ is defined precisely such that $(dl_h(eH))^{-1*}|\omega| = \Delta(h)\cdot|\omega|$ for any $\omega \in \wedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$ and $h \in H$.

The *conjugate adjoint* of the representation σ is the representation (σ^*, V_σ) of H given by

$$\sigma^*(h) = \sigma(h^{-1})^* \quad \text{for } h \in H.$$

The representation σ is unitary if and only if $\sigma = \sigma^*$. We denote by $V_{\sigma\otimes\Delta^{1/2}}$ the space V_σ equipped with the representation $\sigma \otimes \Delta^{1/2}$ of H which is given by $(\sigma\otimes\Delta^{1/2})(h) = \Delta^{1/2}(h)\cdot\sigma(h)$ for $h \in H$. Similarly we denote by $V_{\sigma^*\otimes\Delta^{1/2}}$ the space V_σ equipped with the representation $\sigma^*\otimes\Delta^{1/2}$. We denote by $\mathbb{C}_\Delta$ the space $\mathbb{C}$ equipped with the H-module structure $h\cdot z = \Delta(h)z$. The inner product on V_σ gives an H-equivariant pairing $V_{\sigma\otimes\Delta^{1/2}} \times V_{\sigma^*\otimes\Delta^{1/2}} \to \mathbb{C}_\Delta$. From this pairing we obtain an induced sesquilinear pairing

$$C(G:H:\sigma\otimes\Delta^{1/2}) \times C_c(G:H:\sigma^*\otimes\Delta^{1/2}) \to C_c(G:H:\Delta).$$

The space $C_c(G:H:\Delta)$ can be identified with $\Gamma_c^0(G/H;\mathcal{D}_{G/H})$, the space of compactly supported and continuous densities on G/H. For this we pick an element $\omega \in \wedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})\setminus\{0\}$. Let $f \in C_c(G:H:\Delta)$ and define $\widetilde{\phi}_f\colon G \to \mathcal{D}_{G/H}\colon x \mapsto f(x)(dl_x(eH))^{-1*}|\omega|$. Because $f(xh) = \Delta(h)^{-1}f(x)$ and $(dl_h(eH))^{-1*}|\omega| = \Delta(h)|\omega|$ for $h \in H$ this map is right H-invariant. Hence it descends to a map on G/H and we obtain a section $\phi_f \in \Gamma_c^0(G/H;\mathcal{D}_{G/H})$. The map $C_c(G:H:\Delta) \to \Gamma_c^0(G/H;\mathcal{D}_{G/H})\colon f \mapsto \phi_f$ is a bijection. Using this identification we can view the above pairing as a map into $\Gamma_c^0(G/H;\mathcal{D}_{G/H})$. Elements of this space can be integrated hence we obtain the following sesquilinear pairing

$$C(G:H:\sigma\otimes\Delta^{1/2}) \times C_c(G:H:\sigma^*\otimes\Delta^{1/2}) \to \mathbb{C}\colon (\phi,\psi) \mapsto \int_{G/H} \langle\phi,\psi\rangle_\omega. \tag{1.1}$$

Here $\langle \phi, \psi \rangle_\omega \in \Gamma_c^0(G/H; \mathcal{D}_{G/H})$ is defined as $xH \mapsto \langle \phi(x), \psi(x) \rangle_\sigma \, dl_x(eH)^{-1*} \, |\omega|$. Up to a positive factor this pairing is independent of the choice of $\omega \in \wedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{h})$.

Definition 1.24. The normalized induced representation is defined as $\mathrm{Ind}_H^G(\sigma) := \mathrm{ind}_H^G(\sigma \otimes \Delta^{1/2})$.

Proposition 1.25. The pairing defined by (1.1) is equivariant, i.e.

$$(\pi_{\sigma \otimes \Delta^{1/2}}(g)\phi, \pi_{\sigma^* \otimes \Delta^{1/2}}(g)\psi) = (\phi, \psi) \quad \text{for all } g \in G$$

with $\phi \in C(G : H : \sigma \otimes \Delta^{1/2})$ and $\psi \in C(G : H : \sigma^* \otimes \Delta^{1/2})$.

For a proof see [vdB, Lemma 19.11].
If we assume that σ is unitary then $\sigma^* = \sigma$. For such σ we have the following.

Proposition 1.26. If σ is unitary then the sesquilinear pairing in (1.1) defines a pre-Hilbert structure on the space $C_c(G : H : \sigma \otimes \Delta^{1/2})$. There exists a unique unitary representation on the completion of this space that extends the representation of G.

We will denote this completion by $L^2(G : H : \sigma \otimes \Delta^{1/2})$.

1.6.2 Principal series

In this section we return to the case of G a semisimple Lie group with finite center and use the notation of Section 1.3. Let (ξ, H_ξ) be finite-dimensional unitary representation of M. Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$. Then $\sigma(man) := a^\lambda \xi(m)$ defines a representation of P on H_ξ. We will denote this representation by $\xi \otimes e^\lambda \otimes 1$.

Definition 1.27. The principal series is the series of representations $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ depending on the parameters $\xi \in \widehat{M}$ (irreducible) and $\lambda \in \mathfrak{a}_C^*$. We refer to the series $\mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ depending only on $\lambda \in \mathfrak{a}_\mathbb{C}^*$ as the spherical principal series.

Definition 1.28. The element $\rho \in \mathfrak{a}^*$ is defined as the following weighted sum of the positive roots

$$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)\alpha.$$

or equivalently $\rho(H) = \frac{1}{2}\operatorname{tr}(\mathrm{ad}(H)|_\mathfrak{n})$ for all $H \in \mathfrak{a}$.

Lemma 1.29. The character Δ of P, as defined in the previous section, is given by $\Delta(man) = a^{-2\rho}$ for $man \in MAN$.

For a proof see [vdB, Lemma 20.3]. We see that passing from the induced representation to the normalized induced representation corresponds to a shift in the parameter λ since $(\xi \otimes e^\lambda \otimes 1) \otimes \Delta^{1/2} = \xi \otimes e^{\lambda+\rho} \otimes 1$. We introduce the following shorthand for the representation space $C(G : P : (\xi \otimes e^\lambda \otimes 1) \otimes \Delta^{1/2})$ of the principal series representation

$$C(P : \xi : \lambda) := \{\phi \in C(G; H_\xi) \mid \phi(gman) = a^{-\lambda-\rho}\xi(m)^{-1}\phi(g) \quad \text{for all } g \in G, man \in MAN\}.$$

We denote

$$\pi_{\xi,\lambda} := \mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1),$$

which is the left regular representation of G on this space. Similarly the space of smooth vectors of this representation is denoted by $C^\infty(P : \xi : \lambda) := C(P : \xi : \lambda) \cap C^\infty(G; H_\xi)$. It is easy to see that $(\xi \otimes e^\lambda \otimes 1)^* = \xi \otimes -\overline{\lambda} \otimes 1$ hence this representation is unitary if and only if $\lambda \in i\mathfrak{a}^*$.

1.6.3 Compact picture

From the result of Proposition 1.17 we know that $G/P \cong K/M$. Since we assumed G to have finite center we have that K is compact hence we see that G/P is compact. In this section we will realise the principal series representation $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ as a space of functions on K. This will be called the *compact picture* of the principal series representation. This compact picture has the advantage that the representation space is independent of the parameter λ.

Given (ξ, H_ξ) as in the previous section we can consider the induced representation $\mathrm{Ind}_M^K(\xi)$. Both K and M are compact hence unimodular which implies that the character of M satisfies $\Delta \equiv 1$. This means that $\mathrm{Ind}_M^K(\xi) = \mathrm{ind}_M^K(\xi)$. The corresponding representation space is denoted by

$$C(K : M : \xi) := \{\phi \in C^0(K; H_\xi) \mid \phi(km) = \xi(m)^{-1}\phi(k) \quad \text{for all } k \in K, m \in M\}.$$

Proposition 1.30. The restriction map $r_\lambda \colon C(P : \xi : \lambda) \to C(K : M : \xi)$ is a topological isomorphism of K-modules. Its inverse is given by

$$i_\lambda \colon C(K : M : \xi) \to C(P : \xi : \lambda), \quad i_\lambda(\phi)(kan) = a^{-\lambda-\rho}\phi(k)$$

for $k \in K$, $a \in A$ and $n \in N$.

For a proof see [vdB, Lemma 20.6]. Using this isomorphism the representation $\pi_{\xi,\lambda}$ on $C(P : \xi : \lambda)$ can be transferred to a representation of G on $C(K : M : \xi)$ that extends the left regular representation of K on this space. We will denote this representation also by $\pi_{\xi,\lambda}$ and it is given by

$$[\pi_{\xi,\lambda}(g)\phi](k) = e^{-(\lambda+\rho)H(g^{-1}k)}\phi(k(g^{-1}k)) \quad \text{for } g \in G \text{ and } k \in K.$$

The pairing $C(P : \xi : \lambda) \times C(P : \xi : -\overline{\lambda}) \to \mathbb{C}$ as given in (1.1) corresponds under this isomorphism to the pairing

$$C(K : M : \xi) \times C(K : M : \xi) \to \mathbb{C} \colon (\phi, \psi) \mapsto \int_K \langle \phi(k), \psi(k) \rangle_\xi \, \mathrm{d}k.$$

1.7 Integration

We will denote by $\mathrm{d}g, \mathrm{d}k, \mathrm{d}m, \mathrm{d}a, \mathrm{d}n$ and $\mathrm{d}\overline{n}$ choices of left Haar measures on G, K, M, A, N and $\overline{N}$ respectively. We choose to normalize $\mathrm{d}k$ and $\mathrm{d}m$ such that $\int_K \mathrm{d}k = 1$ and $\int_M \mathrm{d}m = 1$. Taking into account the proposition below we see that all these measures are in fact also right Haar measures.

Proposition 1.31. A Lie group is unimodular if it is either

1. an abelian Lie group
2. a compact Lie group
3. a semisimple Lie group
4. or a nilpotent Lie group.

For a proof see [Kna96, Corollary 8.31].

The below propositions are a collection of facts that will be used throughout the proofs in this text.

Proposition 1.32. The measures $\mathrm{d}a\,\mathrm{d}n$ and $a^{2\rho}\,\mathrm{d}a\,\mathrm{d}n$ define a left invariant Haar measure and a right invariant Haar measure on the group AN respectively.

Proposition 1.33. The measures $\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n$ and $a^{2\rho}\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n$ define a left invariant Haar measure and a right invariant Haar measure on the group MAN respectively.

Proposition 1.34. The Haar measures $\mathrm{d}g, \mathrm{d}a$ and $\mathrm{d}n$ can be normalized such that $\mathrm{d}g = a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$. So for any $f \in L^1(G)$ we have

$$\int_G f(g)\,\mathrm{d}g = \int_{K\times A\times N} f(kan)\cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n.$$

Proposition 1.35. The Haar measures $\mathrm{d}g, \mathrm{d}\overline{n}, \mathrm{d}a$ and $\mathrm{d}n$ can be normalized such that on $\overline{N}MAN$ we have $\mathrm{d}g = a^{2\rho}\,\mathrm{d}\overline{n}\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n$. Since the complement of $\overline{N}MAN$ in G has measure zero (see Proposition 1.22) this means that for all $f \in L^1(G)$ we have

$$\int_G f(g)\,\mathrm{d}g = \int_{\overline{N}\times MAN} f(\overline{n}man)\cdot a^{2\rho}\,\mathrm{d}\overline{n}\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n.$$

Proposition 1.36. If $\mathrm{d}\overline{n}$ is normalized as in Proposition 1.35 then we have for all $f \in L^1(K)$ that

$$\int_K f(k)\,\mathrm{d}k = \int_{\overline{N}}\int_M f(k(\overline{n})m)\cdot e^{-2\rho(H(\overline{n}))}\,\mathrm{d}m\,\mathrm{d}\overline{n}.$$

A proof for all these propositions can be found in [Kna96, Chapter 8.4].

The groups G and N are both unimodular hence the quotient G/N admits a left G-invariant measure $\mathrm{d}(gN)$. The Iwasawa decomposition of G yields a diffeomorphism $G/N \cong K\times A$. Using this diffeomorphism we can express the integral over G/N as an integral over $K\times A$.

Proposition 1.37. Let $\mathrm{d}g, \mathrm{d}a$ and $\mathrm{d}n$ be normalized as in Proposition 1.34. Then $\mathrm{d}(gN)$ can be normalized such that for all $f \in L^1(G/N)$ we have

$$\int_{G/N} f(g)\,\mathrm{d}(gN) = \int_{K\times A} f(ka)\cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a.$$

Proof. Let $\psi \in C^0(N)$ be such that $\int_N \psi(gn)\,\mathrm{d}n = 1$ for all $g \in G$. Then we have, using Theorem 8.34 of [Kna96], that

$$\int_{G/N} f(g)\,\mathrm{d}(gN) = \int_{G/N}\int_N f(gn)\psi(gn)\,\mathrm{d}n\,\mathrm{d}(gN) = \int_G f(g)\psi(g)\,\mathrm{d}g.$$

Applying Proposition 1.34 yields that the right hand side of this equation equals

$$\begin{aligned}\int_{K\times A\times N} f(kan)\psi(kan)\cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n &= \int_{K\times A} f(ka)\left[\int_N \psi(kan)\,\mathrm{d}n\right]\cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a\\ &= \int_{K\times A} f(ka)\cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}a.\end{aligned}$$

□

Proposition 1.38. The measures $\mathrm{d}a, \mathrm{d}n$ and $\mathrm{d}(g\overline{N})$ can be normalized such that for all $f \in L^1(MAN)$ we have

$$\int_{MAN} f(man)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n = \int_{G/\overline{N}} f(g)\,\mathrm{d}(g\overline{N}).$$

Proof. We let $\psi \in C^0(G)$ such that $\int_{\overline{N}} \psi(g\overline{n})\,\mathrm{d}\overline{n} = 1$ for all $g \in G$. Applying Theorem 8.34 of [Kna96] yields

$$\int_{G/\overline{N}} f(g)\,\mathrm{d}(g\overline{N}) = \int_{G/\overline{N}} \int_{\overline{N}} f(g\overline{n})\psi(g\overline{n})\,\mathrm{d}\overline{n}\,\mathrm{d}(g\overline{N}) = \int_G f(g)\psi(g)\,\mathrm{d}g.$$

In the notation of [Kna96] we have $\mathrm{d}g = \mathrm{d}_l(man)\,\mathrm{d}_r\overline{n} = \mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}\overline{n}$ on $MAN\overline{N}$. Using that the complement of $MAN\overline{N}$ has measure zero in G yields

$$\int_G f(g)\psi(g)\,\mathrm{d}g = \int_{MAN\times\overline{N}} f(man\overline{n})\psi(man\overline{n})\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}\overline{n} = \int_{MAN} f(man)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n.$$

□

Since A normalizes N we have that the conjugation map $C_a\colon N \to N\colon n \mapsto ana^{-1}$ is an automorphism of N for every $a \in A$. Using this we find the following 'substitution of variables' result.

Proposition 1.39. Let $a \in A$. The measure $C_a^*\,\mathrm{d}n$ is a left and right Haar measure on N. Furthermore, we have $C_a^*\,\mathrm{d}n = a^{2\rho}\,\mathrm{d}n$ hence

$$\int_N f(ana^{-1})\,\mathrm{d}n = a^{-2\rho}\int_N f(n)\,\mathrm{d}n$$

for all $f \in L^1(N)$.

Proof. The fact that $C_a^*\,\mathrm{d}n$ is a Haar measure follows directly from the fact that C_a is an automorphism and $\mathrm{d}n$ is both a left and a right Haar measure. Since both measures are Haar measures it is enough to show $C_a^*\,\mathrm{d}n = a^{2\rho}\,\mathrm{d}n$ holds at $n = e$. We observe

$$(C_a^*\,\mathrm{d}n)(e) = dC_a(e)^*\,\mathrm{d}n(e) = \mathrm{Ad}(a)^*\,\mathrm{d}n(e) = |\det \mathrm{Ad}(a)|_{\mathfrak{n}}|\,\mathrm{d}n(e) = a^{2\rho}\,\mathrm{d}n(e).$$

□

1.8 Generalized sections

In this section we introduce the notion of a generalized vector of the principal series representation.

Definition 1.40. We define the set of generalized vectors of the principal series representation as the topological antilinear dual of the Fréchet space $C^\infty(G : P : \xi : -\overline{\lambda})$, i.e.

$$C^{-\infty}(G : P : \xi : \lambda) := \overline{C^\infty(G : P : \xi : -\overline{\lambda})^*}.$$

We equip this space with the strong dual topology.

If we consider the sesquilinear pairing $C^\infty(P : \xi : -\overline{\lambda}) \times C(P : \xi : \lambda) \to \mathbb{C}$ as in (1.1) then we see that the map $\phi \mapsto (\cdot, \phi)$ continuously embeds $C(P : \xi : \lambda)$ into $C^{-\infty}(P : \xi : \lambda)$ (remember that the latter space is defined as the antilinear dual hence the given map is linear). We define a representation of G on the space $C^{-\infty}(P : \xi : \lambda)$ by $\pi_{\xi,\lambda}(g)\phi := \phi \circ \pi_{\xi,-\overline{\lambda}}(g^{-1})$. Using this definition we see that

$$(\pi_{\xi,\lambda}(g)\left\langle \cdot, \phi\right\rangle)(\psi) = \left\langle \pi_{\xi,-\overline{\lambda}}(g^{-1})\psi, \phi\right\rangle = \left\langle \psi, \pi_{\xi,\lambda}(g)\phi\right\rangle = \left\langle \cdot, \pi_{\xi,\lambda}(g)\phi\right\rangle(\psi)$$

for $\phi \in C(P : \xi : \lambda)$ and $\psi \in C^\infty(P : \xi : -\overline{\lambda})$. We conclude that this representation of G on $C^{-\infty}(P : \xi : \lambda)$ extends the representation on $C(P : \xi : \lambda)$. Furthermore, the space $C^{-\infty}(P : \xi : \lambda)$ can be equipped with the structure of a $\mathfrak{g}$-module by setting $\pi_{\xi,\lambda}(X)\phi := -\phi \circ \pi_{\xi,-\overline{\lambda}}(X)$ for $X \in \mathfrak{g}$. In this way we uniquely extend the $\mathfrak{g}$-module structure $\pi_{\xi,\lambda}$ on the space $C^\infty(P : \xi : \lambda)$.

For the compact picture we define the space of generalized vectors in a similar way.

Definition 1.41. The set of generalized vectors in the compact picture of the principal series representation is defined as the topological antilinear dual of $C^\infty(K : M : \xi)$, i.e.

$$C^{-\infty}(K : M : \xi) := \overline{C^\infty(K : M : \xi)^*}. \tag{1.2}$$

We equip this space with the strong dual topology.

Taking the dual of the isomorphism i_λ of Proposition 1.30 yields an isomorphism $C^{-\infty}(P : \xi : \lambda) \to C^{-\infty}(K : M : \xi)$.

Following our convention of denoting the space of smooth sections of the trivial bundle $G \times H_\xi$ as $C^\infty(G; H_\xi)$ we denote by $C^{-\infty}(G; H_\xi)$ the space of generalized sections of this bundle, i.e. $C^{-\infty}(G; H_\xi) := \Gamma^{-\infty}(G; G \times H_\xi)$. The space $C^\infty(G; H_\xi)$ can be equipped with the left and right regular representation of G and the representation ξ of M. These three representations can be extended uniquely to representations on $C^{-\infty}(G; H_\xi)$ in a straightforward manner. Per definition the space $C^\infty(P : \xi : \lambda)$ is a subspace of $C^\infty(G; H_\xi)$. This embedding can be extended to the generalized vectors in the following way.

Proposition 1.42. The embedding $C^\infty(P : \xi : \lambda) \hookrightarrow C^\infty(G; H_\xi)$ can uniquely be extended to a continuous embedding $C^{-\infty}(P : \xi : \lambda) \hookrightarrow C^{-\infty}(G; H_\xi)$. The image of this embedding is contained in the closed subspace

$$\{\phi \in C^{-\infty}(G; H_\xi) \mid R_{man}\phi = a^{-\lambda-\rho}\xi(m)^{-1}\phi \quad \text{for all } man \in MAN\}.$$

If we equip $C^{-\infty}(G; H_\xi)$ with the left regular representation this embedding map is G-equivariant.

Proof. We will define a map $T_\lambda \colon \Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G) \to C^\infty(P : \xi : -\overline{\lambda})$ such that the following diagram commutes

$$\begin{tikzcd}
C^\infty(P : \xi : -\overline{\lambda}) & \times & C^\infty(P : \xi : \lambda) \arrow[r] \arrow[d, hook] & \mathbb{C} \arrow[d, equal] \\
\Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G) \arrow[u, "T_\lambda"] & \times & C^\infty(G; H_\xi) \arrow[r] & \mathbb{C}
\end{tikzcd} \tag{1.3}$$

Here we define the lower sesquilinear pairing as follows; Let $\widetilde{\phi} \in \Gamma_c(G; (G \times H_\xi) \otimes \mathcal{D}_G)$ then for suitable $\phi \in C_c^\infty(G; H_\xi)$ we can write $\widetilde{\phi} = \phi \otimes dg$. For $\psi \in C^\infty(G; H_\xi)$ we now set

$$(\widetilde{\phi}, \psi) := \int_G \langle \phi(g), \psi(g) \rangle_\xi \, \mathrm{d}g.$$

Taking the dual of the map $\phi \otimes dg \mapsto \langle \cdot, \phi \rangle_\xi \otimes dg$ provides an isomorphism between $\overline{\Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G)^*}$ and $\Gamma_c^\infty(G; (G \times H_\xi^*) \otimes \mathcal{D}_G)^* = C^{-\infty}(G; H_\xi)$. It is easily verified that under this isomorphism the embedding of $C^\infty(G; H_\xi)$ into $\overline{\Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G)^*}$ via the map $\psi \mapsto (\,\cdot\,, \psi)$ corresponds to the natural embedding of $C^\infty(G; H_\xi)$ into $C^{-\infty}(G; H_\xi)$.

Now we define the map $T_\lambda \colon \Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G) \to C^\infty(P : \xi : -\overline{\lambda})$. Let $\widetilde{\phi} = \phi \otimes dg \in \Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G)$. We set

$$T_\lambda(\widetilde{\phi})(x) := \int_{MAN} a^{-\overline{\lambda}+\rho}\xi(m)\phi(xman) \, \mathrm{d}m \, \mathrm{d}a \, \mathrm{d}n \quad \text{for } x \in G.$$

Since P is closed the set $xP \cap \operatorname{supp} \phi$ is compact hence the above integral expression is finite for every $x \in G$. It is not difficult to check that the above expression defines a smooth function on G and that T_λ

is continuous as a map into $C^\infty(G; H_\xi)$. We now check that T_λ does indeed land in $C^\infty(P : \xi : -\overline{\lambda})$. For this we recall from Proposition 1.33 that $\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n$ defines a left invariant measure on P. We let $x \in G, m' \in M, a' \in A$ and $n' \in N$. For $\widetilde{\phi} = \phi \otimes dg$ as above we have

$$\begin{aligned} T_\lambda(\widetilde{\phi})(xm'a'n') &= \int_{MAN} a^{-\overline{\lambda}+\rho}\xi(m)\phi(xm'a'n'man)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= \int_{MAN} (a'^{-1}a)^{-\overline{\lambda}+\rho}\xi(m'^{-1}m)\phi(xman)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= a^{\overline{\lambda}-\rho}\xi(m')^{-1}\int_{MAN} a^{-\overline{\lambda}+\rho}\xi(m)\phi(xman)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= a^{\overline{\lambda}-\rho}\xi(m')^{-1}T_\lambda(\widetilde{\phi})(x). \end{aligned}$$

So we indeed have that $T_\lambda(\widetilde{\phi}) \in C^\infty(P : \xi : -\overline{\lambda})$.

Now we prove the diagram in (1.3) is commutative. We let $\widetilde{\phi} = \phi \otimes dg$ be as above and let $\psi \in C^\infty(P : \xi : \lambda)$. We observe

$$\begin{aligned} (T_\lambda(\widetilde{\phi}), \psi) &= \int_K \left\langle T_\lambda(\widetilde{\phi})(k), \psi(k) \right\rangle_\xi \\ &= \int_K \int_{MAN} \left\langle a^{-\overline{\lambda}+\rho}\xi(m)\phi(kman), \psi(k) \right\rangle_\xi \mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n\,\mathrm{d}k \\ &= \int_{K\times MAN} \langle \phi(kman), \psi(kman) \rangle_\xi \cdot a^{2\rho}\,\mathrm{d}k\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n. \end{aligned}$$

The result of Proposition 1.34 yields that this equals

$$\int_G \langle \phi(g), \psi(g) \rangle_\xi \,\mathrm{d}g = (\widetilde{\phi}, \psi).$$

We conclude that the diagram in (1.3) is indeed commutative. Taking the dual of T_λ yields a continuous linear map $C^{-\infty}(P : \xi : \lambda) \to \overline{\Gamma_c^\infty(G; (G \times H_\xi) \otimes \mathcal{D}_G)^*} \cong C^{-\infty}(G; H_\xi)$. This map we define to be the embedding of $C^{-\infty}(P : \xi : \lambda)$ into $C^{-\infty}(G; H_\xi)$. The fact that this map is an extension of $C^\infty(P : \xi : \lambda) \hookrightarrow C^\infty(G; H_\xi)$ follows directly from the fact that the diagram in (1.3) commutes.

What remains to be proved is that the extended map is injective. For this it is enough to prove that T_λ is surjective. Let $\psi \in C^\infty(P : \xi : -\overline{\lambda})$ be arbitrary. Let $\chi \in C_c^\infty(AN)$ be such that $\int_{AN} \chi(an)\,\mathrm{d}a\,\mathrm{d}n = 1$. We define $\phi \in C_c^\infty(G; H_\xi)$ as $\phi(kan) = \psi(kan)\chi(an)$. We claim that $T_\lambda(\phi \otimes dg) = \psi$. Since both these functions have the same AN-transformations behaviour on the right it is enough to show these functions coincide on K. We observe, for $k \in K$, that

$$\begin{aligned} T_\lambda(\phi \otimes dg)(k) &= \int_{MAN} a^{\overline{\lambda}+\rho}\xi(m)\phi(kman)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= \int_{MAN} a^{\overline{\lambda}+\rho}\xi(m)\psi(kman) \cdot \chi(an)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= \int_{MAN} \psi(k) \cdot \chi(an)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n \\ &= \psi(k) \int_{AN} \chi(an)\,\mathrm{d}a\,\mathrm{d}n = \psi(k). \end{aligned}$$

We conclude that T_λ is indeed surjective.

The fact that the defined embedding is G-equivariant follows directly from the fact that $C^\infty(P:\xi:\lambda) \hookrightarrow C^\infty(G;H_\xi)$ is G-equivariant and that the fact $C^\infty(P:\xi:\lambda)$ lies dense in $C^{-\infty}(P:\xi:\lambda)$. To see that the image of $C^{-\infty}(P:\xi:\lambda) \hookrightarrow C^{-\infty}(G;H_\xi)$ is contained in the subspace

$$\{\phi \in C^{-\infty}(G;H_\xi) \mid R_{man}\phi = a^{-\lambda-\rho}\xi(m)^{-1}\phi \quad \text{for all } man \in MAN\}.$$

we observe that this subspace is closed in $C^{-\infty}(G;H_\xi)$ and that $C^\infty(P:\xi:\lambda)$ is mapped into this subspace. The statement now follows from again using the fact that $C^\infty(P:\xi:\lambda)$ lies dense in $C^{-\infty}(P:\xi:\lambda)$. This completes the proof. □

1.9 Infinitesimal Characters

In this section we briefly describe the infinitesimal character of a representation and in particular the infinitesimal character of the principal series representation. For this we consider $\mathfrak{g}$ a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$ which is defined as $Z(\mathfrak{g}) := \{Z \in U(\mathfrak{g}) \mid [Z,X] = 0 \quad \text{for all } X \in \mathfrak{g}\}$. A *character of* $Z(\mathfrak{g})$ is a homomorphism of unital algebras $Z(\mathfrak{g}) \to \mathbb{C}$.

Definition 1.43. Let (π, V) be a Lie algebra representation of $\mathfrak{g}$. Suppose that $Z(\mathfrak{g})$ acts on V (viewed as a $U(\mathfrak{g})$-module) by scalars, i.e. there exists a character χ of $Z(\mathfrak{g})$ such that $\pi(Z) = \chi(Z)\cdot I$ for all $Z \in Z(\mathfrak{g})$. If this is the case we say the representation π has infinitesimal character χ.

It is a consequence of Dixmier's Lemma that every irreducible representation of $\mathfrak{g}$ has an infinitesimal character (see [Kna96, Section V.4]).

We denote by $\mathcal{R} = \mathcal{R}(\mathfrak{g};\mathfrak{a})$ the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. The corresponding Weyl-group is denoted by W. We denote by $\mathcal{R}^+$ a choice of positive roots. We write $\mathfrak{g}^+$ for the sum of the positive root spaces and $\mathfrak{g}^-$ for the sum of negative root spaces. The root space decomposition of $\mathfrak{g}$ gives that

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

Using the Poincaré–Birkhoff–Witt Theorem (see [Kna96, Theorem 3.8]) we see that

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{g}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}^+).$$

We denote by ${}'\gamma\colon Z(\mathfrak{g}) \to U(\mathfrak{h})$ the projection along this decomposition. Since $\mathfrak{h}$ is abelian we can canonically identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$ so we can view this projection as a map ${}'\gamma\colon Z(\mathfrak{g}) \to S(\mathfrak{h})$.

We denote by $\delta := \frac{1}{2}\sum_{\alpha\in\mathcal{R}^+}\alpha$ the half sum of all positive roots. The map $\tau\colon \mathfrak{h} \to S(\mathfrak{h})$ defined by $\tau(H) = H - \delta(H)\cdot 1$ uniquely extends to a homomorphism $\tau\colon S(\mathfrak{h}) \to S(\mathfrak{h})$. We define what is called the Harish-Chandra map as follows

$$\gamma = \tau \circ {}'\gamma\colon Z(\mathfrak{g}) \to S(\mathfrak{h}).$$

Although it is not strictly necessary for our current exposition we mention the following important result by Harish-Chandra.

Theorem 1.44. The map γ maps $Z(\mathfrak{g})$ into $S(\mathfrak{h})^W$, the subspace of elements in $S(\mathfrak{h})$ fixed by Weyl-group. As a map $\gamma\colon Z(\mathfrak{g}) \to S(\mathfrak{h})^W$ the Harish-Chandra map is an algebra isomorphism. Furthermore, this map is independent of our choice of positive roots $\mathcal{R}^+$.

For a proof see [Kna96, Theorem 5.44]. The space $\mathcal{S}(\mathfrak{h})$ can be naturally identified with $P(\mathfrak{h}^*)$, the space of polynomials on $\mathfrak{h}^*$. For $Z \in Z(\mathfrak{g})$ and $\Lambda \in \mathfrak{h}^*$ we write $\gamma(Z)(\Lambda)$ for the evaluation of $\gamma(Z)$, seen as an element in $P(\mathfrak{h}^*)$, in the element Λ. For such an element $\Lambda \in \mathfrak{h}^*$ we set $\chi_\Lambda \colon Z(\mathfrak{g}) \to \mathbb{C} \colon Z \mapsto \gamma(Z)(\Lambda)$ which defines a character of $Z(\mathfrak{g})$. The above theorem has the following corollary.

Proposition 1.45. Every character of $Z(\mathfrak{g})$ is of the form χ_Λ for some $\Lambda \in \mathfrak{h}^*$.

For a proof see [Kna96, Theorem 5.62]. From this result it follows that if a $\mathfrak{g}$-representation (π, V) has a infinitesimal character it is given by χ_Λ for a suitable $\Lambda \in \mathfrak{h}^*$. We will refer to this element Λ also as the infinitesimal character of the representation.

It turns out that principal series representation has an infinitesimal character and this character can be calculated from the representation ξ of M. Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Let $P = MAN$ be a minimal parabolic subgroup as introduced in Section 1.3 and let $\mathfrak{t} \subset \mathfrak{m}$ be a θ-stable Cartan subalgebra of $\mathfrak{m}$. Then $\mathfrak{h} = (\mathfrak{t} \oplus \mathfrak{a})_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

First we consider the following lemma.

Lemma 1.46. Suppose (ξ, H_ξ) is a finite-dimensional and irreducible representation of M. Then the associated representation (ξ_*, H_ξ) of $\mathfrak{m}$ is also irreducible.

Proof. Denote by Z_M the center of M. We begin by proving that $M = Z_M M_e$. We denote by Z_G the center of G. By [Kna96, Theorem 6.31] we have that $Z_G \subset K$ hence $Z_G \subset Z_M$. The adjoint map yields an isomorphism $\mathrm{Ad}\colon G/Z_G \to \mathrm{Ad}(G)$. It is readily checked that $M/Z_M \cong \mathrm{Ad}(M)/\mathrm{Ad}(Z_M)$ hence we can pass to the adjoint group and it suffices prove that $\mathrm{Ad}(M)/\mathrm{Ad}(Z_M)$ is connected. This means that we can assume that G has trivial center in which case we have $G \cong \mathrm{Ad}(G)$. The latter group is the real form of the complex connected group $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$. From [Kna96, Theorem 7.53] it now follows that a subgroup $F \subset \mathrm{Ad}(M)$ exists with $F \subset \mathrm{Ad}(Z_M)$ and $\mathrm{Ad}(M) = F\,\mathrm{Ad}(M_e)$. In particular we find that $\mathrm{Ad}(M)/\mathrm{Ad}(Z_M)$ is connected. We conclude that $M = Z_M M_e$.

Now we assume H_ξ is irreducible as an M-module. Let $V \subset H_\xi$ be a subspace invariant under $\xi_*(\mathfrak{m})$. Then V is also invariant under M_e. Let $m \in M$ arbitrary and write $m = zm'$ with $z \in Z_M$ and $m' \in M_e$. Because $z \in Z_M$ we have that $\xi(z)$ is M-intertwining. Schur's lemma now implies that $\xi(z)$ acts as a scalar on V. Hence V is invariant for both $\xi(z)$ and $\xi(m')$. We conclude that V is invariant for all $\xi(m)$ with $m \in M$. Because H_ξ was irreducible as an M-module we conclude V is either 0 or H_ξ. This proves that H_ξ is irreducible as $\mathfrak{m}$-module. $\square$

Combined with the remark made after Definition 1.43 this lemma has as a consequence that if (ξ, H_ξ) is an irreducible representation of M then the representation (ξ_*, H_ξ) of $\mathfrak{m}$ has an infinitesimal character with respect to $\mathfrak{t}$.

Proposition 1.47. Suppose ξ is a finite-dimensional, unitary and irreducible representation of M. Let $\Lambda \in \mathfrak{t}_{\mathbb{C}}^*$ be the infinitesimal character of the representation ξ_* of $\mathfrak{m}$ with respect to $\mathfrak{t}$. Then for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the representation $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ has infinitesimal character $\Lambda + \lambda$ with respect to $(\mathfrak{t} \oplus \mathfrak{a})_{\mathbb{C}}$.

For a proof see [Kna86, Proposition 8.22].

In the case that $G = \mathrm{SL}(n, \mathbb{R})$ we have $\mathfrak{m} = 0$ so we have $\Lambda = 0$ for all $\xi \in \widehat{M}$. The previous result shows that in this case the infinitesimal character of $\mathrm{Ind}_P^G(\xi \otimes e^\lambda \otimes 1)$ is simply λ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

1.9.1 Casimir element

Using the Killing form B of $\mathfrak{g}$ we can identify a special element in $Z(\mathfrak{g})$ called the *Casimir element*. Let $(X_i)_i$ be a basis of $\mathfrak{g}$. Denote by $(\widetilde{X}_j)_j$ the dual basis with respect to B, i.e. $B(X_i, \widetilde{X}_j) = \delta_{ij}$ for all i, j. Then the Casimir element is defined as

$$\Omega = \sum_{i,j} X_i \widetilde{X}_j.$$

The Casimir element is contained in $Z(\mathfrak{g})$ and independent of the basis chosen (see [Kna96, Proposition 5.24]).

For $\mathfrak{sl}(2, \mathbb{R})$ some straightforward computations yield that B is given by

$$B = \begin{array}{cc} \begin{array}{ccc} H & X & Y \end{array} & \\ \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} & \begin{array}{c} H \\ X \\ Y. \end{array} \end{array}$$

It now follows easily that the Casimir element of $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$\frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX.$$

For computational convenience later on we will work with a rescaled Casimir element (this corresponds to a rescaling of the Killing form). For $\mathfrak{sl}(2, \mathbb{R})$ we define $\Omega := H^2 + 2XY + 2YX$ and we will refer to this element of $Z(\mathfrak{sl}(2, \mathbb{R}))$ as the Casimir element of $\mathfrak{sl}(2, \mathbb{R})$. For $\mathfrak{sl}(2, \mathbb{R})$ we have that $Z(\mathfrak{sl}(2, \mathbb{R})) = \mathbb{C}[\Omega]$ (see [Kna96, p.249]). This means that a character of $Z(\mathfrak{sl}(2, \mathbb{R}))$ is completely determined by its value on Ω.

As remarked above in the case $G = \mathrm{SL}(2, \mathbb{R})$ the infinitesimal character of the principal series representation $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ is given by λ. We will now determine $\chi_\lambda(\Omega)$. We observe that

$$\Omega = H^2 + 2XY + 2YX = H^2 + 2H + 4YX.$$

Since $YX \in U(\mathfrak{g})\mathfrak{g}^+$ we see that ${}'\gamma(\Omega) = H^2 + 2H$. We have $\mathcal{R}^+(\mathfrak{g}; \mathfrak{m} \oplus \mathfrak{a}) = \Sigma^+(\mathfrak{g}; \mathfrak{a}) = \{\alpha\}$ hence $\delta = \frac{1}{2}\alpha = \rho$. Using this we see

$$\gamma(\Omega) = (H - \rho(H))^2 + 2(H - \rho(H)) = (H - 1)^2 + 2(H - 1) = H^2 - 1.$$

As a summary of the above discussion we have the following result.

Proposition 1.48. If $G = \mathrm{SL}(2, \mathbb{R})$ the Casimir element $\Omega = H^2 + 2XY + 2YX$ acts on $C^\infty(P : \xi : \lambda)$ by the scalar $\lambda(H)^2 - 1$.

Chapter 2

Whittaker vectors

In this chapter we begin our study of the so called *Whittaker vectors* and the *Whittaker coefficient*. In Section 2.1 we define the concept of Whittaker vectors and we will dedicate the four following sections to studying the space of Whittaker vectors. The space of Whittaker vectors is completely understood and can be given a explicit parametrization. This will be the main result of the first part of this chapter. The results in this first part will be proved for connected semisimple Lie groups with finite center.
In the second part of this chapter, beginning with Section 2.6, we start the study of the Whittaker coefficient. This function on G, defined as a matrix coefficient of the principal series representation, will play a vital role in Chapter 3 when we introduce the Whittaker–Fourier transformation. The definition of the Whittaker coefficient will make use of the description of the space of Whittaker vectors we obtain in the first part of this chapter. Our main objective in the second part of this chapter is to derive several estimates on this function that will be vital in our discussion in Chapter 3. Our discussion of the Whittaker coefficient will be focussed on the case of $G = \mathrm{SL}(2, \mathbb{R})$.

Throughout this chapter G will be be a connected semisimple Lie group with finite center and denote by $\mathfrak{g}$ its Lie algebra. We retain the notation introduced in the previous chapter.

2.1 Whittaker vectors

In this section we introduce the concept of Whittaker vectors in a principal series representation. These are elements of this representation that transform according to a unitary character of $\overline{N}$. We begin by specifying what a character on $\overline{N}$ is.

Definition 2.1. A unitary character on $\overline{N}$ is a group homomorphism $\overline{N} \to S^1$.

We let (ξ, H_ξ) be a unitary representation of M and let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Denote by χ a choice of unitary character on $\overline{N}$. Whittaker vectors are elements of the principal series representation, $\mathrm{Ind}_P^G(\xi \otimes e^{\lambda} \otimes 1)$, that transform according to the character χ when acted on by element of $\overline{N}$, i.e.

$$\pi_{\xi,\lambda}(\overline{n})\phi = \chi(\overline{n})\phi \quad \text{for all } \overline{n} \in \overline{N}. \tag{2.1}$$

It turns out however that the representation space $C(P : \xi : \lambda)$ of continuous functions is, in general, not 'rich' enough and that only for specific choices of λ nonzero elements satisfying (2.1) exist (see Remark 2.12). This is why we turn to generalized sections as described in 1.8.

Definition 2.2. A Whittaker vector is an element $\phi \in C^{-\infty}(P : \xi : \lambda)$ that transforms under $\overline{N}$ as in (2.1). The space of Whittaker vectors is denoted by

$$C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} = \{\phi \in C^{-\infty}(P : \xi : \lambda) \mid \pi_{\xi,\lambda}(\overline{n})\phi = \chi(\overline{n})\phi \quad \text{for all } \overline{n} \in \overline{N}\}.$$

Remark 2.3: These Whittaker vectors are named after the mathematician E. T. Whittaker. In the early 1900s he studied functions on $\mathbb{R}$ satisfying a certain differential equation. The reason for naming these elements after him is that when $G = \mathrm{SL}(2, \mathbb{R})$ several functions related to Whittaker vectors (more specifically the Whittaker coefficient) satisfy this Whittaker differential equation. In Section 2.8 we will study the relation between Whittaker vectors and the Whittaker differential equation.

Remark 2.4: The notation we introduced for the space of Whittaker vectors is an example of a more general piece of notation that we will use. If (π, V) is a representation of G (or possibly only a representation of $\overline{N}$) then we denote

$$V^{\overline{N},\chi} := \{v \in V \mid \pi(\overline{n})v = \chi(\overline{n})v \quad \text{for all } \overline{n} \in \overline{N}\}.$$

As we will see later the space of Whittaker vectors is very well-behaved whenever the unitary character χ is sufficiently non-trivial. To make this precise we introduce the notion of a regular character.

Definition 2.5. We call a unitary character $\chi\colon \overline{N} \to S^1$ regular if for all $\alpha \in \Delta$ we have $d\chi(\mathfrak{g}_{-\alpha}) \neq 0$.

When χ is regular the space of Whittaker vectors is very well-behaved and well understood. In fact we have that when χ is regular the space of Whittaker vectors is, in a natural way, isomorphic to H_ξ, the representation space of ξ. For a precise statement of this result we must defer to Section 2.4 (see Theorem 2.10) because we have not introduced the necessary notation yet.

We have split up the proof of this result into several pieces and we will dedicate the next four sections to it. For the benefit of the reader we will first give a quick overview of the process. First we prove that when restricted to the the set $\overline{N}P$, the *big Bruhat cell*, in G the Whittaker vectors are in fact smooth functions (see Section 2.2). As a consequence the Whittaker vectors are on the big Bruhat cell completely determined by their value in the point e. The second step will be proving that the values of a Whittaker vector on this big Bruhat cell in fact determine the vector on the whole of G. This is done by showing that Whittaker vectors supported in the complement of the big Bruhat cell must vanish (see Section 2.3). As a result we have that the Whittaker vectors are completely determined by their value at the point e. This implies that the evaluation map $\mathrm{ev}_e\colon C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to H_\xi$ is injective. The proof is concluded by showing that a family of maps $j(P : \xi : \lambda)\colon H_\xi \to C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ exists which provides an inverse to ev_e. A special feature of this family of maps is that it depends holomorphically on λ. This gives us a way to holomorphically parametrize the space of Whittaker vectors.

2.2 Whittaker vectors on the big Bruhat cell

In this section we let χ be a, not necessarily regular, character of $\overline{N}$. We consider the set $B := \overline{N}P$ which is, by Proposition 1.22, an open and dense set in G. As discussed in the proof of Proposition 1.22 the set B is a translate of the Bruhat cell that corresponds to the longest element of the Weyl group. We will call B the *big Bruhat cell* of G (although, as we remarked, it is actually a translate of a Bruhat cell).

We note that B is both a left $\overline{N}$-invariant and a right P-invariant set. The right P-invariance means that we can consider the space $C^\infty(B : P : \xi : \lambda)$ of functions $\phi \in C^\infty(B; H_\xi)$ transforming as

$$\phi(xman) = a^{-\lambda-\rho}\xi(m)^{-1}\phi(x) \quad \text{for all } x \in B \text{ and } man \in MAN. \tag{2.2}$$

The restriction map to B yields a continuous linear map $r\colon C^\infty(P : \xi : \lambda) \to C^\infty(B : P : \xi : \lambda)$. This map is $\overline{N}$-intertwining.

We define the generalized vectors on B, analogues to the definitions of Section 1.8, as

$$C^{-\infty}(B : P : \xi : \lambda) := \overline{C_c^\infty(B : P : \xi : -\overline{\lambda})^*}.$$

Restricting to B gives a continuous linear map $r\colon C^{-\infty}(P : \xi : \lambda) \to C^{-\infty}(B : P : \xi : \lambda)$. This map extends $r\colon C^{\infty}(P : \xi : \lambda) \to C^{\infty}(B : P : \xi : \lambda)$ so in particular is also $\overline{N}$-intertwining. Because B is left $\overline{N}$-invariant we can consider the space Whittaker vectors on B which we denote by $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$. Since the map r is $\overline{N}$ intertwining it sends Whittaker vectors to Whittaker vectors.

A Whittaker vector ϕ on B satisfies two transformation properties, namely (2.1) and (2.2). Taking derivatives on both sides of these identities yields that ϕ satisfies

$$\pi_{\xi,\lambda}(Y)\phi = d\chi(Y)\phi \quad \text{for all } Y \in \overline{\mathfrak{n}}$$

and

$$R_{W+H+X}\phi = -[d\xi(W) + (\lambda+\rho)(H)]\phi \quad \text{for all } W \in \mathfrak{m}, H \in \mathfrak{a}, X \in \mathfrak{n}.$$

We see that ϕ is a solution to a certain system of differential equations. The form of this system of differential equations suggests that it might be elliptic and this, by elliptic regularity, suggests that ϕ is in fact smooth on B. This indeed turns out to be the case.

Proposition 2.6. On the big Bruhat cell Whittaker vectors are smooth, i.e. we have $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi} \subset C^{\infty}(B : P : \xi : \lambda)$.

As discussed above the smoothness of these vectors will follow from elliptic regularity. For the proof of this proposition we recall the following result.

Theorem 2.7. Let E be a complex vector bundle over a manifold M. Assume that $P\colon \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$ is an elliptic differential operator. Then for any $u \in \Gamma^{-\infty}(E)$ we have

$$\text{sing supp}\, u = \text{sing supp}\, Pu.$$

The proof of this theorem can be found in [Tay81, Theorem 1.4, p.61]. Here we note that an elliptic differential operator is in fact a properly supported elliptic pseudo-differential operator.

Using this we can give a proof of Proposition 2.6.

Proof of Proposition 2.6. In this proof we use the notation $\sigma(man) = a^{\lambda+\rho}\xi(m)$ for the representation of P on H_ξ as introduced in Section 1.6.2.

Using the same arguments as in the proof of Proposition 1.42 we see that $C^{-\infty}(B : P : \xi : \lambda)$ embeds G-equivariantly into $C^{-\infty}(B; H_\xi)$. The image of this embedding is contained in the subspace of elements $\phi \in C^{-\infty}(B; H_\xi)$ that satisfy

$$R_{man}\phi = \sigma(man)^{-1}\phi \quad \text{for all } man \in MAN. \tag{2.3}$$

Using that the embedding is $\overline{N}$-equivariant we see that the image of the space $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$ is contained in $C^{-\infty}(B; H_\xi)^{\overline{N},\chi}$. Hence we see that it is enough to show that any $\phi \in C^{-\infty}(B; H_\xi)$ satisfying both (2.3) and

$$L_{\overline{n}}\phi = \chi(\overline{n})\phi \quad \text{for all } \overline{n} \in \overline{N} \tag{2.4}$$

is in fact smooth.

Let $Y_1, ..., Y_s$ be a basis of $\overline{\mathfrak{n}}$ and let $X_1, ..., X_r$ be a basis of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. On B we define the operator

$$\mathcal{P} = \sum_{i=1}^{s}(L_{Y_i} - d\chi(Y_i))^2 + \sum_{j=1}^{r}(R_{X_j} + d\sigma(X_j))^2.$$

Here L and R denote the left and right regular representation of G on $C^\infty(G; H_\xi)$ respectively. This $\mathcal{P}$ is in fact a differential operator on B. A straightforward calculation reveals that at the point e the symbol of $\mathcal{P}$ is given by, for $\eta \in T_e^*G$ and $v \in H_\xi$,

$$\sigma^2(\mathcal{P})(\eta, v) = -\left[\sum_{i=1}^{s} \eta(Y_i)^2 + \sum_{j=1}^{r} \eta(X_j)^2\right] \cdot v.$$

Since $(Y_1, ..., Y_s, X_1, ..., X_r)$ is a basis of $\mathfrak{g} = T_eG$ we see that $T_e^*G \ni \eta \neq 0$ implies $\sigma^2(\mathcal{P})(\eta, \cdot) \neq 0$. Since ellipticity is an open condition we conclude that $\mathcal{P}$ is an elliptic operator on an open neighbourhood U of e.

Now we let $\phi \in C^{-\infty}(G; H_\xi)$ be an element that satisfies both (2.3) and (2.4). Taking derivatives on both sides of these equations yields that ϕ satisfies $[L_Y - d\chi(Y)]\phi = 0$ and $[R_X + d\sigma(X)]\phi = 0$ for all $Y \in \overline{\mathfrak{n}}$ and $X \in \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Hence we have, looking at the definition of $\mathcal{P}$, that $\phi \in \ker \mathcal{P}$. Invoking Theorem 2.7 we see that on U the distribution ϕ is smooth. To see ϕ is actually smooth on the whole of B we let $\overline{n}man \in B = \overline{N}MAN$ arbitrary. The transformation properties of ϕ imply that $\phi = \chi(\overline{n})\sigma(man)^{-1} L_{\overline{n}^{-1}} R_{man}\phi$. Using this we observe that

$$\phi|_{\overline{n}Uman} = \chi(\overline{n})\sigma(man)^{-1}\phi|_U \circ R_{(man)^{-1}} \circ L_{\overline{n}}.$$

From the above discussion we know that the right hand side of this expression is a distribution that is smooth on $\overline{n}Uman$. We conclude that ϕ is smooth on the whole of B. □

2.2.1 Evaluation map

From Proposition 2.6 we conclude that if we let $\phi \in C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ and restrict to B then $\phi|_B \in C^\infty(B : P : \xi : \lambda)$. In particular this means that the expression $\phi(e)$ has a well-defined value (since $e \in B$). Hence

$$\mathrm{ev}_e \colon C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to H_\xi \colon \phi \mapsto \phi(e)$$

is a well-defined continuous linear map. The aim of the next section is to show that this map is in fact injective if χ is regular. In preparation for this we note that the smoothness of the Whittaker vectors on the big Bruhat cell implies that a Whittaker vector is completely determined on B by its value at the point e. To make this precise we observe that for $\phi \in C^\infty(B : P : \xi : \lambda)^{\overline{N},\chi}$ we have

$$\phi(\overline{n}man) = (\pi_{\xi,\lambda}(\overline{n}^{-1})\phi)(man) = \chi(\overline{n})^{-1} a^{-\lambda-\rho}\xi(m)^{-1}\phi(e) \quad \text{for all } \overline{n}man \in B$$

We conclude that the evaluation map $\phi \mapsto \phi(e)$ is injective as a map from $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$ to H_ξ.

2.3 Whittaker vectors vanishing on the big Bruhat cell

In this section we show that the evaluation map ev_e, introduced in the previous section, is in fact injective. More precisely we will show that the following holds.

Proposition 2.8. If χ is regular then the evaluation map $\mathrm{ev}_e \colon C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to H_\xi$ is injective.

This evaluation map can be seen as a composition of the restriction map $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$ and the evaluation map $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi} \to H_\xi$. From our discussion in the previous section we know that this second map is in fact injective. Hence the kernel of the evaluation map ev_e is equal to kernel of the restriction map, i.e.

$$\ker \mathrm{ev}_e = \{\phi \in C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \mid \phi|_B = 0\}.$$

The following remarkable result, originally obtained by Harish-Chandra, implies immediately that this kernel equals $\{0\}$. Hence it immediately implies the result of Proposition 2.8.

Theorem 2.9. Assume that χ is regular. A Whittaker vector that vanishes on the big Bruhat cell vanishes identically on the whole of G.

Proof. For the proof of this theorem we refer to the work of Kolk and Varadarajan (see [KV96]). What they prove in the cited reference is the following. Suppose $T \in C^{-\infty}(G)$ satisfies

$$R_n T = T \ (n \in N) \quad \text{and} \quad L_{\overline{n}} T = \chi(\overline{n}) T \ (\overline{n} \in \overline{N}). \tag{2.5}$$

Then if $T|_B = 0$ we have $T = 0$. Note that their formulation of the result the roles of N and $\overline{N}$ are reversed.

In order to use this result in our set up we use Proposition 1.42 to see that any $\phi \in C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ can be seen as an element of $C^{-\infty}(G; H_\xi)$. Using the isomorphism $C^{-\infty}(G; H_\xi) \cong C^{-\infty}(G) \otimes H_\xi$ we can write $\phi = T_1 \otimes v_1 + \cdots + T_n \otimes v_n$ with $T_i \in C^{-\infty}(G)$ and $(v_i)_{i=1}^n$ a basis of H_ξ. The fact that ϕ satisfies (2.5) implies that every T_i also satisfies (2.5). Furthermore, the assumption $\phi|_B = 0$ implies that $T_i|_B = 0$ for every $i = 1, ..., n$. Now the result of Kolk and Varadarajan implies that all T_i's vanish. From this we conclude that ϕ vanishes. This proves the result. ☐

2.4 The $j(P : \xi : \lambda)$ function

In this section we introduce the family of functions $j(P : \xi : \lambda) \colon H_\xi \to C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$, depending on ξ and λ, which will provide an inverse to ev_e. Initially $j(P : \xi : \lambda)$ will be defined only for λ in a particular subset of $\mathfrak{a}_{\mathbb{C}}^*$. We will use that on this subset the family of maps depends on λ in a holomorphic fashion to obtain a holomorphic extension of this family to the whole of $\mathfrak{a}_{\mathbb{C}}^*$. The notion of holomorphic dependence on λ will be made precise later in this section. As a consequence of the existence of this family of maps and the results of the previous sections we will have the following result.

Theorem 2.10. Suppose χ is regular. The map $\mathrm{ev}_e \colon C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to H_\xi$ is a linear isomorphism for every $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

For $v \in H_\xi$ we denote, if it exists, by $j(P : \xi : \lambda : v)$ the element in $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ satisfying $j(P : \xi : \lambda : v)(e) = v$. We should note that as of yet the existence of this object has not been established. We aim to show that this element exists for all choices of v and λ so that we obtain a map $j(P : \xi : \lambda) \colon v \mapsto j(P : \xi : \lambda : v)$ which satisfies $\mathrm{ev}_e \circ j(P : \xi : \lambda) = \mathrm{id}_{H_\xi}$. It is clear from our discussion in Section 2.2.1 that when restricted to B the function $j(P : \xi : \lambda : v)$ must satisfy

$$j(P : \xi : \lambda : v)(\overline{n}man) = \chi(\overline{n})^{-1} a^{-\lambda-\rho} \xi(m)^{-1} v \tag{2.6}$$

for all $\overline{n} \in \overline{N}$ and $man \in MAN$.

For a moment we fix $\xi \in \widehat{M}$, $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Guided by the above observation we define for $v \in H_\xi$ the function $\widetilde{j}_v \colon G \to H_\xi$ as

$$\widetilde{j}_v(x) = \begin{cases} \chi(\overline{n})^{-1} a^{-\lambda-\rho} \xi(m)^{-1} v & \text{if } x = \overline{n}man \in B \\ 0 & \text{if } x \in G \setminus B. \end{cases}$$

Looking at this definition we easily see that $L_{\overline{n}}\widetilde{j}_v = \chi(\overline{n})\widetilde{j}_v$ holds for all $\overline{n} \in \overline{N}$. So if $\widetilde{j}_v$ is a continuous function on G then we have $\widetilde{j}_v \in C(P : \xi : \lambda)^{\overline{N},\chi}$. The next proposition shows this happens only for certain $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The proof given here is inspired by the proof of a similar result in [vdB88, Proposition 5.6].

Proposition 2.11. Define the open subset $\mathcal{A}$ of $\mathfrak{a}_{\mathbb{C}}^*$ as

$$\mathcal{A} := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \operatorname{Re}\lambda + \rho, \alpha\rangle < 0 \quad \text{for all } \alpha \in \Delta\}.$$

We have that $\widetilde{j}_v$, as defined above, is a continuous function on G if and only if $\lambda \in \mathcal{A}$.

Proof. First we assume $\lambda \in \mathcal{A}$. We fix a $v \in H_\xi$. It suffices to show that $\lim_{i\to\infty} \widetilde{j}_v(x_i) = 0$ for every sequence $(x_i)_i$ in B converging to a point $x \in \partial B = G \setminus B$. Let $(x_i)_i$ in B be such a sequence converging to $x \in G \setminus B$. Then for certain $\overline{n}_i \in \overline{N}, m_i \in M, a_i \in A, n_i \in N$ we have $x_i = \overline{n}_i m_i a_i n_i$. Looking at the definition of $\widetilde{j}_v$ we see that

$$\left\|\widetilde{j}_v(x_i)\right\| = \left\|\chi(\overline{n}_i)^{-1} a_i^{-\lambda-\rho}\xi(m_i)^{-1}v\right\| = a_i^{-\operatorname{Re}\lambda-\rho}\left\|v\right\|.$$

We claim that $a_i^{-\operatorname{Re}\lambda-\rho} \to 0$ for $i \to \infty$. The sequence $(m_i^{-1}\overline{n}_i m_i)_i$ is contained in $\overline{N}$ because M normalizes $\overline{N}$. This sequence is not contained in any compact subset of $\overline{N}$. To see this we suppose the contrary is true, so let $C \subset \overline{N}$ compact such that $(m_i^{-1}\overline{n}_i m_i)_i \subset C$. Then $\overline{n}_i \in MCM$ for all i. Since M is compact the set MCM is also compact. Hence by passing to a subsequence we can assume $\overline{n}_i$ converges to an $\overline{n} \in \overline{N}$. Then we have $\lim_{i\to\infty} m_i a_i n_i = \overline{n}^{-1}x$. Since $P = MAN$ is a closed subgroup of G this yields $\overline{n}^{-1}x \in P$, however this is in contradiction with $x \notin B = \overline{N}P$. We conclude that indeed $m_i^{-1}\overline{n}_i m_i$ is not contained in any compact subset of $\overline{N}$. From this and Proposition 1.16 it follows that $H(m_i^{-1}\overline{n}_i m_i) \in \sum_{\alpha\in\Sigma^+}\mathbb{R}_{\geq 0}\cdot H_\alpha$ for all i and $\left\|H(m_i^{-1}\overline{n}_i m_i)\right\| \to \infty$ for $i \to \infty$. We observe that

$$H(x_i) = H(\overline{n}_i m_i a_i n_i) = H(\overline{n_i} m_i) + \log(a_i).$$

Hence $\log(a_i) = H(x_i) - H(\overline{n_i} m_i)$. By continuity of H we have $\lim_{i\to\infty} H(x_i) = H(x)$. So if we combine this with the above we find that an element $R \in \mathbb{R}$ exists such that $\log(a_i) \in \sum_{\alpha\in\Sigma^+}(-\infty, R)\cdot H_\alpha$ for all i and $\|\log(a_i)\| \to \infty$ for $i \to \infty$. Per assumption we have $(-\operatorname{Re}\lambda - \rho)(H_\alpha) > 0$ for all $\alpha \in \Delta$ which implies that $(-\operatorname{Re}\lambda - \rho)\log(a_i) \to -\infty$ for $i \to \infty$. This proves that our claim is true and as a result we have $\left\|\widetilde{j}_v(x_i)\right\| \to 0$ for $i \to \infty$.

To show that the converse holds we assume $\widetilde{j}_v$ is continuous. We fix an $\alpha \in \Delta$ and consider a sequence $(t_i)_i \subset [0,\infty)$ with $t_i \to \infty$ for $i \to \infty$. Define the sequence $a_i := \exp(-t_i H_\alpha)$ in A. Because H maps $\overline{N}$ surjectively onto $\sum_{\alpha\in\Sigma^+}\mathbb{R}_{\geq 0}\cdot H_\alpha$ (see Proposition 1.16) we can find a sequence $(\overline{n}_i)_i$ in $\overline{N}$ with $H(\overline{n}_i) = t_i H_\alpha$ hence $H(\overline{n}_i a_i) = 0$ for all $i \in \mathbb{N}$. We set $n_i := n(\overline{n}_i a_i) \in N$ for $i \in N$. We now consider the sequence $(x_i)_i$ in B defined by $x_i := \overline{n}_i a_i n_i^{-1}$. We observe that

$$a(x_i) = \exp H(\overline{n}_i a_i) = e \quad \text{and} \quad n(x_i) = n(\overline{n}_i a_i) n_i^{-1} = n_i n_i^{-1} = e$$

for all $i \in \mathbb{N}$. From this it follows that the sequence $(x_i)_i$ is contained in K. Hence by passing to a subsequence we can assume this sequence converges to an element $x \in K$. Is is clear that the sequence $(x_i)_i$ is not contained in any compact subset of B hence we must have $x \in G \setminus B$. Since we assumed $\widetilde{j}_v$ to be continuous we must have $\lim_{i\to\infty}\widetilde{j}_v(x_i) = \widetilde{j}_v(x) = 0$. Per construction we have $\left\|\widetilde{j}_v(x_i)\right\| = e^{t_i(\operatorname{Re}\lambda+\rho)(H_\alpha)}$. We see that we must have $e^{t_i(\operatorname{Re}\lambda+\rho)(H_\alpha)} \to 0$ for $i \to \infty$ hence $\langle\operatorname{Re}\lambda + \rho, \alpha\rangle = (\operatorname{Re}\lambda + \rho)(H_\alpha) < 0$ must hold. Since $\alpha \in \Delta$ was arbitrary we conclude that $\lambda \in \mathcal{A}$. $\square$

As a result of the above proposition we have that for $\lambda \in \mathcal{A}$ and all $v \in H_\xi$ the element $j(P : \xi : \lambda : v)$ exists and is equal to $\widetilde{j}_v$. This means in particular that on $\mathcal{A}$ we can view the family of maps $j(P : \xi : \lambda)$ as functions into $C(P : \xi : \lambda)$.

Remark 2.12: The result of this proposition confirms our earlier claim that the space $C(P : \xi : \lambda)$ of continuous functions is not rich enough to contain Whittaker vectors for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We know that for any $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ a Whittaker vector restricted to B is given by the expression in (2.6). The argument in the proof of Proposition 2.11 now shows that a Whittaker vector is continuous on the whole of G if and only if $\lambda \in \mathcal{A}$.

We will now argue that on $\mathcal{A}$ the family of functions $j(P : \xi : \lambda)$ depends in a holomorphic fashion on λ. First we make precise what we mean by a vector-valued holomorphic function.

Definition 2.13. Let $\Omega \subset \mathbb{C}^n$ be an open subset and let V be a complex Banach space. A function $f \colon \Omega \to V$ is called holomorphic if it is continuously differentiable and its derivatives satisfy the Cauchy–Riemann equations, i.e. if $(x^1, y^1, ..., x^n, y^n)$ is the standard real basis of $\mathbb{C}^n$ we have

$$\frac{\partial f}{\partial x^j} = -i\frac{\partial f}{\partial y^j} \quad \text{for all } j = 1, ..., n.$$

Suppose W is a finite-dimensional complex vector space and $\phi \colon \mathbb{C}^n \to W$ a choice of basis. If $\Omega' \subset W$ is open we say $f \colon \Omega' \to V$ is a holomorphic map if $\phi^* f$ is holomorphic as a map from $\phi^{-1}(\Omega') \subset \mathbb{C}^n$ to V.

It is easily seen that this definition is independent of the choice of basis for W.

Lemma 2.14. Let V, V' be complex Banach spaces and W a finite-dimensional complex vector space. Let $A \colon V \to V'$ be a continuous linear map. If $f \colon W \to V$ is holomorphic then $A \circ f \colon W \to V$ is also holomorphic.

Proof. This is a direct consequence of the fact that $D(A \circ f) = A \circ Df$. □

Before we can prove that $j(P : \xi : \lambda)$ depends holomorphically on λ we first prove a more general lemma. If M is a manifold then we denote by $C_b(M)$ the space consisting of all bounded continuous functions on M. We equip this space with the supremum norm.

Lemma 2.15. Let M be a manifold and let $\Omega \subset \mathbb{C}^n$ be an open subset. Furthermore, let $f \colon M \times \Omega \to \mathbb{C}$ be a bounded and continuous function. If $f(m, z)$ is holomorphic in z for every fixed $m \in M$ then the map $F \colon \Omega \to C_b(M) \colon z \mapsto f(\cdot, z)$ is holomorphic.

The proof of this lemma was communicated to me by E.P. van den Ban.

Proof. We can, without loss of generality, assume that Ω is equal to the open polydisk $D(0; 1)^n$. Furthermore it suffices to show that F is holomorphic on the polydisk $D(0; r)^n$ for some $r \in (0, 1)$. We pick an R satisfying $r < R < 1$. Since f is holomorphic in the second variable we have by the Cauchy integral formula that

$$F(z)(m) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1|=R} \cdots \int_{|\zeta_n|=R} \frac{f(m, \zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} \,\mathrm{d}\zeta_1 \cdots \mathrm{d}\zeta_n.$$

for $z \in D(0; r)^n$ and $m \in M$. Since f is bounded on $M \times \Omega$ we see that this integral converges as a $C_b(M)$-valued integral and defines a continuous function $D(0; r)^n \to C_b(M)$. We denote by $(x_1, y_1, ..., x_n, y_n)$ the standard real basis of $\mathbb{C}^n$. By differentiating under the integral sign we see that the partial derivatives $\partial_{x_i} F$ and $\partial_{y_i} F$ exist and are continuous. Furthermore, we see that they satisfy the Cauchy–Riemann equations. We conclude that $F \colon \Omega \to C_b(M)$ is indeed holomorphic. □

We now apply this lemma to show that $j(P : \xi : \lambda)$ depends on λ in a holomorphic fashion on $\mathcal{A}$. We should note that we use Proposition 1.30 to identity $C(P : \xi : \lambda)$ and $C(K : M : \xi)$ in order to view $j(P : \xi : \lambda)$ as a map into the latter space. This is necessary because the space $C(K : M : \xi)$ does not depend on λ whereas $C(P : \xi : \lambda)$ does.

Proposition 2.16. For $v \in H_\xi$ fixed the function $\lambda \mapsto j(P : \xi : \lambda : v)$ is holomorphic as a map $\mathcal{A} \to C(K : M : \xi)$.

Proof. We denote $U = K \cap B$. We leave it to the reader to verify that $U = k(\overline{N})M$ and that $\overline{N} \times M \to U \colon (\overline{n}, m) \mapsto k(\overline{n})m$ is a diffeomorphism. Let $\lambda \in \mathcal{A}$ and $v \in H_\xi$. Using the transformation properties of $j(P : \xi : \lambda : v)$ we have for $k(\overline{n})m \in U$ that

$$\begin{aligned} j(P : \xi : \lambda : v)(k(\overline{n})m) &= j(P : \xi : \lambda : v)(\overline{n}(a(\overline{n})n(\overline{n}))^{-1}m) \\ &= j(P : \xi : \lambda : v)(\overline{n}ma(\overline{n})^{-1}n) \\ &= \chi(\overline{n})^{-1}e^{(\lambda+\rho)H(\overline{n})}\xi(m)^{-1}v. \end{aligned}$$

Here n is some element in N of which the precise value is not important. For $k \in K \setminus U$ we have $j(P : \xi : \lambda : v)(k) = 0$. We first study the λ dependent part of this expression, i.e. $e^{(\lambda+\rho)H(\overline{n})}$, separately.

We define $f \colon K \times \mathcal{A} \to \mathbb{C}$ as

$$f(k, \lambda) = \begin{cases} e^{(\lambda+\rho)H(\overline{n})} & \text{if } k = k(\overline{n})m \in U \\ 0 & \text{if } k \in K \setminus U. \end{cases}$$

We will show that $\lambda \mapsto f(\cdot, \lambda)$ is a holomorphic map from $\mathcal{A}$ into $C(K)$. For this we will apply the result of Lemma 2.15. The simple roots $\Delta = \{\alpha_1, ..., \alpha_n\}$ form a basis of $\mathfrak{a}^*_{\mathbb{C}}$. Let $(\beta_1, ..., \beta_n)$ be the basis of $\mathfrak{a}^*_{\mathbb{C}}$ dual to the former basis with respect to $\langle \cdot, \cdot \rangle$. We denote by $z = (z^1, ..., z^n)$ the coordinates on $\mathfrak{a}^*_{\mathbb{C}}$ with respect to this latter basis. Let $(c^j)_j$ be such that $\rho = c^1\beta_1 + \cdots + c^n\beta_n$. In these coordinates f is given by

$$f(k(\overline{n})m, z) = e^{[(z^1+c^1)\beta_1+\cdots+(z^n+c^n)\beta_n]H(\overline{n})} = \prod_{j=1}^{n} e^{(z^j+c^j)\beta_j(H(\overline{n}))}$$

for $k(\overline{n})m \in U$. We see from this expression that $z \mapsto f(k, z)$ is holomorphic for fixed $k \in K$. It is straightforward to check that in these coordinates on $\mathfrak{a}^*_{\mathbb{C}}$ the set $\mathcal{A}$ corresponds to

$$\widetilde{\mathcal{A}} := \{z \in \mathbb{C}^n \mid z^j + c^j < 0 \quad \text{for all } j = 1, ..., n\}.$$

By Proposition 1.16 we have that $\beta_j(H(\overline{n})) \geq 0$ for every $\overline{n} \in \overline{N}$ and $j = 1, ..., n$ (here we use that if $H \in \mathfrak{a}$ then $H = \beta_1(H)H_{\alpha_1} + \cdots + \beta_n(H)H_{\alpha_n}$). We conclude that $e^{(z^j+c^j)\beta_j(H(\overline{n}))} \leq 1$ for $j = 1, ..., n$ if $\overline{n} \in \overline{N}$ and $z \in \widetilde{\mathcal{A}}$. Hence the function f is uniformly bounded on $K \times \widetilde{\mathcal{A}}$. It remains to check that f defines a continuous function on $K \times \widetilde{\mathcal{A}}$. From the above expression it is clear that f is continuous on $U \times \widetilde{\mathcal{A}}$. So in order to prove that f is continuous on the whole space it is enough to show that if $(k_i, z_i)_i$ is a sequence in $U \times \widetilde{\mathcal{A}}$ converging to a point in $(K \setminus U) \times \widetilde{\mathcal{A}}$ then $\lim_{i\to\infty} f(k_i, z_i) = 0$. This follows from similar arguments as used in the proof of Proposition 2.11. We leave it to the reader to verify this. We are now able to apply Lemma 2.15 and we obtain that $z \mapsto f(\cdot, z)$ is a holomorphic map from $\widetilde{\mathcal{A}}$ into $C(K)$. From this we conclude that $\lambda \mapsto f(\cdot, \lambda)$ is a holomorphic map from $\mathcal{A}$ to $C(K)$.

We now consider the space $F := \{g \in C(K) \mid g|_{K\setminus U} \equiv 0\}$ which is a closed subspace of $C(K)$. We see from the above discussion that $\lambda \mapsto f(\cdot, \lambda)$ maps into F. Since F is a closed subspace this means that we can view this function as a holomorphic map into F. Consider the map $\Phi\colon F \to C(K : M : \xi)$ defined as

$$\Phi(g)(k) = \begin{cases} g(k) \cdot \chi(\overline{n})^{-1}\xi(m)^{-1}v & \text{if } k = k(\overline{n})m \in U \\ 0 & \text{if } k \in K \setminus U. \end{cases}$$

It is easy to check that Φ defines a continuous map. From the expression for $j(P : \xi : \lambda : v)$ derived above we see that the map $\lambda \mapsto j(P : \xi : \lambda : v)$ coincides with the map $\lambda \mapsto f(\cdot, \lambda)$ composed with the linear map Φ. Lemma 2.14 now implies that $\lambda \mapsto j(P : \xi : \lambda : v)$ is indeed a holomorphic map from $\mathcal{A}$ to $C(K : M : \xi)$. $\square$

2.5 Holomorphic continuation of $j(P : \xi : \lambda)$

In the previous section we constructed a holomorphic family of functions $j(P : \xi : \lambda)$, for $\lambda \in \mathcal{A}$, that is inverse to ev_e. In order to obtain such a family for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we extend the domain of definition by constructing a holomorphic extension.

As observed in Remark 2.12 it is not possible to extend $j(P : \xi : \lambda)$ to the whole of $\mathfrak{a}_{\mathbb{C}}^*$ as a map into $C(P : \xi : \lambda)$. Instead we view $j(P : \xi : \lambda)$ as a map into $C^{-\infty}(P : \xi : \lambda)$ by using the natural embedding of $C(P : \xi : \lambda)$ into this space. Because the space of generalized sections is not a Banach space we first need to define what we mean by a holomorphic map into $C^{-\infty}(P : \xi : \lambda)$.

For $k \in \mathbb{N}$ we denote by $C_k^{-\infty}(K : M : \xi)$ the space of generalized vectors on K of order at most k. This space can be realised as the dual of the Banach space $C^k(K : M : \xi)$ (which equals, as one would expect, $C(K : M : \xi) \cap C^k(K; H_\xi)$). Hence $C_k^{-\infty}(K : M : \xi)$ is also a Banach space. We have $C^{-\infty}(K : M : \xi) = \cup_{k\in\mathbb{N}} C_k^{-\infty}(K : M : \xi)$. It turns out that the inductive limit topology on $C^{-\infty}(K : M : \xi)$ obtained from this decomposition coincides with the strong dual topology. This follows from [Kom67, Theorem 11] and the observation that the embeddings $C^{k+1}(K : M : \xi) \hookrightarrow C^k(K : M : \xi)$ are compact.

Definition 2.17. Let W be a finite-dimensional complex vector space and $\Omega \subset W$ an open subset. We say $f\colon \Omega \to C^{-\infty}(K : M : \xi)$ is a holomorphic function if for every $z \in \Omega$ a neighbourhood $U \subset \Omega$ of z and a $k \geq 0$ exists such that f maps U into $C_k^{-\infty}(K : M : \xi)$ and as a map $f\colon U \to C_k^{-\infty}(K : M : \xi)$ is holomorphic in the sense of Definition 2.13.

We now say a map into $C^{-\infty}(P : \xi : \lambda)$ is holomorphic if it is holomorphic viewed as a map into $C^{-\infty}(K : M : \xi)$ (see remark made after Definition 1.41).

In this section we will prove the following result.

Proposition 2.18. Assume χ is regular and let $v \in H_\xi$. As a map into $C^{-\infty}(K : M : \xi)$ the function $\lambda \mapsto j(P : \xi : \lambda : v)$, initially defined on $\mathcal{A}$, can be extended holomorphically to the whole of $\mathfrak{a}_{\mathbb{C}}^*$. For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ the map $j(P : \xi : \lambda)$ maps H_ξ bijectively into $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$. Furthermore, the identity

$$\mathrm{ev}_e \circ j(P : \xi : \lambda) = \mathrm{id}_{H_\xi}$$

holds for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.

We should note that this proposition immediately implies the result of Theorem 2.10. Our approach for the proof of this result was suggested by E.P. van den Ban. We mimic the strategy used in the proof of a very similar result in the context of H-fixed vectors as presented in [vdB88]. The proof will make use of a technical tool called the standard intertwining operator which we will introduce first.

2.5.1 The standard intertwining operator

In this section we introduce the (standard) intertwining operator. We cannot treat the theory of this operator in full detail in the scope of this text. We will give a brief statement of the facts needed for the proof of Proposition 2.18 and refer to the literature (more specifically [VW90]) for the proofs and full details.

We formally define the intertwining operator $A(\overline{P} : P : \xi : \lambda)\colon C^\infty(P : \xi : \lambda) \to C^\infty(\overline{P} : \xi : \lambda)$ as

$$[A(\overline{P} : P : \xi : \lambda)f](x) := \int_{\overline{N}} f(x\overline{n})\,\mathrm{d}\overline{n} \quad \text{for } x \in G. \tag{2.7}$$

A priori it is not clear whether this integral is finite for all $f \in C(P : \xi : \lambda)$. The next proposition shows that this is only the case on a particular region in $\mathfrak{a}^*_{\mathbb{C}}$.

Proposition 2.19. There exists a $c \in \mathbb{R}$ such that for $\lambda \in \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} \mid \langle \operatorname{Re}\lambda, \alpha\rangle > R \quad \text{for all } \alpha \in \Sigma^+\}$ the integral in (2.7) converges absolutely for any $f \in C^\infty(P : \xi : \lambda)$. For fixed λ in this region the map $A(\overline{P} : P : \xi : \lambda)\colon C^\infty(P : \xi : \lambda) \to C^\infty(\overline{P} : \xi : \lambda)$ is continuous. For fixed $f \in C^\infty(P : \xi : \lambda)$ the function $\lambda \mapsto A(\overline{P} : P : \xi : \lambda)f$ is holomorphic as a map into $C^\infty(K : M : \xi)$.

Proof. For the proof of this result we refer to [VW90, Lemma 1.2] and [VW90, Lemma 1.3]. The results in this paper are formulated for *real reductive groups* with the additional property that $\mathrm{Ad}(G)$ maps into the identity component of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We refer to [Wal88, Lemma 2.1.3] for a proof that a connected semisimple Lie group with finite center is in fact a real reductive group. Since G is assumed to be connected we immediately have that $\mathrm{Ad}(G)$ is contained in the identity component of $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We conclude that we are indeed free to use the results of [VW90].

What remains to be checked is that $A(\overline{P} : P : \xi : \lambda)$ maps into $C^\infty(\overline{P} : \xi : \lambda)$. Suppose $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. For $f \in C^\infty(P : \xi : \lambda)$, $x \in G$ and $ma\overline{n} \in MA\overline{N}$ we observe

$$\begin{aligned}
[A(\overline{P} : P : \xi : \lambda)f](xma\overline{n}) &= \int_{\overline{N}} f(xma\overline{n}\,\overline{n}')\,\mathrm{d}\overline{n}' \\
&= \int_{\overline{N}} f(xma\overline{n}')\,\mathrm{d}\overline{n}' \\
&= \int_{\overline{N}} f(x(ma)\overline{n}'(ma)^{-1}ma)\,\mathrm{d}\overline{n}' \\
&= a^{-\lambda-\rho}\xi(m)^{-1}\int_{\overline{N}} f(x(ma)\overline{n}'(ma)^{-1})\,\mathrm{d}\overline{n}'
\end{aligned}$$

By our above discussion we know these integrals are absolutely convergent. By similar arguments as used in the proof of Proposition 1.39 we have $C^*_m\,\mathrm{d}n = |\det \mathrm{Ad}(m)|_{\mathfrak{n}}|\,\mathrm{d}n$. Because M is compact the character $m \mapsto |\det \mathrm{Ad}(m)|_{\mathfrak{n}}|$ is equal to 1 for all $m \in M$. Hence $C^*_m\,\mathrm{d}n = \mathrm{d}n$. We use this and the result of Proposition 1.39 to make the substitution of variables $(ma)\overline{n}'(ma)^{-1} \mapsto \overline{n}'$. We find

$$\begin{aligned}
[A(\overline{P} : P : \xi : \lambda)f](x) &= a^{-\lambda-\rho}a^{2\rho}\xi(m)^{-1}\int_{\overline{N}} f(x\overline{n}')\,\mathrm{d}\overline{n}' \\
&= a^{-\lambda-\rho_{\overline{P}}}\xi(m)^{-1}\int_{\overline{N}} f(x\overline{n}')\,\mathrm{d}\overline{n}'.
\end{aligned}$$

Here used the notation $\rho_{\overline{P}} = \sum_{\alpha\in-\Sigma^+}\dim(\mathfrak{g}_\alpha)\alpha$ for the element as in Definition 1.28 associated to the choice of positive roots $-\Sigma^+$. We conclude that indeed $A(\overline{P} : P : \xi : \lambda)f \in C^\infty(\overline{P} : \xi : \lambda)$. $\square$

For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ in the region where the integral expression for the intertwining operator converges we clearly have

$$\pi_{\overline{P},\xi,\lambda}(g) \circ A(\overline{P} : P : \xi : \lambda) = A(\overline{P} : P : \xi : \lambda) \circ \pi_{P,\xi,\lambda}(g) \quad \text{for all } g \in G. \tag{2.8}$$

Here we denoted $\pi_{P,\xi,\lambda}$ and $\pi_{\overline{P},\xi,\lambda}$ the principal series representations for P and $\overline{P}$ respectively.

As announced earlier we will extend the domain of definition for the intertwining operator to the whole of $\mathfrak{a}_{\mathbb{C}}^*$. However it turns out that this can not be done holomorphically but only meromorphically.

Definition 2.20. Suppose W is a finite-dimensional complex vector space and V is either a complex Banach space or the space of generalized vectors $C^{-\infty}(K : M : \xi)$. Let $\Omega \subset W$ be an open subset. We say that a (densely defined) function $f \colon \Omega \to V$ is meromorphic if for every $z \in \Omega$ a neighbourhood $U \subset \Omega$ of z and a nonzero holomorphic function $\phi \colon U \to \mathbb{C}$ exists such that $\phi f \colon U \to V$ is holomorphic in the sense of Definition 2.13 or Definition 2.17.

Proposition 2.21. *Let $f \in C^\infty(P : \xi : \lambda)$. As a mapping into $C^\infty(K : M : \xi)$ the function $\lambda \mapsto A(\overline{P} : P : \xi : \lambda)f$, initially defined on $\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \mathrm{Re}\,\lambda, \alpha\rangle > c \quad \text{for all } \alpha \in \Sigma^+\}$, can be extended meromorphically to the whole of $\mathfrak{a}_{\mathbb{C}}^*$.*

For the proof of this proposition we refer to [VW90, Theorem 1.6].

We observe that both sides of the identity in (2.8) depend meromorphically on λ. Since we know this identity holds on an open subset of $\mathfrak{a}_{\mathbb{C}}^*$ we conclude that it must hold for all λ such that $A(\overline{P} : P : \xi : \lambda)$ is defined.

If V and W are Banach spaces then we denote by $B(V, W)$ the Banach space of bounded linear maps from V to W.

Proposition 2.22. *Let $R \in \mathbb{R}$ and denote $A_R := \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \mid \langle \mathrm{Re}\,\lambda, \alpha\rangle > R \quad \text{for all } \alpha \in \Sigma^+\}$. There exist constants k and N in $\mathbb{N}$ and a polynomial $q \colon \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ such that*

(1) *For every $f \in C^\infty(K : M : \xi)$ the map $\lambda \mapsto q(\lambda)A(\overline{P} : P : \xi : \lambda)(f)$ is holomorphic on A_R as a map into $C^\infty(K : M : \xi)$.*

(2) *For every $l \in \mathbb{N}$ there exists a $C > 0$ such that the following holds for all $\lambda \in A_R$ and $f \in C^\infty(K : M : \xi)$,*

$$\left\| q(\lambda)A(\overline{P} : P : \xi : \lambda)f \right\|_{C^l(K)} \leq C(1 + \|\lambda\|)^N \|f\|_{C^{l+k}(K)}.$$

As a consequence the map $q(\lambda)A(\overline{P} : P : \xi : \lambda)$ extends uniquely to a continuous map $C^l(K : M : \xi) \to C^{l+k}(K : M : \xi)$ for all $\lambda \in A_R$. For $f \in C^l(K : M : \xi)$ fixed the function $\lambda \mapsto q(\lambda)A(\overline{P} : P : \xi : \lambda)f$ maps holomorphically into $C^{l+k}(K : M : \xi)$. The induced map into $B(C^l(K : M : \xi), C^{l+k}(K : M : \xi))$ is also holomorphic on A_R.

Proof. Statements (1) and (2) are proved in [vdBS12, Corollary 1.4]. What remains to be checked are the last assertions. For this we follow the arguments outlined in [vdB88, Corollary 4.3].

From (2) it follows that if we fix $\lambda \in A_R$ then $q(\lambda)A(\overline{P} : P : \xi : \lambda)$ is a bounded map from $C^\infty(K : M : \xi)$ to $C^\infty(K : M : \xi)$ equipped with the $C^l(K)$ and $C^{l+k}(K)$ topology respectively. Since $C^\infty(K : M : \xi)$ lies dense in $C^l(K : M : \xi)$ we see that $q(\lambda)A(\overline{P} : P : \xi : \lambda)$ uniquely extends to a continuous map $C^l(K : M : \xi) \to C^{l+k}(K : M : \xi)$. We obtain a map $\Psi \colon A_R \times C^l(K : M : \xi) \to C^{l+k}(K : M : \xi) \colon (\lambda, f) \mapsto q(\lambda)A(\overline{P} : P : \xi : \lambda)f$. We prove that for f fixed $\lambda \mapsto \Psi(\lambda, f)$ is holomorphic. Let $\lambda_0 \in A_R$ arbitrary and let $(z^1, ..., z^n)$ be coordinates on $\mathfrak{a}_{\mathbb{C}}^*$ centred around λ_0. The polydisk $D(\lambda_0;\epsilon)^n$ is contained in A_R for some $\epsilon > 0$. For $f \in C^\infty(K : M : \xi)$ we know that $\lambda \mapsto \Psi(\lambda, f)$ is holomorphic hence on $D(\lambda_0;\epsilon)$ it is given by a absolutely convergent power series

$$\sum_{\alpha\in\mathbb{N}^n} z^\alpha c_\alpha(f)$$

with coefficients $c_\alpha(f) \in C^\infty(K : M : \xi)$. An application of the Cauchy integral formula yields that these coefficients are given by

$$c_\alpha(f) = \frac{1}{(2\pi i)^n}\int_{|z_1|=\epsilon}\cdots\int_{|z_n|=\epsilon}\frac{\psi(z,f)}{z_1^{\alpha_1+1}\cdots z_n^{\alpha_n+1}}\,\mathrm{d}z_1\cdots\mathrm{d}z_n.$$

From this expression we conclude that the coefficients c_α are linear maps $c_\alpha\colon C^\infty(K : M : \xi) \to C^\infty(K : M : \xi)$. Furthermore, taking into account that (2) holds, we find that there exists a $C > 0$ such that

$$\|c_\alpha(f)\|_{C^l(K)} \leq C\epsilon^{-|\alpha|}\,\|c_\alpha(f)\|_{C^{l+k}(K)} \tag{$*$}$$

for all $\alpha \in \mathbb{N}^n$ and $f \in C^\infty(K : M : \xi)$. Hence these coefficients can be uniquely extended to continuous maps $c_\alpha\colon C^{l+k}(K : M : \xi) \to C^l(K : M : \xi)$. For these extended maps $(*)$ holds for all $f \in C^{l+k}(K : M : \xi)$. We conclude that if $f \in C^{k+l}(K : M : \xi)$ then on $D(\lambda_0;\epsilon)$ the function $\Psi(\lambda, f)$ is given by the absolutely convergent power series

$$\sum_{\alpha\in\mathbb{N}^n} z^\alpha c_\alpha(f).$$

Since $\lambda_0 \in A_R$ was arbitrary we conclude that $\lambda \mapsto \Psi(\lambda, f)$ is indeed holomorphic as a map into $C^l(K : M : \xi)$. Finally we observe that for every α we have $c_\alpha \in B(C^l(K : M : \xi), C^{l+k}(K : M : \xi))$ and $\|c_\alpha\|_{\mathrm{op}} \leq C\epsilon^{-|\alpha|}$. From this it now follows that the induced map into $B(C^l(K : M : \xi), C^{l+k}(K : M : \xi))$ is given around λ_0 by the absolutely convergent power series $\sum_{\alpha\in\mathbb{N}^n} z^\alpha c_\alpha$. We conclude that this induced map is also holomorphic on A_R. □

Proposition 2.23. There exists a closed and nowhere dense subset $S \subset \mathfrak{a}_\mathbb{C}^*$ such that $A(\overline{P} : P : \xi : \lambda)$ is a bijection between $C^\infty(P : \xi : \lambda)$ and $C^\infty(\overline{P} : \xi : \lambda)$ for all $\lambda \in \mathfrak{a}_\mathbb{C}^* \setminus S$.

Proof. In [VW90, Lemma 5.4 and Lemma 5.5] it is proved that a meromorphic function ϕ on $\mathfrak{a}_\mathbb{C}^*$ exists such that

$$A(\overline{P} : P : \xi : \lambda) \circ A(P : \overline{P} : \xi : \lambda) = \phi(\lambda)\cdot I$$
$$A(P : \overline{P} : \xi : \lambda) \circ A(\overline{P} : P : \xi : \lambda) = \phi(\lambda)\cdot I$$

whenever the left hand sides are defined. We let S be the union of the singular locus of $A(P : \overline{P} : \xi : \lambda)$, the singular locus of $A(\overline{P} : P : \xi : \lambda)$ and the zero set of ϕ. Then for any $\lambda \in \mathfrak{a}_\mathbb{C}^* \setminus S$ we have that $A(\overline{P} : P : \xi : \lambda)$ has an inverse hence is bijective. It is easily seen that S is closed and nowhere dense. □

Proposition 2.24. The transpose of $A(\overline{P} : P : \xi : \lambda)$ with respect to the pairing $C^\infty(P : \xi : -\overline{\lambda}) \times C^\infty(P : \xi : \lambda) \to \mathbb{C}$ equals $A(\overline{P} : P : \xi : -\overline{\lambda})$.

For a proof see [VW90, Lemma 5.7] (recall that we assumed $\xi = \xi^*$).

Definition 2.25. We define the unique extension of the intertwining operator to the generalized vectors, $A(\overline{P} : P : \xi : \lambda)\colon C^{-\infty}(P : \xi : \lambda) \to C^{-\infty}(\overline{P} : \xi : \lambda)$, to be the dual of $A(\overline{P} : P : \xi : -\overline{\lambda})\colon C^{\infty}(\overline{P} : \xi : -\overline{\lambda}) \to C^{\infty}(P : \xi : -\overline{\lambda})$.

It is a consequence of Proposition 2.24 that this definition of the intertwining operator indeed extends our previous definition on the continuous sections. For the extended intertwining operator (2.8) also holds.

The following result is a direct consequence of Proposition 2.22.

Corollary 2.26. For every $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ there exists a neighbourhood $\Omega \subset \mathfrak{a}_{\mathbb{C}}^*$ of λ_0, a nonzero holomorphic map $q\colon \Omega \to \mathbb{C}$ and a $k \in \mathbb{N}$ such that

(1) For $\phi \in C^{-\infty}(K : M : \xi)$ the map $\lambda \mapsto q(\lambda)A(\overline{P} : P : \xi : \lambda)\phi$ is holomorphic on Ω as a map into $C^{-\infty}(K : M : \xi)$.

(2) For $\lambda \in \Omega$ the map $q(\lambda)A(\overline{P} : P : \xi : \lambda)$ continuously maps $C_l^{-\infty}(K : M : \xi)$ into $C_{l+k}^{-\infty}(K : M : \xi)$ for all $l \in \mathbb{N}$

Furthermore, the induced map into $B(C_l^{-\infty}(K : M : \xi), C_{l+k}^{-\infty}(K : M : \xi))$ is holomorphic on Ω.

Proof. We first recall that $C_l^{-\infty}(K : M : \xi)$ in $C^{-\infty}(K : M : \xi)$ can be realised as the dual of $C^l(K : M : \xi)$ for all $l \in \mathbb{N}$. By Proposition 2.22 we have that on some neighbourhood Ω of λ_0 a nonzero holomorphic function q exists such that $(\lambda, f) \mapsto q(\lambda)A(\overline{P} : P : \xi : -\overline{\lambda})$ is regular in λ for f fixed. Furthermore, some $k \in \mathbb{N}$ exists such that this functions maps $C^{k+l}(K : M : \xi)$ into $C^l(K : M : \xi)$ for all $l \in \mathbb{N}$. The map $q(\lambda)A(\overline{P} : P : \xi : \lambda)$ restricted to $C_l^{-\infty}(K : M : \xi)$ can be realised as the dual of the map $q(\lambda)A(\overline{P} : P : \xi : -\overline{\lambda})\colon C^{k+l}(K : M : \xi) \to C^l(K : M : \xi)$. From this the corollary now follows. ☐

2.5.2 Proof of holomorphic continuation

Before we can give the proof of Proposition 2.18 we need a couple of auxiliary lemmas.

Lemma 2.27. Let $\mu \in \mathfrak{a}^*$ such that

$$\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}_{>0} \quad \text{for all } \alpha \in \Sigma^+$$

Then there exists a nonzero element of $C^{\infty}(P : 1 : -\mu - \rho)^{\overline{N},1}$, i.e. a function $\psi\colon G \to \mathbb{C}$ satisfying

$$\psi(\overline{n}xman) = a^{\mu}\psi(x) \quad \text{for all } x \in G, man \in MAN \text{ and } \overline{n} \in \overline{N}.$$

Furthermore, ψ can be chosen such that $\psi(e) = 1$.

Proof. We pick a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$. Then $\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathcal{R} = \mathcal{R}(\mathfrak{g}; \mathfrak{h})$ the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to this Cartan subalgebra. It is easily seen that $\Sigma = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \mathcal{R}, \alpha|_{\mathfrak{a}} \neq 0\}$. It is now possible to make a choice of positive roots $\mathcal{R}^+$ extending the choice of positive reduced roots Σ^+, i.e. $\Sigma^+ = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \mathcal{R}^+, \alpha|_{\alpha} \neq 0\}$. On the strength of Helgason's result on the classification of spherical representations (see [Hel84, Theorem 4.1]) we conclude that a spherical representation (δ, V) of G (this is a representation with a K-fixed vector) exists with highest weight μ (seen as an element of $(\mathfrak{t} \oplus \mathfrak{a})^*$ by extending by zero). Furthermore, any highest weight vector of this representation is M-fixed.

We chose e_μ a nonzero highest weight vector. We know that e_μ is M-fixed and $\delta(H)e = \mu(H)e_\mu$ for $H \in \mathfrak{a}$ implies that $\delta(a)e_\mu = a^\mu e_\mu$ for $a \in A$. Since our choice of positive roots $\mathcal{R}^+$ extends the choice Σ^+ we have that $\mathfrak{n} \subset \mathfrak{g}^+$. Because e_μ is a highest weight vector it follows that $\delta(\mathfrak{n})e_\mu = 0$. Hence e_μ is N-fixed. In conclusion we have $\delta(man)e_\mu = a^\mu e_\mu$ for $man \in MAN$. We now consider the dual representation $(\delta^\vee, V^*)$ of G. The lowest weight of this representation is $-\mu$. It is easily checked that the pairing $V_\mu \times (V^*)_{-\mu} \to \mathbb{C}$ is non-degenerate. Hence we can fix an element $\epsilon \in (V^*)_{-\mu}$ such that $\epsilon(e) = 1$. Since ϵ is a lowest weight vector it follows that $\delta(\overline{\mathfrak{n}})\epsilon = 0$ hence ϵ is $\overline{N}$-fixed.

We now define $\psi\colon G \to \mathbb{C}$ as $\psi(x) = \epsilon(\delta(x)e_\mu)$. Since a matrix coefficient of a finite-dimensional representation is smooth we have $\psi \in C^\infty(G)$. By our choice of ϵ we have $\psi(e) = \epsilon(e_\mu) = 1$. From the transformation properties of e and ϵ it follows that for $x \in G$, $man \in MAN$ and $\overline{n} \in \overline{N}$ we have

$$\psi(\overline{n}xman) = \epsilon(\delta(\overline{n}xman)e_\mu) = (\delta^\vee(\overline{n}^{-1})\epsilon)(\delta(x)(\delta(man)e_\mu)) = a^\mu\epsilon(\delta(x)e_\mu) = a^\mu\psi(x).$$

This proves the lemma. $\square$

Suppose we have a $\mu \in \mathfrak{a}^*$ and $\psi \in C^\infty(P : 1 : -\mu - \rho)^{\overline{N},1}$ as in the lemma above. For any $\lambda \in \mathfrak{a}^*$ we define the mapping $M_\psi\colon C^\infty(P : \xi : \lambda) \to C^\infty(P : \xi : \lambda - \mu)\colon f \mapsto \psi \cdot f$. It is easily seen that this is a continuous map and since ψ is left $\overline{N}$-invariant that it is $\overline{N}$-intertwining.

Lemma 2.28. The map $M_\psi\colon C^\infty(P : \xi : \lambda) \to C^\infty(P : \xi : \lambda - \mu)$ can be uniquely extended to a continuous map $M_\psi\colon C^{-\infty}(P : \xi : \lambda) \to C^{-\infty}(P : \xi : \lambda - \mu)$. This extended map is also $\overline{N}$-intertwining. Furthermore, if χ is regular then the restricted map $M_\psi\colon C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \to C^{-\infty}(P : \xi : \lambda - \mu)^{\overline{N},\chi}$ is injective.

Proof. As discussed above the map $M_\psi\colon C^\infty(P : \xi : -\overline{(\lambda - \mu)}) \to C^\infty(P : \xi : -\overline{\lambda})$ is continuous linear and $\overline{N}$-intertwining. We define the extended map $M_\psi\colon C^{-\infty}(P : \xi : \lambda) \to C^{-\infty}(P : \xi : \lambda - \mu)$ as the dual of this map. It is straightforward to check this map satisfied the requirements.

For the injectivity we note that that $\psi(e) = 1$ implies that the following diagram is commutative

$$\begin{array}{ccc} C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} & \xrightarrow{M_\psi} & C^{-\infty}(P : \xi : \lambda - \mu)^{\overline{N},\chi} \\ \Big\downarrow{\scriptstyle \mathrm{ev}_e} & & \Big\downarrow{\scriptstyle \mathrm{ev}_e} \\ H_\xi & = & H_\xi. \end{array}$$

The injectivity of M_ψ is now a direct consequence of the fact that ev_e is injective when χ is regular (see Proposition 2.8). $\square$

Let $w \in W(\mathfrak{g}; \mathfrak{a})$ be an element of the Weyl group. With a slight abuse of notation we denote by w also a representative of this element in $N_K(\mathfrak{a})$. We introduce the notation

$$w\xi = C^*_{w^{-1}}\xi \quad \text{and} \quad w\lambda = \lambda \circ \mathrm{Ad}(w^{-1})$$

for $\lambda \in \mathfrak{a}^*_\mathbb{C}$. We denote by R_w the action of w on the space $C^\infty(G; H_\xi)$ via the right regular representation.

Lemma 2.29. Suppose $w \in W(\mathfrak{g}; \mathfrak{a})$ is the longest Weyl group element. Then the map R_w restricts to a topological isomorphism of G-modules $R_w\colon C^\infty(P : \xi : \lambda) \to C^\infty(\overline{P} : w\xi : w\lambda)$.

Proof. We first prove that R_w maps $C^\infty(P:\xi:\lambda)$ into $C^\infty(\overline{P}:w\xi:w\lambda)$. Let $f \in C^\infty(P:\xi:\lambda)$, $x \in G$ and $ma\overline{n} \in MAN$. We observe

$$\begin{aligned}(R_w f)(xma\overline{n}) &= f(xma\overline{n}w) = f(xwC_{w^{-1}}(ma\overline{n}))\\ &= (C_{w^{-1}}a)^{-\lambda-\rho}\,\xi(C_{w^{-1}}m)^{-1}f(xw)\\ &= (C_{w^{-1}}a)^{-\lambda-\rho}\,(w\xi)(m)^{-1}(R_w f)(x).\end{aligned}$$

Here we used that $C_{w^{-1}}(\overline{n}) \in N$ which follows because w is the longest Weyl group element. Furthermore, we have

$$(C_{w^{-1}}a)^{-\lambda-\rho} = e^{(-\lambda-\rho)\log C_{w^{-1}}a} = e^{(-\lambda-\rho)\operatorname{Ad}(w^{-1})\log a} = e^{(-w\lambda-w\rho)\log a} = a^{-w\lambda-w\rho}.$$

We use the notation $\rho_{\overline{P}} = \sum_{\alpha\in-\Sigma^+}\dim(\mathfrak{g}_\alpha)\alpha$ for the element as in Definition 1.28 associated to the choice of positive roots $-\Sigma^+$. Because w is the longest Weyl group element we have $w(\Sigma^+) = -\Sigma^+$ hence $w\rho = \rho_{\overline{P}}$. We conclude that

$$(R_w f)(xma\overline{n}) = a^{-w\lambda-\rho_{\overline{P}}}(w\xi)(m)^{-1}(R_w f)(x)$$

so we indeed have $R_w f \in C^\infty(\overline{P}:w\xi:w\lambda)$.

The map R_w is G-intertwining because the left and right regular representations commute. Furthermore, the element w satisfies $w^{-1} = w$ so the inverse to $R_w\colon C^\infty(P:\xi:\lambda) \to C^\infty(\overline{P}:w\xi:w\lambda)$ is readily seen to be $R_w\colon C^\infty(\overline{P}:w\xi:w\lambda) \to C^\infty(P:\xi:\lambda)$. $\square$

Lemma 2.30. Again suppose $w \in W(\mathfrak{g};\mathfrak{a})$ is the longest Weyl group element. The map $R_w\colon C^\infty(P:\xi:\lambda) \to C^\infty(\overline{P}:w\xi:w\lambda)$ extends uniquely to a topological isomorphism of G-modules $R_w\colon C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(\overline{P}:w\xi:w\lambda)$

Proof. As discussed in Section 1.6.3 the pairing $C^\infty(P:\xi:\lambda)\times C^\infty(P:\xi:-\overline{\lambda}) \to \mathbb{C}$ is given by $(f,g) \mapsto \int_K \langle f,g\rangle_\xi\,\mathrm{d}k$. Because $w \in K$ and $\mathrm{d}k$ is a right Haar measure we have $(R_{w^{-1}}f,g) = (f,R_w g)$. This means we can define the extension of R_w to the generalized vectors as the dual of the map $R_w\colon C^\infty(\overline{P}:w\xi:-w\overline{\lambda}) \to C^\infty(P:\xi:-\overline{\lambda})$. The latter map is an topological isomorphism of G-modules hence the extended map $R_w\colon C^{-\infty}(P:\xi:\lambda) \to C^{-\infty}(\overline{P}:w\xi:w\lambda)$ is also a topological isomorphism of G-modules. $\square$

In order to prove Proposition 2.18 we first prove that around every point in $\mathfrak{a}_\mathbb{C}^*$ locally a meromorphic parametrization of $C^{-\infty}(P:\xi:\lambda)^{\overline{N},\chi}$ exists. We will then use this parametrization to construct the holomorphic extension of $j(P:\xi:\lambda)$.

Lemma 2.31. For every $\lambda_0 \in \mathfrak{a}_\mathbb{C}^*$ there exists a neighbourhood $\Omega \subset \mathfrak{a}_\mathbb{C}^*$ of λ_0, a family of maps $\mathcal{J}_\lambda\colon H_\xi \to C^{-\infty}(P:\xi:\lambda)^{\overline{N},\chi}$ for $\lambda \in \Omega$ and a closed and nowhere dense set $S \subset \Omega$ such that

1. If $v \in H_\xi$ is fixed then $\lambda \mapsto \mathcal{J}_\lambda(v)$ is meromorphic on Ω and is holomorphic on $\Omega \setminus S$ as a map into $C^{-\infty}(K:M:\xi)$.
2. The map $\mathcal{J}_\lambda$ is bijective for all $\lambda \in \Omega \setminus S$.

Proof. For any $\mu \in \mathfrak{a}_\mathbb{C}^*$ with $\frac{\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}_{>0}$ we clearly have $\langle\operatorname{Re}\mu,\alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. We denote by $w \in W(\mathfrak{g};\mathfrak{a})$ the longest Weyl group element. Then for all $\alpha \in \Sigma^+$ we have $\langle\operatorname{Re}w\mu,\alpha\rangle = \left\langle\operatorname{Re}\mu,w^{-1}\alpha\right\rangle < 0$ because $w^{-1}\alpha = w\alpha \in -\Sigma^+$. Hence for such μ large enough we have $w\lambda_0 + w\mu \in \mathcal{A}$. Let $\Omega \subset \mathfrak{a}_{\mathbb{C}}^*$ be a neighbourhood of λ_0 that satisfies $w\Omega + w\mu \subset \mathcal{A}$. We consider the operators

$$\begin{aligned} &R_w\colon C^{-\infty}(\overline{P} : w\xi : w\lambda + w\mu) \to C^{-\infty}(P : \xi : \lambda + \mu) \quad \text{and} \\ &M_\psi\colon C^{-\infty}(P : \xi : \lambda + \mu) \to C^{-\infty}(P : \xi : \lambda) \end{aligned}$$

as introduced in Lemmas 2.28 and 2.30. For $\lambda \in \Omega$ we now consider the family of maps

$$\begin{gathered} \mathcal{J}_\lambda\colon H_\xi \to C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}, \\ \mathcal{J}_\lambda = M_\psi \circ R_w \circ A(\overline{P} : P : w\xi : w\lambda + w\mu) \circ j(P : w\xi : w\lambda + w\mu). \end{gathered}$$

Per definition we have that $j(P : w\xi : w\lambda + w\mu)$ maps H_ξ into $C^{-\infty}(P : w\xi : w\lambda + w\mu)$ which is mapped into $C^{-\infty}(\overline{P} : w\xi : w\lambda + w\mu)$ by $A(\overline{P} : P : w\xi : w\lambda + w\mu)$. We see that this space in turn is mapped into $C^{-\infty}(P : \xi : \lambda)$ by $M_\psi \circ R_w$. Since the latter three maps are all $\overline{N}$-intertwining we find that $\mathcal{J}_\lambda$ indeed maps H_ξ into $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$.

By Corollary 2.26 we know that a neighbourhood $\widetilde{\Omega}$ of $w\lambda_0 + w\mu$ exists and a constant $k \in \mathbb{N}$ such that $A(\overline{P} : P : \xi : \lambda)$ induces a meromorphic map $\widetilde{\Omega} \to B(C_0^{-\infty}(K : M : \xi), C_k^{-\infty}(K : M : \xi))$. We shrink Ω such that $w\Omega + w\mu \subset \widetilde{\Omega}$ and fix a $v \in H_\xi$. The function $\lambda \mapsto j(P : w\xi : w\lambda + w\mu : v)$ maps holomorphically into $C(K : M : \xi)$ which embeds continuously into $C_0^{-\infty}(K : M : \xi)$. We see that the composition $A(\overline{P} : P : w\xi : w\lambda + w\mu) \circ j(P : w\xi : w\lambda + w\mu)$ maps meromorphically into $C_k^{-\infty}(K : M : \xi) \subset C^{-\infty}(K : M : \xi)$. We leave it to the reader to check that both M_ψ and R_w seen as maps $C^{-\infty}(K : M : \xi) \to C^{-\infty}(K : M : \xi)$ are independent of λ. From this it follows that $\lambda \mapsto \mathcal{J}_\lambda(v)$ is indeed meromorphic on Ω as a map into $C^{-\infty}(K : M : \xi)$.

We denote by $\widetilde{S}$ the set that is denoted by S in Proposition 2.23. We define S as the set of all $\lambda \in \Omega$ such that $w\lambda + w\mu \in \widetilde{S}$. The set $\widetilde{S}$ contains the singular locus of $A(\overline{P} : P : \xi : \lambda)$ hence we have that $\mathcal{J}_\lambda(v)$ is indeed holomorphic on $\Omega \setminus S$. Now let $\lambda \in \Omega \setminus S$. Then $w\lambda + w\mu \notin \widetilde{S}$ hence $A(\overline{P} : P : w\xi : w\lambda + w\mu)$ is injective. Since $w\lambda + w\mu \in \mathcal{A}$ we know that $j(P : w\xi : w\lambda + w\mu)$ is also injective. From Lemma 2.28 and Lemma 2.30 it follows that $M_\psi \circ R_w$ is injective. We conclude that $\mathcal{J}_\lambda$ maps H_ξ injectively into $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$. Because $\dim C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi} \leq \dim H_\xi$ (see Proposition 2.8) we conclude that $\mathcal{J}_\lambda$ is in fact a bijection. □

Armed with this lemma we can now prove Proposition 2.18.

Proof of Proposition 2.18. We first construct an extension of the family $j(P : \xi : \lambda)$ that is, a priori, only meromorphic in λ. We will then use a uniqueness argument to show that it in fact must be holomorphic.

Per assumption we have that χ is regular. Hence by Proposition 2.8 the map ev_e is injective. Using this we observe the following; A family of maps $f_\lambda\colon H_\xi \to C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ is uniquely determined by the requirement that it satisfies $\mathrm{ev}_e \circ f_\lambda = \mathrm{id}_{H_\xi}$.

Let $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ be arbitrary and let Ω, $S \subset \Omega$ and $\mathcal{J}$ be as in Lemma 2.31. We now consider the family of maps $u_\lambda := \mathrm{ev}_e \circ \mathcal{J}_\lambda\colon H_\xi \to H_\xi$. Since H_ξ is finite-dimensional we can view u_λ as a matrix with coefficients depending meromorphically on λ. On the open and dense set $\Omega \setminus S$ we have that this matrix is invertible. Hence we can consider the family of maps $(u_\lambda)^{-1}$, which also depends meromorphically on λ by Cramer's rule. On the set Ω we declare the family $\mathcal{J}_\lambda \circ (u_\lambda)^{-1}\colon H_\xi \to C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ to be the extension of $j(P : \xi : \lambda)$. We have per construction that $\mathrm{ev}_e \circ (\mathcal{J}_\lambda \circ (u_\lambda)^{-1}) = \mathrm{id}_{H_\xi}$ whenever this expression is defined. Our observation on the uniqueness of families of maps satisfying this requirement show that this is indeed an extension of $j(P : \xi : \lambda)$. It also shows that the extensions of $j(P : \xi : \lambda)$ on different opens patch together. Hence we obtain a meromorphic extension of $j(P : \xi : \lambda)$ to the whole of $\mathfrak{a}_{\mathbb{C}}^*$.

We now argue that the family of maps $j(P : \xi : \lambda)$ is actually holomorphic in λ. Let $v \in H_\xi$. What we will show is that all singularities of $j(P : \xi : \lambda : v)$ are in fact removable. Suppose $\lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ is a singularity of this function. We consider a one-dimensional slice $\mathcal{S}$ of $\mathfrak{a}_{\mathbb{C}}^*$ containing λ_0. We can pick $\mathcal{S}$ such that, when restricted to $\mathcal{S}$, the function $\lambda \mapsto j(P : \xi : \lambda : v)$ has an isolated singularity at λ_0. We denote the restriction of $j(P : \xi : \lambda : v)$ to $\mathcal{S}$ simply by $j\colon \mathcal{S} \to C^{-\infty}(K : M : \xi)$. If the singularity around λ_0 is a pole of positive order then there exists a nonzero holomorphic map $\phi\colon \mathcal{S} \to \mathbb{C}$ which vanishes at λ_0 such that $j_0 := \phi \cdot j$ is holomorphic on an open neighbourhood around λ_0 and such that j_0 is nonzero at λ_0. For $\overline{n} \in \overline{N}$ we have

$$\pi_{\xi,\lambda}(\overline{n}) j_0(\lambda) = \chi(\overline{n}) j_0(\lambda)$$

for $\lambda \neq \lambda_0$ in a neighbourhood around λ_0. Because both sides of this identity are continuous in λ we conclude the identity also holds for $\lambda = \lambda_0$. Hence $j_0(\lambda_0)$ is a nonzero Whittaker vector in $C^{-\infty}(P : \xi : \lambda_0)$. We recall the notation $B = \overline{N}P$ for the big Bruhat cell and $U = B \cap K$. By restricting to B we obtain a map $\lambda \mapsto j(\lambda)|_B$ from $\mathcal{S}$ into $C^{-\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$. In light of Proposition 2.6 we see that this function actually maps into $C^{\infty}(B : P : \xi : \lambda)^{\overline{N},\chi}$. From this it follows that for any $\lambda \in \mathcal{S}$ the function $j(\lambda)|_B\colon B \to H_\xi$ must be given by (2.6). So in particular the map $\lambda \mapsto j(\lambda)|_B$, seen as a map into $C^{-\infty}(U : M : \xi)$, is holomorphic. This means that at λ_0 we have $j_0(\lambda_0)|_B = \phi(\lambda_0) \cdot j(\lambda_0)|_B = 0$ because $\phi(\lambda_0) = 0$. So the Whittaker vector $j_0(\lambda_0)$ vanishes on B. Now Theorem 2.9 yields that $j_0(\lambda_0)$ must vanish on the whole of G. This however is in contradiction with the assumption $j_0(\lambda_0) \neq 0$. We conclude that j cannot have a pole of positive order around λ_0 hence the singularity is removable.

For $v \in H_\xi$ we have per construction that $\mathrm{ev}_e \circ j(P : \xi : \lambda : v) = v$ holds outside the removed singularities of this expression. By continuity we conclude $\mathrm{ev}_e \circ j(P : \xi : \lambda) = \mathrm{id}_{H_\xi}$ holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Since ev_e is injective we conclude that $j(P : \xi : \lambda)$ maps H_ξ bijectively into $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$. This concludes the proof. $\square$

Remark 2.32: Before we finish this section we take some time to point out the correspondence between what we called the $j(P : \xi : \lambda)$ function and the *Jacquet integral*, which is how Whittaker vectors were originally introduced by Jacquet. In our notation the Jacquet integral is defined for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, $\mu \in (H_\xi)^*$ and $f \in C^{\infty}(P : \xi : -\overline{\lambda})$ as

$$J_{\xi,\lambda}(\mu)(f) := \int_{\overline{N}} \chi(\overline{n})^{-1} \mu(f(\overline{n}))\, \mathrm{d}\overline{n}.$$

See for example [Wal92, 15.4.1] where we take $(P_0, A_0) = (\overline{P}, A)$. This integral converges if $\langle \mathrm{Re}\,\lambda, \alpha \rangle < 0$ for all $\alpha \in \Sigma^+$. An analytic continuation argument is then used to define the expression $J_{\xi,\lambda}(\mu)(f)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. The expression $J_{\xi,\lambda}(\mu)(\cdot)$ then defines an element in $C^{-\infty}(P : \xi : \lambda)^{\overline{N},\chi}$ for all $\mu \in (H_\xi)^*$. All this is proved in [Wal92, Section 15.4].

In order to show the correspondence we first let $\lambda \in \mathcal{A}$ and let $v \in H_\xi$ be such that $\mu = \langle \cdot, v \rangle$. We denote for simplicity $j := j(P : \xi : \lambda : v)$, which is a continuous function on G since $\lambda \in \mathcal{A}$, we observe, using Proposition 1.36, that

$$\begin{aligned} j(P:\xi:\lambda:v)(f) = (f,j) &= \int_K \langle f(k), j(k)\rangle \,\mathrm{d}k \\ &= \int_{\overline{N}} \langle f(k(\overline{n})), j(k(\overline{n}))\rangle \cdot e^{-2\rho(H(\overline{n}))}\,\mathrm{d}\overline{n} \\ &= \int_{\overline{N}} \left\langle e^{(-\overline{\lambda}+\rho)H(\overline{n})} f(\overline{n}), e^{(\lambda+\rho)H(\overline{n})} j(\overline{n})\right\rangle \cdot e^{-2\rho(H(\overline{n}))}\,\mathrm{d}\overline{n} \\ &= \int_{\overline{N}} \langle f(\overline{n}), j(\overline{n})\rangle \,\mathrm{d}\overline{n} \\ &= \int_{\overline{N}} \chi(\overline{n})^{-1} \langle f(\overline{n}), v\rangle \,\mathrm{d}\overline{n} = J_{\xi,\lambda}(\mu)(f) \end{aligned}$$

We conclude that the element in $C^{-\infty}(P:\xi:\lambda)^{\overline{N},\chi}$ defined by $J_{\xi,\lambda}(\eta)(\cdot)$ coincides with the element $j(P:\xi:\lambda:v)$. Since both are holomorphic in λ we conclude this holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We conclude that $j(P:\xi:\lambda)$ and the Jacquet integral yield the same parametrization of the Whittaker vectors.

2.6 Whittaker matrix coefficient

In this section we introduce what is called the Whittaker matrix coefficient or Whittaker coefficient for short. We will define the Whittaker coefficient as a matrix coefficient of the principal series representation. In Chapter 3 it will play an important role in the definition of the so called Whittaker–Fourier transformation. In the further sections of this chapter we will study the Whittaker coefficient (with focus on the case $G = \mathrm{SL}(2,\mathbb{R})$) and lay the groundwork for the analysis of the Whittaker–Fourier transformation in Chapter 3.

Let χ be a regular unitary character of $\overline{N}$. For a semisimple Lie group G with finite center we look at the space of Whittaker vectors for the spherical principal representation, i.e. $C^{-\infty}(P:1:\lambda)^{\overline{N},\chi}$. We know that this space is one dimensional and that $j(P:1:\lambda)(1)$ provides a holomorphic parametrization of this space. We denote by $\mathbb{1}_\lambda$ the element in $C^{\infty}(P:1:\lambda)$ that is uniquely determined by $\mathbb{1}_\lambda|_K \equiv 1$.

We now define the Whittaker (matrix) coefficient of the spherical principal series representation. For convenience we denote $j_\lambda := j(P:1:-\overline{\lambda})(1)$.

Definition 2.33. For $\lambda \in \mathfrak{a}_C^*$ we define the Whittaker matrix coefficient $W_\lambda \in C^\infty(G)$ as

$$W_\lambda(x) := \left\langle \pi_{1,\lambda}(x)^{-1}\mathbb{1}_\lambda, j_\lambda\right\rangle.$$

Since the map $\lambda \mapsto j(P:\xi:\lambda)(1)$ is holomorphic and the pairing $C^\infty(P:\xi:\lambda) \times C^{-\infty}(P:\xi:-\overline{\lambda}) \to \mathbb{C}$ is sesquilinear we see that for $x \in G$ fixed the function $\lambda \mapsto W_\lambda(x)$ is a holomorphic.

Using the $\overline{N}$-transformation behaviour of j_λ and the fact that $\mathbb{1}_\lambda$ is K-fixed we see that for any $x \in G, k \in K, \overline{n} \in \overline{N}$ we have

$$\begin{aligned} W_\lambda(kx\overline{n}) = \left\langle \pi_{1,\lambda}(kx\overline{n})^{-1}\mathbb{1}_\lambda, j_{1,\lambda}\right\rangle = \\ = \left\langle \pi_{1,\lambda}(x)^{-1}\pi_{1,\lambda}(k^{-1})\mathbb{1}_\lambda, \pi_{1,-\overline{\lambda}}(\overline{n}) j_{1,\lambda}\right\rangle = \chi(\overline{n})^{-1} W_\lambda(x). \end{aligned}$$

Taking into account the Iwasawa decomposition $G = KA\overline{N}$ we see that the Whittaker coefficient is completely determined by its restriction to A.

As discussed in Section 2.4 we have for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $\langle \operatorname{Re}\lambda - \rho, \alpha\rangle > 0$ for all $\alpha \in \Delta$ that $j_\lambda = j(P : 1 : -\overline{\lambda})(1)$ is simply a continuous function. Using this we can, for such λ, give a more explicit integral expression for the function W_λ. For $a \in A$ we have

$$\begin{aligned} W_\lambda(a) &= \langle \pi_{1,\lambda}(a^{-1})\mathbb{1}_\lambda, j_\lambda\rangle = \left\langle \mathbb{1}_\lambda, \pi_{1,-\overline{\lambda}}(a) j_\lambda\right\rangle \\ &= \int_K \mathbb{1}_\lambda(k)\cdot \overline{j_\lambda(a^{-1}k)}\,\mathrm{d}k = \int_K \overline{j_\lambda(a^{-1}k)}\,\mathrm{d}k. \end{aligned}$$

Applying Proposition 1.36 we see that W_λ is given by

$$W_\lambda(a) = \int_{\overline{N}} \overline{j_\lambda(a^{-1}k(\overline{n}))}\cdot e^{-2\rho H(\overline{n})}\,\mathrm{d}\overline{n}.$$

The transformation properties of the function $j_\lambda = j(P : 1 : -\overline{\lambda})$ imply that for $a \in A$ and $\overline{n} \in \overline{N}$ we have

$$\begin{aligned} j_\lambda(a^{-1}k(\overline{n})) &= j_\lambda(a^{-1}\overline{n}(a(\overline{n})n(\overline{n}))^{-1}) \\ &= j_\lambda(C_{a^{-1}}(\overline{n})a^{-1}a(\overline{n})^{-1}n) \\ &= \chi(C_{a^{-1}}(\overline{n}))^{-1}\cdot a^{-\overline{\lambda}+\rho}\cdot e^{(-\overline{\lambda}+\rho)H(\overline{n})} \end{aligned}$$

Here n is some element in N of which the precise value is not important. By substituting this in the above expression for $W_\lambda(a)$ we find

$$W_\lambda(a) = a^{-\lambda+\rho}\cdot \int_{\overline{N}} \chi(a^{-1}\overline{n}a)\cdot e^{-(\lambda+\rho)H(\overline{n})}\,\mathrm{d}\overline{n}. \tag{2.9}$$

2.6.1 The case of $G = \mathrm{SL}(2,\mathbb{R})$

We will be particularly interested in the Whittaker coefficient when $G = \mathrm{SL}(2,\mathbb{R})$. Therefore, we first spend some time on finding an explicit integral expression for W_λ in this case.

We use the notation as introduced in Section 1.4.1. From the discussion in this section we know that for $G = \mathrm{SL}(2,\mathbb{R})$ we have $\dim\overline{\mathfrak{n}} = 1$. Hence if χ is a (not necessarily unitary) character its derivative $d\chi\colon \overline{\mathfrak{n}} \to \mathbb{C}$ is uniquely determined by the number $\gamma \in \mathbb{C}$ such that $d\chi(Y) = i\gamma$. The corresponding character is then given by $\chi(\overline{n}_x) = e^{i\gamma x}$. We see that is χ unitary if and only if $\gamma \in \mathbb{R}$ and is regular if and only if $\gamma \neq 0$. We assumed χ to be regular so we fix a $\gamma \in \mathbb{R}\setminus\{0\}$.

In order to give an explicit expression for W_λ is this case we consider the expression given in (2.9). This expression holds for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\langle \operatorname{Re}\lambda - \rho, \alpha\rangle > 0$ (or equivalently $\lambda(H) > 1$). As observed in 1.4.1 we have that $\mathbb{R} \cong \overline{n} \xrightarrow{\exp} \overline{N}$ is a group isomorphism. This means in particular that, up to normalization, we have $\exp^*\mathrm{d}\overline{n} = \mathrm{d}x$, with $\mathrm{d}x$ the Lebesque measure on $\mathbb{R}$. Using this to perform a substitution of variables for the integral in (2.9) yields

$$W_\lambda(a) = a^{-\lambda+\rho}\int_{\mathbb{R}} \chi(C_{a^{-1}}(\overline{n}_x))e^{-(\lambda+\rho)H(\overline{n}_x)}\,\mathrm{d}x.$$

By Proposition 1.18 we have $H(\overline{n}_x) = \frac{1}{2}\log(1+x^2)\cdot H$. From this we see that

$$e^{-(\lambda+\rho)H(\overline{n}_x)} = (1+x^2)^{-\lambda(H)/2-1/2}.$$

Furthermore, we observe that, in view of Proposition 1.15,

$$C_{a^{-1}}(\overline{n}_x) = a^{-1}\exp(x\cdot Y)a = \exp(x\cdot \mathrm{Ad}(a^{-1})Y) = \exp(xa^\alpha\cdot Y) = \overline{n}_{a^\alpha x}.$$

By substituting this we now obtain the following expression for the Whittaker coefficient

$$W_\lambda(a) = a^{-\lambda+\rho} \int_{\mathbb{R}} e^{i\gamma a^\alpha x} \cdot (1+x^2)^{-\lambda(H)/2-1/2} \,\mathrm{d}x. \tag{2.10}$$

We note that the Whittaker coefficients of the characters determined by γ and $-\gamma$ coincide. So without loss of generality we can assume that $\gamma > 0$. Furthermore, if we pick $a' \in A$ such that $(a')^\alpha = 1/\gamma$ then we see that

$$\begin{aligned} W_\lambda(aa') &= (a')^{-\lambda+\rho} a^{-\lambda+\rho} \int_{\mathbb{R}} e^{i\gamma (a')^\alpha a^\alpha x} (1+x^2)^{-\lambda(H)/2-1/2} \,\mathrm{d}x \\ &= (a')^{-\lambda+\rho} \left[a^{-\lambda+\rho} \int_{\mathbb{R}} e^{i a^\alpha x} (1+x^2)^{-\lambda(H)/2-1/2} \,\mathrm{d}x \right] \end{aligned}$$

The factor in the brackets is precisely the Whittaker coefficient for $\gamma = 1$. We see that by shifting the Whittaker coefficient on the right we obtain a multiple of the Whittaker coefficient for the case of $\gamma = 1$. For simplicity we will from now on assume $\gamma = 1$ with the knowledge that our results transfer to the general case by reversing the above shifting procedure.

As remarked at the start of the calculation this expression holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda(H) > 1$. However we observe that the integral in (2.10) converges for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda(H) > 0$. A standard application of the Dominated convergence theorem also yields that for $a \in A$ fixed the expression in (2.10) is holomorphic in λ on this region. Since $W_\lambda(a)$ is holomorphic in λ we see that the expression for $W_\lambda(a)$ in (2.10) is valid for all λ with $\operatorname{Re} \lambda(H) > 0$.

Using a partial integration procedure we can find an integral expression for $W_\lambda(a)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. In order to reduce the amount of notation when carrying out this process we introduce the following auxiliary function.

Definition 2.34. We define $w\colon \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\} \times \mathbb{R}_{>0} \to \mathbb{C}$ as

$$w(z,t) := \int_{\mathbb{R}} e^{itx} (1+x^2)^{-z-1/2} \,\mathrm{d}x.$$

On $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ this integral is finite and a standard argument shows that for $t \in \mathbb{R}_{>0}$ fixed the function $z \mapsto w(z,t)$ is holomorphic on this domain. Looking at the expression in Equation (2.10) we see that $W_\lambda(a) = a^{-\lambda+\rho} \cdot w(\lambda(H)/2, a^\alpha)$ holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re}(H) > 0$.

The next two lemmas will establish that the function w can be holomorphically extended to the whole of $\mathbb{C} \times \mathbb{R}_{>0}$.

Lemma 2.35. For every $k \in \mathbb{N}$ and $r \in \mathbb{R}$ there exists a $c = c_{k,r} > 0$ such that

$$\left| \partial_x^k (1+x^2)^{-z-1/2} \right| \leq c \cdot (1+|x|)^{-2z-k-1}$$

for all $x \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\operatorname{Re} z \leq r$.

Proof. A straightforward proof using induction on k shows that for every $k \in \mathbb{N}$

$$\frac{\partial^k}{\partial x^k} (1+x^2)^{-z-1/2} = p_k(x) \cdot (1+x^2)^{-z-1/2-k}$$

with p_k a polynomial of degree at most k. The stated estimate now follows directly from this. □

Lemma 2.36. The function w can be uniquely extended to the whole of $\mathbb{C} \times \mathbb{R}_{>0}$ such that for fixed $t \in \mathbb{R}_{>0}$ the map $z \mapsto w(z,t)$ is a holomorphic function on $\mathbb{C}$.

Proof. We let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $t \in \mathbb{R}_{>0}$. We observe, for $k \in \mathbb{N}$, using partial integration that

$$\begin{aligned} w(z,t) &= \int_{\mathbb{R}} e^{itx} \cdot (1+x^2)^{-z-1/2}\,\mathrm{d}x = (-i)^k t^{-k} \int_{\mathbb{R}} \left(\partial_x^k e^{itx}\right)(1+x^2)^{-z-1/2}\,\mathrm{d}x \\ &= i^k t^{-k} \int_{\mathbb{R}} e^{itx} \cdot \partial_x^k \left((1+x^2)^{-z-1/2}\right)\mathrm{d}x \end{aligned}$$

This expression for $w(z,t)$ is valid for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. However if we take into account the result of Lemma 2.35 we see that this integral converges when $\operatorname{Re} z > -\frac{1}{2}k$. A standard argument using the dominated convergence theorem shows that on this domain the expression is holomorphic in z. Since this expression is holomorphic in z and coincides with w for $\operatorname{Re} z > 0$ we can define the holomorphic extension of w on $\{z \in \mathbb{C} \mid \operatorname{Re} z > -\frac{1}{2}k\}$ to be given by this expression. □

A direct consequence of this lemma is the following.

Corollary 2.37. Suppose $G = \mathrm{SL}(2,\mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$. For every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $a \in A$ the identity

$$W_\lambda(a) = a^{-\lambda+\rho} \cdot w(\lambda(H)/2, a^\alpha) \tag{2.11}$$

holds.

In the next section we will utilize this expression to find several estimates on the Whittaker coefficient.

In Chapter 3 we will be mainly concerned with the Whittaker coefficient for the unitary principal series, i.e. $\lambda \in i\mathfrak{a}^*$. Looking at the proof of the previous lemma we see that on the imaginary axis the simplest expression for w is given by

$$w(z,t) = -(2z+1) \cdot \frac{i}{t} \int_{\mathbb{R}} e^{itx} \cdot x(1+x^2)^{-z-3/2}\,\mathrm{d}x \tag{2.12}$$

for $t \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}$ with $\operatorname{Re} z \geq -1/2$.

Remark 2.38: In Section 2.5 we assumed that χ is regular in order to prove that $j(P : \xi : \lambda)$ can be holomorphically extended. By considering the expression for the Whittaker coefficient we derived in this section we see this assumption is vital. If we take χ to be the non-regular character on $\overline{N}$, i.e. $\chi \equiv 1$, then the Whittaker coefficient, if we were to define it for this case, would for $\lambda(H) > 0$ be given by

$$W_\lambda(a) = a^{-\lambda+\rho} \int_{\mathbb{R}} (1+x^2)^{-\lambda(H)/2-1/2}\,\mathrm{d}x$$

(take (2.10) with $\gamma = 0$). It is clear that this expression can not be extended holomorphically to the whole of $\mathfrak{a}_{\mathbb{C}}^*$ since $W_\lambda(a) \to \infty$ for $\lambda \to 0$.

2.7 Estimates on the Whittaker matrix coefficient

In this section we will study the behaviour of the Whittaker matrix coefficient $W_\lambda(a)$ and its derivatives $\partial_\lambda^k W_\lambda(a)$ as functions of a. We will derive several estimates that will be important in our discussion in Section 3.2 when we introduce the Whittaker–Fourier transformation.

Throughout this section we retain our assumption that $G = \mathrm{SL}(2,\mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$.

It turns out that there is a great difference in the behaviour of W_λ on $A^+ := \exp(\mathbb{R}_{\geq 0} \cdot H)$, the *'positive Weyl chamber'* of A, and $A^- := \exp(\mathbb{R}_{\leq 0} \cdot H)$, the *'negative Weyl chamber'* of A. We first investigate the behaviour on A^+.

Proposition 2.39. Suppose $G = \mathrm{SL}(2,\mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$. Let $s < r \in \mathbb{R}$ and $k \in \mathbb{N}$. Then there exists a $C = C_{k,s,r} > 0$ such that

$$|W_\lambda(a)| \leq C \cdot a^{\operatorname{Re}\lambda - k\rho}$$

holds for all $a \in A^+$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfying $s \leq \operatorname{Re}\lambda(H) \leq r$.

Proof. We fix an $l \in \mathbb{N}$ satisfying $-l < s$ and $2l - 1 > k$. For all $z \in \mathbb{C}$ with $\operatorname{Re} z > -\frac{1}{2}l$ we have, see the proof of Lemma 2.36, the following expression for $w(z,t)$

$$w(z,t) = i^l t^{-l} \int_{\mathbb{R}} e^{itx} \cdot \partial_x^l \left((1+x^2)^{-z-1/2}\right) \mathrm{d}x.$$

On the strength of Lemma 2.35 there exists a constant $c > 0$ such that for $\frac{1}{2}s \leq z \leq \frac{1}{2}r$ we have the estimate

$$|w(z,t)| \leq t^{-l} \int_{\mathbb{R}} \left|\partial_x^l \left((1+x^2)^{-z-1/2}\right)\right| \mathrm{d}x \leq c \cdot t^{-l} \int_{\mathbb{R}} (1+|x|)^{-2\operatorname{Re} z - l - 1} \mathrm{d}x.$$

For $z \in \mathbb{C}$ with $\operatorname{Re} z \geq \frac{1}{2}s$ we have, keeping in mind that $-l < s$ holds, that $-2\operatorname{Re} z - 1 - l \leq -s - l - 1 < -1$. If we set $C = c \cdot \int_{\mathbb{R}} (1+|x|)^{-s-l-1} \mathrm{d}x < \infty$ we see that we obtain the estimate

$$|w(z,t)| \leq C \cdot t^{-k}$$

for all $z \in \mathbb{C}$ with $\frac{1}{2}s \leq \operatorname{Re} z \leq \frac{1}{2}r$.

We use this estimate on w to obtain the stated estimates on W_λ. Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $s \leq \operatorname{Re}\lambda(H) \leq r$. Then using the expression in (2.11) and the above estimate yields for all $a \in A$

$$|W_\lambda(a)| = a^{\operatorname{Re}\lambda + \rho} |w(\lambda(H)/2, a^\alpha)| \leq C \cdot a^{\operatorname{Re}\lambda+\rho} a^{-l\alpha} = C \cdot a^{\operatorname{Re}\lambda - (2l-1)\rho}.$$

Our assumption that $2l - 1 > k$ implies $\operatorname{Re}\lambda(H) - (2l-1) < \operatorname{Re}\lambda(H) - k$. For any $a \in A^+$ we have $\log a \in \mathbb{R}_{\geq 0} \cdot H$ hence we have $(\operatorname{Re}\lambda - (2l-1)\rho)\log a < (\operatorname{Re}\lambda - k\rho)\log a$. From this we conclude that

$$|W_\lambda(a)| \leq C \cdot a^{\operatorname{Re}\lambda - (2l-1)\rho} \leq C \cdot a^{\operatorname{Re}\lambda - k\rho}$$

holds. □

We see from this result that W_λ vanishes to any order $a^{-k\rho}$ when $a \to \infty$ in A^+ (i.e. $a = \exp(tH)$ with $t \to \infty$). In Section 2.8 we will improve this estimate and show that actually W_λ goes to zero at a double exponential rate more specifically for some $c = c_\lambda \in \mathbb{C}$ we have $W_\lambda \sim c \cdot e^{-a^\alpha}$ for $a \to \infty$ in A^+.

Now we investigate the behaviour of $W_\lambda(a)$ on A^-. This case is somewhat more complicated hence we split up the work. We first derive an estimate on the function w.

Lemma 2.40. For every $\epsilon > 0$ there exists a $C = C_\epsilon > 0$ such that

$$\left|t^{-z} w(z,t)\right| \leq C(1+|z|)(1+|\log t|) \cdot t^{-|\operatorname{Re} z|}$$

holds for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq -\frac{1}{2} + \epsilon$ and $t \in (0,1]$.

Proof. From (2.12) we know that for all z with $\operatorname{Re} z > -\frac{1}{2}$ that $w(z,t)$ is given by

$$\begin{aligned} w(z,t) &= -(2z+1) \cdot \frac{i}{t} \int_{\mathbb{R}} e^{itx} \cdot x(1+x^2)^{-z-3/2} \, \mathrm{d}x \\ &= (2z+1) \cdot \frac{2}{t} \int_0^\infty \sin(tx) \cdot x(1+x^2)^{-z-3/2} \, \mathrm{d}x \end{aligned}$$

We fix a $z \in \mathbb{C}$ with $\operatorname{Re} z \geq -\frac{1}{2} + \epsilon$ and a $t \in (0,1]$. Using the above expression we obtain the following estimate

$$\left| t^{-z} w(z,t) \right| \leq 4(1+|z|) \cdot t^{-\operatorname{Re} z - 1} \int_0^\infty |\sin(tx)| \, x(1+x^2)^{-\operatorname{Re} z - 3/2} \, \mathrm{d}x.$$

For simplicity we denote $s = -\operatorname{Re} z$, so s satisfies $s \leq \frac{1}{2} - \epsilon$. We observe

$$\begin{aligned} & t^{s-1} \int_0^\infty |\sin(tx)| \, x(1+x^2)^{s-3/2} \, \mathrm{d}x \\ &= t^s \int_0^1 \left| \frac{\sin(xt)}{xt} \right| x^2(1+x^2)^{s-3/2} \, \mathrm{d}x + t^{s-1} \int_1^\infty |\sin(tx)| \, x(1+x^2)^{s-3/2} \, \mathrm{d}x \\ &= A_1 + A_2. \end{aligned}$$

In order to estimate the first term, A_1, we use that $|\sin(y)/y| \leq 1$ for all $y \in \mathbb{R}$ to find

$$A_1 \leq t^s \int_0^1 x^2(1+x^2)^{s-3/2} \, \mathrm{d}x = C_1 \cdot t^s \leq C_1 \cdot t^{-|s|}$$

Here we set $C_1 = \int_0^1 x^2(1+x^2)^{s-3/2} \, \mathrm{d}x$ which is clearly finite.

Now we estimate the second term A_2. Since $1 + x^2 \geq x^2$ and $s - 3/2 < 0$ we can estimate $x(1+x^2)^{s-3/2} \leq x^{2x-2}$ for all $x \geq 1$. We find

$$\begin{aligned} A_2 &\leq t^{s-1} \int_1^\infty |\sin(tx)| \, x^{2s-2} \, \mathrm{d}x & \text{(substituting } y = tx\text{)} \\ &= t^{-s} \int_t^\infty |\sin(y)| \, y^{2s-2} \, \mathrm{d}y \\ &= t^{-s} \int_t^1 |\sin(y)| \, y^{2s-2} \, \mathrm{d}y + t^{-s} \int_1^\infty |\sin(y)| \, y^{2s-2} \, \mathrm{d}y \\ &= B_1 + B_2. \end{aligned}$$

To estimate the second term, B_2, we use that $s \leq \frac{1}{2} - \epsilon$ hence

$$B_2 \leq t^{-s} \int_1^\infty y^{-1-2\epsilon} \, \mathrm{d}y = C_2 \cdot t^{-s} \leq C_2 \cdot t^{-|s|}$$

were we defined $C_2 = \int_1^\infty y^{-1-2\epsilon} \, \mathrm{d}y < \infty$. In order to estimate the first term, B_1, we again use that $|\sin(y)/y| \leq 1$ for all $y \in \mathbb{R}$ to find

$$B_1 = t^{-s} \int_t^1 \left| \frac{\sin(y)}{y} \right| y^{2s-1} \, \mathrm{d}y \leq t^{-s} \int_t^1 y^{2s-1} \, \mathrm{d}y = \begin{cases} \frac{t^{-s} - t^s}{2s} & s \neq 0 \\ -t^{-s} \log(t) & s = 0 \end{cases}$$

We claim that $B_1 \leq |\log t|\, t^{-|s|}$ holds for all s. For $s = 0$ this is immediate from the above expression. For $s \neq 0$ we observe that

$$\left|\frac{t^{-s} - t^{s}}{2s \cdot \log(t)}\right| = \frac{t^{-|s|}}{2\,|s|}\left|\frac{t^{2|s|} - 1}{\log t}\right|.$$

If we write $\tau = t^{2|s|}$ then $\tau \in (0, 1]$ and the above expression equals

$$\frac{t^{-|s|}}{2\,|s|}\left|\frac{\tau - 1}{\log \tau^{1/(2|s|)}}\right| = \frac{t^{-|s|}}{2\,|s|} \cdot 2\,|s| \cdot \left|\frac{\tau - 1}{\log \tau}\right| \leq t^{-|s|}.$$

We leave it to the reader to check that indeed $|(\tau - 1)/\log \tau| \leq 1$ for $\tau \in (0, 1]$. Taking the factor $|\log t|$ to the other side yields the desired estimate for B_2 in the case $s \neq 0$.

Now combining all these estimates and substituting back $s = -\operatorname{Re} z$ we obtain

$$\begin{aligned}\left|t^{-z} w(z, t)\right| &\leq 4(1 + |z|)(A_1 + B_1 + B_2) \leq 4(1 + |z|)\left[(C_1 + C_2)t^{-|\operatorname{Re} z|} + |\log t|\, t^{-|\operatorname{Re} z|}\right] \\ &\leq 4(C_1 + C_2 + 1)(1 + |z|)(1 + |\log t|)t^{-|\operatorname{Re} z|}.\end{aligned}$$

This concludes the proof. □

Using this we obtain the following estimate for W_λ on A^-.

Proposition 2.41. Suppose $G = \mathrm{SL}(2, \mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$. For every $\epsilon > 0$ there exists a $C = C_\epsilon > 0$ such that for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\operatorname{Re} \lambda(H) \geq -1 + \epsilon$ and $a \in A^-$ the following estimate holds

$$|W_\lambda(a)| \leq C a^\rho (1 + \|\lambda\|)(1 + \|\log a\|) e^{\|\operatorname{Re} \lambda\| \|\log a\|}.$$

Here we use the normalized norms on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ as introduced in Section 1.4.1.

Proof. We fix a $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda(H) \geq -1 + \epsilon$ and let $a \in A^-$. We observe that since $\lambda = \frac{1}{2}\lambda(H) \cdot \alpha$ we have

$$a^{-\lambda} = a^{-\lambda(H)/2 \cdot \alpha} = (a^\alpha)^{-\lambda(H)/2}.$$

Furthermore, since a is such that $\log a \in \mathbb{R}_{\leq 0} \cdot H$ we have $a^\alpha \in (0, 1]$. Applying Lemma 2.40 with $z = \frac{1}{2}\lambda(H)$ and $t = a^\alpha$ yields that a $C > 0$ exists such the following holds

$$\begin{aligned}|W_\lambda(a)| &= \left|a^{-\lambda + \rho} w(\lambda(H)/2, a^\alpha)\right| = a^\rho \cdot \left|(a^\alpha)^{-\lambda(H)/2} w(\lambda(H)/2, a^\alpha)\right| \\ &\leq C \cdot a^\rho (1 + \frac{1}{2}\,|\lambda(H)|)(1 + |\log a^\alpha|)(a^\alpha)^{-|\operatorname{Re} \lambda(H)|/2}.\end{aligned}$$

By our chosen normalization of the norms on $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$ we have $|\lambda(H)| = \|\lambda\|$. Furthermore, we have $\alpha(\log a) < 0$ hence

$$-\alpha(\log a) = |\alpha(\log a)| \leq \|\alpha\|\,\|\log a\| = 2\,\|\log a\|\,.$$

From this it follows that $|\log a^\alpha| = |\alpha(\log a)| \leq 2\,\|\log a\|$ and

$$(a^\alpha)^{-|\operatorname{Re} \lambda(H)|/2} = e^{-\alpha(\log a) \cdot |\operatorname{Re} \lambda(H)|/2} \leq e^{\|\operatorname{Re} \lambda\| \|\log a\|}.$$

Combining all this yields

$$|W_\lambda(a)| \leq 2C a^\rho (1 + \|\lambda\|)(1 + \|\log a\|) e^{\|\operatorname{Re} \lambda\| \|\log a\|}$$

□

We conclude that on A^+ the Whittaker coefficient W_λ is bounded and rapidly decreases to zero for $a \to \infty$ in A^+. On A^- the behaviour is completely different and W_λ does not go to zero for $a \to \infty$ in A^-. We summarize these two results in the following proposition. The estimate in this proposition is stated such that it holds on the whole of A. It should be noted that the estimate is very bad on A^+ but it has the advantage that we do not need to distinguish between A^+ and A^- in future proofs. Furthermore, we also return to the case of arbitrary unitary character χ of $\overline{N}$.

Proposition 2.42. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. There exists $C > 0$ such that for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $-\frac{1}{2} \leq \operatorname{Re}\lambda(H) \leq \frac{1}{2}$ and $a \in A$ the following estimate is valid

$$|W_\lambda(a)| \leq C a^\rho (1 + \|\lambda\|)(1 + \|\log a\|) e^{\|\operatorname{Re}\lambda\| \|\log a\|}.$$

Proof. We let $\gamma \in \mathbb{R} \setminus \{0\}$ be such that $\chi(\overline{n}_x) = e^{i\gamma x}$. In order to avoid confusion we write in this proof W_λ^γ for the Whittaker coefficient associated to this character χ.

We first prove this statement for the case $\chi(\overline{n}_x) = e^{ix}$, i.e. $\gamma = 1$. For $a \in A^-$ this result is a restatement of Proposition 2.41. On A^+ this estimate follows easily from Proposition 2.39. By this proposition there exists a $C > 0$ such that $|W_\lambda(a)| \leq C a^{\operatorname{Re}\lambda}$ for all $a \in A^+$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfying $-\frac{1}{2} \leq \operatorname{Re}\lambda(H) \leq \frac{1}{2}$. So for $a \in A^+$ we have

$$\left|W_\lambda^1(a)\right| \leq C e^{\operatorname{Re}\lambda(\log a)} \leq C e^{|\operatorname{Re}\lambda(\log a)|} \leq C e^{\|\operatorname{Re}\lambda\| \|\log a\|}$$

Since $a^\rho \geq 1$ on A^+ the estimate follows.

We will infer from this special case the statement for arbitrary γ. From our discussion in Section 2.6.1 we know $W_\lambda^\gamma = W_\lambda^{-\gamma}$ so we can, without loss of generality, assume $\gamma > 0$. We fix $a' \in A$ such that $(a')^\alpha = 1/\gamma$. As shown in Section 2.6.1 we have $W_\lambda^\gamma(aa') = (a')^{-\lambda+\rho} W_\lambda^1(a)$ for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $\operatorname{Re}\lambda(H) > 0$. Since both sides of this expression are holomorphic we conclude this holds for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. We can now apply the above estimate for W_λ^1 to find

$$\begin{aligned}
\left|W_\lambda^\gamma(a)\right| &= \left|W_\lambda^\gamma((aa'^{-1})a')\right| = (a')^{-\operatorname{Re}\lambda+\rho} \left|W_\lambda^1(aa'^{-1})\right| \\
&\leq C (a')^{-\operatorname{Re}\lambda+\rho} (aa'^{-1})^\rho (1 + \|\lambda\|)(1 + \left\|\log(aa'^{-1})\right\|) e^{\|\operatorname{Re}\lambda\| \|\log a\|} \\
&\leq C (a')^{-\operatorname{Re}\lambda} a^\rho (1 + \|\lambda\|)(1 + \|\log a\| + \left\|\log a'\right\|) e^{\|\operatorname{Re}\lambda\| \|\log a\|} \\
&\leq C' a^\rho (1 + \|\lambda\|)(1 + \|\log a\|) e^{\|\operatorname{Re}\lambda\| \|\log a\|}.
\end{aligned}$$

Here $C' > 0$ is a constant large enough such that $C(a')^{-\operatorname{Re}\lambda} \leq C'$ for all $-\frac{1}{2} \leq \operatorname{Re}\lambda(H) \leq \frac{1}{2}$. $\square$

Finally we exploit the fact that $W_\lambda(a)$ depends holomorphically on λ to show, using the Cauchy integral formula, that the derivatives of the Whittaker coefficient $\partial_\lambda^k W_\lambda$ satisfy similar estimates on A.

Proposition 2.43. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. For every $k \in \mathbb{N}$ there exists a $C_k > 0$ such that

$$\left|\partial_\lambda^k W_\lambda(a)\right| \leq C_k a^\rho (1 + \|\lambda\|)(1 + \|\log a\|)^{k+1}$$

for all $\lambda \in i\mathfrak{a}^*$ and $a \in A$.

Proof. We fix an $a \in A$ and $\lambda_0 \in i\mathfrak{a}^*$. We know that the map $\lambda \mapsto W_\lambda(a)$ is holomorphic on $\mathfrak{a}^*_{\mathbb{C}}$ hence its derivative $\partial_\lambda^k W_\lambda(a)\big|_{\lambda=\lambda_0}$ can be expressed using the Cauchy formula as

$$\partial_\lambda^k W_\lambda(a)\big|_{\lambda=\lambda_0} = \frac{1}{2\pi i} \int_{\|\lambda-\lambda_0\|=r} W_\lambda(a)(\lambda - \lambda_0)^{-k-1} d\lambda$$

for all $r > 0$. If we impose that $r \leq \frac{1}{2}$ we can make, using the result of Proposition 2.42, the following estimate for all $a \in A$

$$\begin{aligned}
\left|\partial_\lambda^k W_\lambda(a)\big|_{\lambda=\lambda_0}\right| &\leq \frac{1}{2\pi}\int_{\|\lambda-\lambda_0\|=r} |W_\lambda(a)|\,\|\lambda-\lambda_0\|^{-k-1}\,d\lambda \\
&\leq \frac{1}{2\pi}\cdot \mathrm{Length}(\|\lambda-\lambda_0\|=r)\cdot r^{-k-1}\sup_{|\lambda-\lambda_0|=r}|W_\lambda(a)| \\
&\leq C\cdot r^{-k}a^\rho(1+\|\log a\|)\sup_{|\lambda-\lambda_0|=r}(1+\|\lambda\|)e^{\|\mathrm{Re}\,\lambda\|\|\log a\|}
\end{aligned}$$

with $C > 0$ a suitable constant independent of λ_0.

We now select a particular value for r namely $r = \frac{1}{2}(1+\|\log a\|)^{-1}$. Then we have $0 < r \leq \frac{1}{2}$. Keeping in mind that $\lambda_0 \in i\mathfrak{a}^*$ we have for all λ with $\|\lambda-\lambda_0\| = r$ that $\|\mathrm{Re}\,\lambda\| \leq \frac{1}{2}(1+\|\log a\|)^{-1}$. So for all such λ the inequality $\|\mathrm{Re}\,\lambda\|\,\|\log a\| \leq 1$ holds. Furthermore, we have $1+\|\lambda\| \leq 1+\|\lambda_0\|+r \leq 2(1+\|\lambda_0\|)$. Using this the estimate

$$\left|\partial_\lambda^k W_\lambda(a)\big|_{\lambda_0}\right| \leq 2C\cdot e\cdot a^\rho(1+\|\lambda_0\|)(1+\|\log a\|)^{k+1}$$

follows. □

2.8 Whittaker functions

We begin this section by showing that the Whittaker coefficient satisfies a certain differential equation on G. This will follow from the fact that we know precisely how the Casimir element acts on $C^\infty(P : \xi : \lambda)$, namely by a scalar. The fact that W_λ satisfies this differential equation will be used in several proofs in Chapter 3. In this section however we use it to investigate the relationship between W_λ and the classical notion of Whittaker functions.

In this section we again assume that $G = \mathrm{SL}(2,\mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$.

Proposition 2.44. The Whittaker coefficient satisfies the following differential equation on G

$$L_\Omega W_\lambda = (\lambda(H)^2 - 1)W_\lambda.$$

Here $\Omega = H^2 + 2XY + 2YX$ is the Casimir element of $\mathfrak{sl}(2,\mathbb{R})$ (see Section 1.9.1).

Proof. First, for any $Z \in \mathfrak{g}$, we observe

$$\begin{aligned}
(L_Z W_\lambda)(x) &= \frac{d}{dt}\Big|_{t=0} W_\lambda(\exp(-tZ)x) \\
&= \frac{d}{dt}\Big|_{t=0}\left\langle \pi(x^{-1})\pi(\exp(tZ))\mathbb{1}_\lambda, j_\lambda\right\rangle \\
&= \left\langle \pi(x^{-1})\frac{d}{dt}\Big|_{t=0}\pi(\exp(tZ))\mathbb{1}_\lambda, j_\lambda\right\rangle \\
&= \left\langle \pi(x^{-1})(\pi(Z)\mathbb{1}_\lambda), j_\lambda\right\rangle.
\end{aligned}$$

By repeated application of this calculation we find

$$(L_\Omega W_\lambda)(x) = \left\langle \pi(x^{-1})(\pi(\Omega)\mathbb{1}_\lambda), j_\lambda\right\rangle.$$

From Proposition 1.48 we know that the Casimir element Ω acts on $C^\infty(P : 1 : \lambda)$ by the scalar $\lambda(H)^2 - 1$. We conclude that

$$(L_\Omega W_\lambda)(x) = \left\langle \pi(x^{-1})(\pi(\Omega)\mathbb{1}_\lambda), j_\lambda\right\rangle = (\lambda(H)^2-1)W_\lambda(x).$$

□

2.8.1 Relation between W_λ and Whittaker functions.

We now make a brief digression to investigate the relation between the Whittaker coefficient and what are called *Whittaker functions* on $\mathbb{R}$. As a result we will derive an interesting result about the asymptotic behaviour of W_λ on A^+. It should be noted that this result is not required for our future proofs so in principle this section may be safely skipped.

On $\mathbb{R}$ a *Whittaker function* is a function $F\colon \mathbb{R} \to \mathbb{R}$ satisfying the *Whittaker differential equation*:

$$F''(x) + \left(\frac{1/4 - \nu^2}{x^2} + \frac{\mu}{x} - \frac{1}{4} \right) F(x) = 0. \tag{2.13}$$

Here ν, μ are parameters in $\mathbb{C}$. These functions where first introduced by E.T. Whittaker and are studied in detail in the classic text Whittaker and Whatson, "A course of modern analysis" ([WW27]). In this section we will show that the Whittaker coefficient W_λ satisfies this differential equation if we make the identification $A \cong (0, \infty)$.

By the previous proposition we know that W_λ satisfies

$$L_\Omega W_\lambda = (\lambda(H)^2 - 1) W_\lambda$$

with $\Omega = H^2 + 2XY + 2YX$. As noted in Section 2.6 the Whittaker coefficient is determined by its values on A hence it suffices to only consider the radial part of this differential equation. We introduce the following notation

$$Z := Y - X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easily seen that $Z \in \mathfrak{k}$ and in fact spans this subspace. As discussed in Section 2.6 we have that W_λ is left K-invariant and on the right transforms as $W_\lambda(x\overline{n}) = \chi(\overline{n})^{-1} W_\lambda(x)$ for $x \in G, \overline{n} \in \overline{N}$. The left K-invariance of W_λ immediately yields $L_W W_\lambda = 0$. For any $a \in A$ we observe using transformation behaviour on the right that

$$(L_Y W_\lambda)(a) = (L_{a^{-1}} L_Y W_\lambda)(e) = (L_{\mathrm{Ad}(a^{-1})Y} L_{a^{-1}} W_\lambda)(e),$$

which equals, applying the result of Proposition 1.15,

$$\begin{aligned} &= a^\alpha (L_Y L_{a^{-1}} W_\lambda)(e) = -a^\alpha (R_Y L_{a^{-1}} W_\lambda)(e) \\ &= -a^\alpha \frac{d}{dt}\Big|_{t=0} (L_{a^{-1}} W_\lambda)(\exp(tY)) = -a^\alpha \frac{d}{dt}\Big|_{t=0} \chi(\exp(tY))^{-1} (L_{a^{-1}} W_\lambda)(e) \\ &= a^\alpha d\chi(Y) W_\lambda(a). \end{aligned}$$

We made the assumption that χ is given by $\chi(\overline{n}_x) = e^{ix}$ hence

$$(L_Y W_\lambda)(a) = i a^\alpha W_\lambda(a).$$

In order to make use of the fact that we know the action of L_Y and L_Z on W_λ we rewrite Ω as

$$\Omega = H^2 + 2XY + 2YX = H^2 + 2H + 4YX = H^2 + 2H + 4Y^2 - 4YZ.$$

We apply this to find that L_Ω acts on W_λ as

$$\begin{aligned} (L_\Omega W_\lambda)(a) &= (L_H^2 + 2L_H + 4L_Y^2 - 4L_Y L_Z) W_\lambda(a) \\ &= (L_H^2 + 2L_H) W_\lambda(a) + 4(ia^\alpha)^2 W_\lambda(a) \\ &= (L_H^2 + 2L_H) W_\lambda(a) - 4a^{2\alpha} W_\lambda(a) \end{aligned}$$

We conclude that on A the Whittaker coefficient satisfies

$$[L_H^2 + 2L_H - 4a^{2\alpha}]W_\lambda = (\lambda(H)^2 - 1)W_\lambda.$$

We recall that $A \cong \mathfrak{a}$ via the map $\exp\colon \mathfrak{a} \to A$ and that $\dim \mathfrak{a} = 1$. Hence we have that $a \in A$ is uniquely determined by the value of $a^\alpha \in (0, \infty)$. So on A we can introduce the coordinate system $x = 2a^\alpha$. A straightforward calculation reveals that in this coordinate system L_H corresponds to $-2x\frac{d}{dx}$. We see that in this coordinate system W_λ satisfies the differential equation

$$\begin{aligned} 0 &= \left[2x\left(\frac{d}{dx}\right)^2 - 2x\frac{d}{dx} - x^2 + (1 - \lambda(H)^2)\right] W_\lambda(x) \\ &= \left[4x^2\frac{d^2}{dx^2} - x^2 + (1 - \lambda(H)^2)\right] W_\lambda(x) \end{aligned}$$

We conclude, after dividing by $4x^2$, that W_λ satisfies the Whittaker differential equation with parameters $\nu = \lambda(H)/2$ and $\mu = 0$, i.e.

$$\frac{d^2W_\lambda(x)}{dx^2} + \left[\frac{1/4 - (\lambda(H)/2)^2}{x^2} - \frac{1}{4}\right] W_\lambda(x) = 0.$$

The Whittaker differential equation has been well studied. We will take some of the results on Whittaker functions for granted and use them to derive a result on the asymptotic behaviour of W_λ on A^+. The Whittaker differential equation has two singularities, one at $x = 0$ and one at $x = \infty$. The former is a regular singularity and the latter is an irregular singularity. The solution space for the Whittaker differential equations is two dimensional. On $(0, \infty)$ there exists a basis $\{W^+_{\mu,\nu}, W^-_{\mu,\nu}\}$ of this solution space of which the asymptotic behaviour at the irregular singularity is particularly nice. In [WW27, Chapter XVI, Section 16.4] the function $W_{\mu,\nu}(z)$, which is a solution to the Whittaker equation, is defined as a multi-valued function on $\mathbb{C}$ for $|\arg z| < \frac{3}{2}\pi$. We set $W^-_{\mu,\nu}(x) := W_{\mu,\nu}(x)$ for $x \in (0, \infty)$. The function $W^+_{\mu,\nu}$ we define as $W^+_{\mu,\nu}(x) := W_{-\mu,\nu}(-x)$ for $x \in (0, \infty)$. Here we make the choice $\arg(-x) = \pi$. These functions both satisfy the Whittaker differential equation and are linearly independent. Furthermore, for $x \to \infty$ we have

$$W^-_{\mu,\nu}(x) \sim e^{-x/2}x^\nu \quad \text{and} \quad W^+_{\mu,\nu}(x) \sim e^{x/2}x^{-\nu}$$

See for more details [WW27, Chapter XVI].

By making use of these facts we make the following interesting observation.

Proposition 2.45. Suppose $G = \mathrm{SL}(2, \mathbb{R})$ and $\chi(\overline{n}_x) = e^{ix}$. For every $\lambda \in \mathfrak{a}^*_\mathbb{C}$ there exists a $c = c_\lambda \in \mathbb{C}$ such that

$$W_\lambda(a) \sim c \cdot e^{-a^\alpha} a^\lambda$$

for $a \to \infty$ in A^+ (i.e. $a = \exp(tH)$ for $t \to \infty$).

Proof. The function $W_\lambda(x)$ satisfies (2.13) with $\nu = \lambda(H)/2$ and $\mu = 0$. Since $\{W^+_{0,\nu}, W^-_{0,\nu}\}$ is a basis of solutions for this differential equation there exist constants c and c' such that

$$W_\lambda(x) = c \cdot W^-_{0,\nu} + c' \cdot W^+_{0,\nu}.$$

From Proposition 2.39 we know that $W_\lambda(x) \to 0$ for $x \to \infty$. This implies that we must have $c' = 0$ since $W^+_{0,\nu}(x) \sim e^{x/2}x^\lambda$ which clearly does not go to zero. We conclude that $W_\lambda = c' \cdot W^+_{0,\nu}$. By substituting back $x = 2a^\alpha$ we find $W_\lambda(a) \sim c \cdot e^{-a^\alpha}(a^\alpha)^\nu = c \cdot e^{-a^\alpha} a^\lambda$. ☐

Chapter 3

Transformations

In this chapter we introduce and study the Whittaker-Fourier transformation and the Harish-Chandra transformation. First, in Section 3.1, we introduce two notions of Schwartz spaces on Lie groups that are analogous to the familiar notion of Schwartz functions on $\mathbb{R}^n$. These Schwartz spaces will be the function spaces on which both the Whittaker–Fourier and the Harish-Chandra transformation are defined. In Section 3.2 we introduce the Whittaker–Fourier transformation. In this section we will prove some properties of this transformation. For this we make critical use of the estimates on the Whittaker coefficient we derived in the previous chapter. In Section 3.3 we introduce the Harish-Chandra transformation. The main question we study in this chapter is under what assumptions this Harish-Chandra transformation maps into the space of Schwartz functions. Our main result is Proposition 3.11 which shows that under suitable conditions the Harish-Chandra transformation does map into the space of Schwartz functions when $G = \mathrm{SL}(2, \mathbb{R})$.

As in the previous chapter we let G be a connected semisimple Lie group with finite center. Whenever necessary we will specialize to the case $G = \mathrm{SL}(2, \mathbb{R})$.

3.1 Schwartz spaces

In this section we introduce the notion of Schwartz functions on A and the notion of Schwartz functions on G that transform on the right according to a unitary character of $\overline{N}$.

On $\mathbb{R}^n$ we have the familiar Schwartz seminorms

$$p_{N,k}(f) = \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + \|x\|)^N |\partial^\alpha f(x)| \quad \text{for } N, k \in N$$

and the Schwartz space

$$\mathcal{S}(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) \mid p_{N,k}(f) < \infty \quad \text{for all } N, k \in \mathbb{N}\}$$

which is equipped with the locally convex topology induced by the these seminorms.

If we pick a basis of $\mathfrak{a}$, i.e. an isomorphism $\phi \colon \mathbb{R}^n \to \mathfrak{a}$, then we can transport the notion of Schwartz functions to $\mathfrak{a}$. Keeping with the standard notation for Schwartz spaces on Lie groups and Lie algebras we denote this Schwartz space by $\mathscr{C}(\mathfrak{a})$. This space consists of functions f on $\mathfrak{a}$ such that $\phi^* f \in \mathcal{S}(\mathbb{R}^n)$ and is equipped with the topology induced by the seminorms $p_{N,k} \circ \phi^*$. It is easily checked that this space is defined independently of the basis chosen.

In light of Proposition 1.14 we see that $\exp \colon \mathfrak{a} \to A$ is an isomorphism. We introduce a notion of Schwartz space on A that corresponds to the Schwartz space $\mathscr{C}(\mathfrak{a})$ under this isomorphism.

Definition 3.1. We define the following collection of seminorms

$$p_{N,X}(f) := \sup_{a\in A}(1+\|\log a\|)^N \left|(L_X f)(a)\right| \quad \text{for } N\in\mathbb{N}, X\in U(\mathfrak{a}), f\in C^\infty(A).$$

Then the space of Schwartz functions on A is defined as

$$\mathscr{C}(A) := \{f\in C^\infty(A) \mid p_{N,X}(f)<\infty \quad \text{for all } N\in\mathbb{N}, X\in U(\mathfrak{a})\}.$$

We equip this space with the locally convex topology induced by the seminorms $p_{N,X}$.

It is a matter of routine verification to see that this space is a Fréchet space. It is also easily verified that $\exp^*\colon \mathscr{C}(A)\to\mathscr{C}(\mathfrak{a})$ is an isomorphism.

Let χ be a unitary character of $\overline{N}$. In this section we introduce, following [Wal92, Section 15.3], the notion of a Schwartz space for functions $f\in C^\infty(G)$ that transform according to χ when acted on from the right by $\overline{N}$. We denote

$$C^\infty(G/\overline{N};\chi) := \{f\in C^\infty(G) \mid f(x\overline{n}) = \chi(\overline{n})f(x) \quad \text{for all } x\in G, \overline{n}\in\overline{N}\}.$$

Furthermore, we denote by $a_{\overline{P}}\colon G\to A$ the Iwasawa projection with respect to the decomposition $G = KA\overline{N}$. We added the subscript in order to avoid confusion with the Iwasawa projection $a = a_P\colon G\to A$ associated to the decomposition $G = KAN$.

Definition 3.2. We define the following collection of seminorms

$$q_{N,X}(f) := \sup_{g\in G} a_{\overline{P}}(g)^{-\rho}(1+\left\|\log a_{\overline{P}}(g)\right\|)^N \left|(L_X f)(g)\right| \quad \text{for } N\in\mathbb{N}, X\in U(\mathfrak{g}), f\in C^\infty(G/\overline{N};\chi).$$

The corresponding Schwartz space is defined as

$$\mathscr{C}(G/\overline{N};\chi) := \{f\in C^\infty(G/\overline{N};\chi) \mid q_{N,X}(f)<\infty \quad \text{for all } N\in\mathbb{N}, X\in U(\mathfrak{g})\}.$$

We equip this space with the locally convex topology induced by the seminorms $q_{N,X}$.

Again it is a routine verification that the topology on $\mathscr{C}(G/\overline{N};\chi)$ is complete so this space is in fact a Fréchet space.

Remark 3.3: For $f\in\mathscr{C}(G/\overline{N};\chi)$ and $X\in U(\mathfrak{g})$ we see that $|L_X f(x\overline{n})| = |L_X f(x)|$ for $x\in G$ and $\overline{n}\in\overline{N}$. Using this and the Iwasawa decomposition $G = KA\overline{N}$ we observe that the Schwartz seminorms on $\mathscr{C}(G/\overline{N};\chi)$ can alternatively be written as

$$q_{N,X}(f) = \sup_{(k,a)\in K\times A} a^{-\rho}(1+\|\log a\|)^N \left|(L_X f)(ka)\right|.$$

We denote by $L^2(G/\overline{N};\chi)$ the space of measurable functions f on G that satisfy $f(x\overline{n}) = \chi(\overline{n})f(x)$ for all $x\in G, \overline{n}\in\overline{N}$ and $|f|\in L^2(G/\overline{N})$. This space we equip with the obvious L^2-norm.

Proposition 3.4. The space $\mathscr{C}(G/\overline{N};\chi)$ is continuously included in $L^2(G/\overline{N};\chi)$.

Proof. Let $N\in\mathbb{N}$ be large enough such that

$$\int_A (1+\|\log a\|)^{-2N}\,\mathrm{d}a = C < \infty.$$

We let $f \in \mathscr{C}(G/\overline{N};\chi)$ and observe, using Proposition 1.37, that

$$\int_{G/\overline{N}} |f(g)|^2 \,\mathrm{d}(g\overline{N}) = \int_{K\times A} |f(ka)|^2 \cdot a^{-2\rho} \,\mathrm{d}k \,\mathrm{d}a$$
$$\leq q_{N,1}(f)^2 \int_{K\times A} (1+\|\log a\|)^{-2N} \,\mathrm{d}k \,\mathrm{d}a \leq C \cdot q_{N,1}(f)^2.$$

We conclude that $f \in L^2(G/\overline{N};\chi)$ and that $\mathscr{C}(G/\overline{N};\chi) \hookrightarrow L^2(G/\overline{N};\chi)$ is continuous. $\square$

We denote by $C_c^\infty(G/\overline{N};\chi)$ the space of functions $f \in C^\infty(G/\overline{N};\chi)$ that satisfy $|f| \in C_c(G/\overline{N})$. If $C \subset G/\overline{N}$ is compact we denote by $C_C^\infty(G/\overline{N};\chi)$ the subspace of functions that satisfy $\operatorname{supp}|f| \subset C$.

Proposition 3.5. The set $C_c^\infty(G/\overline{N};\chi)$ is a dense subset of $\mathscr{C}(G/\overline{N};\chi)$. Furthermore, for the subspace of left K-invariant functions, we have that $C_c^\infty(G/\overline{N};\chi)^K$ is dense in $\mathscr{C}(G/\overline{N};\chi)^K$.

Proof. For $f \in C_c^\infty(G/\overline{N};\chi)$ we have, by Remark 3.3, that

$$q_{N,X}(f) = \sup_{(k,a)\in K\times A} a^{-\rho}(1+\|\log a\|)^N \left|(Xf)(ka)\right|.$$

This expression is finite for all $N \in \mathbb{N}$ and $X \in U(\mathfrak{g})$ since f is compactly supported in $K \times A$. We conclude that $C_c^\infty(G/\overline{N};\chi) \subset \mathscr{C}(G/\overline{N};\chi)$.

We now prove the density assertion. Let $f \in \mathscr{C}(G/\overline{N};\chi)$ be arbitrary. In this proof we let $\widetilde{\psi}$ be a smooth bump function in $C_c^\infty(\mathfrak{a})$ such that $\widetilde{\psi}|_{B^{\mathfrak{a}}(0;1)} \equiv 1$ and $\operatorname{supp}\widetilde{\psi} \subset B^{\mathfrak{a}}(0;2)$. Define $\psi_j \in C^\infty(G)$ as $\psi_j(ka\overline{n}) = \widetilde{\psi}(\frac{1}{j}\log a)$ for $j \in \mathbb{N}$. By the right $\overline{N}$-invariance of ψ_j we see that $\psi_j \cdot f \in C^\infty(G/\overline{N};\chi)$. Per construction we have that $\operatorname{supp}|\psi_i|$ is compact in $G/\overline{N}$. Hence we find $\psi_j \cdot f \in C_c^\infty(G/\overline{N};\chi)$. We now prove that $\psi_j \cdot f \to f$ in $\mathscr{C}(G/\overline{N};\chi)$.

Consider $X \in U(\mathfrak{g})$ of the form $X = X_1 \cdots X_n$ with $X_i \in \mathfrak{g}$. Since any element in $U(\mathfrak{g})$ is a finite linear combination of such terms it is enough to consider the seminorms $q_{N,X}$ with X of this form. For any subset $I \subset \{1,...,n\}$ we write $X_I = X_{i_1}\cdots X_{i_k}$ where $I = \{i_1 < \cdots < i_k\}$. Repeated use of the Leibniz rule gives that for any $f, g \in C^\infty(G)$ we have

$$L_X(f\cdot g) = L_{X_1\cdots X_n}(f\cdot g) = \sum_{I\subset\{1,\ldots,n\}} L_{X_I}f \cdot L_{X_{I^c}}g.$$

Using this we find

$$\begin{aligned} q_{N,X}(f-\psi_j f) &= \sup_{(k,a)\in K\times A} a^{-\rho}(1+\|\log a\|)^N \left|L_X((1-\psi_j)f)(ka)\right| \\ &\leq \sum_{I\subset\{1,\ldots,n\}} \sup_{(k,a)\in K\times A} a^{-\rho}(1+\|\log a\|)^N \left|L_{X_I}(1-\psi_j)(ka)\right| \left|L_{X_{I^c}}f(ka)\right| \\ &\leq \sum_{I\subset\{1,\ldots,n\}} \left[\sup_{(k,a)\in K\times A} (1+\|\log a\|)^{-1} \left|L_{X_I}(1-\psi_j)(ka)\right|\right] \\ &\quad\cdot \left[\sup_{(k,a)\in K\times A} a^{-\rho}(1+\|\log a\|)^{N+1} \left|L_{X_{I^c}}f(ka)\right|\right] \end{aligned}$$

We observe that for $a \in A$ such that $\|\log a\| < j$ we have $\psi_j \equiv 1$ on a neighbourhood around $ka\overline{n}$ (for any $k \in K, \overline{n} \in \overline{N}$) this gives that $L_Y(1-\psi_j)(ka) = 0$ for any $Y \in U(\mathfrak{g}), k \in K$

and $a \in A$ with $\|\log a\| < j$. Hence the first factor in the above estimate can be estimated by $(1+j)^{-1}\sup_{(k,a)\in K\times A}|L_{X_I}(1-\psi_j)(ka)|$. We obtain the estimate

$$q_{N,X}(f-\psi_j f) \leq (1+j)^{-1}\sum_{I\subset\{1,\dots,n\}}\left[\sup_{(k,a)\in K\times A}|L_{X_I}(1-\psi_j)(ka)|\right]\cdot q_{N+1,L_{X_{I^c}}}(f).$$

To finish the proof we need to show that for any $Y \in U(\mathfrak{g})$ the value of $\sup_{(k,a)\in K\times A}\|L_Y(1-\psi_j)(ka)\|$ is uniformly bounded in j.

Let $Y \in U(\mathfrak{g})$. For any $(k,a) \in K\times A$ we have, using that ψ_j is left K-invariant, that

$$L_Y\psi_j(ka) = L_{k^{-1}}(L_Y\psi_j)(a) = L_{\mathrm{Ad}(k)^{-1}X}(L_{k^{-1}}\psi_j)(a) = L_{\mathrm{Ad}(k)^{-1}X}\psi_j(a).$$

From the infinitesimal Iwasawa decomposition $\mathfrak{g} = \mathfrak{a}\oplus(\mathfrak{k}\oplus\overline{\mathfrak{n}})$ and by the Poincaré–Birkhoff–Witt theorem we know that we can decompose $U(\mathfrak{g})$ as $U(\mathfrak{g}) = U(\mathfrak{a})\oplus U(\mathfrak{g})(\mathfrak{k}\oplus\overline{\mathfrak{n}})$. We denote by $P\colon U(\mathfrak{g})\to U(\mathfrak{a})$ the projection along this decomposition. We observe that for any $W\in\mathfrak{k}$ we have $(L_W\psi_j)(a) = 0$ since ψ_j is left K-invariant. Furthermore, we have for any $Y\in\overline{\mathfrak{n}}$ that

$$(L_Y\psi_j)(a) = L_{a^{-1}}L_Y\psi_j(e) = R_{(\mathrm{Ad}(a)^{-1}Y)^\vee}(L_{a^{-1}}\psi_j)(e) = 0$$

since $\mathrm{Ad}(a)^{-1}Y \in \overline{\mathfrak{n}}$ and $L_{a^{-1}}\psi_j$ is right $\overline{N}$-invariant. Here we denote by $U\mapsto U^\vee$ the canonical anti-automorphism of $U(\mathfrak{g})$. From this it follows that for any $Z\in U(\mathfrak{g})$ we have $(L_Z\psi_j)(a) = (L_{P(Z)}\psi_j)(a)$. Hence for $Y\in U(\mathfrak{g})$ we have

$$\sup_{(k,a)\in K\times A}|L_Y\psi_j(ka)| = \sup_{(k,a)\in K\times A}\left|L_{\mathrm{Ad}(k)^{-1}Y}\psi_j(a)\right| = \sup_{(k,a)\in K\times A}\left|L_{P(\mathrm{Ad}(k)^{-1}Y)}\psi_j(a)\right|.$$

Looking at the definition of ψ_j it is clear that for $H\in\mathfrak{a}$ nonzero we have $(L_H\psi_j)(\exp x) = \frac{1}{j}(L_H\psi_1)(\exp(x/j))$ for all $x\in\mathfrak{a}$ and $j\in\mathbb{N}$. Hence $\sup_{a\in A}|L_H\psi_j(a)| = \frac{1}{j}\sup_{a\in A}|L_H\psi_1(a)|$. From this observation it follows that for any $W\in U(\mathfrak{a})$ we have $\sup_{a\in A}|L_W\psi_j(a)| \leq \sup_{a\in A}|L_W\psi_1(a)|$. Applying this observation for $W = P(\mathrm{Ad}(k)^{-1}Y)$ we obtain the estimate

$$\sup_{(k,a)\in K\times A}\left|L_{P(\mathrm{Ad}(k)^{-1}Y)}\psi_j(a)\right| \leq \sup_{(k,a)\in K\times A}\left|L_{P(\mathrm{Ad}(k)^{-1})Y}\psi_1(a)\right| = \sup_{(k,a)\in K\times A}|L_Y\psi_1(ka)|.$$

Finally we observe that because ψ_1 is compactly supported in $K\times A$ the right hand side of this inequality is finite for all $Y\in U(\mathfrak{g})$. This proves that $\sup_{(k,a)\in K\times A}|L_Y(1-\psi_j)(ka)|$ can be bounded uniformly in j. We conclude that indeed $\psi_j\cdot f\to f$ in $\mathscr{C}(G/\overline{N};\chi)$.

Since ψ_j is left K-invariant we see for every $j\in\mathbb{N}$ that $\psi_j\cdot f\in C_c^\infty(G/\overline{N};\chi)^K$ if $f\in\mathscr{C}(G/\overline{N};\chi)^K$. From this the second statement of the proposition follows. This concludes the proof. □

3.2 Whittaker–Fourier transformation

Let χ be a regular unitary character of $\overline{N}$. We define the Whittaker–Fourier transformation of a function $f\in\mathscr{C}(G/\overline{N};\chi)$ as a function on $i\mathfrak{a}^*$ given by

$$(\mathcal{F}_{\mathrm{wh}}f)(\lambda) = \int_{G/\overline{N}} f(g)\cdot W_\lambda(g)\,\mathrm{d}(g\overline{N}) \quad \text{for } \lambda\in i\mathfrak{a}^*.$$

Here W_λ denotes the Whittaker coefficient associated to χ as introduced in Section 2.6. The transformation behaviour of both f and W_λ imply that $f\cdot W_\lambda$ is right $\overline{N}$-invariant so we can indeed consider the integral of this quantity over $G/\overline{N}$.

A priori it is not clear whether this integral is convergent. In the next two propositions we will show that the Whittaker–Fourier transformation is well-defined for $f \in \mathscr{C}(G/\overline{N};\chi)$ and actually defines a continuous map $\mathcal{F}_{\mathrm{wh}}\colon \mathscr{C}(G/\overline{N};\chi) \to \mathscr{C}(i\mathfrak{a}^*)$. In this section we again restrict to the case $G = \mathrm{SL}(2,\mathbb{R})$.

Proposition 3.6. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. The map $\mathcal{F}_{\mathrm{wh}}$ maps $\mathscr{C}(G/\overline{N};\chi)$ continuously into $C^\infty(i\mathfrak{a}^*)$.

We equip $C^\infty(i\mathfrak{a}^*)$ with the topology of uniform convergence of all derivatives on compact sets.

Proof. Let $f \in \mathscr{C}(G/\overline{N};\chi)$. We consider the integral

$$\int_{G/\overline{N}} |f(g)| \left|\partial_\lambda^k W_\lambda(g)\right| \mathrm{d}(g\overline{N}).$$

We apply Proposition 1.37 but with the subgroup N replaced by $\overline{N}$. This corresponds to replacing Σ^+ by $-\Sigma^+$ as choice of positive roots hence we find that ρ is replaced by $\rho_{\overline{P}} = -\rho$. We obtain

$$\int_{G/\overline{N}} |f(g)| \left|\partial_\lambda^k W_\lambda(g)\right| \mathrm{d}(g\overline{N}) = \int_{K\times A} |f(ka)| \left|\partial_\lambda^k W_\lambda(a)\right| \cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a.$$

Using the results of Proposition 2.43 we see that for all $N \in \mathbb{N}$ this can be estimated by

$$\begin{aligned} &\leq q_{N,1}(f)\cdot \int_{K\times A} [a^\rho(1+\|\log a\|)^{-N}][a^\rho(1+\|\log a\|)^{k+1}(1+\|\lambda\|)]\cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a \\ &\leq q_{N,1}(f)(1+\|\lambda\|)\int_A (1+\|\log a\|)^{-N+k+1}\,\mathrm{d}a = C\cdot q_{N,1}(f)(1+\|\lambda\|). \end{aligned} \tag{3.1}$$

Here $C = \int_A (1+\|\log a\|)^{-N+k+1}\,\mathrm{d}a$ which is finite for N large enough. From this we conclude, using that the dominated convergence theorem, that $\mathcal{F}_{\mathrm{wh}} \in C^\infty(i\mathfrak{a}^*)$ and that for $k \in \mathbb{N}$ we have

$$\partial_\lambda^k(\mathcal{F}_{\mathrm{wh}} f)(\lambda) = \int_{G/\overline{N}} f(g)\cdot \partial_\lambda^k W_\lambda(g)\,\mathrm{d}(g\overline{N}).$$

From the estimate it also follows that on any compact in $i\mathfrak{a}^*$ the expression $\left|\partial_\lambda^k \mathcal{F}_{\mathrm{wh}} f(\lambda)\right|$ is bounded by $C' q_{N,1}(f)$ for some $C' > 0$ and $N \in \mathbb{N}$. We conclude that $\mathcal{F}_{\mathrm{wh}}$ does indeed map $\mathscr{C}(G/\overline{N};\chi)$ continuously into $C^\infty(i\mathfrak{a}^*)$. □

Proposition 3.7. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. The map $\mathcal{F}_{\mathrm{wh}}$ maps $\mathscr{C}(G/\overline{N};\chi)$ continuously into $\mathscr{C}(i\mathfrak{a}^*)$.

For the proof of this proposition we need the following observation.

Lemma 3.8. For $\phi \in C^\infty(G/\overline{N})$ and $\psi \in C_c^\infty(G/\overline{N})$ the following 'partial integration rule' holds

$$\int_{G/\overline{N}} (L_X\phi)(g)\cdot\psi(g)\,\mathrm{d}(g\overline{N}) = \int_{G/\overline{N}} \phi(g)\cdot(L_{X^\vee}\psi(g))\,\mathrm{d}(g\overline{N}).$$

Proof. Because the measure $d(g\overline{N})$ is left G-invariant we have

$$\int_{G/\overline{N}} (L_x\phi)(g)\cdot(L_x\psi)(g)\,\mathrm{d}(g\overline{N}) = \int_{G/\overline{N}} \phi(g)\cdot\psi(g)\,\mathrm{d}(g\overline{N})$$

for all $x \in G$. The function ψ is compactly supported in $G/\overline{N}$ hence we can differentiate with respect to x under the integral sign. If we set $x = \exp(tX)$, for $X \in \mathfrak{g}$, and take the derivative with respect to t we obtain

$$\int_{G/\overline{N}} (L_X\phi)(g) \cdot \psi(g)\, \mathrm{d}(g\overline{N}) = -\int_{G/\overline{N}} \phi(g) \cdot (L_X\psi(g))\, \mathrm{d}(g\overline{N}).$$

From this the lemma follows. □

Proof of Proposition 3.7. For this proof we will make use of the fact that the function W_λ satisfies the differential equation $L_\Omega W_\lambda = (\lambda(H)^2 - 1)W_\lambda$ (see Proposition 2.44). For simplicity we will denote $s_\lambda := \lambda(H)^2 - 1$.

For $\lambda \in i\mathfrak{a}^*$ we have $\lambda(H) \in i\mathbb{R}$ and using this we observe

$$|s_\lambda| = \left|\lambda(H)^2 - 1\right| = 1 - \lambda(H)^2 = 1 + |\lambda(H)|^2 = 1 + \|\lambda\|^2 .$$

From this it follows that $|s_\lambda| \geq 1$ and a constant $c > 0$ exists such that $(1 + \|\lambda\|) \leq c\,|s_\lambda|$ holds for all $\lambda \in i\mathfrak{a}^*$. We conclude that for any $k, N \in \mathbb{N}$ and $\phi \in C^\infty(i\mathfrak{a}^*)$ we have

$$\sup_{\lambda \in i\mathfrak{a}^*} (1 + \|\lambda\|)^N \left|\partial_\lambda^k \phi(\lambda)\right| \leq c^N \sup_{\lambda \in i\mathfrak{a}^*} \left|s_\lambda^N \cdot \partial_\lambda^k \phi(\lambda)\right| .$$

So in order to prove the result it suffices to show that for $f \in \mathscr{C}(G/\overline{N};\chi)$ the expression $\sup_{\lambda \in i\mathfrak{a}^*} \left|s_\lambda^N \partial_\lambda^k (\mathcal{F}_{\mathrm{wh}} f)(\lambda)\right|$ is bounded for all $k, N \in \mathbb{N}$.

First we make the following observation. Suppose $f \in C_c^\infty(G/\overline{N};\chi)$. For $k, N \in \mathbb{N}$ we have, using $L_\Omega W_\lambda = s_\lambda W_\lambda$, that

$$\begin{aligned}
\partial_\lambda^k (s_\lambda^N \mathcal{F}_{\mathrm{wh}} f)(\lambda) &= \int_{G/\overline{N}} f(g) \cdot \partial_\lambda^k (s_\lambda^N W_\lambda(g))\, \mathrm{d}(g\overline{N}) \\
&= \int_{G/\overline{N}} f(g) \cdot \partial_\lambda^k (L_\Omega^N W_\lambda(g))\, \mathrm{d}(g\overline{N}) \\
&= \int_{G/\overline{N}} f(g) \cdot L_\Omega^N (\partial_\lambda^k W_\lambda(g))\, \mathrm{d}(g\overline{N}).
\end{aligned}$$

Because f is compactly supported we can apply the 'partial integration rule' of Lemma 3.8. Using that $\Omega^\vee = \Omega$ we obtain

$$\partial_\lambda^k (s_\lambda^N \mathcal{F}_{\mathrm{wh}} f)(\lambda) = \int_{G/\overline{N}} (L_\Omega^N f)(g) \cdot \partial_\lambda^k W_\lambda(g)\, \mathrm{d}(g\overline{N}).$$

By applying the estimate in (3.1), that was obtained in the proof of Proposition 3.6, to the above integral we obtain the following

$$\left|\partial_\lambda^k (s_\lambda^N \mathcal{F}_{\mathrm{wh}} f)(\lambda)\right| \leq C \cdot q_{M,1}(L_\Omega^N f)(1 + \|\lambda\|) = C \cdot q_{M,\Omega^N}(f)(1 + \|\lambda\|). \tag{$*$}$$

With constants $C > 0$ and $M \in \mathbb{N}$ depending only on k and N. For $\lambda \in i\mathfrak{a}^*$ fixed both sides of this inequality depend continuously on f with respect to the Schwartz topology (see Proposition 3.6). Because $C_c^\infty(G/\overline{N};\chi)$ lies dense in $\mathscr{C}(G/\overline{N};\chi)$ we conclude that this estimate holds for all $f \in \mathscr{C}(G/\overline{N};\chi)$.

We now make the following claim.

Claim: For every $k, N \in \mathbb{N}$ there exists a continuous seminorm p on $\mathscr{C}(G/\overline{N};\chi)$ such that

$$\sup_{\lambda \in i\mathfrak{a}^*} \left|s_\lambda^N \partial_\lambda^k (\mathcal{F}_{\mathrm{wh}} f)(\lambda)\right| \leq p(f)$$

for all $f \in \mathscr{C}(G/\overline{N};\chi)$.

From the above discussion it follows that this claim immediately implies the statement of the proposition. In order to prove this claim we proceed by induction on k. For $k = 0$ we obtain from the above estimate $(*)$ that for any $N \in \mathbb{N}$

$$\left|s_\lambda^{N+1}\mathcal{F}_{\mathrm{wh}}f(\lambda)\right| \leq C \cdot q_{M,\Omega^{N+1}}(f) \cdot (1+\|\lambda\|)$$

Dividing by s_λ and using that $(1+\|\lambda\|)/\left|s_\lambda\right| \leq c$ shows that the claim indeed holds for $k = 0$. Now let $k > 0$ and assume our claim holds for all $k' < k$. We let $N \in \mathbb{N}$ arbitrary. We observe that

$$\partial_\lambda^k(s_\lambda^{N+1}\mathcal{F}_{\mathrm{wh}}f)(\lambda) = s_\lambda^{N+1} \cdot \partial_\lambda^k(\mathcal{F}_{\mathrm{wh}}f)(\lambda) + \sum_{i=1}^{k} \partial_\lambda^i(s_\lambda^{N+1}) \cdot \partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda).$$

From this we obtain the following estimate

$$\left|s_\lambda^{N+1}\partial_\lambda^k(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| \leq \left|\partial_\lambda^k(s_\lambda^{N+1}\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| + \sum_{i=1}^{k} \left|\partial_\lambda^i(s_\lambda^{N+1}) \cdot \partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right|.$$

For the first term on the right hand side we see from the estimate $(*)$ with $k = 0$ that a continuous seminorm p on $\mathscr{C}(G/\overline{N};\chi)$, depending N, exists such that

$$\left|\partial_\lambda^k(s_\lambda^{N+1}\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| \leq p(f) \cdot (1+\|\lambda\|).$$

In order to estimate the second term we observe, recalling that $s_\lambda = \lambda(H)^2 - 1$, that for $i = 1,...,k$ the expression $\partial_\lambda^i(s_\lambda^{N+1})$ is a polynomial in $\lambda(H)$ of degree at most $2N+1$. From this it follows that a $C' > 0$ exists such that $\left|\partial_\lambda^i(s_\lambda^{N+1})\right| \leq C'(1+\|\lambda\|^2)^{N+1} = C'\left|s_\lambda^{N+1}\right|$ for all $\lambda \in i\mathfrak{a}^*$. We obtain the bound

$$\sum_{i=1}^{k} \left|\partial_\lambda^i(s_\lambda^{N+1}) \cdot \partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| \leq C' \sum_{i=1}^{k} \left|s_\lambda^{N+1}\partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right|$$

Applying the induction hypothesis to the terms $\left|s_\lambda^{N+1}\partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right|$ yields that a continuous seminorm p' of $\mathscr{C}(G/\overline{N};\chi)$ exist such that

$$C' \sum_{i=1}^{k} \left|s_\lambda^{N+1}\partial_\lambda^{k-i}(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| \leq p'(f)$$

for all $f \in \mathscr{C}(G/\overline{N};\chi)$. Combining the estimates for both terms yields

$$\left|s_\lambda^{N+1}\partial_\lambda^k(\mathcal{F}_{\mathrm{wh}}f)(\lambda)\right| \leq p(f) \cdot (1+\|\lambda\|) + p'(f) = [p(f)+p'(f)](1+\|\lambda\|).$$

Dividing by $|s_\lambda|$ and again using that $(1+\|\lambda\|)/\left|s_\lambda\right| \leq c$ show that the claim holds true for k. We conclude that the claim holds for all $k \in \mathbb{N}$. This finishes the proof. □

3.3 Harish-Chandra transformation

In this section we introduce what is called the Harish-Chandra transformation. We return to the general case that G is a connected semisimple Lie group with finite center. We let χ be a, not necessarily regular, character of $\overline{N}$. For $f \in \mathscr{C}(G/\overline{N};\chi)$ we define the Harish-Chandra transformation of f as a function on MA given by

$$(\mathcal{H}f)(ma) := a^\rho \int_N f(man)\,\mathrm{d}n \quad \text{for } ma \in MA.$$

A priori it is not clear that this integral converges for all $ma \in MA$. Our first objective is to show that this is the case and that the function defined in this way is in fact smooth on MA. The proofs presented for the following two lemmas are based on [Wal92, Section 15.3.2].

Lemma 3.9. The map $\mathcal{H}$ maps $\mathscr{C}(G/\overline{N};\chi)$ continuously into $C^\infty(MA)$.

Proof. Again we denote $a_{\overline{P}}\colon G \to A$ the Iwasawa projection associated to the decomposition $G = KA\overline{N}$. We let $f \in \mathscr{C}(G/\overline{N};\chi)$ and $X \in U(\mathfrak{m}\oplus\mathfrak{a})$ arbitrary. We observe that for $ma \in MA$ and $N \in \mathbb{N}$ we have

$$a^\rho \int_N |L_X f(man)|\,\mathrm{d}n \le q_{N,X}(f)a^\rho \int_N (a_{\overline{P}}(man))^\rho(1+\big\|\log a_{\overline{P}}(man)\big\|)^{-N}\,\mathrm{d}n.$$

Since $m \in K$ we have $a_{\overline{P}}(man) = a_{\overline{P}}(an)$. Furthermore, we have $a_{\overline{P}}(an) = a_{\overline{P}}(C_a(n))a$. Substituting this yields that the the above integral equals

$$q_{N,X}(f)\cdot a^{2\rho}\int_N a_{\overline{P}}(C_a(n))^\rho(1+\big\|\log a_{\overline{P}}(C_a(n)) + \log a\big\|)^{-N}\,\mathrm{d}n.$$

By applying the substitution of variables $ana^{-1} \mapsto n$ (see Proposition 1.39) we see this equals

$$\begin{aligned} &q_{N,X}(f)\int_N a_{\overline{P}}(n)^\rho(1+\big\|\log a_{\overline{P}}(n)+\log a\big\|)^{-N}\,\mathrm{d}n \\ \le &q_{N,X}(f)(1+\|\log a\|)^N \int_N a_{\overline{P}}(n)^\rho(1+\big\|\log a_{\overline{P}}\big\|)^{-N}\,\mathrm{d}n. \end{aligned}$$

Here we used the inequality $(1+\|x+y\|)^{-N} \le (1+\|x\|)^N(1+\|y\|)^{-N}$ for all $x, y \in \mathfrak{a}$ to obtain the estimate. In Lemma 3.10 below we prove that the integral under consideration

$$\int_N a_{\overline{P}}(n)^\rho(1+\big\|\log a_{\overline{P}}(n)\big\|)^{-N}\,\mathrm{d}n$$

is finite for N large enough. We conclude that a constant $C > 0$ exists such that

$$\int_N |L_X f(man)|\,\mathrm{d}n \le C(1+\|\log a\|)^N q_{N,X}(f). \tag{3.2}$$

Since $X \in U(\mathfrak{m}\oplus\mathfrak{a})$ was arbitrary it follows from the dominated convergence theorem that $ma \mapsto \int_N f(man)$ defines a smooth function on MA. The factor $a^\rho(1+\|\log a\|))^N$ is bounded on compacts subsets of MA. Hence for each compact subset a $C' > 0$ exists such that on this compact set $|(L_X\mathcal{H}f)(ma)| \le C'q_{N,X}(f)$ for all $f \in \mathscr{C}(G/\overline{N};\chi)$. We conclude that $\mathcal{H}\colon \mathscr{C}(G/\overline{N};\chi) \to C^\infty(MA)$ is indeed continuous. $\square$

Lemma 3.10. For $d > 0$ sufficiently large the integral

$$\int_N a_{\overline{P}}(n)^\rho (1 + \left\|\log a_{\overline{P}}(n)\right\|)^{-d} \, \mathrm{d}n$$

is finite.

Proof. We derive this fact from a result that was originally obtained by Harish-Chandra. In [HC58] it is proved that the integral

$$\int_{\overline{N}} e^{-\rho(H(\overline{n}))} (1 + \rho(H(\overline{n})))^{-d} \, \mathrm{d}\overline{n}$$

is finite for $d > 0$ sufficiently large. A straightforward computation reveals that $a_{\overline{P}}(g) = a_P(\theta g)^{-1}$ for $g \in G$. Using this we can write the above integral as

$$\int_{\overline{N}} e^{-\rho(H(\overline{n}))} (1 + \rho(H(\overline{n})))^{-d} \, \mathrm{d}\overline{n} = \int_{\overline{N}} (a_{\overline{P}}(\theta\overline{n}))^\rho (1 - \rho(\log a_{\overline{P}}(\theta\overline{n})))^{-d} \, \mathrm{d}\overline{n}.$$

The Cartan involution θ is an isomorphism between N and $\overline{N}$ hence $\theta^* \, \mathrm{d}\overline{n} = \mathrm{d}n$. Using this we make the substitution of variables $\theta\overline{n} \mapsto n$ and obtain that the above integral equals

$$\int_N (a_{\overline{P}}(n))^\rho (1 - \rho(\log a_{\overline{P}}(n)))^{-d} \, \mathrm{d}n.$$

On $\mathfrak{a}$ we define the norm

$$\|H\|_\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha) \, |\alpha(H)| \, .$$

For $n \in N$ and $\alpha \in \Sigma^+$ we have $\alpha(\log a_{\overline{P}}(n)) = -\alpha(H(\theta n)) < 0$ (the inequality follows from Lemma 1.16). Per definition of the norm $\|\cdot\|_\rho$ we now have

$$-\rho(\log a_{\overline{P}}(n)) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)[-\alpha(\log a_{\overline{P}}(n))] = \left\|\log a_{\overline{P}}(n)\right\|_\rho .$$

Since $\mathfrak{a}$ is finite-dimensional the norms $\|\cdot\|$ and $\|\cdot\|_\rho$ are equivalent. We conclude that a constant $c > 0$ exists such that

$$a_{\overline{P}}(n)^\rho (1 + \left\|\log a_{\overline{P}}(n)\right\|)^{-d} \leq c \cdot (a_{\overline{P}}(n))^\rho (1 + \left\|\log a_{\overline{P}}(n)\right\|_\rho)^{-d} = c \cdot (a_{\overline{P}}(n))^\rho (1 - \rho(\log a_{\overline{P}}(n)))^{-d}.$$

Integrating both sides over N yields

$$\int_N a_{\overline{P}}(n)^\rho (1 + \left\|\log a_{\overline{P}}(n)\right\|)^{-d} \, \mathrm{d}n \leq \int_N (a_{\overline{P}}(n))^\rho (1 - \rho(\log a_{\overline{P}}(n)))^{-d} \, \mathrm{d}n < \infty.$$

$\square$

We have now established that $\mathcal{H}$ maps $\mathscr{C}(G/\overline{N};\chi)$ into $C^\infty(MA)$. As a result $f \mapsto \mathcal{H}f|_A$ maps into $C^\infty(A)$. We may ask ourselves whether the Harish-Chandra transformation actually maps into $\mathscr{C}(A)$. It is claimed in Lemma 15.3.2 of [Wal92] that this is indeed the case. However it turns out that an amendment must be made to the statement of this lemma. For the result to hold we must assume χ to be regular. The fact that this lemma did not hold for χ non-regular was first pointed out in [vdBK]. In Remark 3.3.1 we confirm that this is the case by constructing a counterexample. Recently it was shown in a preprint by Wallach (see [Wal]) that the lemma, with the amendment that χ is regular, does hold. It must be noted that this thesis project was started before the publication of this preprint. By different means than used by Wallach we will prove the following partial result for $\mathrm{SL}(2,\mathbb{R})$.

Proposition 3.11. Suppose $G = \mathrm{SL}(2,\mathbb{R})$ and that χ is regular. The Harish-Chandra transformation $\mathcal{H}$ maps $\mathscr{C}(G/\overline{N};\chi)^K$, the left K-invariant elements of $\mathscr{C}(G/\overline{N};\chi)$, continuously into $\mathscr{C}(A)$.

3.3.1 Counterexample

In this section we illustrate that the Harish-Chandra transformation does not necessarily map $\mathscr{C}(G/\overline{N};\chi)$ into $\mathscr{C}(A)$ if the character χ is not regular.

We consider the case $G = \mathrm{SL}(2,\mathbb{R})$ and chose χ to be the only non-regular character of $\overline{N}$, i.e. $\chi \equiv 1$. We let ψ be a compactly supported smooth function on $\mathfrak{a}$ such that $\psi \geq 0$ and $\psi \equiv 1$ on the subset $[-1,1]\cdot H \subset \mathfrak{a}$. Then if we set $f(ka\overline{n}) = \psi(\log a)$ we easily see that $f \in C_c^\infty(G/\overline{N};\chi) \subset \mathscr{C}(G/\overline{N};\chi)$. For $a \in A$ we observe, using Proposition 1.39, that

$$\begin{aligned}(\mathcal{H}f)(a) &= a^\rho \int_N f(an)\,\mathrm{d}n = a^\rho \int_N f((ana^{-1})a)\,\mathrm{d}n \\ &= a^{-\rho} \int_N f(na)\,\mathrm{d}n = a^{-\rho}\int_N \psi(\log a_{\overline{P}}(n) + \log a)\,\mathrm{d}n.\end{aligned}$$

Since $\psi \geq 0$ and $\psi|_{[-1,1]\cdot H} \equiv 1$ we see that this expression can be bounded from below as follows

$$(\mathcal{H}f)(a) \geq a^{-\rho}\int_N \mathbb{1}_{[-1,1]\cdot H}(\log a_{\overline{P}}(n) + \log a)\,\mathrm{d}n = a^{-\rho}\cdot \mathrm{Vol}(R_a)$$

with $R_a := \{n \in N \mid -1 \leq \rho(\log a_{\overline{P}}(n) + \log a) \leq 1\}$. If we write $a = \exp(tH)$ and $n = n_x$ for $t, x \in \mathbb{R}$ we see

$$\begin{aligned}& -1 \leq \rho(\log a_{\overline{P}}(n) + \log a) \leq 1 \Longleftrightarrow -1 \leq -\frac{1}{2}\log(1+x^2) + t \leq 1 \\ \Longleftrightarrow & -2(1+t) \leq \log(1+x^2) \leq 2(1+t) \Longleftrightarrow e^{-2(1+t)} - 1 \leq x^2 \leq e^{2(1+t)} - 1.\end{aligned}$$

For $t \geq 1$ we have $e^{-2(1+t)} - 1 \leq 0$ and $e^{2(1+t)} - 1 \geq 0$ hence $R_a = \{n_x \mid 0 \leq x \leq (e^{2(1+t)} - 1)^{1/2}\}$ for $a = \exp(tH)$ with $t \geq 1$. We see that $\mathrm{Vol}(R_a) = (e^{2(1+t)} - 1)^{1/2}$ for such a. It now follows that

$$\mathcal{H}f(a) \geq a^{-\rho}\cdot \mathrm{Vol}(R_a) = e^{-t}(e^{2(1+t)} - 1)^{1/2} \xrightarrow{t\to\infty} e.$$

From this we conclude that $\lim_{a\to\infty, A^+}(\mathcal{H}f)(a) \neq 0$ so we have in particular that $\mathcal{H}f \notin \mathscr{C}(A)$.

We note that in the construction of this counterexample the assumption that $\chi = 1$ was vital. In order to obtain a lower bound for the expression $\mathcal{H}f$ we used that χ does not oscillate and is everywhere positive. This also sheds some light on why $\mathcal{H}$ might map into $\mathscr{C}(A)$ if χ is regular, since in this case oscillating behaviour of χ will average out contributions from different parts of the integral resulting in better behaviour of $\mathcal{H}f(a)$ in the variable a.

3.3.2 Proof of Proposition 3.11

In this section we will show that if $G = \mathrm{SL}(2,\mathbb{R})$ and χ is regular then the Harish-Chandra transformation maps $\mathscr{C}(G/\overline{N};\chi)^K$ into $\mathscr{C}(A)$. We will do this by exhibiting the map $\mathcal{H}$ as a composition of the Fourier transformation and the Whittaker–Fourier transformation. From our knowledge that both these functions do map Schwartz functions to Schwartz functions we will be able to conclude our result.

In this section we assume that χ is regular. Our first step will be to investigate the behaviour of the function $\mathcal{H}f$ when $f \in C_c^\infty(G/\overline{N};\chi)$. For this discussion the assumption $G = \mathrm{SL}(2,\mathbb{R})$ is not yet necessary so for the sake of generality we assume that G is a connected semisimple Lie group with finite center. We will specialize to $\mathrm{SL}(2,\mathbb{R})$ later. We do however assume that χ is a regular unitary character.

The set $\Delta = \{\alpha_1, ..., \alpha_n\}$ of simple roots is a basis of $\mathfrak{a}^*$. In this section we denote by $(\beta_1, ..., \beta_n)$ the dual basis of $\mathfrak{a}^*$ relative to $\langle\cdot,\cdot\rangle$. Furthermore, we denote for $r \in \mathbb{R}$,

$$A_r := \{a \in A \mid \beta_i(\log a) \geq r \quad \text{for all } i = 1, ..., n\}.$$

Lemma 3.12. Suppose $C \subset G/\overline{N}$ is compact. Then there exists a constant $r \in \mathbb{R}$ such that all $f \in C_C^\infty(G/\overline{N};\chi)$ satisfy $\operatorname{supp}\mathcal{H}f \subset MA_r$.

Proof. Since $|f| \in C^\infty(G/\overline{N})$ is supported in C there exists a number $R > 0$ (depending only on C) such that $\operatorname{supp} f \subset K \times \exp(B^{\mathfrak{a}}(0;R)) \times \overline{N}$. Here $B^{\mathfrak{a}}(0;R)$ denotes the open ball in $\mathfrak{a}$ of radius R with respect to the norm induced by the Cartan inner product. Hence $\log a_{\overline{P}}(g) \notin B^{\mathfrak{a}}(0;R)$ implies $f(g) = 0$. Because the set $B^{\mathfrak{a}}(0;R)$ is bounded there exists a constant $r \in \mathbb{R}$ such that $r < \beta_i(H)$ for all $H \in B^{\mathfrak{a}}(0;R)$ and $i = 1,...,n$.

Now if $ma \in MA \setminus MA_r$ then $\beta_i(\log a) < r$ for some $i \in \{1,...,n\}$. From the identity $\log a_{\overline{P}}(n) = -H(\theta n)$ for $n \in N$ and Lemma 1.16 it follows that

$$\log a_{\overline{P}}(N) \subset -\sum_{\alpha\in\Sigma^+} \mathbb{R}_{\geq 0} H_\alpha.$$

The element $\beta_i \in \mathfrak{a}^*$ is such that $\beta_i(H_{\alpha_j}) = \langle \beta_i, \alpha_j\rangle = \delta_{ij}$ for $j = 1,...,n$ hence $\beta_i(\log a_{\overline{P}}(n)) \leq 0$ for all $n \in N$. Now we observe that for any $n \in N$,

$$\log a_{\overline{P}}(man) = \log a_{\overline{P}}(C_a(n)a) = \log a_{\overline{P}}(C_a(n)) + \log a.$$

Because $C_a(n) \in N$ we have $\beta_i(\log a_{\overline{P}}(C_a(n))) \leq 0$. Combining this with $\beta_i(\log a) < r$ gives

$$\beta_i(\log a_{\overline{P}}(man)) = \beta_i(\log a) + \beta_i(\log a_{\overline{P}}(C_a(n))) < r.$$

Since every $H \in B^{\mathfrak{a}}(0;R)$ satisfies $\beta_i(H) > r$ we conclude $\log a_{\overline{P}}(man) \notin B^{\mathfrak{a}}(0;R)$. From this it follows that if $ma \in MA \setminus MA_r$ then $f(man) = 0$ for all $n \in N$ hence

$$\mathcal{H}f(man) = a^\rho \int_N f(man)\,\mathrm{d}n = 0.$$

We conclude that indeed $\operatorname{supp}\mathcal{H}f \subset MA_r$. □

Lemma 3.13. Suppose $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is such that $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. Then for $f \in C_c^\infty(G/\overline{N};\chi)$ we have $a^{-\lambda}\mathcal{H}f \in \mathscr{C}(A)$.

Here we used the shorthand $a^{-\lambda}$ for the function $a \mapsto a^{-\lambda}$ on A.

Proof. We fix an $f \in C_c^\infty(G/\overline{N};\chi)$. We want to show that for every $X \in U(\mathfrak{a})$ and $N \in \mathbb{N}$ we have $q_{N,X}(a^{-\lambda}\mathcal{H}f) < \infty$. Using the Leibniz rule and the fact that $L_H a^{-\lambda} = -\lambda(H)a^{-\lambda}$ for all $H \in \mathfrak{a}$ we see that $L_X(a^{-\lambda}\mathcal{H}f)$ is a finite linear combination of terms of the form $a^{-\lambda}L_Y(\mathcal{H}f)$ with $Y \in U(\mathfrak{a})$. Hence we find it suffices to show that $\sup_{a\in A}(1 + \|\log a\|)^N \left|a^{-\lambda}L_Y(\mathcal{H}f)\right| < \infty$ for all $N \in \mathbb{N}$.

From the estimate in (3.2), which was obtained in the proof of Lemma 3.9, we see that constants $C > 0$ and $N \in \mathbb{N}$ exists such that

$$|L_Y\mathcal{H}f(a)| = |(\mathcal{H}L_Y f)(a)| \leq C(1 + \|\log a\|)^N.$$

From Lemma 3.12 we know that a constant $r \in \mathbb{R}$ exists such that $\operatorname{supp}\mathcal{H}f \subset A_r$. Combining this we see that

$$\sup_{a\in A}(1 + \|\log a\|)^N \left|a^{-\lambda}L_Y(\mathcal{H}f)\right| \leq C \sup_{a\in A_r}(1 + \|\log a\|)^{N+d} a^{-\operatorname{Re}\lambda}.$$

We will prove that the expression on the right hand side is finite. For convenience we set $M = N + d$ and $\mathfrak{a}_r := \{H \in \mathfrak{a} \mid \beta_i(H) \geq r \quad \text{for all } i = 1,...,n\}$ (so that $A_r = \exp\mathfrak{a}_r$).

We retain the enumeration $\Delta = \{\alpha_1, ..., \alpha_n\}$ of simple roots introduced in the above discussion. If we set $H_i := H_{\alpha_i}$ then the set $(H_1, ..., H_n)$ is a basis of $\mathfrak{a}$. We denote by $(x^1, ..., x^n)$ the coordinates on $\mathfrak{a}$ induced by this basis. It is readily verified that if $H = x^1 H_1 + \cdots + x^n H_n$ then $\beta_i(H) = x^i$ for $i = 1, ..., n$. From this it follows that if $H \in \mathfrak{a}_r$ then $x^i \geq r$ for all $i = 1, ..., n$. We set $c_i := \langle \operatorname{Re}\lambda, \alpha_i \rangle > 0$. Using this notation we have $\operatorname{Re}\lambda(H) = \sum_i x_i \operatorname{Re}\lambda(H_i) = \sum_i x_i \langle \operatorname{Re}\lambda, \alpha_i \rangle = \sum_i c_i \cdot x^i$.

For $x \in \mathbb{R}$ with $x \geq r$ we have

$$x = (x - r) + r = |x - r| + r \geq |x| - |r| + r \geq |x| - 2|r|.$$

Now for $H = \sum_i x^i H_i \in \mathfrak{a}_r$ (hence $x^i \geq r$) we find the following

$$\operatorname{Re}\lambda(H) = c_1 x^1 + \cdots + c_n x^n \geq c_1 \left|x^1\right| + \cdots + c_n |x^n| - 2(c_1 + \cdots + c_n)|r|.$$

Setting $C = 2(c_1 + \cdots + c_n)|r|$ gives

$$\operatorname{Re}\lambda(H) \geq (\min_i c_i)(\left|x^1\right| + \cdots + |x^n|) - C.$$

On $\mathfrak{a}$ we define the norm $\left\|\sum_i y^i H_i\right\|_1 := \sum_i \left|y^i\right|$, so we have

$$\operatorname{Re}\lambda(H) \geq (\min_i c_i)\|H\|_1 - C.$$

Since $\mathfrak{a}$ is finite-dimensional the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent. Hence there exists a $\delta > 0$ such that

$$\operatorname{Re}\lambda(H) \geq \delta\|H\| - C.$$

Using this estimate we now find

$$\begin{aligned}
\sup_{a \in A_r} (1 + \|\log a\|)^M a^{-\operatorname{Re}\lambda} &= \sup_{H \in \mathfrak{a}_r} (1 + \|H\|)^M e^{-\operatorname{Re}\lambda(H)} \\
&\leq \sup_{H \in \mathfrak{a}_r} (1 + \|H\|)^M e^{-\delta\|H\| + C} \\
&= e^C \sup_{H \in \mathfrak{a}_r} (1 + \|H\|)^M e^{-\delta\|H\|} < \infty.
\end{aligned}$$

This concludes the proof. □

In Section 3.2 we showed that for $f \in \mathscr{C}(G/\overline{N}; \chi)$ the Whittaker–Fourier transformation $\mathcal{F}_{\mathrm{wh}} f$ is defined on $i\mathfrak{a}^*$ and is of Schwartz type as a function on $i\mathfrak{a}^*$. For this we used various estimates on the growth behaviour of W_λ on A. If we instead take f to be compactly supported, i.e. $f \in C_c^\infty(G/\overline{N}; \chi)$ then it turns out that the expression $\mathcal{F}_{\mathrm{wh}} f$ can be interpreted on the whole of $\mathfrak{a}_{\mathbb{C}}^*$. Furthermore, by exploiting that the support of $f \cdot W_\lambda$ is compact in $G/\overline{N}$ no assumptions on the behaviour of W_λ are needed.

Lemma 3.14. The Whittaker–Fourier transformation $\mathcal{F}_{\mathrm{wh}}$ maps $C_c^\infty(G/\overline{N}; \chi)$ into $\mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)$, the space of holomorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$.

Proof. Let $f \in C_c^\infty(G/\overline{N}; \chi)$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we consider the expression

$$(\mathcal{F}_{\mathrm{wh}} f)(\lambda) = \int_{G/\overline{N}} f(g) W_\lambda(g) \, \mathrm{d}(g\overline{N}).$$

Because $|f|$ is compactly supported in $G/\overline{N}$ this integral is absolutely convergent for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Furthermore, we can differentiate with respect to λ under the integral sign. Because, for $g \in G$ fixed, $W_\lambda(g)$ depends on λ in a holomorphic fashion we conclude that $\mathcal{F}_{\mathrm{wh}} f$ defines a holomorphic function on $\mathfrak{a}_{\mathbb{C}}^*$. □

We define $\mathcal{F}\colon L^1(A) \to C^0(i\mathfrak{a}^*)$, an analogue of the classical Fourier transformation, as follows

$$(\mathcal{F}f)(\lambda) = \int_A a^{-\lambda} f(a)\,\mathrm{d}a.$$

Identifying $A \cong \mathfrak{a} \cong \mathbb{R}^n$ and $i\mathfrak{a}^* \cong \mathbb{R}^n$ we see that this map corresponds to the familiar Fourier transformation map $\mathcal{F}\colon L^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$. Keeping this in mind we easily see that $\mathcal{F}$ as defined above restricts to an isomorphism $\mathcal{F}\colon \mathscr{C}(A) \to \mathscr{C}(i\mathfrak{a}^*)$. Its inverse is given by

$$(\mathcal{F}^{-1}f)(a) = \int_{i\mathfrak{a}^*} a^{\lambda} f(a)\,\mathrm{d}\lambda.$$

The next step toward the proof of Proposition 3.11 is to show the following identity

$$(\mathcal{F}\mathcal{H}f)(\lambda) = (\mathcal{F}_{\mathrm{wh}}f)(\lambda) \tag{3.3}$$

holds for $f \in C_c^\infty(G/\overline{N};\chi)^K$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$.

From now on we must again assume that $G = \mathrm{SL}(2,\mathbb{R})$. The reader should take note however that throughout the following arguments this assumption is only used when properties of the Whittaker coefficient (and by extension, properties of $\mathcal{F}_{\mathrm{wh}}$) are used. Most calculations however go through (at least formally) for general G.

Lemma 3.15. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. For all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ and $f \in C_c^\infty(G/\overline{N};\chi)^K$ the identity (3.3) holds.

Proof. We fix λ and f as in the statement of the lemma. From Lemma 3.13 we know that $a^{-\lambda}\mathcal{H}f \in \mathscr{C}(A)$. So in particular we have $a^{-\lambda}\mathcal{H}f \in L^1(A)$. Using that f is left K-invariant we see that $a^{-\lambda}\mathcal{H}(ma) = a^{-\lambda}\mathcal{H}(a)$ so the expression

$$ma \mapsto a^{-\lambda+\rho} \int_N f(man)\,\mathrm{d}n$$

defines an element of $L^1(MA)$. From Fubini's theorem it now follows that $a^{-\lambda+\rho}f \in L^1(MAN)$.

Now we observe, using the left K-invariance of f, that

$$\begin{aligned}(\mathcal{F}\mathcal{H}f)(\lambda) &= \int_A a^{-\lambda}(\mathcal{H}f)(a)\,\mathrm{d}a = \int_{AN} a^{-\lambda+\rho} f(an)\,\mathrm{d}a\,\mathrm{d}n \\ &= \int_{MAN} a^{-\lambda+\rho} f(man)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n.\end{aligned}$$

We note that this integral converges absolutely since $a^{-\lambda+\rho}f \in L^1(MAN)$. We define the measureable function $\psi\colon G \to \mathbb{C}$ as $\psi(man\overline{n}) = a^{-\lambda+\rho}\chi(\overline{n})^{-1}$ on the big Bruhat cell $MAN\overline{N}$ and set it zero outside (recall that the complement of $MAN\overline{N}$ is of measure zero in G). We define Φ, a measurable function on G, as $\Phi = \psi \cdot f$. We observe that $\Phi(man) = \psi(man)f(man) = a^{-\lambda+\rho}f(man)$ on MAN so we can write

$$(\mathcal{F}\mathcal{H}f)(\lambda) = \int_{MAN} \Phi(man)\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n.$$

We have $\Phi(g\overline{n}) = \Phi(g)$ for all $g \in G$ and $\overline{n} \in \overline{N}$. So applying Proposition 1.38 and then Proposition 1.37 yields

$$(\mathcal{F}\mathcal{H}f)(\lambda) = \int_{G/\overline{N}} \Phi(g)\,\mathrm{d}(g\overline{N}) = \int_{K\times A} a^{-2\rho}\Phi(ka)\,\mathrm{d}k\,\mathrm{d}a.$$

Per assumption we have that f is left K-invariant hence

$$(\mathcal{F}\mathcal{H}f)(\lambda) = \int_{K\times A} \psi(ka)f(ka)\cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a = \int_A \left[\int_K \psi(ka)\,\mathrm{d}k\right] f(a)\cdot a^{-2\rho}\,\mathrm{d}a$$

We now investigate the integral $\int_K \psi(ka)\,\mathrm{d}k$. We temporarily make the assumption that $\langle \operatorname{Re}\lambda - \rho, \alpha\rangle > 0$. In this case the function $j_\lambda = j(P:1:-\overline{\lambda})(1)$ is given by $j_\lambda(\overline{n}man) = \chi(\overline{n})^{-1}a^{\overline{\lambda}-\rho}$ on $\overline{N}MAN$ (see Section 2.4). For any $x = man\overline{n}$ in $MAN\overline{N}$ we observe

$$\psi(x) = \psi(man\overline{n}) = a^{-\lambda+\rho}\chi(\overline{n})^{-1} = \overline{j_\lambda(\overline{n}^{-1}m^{-1}a^{-1}n^{-1})} = \overline{j_\lambda(x^{-1})}.$$

Since the complement of $\overline{N}P$ in G has measure zero we conclude that $\psi(x) = \overline{j_\lambda(x^{-1})}$ holds almost everywhere. Substituting this yields

$$\int_K \psi(ka)\,\mathrm{d}k = \int_K \overline{j_\lambda(a^{-1}k^{-1})}\,\mathrm{d}k.$$

We note that K is unimodular so $\mathrm{d}(k^{-1}) = \mathrm{d}k$. The substitution of variables $k \mapsto k^{-1}$ yields that this integral is equal to

$$\int_K \overline{j_\lambda(a^{-1}k)}\,\mathrm{d}k = \left\langle \mathbb{1}_\lambda, \pi_{1,-\overline{\lambda}}(a)j_\lambda\right\rangle = W_\lambda(a).$$

We conclude that $\int_K \psi(ka)\,\mathrm{d}k$ equals the integral expression for W_λ as derived in Section 2.6.1. In the discussion in this section it is observed that this integral expression for W_λ, although initially only defined for $\langle \operatorname{Re}\lambda - \rho, \alpha\rangle > 0$, actually holds for all $\lambda \in \mathfrak{a}^*_\mathbb{C}$ with $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$. From this we conclude that

$$\int_K \psi(ka)\,\mathrm{d}k = W_\lambda(a)$$

holds for all $\lambda \in \mathfrak{a}^*_\mathbb{C}$ with $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$.

We now substitute this in our expression for $\mathcal{F}\mathcal{H}f$. We use that both W_λ and f are left K-invariant to find

$$\begin{aligned}\int_A \left[\int_K \psi(ka)\,\mathrm{d}k\right] f(a)\cdot a^{-2\rho}\,\mathrm{d}a &= \int_A W_\lambda(a)f(a)\cdot a^{-2\rho}\,\mathrm{d}a\\ = \int_{K\times A} W_\lambda(ka)f(ka)\cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a &= \int_{G/\overline{N}} W_\lambda(g)f(g)\,\mathrm{d}(g\overline{N}) = (\mathcal{F}_{\mathrm{wh}}f)(\lambda).\end{aligned}$$

□

The assumption that $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$ is crucially used in the proof of this lemma to ensure that $a^{-\lambda+\rho}f \in L^1(MAN)$. Hence we can not use the techniques used in this proof to show that the identity (3.3) holds for $\lambda \in i\mathfrak{a}^*$. We can however, using an approximation argument, show that $\mathcal{H}f = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}f$ holds.

Lemma 3.16. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. The identity

$$\mathcal{H}f = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}f$$

holds for all $f \in C_c^\infty(G/\overline{N};\chi)^K$.

Proof. We let $f \in C_c^\infty(G/\overline{N};\chi)$ and let $0 < \epsilon < 1/2$ arbitrary. Then by Lemma 3.15 we have for all $\lambda \in i\mathfrak{a}^*$ that
$$(\mathcal{F}\mathcal{H}f)(\lambda + \epsilon\rho) = (\mathcal{F}_{\mathrm{wh}}f)(\lambda + \epsilon\rho). \tag{$*$}$$
In light of Lemma 3.13 we have that $a^{-\epsilon\rho}\mathcal{H}f \in \mathscr{C}(A)$. Using this it is readily seen that
$$(\mathcal{F}\mathcal{H}f)(\lambda + \epsilon\rho) = \mathcal{F}(a^{-\epsilon\rho}\mathcal{H}f)(\lambda).$$
If we set $\phi_\epsilon \colon i\mathfrak{a}^* \to \mathbb{C} \colon \lambda \mapsto (\mathcal{F}_{\mathrm{wh}}f)(\lambda+\epsilon\rho)$ then $(*)$ implies that $\phi_\epsilon = \mathcal{F}(a^{-\epsilon\rho}\mathcal{F}f)$ on $i\mathfrak{a}^*$. Because $\mathcal{F}$ maps $\mathscr{C}(A)$ into $\mathscr{C}(i\mathfrak{a}^*)$ we find $\phi_\epsilon = \mathcal{F}(a^{-\epsilon\rho}\mathcal{H}f) \in \mathscr{C}(i\mathfrak{a}^*)$. We can now apply the Fourier inverse to this to find
$$a^{-\epsilon\rho}\mathcal{H}f(a) = (\mathcal{F}^{-1}\phi_\epsilon)(a) \quad \text{for all } a \in A.$$
We now fix an $a \in A$. It is clear that $a^{-\epsilon\rho}\mathcal{H}f(a) \xrightarrow{\epsilon\downarrow 0} \mathcal{H}f(a)$. So in order to finish the proof it is enough to show $(\mathcal{F}^{-1}\phi_\epsilon)(a) \xrightarrow{\epsilon\downarrow 0} (\mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}f)(a)$.

We observe that
$$(\mathcal{F}^{-1}\phi_\epsilon)(a) = \int_{i\mathfrak{a}^*} a^\lambda \phi_\epsilon(\lambda)\,\mathrm{d}\lambda = \int_{i\mathfrak{a}^*} a^\lambda (\mathcal{F}_{\mathrm{wh}}f)(\lambda+\epsilon\rho)\,\mathrm{d}\lambda = a^{-\epsilon\rho}\int_{i\mathfrak{a}^*} a^{\lambda+\epsilon\rho}(\mathcal{F}_{\mathrm{wh}}f)(\lambda+\epsilon\rho)\,\mathrm{d}\lambda$$
We consider the integral
$$\int_{i\mathfrak{a}^*} a^{\lambda+\epsilon\rho}(\mathcal{F}_{\mathrm{wh}}f)(\lambda+\epsilon\rho)\,\mathrm{d}\lambda = \int_{i\mathfrak{a}^*+\epsilon\rho} a^{\mu}(\mathcal{F}_{\mathrm{wh}}f)(\mu)\,\mathrm{d}\mu.$$
We will show that the above integral is independent of ϵ using the Cauchy integral formula. For this we note that because $G = \mathrm{SL}(2,\mathbb{R})$ we have that $\mathfrak{a}_{\mathbb{C}}^*$ is of complex dimension one. We consider the closed rectangular path γ_R in $\mathfrak{a}_{\mathbb{C}}^*$ going from $(\epsilon - iR)\rho$ to $(\epsilon + iR)\rho$, to $iR\rho$, to $-iR\rho$ and back to $(\epsilon - iR)\rho$. We observe that $a^\mu(\mathcal{F}_{\mathrm{wh}}f)(\mu)$ is holomorphic in μ so $\int_{\gamma_R} a^\mu(\mathcal{F}_{\mathrm{wh}}f)(\mu)\,\mathrm{d}\mu = 0$ by the Cauchy integral formula. From the estimate we obtain in Lemma 3.17 (see below, recall $\epsilon < 1/2$) we get that on the smaller sides of this rectangle (those are the segments from $(\epsilon + iR)\rho$ to $iR\rho$ and $-iR\rho$ to $(\epsilon - iR)\rho$) the integrand $a^\mu(\mathcal{F}_{\mathrm{wh}}f)(\mu)$ can be estimated by
$$|a^\mu(\mathcal{F}_{\mathrm{wh}}f)(\mu)| \le a^{\epsilon\rho}\,|\mathcal{F}_{\mathrm{wh}}f(\mu)| \le C(1+\|\mathrm{Im}\,\mu\|)^{-1} = C(1+R)^{-1}$$
for some $C > 0$. Since the length of these sides equals ϵ (and as a result is independent of R) we find that the contributions of these sides to the integral $\int_{\gamma_R} a^\mu(\mathcal{F}_{\mathrm{wh}}f)(\mu)\,\mathrm{d}\mu$ vanishes for $R \to \infty$. Hence taking the limit $R \to \infty$ yields
$$\int_{i\mathfrak{a}^*} a^\mu(\mathcal{F}_{\mathrm{wh}})(\mu)\,\mathrm{d}\mu = \int_{\epsilon\rho+i\mathfrak{a}^*} a^\mu(\mathcal{F}_{\mathrm{wh}})(\mu)\,\mathrm{d}\mu.$$
From this it now follows that
$$(\mathcal{F}^{-1}\phi_\epsilon)(a) = a^{-\epsilon\rho}\int_{i\mathfrak{a}^*} a^\mu(\mathcal{F}_{\mathrm{wh}})(\mu)\,\mathrm{d}\mu = a^{-\epsilon\rho}(\mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}f)(a) \xrightarrow{\epsilon\downarrow 0} (\mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}f)(a).$$
This proves the lemma. □

Lemma 3.17. Suppose $G = \mathrm{SL}(2,\mathbb{R})$. For every $f \in C_c^\infty(G/\overline{N};\chi)$ there exists a $C > 0$ such that
$$|\mathcal{F}_{\mathrm{wh}}f(\lambda)| \le C(1+\|\mathrm{Im}\,\lambda\|)^{-1}$$
holds for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ satisfying $\|\mathrm{Re}\,\lambda\| \le 1/2$.

Proof. For the proof of this lemma we will adapt some techniques used in the proof of Proposition 3.7 to our current situation. We know that W_λ satisfies $L_\Omega W_\lambda = (\lambda(H)^2 - 1)W_\lambda$ (see Proposition 2.44). Using this we observe

$$\begin{aligned}(\lambda(H)^2 - 1)(\mathcal{F}_{\mathrm{wh}} f)(\lambda) &= \int_{G/\overline{N}} [(\lambda(H)^2 - 1)W_\lambda(g)]f(g)\,\mathrm{d}(g\overline{N}) \\ &= \int_{G/\overline{N}} (L_\Omega W_\lambda)(g) \cdot f(g)\,\mathrm{d}(g\overline{N}).\end{aligned}$$

Because $|f|$ is compactly supported in $G/\overline{N}$ we can appply the 'partial integration rule' of Lemma 3.8. Hence the above integral can be written as

$$\begin{aligned}&\int_{G/\overline{N}} W_\lambda(g) \cdot (L_\Omega f)(g)\,\mathrm{d}(g\overline{N}) \\ &= \int_{K\times A} W_\lambda(ka) \cdot (L_\Omega f)(ka) \cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a.\end{aligned}$$

Her we used the result of Proposition 1.37. Using this expression and the estimate of Proposition 2.42 (here the assumption $\|\mathrm{Re}\,\lambda\| \leq 1/2$ is needed) we find

$$\begin{aligned}\left|\lambda(H)^2 - 1\right| |\mathcal{F}_{\mathrm{wh}} f(\lambda)| &\leq \int_{K\times A} |W_\lambda(ka)|\,|(L_\Omega f)(ka)| \cdot a^{-2\rho}\,\mathrm{d}k\,\mathrm{d}a \\ &\leq C(1 + \|\lambda\|) \int_{K\times A} a^{-\rho}(1 + \|\log a\|)e^{\|\mathrm{Re}\,\lambda\|\|\log a\|}\,|(L_\Omega f)(ka)|\,\mathrm{d}k\,\mathrm{d}a\end{aligned}$$

for some constant $C > 0$ large enough. Since per assumption $\|\mathrm{Re}\,\lambda\| \leq 1/2$ and f is compactly supported in $K \times A$ this integral is finite and can be bounded independently of λ. We conclude

$$\left|\lambda(H)^2 - 1\right| |\mathcal{F}_{\mathrm{wh}} f(\lambda)| \leq C'(1 + \|\lambda\|)$$

for some constant $C' > 0$. We observe that the real part of $\lambda(H)^2 - 1$ is given by $(\mathrm{Im}\,\lambda(H))^2 - 1$ hence $\left|\lambda(H)^2 - 1\right| \geq \left|(\mathrm{Im}\,\lambda(H))^2 - 1\right| = 1 + \|\mathrm{Im}\,\lambda\|^2$. Furthermore, $1 + \|\lambda\| \leq 1 + \|\mathrm{Re}\,\lambda\| + \|\mathrm{Im}\,\lambda\| \leq 2(1 + \|\mathrm{Im}\,\lambda\|)$ since $\|\mathrm{Re}\,\lambda\| \leq 1/2$. Using these observations we find

$$|\mathcal{F}_{\mathrm{wh}} f(\lambda)| \leq C' \frac{1 + \|\lambda\|}{|\lambda(H)^2 - 1|} \leq 2C' \frac{1 + \|\mathrm{Im}\,\lambda\|}{1 + \|\mathrm{Im}\,\lambda\|^2} \leq 4C'(1 + \|\mathrm{Im}\,\lambda\|)^{-1}.$$

This proves the lemma. □

The proof of Proposition 3.11 now is a simple corollary of this identity.

Proof of Proposition 3.11. On the subspace $C_c^\infty(G/\overline{N};\chi)^K$ of $\mathscr{C}(G/\overline{N};\chi)^K$ we have the equality

$$\mathcal{H} = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}.$$

Let $f \in \mathscr{C}(G/\overline{N};\chi)^K$ and $(f_n)_n$ a sequence in $C_c^\infty(G/\overline{N};\chi)^K$ converging to f (see Proposition 3.5). Then using that $\mathcal{H}$ is a continuous map $\mathscr{C}(G/\overline{N};\chi) \to C^\infty(A)$ we find, for any $a \in A$ that

$$(\mathcal{H}f)(a) = \lim_{n\to\infty} (\mathcal{H}f_n)(a) = \lim_{n\to\infty} (\mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}} f_n)(a) = (\mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}} f)(a).$$

Since a was arbitrary we conclude that $\mathcal{H}f = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}} f$ holds for all $f \in \mathscr{C}(G/\overline{N};\chi)$. Of the right hand side we know that it is a continuous map from $\mathscr{C}(G/\overline{N};\chi)$ to $\mathscr{C}(A)$ so we find that the same holds for $\mathcal{H}$. □

Remark 3.18: We would like to end this section by taking a step back and point out how the assumption that χ is regular was used in the proof of Proposition 3.11. For this we recall that all properties of the Whittaker–Fourier transformation $\mathcal{F}_{\mathrm{wh}}$ used in the above proof depend vitally on the assumption that χ is regular. In fact the Whittaker coefficient (and by extension $\mathcal{F}_{\mathrm{wh}}$) can only be defined under the assumption that χ is regular. So by exploiting the identity $\mathcal{H} = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}$ and the fact that $\mathcal{F}_{\mathrm{wh}}$ is a continuous map into $\mathscr{C}(i\mathfrak{a}^*)$ we make use of the assumption that χ is regular.

3.4 Further questions

We have given a partial answer to the question whether $\mathcal{H}$ maps $\mathscr{C}(G/\overline{N};\chi)$ into $\mathscr{C}(A)$ when χ is regular. It is of course interesting to see whether the techniques we used can be extended to answer this question in more cases.

The first natural place for extension is to look at elements in $\mathscr{C}(G/\overline{N};\chi)$ that have K-type other than the trivial one. For $G = \mathrm{SL}(2,\mathbb{R})$ we have $K \cong SO(1)$ and we set

$$\phi_l\colon K \to \mathbb{C}\colon \phi_l(k_\theta) = e^{il\theta}.$$

The subspace of elements in $\mathscr{C}(G/\overline{N};\chi)$ of K-type ϕ_l is then given by

$$\mathscr{C}(G/\overline{N};\chi)^{K,\phi_l} = \{f \in \mathscr{C}(G/\overline{N};\chi) \mid f(kg) = \phi_l(k)f(g) \quad \text{for all } g \in G, k \in K\}$$

The questions we can now ask is whether $\mathcal{H}$ maps $\mathscr{C}(G/\overline{N};\chi)^{K,\phi_l}$ into $\mathscr{C}(A)$ for all $l \in \mathbb{Z}$. Our proof of this result for $l = 0$ hinged on showing that the identity $\mathcal{H} = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh}}$ holds on $\mathscr{C}(G/\overline{N};\chi)^K$. In the proof of this identity, more specifically in the proof of Lemma 3.15, it was used that both $f \in C^\infty(G/\overline{N};\chi)^K$ and W_λ are both left K-fixed (so in particular have the same K-type). So a logical way to approach the other K-types in $\mathscr{C}(G/\overline{N};\chi)$ is to introduce a family of Whittaker coefficients $W_{\lambda,l}$ which have left K-type ϕ_l. For $l \in \mathbb{Z}$ we set

$$W_{\lambda,l}(x) := \left\langle \pi_{1,\lambda}(x)^{-1}\phi_l, j(P:1:-\overline{\lambda})(1)\right\rangle$$

if l even (in this case $\phi_l \in C(K:M:1)$) or

$$W_{\lambda,l} := \left\langle \pi_{\epsilon,\lambda}(x)^{-1}\phi_l, j(P:\epsilon:-\overline{\lambda})(1)\right\rangle$$

if l odd (in this case $\phi_l \in C(K:M:\epsilon)$). Here ϵ is the non-trivial element of $\widehat{M}$, i.e. $\epsilon(\pm I) = \pm 1$. These Whittaker coefficients satisfy $W_{\lambda,l}(kx) = \phi_l(k)W_{\lambda,l}(x)$ for $k \in K$. To this family of Whittaker coefficients corresponds a family of transformations defined as

$$\mathcal{F}_{\mathrm{wh},l}f(\lambda) := \int_{G/\overline{N}} W_{\lambda,l}(g)\cdot f(g)\,\mathrm{d}(g\overline{N}).$$

The reader is invited to check that the arguments used in the proof of Lemma 3.15 can be used, with minor adjustments, to show that $(\mathcal{F}\mathcal{H}f)(\lambda) = (\mathcal{F}_{\mathrm{wh},-l}f)(\lambda)$ for all $f \in C_c^\infty(G/\overline{N};\chi)^{K,\phi_l}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ with $\langle \operatorname{Re}\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+$. Then one could hope to show that $\mathcal{H} = \mathcal{F}^{-1}\mathcal{F}_{\mathrm{wh},-l}$ holds on $\mathscr{C}(G/\overline{N};\chi)^{K,\phi_l}$ using the techniques as used in the proof of Lemma 3.16. If it is then possible to prove that $\mathcal{F}_{\mathrm{wh},-l}$ is a continuous map $\mathscr{C}(G/\overline{N};\chi) \to \mathscr{C}(A)$ (as Proposition 3.7 states for $l = 0$) then we could extend the result of Proposition 3.11 to all K-types. It must be noted however that the proofs of these two last steps for the case $l = 0$ (i.e. Proposition 3.7 and Proposition 3.11) depend heavily on the estimates derived on W_λ in Section 2.7. Hence the above proof strategy is only viable if similar estimates for $W_{\lambda,l}$ on A can be found. Unfortunately the techniques employed in Section 2.7 do not carry over in a straightforward manner to the general case of $W_{\lambda,l}$ for $l \neq 0$. It is therefore an interesting question whether such estimates for $W_{\lambda,l}$ can perhaps be derived by different means. The author would like to point out that the difficulties in applying the techniques of Section 2.7 to $W_{\lambda,l}$ lie mostly in adapting the result of Proposition 2.41 (estimates on A^-) because its proof relies on the specific form of the integral expression for W_λ. On A^+ however an analogue of Proposition 2.39 is easily seen to be true also for $W_{\lambda,l}$.

Another possible avenue for extending our result is to move away from our assumption $G = \mathrm{SL}(2,\mathbb{R})$ and consider all semisimple Lie groups of split rank one. As pointed out throughout the text the precise structure of $G = \mathrm{SL}(2,\mathbb{R})$ is not used in the proofs of our results. The assumption $G = \mathrm{SL}(2,\mathbb{R})$ only comes into play when the properties of W_λ (and by extension $\mathcal{F}_{\mathrm{wh}}$) are used. Therefore it follows that if estimates on W_λ similar to the estimates in Proposition 2.42 can be found in the general case of groups of split rank one then our arguments are easily adapted to show an analogue of holds for such groups. The author is hopeful that this should be possible.

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