Universiteit Utrecht

Master's Thesis

Special values of the hypergeometric function

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Introduction

In this thesis, we want to find special values of the hypergeometric function, so we first define the hypergeometric function. The *Pochhammer symbol* is defined as

$$(a)_n = \prod_{i=0}^{n-1}(a+i) = a(a+1)\cdots(a+n-1), \tag{1}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}_{\geq 1}$, we also define $(a)_0 = 1$. Now we define the *Gauss hypergeometric function* by ${}_2F_1(a,b;c|x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n$, where $a,b,c \in \mathbb{C}$ and $c \notin \{k \in \mathbb{Z}; k \leq 0\}$. The given power series has a radius of convergence equal to ∞ if a or b is a non-negative integer, and radius of convergence equal to 1 otherwise. The function ${}_2F_1(a,b;c|x)$ is a solution of the hypergeometric differential equation

$$\left(x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (c-(a+b+1)x)\frac{\mathrm{d}}{\mathrm{d}x} - ab\right) f(x) = 0. \tag{2}$$

Note that equation (2) is a linear differential equation of degree 2, so its solution space is a 2-dimensional $\mathbb{C}$-vector space, but we do not consider the other solution of equation (2) yet.

A theorem of Wolfart, see [1, chapter 5], states that for $F(x) = {}_2F_1\left(a,b;c\,\middle|\,x\right)$ there are three cases with respect to the parameters a,b,c:

- the function $F(x)$ is algebraic over $\mathbb{C}(x)$, so if x is algebraic, then $F(x)$ is algebraic;
- the monodromy group of $F(x)$ is an arithmetic hyperbolic triangle group and $0 < a < c < 1$ and $0 < b < c < 1$ and $1 - c + |a-b| + |c-a-b| < 1$; then there is a subset E of the algebraic numbers, dense in $\mathbb{C}$, such that $F(x)$ is algebraic for $x \in E$;
- otherwise, there are only finitely many algebraic numbers x with $F(x)$ algebraic.

The most suprising statement in Wolfart's theorem is that there exists a class of transcendental functions, such that there exists a set E of algebraic numbers such that if $x \in E$, then $F(x)$ is algebraic. Moreover, this set is not only infinite, but is dense in $\mathbb{C}$. In the classification of Wolfart's theorem, we are only interested in the second case: the first case can be calculated and the third case is beyond this stage of this thesis.

In chapter 1 we start with a theorem of Schwarz[1], which states that the image of the upper half plane $\mathbb{H}$ by a quotient of hypergeometric functions is equal to a curvilinear triangle; this statement will be made precise in this chapter. The proof of Schwarz' theorem is classical and not the main part of this thesis; it is written out in appendix A. The theorem of Schwarz gives a relation between hypergeometric functions and triangle groups, where the triangle group follows

[1]Hermann Amandus Schwarz, 1843–1921, son in law of Ernst Kummer

from the given curvilinear triangle. It turns out that, after some choices, this triangle group is commensurable with the modular group $\mathrm{SL}_2(\mathbb{Z})$, and this is where modular functions and modular curves enter this thesis. In chapter 1 we do some preliminary calculations to be able to calculate special values of the hypergeometric function later on: for a given finite list of parameters, we calculate their triangle groups, this result is given in table 1.1. It turns out that the modular curves we consider in this thesis are all isomorphic to $\mathbb{P}^1(\mathbb{C})$, and the function field of a modular curve isomorphic to $\mathbb{P}^1(\mathbb{C})$ is generated by a single element, called a Hauptmodul. We also calculate these Hauptmoduln for the given list of parameters, which gives the tools we need to actually calculate special values of the hypergeometric function, this result is given in table 1.2.

In chapter 2 we calculate some special values of the hypergeometric function, using the triangle groups and Hauptmoduln from chapter 1. The key idea is that for the Hauptmoduln in this thesis it is the case that a Hauptmodul is a biholomorphic function from a curvilinear triangle to the upper half plane, and the quotient of hypergeometric functions is a biholomorphic function from the upper half plane to a curvilinear triangle. The composition of these two functions is a Möbius transformation of the triangle we started with, we also calculate this transformation. From differentiating this relation follows a formula to calculate special values of the hypergeometric function. The main interesting point of this formula is that it involves the derivative of the Hauptmodul, and the derivative of a modular function is not a modular function. To calculate the derivative of a modular function is an interesting problem, but it is possible and we did this in our thesis. Using the formula we proved in this chapter, we succeeded in giving two proofs, see theorem 2.15 and theorem 2.20, of the identity

$$ {}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6} \,\middle|\, \frac{4}{5}\right) = \frac{3}{\sqrt{5}}, \tag{3} $$

which is given as a conjecture in [4]. Using the same method, we have found some other identities, which are given in theorem 2.16.

In chapter 3 we do some more calculations on special values of the hypergeometric function without using the theorem of Schwarz. The method we use is a little bit technical; it uses the analytic continuation of a hypergeometric function along a curve. Using this method we found some other special values, one of which is

$$ \begin{aligned} &{}_2F_1\left(\frac{1}{84}, \frac{43}{84}; \frac{2}{3} \,\middle|\, \frac{38241952\left(5289411798647305 - 672452454064707\sqrt{21}\right)}{8443412305470285118248 1}\right) \\ &= \sqrt[84]{\frac{297568118001823519028019 2\left(3224592092541346723\sqrt{21} - 14673095170014395553\right)}{16748026101230266787394084996662321742373478233948229116033593 75}} \\ &\qquad \cdot \Re\left(e^{-\pi i/6}\sqrt[12]{2343 + 1287i\sqrt{3} + 1521i\sqrt{7} + 923\sqrt{21}}\right), \end{aligned} \tag{4} $$

which is equation (3.60). This identity has an algebraic argument, and an algebraic value; moreover the absolute value of its argument is less than 1, so this special value falls inside the region of convergence of the function ${}_2F_1\left(1/84, 13/84; 2/3 \,|\, x\right)$.

I would like to thank my supervisor Frits Beukers for advising me while I wrote my thesis. Several times he pointed out possible simplifications in my proofs and calculations, and for this clarity I am especially grateful. I would like to thank my second reader Gunther Cornelissen for taking the time to read this thesis. Also I want to thank my fellow students who participated in the master's thesis seminar; I have learned a lot from their talks and they listened to my talks, even when I did calculations on the blackboard.

Contents

Chapter 1

Triangle groups and Hauptmoduln

In this chapter we will study triangle groups and Hauptmoduln (these objects will be defined later), which will give special values of the hypergeometric function in the next chapter, see chapter 2 on page 16. To see the connection between triangle groups and hypergeometric functions, we start with Schwarz' theorem. By definition, a curvilinear triangle has edges which are line segments or circle segments. It can be shown that a curvilinear triangle with given angles is unique up to a Möbius transformation.

Theorem 1.1 (Schwarz,[12, page 311],[6, page 207])**.** *Let $a, b, c \in \mathbb{R}$ and define $\lambda = |1-c|$ and $\mu = |c-a-b|$ and $\nu = |a-b|$. Suppose that $0 \leq \lambda, \mu, \nu \leq 1$. Let f, g be linearly independent solutions of the hypergeometric differential equation with parameters a, b, c, then the function $D = f/g$ maps $\mathbb{H}$ one-to-one and conformal onto the interior of a curvilinear triangle T with vertices A with angle $\lambda\pi$ and B with angle $\mu\pi$ and C with angle $\nu\pi$. Moreover, it holds that $D(0) = A$ and $D(1) = B$ and $D(\infty) = C$.*

Note that choosing other linearly independent solutions of the hypergeometric differential equation gives a Möbius transformation of the triangle T: let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\alpha\delta - \beta\gamma \neq 0$, then from considering the functions $\alpha f + \beta g$ and $\gamma f + \delta g$ instead of f and g follows that $\frac{\alpha f+\beta g}{\gamma f+\delta g} = \frac{\alpha(f/g)+\beta}{\gamma(f/g)+\delta}$. From now on we will assume $\lambda = 1/k$ and $\mu = 1/l$ and $\nu = 1/m$ and $1/k + 1/l + 1/m < 1$ for $k, l, m \in \mathbb{N}_{\geq 1}$.

Definition 1.2 (Definition and conventions)**.** Let T be a curvilinear triangle with angles $\pi/k, \pi/l, \pi/m$ and $1/k + 1/l + 1/m < 1$. Let K_0, L_0, M_0 be, respectively, the edges of T opposite to the angles $\pi/k, \pi/l, \pi/m$. Let K, L, M be, respectively, be the reflections of the hyperbolic plane in the edges K_0, L_0, M_0. Define $\Gamma^*(k, l, m) = \langle K, L, M\rangle$ and $\Gamma(k, l, m) \subset \Gamma^*(k, l, m)$ as the index 2 subgroup consisting of words of even length. Note that $\Gamma(k, l, m)$ is contained in $\mathrm{SL}_2(\mathbb{C})$, because the composition of two reflections is a Möbius transformation. Because the matrix $-I_2$ gives the identity map, we identify $\Gamma(k, l, m)$ with $\Gamma(k, l, m) \cup -\Gamma(k, l, m)$, which is its lift from $\mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$. If no confusion arises, we write Γ instead of $\Gamma(k, l, m)$.

From [5, page 34] we know that given three circles in $\mathbb{C}$, there exists a unique circle C which intersects all three circles orthogonally, which is called the orthogonal circle or the radical circle. Note that this orthogonal circle is invariant under Γ, so for $z \in C$ and $\gamma \in \Gamma$, it follows that $\gamma z \in C$. Later on it will be the case that $C = \mathbb{R}$, from which follows that $\Gamma \subset \mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$.

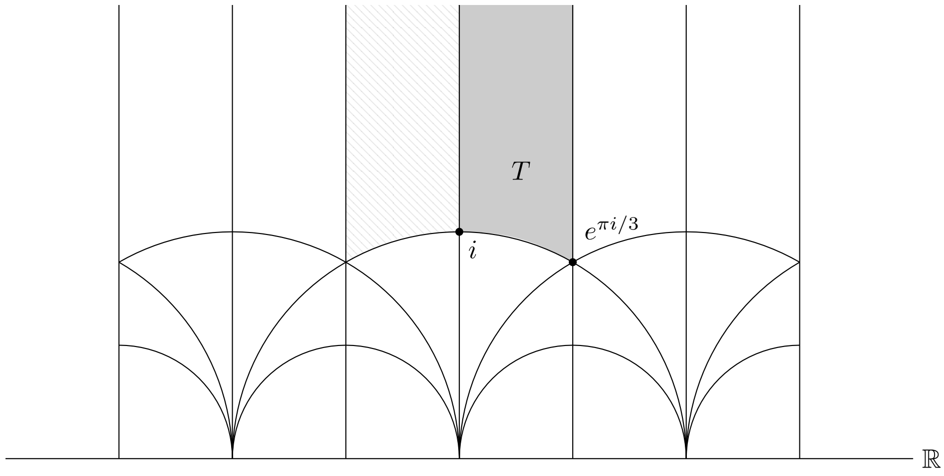

Figure 1.1: Picture of the triangle $T(2, 3, \infty)$ with its reflections; some triangles of the tesselation of $\mathbb{H}$ are drawn. One reflection of T is drawn in stripes, which together with T is a fundamental domain of $\Gamma(2, 3, \infty)\backslash\mathbb{H}$.

Theorem 1.3 ([9, theorem 2.8])**.** *Let K, L, M be the reflections in the edges of a hyperbolic triangle T with angles $\pi/k, \pi/l, \pi/m$. The images of (the interior of) T under the action of the distinct elements of the group $\Gamma^*(k, l, m) = \langle K, L, M\rangle$ fill the hyperbolic plane without gaps and overlappings. Moreover, $\Gamma^*(k, l, m)$ is defined by the relations $K^2 = L^2 = M^2 = (KL)^m = (MK)^l = (LM)^k = e$.*

Proof. A partial proof of this theorem is given in [9, theorem 2.8]. For the parts omitted, a reference to [2, paragraph 398–402] is given. □

Because the images of T with respect to the action of Γ do not overlap, it follows that elements of the interior of T have a trivial stabilizer. Moreover, let z be an element of the closure of T, not equal to one of the vertices of T, then z has a trivial stabilizer in Γ: there are only two images of T under Γ^* which are neighbours of the edge on which z lies, which are reflections of each other. On the other hand, if z is a vertex of T, then z has a non-trivial stabilizer in Γ. From this follows that the union of T and one of its reflections (with edges identified) is equal to the hyperbolic space modulo the action of Γ. If we make the assumption that the orthogonal circle of T is equal to $\mathbb{R}$, we can identify the hyperbolic space with $\mathbb{H}$.

For the function D in theorem 1.1, we have that D^{-1} is a biholomorphic function from T to $\mathbb{H}$. Using the Schwarz reflection principle, see appendix A.4, it follows that D^{-1} extends to a modular function with respect to $\Gamma(k, l, m) \subset \mathrm{SL}_2(\mathbb{R})$. We want at least one of $|1 - c|$ and $|c - a - b|$ and $|a - b|$ to be equal to 0: with a Möbius transformation the vertex of T with angle 0 can be sent to ∞ such that the two edges from ∞ are parallel to the imaginary axis $\{z \in \mathbb{C}|\Re(z) = 0\}$. From this follows that the function D is periodic, which is necessary if we want D to be a modular function with respect to a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. The requirement that at least one of the vertices of $T(k, l, m)$ has an angle equal to 0 means that $\Gamma(k, l, m)$ is a non-cocompact Fuchsian triangle group. In [14, theorem 3ii] all non-cocompact arithmetic Fuchsian triangle groups are given, where $\{k, l, m\}$ is equal to one of the following sets:

$$\{2,3,\infty\},\{2,4,\infty\},\{2,6,\infty\},\{2,\infty,\infty\},\{3,3,\infty\},\{3,\infty,\infty\},\{4,4,\infty\},\{6,6,\infty\},\{\infty,\infty,\infty\}. \tag{1.1}$$

For the list in equation (1.1), we calculate $\Gamma(k,l,m)$, which we summarize in table 1.1. In the following calculations, we use that the triangle group of a triangle is generated by rotations around vertices, where only two generators are necessary. Moreover, a rotation can be obtained from two reflections. Note that not always holds that $\Gamma(k,l,m) \subset \mathrm{PSL}_2(\mathbb{Z})$, because elements of $\mathrm{PSL}_2(\mathbb{Z})$ with finite order have order at most 3, so a rotation with order at least 4 cannot be written as an element of $\mathrm{PSL}_2(\mathbb{Z})$. We choose the triangles T such that the orthogonal circle of T is equal to $\mathbb{R}$, and the group Γ is straightforward to write down.

1.1 Triangle groups

Let $\{k,l,m\} = \{2,3,\infty\}$ and define $T(2,3,\infty) = \{z \in \mathbb{H}; 0 < \Re(z) < \frac{1}{2}, |z| > 1\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection along the line $\Re(z) = 1$ gives the map $\tau : z \mapsto z+1$. Reflection along the line $\Re(z) = 0$, followed by reflection with respect to the circle $|z| = 1$ gives the map $\sigma : z \mapsto \frac{-1}{z}$. From this follows that $\Gamma(2,3,\infty) = \langle \tau, \sigma \rangle$. Because $\mathrm{PSL}_2(\mathbb{Z})$ is generated by $z \mapsto z+1$, which is equal to τ, and $z \mapsto \frac{-1}{z}$, which is equal to σ, it follows that

$$\Gamma(2,3,\infty) = \mathrm{PSL}_2(\mathbb{Z}). \tag{1.2}$$

Let $\{k,l,m\} = \{2,4,\infty\}$ and define $T(2,4,\infty) = \{z \in \mathbb{H}; 0 < \Re(z) < \frac{1}{2}, |z| > \frac{\sqrt{2}}{2}\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection along the line $\Re(z) = 1$ gives the map $\tau : z \mapsto z+1$. Reflection along the line $\Re(z) = 0$, followed by reflection with respect to the circle $|z| = \frac{\sqrt{2}}{2}$ gives the map $\alpha : z \mapsto \frac{-1}{2z}$. From this follows that $\Gamma(2,4,\infty) = \langle \tau, \alpha \rangle$. Note that the map α cannot be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$, but $\sigma = \alpha\tau\alpha^{-1} : z \mapsto \frac{z}{1-2z}$ can be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$ and α^2 is the identity map. From this follows that a word in α, τ with an even number of factors α in it can be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$, so the subgroup $\Gamma(2,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ has index 2 in $\Gamma(2,4,\infty)$, with the identity and α as coset representatives. Furthermore $\Gamma(2,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ is generated by σ, τ. We know that $\Gamma_0(2) = \Gamma_1(2)$ is generated by $z \mapsto z+1$, which is equal to τ, and $z \mapsto \frac{z-1}{2z-1}$, which is equal to $\tau\sigma$. From this follows that $\Gamma_0(2) \subset \Gamma(2,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$. By considering the image of $\Gamma(2,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ modulo 2 it follows that $\Gamma(2,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(2)$ and

$$\Gamma(2,4,\infty) = \Gamma_0(2) \cup \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \Gamma_0(2). \tag{1.3}$$

Let $\{k,l,m\} = \{2,6,\infty\}$ and define $T(2,6,\infty) = \{z \in \mathbb{H}; 0 < \Re(z) < \frac{1}{2}, |z| > \frac{\sqrt{3}}{3}\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection along the line $\Re(z) = 1$ gives the map $\tau : z \mapsto z+1$. Reflection along the line $\Re(z) = 0$, followed by reflection with respect to the circle $|z| = \frac{\sqrt{3}}{3}$ gives the map $\alpha : z \mapsto \frac{-1}{3z}$. From this follows that $\Gamma(2,6,\infty) = \langle \tau, \alpha \rangle$. Note that the map α cannot be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$, but $\sigma = \alpha\tau\alpha^{-1} : z \mapsto \frac{z}{1-3z}$ can be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$ and we have $\alpha^2 : z \mapsto z$. From this follows that a word in α, τ with an even number of factors α in it can be defined by a matrix in $\mathrm{PSL}_2(\mathbb{Z})$, so the subgroup $\Gamma(2,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ has index 2 in $\Gamma(2,6,\infty)$, with the identity and α as coset representatives. Furthermore $\Gamma(2,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ is generated by σ, τ. We know that $\Gamma_0(3)$ is generated by $z \mapsto z+1$, which is equal to τ, and $z \mapsto \frac{z-1}{3z-2}$, which is equal to $\sigma^{-1}\tau^{-1}$. From this follows that

$\Gamma_0(3) \subset \Gamma(2,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$. By considering the image of $\Gamma(2,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z})$ modulo 3 it follows that $\Gamma(2,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(3)$ and

$$\Gamma(2,6,\infty) = \Gamma_0(3) \cup \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \Gamma_0(3). \tag{1.4}$$

Let $\{k,l,m\} = \{2,\infty,\infty\}$ and define $T(2,\infty,\infty) = \{z \in \mathbb{H}; 0 < \Re(z) < \frac{1}{2}, |z - \frac{1}{2}| > \frac{1}{2}\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by a reflection along the line $\Re(z) = 1$ gives the map $\tau : z \mapsto z+1$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection with respect to the circle $|z - \frac{1}{2}| = \frac{1}{2}$ gives the map $\sigma : z \mapsto \frac{z-1}{2z-1}$. From this follows that $\Gamma(2,\infty,\infty) = \langle \tau, \sigma \rangle$. We know that $\Gamma_0(2)$ is generated by $z \mapsto z+1$, which is equal to τ, and $z \mapsto \frac{z-1}{2z-1}$, which is equal to σ. From this follows that

$$\Gamma(2,\infty,\infty) = \Gamma_0(2). \tag{1.5}$$

Let $\{k,l,m\} = \{3,3,\infty\}$ and define $T(3,3,\infty) = \{z \in \mathbb{H}; -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > 1\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection along the line $\Re(z) = \frac{3}{2}$ gives the map $\tau : z \mapsto z+2$. Reflection along the line $\Re(z) = -\frac{1}{2}$, followed by reflection with respect to the circle $|z+1| = 1$ gives the map $\sigma : z \mapsto \frac{-z-1}{z}$. From this follows that $\Gamma(3,3,\infty) = \langle \tau, \sigma \rangle$. We know that $\Gamma(2)$ is generated by $z \mapsto z+2$, which is equal to τ, and $z \mapsto \frac{3z-2}{2z-1}$, which is equal to $\tau\sigma^{-1}\tau\sigma^{-2}$. From this follows that $\Gamma(2) \subset \Gamma(3,3,\infty)$. Reduction of $\Gamma(3,3,\infty)$ modulo 2 gives an image of $\langle \sigma \rangle \subset \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$, from which follows that

$$M(3,3,\infty) = \Gamma(2) \cup \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \Gamma(2) \cup \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \Gamma(2). \tag{1.6}$$

Let $\{k,l,m\} = \{3,\infty,\infty\}$ and define $T = \{z \in \mathbb{H}; 0 < \Re(z) < \frac{1}{2}, |z - \frac{1}{3}| > \frac{1}{3}\}$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection along the line $\Re(z) = 1$ gives the map $\tau : z \mapsto z+1$. Reflection along the line $\Re(z) = \frac{1}{2}$, followed by reflection with respect to the circle $|z - \frac{2}{3}| = \frac{1}{3}$ gives the map $\sigma : z \mapsto \frac{2z-1}{3z-1}$. From this follows that $\Gamma(3,\infty,\infty) = \langle \tau, \sigma \rangle$. We know that $\Gamma_0(3)$ is generated by $z \mapsto z+1$, which is equal to τ, and $z \mapsto \frac{-z+1}{-3z+2}$, which is equal to σ^{-1}. Reduction of $\Gamma(3,\infty,\infty)$ modulo 3 gives an image $\langle \tau \rangle \subset \mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$, from which follows that

$$M(3,\infty,\infty) = \Gamma_0(3). \tag{1.7}$$

Let $\{k,l,m\} = \{4,4,\infty\}$ and define $T = \{z \in \mathbb{H}; -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > \frac{\sqrt{2}}{2}\}$. In the calculations of $\Gamma(2,4,\infty)$ we found that the rotation of $T(2,4,\infty)$ around ∞ is given by $\tau : z \mapsto z+1$ and the rotation around $\frac{\sqrt{2}}{2}i$ is given by $\alpha : z \mapsto \frac{-1}{2z}$. From this follows that the rotation around $\frac{1}{2} + \frac{i}{2}$ is given by $\tau\alpha : z \mapsto \frac{2z-1}{2z}$. Because around the point $\frac{1}{2} + \frac{i}{2}$ the triangles $T(2,4,\infty)$ and $T(4,4,\infty)$ are equal, it follows that the rotation of $T(2,4,\infty)$ around $\frac{1}{2} + \frac{i}{2}$ is the same map as rotation of $T(4,4,\infty)$ around $\frac{1}{2} + \frac{i}{2}$. Because $T(4,4,\infty)$ consists of $T(2,4,\infty)$ and its reflection in the axis $\Re(z) = 0$, it follows that the rotation of $T(4,4,\infty)$ around ∞ is given by $\tau^2 : z \mapsto z+2$. From this follows that $\Gamma(4,4,\infty) = \langle \tau^2, \tau\alpha \rangle$. We already know that $\Gamma(2,4,\infty) = \langle \tau, \alpha \rangle = \Gamma_0(2) \cup \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \Gamma_0(2)$. Because α and $\tau\alpha$ have even order, the length of a word in $\langle \tau, \alpha \rangle$ can be defined, from this follows that $\Gamma(4,4,\infty)$ consists of the words in $\Gamma(2,4,\infty)$ with even length: here we use that all words of even length in $\langle \tau, \alpha \rangle$ can be written as words in $\langle \tau^2, \tau\alpha \rangle$, because $\alpha^{-1} = \alpha$ and if a word w starts with τ^{-1}, then $\tau^2 w$ starts with τ. From this follows that $\Gamma(4,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z}) = \langle \tau^2, (\tau\alpha)^2, (\tau\alpha)\tau^2(\tau\alpha)^{-1} \rangle$. Now consider the reduction of $\Gamma_0(2)$ modulo 4. Because the identity matrix modulo 4 is a word of even length, it follows that $\Gamma(4) \subset \Gamma_0(2)$ consists of words of even length, from which follows that $\Gamma(4) \subset \Gamma(4,4,\infty)$. Reduction modulo 4 of $(\tau\alpha)\tau^2(\tau\alpha)^{-1} : z \mapsto \frac{-3z+4}{-4z+5}$ gives the identity matrix, so the image of $\langle \tau^2, \tau\alpha, (\tau\alpha)\tau^2(\tau\alpha)^{-1}\rangle = \langle \tau^2, \tau\alpha\rangle \subset \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ gives a group of order 8, and because $\Gamma(4)$ has index 48 in $\mathrm{SL}_2(\mathbb{Z})$, it follows that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(4,4,\infty) \cap \mathrm{SL}_2(\mathbb{Z})] = \frac{48}{8} = 6$, where $(\tau\alpha)\tau^2(\tau\alpha)^{-1} : z \mapsto \frac{z-1}{2z-1}$. To sum up, it follows that

$$\Gamma(4,4,\infty) = \left\langle \pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}, \Gamma(4) \right\rangle. \tag{1.8}$$

Let $\{k,l,m\} = \{6,6,\infty\}$ and define $T = \{z \in \mathbb{H}; -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > \frac{\sqrt{3}}{3}\}$. In the calculations of $\Gamma(2,6,\infty)$ we found that the rotation of $T(2,6,\infty)$ around ∞ is given by $\tau : z \mapsto z+1$ and the rotation around $\frac{\sqrt{3}i}{3}$ is given by $\alpha : z \mapsto \frac{-1}{3z}$. From this follows that the rotation around $\frac{1}{2} + \frac{\sqrt{3}i}{6}$ is given by $\tau\alpha : z \mapsto \frac{3z-1}{3z}$. Because around the point $\frac{1}{2} + \frac{\sqrt{3}i}{6}$ the triangles $T(2,6,\infty)$ and $T(6,6,\infty)$ are equal, it follows that the rotation of $T(2,6,\infty)$ around $\frac{1}{2} + \frac{\sqrt{3}i}{6}$ is the same map as rotation of $T(6,6,\infty)$ around $\frac{1}{2} + \frac{\sqrt{3}i}{6}$. Because $T(6,6,\infty)$ consists of $T(2,6,\infty)$ and its reflection in the axis $\Re(z) = 0$, it follows that the rotation of $T(6,6,\infty)$ around ∞ is given by $\tau^2 : z \mapsto z+2$. From this follows that $\Gamma(6,6,\infty) = \langle \tau^2, \tau\alpha\rangle$. We already know that $\Gamma(2,6,\infty) = \langle \tau, \alpha\rangle = \Gamma_0(3) \cup \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \Gamma_0(3)$. Because α and $\tau\alpha$ have even order, the length of a word in $\langle \tau, \alpha\rangle$ can be defined, from this follows that $\Gamma(6,6,\infty)$ consists of the words in $\Gamma(2,6,\infty)$ with even length: here we use that all words of even length in $\langle \tau, \alpha\rangle$ can be written as words in $\langle \tau^2, \tau\alpha\rangle$, because $\alpha^{-1} = \alpha$ and if a word w starts with τ^{-1}, then $\tau^2 w$ starts with τ. From this follows that $\Gamma(6,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z}) = \langle \tau^2, (\tau\alpha)^2, (\tau\alpha)\tau^2(\tau\alpha)^{-1}\rangle$. Now consider the reduction of $\Gamma_0(3)$ modulo 6. Because the identity matrix modulo 6 is a word of even length, it follows that $\Gamma(6) \subset \Gamma_0(3)$ consists of words of even length, from which follows that $\Gamma(6) \subset \Gamma(6,6,\infty)$. Reduction modulo 6 of $(\tau\alpha)\tau^2(\tau\alpha)^{-1} : z \mapsto \frac{-5z+6}{-6z+7}$ gives the identity matrix, so the image of $\langle \tau^2, \tau\alpha, (\tau\alpha)\tau^2(\tau\alpha)^{-1}\rangle = \langle \tau^2, \tau\alpha\rangle \subset \mathrm{SL}_2(\mathbb{Z}/6\mathbb{Z})$ gives a group of order 18, and because $\Gamma(6)$ has index 144 in $\mathrm{SL}_2(\mathbb{Z})$, it follows that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(6,6,\infty) \cap \mathrm{SL}_2(\mathbb{Z})] = \frac{144}{18} = 8$, where $(\tau\alpha)\tau^2(\tau\alpha)^{-1} : z \mapsto \frac{2z-1}{3z-1}$. To sum up, it follows that

$$\Gamma(6,6,\infty) = \left\langle \pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 3 & -1 \\ 3 & 0 \end{pmatrix}, \Gamma(6) \right\rangle. \tag{1.9}$$

Let $\{k,l,m\} = \{\infty,\infty,\infty\}$ and define $T(\infty,\infty,\infty) = \{z \in \mathbb{H}; 0 < \Re(z) < 1, |z - \frac{1}{2}| > \frac{1}{2}\}$. In the calculations of $\Gamma(2,\infty,\infty)$ we found that the rotation of $T(2,\infty,\infty)$ around ∞ is given by $\tau : z \mapsto z+1$ and the rotation around 1 is given by $\alpha : z \mapsto \frac{3z-2}{2z-1}$. Because around the point 1 the triangles $T(2,\infty,\infty)$ and $T(\infty,\infty,\infty)$ are equal, it follows that the rotation of $T(2,\infty,\infty)$ around 1 is the same map as rotation of $T(\infty,\infty,\infty)$ around 1. Furthermore, because $T(\infty,\infty,\infty)$ consists of $T(2,\infty,\infty)$ and its reflection with respect to $\Re(z) = \frac{1}{2}$, it follows that the rotation around ∞ of $T(\infty,\infty,\infty)$ is given by $\tau^2 : z \mapsto z+2$. From this follows that $\Gamma(\infty,\infty,\infty) = \langle \tau^2, \alpha\rangle$. We know that $\Gamma(2)$ is generated by $z \mapsto z+2$, which is equal to τ^2, and $z \mapsto \frac{3z-2}{2z-1}$, which is equal to α. From this follows that

$$\Gamma(\infty,\infty,\infty) = \Gamma(2). \tag{1.10}$$

As a check of the results in this section, we consider the volume of the triangle T. The volume of a curvilinear triangle with angles $\pi/k, \pi/l, \pi/m$ is given by $\pi - \pi/k - \pi/l - \pi/m$. The fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is given by $\{z \in \mathbb{H}; -\frac{1}{2} < \Re(z) < \frac{1}{2}, |z| > 1\}$, which has angles $0, \pi/3, \pi/3$ and volume $\pi/3$. From this follows that

$$\mathrm{Vol}(T(k,l,m)) = \frac{\pi}{6} \cdot \frac{[\mathrm{SL}_2(\mathbb{Z}) : \mathrm{SL}_2(\mathbb{Z}) \cap \Gamma(k,l,m)]}{[\Gamma(k,l,m) : \mathrm{SL}_2(\mathbb{Z}) \cap \Gamma(k,l,m)]}; \tag{1.11}$$

here we wrote $\pi/6$, because the volume of half the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ is equal to $\pi/6$.

k, l, m	Triangle T	Left Vertex	Right Vertex	$\Gamma(k, l, m) \subset \mathrm{SL}_2(\mathbb{R})$
$2, 3, \infty$	$\Re(z) \in (0, \frac{1}{2}), \lvert z\rvert > 1$	i	$1/2 + \sqrt{3}i/2$	$\mathrm{SL}_2(\mathbb{Z})$
$2, 4, \infty$	$\Re(z) \in (0, \frac{1}{2}), \lvert z\rvert > \frac{1}{2}\sqrt{2}$	$i/\sqrt{2}$	$1/2 + i/2$	$\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \Gamma_0(2) \right\rangle$
$2, 6, \infty$	$\Re(z) \in (0, \frac{1}{2}), \lvert z\rvert > \frac{1}{3}\sqrt{3}$	$i/\sqrt{3}$	$1/2 + \sqrt{3}i/6$	$\left\langle \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}, \Gamma_0(3) \right\rangle$
$2, \infty, \infty$	$\Re(z) \in (0, \frac{1}{2}), \lvert z - \frac{1}{2}\rvert > \frac{1}{2}$	0	$1/2 + i/2$	$\Gamma_0(2)$
$3, 3, \infty$	$\Re(z) \in (-\frac{1}{2}, \frac{1}{2}), \lvert z\rvert > 1$	$-1/2 + \sqrt{3}i/2$	$1/2 + \sqrt{3}i/2$	$\left\langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \Gamma(2) \right\rangle$
$3, \infty, \infty$	$\Re(z) \in (0, \frac{1}{2}), \lvert z - \frac{1}{3}\rvert > \frac{1}{3}$	0	$1/2 + \sqrt{3}i/6$	$\Gamma_0(3)$
$4, 4, \infty$	$\Re(z) \in (-\frac{1}{2}, \frac{1}{2}), \lvert z\rvert > \frac{1}{2}\sqrt{2}$	$-1/2 + i/2$	$1/2 + i/2$	$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}, \Gamma(4) \right\rangle$
$6, 6, \infty$	$\Re(z) \in (-\frac{1}{2}, \frac{1}{2}), \lvert z\rvert > \frac{1}{3}\sqrt{3}$	$-1/2 + \sqrt{3}i/6$	$1/2 + \sqrt{3}i/6$	$\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 3 & 0 \end{pmatrix}, \Gamma(6) \right\rangle$
∞, ∞, ∞	$\Re(z) \in (0, 1), \lvert z - \frac{1}{2}\rvert > \frac{1}{2}$	0	1	$\Gamma(2)$

Table 1.1: Table of non-compact arithmetic Fuchsian triangle groups; two of the three vertices of the triangles are given: the third vertex is equal to ∞. Here the left and right vertices are such that the real part of the left vertex is smaller than the real part of the right vertex. Note that all given groups contain the matrix $-I_2$.

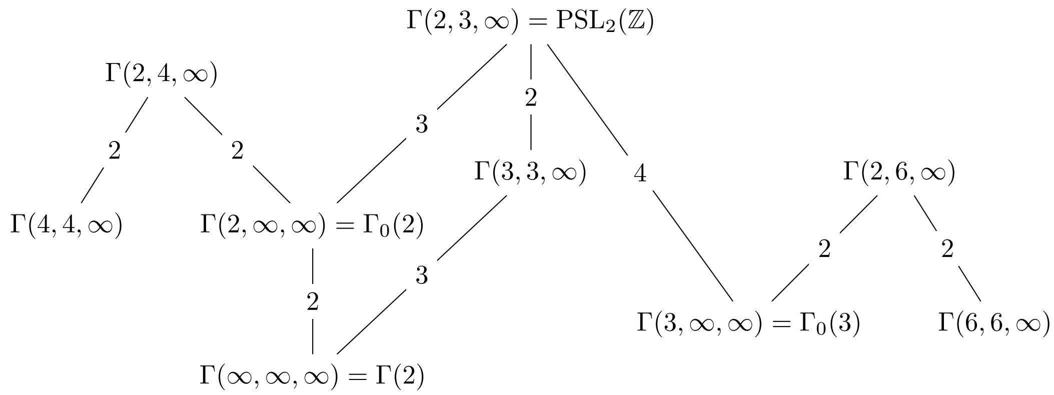

Figure 1.2: Subgroup inclusions of the groups in table 1.1

1.2 Hauptmoduln

From now on we write $X(\Gamma)$ for the Riemann surface $\Gamma\backslash(\mathbb{H} \cup \mathbb{Q} \cup \{\infty\})$; in [10, chapter I.2] it is explained how the quotient of $\mathbb{H}$ modulo a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ can be seen as a Riemann surface. Note that only for congruence subgroups $G \subset \mathrm{SL}_2(\mathbb{Z})$ it is guaranteed that $X(G)$ is compact. Because the groups Γ from table 1.1 all contain a congruence subgroup, it follows that $X(\Gamma \cap \mathrm{SL}_2(\mathbb{Z}))$ is a compact Riemann surface, and after taking the quotient modulo

Γ it follows that $X(\Gamma)$ is compact, because the quotient space of a compact topological space is compact. Now we know that $X(\Gamma)$ is a Riemann surface, we can calculate its isomorphism class:

Lemma 1.4. *Let* Γ *as in table 1.1. Then* $X(\Gamma) \cong \mathbb{P}^1(\mathbb{C})$ *as Riemann surfaces.*

Proof. First note that $X(\Gamma)$ is a compact Riemann surface. Now we calculate the genus of $X(\Gamma)$, which follows from the triangulation of $X(\Gamma)$. Let T^* be a reflection of the triangle T from table 1.1 and take the triangles T and T^* as faces (for an example, see figure 1.1), and their edges as edges of the triangulation, and the vertices of T and T^* as vertices of the triangulation. Taking equivalences modulo Γ into account, it follows that there are 3 vertices and 3 edges and 2 faces, from which follows that the Euler characteristic of $X(\Gamma)$ is equal to $3-3+2=2=2-2g$, from which follows that $X(\Gamma)$ has genus equal to 0. Because the only compact Riemann surface of genus 0 is $\mathbb{P}^1(\mathbb{C})$, it follows that $X(\Gamma) \cong \mathbb{P}^1(\mathbb{C})$. □

Recall that a meromorphic function f on a Riemann surface is a function with values in $\mathbb{C}$ with a discrete set of poles. If the codomain of f is equal to $\mathbb{P}^1(\mathbb{C})$ and f is not identically equal to ∞, then f is called holomorphic instead of meromorphic.

Lemma 1.5. *Let* X *be a compact Riemann surface. Then a meromorphic function* $f : X \to \mathbb{P}^1(\mathbb{C})$ *has a unique pole of order* 1 *if and only if* f *is an isomorphism.*

Proof. Suppose that $f : X \to \mathbb{P}^1(\mathbb{C})$ is an isomorphism. From this follows that f is injective, so f has a unique pole of order 1; here we use that a function with a pole of order ≥ 2 is not injective.

Suppose that $f : X \to \mathbb{P}^1(\mathbb{C})$ has a unique pole of order 1. From this follows that f is injective, because a meromorphic function has the same number of poles as it has zeroes, counting multiplicities, see [10, proposition 1.12a]. On the other hand, because f is continuous and X is compact, it follows that $f(X) \subset \mathbb{P}^1(\mathbb{C})$ is compact. Because $\mathbb{P}^1(\mathbb{C})$ is Hausdorff, it follows that $f(X)$ is closed. Because X is open, and f is an open map, it follows that $f(X)$ is open. Because f has a unique pole of order 1, it follows that f is non-constant, so $f(X) = \mathbb{P}^1(\mathbb{C})$. From this follows that f is bijective, so f^{-1} is holomorphic. From this follows that f is an isomorphism. □

Lemma 1.6. *Let* $f : X \to \mathbb{P}^1(\mathbb{C})$ *be an isomorphism. Let* $\mathbb{C}(X)$ *be the field of meromorphic functions on* X*, then it follows that* $\mathbb{C}(X) = \mathbb{C}(f(z))$*.*

Proof. Suppose that f is an isomorphism. Let g be a meromorphic function on X, then it follows that $g \circ f^{-1}$ is a meromorphic function on $\mathbb{P}^1(\mathbb{C})$. Because the field of meromorphic functions of $\mathbb{P}^1(\mathbb{C})$ is equal to $\mathbb{C}(z)$, it follows that $(g \circ f^{-1})(z) = R(z)$ for some rational function $R \in \mathbb{C}(z)$. From this follows that $g(z) = R(f(z))$, which proves that $g \in \mathbb{C}(f(z))$, so $\mathbb{C}(X) = \mathbb{C}(f(z))$. □

Corollary 1.7. *Let* $f \in \mathbb{C}(X(\Gamma))$ *be a meromorphic function with a unique pole. Then* f *is a Hauptmodul of* $X(\Gamma)$*.*

In the situation of lemma 1.6 the function field $\mathbb{C}(X)$ is generated by one element, which is called a *Hauptmodul* of $X(\Gamma)$. To find Hauptmoduln with respect to the groups in table 1.1, we use lemma 1.5 to prove that the functions we find indeed are Hauptmoduln. Now we want to define functions and their orders on $X(\Gamma)$. Let $a \in \mathbb{Z}_{>0}$ be minimal such that every element of $\mathbb{C}(X(\Gamma))$ is periodic with period a, note that a is the width of the cusp ∞ of Γ. Then for $f \in \mathbb{C}(X(\Gamma))$ we can write $f(z) = \sum_{k=k_0}^{\infty} a_k q^k$ with $q = e^{2\pi i z/a}$. The order of f at $z \to \infty$ is equal to the minimal $k \in \mathbb{Z}$ such that $a_k \neq 0$. Note that if $f(z) \to 0$ for $z \to \infty$, then f has a positive order, and f has a negative order if $f(z) \to \infty$ for $z \to \infty$. The order of a pole of $f(z)$ at $z \to \infty$ is equal to minus the order of $f(z)$ at $z \to \infty$.

Let $\{k, l, m\} = \{2, 3, \infty\}$, then we know that the j-function is a Hauptmodul with respect to the group $\mathrm{PSL}_2(\mathbb{Z})$.

Let $\{k, l, m\} = \{2, 4, \infty\}$. We know that $\Delta(z)/\Delta(2z)$ is a modular function with respect to $\Gamma_0(2)$. We have $\Delta\left(-\frac{1}{2z}\right)/\Delta\left(-2\frac{1}{2z}\right) = \frac{(2z)^{12}\Delta(2z)}{z^{12}\Delta(z)} = 2^{12}\Delta(2z)/\Delta(z)$, from which follows that $h(z) = \Delta(z)/\Delta(2z) + 2^{12}\Delta(2z)/\Delta(z)$ is a modular function with respect to Γ, see table 1.1. Because $h(z)$ is holomorphic on $\mathbb{H}$ and has a pole of order 1 for $z \to \infty$, it follows from corollary 1.7 that

$$h(2, 4, \infty)(z) = \frac{\Delta(z)}{\Delta(2z)} + 2^{12}\frac{\Delta(2z)}{\Delta(z)}. \tag{1.12}$$

Let $\{k, l, m\} = \{2, 6, \infty\}$. We know that $\eta^{12}(z)/\eta^{12}(3z)$ is a modular function with respect to $\Gamma_0(3)$. We have $\eta^{12}\left(-\frac{1}{3z}\right)/\eta^{12}\left(-3\frac{1}{3z}\right) = \frac{(3z)^6\eta^{12}(3z)}{z^6\eta^{12}(z)} = 3^6\eta^{12}(3z)/\eta^{12}(z)$, from which follows that $h(z) = \eta^{12}(z)/\eta^{12}(3z) + 3^6\eta^{12}(3z)/\eta^{12}(z)$ is a modular function with respect to Γ, see table 1.1. Because h is holomorphic on $\mathbb{H}$ and has a pole of order 1 for $z \to \infty$, it follows from corollary 1.7 that

$$h(2, 6, \infty)(z) = \frac{\eta^{12}(z)}{\eta^{12}(3z)} + 3^6\frac{\eta^{12}(3z)}{\eta^{12}(z)}. \tag{1.13}$$

Let $\{k, l, m\} = \{2, \infty, \infty\}$. We know that $h(z) = \Delta(z)/\Delta(2z)$ is a modular function with respect to $\Gamma = \Gamma_0(2)$, see table 1.1. Note that h is holomorphic on $\mathbb{H}$ and has a pole of order 1 for $z \to \infty$. From corollary 1.7 follows that

$$h(2, \infty, \infty)(z) = \frac{\Delta(z)}{\Delta(2z)}. \tag{1.14}$$

Let $\{k, l, m\} = \{3, 3, \infty\}$. We already know that the j-function is a Hauptmodul in the case $\{k, l, m\} = \{2, 3, \infty\}$. The triangle $T(3, 3, \infty)$ consists of $T(2, 3, \infty)$ and its reflection with respect to the line $\Re(z) = 0$. From Schwarz reflection follows that the image of the j-function on the triangle $T(3, 3, \infty)$, see table 1.1, is equal to $\mathbb{H} \cup (1728, \infty) \cup \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} = \{z \in \mathbb{C}; \Im(z) < 0\}$. From this follows that $1728 - j(z)$ is a biholomorphic function from T to $\mathbb{C}\backslash[0, \infty)$, which is invariant under $\Gamma(3, 3, \infty)$. We define the square root on $\mathbb{C}\backslash[0, \infty)$ such that $\sqrt{-1} = i$. Note that $h(z) = \sqrt{1728 - j(z)}$ has a pole of order 1 for $z \to \infty$, so from corollary 1.7 that

$$h(3, 3, \infty)(z) = \sqrt{1728 - j(z)}. \tag{1.15}$$

Let $\{k, l, m\} = \{3, \infty, \infty\}$. We know that $\eta^{12}(z)/\eta^{12}(3z)$ is a modular function with respect to $\Gamma = \Gamma_0(3)$, see table 1.1. From corollary 1.7 follows that

$$h(3, \infty, \infty)(z) = \frac{\eta^{12}(z)}{\eta^{12}(3z)}. \tag{1.16}$$

Let $\{k, l, m\} = \{4, 4, \infty\}$. We already know that $\Delta(z)/\Delta(2z) + 2^{12}\Delta(2z)/\Delta(z)$ is a Hauptmodul in the case $\{k, l, m\} = \{2, 4, \infty\}$. The triangle $T(4, 4, \infty)$ consists of $T(2, 4, \infty)$ and its reflection with respect to the line $\Re(z) = 0$. From Schwarz reflection follows that the image of $h(2, 4, \infty)$ on the triangle $T(4, 4, \infty)$, see table 1.1, is equal to $\mathbb{H} \cup (128, \infty) \cup \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} = \{z \in \mathbb{C}; \Im(z) < 0\}$. From this follows that $128 - h(2, 4, \infty)(z)$ is a biholomorphic function from T to $\mathbb{C}\backslash[0, \infty)$, which is invariant under $\Gamma(4, 4, \infty)$. We define the square root on $\mathbb{C}\backslash[0, \infty)$ such that $\sqrt{-1} = i$. Note that $h(z) = \sqrt{128 - \Delta(z)/\Delta(2z) - 2^{12}\Delta(2z)/\Delta(z)}$ has a pole of order 1 for $z \to \infty$, so from corollary 1.7 that

$$h(4, 4, \infty)(z) = \sqrt{128 - \frac{\Delta(z)}{\Delta(2z)} - 2^{12}\frac{\Delta(2z)}{\Delta(z)}}. \tag{1.17}$$

Let $\{k,l,m\} = \{6,6,\infty\}$. We already know that $\eta^{12}(z)/\eta^{12}(3z) + 3^6\eta^{12}(3z)/\eta^{12}(z)$ is a Hauptmodul in the case $\{k,l,m\} = \{2,6,\infty\}$. The triangle $T(6,6,\infty)$ consists of $T(2,6,\infty)$ and its reflection with respect to the line $\Re(z) = 0$. From Schwarz reflection follows that the image of $h(2,6,\infty)$ on the triangle $T(6,6,\infty)$, see table 1.1, is equal to $\mathbb{H} \cup (54,\infty) \cup \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} = \{z \in \mathbb{C}; \Im(z) < 0\}$. From this follows that $54 - h(2,6,\infty)(z)$ is a biholomorphic function from T to $\mathbb{C}\backslash[0,\infty)$, which is invariant under $\Gamma(6,6,\infty)$. We define the square root on $\mathbb{C}\backslash[0,\infty)$ such that $\sqrt{-1} = i$. Note that $h(z) = \sqrt{54 - \eta^{12}(z)/\eta^{12}(3z) - 3^6\eta^{12}(3z)/\eta^{12}(z)}$ has a pole of order 1 for $z \to \infty$, so from corollary 1.7 that

$$h(6,6,\infty)(z) = \sqrt{54 - \frac{\eta^{12}(z)}{\eta^{12}(3z)} - 3^6\frac{\eta^{12}(3z)}{\eta^{12}(z)}}. \tag{1.18}$$

Let $\{k,l,m\} = \{\infty,\infty,\infty\}$. The Hauptmodul of $\Gamma(2)$ is known as the modular λ-function, which is defined as $\lambda(z) = 16\frac{\eta^8(z/2)\eta^{16}(2z)}{\eta^{24}(z)}$, which gives

$$h(\infty,\infty,\infty)(z) = \lambda(z). \tag{1.19}$$

k,l,m	a	b	$h_{k,l,m}$	∞	$h(l)$	$h(r)$
$2,3,\infty$	$1/12$	$5/12$	$j(z)$	∞	1728	0
$2,4,\infty$	$1/8$	$3/8$	$\Delta(z)/\Delta(2z) + 2^{12}\Delta(2z)/\Delta(z)$	∞	128	-128
$2,6,\infty$	$1/6$	$1/3$	$\eta^{12}(z)/\eta^{12}(3z) + 3^6\eta^{12}(3z)/\eta^{12}(z)$	∞	54	-54
$2,\infty,\infty$	$1/4$	$3/4$	$\Delta(z)/\Delta(2z)$	∞	0	-64
$3,3,\infty$	$1/6$	$1/2$	$\sqrt{1728 - j(z)}$	∞	$24\sqrt{3}$	$-24\sqrt{3}$
$3,\infty,\infty$	$1/3$	$2/3$	$\eta^{12}(z)/\eta^{12}(3z)$	∞	0	-27
$4,4,\infty$	$1/4$	$1/2$	$\sqrt{128 - \Delta(z)/\Delta(2z) - 2^{12}\Delta(2z)/\Delta(z)}$	∞	16	-16
$6,6,\infty$	$1/3$	$1/2$	$\sqrt{54 - \eta^{12}(z)/\eta^{12}(3z) - 3^6\eta^{12}(3z)/\eta^{12}(z)}$	∞	$6\sqrt{3}$	$-6\sqrt{3}$
∞,∞,∞	$1/2$	$1/2$	$16\eta^8(z/2)\eta^{16}(2z)/\eta^{24}(z)$	0	1	∞

Table 1.2: Table of Hauptmoduln, where the functions η, j, Δ respectively denote the Dedekind eta function, the modular j-function and the unique weight 12 normalized cusp form. The values of h at the left and right vertices from table 1.1 are written as $h(l)$ and $h(r)$.

1.3 Algebraic relations between Hauptmoduln

The Hauptmoduln of the groups from table 1.1 are given in table 1.2. Now we want to find the equations which relate them to each other, using the subgroup inclusions from figure 1.2: if $\Gamma_1 \subset \Gamma_2$, it follows that $\mathbb{C}(X(\Gamma_2)) \subset \mathbb{C}(X(\Gamma_1))$. The results will be summarized in table 1.3. In the calculations in section 1.2 we already found some algebraic relations, which gives the first five rows of table 1.3. Because figure 1.2 gives nine subgroup inclusions, and we obtained algebraic relations following from five subgroup inclusions, we want to calculate the algebraic relations between Hauptmoduln which follow from the remaining four subgroup inclusions. Note that in these four subgroup inclusions both groups are contained in $\mathrm{SL}_2(\mathbb{Z})$. These results are summarized in the last four rows of table 1.3. It is possible to calculate these relations as in the second proof of theorem 2.11, by calculating q-expansions, but we will use a more theoretical calculation, starting with a lemma.

Lemma 1.8. *Let $\Gamma_1 \subset \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z})$ with index n such that $\mathbb{C}(X(\Gamma_1)) = \mathbb{C}(h_1(z))$ and $\mathbb{C}(X(\Gamma_2)) = \mathbb{C}(h_2(z))$. Write $h_2(z) = R(h_1(z))$ with $R(z) \in \mathbb{C}(z)$. Then it follows that the degree of R is* *equal to n. Moreover, if z_0 is not equal to $h_2(w_0)$ where $w_0 \in \mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i or $e^{\pi i/3}$, it follows that $\#R^{-1}(z_0) = n$, counted without multiplicity.*

Proof. Let $z_0 \in \mathbb{C}$, then there exists a $w_0 \in \mathbb{H}$ such that $h_2(w_0) = z_0$; otherwise the function $1/(h_2(z) - z_0)$ has no poles and therefore is constant, which is a contradiction. Let $\gamma \in \Gamma_2$, then it follows that $z_0 = h_2(w_0) = h_2(\gamma w_0) = R(h_1(\gamma w_0))$, from which follows that $\{h_1(\gamma w_0) \in \mathbb{H} | \gamma \in \Gamma_2\} \subset R^{-1}(z_0)$. On the other hand, assume that $R(h_1(w_1)) = h_2(w_1) = h_2(w_0) = z_0$ for a $w_1 \in \mathbb{H}$, then it follows that there exists a $\gamma \in \Gamma_2$ such that $\gamma w_0 = w_1$: if there are $w_0, w_1 \in \mathbb{H}$ which are not equivalent modulo Γ_2 such that $h_2(w_0) = h_2(w_1) = z_0$, the function $1/(h(z) - z_0)$ has two poles, which is a contradiction. From this follows that

$$R^{-1}(z_0) = \{h_1(\gamma w_0) \in \mathbb{H} | \gamma \in \Gamma_2\}. \tag{1.20}$$

For $w_0 \in \mathbb{H}$, the set $\{h_1(\gamma w_0) \in \mathbb{H} | \gamma \in \Gamma_2\}$ has at most n elements, because Γ_1 has index n in Γ_2. Now assume that $\#R^{-1}(z_0) < n$, note that there exists a $w_0 \in \mathbb{H}$ such that $h_2(w_0) = z_0$. From this follows that there exist $\gamma_1, \gamma_2 \in \Gamma_2$ which are not equivalent modulo Γ_1 such that $h_1(\gamma_1 w_0) = h_1(\gamma_2 w_0)$. From this follows that there exists a $\gamma \in \Gamma_1$ such that $\gamma\gamma_1 w_0 = \gamma_2 w_0$, so $\gamma_2^{-1}\gamma\gamma_1 w_0 = w_0$. If $\gamma_2^{-1}\gamma\gamma_1 = \pm I_2$, it follows that $\gamma\gamma_1 = \pm\gamma_2$, which contradicts the assumption that γ_1, γ_2 are not eqiuvalent modulo Γ_1. From this follows that w_0 has a non-trivial stabilisator in $\mathrm{SL}_2(\mathbb{Z})$, from which follows that $z_0 = h_2(w_0)$ where w_0 is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i or $e^{2\pi i/3}$. □

The group $\Gamma(\infty,\infty,\infty)$ is an index 2 subgroup of $\Gamma(2,\infty,\infty)$, so $h(2,\infty,\infty)(z) \in \mathbb{C}(h(\infty,\infty,\infty)(z))$, write $h(2,\infty,\infty)(z) = h(z)$ and $h(\infty,\infty,\infty)(z) = \lambda(z)$. The triangle $T(\infty,\infty,\infty)$ consists of $T(2,\infty,\infty)$ and its reflection with respect to the line $\Re(z) = 1/2$. From Schwarz reflection follows that the image of the function h on the triangle $T(\infty,\infty,\infty)$, see table 1.1, is equal to $\mathbb{H} \cup (-\infty,-64) \cup \overline{\mathbb{H}}$, where $\overline{\mathbb{H}} = \{z \in \mathbb{C}; \Im(z) < 0\}$. From this follows that $64 + h(z)$ is a biholomorphic function from T to $\mathbb{C}\backslash[0,\infty)$. We define the square root on $\mathbb{C}\backslash[0,\infty)$ such that $\sqrt{-1} = i$. From this follows that $\sqrt{h(z)+64}$ is a biholomorphic function from $T(\infty,\infty,\infty)$ to $\mathbb{H}$, which is a Hauptmodul, see corollary 1.7. Because λ also is a Hauptmodul of $X(\Gamma(2))$, it must be a Möbius transformation of $\sqrt{h(z)+64}$. We have $\lim_{z\to 0}\sqrt{h(z)+64} = 8$ and $\lim_{z\to 1}\sqrt{h(z)+64} = -8$ and $\lim_{z\to\infty}\sqrt{h(z)+64} = \infty$ and $\lim_{z\to 0}\lambda(z) = 1$ and $\lim_{z\to 1}\lambda(z) = \infty$ and $\lim_{z\to\infty}\lambda(z) = 0$, from which follows that $\lambda(z) = \frac{16}{8+\sqrt{h(z)+64}}$ and

$$h(2,\infty,\infty)(z) = \frac{256}{\lambda(z)^2} - \frac{256}{\lambda(z)}. \tag{1.21}$$

The group $\Gamma(\infty,\infty,\infty)$ is an index 3 subgroup of $\Gamma(3,3,\infty)$, from which follows that $\sqrt{1728 - j(z)} \in \mathbb{C}(\lambda(z))$. Now we apply lemma 1.8 with $\Gamma_1 = \Gamma(\infty,\infty,\infty)$ and $\Gamma_2 = \Gamma(3,3,\infty)$ and $h_1(z) = \lambda(z)$ and $h(z) = h_2(z) = \sqrt{1728 - j(z)}$, from which follows a $R \in \mathbb{C}(z)$ such that $h(z) = R(\lambda(z))$. We have $h(z+1) = -h(z)$ and $\lambda(z+1) = 1 - \lambda(z)$, from which follows that $R(1-z) = -R(z)$. A set of coset representatives of $\Gamma(\infty,\infty,\infty)\backslash\Gamma(3,3,\infty)$ is given by $\{z \mapsto z, z \mapsto -1 - 1/z, z \mapsto -\frac{1}{z+1}\}$ and a set of coset representatives of $\Gamma(3,3,\infty)\backslash\mathrm{SL}_2(\mathbb{Z})$ is given by $\{z \mapsto z, z \mapsto z+1\}$. We have $h(i) = h(i+1) = 0$ and $\lambda(i) = 1/2$ and $\lambda(-1-1/i) = -1$ and $\lambda\left(\frac{-1}{i+1}\right) = 2$, from which follows that $R^{-1}(0) = \{1/2, -1, 2\}$. From $h(\rho) = -24\sqrt{3}$ and $\lambda(\rho) = \lambda(-1-1/\rho) = \lambda(\frac{-1}{\rho+1}) = \rho$ follows that $R^{-1}(-24\sqrt{3}) = \{\rho, \rho, \rho\}$. From $h(\rho+1) = 24\sqrt{3}$ and $\lambda(\rho+1) = \lambda\left(-1 - \frac{1}{\rho+1}\right) = \lambda(\frac{-1}{\rho+2}) = 1 - \rho$ follows that $R^{-1}(24\sqrt{3}) = \{1-\rho, 1-\rho, 1-\rho\}$. Moreover, we have $\lim_{z\to\infty} h(z) = \infty$ and $\lim_{z\to\infty}\lambda(z) = 0$ and $\lim_{z\to\infty}\lambda(-1-1/z) = \infty$ and $\lim_{z\to\infty}\lambda\left(\frac{-1}{z+1}\right) = 1$. From this follows that the numerator of R is equal to $(z-1/2)(z+1)(z-2)$, and the denominator of R is equal to a constant times $z(z-1)$. A calculation gives that

$$h(3,3,\infty)(z) = 8i\frac{(2\lambda(z)+1)(\lambda(z)-2)(\lambda(z)+1)}{\lambda(z)(\lambda(z)-1)}. \tag{1.22}$$

The group $\Gamma(2,\infty,\infty) = \Gamma_0(2)$ is an index 3 subgroup of $\Gamma(2,3,\infty)$. From this follows that $j(z) \in \mathbb{C}(h(z))$, where j is the j-function and $h(z) = h(2,\infty,\infty)(z) = \Delta(z)/\Delta(2z)$. Write $j(z) = R(h(z))$. Now we apply lemma 1.8 with $\Gamma_1 = \Gamma(2,\infty,\infty)$ and $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})$ and $h_1(z) = h(z)$ and $h_2(z) = j(z)$. A set of coset representatives of $\Gamma_0(2)\backslash \mathrm{SL}_2(\mathbb{Z})$ is given by $\{z \mapsto z, z \mapsto -\frac{1}{z}, z \mapsto -\frac{1}{z+1}\}$. We have $j(i) = 1728$ and $h(i) = h(-1/i) = 512$ and $h\left(\frac{-1}{i+1}\right) = -64$, from which follows that $R^{-1}(1728) = \{512, 512, -64\}$. Also we have $j(\rho) = 0$ and $h(\rho) = h(-1/\rho) = h\left(\frac{-1}{\rho+1}\right) = -256$ with $\rho = e^{\pi i/3}$, from which follows that $R^{-1}(0) = \{-256, -256, -256\}$. Moreover, we have $\lim_{z\to\infty} j(z) = \infty$ and $\lim_{z\to\infty} h(z) = \infty$ and $\lim_{z\to\infty} h(-1/z) = \lim_{z\to\infty} h\left(\frac{-1}{z+1}\right) = 0$. From this follows that the numerator of R is equal to $(z+256)^3$, and the denominator of R is equal to a constant times z^2. A calculation gives that

$$h(2,3,\infty)(z) = j(z) = \frac{(h(z)+256)^3}{h(z)^2}. \tag{1.23}$$

The group $\Gamma(3,\infty,\infty) = \Gamma_0(3)$ is an index 4 subgroup of $\Gamma(2,3,\infty)$. From this follows that $j(z) \in \mathbb{C}(h(z))$, where j is the j-function and $h(z) = h(3,\infty,\infty)(z) = \eta^{12}(z)/\eta^{12}(3z)$. Write $j(z) = R(h(z))$. Now we apply lemma 1.8 with $\Gamma_1 = \Gamma(3,\infty,\infty)$ and $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})$ and $h_1(z) = h(z)$ and $h_2(z) = j(z)$. A set of coset representatives of $\Gamma_0(3)\backslash \mathrm{SL}_2(\mathbb{Z})$ is given by $\{z \mapsto z, z \mapsto -\frac{1}{z}, z \mapsto -\frac{1}{z+1}, z \mapsto -\frac{1}{z+2}\}$. We have $j(i) = 1728$ and $h(i) = h(-1/i) = 243 + 162\sqrt{3}$ and $h\left(\frac{-1}{i+1}\right) = h\left(\frac{-1}{i+2}\right) = 243 - 162\sqrt{3}$, from which follows that $R^{-1}(1728) = \{243+162\sqrt{3}, 243+162\sqrt{3}, 243-162\sqrt{3}, 243-162\sqrt{3}\}$. Also we have $j(\rho) = 0$ and $h(\rho) = h(-1/\rho) = h\left(\frac{-1}{\rho+2}\right) = -243$ and $h\left(\frac{-1}{\rho+2}\right) = -27$ with $\rho = e^{\pi i/3}$, from which follows that $R^{-1}(0) = \{-243, -243, -243, -27\}$. Moreover, we have $\lim_{z\to\infty} j(z) = \infty$ and $\lim_{z\to\infty} h(z) = \infty$ and $\lim_{z\to\infty} h(-1/z) = \lim_{z\to\infty} h\left(\frac{-1}{z+1}\right) = \lim_{z\to\infty} h\left(\frac{-1}{z+2}\right) = 0$. From this follows that the numerator of R is equal to $(z+243)^3(z+27)$, and the denominator of R is equal to a constant times z^3. A calculation gives that

$$h(2,3,\infty)(z) = j(z) = \frac{(h(z)+243)^3(h(z)+27)}{h(z)^3}. \tag{1.24}$$

Inclusion	Index	Function
$\Gamma(3,3,\infty) \subset \Gamma(2,3,\infty)$	2	$1728 - z^2$
$\Gamma(2,\infty,\infty) \subset \Gamma(2,4,\infty)$	2	$z + 2^{12}/z$
$\Gamma(4,4,\infty) \subset \Gamma(2,4,\infty)$	2	$128 - z^2$
$\Gamma(3,\infty,\infty) \subset \Gamma(2,6,\infty)$	2	$z + 3^6/z$
$\Gamma(6,6,\infty) \subset \Gamma(2,6,\infty)$	2	$54 - z^2$
$\Gamma(\infty,\infty,\infty) \subset \Gamma(2,\infty,\infty)$	2	$256/z^2 - 256/z$
$\Gamma(\infty,\infty,\infty) \subset \Gamma(3,3,\infty)$	3	$8i\frac{(2z-1)(z-2)(z+1)}{z(z-1)}$
$\Gamma(2,\infty,\infty) \subset \Gamma(2,3,\infty)$	3	$(z+256)^3/z^2$
$\Gamma(3,\infty,\infty) \subset \Gamma(2,3,\infty)$	4	$(z+243)^3(z+27)/z^3$

Table 1.3: Table of algebraic relations between Hauptmoduln. Here the given rational function expresses the Hauptmodul of the second group in terms of the Hauptmodul of the subgroup, see table 1.2 for the Hauptmoduln.

Chapter 2

Special values of the hypergeometric function

2.1 The image of the function D of Schwarz' theorem

In theorem 1.1 it is given that f/g is a biholomorphic function from $\mathbb{H}$ to a curvilinear triangle; here f, g are linearly indepedent solutions of the hypergeometric differential equation. We do not know the image of f/g yet, we only know it is a Möbius transformation of the triangle in table 1.1. To calculate the image of f/g, we start with some calculations on the hypergeometric function, but first define basis elements of the vector space of solutions of the hypergeometric differential equation.

Definition 2.1. Let $0 < a, b < 1$. Then we define the following basis of local solutions around $x = 0$ of the hypergeometric differential equation:

$$\begin{aligned} F_0(x) &:= {}_2F_1\left(a, b; 1 \,|x\right) && |x| < 1 \\ G_0(x) &:= \log(x)F_0(x) + \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} \left(\sum_{j=0}^{n-1} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} \right) x^n && |x| < 1. \end{aligned} \tag{2.1}$$

Define $D(x) = G_0(x)/F_0(x)$. Assume that $a + b < 1$. Then we define the following basis of local solutions around $x = 1$ of the hypergeometric differential equation:

$$\begin{aligned} F_1(x) &:= {}_2F_1\left(a, b; a+b \,|1-x\right) && |x-1| < 1 \\ G_1(x) &:= (1-x)^{1-a-b} {}_2F_1\left(1-b, 1-a; 2-a-b \,|1-x\right) && |x-1| < 1. \end{aligned} \tag{2.2}$$

Assume that $a + b = 1$. Then we define the following basis of local solutions around $x = 1$ of the hypergeometric differential equation:

$$\begin{aligned} F_1(x) &:= {}_2F_1\left(a, b; 1 \,|1-x\right) && |x-1| < 1 \\ G_1(x) &:= \log(1-x)F_1(x) + \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} \left(\sum_{j=0}^{n-1} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} \right) (1-x)^n && |x-1| < 1. \end{aligned} \tag{2.3}$$

The element ∞ is contained in all sets $\{k, l, m\}$ in table 1.1, so we can take $c = 1$, from which follows that $\lim_{x\to 0} D(x) = G_0(x)/F_0(x) = \infty$. Note that later in this chapter we will calculate ${}_2F_1(a, b; a + b\,|x)$ for some values of a, b, x, but we choose parameters $a, b, 1$, and do not choose $a, b, a+b$: if we choose parameters $a, b, 1$, we do not have to consider different cases in the definitions of $F_0(x)$ and $G_0(x)$. Moreover, with parameters $a, b, 1$ we have $\lim_{x\to 0} D(x) = \infty$, which is not always the case if we choose a basis of local solutions around $x = 0$ with parameters $a, b, a + b$.

For $0 < x < 1$ we have $D(x) \in \mathbb{R}$ and $\lim_{x\to 0} D(x) = -\infty$. For $-1 < x < 0$ we have $D(x) \in \pi i + \mathbb{R}$ and $\lim_{x\to 0} D(x) = \pi i - \infty$, where the logarithm of a negative real number has imaginary part π, this follows from analytic continuation along $\mathbb{H}$. From this follows that the edge of $D(\mathbb{H}) = \{D(x)|x \in \mathbb{H}\}$ between $D(0)$ and $D(1)$ is contained in $\mathbb{R}$, and the edge of $D(\mathbb{H})$ between $D(0)$ and $D(\infty)$ is contained in $\pi i + \mathbb{R}$. If we choose the parameters a, b such that the angles and orientation of $D(\mathbb{H})$ and T, see table 1.1, are equal, then it follows that $D(\mathbb{H})$ and T are Möbius transformations of each other: here we use that a curvilinear triangle is unique up to a Möbius transformation.

Because both T and $D(\mathbb{H})$ have one vertex at ∞ with angle 0, it follows that the Möbius transformation from $D(\mathbb{H})$ to T is a linear transformation: it sends ∞ to ∞. The triangle $D(\mathbb{H})$ has distance π between the parallel lines, which we call its width. Let v be the width of the triangle in table 1.1, then it follows that $M(z) = \frac{\pi}{v} z + w$ for some $w \in \mathbb{C}$. Now we will calculate $D(1) = \lim_{x\to 1} D(x)$, from which we can calculate the number w, because we want that M sends the left vertex of T to $D(1)$. The results will be summarized in table 2.1 and an example is given in figure 2.1 on page 22. In our calculations we will use the following lemma's.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{R}\backslash\{r \in \mathbb{Z}; r \leq 0\}$. Then it follows that $(\alpha)_n/(\beta)_n = \frac{\Gamma(\beta)}{\Gamma(\alpha)} n^{\alpha-\beta}(1 + O(1/n))$, where for a function $f : \mathbb{N} \to \mathbb{R}$ we have that $f = O(1/n)$ if the function $n \cdot f$ is bounded for $n \to \infty$.*

Proof. The Stirling approximation reads $\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z))$, from which follows for $\alpha, \beta \in \mathbb{R}$ that

$$\begin{aligned}
\frac{(\alpha)_n}{(\beta)_n} &= \frac{\Gamma(\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\beta+n)} = \frac{\Gamma(\beta)\sqrt{\frac{2\pi}{n+\alpha}}\left(\frac{n+\alpha}{e}\right)^{n+\alpha}}{\Gamma(\alpha)\sqrt{\frac{2\pi}{n+\beta}}\left(\frac{n+\beta}{e}\right)^{n+\beta}} \frac{1+O(1/n)}{1+O(1/n)} \\
&= \frac{\Gamma(\beta)\sqrt{\frac{n+\beta}{n+\alpha}}}{\Gamma(\alpha)(n+\alpha)^{\beta-\alpha}} \frac{e^{\beta-\alpha}(n+\alpha)^{n+\beta}}{(n+\beta)^{n+\beta}} \frac{1+O(1/n)}{1+O(1/n)} \\
&= \frac{\Gamma(\beta)\sqrt{\frac{n+\beta}{n+\alpha}}}{\Gamma(\alpha)(n+\alpha)^{\beta-\alpha}} e^{\beta-\alpha} \left(1 + \frac{\alpha-\beta}{n+\beta}\right)^{n+\beta} \frac{1+O(1/n)}{1+O(1/n)} \overset{n\to\infty}{\to} \frac{\Gamma(\beta)}{\Gamma(\alpha)} n^{\alpha-\beta}(1+O(1/n)),
\end{aligned} \tag{2.4}$$

where we use that $\sqrt{\frac{n+\beta}{n+\alpha}} = 1+O(1/n)$ and $(n+\alpha)^{\alpha-\beta} = n^{\alpha-\beta}(1+O(1/n))$ and $\left(1 + \frac{\alpha-\beta}{n+\beta}\right)^{n+\beta} = \exp(\alpha-\beta)(1 + O(1/n))$ and $\frac{1+O(1/n)}{1+O(1/n)} = 1 + O(1/n)$, note that $(\alpha)_n/(\beta)_n$ is defined, because $\alpha, \beta \notin \{r \in \mathbb{Z}; r \leq 0\}$. □

Lemma 2.3. *Suppose that $0 < a, b < 1$ and $0 < a + b \leq 1$ and let D as in definition 2.1. Then*

$$\lim_{x\to 1} D(x) = \sum_{j=1}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1}. \tag{2.5}$$

Proof. From comparing Riemann schemes follows that ${}_2F_1\left(a, b; c\,|x\right) = (1-x)^{c-a-b}{}_2F_1\left(c-a, c-b; c\,|x\right)$, from which follows that $(1-x)^{1-a-b}F_(x) = {}_2F_1\left(1-a, 1-b; 1\,|x\right)$.

$$S_{m,n} = \sum_{j=m}^{n} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1} = \sum_{j=m}^{n} \frac{(a+b)j+a+b-2ab}{(1-a+j)(1-b+j)(j+1)}, \tag{2.6}$$

then it follows that

$$\begin{aligned}(1-x)^{1-a-b}G_0(x) = \log(x){}_2F_1\left(1-a, 1-b; 1\,|x\right)& \\ + \sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{(n!)^2} \left(\sum_{j=0}^{n-1} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1}\right) x^n& \\ = \log(x){}_2F_1\left(1-a, 1-b; 1\,|x\right)& \\ + S_{0,\infty}{}_2F_1\left(1-a, 1-b; 1\,|x\right) - \sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{(n!)^2} S_{n,\infty} x^n& \quad (2.7)\end{aligned}$$

and

$$D(x) = \frac{G_0(x)}{F_0(x)} = \frac{(1-x)^{1-a-b}G_0(x)}{(1-x)^{1-a-b}F_0(x)} = \log(x) + S_{0,\infty} - \frac{\sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{(n!)^2} S_{n,\infty} x^n}{{}_2F_1\left(1-a, 1-b; 1\,|x\right)}. \tag{2.8}$$

For $n \to \infty$ we have $\frac{(1-a)_n(1-b)_n}{(n!)^2} = O(n^{-a-b})$ and $S_{n,\infty} = O(1/n)$, because the summand in the definition of $S_{n,m}$ is in $O(1/j^2)$. Because $-a-b < 0$, it follows that $\lim_{x\to 1} \sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{(n!)^2} S_{n,\infty} x^n$ is finite. On the other hand, because $\frac{(1-a)_n(1-b)_n}{(n!)^2} = O(n^{-a-b})$, it follows that $\lim_{x\to 1} {}_2F_1\left(1-a, 1-b; 1\,|x\right)$ is infinite. From this follows that

$$\lim_{x\to 1} D(x) = \lim_{x\to 1} \left(\log(x) + S_{0,\infty} - \frac{\sum_{n=0}^{\infty} \frac{(1-a)_n(1-b)_n}{(n!)^2} S_{n,\infty} x^n}{{}_2F_1\left(1-a, 1-b; 1\,|x\right)}\right) = S_{0,\infty}, \tag{2.9}$$

which is what we wanted to prove. □

2.2 Calculations with given parameters

We will calculate the result in lemma 2.3 for the parameters which follow from table 1.1; we still have to choose parameters $a, b \in \mathbb{R}$ such that $\{0, |1-a-b|, |a-b|\} = \{1/k, 1/l, 1/m\}$ for $\{k, l, m\}$ as in equation (1.1). In our calculations we will use that $\sum_{j=1}^{n} \frac{1}{j} = \log(n) + \gamma + O(1/n)$ where γ is the Euler-Mascheroni constant. It can be shown that $\pi\frac{\cos(\pi z)}{\sin(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$, from which follows for $a \in \mathbb{R}$ that

$$\sum_{j=0}^{\infty} \frac{1}{a+j} - \frac{1}{1-a+j} = \sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{a-1-j} = \sum_{j\in\mathbb{Z}} \frac{1}{a+j} = \pi\frac{\cos(\pi a)}{\sin(\pi a)}. \tag{2.10}$$

Define

$$S_m = \sum_{j=0}^{\infty} \left(\sum_{k=1}^{m-1} \frac{1}{\frac{k}{m}+j} - \frac{m-1}{j+1}\right), \tag{2.11}$$

then it follows that

$$\begin{aligned}
S_m &= m\sum_{j=0}^{\infty}\left(\sum_{k=1}^{m}\frac{1}{k+mj}-\frac{m}{m+mj}\right) = m\lim_{n\to\infty}\int_0^1\sum_{j=0}^{n}x^{mj}\left(\sum_{k=1}^{m}x^{k-1}-mx^{m-1}\right)\mathrm{d}x \\
&= m\lim_{n\to\infty}\int_0^1\frac{1-x^{m(n+1)}}{1-x^m}\left(\sum_{k=1}^{m}x^{k-1}-mx^{m-1}\right)\mathrm{d}x = m\int_0^1\frac{\sum_{k=1}^{m}x^{k-1}-mx^{m-1}}{1-x^m}\mathrm{d}x \\
&= m\int_0^1\left(\frac{1}{1-x}-\frac{mx^{m-1}}{1-x^m}\right)\mathrm{d}x = m\left[\log\left(\frac{1-x^m}{1-x}\right)\right]_0^1 = m\log(m). \qquad (2.12)
\end{aligned}$$

Let $\{k,l,m\}=\{2,3,\infty\}$, so $a=\frac{1}{12}$ and $b=\frac{5}{12}$ and

$$\begin{aligned}
\lim_{x\to1}D(x) &= \sum_{j=0}^{\infty}\frac{1}{\frac{11}{12}+j}+\frac{1}{\frac{7}{12}+j}-\frac{2}{j+1} \\
&= \frac{\pi\cos(7\pi/12)}{2\sin(7\pi/12)}+\frac{\pi\cos(11\pi/12)}{2\sin(11\pi/12)}+\sum_{j=0}^{\infty}\frac{1/2}{\frac{1}{12}+j}+\frac{1/2}{\frac{5}{12}+j}+\frac{1/2}{\frac{7}{12}+j}+\frac{1/2}{\frac{11}{12}+j}-\frac{2}{j+1} \\
&= -2\pi+\frac{1}{2}\left(S_{12}-S_6-S_4+S_2\right) = -2\pi+\log(1728). \qquad (2.13)
\end{aligned}$$

Because the triangle T from table 1.1 has width $\frac{1}{2}$, it follows that $M(z)=2\pi iz+w$ for some $w\in\mathbb{C}$. Because $M(i)=-2\pi+\log(1728)=-2\pi+w$, it follows that $M(z)=2\pi iz+\log(1728)$.

Let $\{k,l,m\}=\{2,4,\infty\}$, so $a=\frac{1}{8}$ and $b=\frac{3}{8}$ and

$$\begin{aligned}
\lim_{x\to1}D(x) &= \sum_{j=0}^{\infty}\frac{1}{\frac{7}{8}+j}+\frac{1}{\frac{5}{8}+j}-\frac{2}{j+1} \\
&= \frac{\pi\cos(5\pi/8)}{2\sin(5\pi/8)}+\frac{\pi\cos(7\pi/8)}{2\sin(7\pi/8)}+\sum_{j=0}^{\infty}\frac{1/2}{\frac{1}{8}+j}+\frac{1/2}{\frac{3}{8}+j}+\frac{1/2}{\frac{5}{8}+j}+\frac{1/2}{\frac{7}{8}+j}-\frac{2}{j+1} \\
&= -\sqrt{2}\pi+\frac{1}{2}\left(S_8-S_4\right) = -2\pi+\log(256). \qquad (2.14)
\end{aligned}$$

Because the triangle T from table 1.1 has width $\frac{1}{2}$, it follows that $M(z)=2\pi iz+w$ for some $w\in\mathbb{C}$. Because $M(\frac{1}{2}\sqrt{2}i)=-\sqrt{2}\pi+\log(256)=-\sqrt{2}\pi+w$, it follows that $M(z)=2\pi iz+\log(256)$.

Let $\{k,l,m\}=\{2,6,\infty\}$, so $a=\frac{1}{6}$ and $b=\frac{1}{3}$ and

$$\begin{aligned}
\lim_{x\to1}D(x) &= \sum_{j=0}^{\infty}\frac{1}{\frac{5}{6}+j}+\frac{1}{\frac{2}{3}+j}-\frac{2}{j+1} \\
&= \frac{\pi\cos(2\pi/3)}{2\sin(2\pi/3)}+\frac{\pi\cos(5\pi/6)}{2\sin(5\pi/6)}+\sum_{j=0}^{\infty}\frac{1/2}{\frac{1}{6}+j}+\frac{1/2}{\frac{1}{3}+j}+\frac{1/2}{\frac{2}{3}+j}+\frac{1/2}{\frac{5}{6}+j}-\frac{2}{j+1} \\
&= -\frac{2\pi}{\sqrt{3}}+\frac{1}{2}\left(S_6-S_2\right) = -\frac{2\pi}{\sqrt{3}}+\log(108). \qquad (2.15)
\end{aligned}$$

Because the triangle T from table 1.1 has width $\frac{1}{2}$, it follows that $M(z)=2\pi iz+w$ for some $w\in\mathbb{C}$. Because $M(\frac{1}{3}\sqrt{3}i)=-2\pi/\sqrt{3}+\log(108)=-\frac{2}{3}\sqrt{3}\pi+w$, it follows that $M(z)=2\pi iz+\log(108)$.

Let $\{k,l,m\}=\{2,\infty,\infty\}$, so $a=\frac{1}{4}$ and $b=\frac{3}{4}$ and

$$\lim_{x\to1}D(x)=\sum_{j=0}^{\infty}\frac{1}{\frac{1}{4}+j}+\frac{1}{\frac{3}{4}+j}-\frac{2}{j+1}=S_4-S_2=\log(64). \qquad (2.16)$$

Because the triangle T from table 1.1 has width $\frac{1}{2}$, it follows that $M(z) = 2\pi iz + w$ for some $w \in \mathbb{C}$. Because $M(0) = \log(64) = w$, it follows that $M(z) = 2\pi iz + \log(64)$.

Let $\{k, l, m\} = \{3, 3, \infty\}$, so $a = \frac{1}{6}$ and $b = \frac{1}{2}$ and

$$\begin{aligned}\lim_{x\to 1} D(x) &= \sum_{j=0}^{\infty} \frac{1}{\frac{5}{6}+j} + \frac{1}{\frac{1}{2}+j} - \frac{2}{j+1} = \frac{\pi\cos(5\pi/6)}{2\sin(5\pi/6)} + \sum_{j=0}^{\infty} \frac{1/2}{\frac{1}{6}+j} + \frac{1}{\frac{1}{2}+j} + \frac{1/2}{\frac{5}{6}+j} - \frac{2}{j+1} \\ &= -\frac{\sqrt{3}\pi}{2} + \frac{1}{2}(S_6 - S_3 + S_2) = -\frac{\sqrt{3}\pi}{2} + \log(48\sqrt{3}). \end{aligned} \tag{2.17}$$

Because the triangle T from table 1.1 has width 1, it follows that $M(z) = \pi iz + w$ for some $w \in \mathbb{C}$. Because $M(-1/2 + \sqrt{3}i/2) = -\sqrt{3}\pi/2 + \log(48\sqrt{3}) = -\pi i/2 - \sqrt{3}\pi/2 + w$, it follows that $M(z) = \pi iz + \log(48\sqrt{3}) + \pi i/2$.

Let $\{k, l, m\} = \{3, \infty, \infty\}$, so $a = \frac{1}{3}$ and $b = \frac{2}{3}$ and

$$\lim_{x\to 1} D(x) = \sum_{j=0}^{\infty} \frac{1}{\frac{1}{3}+j} + \frac{1}{\frac{2}{3}+j} - \frac{2}{j+1} = S_3 = 3\log(3). \tag{2.18}$$

Because the triangle T from table 1.1 has width $\frac{1}{2}$, it follows that $M(z) = 2\pi iz + w$ for some $w \in \mathbb{C}$. Because $M(0) = \log(27) = w$, it follows that $M(z) = 2\pi iz + \log(27)$.

Let $\{k, l, m\} = \{4, 4, \infty\}$, so $a = \frac{1}{4}$ and $b = \frac{1}{2}$ and

$$\begin{aligned}\lim_{x\to 1} D(x) &= \sum_{j=0}^{\infty} \frac{1}{\frac{3}{4}+j} + \frac{1}{\frac{1}{2}+j} - \frac{2}{j+1} = \frac{\pi\cos(3\pi/4)}{2\sin(3\pi/4)} + \sum_{j=0}^{\infty} \frac{1/2}{\frac{1}{4}+j} + \frac{1}{\frac{1}{2}+j} + \frac{1/2}{\frac{3}{4}+j} - \frac{2}{j+1} \\ &= -\frac{\pi}{2} + \frac{1}{2}(S_4 + S_2) = -\frac{\pi}{2} + \log(32). \end{aligned} \tag{2.19}$$

Because the triangle T from table 1.1 has width 1, it follows that $M(z) = \pi iz + w$ for some $w \in \mathbb{C}$. Because $M(-1/2 + i/2) = -\pi/2 + \log(32) = -\pi i/2 - \pi/2 + w$, it follows that $M(z) = \pi iz + \log(32) + \pi i/2$.

Let $\{k, l, m\} = \{6, 6, \infty\}$, so $a = \frac{1}{3}$ and $b = \frac{1}{2}$ and

$$\begin{aligned}\lim_{x\to 1} D(x) &= \sum_{j=0}^{\infty} \frac{1}{\frac{2}{3}+j} + \frac{1}{\frac{1}{2}+j} - \frac{2}{j+1} = \frac{\pi\cos(2\pi/3)}{2\sin(2\pi/3)} + \sum_{j=0}^{\infty} \frac{1/2}{\frac{1}{3}+j} + \frac{1}{\frac{1}{2}+j} + \frac{1/2}{\frac{2}{3}+j} - \frac{2}{j+1} \\ &= -\frac{\sqrt{3}\pi}{6} + \frac{1}{2}S_3 + S_2 = -\frac{\sqrt{3}\pi}{6} + \log(12\sqrt{3}). \end{aligned} \tag{2.20}$$

Because the triangle T from table 1.1 has width 1, it follows that $M(z) = \pi iz + w$ for some $w \in \mathbb{C}$. Because $M(-1/2 + \sqrt{3}i/6) = -\sqrt{3}\pi/6 + \log(12\sqrt{3}) = -\pi i/2 - \sqrt{3}\pi/6 + w$, it follows that $M(z) = \pi iz + \log(12\sqrt{3}) + \pi i/2$.

Let $\{k, l, m\} = \{\infty, \infty, \infty\}$, so $a = b = \frac{1}{2}$ and

$$\lim_{x\to 1} D(x) = \sum_{j=0}^{\infty} \frac{1}{\frac{1}{2}+j} + \frac{1}{\frac{1}{2}+j} - \frac{2}{j+1} = 2S_2 = \log(16). \tag{2.21}$$

Because the triangle T from table 1.1 has width 1, it follows that $M(z) = \pi iz + w$ for some $w \in \mathbb{C}$. Because $M(0) = \log(16) = w$, it follows that $M(z) = \pi iz + \log(16)$.

k,l,m	a	b	$M(z)$
$2,3,\infty$	$1/12$	$5/12$	$2\pi iz+\log(1728)$
$2,4,\infty$	$1/8$	$3/8$	$2\pi iz+\log(256)$
$2,6,\infty$	$1/6$	$1/3$	$2\pi iz+\log(108)$
$2,\infty,\infty$	$1/4$	$3/4$	$2\pi iz+\log(64)$
$3,3,\infty$	$1/6$	$1/2$	$\pi iz+\log(48\sqrt{3})+\pi i/2$
$3,\infty,\infty$	$1/3$	$2/3$	$2\pi iz+\log(27)$
$4,4,\infty$	$1/4$	$1/2$	$\pi iz+\log(32)+\pi i/2$
$6,6,\infty$	$1/3$	$1/2$	$\pi iz+\log(12\sqrt{3})+\pi i/2$
∞,∞,∞	$1/2$	$1/2$	$\pi iz+\log(16)$

Table 2.1: Table of Möbius transformations M which send the triangle from table 1.1 to the image of the function D from definition 2.1.

2.3 A formula to calculate special values of the hypergeometric function

Now we want to calculate special values of the hypergeometric function, using the composition of the functions h from table 1.2 and D from definition 2.1. However, because the function h does not send the vertices of T from table 1.1 to $0,1,\infty$, the function $D(h(z))$, with h from table 1.2 does not send the vertices of T to the vertices of $D(\mathbb{H})$. To send the vertices of T to the vertices of $D(\mathbb{H})$, define $H(z)=m(h(z))$ as a Möbius transformation of h such that $H(\infty)=m(h(\infty))=0$ and $H(l)=m(h(l))=1$ and $H(r)=m(h(r))=\infty$, where l and r are the left and right vertex of the triangle T from table 1.1. For an example, see figure 2.2 on page 22.

k,l,m	a	b	$h(z)$	$H(z)=m(h(z))$
$2,3,\infty$	$1/12$	$5/12$	$j(z)$	$1728/z$
$2,4,\infty$	$1/8$	$3/8$	$\Delta(z)/\Delta(2z)+2^{12}\Delta(2z)/\Delta(z)$	$256/(z+128)$
$2,6,\infty$	$1/6$	$1/3$	$\eta^{12}(z)/\eta^{12}(3z)+3^6\eta^{12}(3z)/\eta^{12}(z)$	$108/(z+54)$
$2,\infty,\infty$	$1/4$	$3/4$	$\Delta(z)/\Delta(2z)$	$64/(z+64)$
$3,3,\infty$	$1/6$	$1/2$	$\sqrt{1728-j(z)}$	$48\sqrt{3}/(z+24\sqrt{3})$
$3,\infty,\infty$	$1/3$	$2/3$	$\eta^{12}(z)/\eta^{12}(3z)$	$27/(z+27)$
$4,4,\infty$	$1/4$	$1/2$	$\sqrt{128-\Delta(z)/\Delta(2z)-2^{12}\Delta(2z)/\Delta(z)}$	$32/(z+16)$
$6,6,\infty$	$1/3$	$1/2$	$\sqrt{54-\eta^{12}(z)/\eta^{12}(3z)-3^6\eta^{12}(3z)/\eta^{12}(z)}$	$12\sqrt{3}/(z+6\sqrt{3})$
∞,∞,∞	$1/2$	$1/2$	$16\eta^8(z/2)\eta^{16}(2z)/\eta^{24}(z)$	z

Table 2.2: Table of Hauptmoduln as in table 1.2; the Möbius transformation $m(z)$ is such that $H(\infty)=m(h(\infty))=0$ and $H(l)=m(h(l))=1$ and $H(r)=m(h(r))=\infty$, where l and r are the left and right vertex of the triangle T from table 1.1.

Lemma 2.4. *Let D as in definition 2.1, let H as in table 2.2, let M as in table 2.1, let T as in table 1.1. Then it follows that $D(H(z))=M(z)$.*

Proof. The function D is a biholomorphic function from $\mathbb{H}$ to $D(\mathbb{H})$. We also have that H is a biholomorphic function from T to $\mathbb{H}$: using $H(-\overline{z})=\overline{H(z)}$ it can be shown that H sends the edges of T to $\mathbb{R}$. Because H sends the vertices of T to $0,1,\infty$ in counterclockwise order, it follows that H sends T to $\mathbb{H}$ and the reflection of T to $\overline{\mathbb{H}}=\{z\in\mathbb{C}|\Im(z)<0\}$. From this follows that $D\circ H$ is a biholomorphic function from T to $D(\mathbb{H})$. On the other hand, the function M is also a biholomorphic function from T to D.

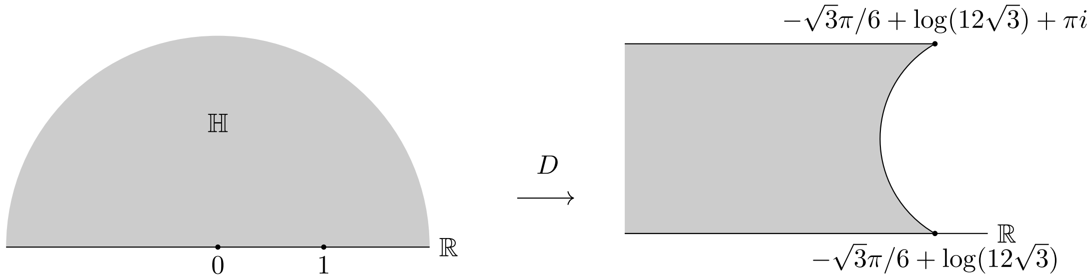

Figure 2.1: The function D from definition 2.1 sends the upper half plane $\mathbb{H}$ biholomorphically to the triangle in the right picture. Here the parameters are equal to $a = 1/3$ and $b = 1/2$ and the inverse angles are equal to $\{6, 6, \infty\}$.

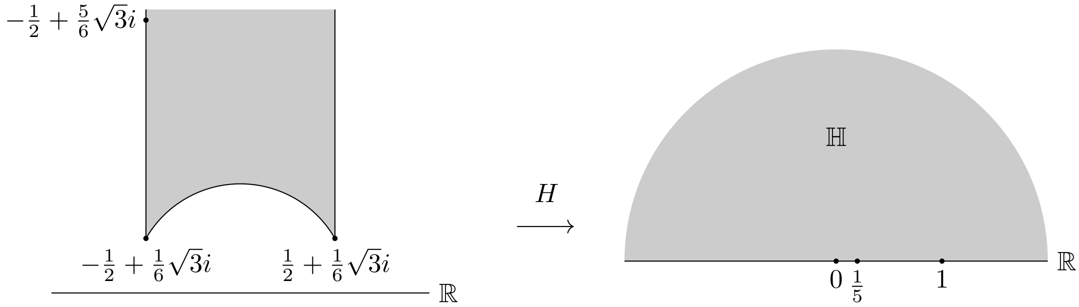

Figure 2.2: The function H from table 2.2 sends the triangle T from table 1.1 biholomorphically to the upper half plane $\mathbb{H}$. Here the parameters are equal to $a = 1/3$ and $b = 1/2$ and the inverse angles are equal to $\{6, 6, \infty\}$. Later in this chapter we will show that $H(-1/2 + 5\sqrt{3}i/6) = 1/5$, which we already indicate in this picture.

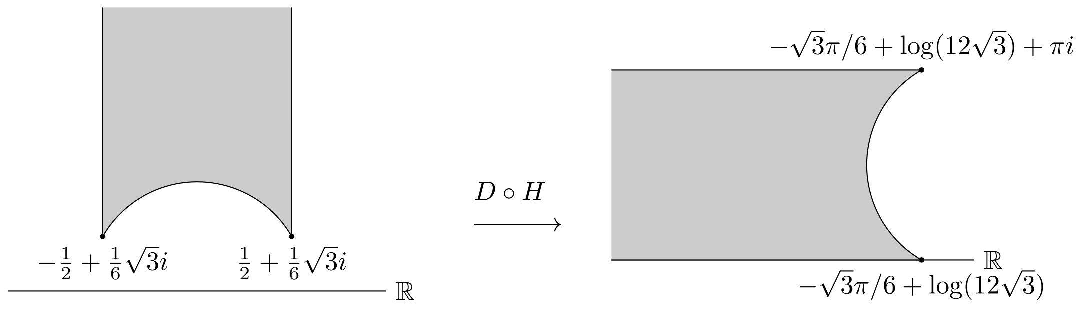

Figure 2.3: The function $D \circ H$ with D from definition 2.1 and H from table 2.2 sends the triangle T from table 1.1 biholomorphically to the triangle in the right picture. Here the parameters are equal to $a = 1/3$ and $b = 1/2$ and the inverse angles are equal to $\{6, 6, \infty\}$.

The function $D^{-1} \circ M \circ H^{-1}$ is biholomorphic function from $\mathbb{H}$ to $\mathbb{H}$, so it is a Möbius transformation. We have $(D^{-1} \circ M \circ H^{-1})(0) = 0$ and $(D^{-1} \circ M \circ H^{-1})(1) = 1$ and $(D^{-1} \circ M \circ H^{-1})(\infty) = \infty$, because $H(\infty) = 0$ and H sends the vertices of T to $0, 1, \infty$ and $D(0) = \infty$ and D sends $0, 1, \infty$ to the vertices of $D(\mathbb{H})$. From this follows that $D^{-1} \circ M \circ H^{-1}$ is the identity function, so $M(z) = D(H(z))$. □

Now we know that $M(z) = D(H(z))$, we can differentiate this relation, from which a formula follows. To find this formula, we start with calculating the derivative of D.

Lemma 2.5. *Let F_0, G_0 as in definition 2.1, then it follows that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{G_0(x)}{F_0(x)}\right) = \frac{x^{-1}(1-x)^{-a-b}}{F_0(x)^2}. \tag{2.22}$$

Proof. Define the Wronskian determinant

$$W(x) := \det\begin{pmatrix} G_0(x) & F_0(x) \\ G_0'(x) & F_0'(x) \end{pmatrix} = G_0'(x)F_0(x) - F_0'(x)G_0(x), \tag{2.23}$$

then it follows that

$$\begin{aligned} W'(x) &= G_0''(x)F_0(x) - F_0''(x)G_0(x) \\ &= \left(-\frac{1-(a+b+1)x}{x(1-x)}G_0'(x) + abG_0(x)\right)F_0(x) - \left(-\frac{1-(a+b+1)x}{x(1-x)}F_0'(x) + abG_0(x)\right)F_0(x) \\ &= -\frac{1-(a+b+1)x}{x(1-x)}W(x) = \left(-\frac{1}{x} + \frac{a+b}{1-x}\right)W(x). \end{aligned} \tag{2.24}$$

From this follows that $W(x)$ is a multiple of $x^{-1}(1-x)^{-a-b}$. Around $x = 0$ we have $G_0(x) \approx \log(x)F_0(x)$, from which follows that $G_0'(x)F_0(x) - G_0(x)F_0'(x) \approx x^{-1}F_0(x)^2 + \log(x)F_0'(x)F_0(x) - \log(x)F_0(x)F_0'(x) = F_0(x)^2/x \approx 1/x$, which is what we wanted to prove. □

Lemma 2.6. *Let D and F_0 as in definition 2.1, let H as in table 2.2, let M as in table 2.1, let T as in table 1.1. Let z be an element of T or one of its reflections such that $|H(z)| < 1$, then it follows that*

$$F_0(H(z))^2 = \frac{H(z)^{-1} \cdot (1-H(z))^{-a-b} \cdot H'(z)}{M'(z)}. \tag{2.25}$$

Proof. We have $D(H(z)) = M(z)$ for $z \in T$, note that this relation does not always hold if z is not an element of T, because D is a multivalued function. From differentiating $D(H(z)) = M(z)$ follows that $M'(z) = D'(H(z)) \cdot H'(z)$. Using lemma 2.5 gives the result. □

Lemma 2.7. *Let D, F_0, G_0 as in definition 2.1, let H as in table 2.2, let M as in table 2.1, let T as in table 1.1. Let z be an element of T or one of its reflections. Let $L(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be a Möbius transformation, then it follows that*

$$(\gamma G_0(H(z)) + \delta F_0(H(z)))^2 = (\gamma M(z) + \delta)^2 \frac{H(z)^{-1}(1-H(z))^{-a-b}H'(z)}{M'(z)}. \tag{2.26}$$

Proof. From $M(z) = D(H(z))$ follows that $LM(z) = LD(H(z))$ and

$$L'(M(z)) \cdot M'(z) = \frac{\det(L)M'(z)}{(\gamma M(z)+\delta)^2} = (LD)'(H(z)) \cdot H'(z) = \frac{\det(L)H(z)^{-1}(1-H(z))^{-a-b}H'(z)}{(\gamma G_0(H(z)) + \delta F_0(H(z)))^2}, \tag{2.27}$$

from which the result follows. □

Lemma 2.8. *Let $0 < a, b, a+b < 1$ and F_0, G_0, F_1 as in definition 2.1. Then it follows for $z \in \mathbb{C}$ with $|x|, |1-x| < 1$ that*

$$ {}_2F_1\left(a, b; a+b \,|1-x\right) = F_1(x) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} G_0(x) + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} \right) F_0(x). \tag{2.28} $$

Proof. We have

$$ \frac{(a)_n (b)_n}{(a+b)_n \cdot n!} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \frac{1}{n} + O(1/n^2). \tag{2.29} $$

From this follows that around $x = 0$ we have $F_1(x) \approx -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(x)$, which gives us the coefficient of $G_0(x)$, here the approximation means that $F_1(x) + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(x)$ is finite for $x \to 0$. Now write $F_1(x) + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} G_0(x) = AF_0(x)$ for some $A \in \mathbb{C}$, then it follows that

$$ A = \frac{F_1(x)}{F_0(x)} + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{G_0(x)}{F_0(x)}. \tag{2.30} $$

We have $F_1(1) = 1$ and $F_0(1) = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}$, from which follows that the limit $x \to 1$ of equation (2.30) is equal to

$$ \begin{aligned} A &= \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-a-b)} + \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\sum_{j=0}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1} \right) \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\frac{\pi \sin(\pi a + \pi b)}{\sin(\pi a)\sin(\pi b)} + \sum_{j=0}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1} \right) \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} \right). \end{aligned} \tag{2.31} $$

Here we used lemma 2.3, and the identities $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and equation (2.10). □

Corollary 2.9. *Let $0 < a, b, a+b < 1$, let T as in table 1.1, let M as in table 2.1, let H as in table 2.2, let v be the left vertex from table 1.1. Let z be an element of T or one of its reflections such that $|1 - H(z)| < 1$, then it follows that*

$$ {}_2F_1\left(a, b; a+b \,|1 - H(z)\right)^2 \tag{2.32} $$

is equal to

$$ \begin{aligned} &\frac{(1-H(z))^{-a-b}}{H(z) \cdot M'(z)} \cdot \frac{\Gamma(a+b)^2}{\Gamma(a)^2\Gamma(b)^2} \left(M(z) + \sum_{j=0}^{\infty} -\frac{1}{a+j} - \frac{1}{b+j} + \frac{2}{j+1} \right)^2 \cdot H'(z) \\ =&\frac{(1-H(z))^{-a-b}}{H(z) \cdot M'(z)} \cdot \frac{\Gamma(a+b)^2}{\Gamma(a)^2\Gamma(b)^2} \left(M' \cdot (z-v) - \pi\frac{\cos(\pi a)}{\sin(\pi a)} - \pi\frac{\cos(\pi b)}{\sin(\pi b)} \right)^2 \cdot H'(z). \end{aligned} \tag{2.33} $$

Proof. The proof follows from lemma 2.7 and using a Möbius transformation with lower entries equal to the numbers given in lemma 2.8. To prove the last equality, note that $M(z) = M' \cdot (z - v) + \sum_{j=0}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1}$, where M' is a constant, because $M'(z)$ is a linear function. □

Remark 2.10. Another method to give a proof of corollary 2.9 starts with the observation that $\frac{G_1(H(z))}{F_1(H(z))}$ is a Möbius transformation of the triangle T, where G_1, F_1 are as in definition 2.1. Because $a+b<1$, it follows that $G_1(H(v))/F_1(H(v)) = G_1(1)/F_1(1) = 0$ and

$$\lim_{z\to\infty}\frac{G_1(H(z))}{F_1(H(z))} = \lim_{x\to 0}\frac{G_1(x)}{F_1(x)} = \frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)}\Big/\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} = \frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}. \tag{2.34}$$

From this follows that

$$\frac{G_1(H(z))}{F_1(H(z))} = \frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}\cdot\frac{z-v}{z+C}, \tag{2.35}$$

for some still unknown complex number C. It can be shown that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{G_1(x)}{F_1(x)}\right) = (a+b-1)\frac{x^{-1}(1-x)^{-a-b}}{F_1(x)^2}, \tag{2.36}$$

from which follows that

$$\begin{aligned}\lim_{z\to v}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{G_1(H(z))}{F_1(H(z))}\right) &= \lim_{z\to v}(a+b-1)\frac{H(z)^{-1}(1-H(z))^{-a-b}}{F_1(H(z))^2}H'(z)\\ &= (a+b-1)\lim_{z\to v}(1-H(z))^{-a-b}H'(z).\end{aligned} \tag{2.37}$$

From lemma 2.6 we know that

$$\lim_{z\to v}F_0(H(z))^2 = F_0(1)^2 = \lim_{z\to v}\frac{H(z)^{-1}\cdot(1-H(z))^{-a-b}\cdot H'(z)}{M'(z)} = \lim_{z\to v}\frac{(1-H(z))^{-a-b}\cdot H'(z)}{M'(z)}, \tag{2.38}$$

from which follows that

$$\lim_{z\to v}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{G_1(H(z))}{F_1(H(z))}\right) = (a+b-1)M'\cdot F_0(1)^2 = (a+b-1)\cdot M'\frac{\Gamma(1-a-b)^2}{\Gamma(1-a)^2\Gamma(1-b)^2}. \tag{2.39}$$

On the other hand, we have

$$\lim_{z\to v}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}\cdot\frac{z-v}{z+C}\right) = \frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}\cdot\frac{1}{v+C}. \tag{2.40}$$

From comparing equation (2.39) and equation (2.40) follows the value of C, which gives

$$\frac{G_1(H(z))}{F_1(H(z))} = \frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}\cdot\frac{z-v}{z-v-\frac{\pi}{M'}\left(\frac{\cos(a\pi)}{\sin(a\pi)}+\frac{\cos(b\pi)}{\sin(b\pi)}\right)}. \tag{2.41}$$

From differentiating both sides of equation (2.41) follows that

$$(a+b-1)\frac{(1-H(z))^{-a-b}}{H(z)\cdot F_1(H(z))^2}H'(z) = \frac{\Gamma(2-a-b)\Gamma(a)\Gamma(b)}{\Gamma(1-a)\Gamma(1-b)\Gamma(a+b)}\cdot\frac{-\frac{\pi}{M'}\left(\frac{\cos(a\pi)}{\sin(a\pi)}+\frac{\cos(b\pi)}{\sin(b\pi)}\right)}{\left(z-v-\frac{\pi}{M'}\left(\frac{\cos(a\pi)}{\sin(a\pi)}+\frac{\cos(b\pi)}{\sin(b\pi)}\right)\right)^2}, \tag{2.42}$$

from which the result follows.

2.4 First example

Theorem 2.11 ([1, chapter 5, theorem 3]). *We have*

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\middle|\frac{1323}{1331}\right)=\frac{3}{4}\sqrt[4]{11}. \tag{2.43}$$

We will give two proofs of this result.

First proof. We use corollary 2.9 with $a=1/12$ and $b=5/12$ and $H(z)=1728/j(z)$ and $M(z)=2\pi iz+\log(1728)$, see table 2.1 and table 2.2. We have $\sum_{j=0}^{\infty}\frac{1}{a+j}+\frac{1}{b+j}-\frac{2}{j+1}=-2\pi-\log(1728)$. From this follows that

$$\begin{aligned} {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\middle|1-\frac{1728}{j(z)}\right)^2 &= \frac{\left(1-\frac{1728}{j(z)}\right)^{-1/2}}{2\pi i\cdot 1728/j(z)}\cdot\frac{\pi}{\Gamma(1/12)^2\Gamma(5/12)^2}\left(2\pi iz-2\pi\right)^2\cdot\left(\frac{1728}{j(z)}\right)' \\ &= \frac{2\pi^2 i\left(1-\frac{1728}{j(z)}\right)^{-1/2}(iz-1)^2}{\Gamma(1/12)^2\Gamma(5/12)^2 j(z)}\cdot j'(z). \end{aligned} \tag{2.44}$$

Let $z=2i$, we have $j(2i)=66^3=287\,496$, from which follows that

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\middle|\frac{1323}{1331}\right)^2=\frac{\pi^2 i\sqrt{\frac{1331}{1323}}}{15972\Gamma(1/12)^2\Gamma(5/12)^2}\cdot j'(z) \tag{2.45}$$

Now we want to calculate $j'(z)$, note that for a Möbius transformation $\gamma(z)=\frac{az+b}{cz+d}\in \mathrm{SL}_2(\mathbb{Z})$ we have $j(\gamma z)=j(z)$, from which follows that

$$ j'(\gamma z)\gamma'(z)=(cz+d)^{-2}j'(\gamma z)=j'(z). \tag{2.46}$$

Let Δ be the unique normalized cusp form of weight 12, then it follows that $\Delta(z)\cdot j'(z)$ is a modular form of weight 14, because $\lim_{z\to\infty}\Delta(z)j'(z)=-2\pi i$, this can be calulated using the q-expansion of $j'(z)$ and $\Delta(z)$. The space of modular forms of weight 14 with respect to $\mathrm{SL}_2(\mathbb{Z})$ is onedimensional, and spanned by $E_4^2(z)E_6(z)$, with

$$ E_k(z)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n. \tag{2.47}$$

Because the constant coefficient of $E_4^2(z)E_6(z)$ is equal to 1, it follows that

$$ \Delta(z)j'(z)=-2\pi iE_4^2(z)E_6(z). \tag{2.48}$$

It is known that $\eta(2i)=2^{-11/8}\pi^{-3/4}\Gamma(1/4)$, from which follows that $\Delta(2i)=2^{-33}\pi^{-18}\Gamma(1/4)^{24}$. We have

$$ E_4(z)^3=\Delta(z)j(z) \quad\text{and}\quad E_6^2(z)=E_4(z)^3-1728\Delta(z)=\Delta(z)(j(z)-1728). \tag{2.49}$$

From this follows that $E_4(2i)^3=2^{-30}\cdot 3^3\cdot 11^3\pi^{-18}\Gamma(1/4)^{24}$ and $E_6(2i)^2=2^{-30}\cdot 3^6\cdot 7^2\pi^{-18}\Gamma(1/4)^{24}$. We know that $E_4(2i)$ and $E_6(2i)$ are real and positive, so

$$ E_4(2i)^2=2^{-20}\cdot 3^2\cdot 11^2\pi^{-12}\Gamma(1/4)^{16} \quad\text{and}\quad E_6(2i)=2^{-15}\cdot 3^3\cdot 7\pi^{-9}\Gamma(1/4)^{12} \tag{2.50}$$

and

$$j'(2i) = -2\pi i \frac{E_4(2i)^2 E_6(2i)}{\Delta(2i)} = -2\pi i \frac{2^{-35}\cdot 3^3 \cdot 7 \cdot 11^2 \pi^{-21}\Gamma(1/4)^{28}}{2^{-33}\pi^{-18}\Gamma(1/4)^{24}} = -i\frac{205821\Gamma(1/4)^4}{2\pi^2}. \quad (2.51)$$

From this follows that

$$\begin{aligned} {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2 &= \frac{\pi^2 i\sqrt{\frac{1331}{1323}}}{15972\Gamma(1/12)^2\Gamma(5/12)^2}\cdot -i\frac{205821\Gamma(1/4)^4}{2\pi^2} \\ &= \frac{205821\sqrt{\frac{1331}{1323}}}{2\cdot 15972}\cdot\frac{\Gamma(1/4)^4}{\Gamma(1/12)^2\Gamma(5/12)^2} = \frac{9\sqrt{33}}{8}\frac{\Gamma(1/4)^4}{\Gamma(1/12)^2\Gamma(5/12)^2}. \end{aligned} \quad (2.52)$$

From $\prod_{k=0}^{m-1}\Gamma(z+k/m) = (2\pi)^{(m-1)/2}m^{1/2-mz}\Gamma(mz)$ with $z = 1/12$ and $m = 3$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ with $z = 1/4$ follows that

$$\frac{\Gamma(1/4)^2}{\Gamma(1/12)\Gamma(5/12)} = \frac{\Gamma(1/4)^2\Gamma(3/4)}{\Gamma(1/12)\Gamma(5/12)\Gamma(3/4)} = \frac{\Gamma(1/4)^2\Gamma(3/4)}{2\pi\sqrt[4]{3}\Gamma(1/4)} = \frac{\Gamma(1/4)\Gamma(3/4)}{2\pi\sqrt[4]{3}} = \frac{1}{\sqrt{2}\sqrt[4]{3}} \quad (2.53)$$

and

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2 = \frac{9\sqrt{33}}{8}\cdot\frac{1}{2\sqrt{3}} = \frac{9\sqrt{11}}{16}. \quad (2.54)$$

The result follows, because ${}_2F_1\left(1/12, 5/12; 1/2\,|1323/1331\right)$ is real and positive. □

Second proof. We will evaluate equation (2.44) in $z = 2i$ and $z = i$. Note that $j(i) = 1728$ and $j(2i) = 66^3 = 287\,496$, from which follows that

$$ {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2 = \frac{{}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|1-\frac{1728}{j(2i)}\right)^2}{{}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|1-\frac{1728}{j(i)}\right)^2} = \frac{18\left(1-\frac{1728}{j(2i)}\right)^{-1/2}\cdot j'(2i)}{11^3\left(1-\frac{1728}{j(z)}\right)^{-1/2}\cdot j'(z)}. \quad (2.55)$$

The function $j(2z)$ is a modular function with respect to $\Gamma_0(2)$, so the polynomial

$$P(x,z) = (x-j(2z))\left(x-j\left(2\frac{-1}{z}\right)\right)\left(x-j\left(2\frac{z}{z+1}\right)\right) = (x-j(2z))(x-j(z/2))(x-j(z/2+1/2)) \quad (2.56)$$

has coefficients which are modular with respect to $\mathrm{SL}_2(\mathbb{Z})$. From this follows that the coefficients of $P(x,z)$ are in $\mathbb{C}(j(z))$. Moreover, because $j(2z)$ is holomorphic on $\mathbb{H}$, it follows that the coefficients of $P(x,z)$ are in $\mathbb{C}[j(z)]$. We will calculate the q-expansions of the coefficients of $P(x,z)$, where we omit all positive powers of q; here we use that a modular function with respect to $\mathrm{SL}_2(\mathbb{Z})$ which is holomorphic on $\mathbb{H}$ and goes to 0 for $z\to\infty$ is the zero function. The q-expansion of the coefficient of x^2 is equal to

$$-\left(q^{-2}+744\right)-\left(q^{1/2}+744\right)-\left(-q^{-1/2}+744\right)\equiv -q^{-2}-2232\equiv -j(z)^2+1488j(z)-162000. \quad (2.57)$$

The coefficient of x is equal to

$$\begin{aligned} &\left(q^{-2}+744\right)\left(q^{1/2}+744\right)+\left(q^{-2}+744\right)\left(-q^{-1/2}+744\right)+\left(q^{1/2}+744\right)\left(-q^{-1/2}+744\right) \\ &\equiv 1488q^{-2}+42987519q^{-1}+40492979352\equiv 1488j(z)^2+40773375j(z)+8748000000. \end{aligned} \quad (2.58)$$

The constant coefficient is equal to

$$\begin{aligned}&-\left(q^{-2}+744\right)\left(q^{1/2}+744\right)\left(-q^{-1/2}+744\right)\equiv q^{-3}-159768q^{-2}+8509195260q^{-1}-151107596045760\\&\equiv j(z)^3-162000j(z)^2+8748000000j(z)-157464000000000. \qquad (2.59)\end{aligned}$$

From this follows that $P(j(2z),j(z))$ is equal to the zero function with

$$\begin{aligned}P(x,y)=x^3+(-y^2+1488y-162000)x^2+(1488y^2+40773375y+8748000000)x\\+(y^3-162000y^2+8748000000y-157464000000000). \qquad (2.60)\end{aligned}$$

Differentiating with respect to z gives that $2P_x(z)j'(2z)+P_y(z)j'(z):=2P_x(x,y)j'(2z)+P_y(x,y)j'(z)=0$, where P_x and P_y denote the derivatives of P with respect to the variables x and y; we also write $P_x(z)$ instead of $P_x(j(2z),j(z))$ and $P_y(z)$ instead of $P_y(j(2z),j(z))$. From this follows that

$$\frac{j'(2z)}{j'(z)}=-\frac{P_y(z)}{2P_x(z)} \qquad (2.61)$$

and

$$\begin{aligned}{}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2&=-\frac{18}{11^3}\lim_{z\to i}\sqrt{\frac{1-1728/j(z)}{1-1728/j(2z)}}\frac{P_y(z)}{2P_x(z)}\\&=2^3\cdot3^{12}\cdot7^3\cdot11^2\sqrt{33}\lim_{z\to i}\frac{\sqrt{1-\frac{1728}{j(z)}}}{P_x(z)}, \qquad (2.62)\end{aligned}$$

where we calculated the nonzero limit terms in the quotient. Because in this limit both the numerator and denominator go to 0, we use the rule of l'Hôpital, from which follows that

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2=2^{-4}\cdot3^9\cdot7^3\cdot11^2\sqrt{33}\lim_{z\to i}\frac{\left(1-\frac{1728}{j(z)}\right)^{-1/2}j'(z)}{2P_{xx}(z)j'(2z)+P_{xy}(z)j'(z)}. \qquad (2.63)$$

Dividing by $j'(z)$ and using equation (2.61) gives that

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2=2^{-4}\cdot3^9\cdot7^3\cdot11^2\sqrt{33}\lim_{z\to i}\frac{\left(1-\frac{1728}{j(z)}\right)^{-1/2}P_x(z)}{-P_y(z)P_{xx}(z)+P_x(z)P_{xy}(z)}. \qquad (2.64)$$

The denominator in equation (2.64) has a nonzero limit, from which follows that

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^2=2^{-11}\cdot3^{-9}\cdot7^{-3}\cdot11^{-2}\sqrt{33}\lim_{z\to i}\frac{P_x(z)}{\sqrt{1-\frac{1728}{j(z)}}}. \qquad (2.65)$$

Multiplying equation (2.62) and equation (2.65) gives that

$${}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\,\middle|\,\frac{1323}{1331}\right)^4=\frac{3^4\cdot11}{2^8}. \qquad (2.66)$$

The result follows, because ${}_2F_1\left(1/12,5/12;1/2\,|1323/1331\right)$ is real and positive. □

Remark 2.12. It is possible to finish the second proof of theorem 2.11 in another way, which does not use the rule of l'Hôpital. We know that $j'(z)$ has a root of order 2 at $z = e^{\pi i/3}$ and a root of order 1 at $z = i$, but not other roots. From this follows that we can write $j(2z+2i) = 66^3 + \alpha z + O(z^2)$ and $j(z+i) = 1728 + \beta z^2 + O(z^3)$ with $\alpha \in i\mathbb{R}_{<0}$ and $\beta \in \mathbb{R}_{<0}$. From this follows that $\lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z} j(2z+2i) = 2j'(2z+2i) = \alpha$ and $\lim_{z\to 0} j'(2z+2i) = \alpha/2$. We also have

$$\frac{j'(z+i)}{\sqrt{1-\frac{1728}{j(z+i)}}} = \frac{2\beta z + O(z^2)}{\frac{\sqrt{\beta}z}{24\sqrt{3}} + O(z^2)} = \frac{2\beta + O(z)}{\frac{\sqrt{\beta}}{24\sqrt{3}} + O(z)} = 48\sqrt{3\beta} + O(z), \tag{2.67}$$

where we have $\sqrt{3\beta} \in i\mathbb{R}_{<0}$; this follows from the fact that for $z \in i\mathbb{R}_{>1}$ we have $\sqrt{1-1728/j(z)} \in \mathbb{R}_{>0}$. From this follows that

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,\middle|\, \frac{1323}{1331}\right)^2 = \lim_{z\to i} \frac{18\left(1-\frac{1728}{j(2i)}\right)^{-1/2} \cdot j'(2i)}{11^3\left(1-\frac{1728}{j(z)}\right)^{-1/2} \cdot j'(z)} = \frac{18\sqrt{1331}}{11^3\sqrt{1323}} \cdot \frac{\alpha/2}{48\sqrt{3\beta}}. \tag{2.68}$$

Because the coefficient of z^2 in $P(66^3+\alpha z, 1728+\beta z^2)$ is equal to 0, it follows that $\alpha/\sqrt{\beta} = 22\,869$, from which follows that

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,\middle|\, \frac{1323}{1331}\right)^2 = \frac{18\sqrt{1331}}{11^3\sqrt{1323}} \cdot \frac{22\,869}{48\sqrt{3}} = \frac{18\sqrt{1331}}{11^3\sqrt{1323}} \cdot \frac{22\,869/2}{48\sqrt{3}} = \frac{9}{16}\sqrt{11}. \tag{2.69}$$

Corollary 2.13. *We have*

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,\middle|\, \frac{8}{1331}\right) = \frac{\sqrt[4]{33}\Gamma(1/4)^2}{4\sqrt{2}\pi^{3/2}}. \tag{2.70}$$

Note that this special value of the hypergeometric function is not algebraic, which falls outside this thesis.

Proof. From lemma 2.6 follows that

$$F_0(H(z))^2 = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,\middle|\, \frac{1728}{j(z)}\right)^2 = -\frac{1}{2\pi i \cdot j(z)}\left(1-\frac{1728}{j(z)}\right)^{-1/2} j'(z), \tag{2.71}$$

where $M(z) = 2\pi i z + \log(1728)$ and $H(z) = 1728/j(z)$. We have $j(2i) = 66^3$, and from the first proof of theorem 2.11 we know that $j'(2i) = -i\frac{205821\Gamma(1/4)^4}{2\pi^2}$. Because ${}_2F_1\left(1/12, 5/12; 1 \,\middle|\, \frac{8}{1331}\right)$ is real and positive, the result follows. □

In the first proof of theorem 2.11 we do not have to calculate a polynomial in two variables, but is is a transcendental proof in the sense that the proof ends with some combination of Γ-function factors which happens to be algebraic. The second proof is algebraic, in the sense that we calculate a limit of the square root of a rational function, which arguments have a rational limit.

2.5 Transcendental method

In theorem 2.15 we prove the identity which is conjectured in [4, equation 5.3], where we also use theorem 2.14.

Theorem 2.14. *[Chowla-Selberg formula, [3, Page 110, formula 2]]For a fundamental discriminant d it follows that*

$$\prod_{[a,b,c]\in H(d)} a^{-1/4}\left|\eta\left(\frac{b+\sqrt{d}}{2a}\right)\right| = (2\pi|d|)^{-h(d)/4}\left(\prod_{m=1}^{|d|}\Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right)^{w(d)/8}, \tag{2.72}$$

where $H(d)$ is a complete set of non-equivalent primitive binary quadratic forms of discriminant d, and $h(d)$ is the class number of the discrimimant d. Moreover, $w(-3)=6$ and $w(-4)=4$ and $w(d)=2$ for $d<-4$ and $\left(\frac{d}{m}\right)$ is the Jacobi symbol.

Theorem 2.15. *We have*

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\middle|\frac{4}{5}\right) = \frac{3}{\sqrt{5}}. \tag{2.73}$$

Proof. Let $a=1/3$ and $b=1/2$ and $M(z)=\pi iz+\log(12\sqrt{3})+\pi i/2$ and

$$H(z) = \frac{12\sqrt{3}}{6\sqrt{3}+\sqrt{54-\frac{\eta^{12}(z)}{\eta^{12}(3z)}-3^6\frac{\eta^{12}(3z)}{\eta^{12}(z)}}}, \tag{2.74}$$

see table 2.1 and table 2.2. From this follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}|1-H(z)\right)^2 = \frac{(1-H(z))^{-5/6}}{\pi i\cdot H(z)}\cdot\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2\Gamma(1/2)^2}\left(\pi iz+\pi i/2-\sqrt{3}\pi/6\right)^2 H'(z). \tag{2.75}$$

Let $y=-\frac{1}{2}+\frac{5}{6}\sqrt{3}i$. We know that $j(y)$ is an algebraic integer, and using the theory of complex multiplication we will calculate its value; in this situation we can calculate this without difficulties, the general method is by searching for the Hilbert class polynomial of a given discriminant. The lattice $3\mathbb{Z}+\mathbb{Z}3y$ is an ideal in the order $\mathcal{O}=\mathbb{Z}+5\mathbb{Z}(\frac{1}{2}+\frac{1}{2}\sqrt{3}i)$ in the ring of integers $\mathcal{O}_K$ of $K=\mathbb{Q}(\sqrt{-3})$. The class number of $\mathcal{O}$ is equal to

$$h(\mathcal{O}) = \frac{h(\mathcal{O}_K)f}{[\mathcal{O}_K^\times:\mathcal{O}^\times]}\prod_{p|f}\left(1-\left(\frac{d_K}{p}\right)\frac{1}{p}\right) = \frac{1\cdot 5}{3}\prod_{p|5}\left(1-\left(\frac{-3}{p}\right)\frac{1}{p}\right) = 2, \tag{2.76}$$

where f is the conductor of the order $\mathcal{O}=\mathbb{Z}+f\mathcal{O}_K$. The ideal $3\mathbb{Z}+3y\mathbb{Z}=3\mathbb{Z}+\left(-\frac{3}{2}+\frac{5}{2}\sqrt{3}i\right)\mathbb{Z}$ is not a principal ideal in $\mathcal{O}$, because 3 has norm 9 and $3y$ has norm 21, but $\mathcal{O}$ does not contain elements of order 3. The square of this ideal is equal to $\mathbb{Z}+\mathbb{Z}(3y+2)=\mathbb{Z}+3y\mathbb{Z}$, which is a principal ideal. Because the ideal classes in $\mathcal{O}$ are given by $3\mathbb{Z}+3y\mathbb{Z}$ and $\mathbb{Z}+3y\mathbb{Z}$, it follows that the Galois conjugates of $j(y)$ are given by $j(y)$ and $j(3y)$. The polynomial $(x-j(y))(x-j(3y))$ has integer coefficients because $j(y)$ is an algebraic integer. A numerical calculation gives that

$$j(y) = 1728\cdot 512(-369830+165393\sqrt{5}) = -1728\cdot 512\cdot\sqrt{5}(2-\sqrt{5})^6\left(\frac{3}{2}\sqrt{5}-\frac{1}{2}\right). \tag{2.77}$$

We know that $\frac{\Delta(z)}{\Delta(3z)}$ is a modular function with respect to $\Gamma_0(3)$. From this follows that the polynomial

$$P(x,z) = \prod_{\gamma\in\Gamma_0(3)\backslash\mathrm{SL}_2(\mathbb{Z})}\left(x-\frac{\Delta(\gamma z)}{\Delta(3\gamma z)}\right) \tag{2.78}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$, so has coefficients in $\mathbb{C}(j(z))$. Because the functions $\Delta(\gamma z)/\Delta(3\gamma z)$ are holomorphic on $\mathbb{H}$, it follows that $P(x,z)$ has coefficients in $\mathbb{C}[j(z)]$. Calculating the q-expansion of the coefficients of $P(x,z)$ gives that $P(x,z)$ has coefficients in $\mathbb{Z}[j(z)]$. Because

$j(y) = 1728 \cdot 512(-369830 + 165393\sqrt{5})$, it follows that $P(x, y)$ has coefficients in $\mathbb{Q}(\sqrt{5})$. The algebraic conjugate of $j(y)$ is equal to $j(3y)$, so $P(x, y)P(x, 3y)$ has integer coefficients, because $j(y)$ is an algebraic integer. A calculation in Mathematica gives that a factor of $P(x, y)P(x, 3y)$ is equal to $531441 - 75584178x + x^2$, from which follows that

$$\frac{\Delta\left(-\frac{1}{2} + \frac{5}{6}\sqrt{3}i\right)}{\Delta\left(-\frac{3}{2} + \frac{5}{2}\sqrt{3}i\right)} = \frac{\Delta\left(-\frac{1}{2} + \frac{5}{6}\sqrt{3}i\right)}{\Delta\left(-\frac{5}{2} + \frac{5}{2}\sqrt{3}i\right)} = 729(51841 + 23184\sqrt{5}) = 2^{-24}3^6(\sqrt{5} + 1)^{24}. \tag{2.79}$$

From equation (2.79) we know that $\eta^{12}(y)/\eta^{12}(3y) = -27(2 + \sqrt{5})^4$, where the sign follows from the definition of the η function. From this follows that $H(y) = 1/5$ and

$$_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = -2^{-5/3} \cdot 3 \cdot 5^{11/6} i \frac{\Gamma(5/6)^2}{\Gamma(1/3)^2} H'(y). \tag{2.80}$$

We have

$$H'(z) = \frac{-12\sqrt{3}}{\left(6\sqrt{3} + \sqrt{54 - \frac{\eta^{12}(z)}{\eta^{12}(3z)} - 3^6\frac{\eta^{12}(3z)}{\eta^{12}(z)}}\right)^2} \cdot \frac{-1 + 3^6\left(\frac{\eta^{12}(3z)}{\eta^{12}(z)}\right)^2}{2\sqrt{54 - \frac{\eta^{12}(z)}{\eta^{12}(3z)} - 3^6\frac{\eta^{12}(3z)}{\eta^{12}(z)}}} \cdot \left(\frac{\eta^{12}(z)}{\eta^{12}(3z)}\right)', \tag{2.81}$$

from which follows that

$$\begin{aligned} h(y)' &= \frac{1}{675}\left(-360 + 161\sqrt{5}\right) \cdot \left(\frac{\eta^{12}(y)}{\eta^{12}(3y)}\right)' \\ &= \frac{1}{675}\left(-360 + 161\sqrt{5}\right) \cdot 12 \cdot -27(2 + \sqrt{5})^4 \frac{(\eta(y)/\eta(3y))'}{\eta(y)/\eta(3y)} \end{aligned} \tag{2.82}$$

and

$$_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = 2^{1/3} \cdot 3^2 \cdot 5^{1/3} i \cdot \frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\left(\frac{\eta'(y)}{\eta(y)} - 3\frac{\eta'(3y)}{\eta(3y)}\right). \tag{2.83}$$

We know that $\frac{\eta'(z)}{\eta(z)} = \frac{\pi i}{12}E_2(z)$ with $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$. Moreover, $E(z) = \frac{3}{2}E_2(3z) - \frac{1}{2}E_2(z)$ is a modular form with constant coefficient 1, and of weight 2 with respect to the group $\Gamma_0(3)$. From this follows that

$$\frac{\eta'(z)}{\eta(z)} - 3\frac{\eta'(3z)}{\eta(3z)} = \frac{\pi i}{12}E_2(z) - 3\frac{\pi i}{12}E_2(3z) = -\frac{\pi i}{6}E(z), \tag{2.84}$$

from which follows that

$$_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = 2^{-2/3} \cdot 3 \cdot 5^{1/3}\pi \cdot \frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}E(y). \tag{2.85}$$

A calculation in SageMath gives that the space of modular forms of weight 8 with respect to the group $\Gamma_0(3)$ is 3-dimensional, from which follows that $E^4, E^2 \cdot E_4, E \cdot E_6, E_4^2$ are linearly dependent, with $E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$. A calculation gives $-27E^4(y) + 18E^2(y)E_4(y) + 8E(y)E_6(y) + E_4^2(y) = 0$, from which after dividing by $|\eta^{16}(y)|$ follows that

$$-27\left(\frac{E(y)}{|\eta^4(y)|}\right)^4 + 18\left(\frac{E(y)}{|\eta^4(y)|}\right)^2\frac{E_4(y)}{|\eta^8(y)|} + 8\frac{E(y)}{|\eta^4(y)|}\frac{E_6(y)}{|\eta^{12}(y)|} + \left(\frac{E_4(y)}{|\eta^8(y)|}\right)^2 = 0. \tag{2.86}$$

We know that $E(y)$ and $E_4(y)$ and $E_6(y)$ are real and positive, from which follows that

$$\begin{aligned}\frac{E_4(y)}{|\eta^8(y)|} &= \sqrt[3]{j(y)} = 48\sqrt[6]{5}(-69+31\sqrt{5})\\ \frac{E_6(y)}{|\eta^{12}(y)|} &= \sqrt{j(y)-1728} = -233496\sqrt{3}+104448\sqrt{15}. \end{aligned} \tag{2.87}$$

From this follows that

$$\begin{aligned}&-27\left(\frac{E(y)}{|\eta^4(y)|}\right)^4 + 864\sqrt[6]{5}(-69+31\sqrt{5})\left(\frac{E(y)}{|\eta^4(y)|}\right)^2\\ &\quad + 192\sqrt{3}(-9729+4352\sqrt{5})\frac{E(y)}{|\eta^4(y)|} + 2304\cdot\sqrt[3]{5}\cdot(69-31\sqrt{5})^2 = 0 \end{aligned} \tag{2.88}$$

and

$$\begin{aligned}&-45\left(\frac{E(y)}{\sqrt[3]{5}|\eta^4(y)|}\right)^4 + 288(155-69\sqrt{5})\left(\frac{E(y)}{\sqrt[3]{5}|\eta^4(y)|}\right)^2\\ &\quad + 64\sqrt{3}(-9729+4352\sqrt{5})\frac{E(y)}{\sqrt[3]{5}|\eta^4(y)|} + 1536(4783-2139\sqrt{5}) = 0. \end{aligned} \tag{2.89}$$

Solving this equation gives $\frac{E(y)}{\sqrt[3]{5}|\eta^4(y)|} = 2(3\sqrt{3}-\sqrt{15})$ and $E(y) = 2\sqrt[3]{5}(3\sqrt{3}-\sqrt{15})|\eta^4(y)|$. From this follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = 2^{1/3}\cdot 3^{3/2}\cdot 5^{2/3}(3-\sqrt{5})\pi\cdot\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}|\eta^4(y)|. \tag{2.90}$$

From theorem 2.14 follows that

$$\eta\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right) = e^{\pi i/24}2^{-1/4}3^{-1/4}\pi^{-1/4}\frac{\Gamma(1/3)^{3/4}}{\Gamma(2/3)^{3/4}}. \tag{2.91}$$

We know that $\frac{\Delta(z)}{\Delta(5z)}$ is a modular function with respect to $\Gamma_0(5)$. From this follows that the polynomial

$$P(x,z) = \prod_{\gamma\in\Gamma_0(5)\backslash\mathrm{SL}_2(\mathbb{Z})}\left(x-\frac{\Delta(\gamma z)}{\Delta(5\gamma z)}\right) \tag{2.92}$$

is invariant under $\mathrm{SL}_2(\mathbb{Z})$, so has coefficients in $\mathbb{C}(j(z))$. Because $\Delta(\gamma z)/\Delta(5\gamma z)$ is holomorphic on $\mathbb{H}$, it follows that $P(x,z)$ has coefficients in $\mathbb{C}[j(z)]$, and from the q-expansions of the coefficients of $P(x,z)$ follows that $P(x,z)$ has coefficients in $\mathbb{Z}[j(z)]$. Because $j(1/2+\sqrt{3}i/2)=0$, it follows that $P(x,\frac{1}{2}+\frac{1}{2}\sqrt{3}i)$ has integer coefficients. A calculation in Mathematica gives that $P(x,\frac{1}{2}+\frac{1}{2}\sqrt{3}i) = (95367431640625-3144531250x+x^2)^3$, from which follows that

$$\frac{\Delta\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)}{\Delta\left(\frac{5}{2}+\frac{5}{2}\sqrt{3}i\right)} = \frac{\Delta\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)}{\Delta\left(-\frac{3}{2}+\frac{5}{2}\sqrt{3}i\right)} = 9765625(161+72\sqrt{5}) = 2^{-12}5^{10}(\sqrt{5}+1)^{12}. \tag{2.93}$$

From equation (2.79) and equation (2.93) follows that

$$\begin{aligned}\Delta\left(-\frac{1}{2}+\frac{5}{6}\sqrt{3}i\right) &= \Delta\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)\frac{\Delta\left(-\frac{3}{2}+\frac{5}{2}\sqrt{3}i\right)}{\Delta\left(\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)}\frac{\Delta\left(-\frac{1}{2}+\frac{5}{6}\sqrt{3}i\right)}{\Delta\left(-\frac{3}{2}+\frac{5}{2}\sqrt{3}i\right)}\\ &= -2^{-18}5^{-10}\pi^{-6}\frac{\Gamma(1/3)^{18}}{\Gamma(2/3)^{18}}(\sqrt{5}+1)^{12} \end{aligned} \tag{2.94}$$

and

$$\left|\eta^4\left(-\frac{1}{2}+\frac{5}{6}\sqrt{3}i\right)\right| = 2^{-3}5^{-5/3}(\sqrt{5}+1)^2\pi^{-1}\frac{\Gamma(1/3)^3}{\Gamma(2/3)^3}. \tag{2.95}$$

From this follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = 2^{1/3}\cdot 3^{3/2}\cdot 5^{-1}\frac{\Gamma(5/6)^2\Gamma(1/3)}{\Gamma(2/3)^3}. \tag{2.96}$$

We have

$$\frac{\Gamma(5/6)^2\Gamma(1/3)}{\Gamma(2/3)^3} = \frac{\Gamma(5/6)^2\Gamma(1/3)^2}{\Gamma(1/3)\Gamma(2/3)^3} = \frac{2^{2/3}\pi\Gamma(2/3)^2}{\Gamma(1/3)\Gamma(2/3)^3} = \frac{2^{2/3}\pi}{\Gamma(1/3)\Gamma(2/3)} = 2^{-1/3}3^{1/2}, \tag{2.97}$$

from which follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = \frac{9}{5}. \tag{2.98}$$

Because ${}_2F_1\left(1/3,1/2;5/6\,|4/5\right)$ is real and positive, the result follows. □

Using the method of theorem 2.15, we proved some other results: first we searched for imaginary quadratic $z \in \mathbb{H}$ such that $j(z)$ is algebraic of degree at most 2. Then we selected the values of z such that $H(z)$ from table 2.1 is rational and $|H(z)-1| \le 1$, because the convergence radius of a hypergeometric series is equal to 1. Moreover, the outcome should be algebraic, so we do not include answers with the Γ-function in it.

Theorem 2.16. *We have*

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{27}{28}\right) = \sqrt[3]{\frac{256}{49}} \qquad {}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{3}{4}\,\middle|\,\frac{3}{4}\right) = \sqrt[4]{\frac{64}{27}} \tag{2.99}$$

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{1}{4}\right) = \frac{8}{9}\sqrt[3]{2} \qquad {}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{3}{4}\,\middle|\,\frac{80}{81}\right) = \frac{9}{5} \tag{2.100}$$

$${}_2F_1\left(\frac{1}{6},\frac{1}{3};\frac{1}{2}\,\middle|\,\frac{25}{27}\right) = \frac{3}{4}\sqrt{3} \qquad {}_2F_1\left(\frac{1}{8},\frac{3}{8};\frac{1}{2}\,\middle|\,\frac{2400}{2401}\right) = \frac{2}{3}\sqrt{7}. \tag{2.101}$$

Proof. To prove the first equality in equation (2.99), let $a = 1/3$ and $b = 1/2$ and $M(z) + \sum_{j=0}^{\infty}\frac{1}{a+j}+\frac{1}{b+j}-\frac{2}{j+1} = \pi iz + \frac{1}{2}\pi i - \frac{1}{6}\sqrt{3}\pi$. let $y = -\frac{1}{2}+\frac{7}{6}\sqrt{3}i$. The lattice $3\mathbb{Z}+\mathbb{Z}3y$ is an ideal in the order $\mathcal{O} = \mathbb{Z}+7\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{3}i]$, which has class number 2. From this follows that $j(y) = 331776000(-52518123+11460394\sqrt{21})$, and $j(3y)$ is the algebraic conjugate of $j(y)$. We have $\eta^{12}(y)/\eta^{12}(3y) = -27(6049+1320\sqrt{21})$, from which follows that $H(y) = 1/28$, see table 2.2. Solving equation (2.86) gives that $E(y)/|\eta(y)|^4 = 6\sqrt{3}\sqrt[3]{7}(\sqrt{21}-5)$. We also have $\Delta(-\frac{1}{2}+\frac{1}{2}\sqrt{3}i)/\Delta(-\frac{7}{2}+\frac{7}{2}\sqrt{3}i) = 13841287201(6049+1320\sqrt{21})$ and $\eta(\frac{1}{2}+\frac{1}{2}\sqrt{3}i) = e^{\pi i/24}2^{-1/4}3^{-1/4}\pi^{-1/4}\frac{\Gamma(1/3)^{3/4}}{\Gamma(2/3)^{3/4}}$, from which the result follows.

To prove the first equality in equation (2.100), let $a = 1/3$ and $b = 1/2$ and $M(z)+\sum_{j=0}^{\infty}\frac{1}{a+j}+\frac{1}{b+j}-\frac{2}{j+1} = \pi iz+\frac{1}{2}\pi i-\frac{1}{6}\sqrt{3}\pi$. let $y = -\frac{1}{2}+\frac{1}{2}\sqrt{3}i$, so $j(y) = 0$. We have $\eta^{12}(y)/\eta^{12}(3y) = -243$, from which follows that $H(y) = 3/4$, see table 2.2. Solving equation (2.86) gives that $E(y)/|\eta(y)|^4 = 4/\sqrt{3}$. We also have $\eta(\frac{1}{2}+\frac{1}{2}\sqrt{3}i) = e^{\pi i/24}2^{-1/4}3^{-1/4}\pi^{-1/4}\frac{\Gamma(1/3)^{3/4}}{\Gamma(2/3)^{3/4}}$, from which the result follows.

To prove the first equality in equation (2.101), let $a = 1/6$ and $b = 1/3$ and $M(z) + \sum_{j=0}^{\infty}\frac{1}{a+j}+\frac{1}{b+j}-\frac{2}{j+1} = 2\pi iz - \frac{2}{3}\sqrt{3}\pi$. let $y = \frac{2}{3}\sqrt{3}i$. The lattice $3\mathbb{Z}+\mathbb{Z}3y$ is an ideal in the order $\mathcal{O} = \mathbb{Z} + 4\mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{3}i]$, which has class number 2. From this follows that $j(y) = -40500(-35010 + 20213\sqrt{3})$, and $j(3y)$ is the algebraic conjugate of $j(y)$. We have $\eta^{12}(y)/\eta^{12}(3y) = 702 + 405\sqrt{3}$, from which follows that $H(y) = 2/27$, see table 2.2. Solving equation (2.86) gives that $E(y)/|\eta(y)|^4 = 3(3 - \sqrt{3})/\sqrt[6]{2}$. We also have $\Delta(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i)/\Delta(-\frac{4}{2} + \frac{4}{2}\sqrt{3}i) =$ and $\eta(\frac{1}{2} + \frac{1}{2}\sqrt{3}i) = e^{\pi i/24}2^{-1/4}3^{-1/4}\pi^{-1/4}\frac{\Gamma(1/3)^{3/4}}{\Gamma(2/3)^{3/4}}$, from which the result follows.

To prove the second equality in equation (2.99), let $a = 1/4$ and $b = 1/2$ and $M(z) + \sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} = \pi iz + \frac{1}{2}\pi i - \frac{1}{2}\pi$. let $y = -\frac{1}{2} + \frac{3}{2}i$. The lattice $2\mathbb{Z} + \mathbb{Z}2y$ is an ideal in the order $\mathcal{O} = \mathbb{Z} + 3\mathbb{Z}[i]$, which has class number 2. From this follows that $j(y) = 76771008 - 44330496\sqrt{3}$, and $j(2y)$ is the algebraic conjugate of $j(y)$. We have $\Delta(y)/\Delta(2y) = -64(97 + 56\sqrt{3})$, from which follows that $H(y) = 1/4$, see table 2.2. In this case we consider $E(z) = 2E_2(z) - E_2(z)$ instead of the previously used definition. The space of modular forms of weight 6 with respect to the group $\Gamma_0(2)$ turns out to be twodimensional, from which follows that

$$-4E(z)^3 + 3E(z)E_4(z) + E_6(z) = 0 \tag{2.102}$$

and $E(y)/|\eta(y)|^4 = 4\sqrt{2}\sqrt[6]{21\sqrt{3} - 36}$. We also have $\Delta(i)/\Delta(3i) = 19683(7 + 4\sqrt{3})$ and $\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}$, from which the result follows.

To prove the second equality in equation (2.100), let $a = 1/4$ and $b = 1/2$ and $M(z) + \sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} = \pi iz + \frac{1}{2}\pi i - \frac{1}{2}\pi$. let $y = -\frac{1}{2} + \frac{5}{2}i$. The lattice $2\mathbb{Z} + \mathbb{Z}2y$ is an ideal in the order $\mathcal{O} = \mathbb{Z} + 5\mathbb{Z}[i]$, which has class number 2. From this follows that $j(y) = 1728(12740595841 - 5697769392\sqrt{5})$, and $j(2y)$ is the algebraic conjugate of $j(y)$. We have $\Delta(y)/\Delta(2y) = -64(51841 + 23184\sqrt{5})$, from which follows that $H(y) = 1/81$, see table 2.2. Solving equation (2.102) gives and $E(y)/|\eta(y)|^4 = 12\sqrt[4]{5}(\sqrt{5} - 3)$. We also have $\Delta(i)/\Delta(5i) = 244140625(161 + 72\sqrt{5})$ and $\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}$, from which the result follows.

To prove the second equality in equation (2.101), let $a = 1/8$ and $b = 3/8$ and $M(z) + \sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} = 2\pi iz - \sqrt{2}\pi$. let $y = \frac{3}{2}\sqrt{2}i$. The lattice $2\mathbb{Z}+\mathbb{Z}2y$ is an ideal in the order $\mathcal{O} = \mathbb{Z} + 3\mathbb{Z}[\sqrt{2}i]$, which has class number 2. From this follows that $j(y) = 8000(23604673 - 9636536\sqrt{6})$, and $j(2y)$ is the algebraic conjugate of $j(y)$. We have $\Delta(y)/\Delta(2y) = 64(4801 + 1960\sqrt{6})$, from which follows that $H(y) = 1/2401$, see table 2.2. Solving equation (2.102) gives that and $E(y)/|\eta(y)|^4 = 14\sqrt{2}\sqrt[3]{5 - 2\sqrt{6}}$. We have $\Delta(\sqrt{2}i)/\Delta(3\sqrt{2}i) = 531441(49 + 20\sqrt{6})$ and from theorem 2.14 follows that $\left|\eta^4(\sqrt{2}i)\right| = \frac{\Gamma(1/8)\Gamma(3/8)}{16\pi\Gamma(5/8)\Gamma(7/8)}$, from which the answer follows. □

2.6 Remarks on the transcendental method

It is possible to generalize theorem 2.15, from which follows that there exists a subset E of the algebraic numbers, dense in $\mathbb{C}$ such that if $x \in E$, then ${}_2F_1\left(1/3, 1/2; 5/6\,|x\right)$ is also algebraic. Note that here we use that H is a continuous surjective function from $\mathbb{H}$ to $\mathbb{C}$, so the image of a dense subset of $\mathbb{H}$ under H is a dense subset of $\mathbb{C}$.

Theorem 2.17. *Let $a = 1/3$ and $b = 1/2$, let $H(z)$ be defined as in table 2.2, let $z \in \mathbb{Q}(\sqrt{3}i)\cap\mathbb{H}$ such that $|1 - H(z)| < 1$. Then ${}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,|1 - H(z)\right)$ is algebraic.*

Proof. From $a = 1/3$ and $b = 1/2$ follows that

$$M(z) + \sum_{j=0}^{\infty} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1} = -\frac{1}{6}\pi(-3i + \sqrt{3} - 6iz). \tag{2.103}$$

From table 2.2 we know that $H(z)$ is an algebraic function of $\eta^{12}(z)/\eta^{12}(3z)$, write $H(z) = f(\eta^{12}(z)/\eta^{12}(3z))$. Then it follows from corollary 2.9 that

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,|1-h(z)\right)^2 = \frac{(1-H(z))^{-5/6}}{36i\cdot H(z)}\cdot\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\left(-3i+\sqrt{3}-6iz\right)^2\cdot H'(z). \tag{2.104}$$

From equation (2.78) follows that $\eta^{12}(z)/\eta^{12}(3z)$ is algebraic, because $j(z)$ is algebraic. From this follows that $f'(\eta^{12}(z)/\eta^{12}(3z))$ is algebraic, so ${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,|1-H(z)\right)$ is algebraic if and only if

$$\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\frac{(\eta^{12}(z)/\eta^{12}(3z))'}{\eta^{12}(z)/\eta^{12}(3z)} \tag{2.105}$$

is algebraic. We have

$$\begin{aligned}\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\frac{(\eta^{12}(z)/\eta^{12}(3z))'}{\eta^{12}(z)/\eta^{12}(3z)} &= -2\pi i\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}E(z)\\ &= -2\pi i\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\cdot\frac{E(z)}{|\eta^4(z)|}\cdot\frac{|\eta^4(z)|}{\left|\eta^4\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)\right|}\cdot\left|\eta^4\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)\right|,\end{aligned} \tag{2.106}$$

where $E(z)=\frac{3}{2}E_2(3z)-\frac{1}{2}E_2(z)$ and $E_2(z)=1-24\sum_{n=1}^{\infty}\sigma(n)q^n$. From equation (2.86) follows that $\frac{E(z)}{|\eta^4(z)|}$ is algebraic, and because $z\in\mathbb{Q}(\sqrt{3}i)$, it follows that $|\eta^4(z)|/\left|\eta^4\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)\right|$ is algebraic. We have $\left|\eta^4\left(-\frac{1}{2}+\frac{1}{2}\sqrt{3}i\right)\right| = 9\Gamma(7/6)^3\pi^{-5/2}$, so ${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,|1-H(z)\right)$ is algebraic if and only if

$$\pi\frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\frac{\Gamma(7/6)^3}{\pi^{5/2}} = \frac{\Gamma(5/6)^2}{\Gamma(1/3)^2}\frac{\Gamma(1/6)^3}{216\pi^{3/2}} = \frac{\sqrt{\pi}\Gamma(1/6)\Gamma(2/3)}{54\Gamma(1/3)^2\Gamma(2/3)} = \frac{\sqrt{3}\Gamma(1/6)\Gamma(2/3)}{108\sqrt{\pi}\Gamma(1/3)} = \frac{\sqrt{3}2^{2/3}}{108} \tag{2.107}$$

is algebraic, which is the case: here we used that $\Gamma(1/6)\Gamma(2/3) = 2^{2/3}\sqrt{\pi}\Gamma(1/3)$. From this follows that if $z\in\mathbb{Q}(\sqrt{3}i)$, then ${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,|1-H(z)\right)$ is algebraic. □

The proof of theorem 2.15 ends with a combination of Γ-function factors which happens to be algebraic, like the first proof of theorem 2.11. Although the proof of theorem 2.15 gives us an algebraic special value of the hypergeometric function, it is not an elegant proof, also because its beginning does not indicate that it will actually produce an algebraic number. We want to find a more elegant proof, like the second proof of theorem 2.11, so we want an algebraic proof instead of a transcendental proof which at the very end happens to produce an algebraic number.

The method in theorem 2.16 relies on the cancellation of some Γ-function factors, so for given values of the parameters a, b it follows from theorem 2.14 that there are very few quadratic imaginary fields in which the number y can be chosen, and the expression in theorem 2.14 gets more complicated with higher absolute value of the discriminant. In theorem 2.15 we only considered rational arguments, but for non-rational arguments the method also works: without proof we state

$$\begin{aligned}{}_2F_1\left(\frac{1}{2},\frac{1}{3};\frac{5}{6}\middle|\frac{23}{27}-\frac{10}{27}\sqrt{2}i\right) &= \frac{1}{16}\left(i+\sqrt{2}\right)\left(-2i+\sqrt{3}\right)\\ {}_2F_1\left(\frac{1}{6},\frac{1}{3};\frac{1}{2}\middle|1-\frac{9\left(2677+2284\sqrt[3]{2}-3497\sqrt[3]{4}\right)}{15625}\right) &= \frac{2}{3}\left(1+\sqrt[3]{4}\right)\\ {}_2F_1\left(\frac{1}{8},\frac{3}{8};\frac{1}{2}\middle|1-\frac{32\left(325\sqrt{2}-457\right)}{2401}\right) &= \frac{3}{8}\sqrt{4+6\sqrt{2}},\end{aligned} \tag{2.108}$$

where we respectively used $y = 2i/\sqrt{3}$ and $y = \sqrt{3}i$ and $y = \sqrt{2}i$.

Until here we only could produce a special value if $|1 - H(z)| \leq 1$, but for $|1 - H(z)| > 1$ there is also a method: we will calculate $A, B \in \mathbb{C}$ such that $x^{-a}{}_2F_1(a, a; a+1-b\,|1/x) = AF_0(x) + BG_0(x)$, because $x^{-a}{}_2F_1(a, a; a+1-b\,|1/x)$ is a solution of around $x = \infty$ of the hypergeometric differential equation with parameters $a, b, 1$. Note that $x^{-a}{}_2F_1(a, a; a+1-b\,|1/x)$ and $F_0(x), G_0(x)$ have a disjoint domain, so this equality means that their analytic continuations to $|x - 1| < 1$ are equal.

Lemma 2.18. *Let $0 < a, b, a+b < 1$, let F_0, G_0 as in definition 2.1. Let $F_\infty(x) = x^{-a}{}_2F_1(a, a; a+1-b\,|1/x)$ be a solution of the hypergeometric differential equation with parameters $a, b, 1$ around $x = \infty$. Then it follows that*

$$F_\infty(x) = \frac{\Gamma(1-a)\Gamma(a-b+1)}{2\pi\Gamma(1-b)}\left(i\cdot(1-e^{-2\pi ia})\cdot\left(\sum_{j=0}^{\infty}\frac{1}{1-a+j}+\frac{1}{1-b+j}-\frac{2}{j+1}\right)+2\pi\right)F_0(x)$$
$$+\frac{\Gamma(1-a)\Gamma(a-b+1)}{2\pi\Gamma(1-b)}\cdot i\cdot\left(1-e^{-2\pi ia}\right)G_0(x). \tag{2.109}$$

Proof. Write $F_\infty(x) = AF_0(x) + BG_0(x)$ for some $A, B \in \mathbb{C}$. We have $F_0(1) = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}$ and $F_\infty(1) = \frac{\Gamma(1-a-b)\Gamma(a+1-b)}{\Gamma(1-b)^2}$. From lemma 2.3 follows that

$$G_0(1) = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}\left(\sum_{j=0}^{\infty}-\frac{1}{a+j}-\frac{1}{b+j}+\frac{2}{j+1}\right), \tag{2.110}$$

from which a linear relation follows. Now consider the analytic continuation of $F_\infty(x) = AF_0(x) + BG_0(x)$, starting and ending at $x = 1$, with one counterclockwise loop around $x = 0$. Because analytic contination along a curve is invariant with respect to homotopy, for the function $F_\infty(x)$ a path in $|x| > 1$ can be followed, and for $AF_0(x) + BG_0(x)$ a path in $|x| < 1$ can be followed. From this follows that $e^{2\pi ia}F_\infty(x) = AF_0(x) + B(G_0(x) + 2\pi iF_0(x))$, from which another linear relation follows. From this follows that

$$\frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)}\begin{pmatrix}1 & \sum_{j=0}^{\infty}-\frac{1}{a+j}-\frac{1}{b+j}+\frac{2}{j+1}\\ 1 & 2\pi i+\sum_{j=0}^{\infty}-\frac{1}{a+j}-\frac{1}{b+j}+\frac{2}{j+1}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix} = \begin{pmatrix}\frac{\Gamma(1-a-b)\Gamma(a+1-b)}{\Gamma(1-b)^2}\\ e^{-2\pi ia}\frac{\Gamma(1-a-b)\Gamma(a+1-b)}{\Gamma(1-b)^2}\end{pmatrix}, \tag{2.111}$$

from which the result follows. □

Proposition 2.19 ([1, chapter 5, theorem 3])**.** *It holds that*

$${}_2F_1\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}\,\middle|\,\frac{64000}{64009}\right) = \frac{2}{3}\sqrt[6]{253}. \tag{2.112}$$

Proof. Let $a = 1/12$ and $b = 5/12$, then it follows from table 2.2 that $H(z) = 1728/j(z)$, and from table 2.1 follows that $M(z) = 2\pi iz + \log(1728)$. Let $y = 1/2 + 3\sqrt{3}i/2$, then it follows that $j(z) = -12\,288\,000$ and $H(y) = -9/64000$. From lemma 2.7 follows that

$$F_\infty(x) = \frac{\Gamma(11/12)\Gamma(2/3)}{2\pi\Gamma(7/12)}\left(\left(i - e^{\pi i/3}\right)(2\pi - \log(1728)) + 2\pi\right)F_0(x)$$
$$+\frac{\Gamma(11/12)\Gamma(2/3)}{2\pi\Gamma(7/12)}\left(i - e^{\pi i/3}\right)G_0(x). \tag{2.113}$$

Note that $\sum_{j=0}^{\infty} \frac{1}{11/12+j} + \frac{1}{7/12+j} - \frac{2}{j+1} = 2\pi - \log(1728)$, then from lemma 2.18 it follows that

$$H(y)^{-1/6}{}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3} \middle| \frac{1}{H(y)}\right)^2 \tag{2.114}$$

is equal to

$$= \frac{H(y)^{-1}(1-H(y))^{-a-b}H'(y)}{2\pi i} \cdot \left(\frac{\Gamma(11/12)\Gamma(2/3)}{2\pi\Gamma(7/12)}\right)^2$$
$$\cdot \left(\left(i - e^{\pi i/3}\right)(2\pi i y + \log(1728)) + \left(i - e^{\pi i/3}\right)(2\pi - \log(1728)) + 2\pi\right)^2 \tag{2.115}$$

and

$${}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3} \middle| \frac{-64000}{9}\right)^2 = -\frac{3^{7/3} i\Gamma(2/3)^2\Gamma(11/6)^2}{161\,920\,000 \cdot \sqrt[3]{4}\pi^2} j'(y). \tag{2.116}$$

We have $j'(z) = -2\pi i E_4(z)^2 E_6(z)/\Delta(z)$ and

$$E_4(z) = \sqrt[3]{\Delta(z)j(z)} \qquad \text{and} \qquad E_6(z) = \sqrt{\Delta(z)\cdot(j(z)-1728)}, \tag{2.117}$$

from which follows that $j'(z) = -2\pi i \cdot \Delta(z)^{1/6} \cdot j(z)^{1/3}\sqrt{j(z)-1728}$, so we first calculate $\Delta(y)$. Note that $E_4(y) \in \mathbb{R}$ and $E_6(y) \in \mathbb{R}_{>0}$ and $\Delta(y) \in \mathbb{R}_{<0}$, so we have $j'(y) \in i\mathbb{R}_{>0}$. We have

$$\Delta\left(-\frac{1}{2} + \frac{3}{2}\sqrt{3}i\right) = \Delta\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i\right) \cdot \frac{\Delta\left(-\frac{1}{2} + \frac{3}{2}\sqrt{3}i\right)}{\Delta\left(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i\right)} = -\frac{\Gamma(1/3)^{18}}{6^6\pi^6\Gamma(2/3)^{18}} \cdot 3^{-10}, \tag{2.118}$$

from which follows that

$$j'(y) = 2\pi i \frac{\Gamma(1/3)^3}{2\cdot 3^{8/3}\cdot\Gamma(2/3)^3} \cdot 12\,288\,000^{2/3} \cdot \sqrt{12\,289\,728} = \frac{310\,886\,400 i\sqrt{3}\Gamma(7/6)^3}{\pi^{3/2}} \tag{2.119}$$

and

$${}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{2}{3} \middle| \frac{-64000}{9}\right)^2 = -\frac{3^{7/3} i\Gamma(2/3)^2\Gamma(11/6)^2}{161\,920\,000 \cdot \sqrt[3]{4}\pi^2} \cdot \frac{310\,886\,400 i\sqrt{3}\Gamma(7/6)^3}{\pi^{3/2}} = 2^2 \cdot 3^{-5/3}. \tag{2.120}$$

We have

$$(1-x)^a {}_2F_1\left(a, a; a+1-b \middle| x\right) = {}_2F_1\left(a, 1-b; a+1-b \middle| \frac{x}{x-1}\right), \tag{2.121}$$

from which follows that

$${}_2F_1\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3} \middle| \frac{64000}{64009}\right)^2 = 2^2 \cdot 3^{-5/3} \cdot \left(\frac{64009}{9}\right)^{1/6} = \frac{4}{9}\sqrt[3]{253}. \tag{2.122}$$

The result follows, because ${}_2F_1\left(1/12, 7/12; 2/3 \middle| 64000/64009\right)$ is real and positive. □

2.7 Algebraic method

The proof of theorem 2.15 ended with a combination of Γ-factors which happens to be algebraic. Now we want to produce an alternative proof of theorem 2.15, which looks like the second proof of theorem 2.11. To find this proof, we first want to find a polynomial $P(x,y)$ such that $P(j(z),H(z))=0$, with H as in table 2.2; from the subgroup inclusions from figure 1.2 we know that we can find such a polynomial. Because $\Gamma(3,\infty,\infty)$ is contained in both $\Gamma(2,3,\infty)$ and $\Gamma(2,6,\infty)$, we can express $h(2,3,\infty)$ and $h(2,6,\infty)$ as a rational function in $h(3,\infty,\infty)$, see table 1.3.

Let

$$P_1(x,y)=803894544+24690528y+221400y^2+792y^3+y^4+50976x+1916xy-36xy^2-xy^3+x^2, \tag{2.123}$$

then it follows that

$$P_1(z)=P_1\left(\frac{(z+243)^3(z+27)}{z^3}, z+\frac{3^6}{z}\right)=0, \tag{2.124}$$

so $P_1(z)$ is the zero function; note that the arguments of $P_1(x,y)$ come from table 1.3. It is also possible to give a nonconstructive proof of the existence of the polynomial P_1: let $k,l\in\mathbb{Z}_{\geq 0}$ and consider the set $S=\{(a,b)\in(\mathbb{Z}_{\geq 0})^2|3a+b\leq k; a+b\leq l\}$. Then it follows that the function $\left((z+243)^3(z+27)/z^3\right)^a\left(z+3^6/z\right)^b$ with $(a,b)\in S$ has, seen as a Laurent series around $z=0$, only terms z^m with $-k\leq m\leq l$. The elements $\{z^m|-k\leq m\leq l\}$ span a $k+l+1$-dimensional $\mathbb{C}$-vector space, so if $\#S>k+l+1$, it follows that there must be a linear relation between the elements $\left((z+243)^3(z+27)/z^3\right)^a\left(z+3^6/z\right)^b$ with $(a,b)\in S$, which gives the polynomial P_1. In this case, if $k=8$ and $l=6$, it follows that $\#S=16>k+l+1$, which proves the existence of a linear relation.

Now we insert $h(3,\infty,\infty)(z)$ from table 1.2 into equation (2.124), from which follows that

$$P_1(j(z),h(2,6,\infty)(z)=0, \tag{2.125}$$

From $h(2,6,\infty)(z)=54-h(6,6,\infty)(z)^2$ follows that $P_1(j(z),54-h(6,6,\infty)(z)^2)=0$, so $P_2(j(z),h_{6,6,\infty}(z))=0$, where $P_2(x,y)=P_1(x,54-y^2)$ and

$$P_2(x,y)=2916000000-108000x+x^2-56160000y^2+10720xy^2+367200y^4-198xy^4-1008y^6+xy^6+y^8. \tag{2.126}$$

From table 2.2 we know that $H(6,6,\infty)(z)=\frac{12\sqrt{3}}{h(6,6,\infty)(z)+6\sqrt{3}}$, from which follows that $h_{6,6,\infty}(z)=\sqrt{108}\frac{-H(6,6,\infty)(z)+2}{H(6,6,\infty)(z)}$. From this follows that $P_2\left(j(z),\sqrt{108}\frac{-h(z)+2}{h(z)}\right)=0$, so $P(j(z),H(6,6,\infty)(z))=0$ with $P(x,y)=y^8P_2\left(x,\sqrt{108}\frac{y-2}{y}\right)$ and

$$\begin{aligned}P(x,y)=&\ 34828517376-139314069504y+162533081088y^2+80621568xy^2\\&-241864704xy^3-83846430720y^4+265379328xy^4+5159780352y^5-127650816xy^5\\&+17581473792y^6+24786432xy^6+3057647616y^7-1271808xy^7+x^2y^8.\end{aligned} \tag{2.127}$$

Using this fact we can produce a second proof of theorem 2.15.

Theorem 2.20 (Same statement as theorem 2.15)**.** *We have*

$$_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\middle|\frac{4}{5}\right)=\frac{3}{\sqrt{5}}. \tag{2.128}$$

Proof. We use corollary 2.9 with $a=1/3$ and $b=1/2$ and $M(z)+\sum_{j=0}^{\infty}\frac{1}{a+j}+\frac{1}{b+j}-\frac{2}{j+1}=\pi/6\cdot(6iz+3i-\sqrt{3})$ and $M'(z)=\pi i$ and $H(z)$ as in table 2.2. From this follows that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\left|1-H(5z)\right.\right)^2}{{}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\left|1-H(z)\right.\right)^2}=\frac{(1-H(5z))^{-5/6}\cdot H(z)\cdot\left(6i(5z)+3i-\sqrt{3}\right)^2\cdot H'(5z)}{(1-H(z))^{-5/6}\cdot H(5z)\cdot\left(6iz+3i-\sqrt{3}\right)^2\cdot H'(z)}. \tag{2.129}$$

Note that we consider $1-H(z)$ on the line segment $\{z \in \mathbb{H}; \Re(z) = -1/2, \Im(z) > \sqrt{3}i/6\}$ where it is real and positive, so we can define its fractional powers. Let $y = -\frac{1}{2} + \frac{1}{6}\sqrt{3}i$, then we have $H(y) = 1$ and $H(5y) = 1/5$. The function $M(z)$ is only defined on the triangle from table 1.1, so instead of $5y = -\frac{5}{2} + \frac{5}{6}\sqrt{3}i$ we consider $-\frac{1}{2} + \frac{5}{6}\sqrt{3}i$, from which follows that

$$ {}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = \lim_{z \to y} \frac{{}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,|1-H(5z)\right)^2}{{}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,|1-H(z)\right)^2} = 45 \lim_{z \to y} \frac{(1-H(5z))^{-5/6} \cdot H'(5z)}{(1-H(z))^{-5/6} \cdot H'(z)}. \tag{2.130} $$

Let $P(x,y)$ as in equation (2.127). Differentiating $P(z) := P(j(z), H(z)) = 0$ gives that

$$ P_x(j(z), H(z)) \cdot j'(z) + P_y(j(z), H(z)) \cdot h'(z) = P_x(z) \cdot j'(z) + P_y(z) \cdot H'(z) = 0 \tag{2.131} $$

and differentiating $P(5z) = P(j(5z), H(5z)) = 0$ gives

$$ P_x(j(5z), H(5z)) \cdot j'(5z) + P_y(j(5z), H(5z)) \cdot H'(5z) = P_x(5z) \cdot j'(5z) + P_y(5z) \cdot H'(5z) = 0, \tag{2.132} $$

where P_x and P_y denote the partial derivatives of $P(x,y)$. From this follows that

$$ \frac{H'(5z)}{H'(z)} = \frac{P_x(5z) \cdot j'(5z) \cdot P_y(z)}{P_x(z) \cdot j'(z) \cdot P_y(5z)} \tag{2.133} $$

and

$$ {}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = 45 \lim_{z \to y} \frac{(1-H(5z))^{-5/6}}{(1-H(z))^{-5/6}} \cdot \frac{P_x(5z) \cdot j'(5z) \cdot P_y(z)}{P_x(z) \cdot j'(z) \cdot P_y(5z)}. \tag{2.134} $$

We already knew that $H(y) = 1$ and $H(5y) = 1/5$, we also have $j(y) = 0$ because $-1/y = \frac{3}{2} + \sqrt{3}i/2$, and we have $j(5y) = 1728 \cdot 512(-369830 + 165393\sqrt{5})$. Extracting nonzero limit terms gives

$$ {}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2 = \frac{45 P_y(y) \cdot P_x(5y)}{(4/5)^{5/6} P_y(5y)} \lim_{z \to y} \frac{(1-H(z))^{5/6}}{P_x(z)} \cdot \frac{j'(5z)}{j'(z)}. \tag{2.135} $$

To calculate the limit in equation (2.135), we first want to find an expression for $j'(5z)/j'(z)$. A set of coset representatives of $\Gamma_0(5)\backslash\mathbb{H}$ is given by $z \mapsto z$, $z \mapsto -1/z$, $z \mapsto -\frac{1}{z+1}$, $z \mapsto -\frac{1}{z+2}$, $z \mapsto -\frac{1}{z+3}$, $z \mapsto -\frac{1}{z+4}$. From this follows that the polynomial

$$ \begin{aligned} f(x) = (x - j(5z)) \left(x - j\left(\frac{z}{5}\right)\right) \left(x - j\left(\frac{z+1}{5}\right)\right) \\ \cdot \left(x - j\left(\frac{z+2}{5}\right)\right) \left(x - j\left(\frac{z+3}{5}\right)\right) \left(x - j\left(\frac{z+4}{5}\right)\right) \end{aligned} \tag{2.136} $$

has coefficients in the ring $\mathbb{C}[j(z)] \subset \mathbb{C}(j(z))$: here we use that $j(z)$ is holomorphic on $\mathbb{H}$, so the coefficients of $f(x)$ are polynomials in $j(z)$. A calculation gives a polynomial $Q(x,y)$ such that

$Q(j(5z), j(z))$ is the zero function, with

$$\begin{aligned}
Q(x,y) =&141359947154721358697753474691071362751004672000 + 53274330803424425450420160273356509151232000x \\
&+ 6692500042627997708487149415015068467200x^2 + 280244777828439527804321565297868800x^3 \\
&+ 1284733132841424456253440x^4 + 1963211489280x^5 + x^6 + 53274330803424425450420160273356509151232000y \\
&- 264073457076620596259715790247978782949376xy + 36554736583949629295706472332656640000x^2y \\
&- 192457934618928299655108231168000x^3y + 128541798906828816384000x^4y - 246683410950x^5y \\
&+ 6692500042627997708487149415015068467200y^2 + 36554736583949629295706472332656640000xy^2 \\
&+ 5110941777552418083110765199360000x^2y^2 + 26898488858380731577417728000x^3y^2 \\
&+ 383083609779811215375x^4y^2 + 2028551200x^5y^2 + 280244777828439527804321565297868800y^3 \\
&- 192457934618928299655108231168000xy^3 + 26898488858380731577417728000x^2y^3 \\
&- 441206965512914835246100x^3y^3 + 107878928185336800x^4y^3 - 4550940x^5y^3 + 1284733132841424456253440y^4 \\
&+ 128541798906828816384000xy^4 + 383083609779811215375x^2y^4 + 107878928185336800x^3y^4 \\
&+ 1665999364600x^4y^4 + 3720x^5y^4 + 1963211489280y^5 - 246683410950xy^5 + 2028551200x^2y^5 \\
&- 4550940x^3y^5 + 3720x^4y^5 - x^5y^5 + y^6. \qquad (2.137)
\end{aligned}$$

Differentiating $Q(z) := Q(j(5z), j(z)) = 0$ gives

$$5Q_x(j(5z), j(z)) \cdot j'(5z) + Q_y(j(5z), j(z)) \cdot j'(z) = 5Q_x(z) \cdot j'(5z) + Q_y(z) \cdot j'(z) = 0, \quad (2.138)$$

where Q_x and Q_y denote the partial derivatives of Q with respect to x and y. From equation (2.138) follows that

$${}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6} \,\middle|\, \frac{4}{5}\right)^2 = -\frac{9P_y(y) \cdot P_x(5y) \cdot Q_y(y)}{(4/5)^{5/6} P_y(5y)} \cdot \lim_{z \to y} \frac{(1 - H(z))^{5/6}}{P_x(z) \cdot Q_x(z)}. \quad (2.139)$$

It is possible to evaluate the limit in equation (2.139) using the rule of l'Hôpital, but there is a more simple method. From the fact that that $T(6, 6, \infty)$ has angle $\pi/6$ at y, it follows that $H(z) - 1$ has a root of order 6 at $z = y$, so $(1 - H(z))^{5/6}$ has a root of order 5 at $z = y$. If we know the root orders of $P_x(z)$ and $Q_x(z)$ at $z = y$, we can factor the limit in equation (2.139), which gives a limit with roots of lower order in numerator and denominator. From the valence formula follows that $j(z)$ has a third order root at $\rho = e^{2\pi i/3}$. On the other hand, we have $y = 1/(-\rho - 2)$, so $j(z)$ also has a third order root at y. The function $j'(z)$ is a modular function of weight 2, which has a pole of order 1 at $z = \infty$, and a root of order 2 at $z = \rho$, and a root of order 1 at $z = i$. To calculate the order of $Q_x(z)$, we have $Q_x(z)' = 5Q_{xx}(z) \cdot j'(5z) + Q_{xy}(z) \cdot j'(z)$, where the second term has a root of order 2. To calculate the order of $Q_{xx}(z)$ at $z = y$, we have

$$Q_{xx}(z)' = 5Q_{xxx}(z) \cdot j'(5z) + Q_{xxy}(z) \cdot j'(z), \quad (2.140)$$

which has a nonzero limit, because $j'(y) = 0$ and $Q_{xxx}(y) \neq 0$ and $j'(5y) \neq 0$. From this follows that $Q_x(z)$ has a second order root at $z = y$.

Because both sides of equation (2.139) are nonzero and finite, it follows that $P_x(z)$ has a third order root at $z = y$. From this follows that we can factor the limit in equation (2.139) in two nonzero limits, which have lower order roots in numerator and denominator, which hopefully gives a shorter calculation:

$${}_2F_1\left(\frac{1}{3}, \frac{1}{2}; \frac{5}{6} \,\middle|\, \frac{4}{5}\right)^2 = -\frac{9P_y(y) \cdot P_x(5y) \cdot Q_y(y)}{(4/5)^{5/6} P_y(5y)} \cdot \lim_{z \to y} \frac{(1 - H(z))^{1/2}}{P_x(z)} \cdot \lim_{z \to y} \frac{(1 - H(z))^{1/3}}{Q_x(z)}. \quad (2.141)$$

In both these limits in equation (2.141) the denominator and numerator go to 0, so we use the rule of l'Hôpital, from which follows that the first limit from equation (2.141) is equal to

$$-\lim_{z\to y}\frac{\frac{1}{2}(1-H(z))^{-1/2}}{P_{xx}(z)\cdot\frac{j'(z)}{H'(z)}+P_{xy}(z)}=\lim_{z\to y}\frac{\frac{1}{2}(1-H(z))^{-1/2}\cdot P_x(z)}{P_{xx}(z)\cdot P_y(z)-P_{xy}(z)\cdot P_x(z)}$$
$$=\frac{1}{12230590464}\lim_{z\to y}\frac{P_x(z)}{(1-H(z))^{1/2}}, \tag{2.142}$$

where in the first equality we used equation (2.131) and in the second equality we calculated the nonzero limit in the denominator. Because the limit in the last step is the inverse of the limit we started with, it follows that $\lim_{z\to y}\frac{(1-H(z))^{1/2}}{P_x(z)}=1/\sqrt{12230590464}=48^{-3}$.

The second limit from equation (2.141) is equal to

$$-\lim_{z\to y}\frac{\frac{1}{3}(1-H(z))^{-2/3}\cdot\frac{H'(z)}{j'(z)}}{5Q_{xx}(z)\cdot\frac{j'(5z)}{j'(z)}+Q_{xy}(z)}=\lim_{z\to y}\frac{\frac{1}{3}(1-H(z))^{-2/3}\cdot P_x(z)\cdot Q_x(z)}{Q_{xx}(z)\cdot Q_y(z)\cdot P_y(z)}, \tag{2.143}$$

here we used that $Q_{xx}(z)$ has a root of order 1, see equation (2.140), and $j'(z)$ has a root of order 2 at $z=y$, and $Q_{xy}(z)$ has a finite limit at $z=y$. Evaluating nonzero limits gives that the second limit from equation (2.141) is equal to

$$\frac{1}{3Q_y(y)\cdot P_y(y)}\lim_{z\to y}\frac{P_x(z)\cdot Q_x(z)}{(1-H(z))^{2/3}\cdot Q_{xx}(z)}=\frac{48^3}{3Q_y(y)\cdot P_y(y)}\lim_{z\to y}\frac{Q_x(z)}{(1-H(z))^{1/6}\cdot Q_{xx}(z)}. \tag{2.144}$$

Multiplying the second limit from equation (2.141) by the square of equation (2.144) gives that

$$\left(\lim_{z\to y}\frac{(1-H(z))^{1/3}}{Q_x(z)}\right)^3=\left(\frac{48^3}{3Q_y(y)\cdot P_y(y)}\right)^2\lim_{z\to y}\frac{Q_x(z)}{Q_{xx}(z)^2}, \tag{2.145}$$

from which follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2=-\frac{9\sqrt[3]{P_y(y)}\cdot P_x(5y)\cdot\sqrt[3]{Q_y(y)}}{48\cdot 3^{2/3}\cdot(4/5)^{5/6}P_y(5y)}\cdot\sqrt[3]{\lim_{z\to y}\frac{Q_x(z)}{Q_{xx}(z)}}. \tag{2.146}$$

To calculate the limit in equation (2.146), we use the rule of l'Hôpital, which gives

$$\lim_{z\to y}\frac{Q_x(z)}{Q_{xx}(z)}=\lim_{z\to y}\frac{5Q_{xx}(z)\cdot j'(5z)+Q_{xy}(z)\cdot j'(z)}{10Q_{xx}(z)\cdot Q_{xxx}(z)\cdot j'(5z)+2Q_{xx}(z)\cdot Q_{xxy}(z)\cdot j'(z)}=\lim_{z\to y}\frac{1}{2Q_{xxx}(z)}, \tag{2.147}$$

here we used that $Q_{xx}(z)\cdot j'(z)$ and $Q_{xx}(z)\cdot Q_{xxx}(z)\cdot j'(z)$ both have a root of order 1 at $z=y$, but the terms with $j'(z)$ have higher order roots. From this follows that

$${}_2F_1\left(\frac{1}{3},\frac{1}{2};\frac{5}{6}\,\middle|\,\frac{4}{5}\right)^2=-\frac{9\sqrt[3]{P_y(y)}\cdot P_x(5y)\cdot\sqrt[3]{Q_y(y)}}{48\cdot\sqrt[3]{2}\cdot 3^{2/3}\cdot(4/5)^{5/6}P_y(5y)\cdot\sqrt[3]{Q_{xxx}(y)}}=\frac{9}{5}. \tag{2.148}$$

Because ${}_2F_1\left(1/3,1/2;5/6\,|4/5\right)^2$ is real and positive, the result follows. □

Chapter 3

Belyi functions

In this chapter we construct another method to calculate special values of the hypergeometric function. Because in this chapter we do not use modular functions, we use the symbol z to denote a complex variable. We start with a result on Fuchsian differential equations, which we define first.

Definition 3.1. Let $z_1, \ldots, z_k \in \mathbb{C}$ be distinct points. Let p, q be meromorphic functions such that: p only can have poles of order at most 1 at the points $z_1, \ldots, z_k$ and no other poles; q only can have poles of order at most 2 at the points $z_1, \ldots, z_k$ and no other poles. Moreover, for $z \to \infty$ it should be the case that $p(z) = O(1/z)$ and $q(z) = O(1/z^2)$. A *Fuchsian differential equation* (of degree 2) is a linear differential equation $y''(z) + p(z) \cdot y'(z) + q(z) \cdot y(z) = 0$, such that p, q have the properties mentioned before. The points $z_1, \ldots, z_k, \infty$ are called the singular points of this equation. In this chapter, we only consider Fuchsian differential equations of degree 2. It is possible to define conditions on p and q such that ∞ is a regular point, but we always see ∞ as a singular point.

Proposition 3.2. *Let f be a solution of a Fuchsian differential equation with singular points $z_1, \ldots, z_k, \infty$ of degree 2. Let Q be a rational function which only ramifies above the singularities of the equation of f. Then $F(z) := f(Q(z))$ satisfies a Fuchsian differential equation with singular points $Q^{-1}(\{z_1, \ldots, z_k, \infty\})$. Moreover, let $y \in Q^{-1}(\{z_1, \ldots, z_k, \infty\})$ with $Q(y) = z_i$ such that $Q(z) - Q(y)$ has a root of degree d at $z = y$. Let a, b be the local exponents of z_i, then y has local exponents da, db. Let $y \in Q^{-1}(\infty)$ such that $Q(z)$ has a pole of order d at $z = y$. Let a, b be the local exponents of ∞, then y has local exponents da, db.*

Proof. By assumption we know that f is a solution of the equation $y''(z)+p(z)\cdot y'(z)+q(z)\cdot y(z) = 0$, from which follows that

$$Q'(z)^2 \cdot f''(Q(z)) + Q'(z)^2 \cdot p(Q(z)) \cdot f'(Q(z)) + Q'(z)^2 \cdot q(Q(z)) \cdot f(Q(z)) = 0. \quad (3.1)$$

We have $F'(z) = f'(Q(z)) \cdot Q'(z)$ and $F''(z) = f''(Q(z)) \cdot Q'(z)^2 + f'(Q(z)) \cdot Q''(z)$. From this follows that

$$F''(z) + \left(p(Q(z)) \cdot Q'(z) - \frac{Q''(z)}{Q'(z)}\right) \cdot F'(z) + \left(q(Q(z)) \cdot Q'(z)^2\right) \cdot F(z) = 0. \quad (3.2)$$

We want to prove that the function $p(Q(z)) \cdot Q'(z) - \frac{Q''(z)}{Q'(z)}$ only has poles of order at most 1 in the points $Q^{-1}(\{z_1, \ldots, z_k, \infty\})$, and no other poles. The function $\frac{Q''(z)}{Q'(z)}$ only has a pole at

$z = y$ if $Q'(y) = 0$ or $Q''(y) = \infty$, and then this pole has order 1. If $Q'(y) = 0$, then the function $Q(z) - Q(y)$ has a multiple root at $z = y$, which implies that $Q(y)$ is one of the singularities of f; if $Q''(y) = \infty$, it follows that $Q(y) = \infty$. For $z \to \infty$, from writing $Q(z)$ as a Laurent series around $z = \infty$, it follows that $Q''(z)/Q'(z) = O(1/z)$.

The function $p(Q(z)) \cdot Q'(z)$ only has a pole at $z = y$ if p has a pole at $Q(y)$ or if $Q(y) = \infty$, from which follows that $y \in Q^{-1}(\{z_1, \ldots, z_k, \infty\})$. First assume that Q has a pole of order d at $z = y$, then $p(Q(z))$ has a root of order at least d at $z = y$, and $Q'(z)$ has a pole of order $d + 1$ at $z = y$, so $p(Q(z)) \cdot Q'(z)$ has a pole of order at most 1 at $z = y$. Now assume that $Q(y)$ is finite. If $Q(z) - Q(y)$ has a root of degree d at $z = y$, it follows that the $p(Q(z))$ has a pole of order at most d. On the other hand, the function $Q'(z)$ has a root of order $d - 1$ at $z = y$, so the function $p(Q(z)) \cdot Q'(z)$ has a pole of order at most 1 at $z = y$.

Now we want to prove that $p(Q(z)) \cdot Q'(z) = O(1/z)$, first assume that Q goes to ∞ with order d, then $p(Q(z))$ has a root of order at least d, and $Q'(z)$ has a pole of order $d-1$ at $z = \infty$. If $Q(\infty)$ is finite and $Q(z)-Q(y)$ has order d, it follows that $p(Q(z))$ has a pole of order at most d for $z \to \infty$, but $Q'(z)$ has a root of order $d - 1$ for $z \to \infty$. From this follows that the function $p(Q(z)) \cdot Q'(z) - \frac{Q''(z)}{Q'(z)}$ only has poles of order at most 1 in the points $Q^{-1}(\{z_1, \ldots, z_k, \infty\})$, and no other poles. Moreover, for $z \to \infty$ we have $p(Q(z)) \cdot Q'(z) - \frac{Q''(z)}{Q'(z)} = O(1/z)$.

The function $q(Q(z)) \cdot Q'(z)^2$ only has a pole at $z = y$ if q has a pole at $Q(y)$ or if $Q(y) = \infty$, from which follows that $y \in Q^{-1}(\{z_1, \ldots, z_k, \infty\})$. First assume that Q has a pole of order d at $z = y$, then $q(Q(z))$ has a root of order at least $2d$ at $z = y$, and $Q'(z)^2$ has a pole of order $2d+2$ at $z = y$, so $q(Q(z)) \cdot Q'(z)^2$ has a pole of order at most 2 at $z = y$. Now assume that $Q(y)$ is finite. If $Q(z) - Q(y)$ has a root of degree d at $z = y$, it follows that the pole of $q(Q(z))$ has a pole of order at most $2d$. On the other hand, the function $Q'(z)^2$ has a root of order $2d - 2$ at $z = y$, so the function $q(Q(z)) \cdot Q'(z)$ has a pole of order at most 2 at $z = y$.

Now we want to prove that $q(Q(z)) \cdot Q'(z)^2 = O(1/z^2)$, first assume that Q goes to ∞ with order d, then $q(Q(z))$ has a root of order at least $2d$, and $Q'(z)$ has a pole of order $2d - 2$ at $z = \infty$. If $Q(\infty)$ is finite and $Q(z) - Q(y)$ has order d, it follows that $q(Q(z))$ has a pole of order at most $2d$ for $z \to \infty$, but $Q'(z)^2$ has a root of order $2d - 2$ for $z \to \infty$. From this follows that the function $q(Q(z)) \cdot Q'(z)^2$ only has poles of order at most 2 in the points $Q^{-1}(\{z_1, \ldots, z_k, \infty\})$, and no other poles. Moreover, for $z \to \infty$ we have $q(Q(z)) \cdot Q'(z)^2 = O(1/z^2)$.

The statement about local exponents follows from the composition $f \circ Q$, we omit the complete proof. □

3.1 First example

In this section we will give a third proof of theorem 2.11. Our first goal is to prove the identity in equation (3.11). Consider the function $P(z) = (z + 256)^3/z^2$. We have $P^{-1}(0) = \{-256, -256, -256\}$ and $P^{-1}(1728) = \{-64, 512, 512\}$ and $P^{-1}(\infty) = \{0, 0, \infty\}$, counting multiplicities. From this follows that P only ramifies above the points $0, 1728, \infty$, this can be proven using the Riemann–Hurwitz formula: let S', S be Riemann surfaces and let an analytic map $S' \to S$ of degree N be given. Then it follows that

$$2g(S') - 2 = N \cdot (2g(S) - 2) + \sum_{p \in S'} (e_P - 1), \tag{3.3}$$

where e_P is the ramification index of P. We can apply this formula with $P : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, which is a map of degree 3. Because $g(\mathbb{P}^1(\mathbb{C})) = 0$, it follows that $\sum_{p \in \mathbb{P}^1(\mathbb{C})} (e_P - 1) = 4$, but we already found the points $-256, 512, 0$ with e_P respectively equal to $2, 1, 1$. From this follows that P only ramifies above $\{P(-256), P(512), P(0)\} = \{0, 1728, \infty\}$.

First recall that a Fuchsian differential equation with 3 singular points is determined uniquely by its local exponents. Consider

$$Q(z) = 1 - \frac{1728}{P\left(64z - 64\right)}, \tag{3.4}$$

then it then follows that $Q^{-1}(0) = \{0, 9, 9\}$ and $Q^{-1}(1) = \{1, 1, \infty\}$ and $Q^{-1}(\infty) = \{-3, -3, -3\}$. Consider

$$ {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|Q(z)\right), \qquad \text{which has Riemann scheme} \qquad \begin{array}{ccccc} 0 & 9 & 1 & \infty & -3 \\ \hline 0 & 0 & 0 & 0 & 1/4 \\ 1/2 & 1 & 0 & 0 & 5/4 \end{array}. \tag{3.5}$$

From this follows that

$$(z+3)^{-1/4}{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|Q(z)\right) \quad \text{has Riemann scheme} \quad \begin{array}{ccccc} 0 & 9 & 1 & \infty & -3 \\ \hline 0 & 0 & 0 & 1/4 & 0 \\ 1/2 & 1 & 0 & 1/4 & 1 \end{array}. \tag{3.6}$$

Because in equation (3.6) the points $z = -3$ and $z = 9$ have local exponents $0, 1$, these points are regular points of this function, which leaves $z = 0, 1, \infty$ as singular points. From this follows that

$$\left(\frac{z}{3} + 1\right)^{-1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|Q(z)\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2} \,|z\right) \tag{3.7}$$

as an equation of power series around $z = 0$, because both functions have the same Riemann scheme, and evaluate to 1 for $z = 0$, and are holomorphic at $z = 0$. From inserting $\frac{z}{z-1}$ into equation (3.7) follows that

$$\left(\frac{z}{3(z-1)} + 1\right)^{-1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| Q\left(\frac{z}{z-1}\right)\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2} \middle| \frac{z}{z-1}\right). \tag{3.8}$$

Comparing Riemann schemes gives that

$$(1-z)^{-1/4}{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2} \middle| \frac{z}{z-1}\right) = {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2} \,|z\right), \tag{3.9}$$

from which follows that

$$\left(1 - \frac{4}{3}z\right)^{-1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| Q\left(\frac{z}{z-1}\right)\right) = \left(\frac{z}{3} + 1\right)^{-1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|Q(z)\right) \tag{3.10}$$

and

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| \frac{z(8z-9)^2}{(4z-3)^3}\right) = \left(\frac{3-4z}{z+3}\right)^{1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| \frac{z(z-9)^2}{(z+3)^3}\right). \tag{3.11}$$

Now we know the identity in equation (3.11), we can plug in $z = 9/8$, which gives

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|0\right) = 1 \stackrel{?}{=} \left(-\frac{4}{11}\right)^{1/4} {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \middle| \frac{1323}{1331}\right), \tag{3.12}$$

which is not true, because we know that ${}_2F_1\left(1/12, 5/12; 1/2 \,|1323/1331\right)$ is real and positive, and also is not equal to $\sqrt[4]{11/4}$. The reason that the possible equality in equation (3.12) is not true, is that equation (3.11) is an equality of power series around $z = 0$. If we want to plug in $z = 9/8$, we should construct a curve $\gamma : [0, 1] \to \mathbb{C}$ which starts at $z = 0$ and ends at $z = 9/8$, because for both these values of z the function $\frac{z(8z-9)^2}{(4z-3)^3}$ evaluates to 0. But along the curve $\frac{\gamma(t)(8\gamma(t)-9)^2}{(4\gamma(t)-3)^3}$ the functions in equation (3.11) change because the hypergeometric function is multivalued, and changes under monodromy.

Consider the path

$$\gamma(t) = \frac{9}{16} + \frac{9}{16} e^{\pi i(1-t)}, \tag{3.13}$$

we want to calculate the analytic continuation of equation (3.11) along γ. Substituting $\gamma(t)$ into $\frac{z(8z-9)^2}{(4z-3)^3}$ gives a loop which starts and ends at $z = 0$, and is homotopic to a clockwise loop around $z = 1$, for a picture see figure 3.1. Inserting $\gamma(t)$ into $\frac{z(z-9)^2}{(z+3)^3}$ gives a path, starting at $z = 0$ and ending at $z = 1323/1331$, which is homotopic to one clockwise loop around $z = 1$, see figure 3.1 for a picture; formally we should extend the path we found with a line segment from $z = 1323/1331$ to $z = 0$, but along this line segment the functions F_1 and G_1 are invariant under analytic continuation. From this follows that if we want to calculate the analytic continuation of equation (3.11), we have to know the function ${}_2F_1(1/12, 5/12; 1/2 \,|\, x)$ after one clockwise loop around $x = 1$. Unfortunately, we do not know the monodromy of this function, but in chapter 2 we calculated the monodromy of some other functions, so we want to use these results.

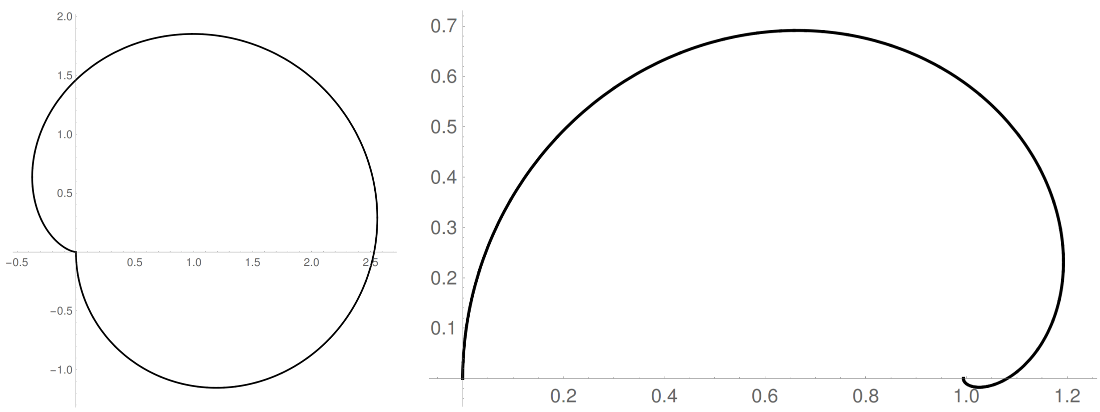

Figure 3.1: Left: picture of the loop $\frac{z(8z-9)^2}{(4z-3)^3}$ with $z = \gamma(t)$; this loop starts at $z = 0$, goes clockwise around $z = 1$, and ends at $z = 0$. Right: picture of the loop $\frac{z(z-9)^2}{(z+3)^3}$ with $z = \gamma(t)$; this loop starts at $z = 0$, goes clockwise around $z = 1$, and ends at $z = 1323/1331$. Here $\gamma : [0, 1] \to \mathbb{C}$ is given by $t \mapsto \frac{9}{16} + \frac{9}{16} e^{\pi i(1-t)}$.

In chapter 2 we already found some formulas, which we can use to calculate the monodromy of ${}_2F_1(1/12, 5/12; 1/2 \,|\, x)$. In definition 2.1 we defined the following bases of local solutions of the hypergeometric differential equation with parameters $1/12, 5/12, 1$:

$$\begin{aligned}
F_0(x) &:= {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,|x\right) && |x| < 1\\
G_0(x) &:= \log(x)F_0(x) + \sum_{n=0}^{\infty} \frac{(1/12)_n(5/12)_n}{(n!)^2}\left(\sum_{j=0}^{n-1} \frac{1}{1/12+j} + \frac{1}{5/12+j} - \frac{2}{j+1}\right)x^n && |x| < 1\\
F_1(x) &:= {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|1-x\right) && |x-1| < 1\\
G_1(x) &:= (1-x)^{1/2}{}_2F_1\left(\frac{7}{12}, \frac{11}{12}; \frac{3}{2} \,|1-x\right) && |x-1| < 1.
\end{aligned} \tag{3.14}$$

Note that in this definition we have ${}_2F_1\left(1/12, 5/12; 1/2 \,|x\right) = F_1(1-x)$, so in our calculations we sometimes have to insert $1-x$ instead of x. We follow these definitions to be consistent with chapter 2, so we can use the results in this chapter. In lemma 2.8 we found that

$${}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|1-x\right) = F_1(x) = -\frac{\sqrt{\pi}}{\Gamma(1/12)\Gamma(5/12)}G_0(x) + \frac{\sqrt{\pi}(2\pi+\log(1728))}{\Gamma(1/12)\Gamma(5/12)}F_0(x). \tag{3.15}$$

We also want to express G_1 in terms of F_0 and G_0; we first state and prove this formula as a lemma:

Lemma 3.3. *Let $0 < a, b, a+b < 1$ and F_0, G_0, G_1 as in definition 2.1. Then it follows for $x \in \mathbb{C}$ with $|x|, |1-x| < 1$ that*

$$G_1(x) = -\frac{\Gamma(2-a-b)}{\Gamma(1-b)\Gamma(1-a)}G_0(x) + \frac{\Gamma(2-a-b)}{\Gamma(1-b)\Gamma(1-a)}\left(\sum_{j=0}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1}\right)F_0(x). \tag{3.16}$$

Proof. The unbounded part of $G_1(x)$ around $x = 0$ is equal to $-\frac{\Gamma(2-a-b)}{\Gamma(1-b)\Gamma(1-a)}\log(x)$, which gives us the coefficient of $G_0(x)$. Write $G_1(x) = -\frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)}G_0(x) + AF_0(x)$ for some $A \in \mathbb{C}$, then it follows that

$$A = \frac{G_1(x)}{F_0(x)} + \frac{\Gamma(2-a-b)}{\Gamma(1-a)\Gamma(1-b)}\frac{G_0(x)}{F_0(x)}. \tag{3.17}$$

We have $\lim_{x\to1} G_1(x) = 0$ and $\lim_{x\to1} \frac{G_0(x)}{F_0(x)} = \sum_{j=0}^{\infty} \frac{1}{1-a+j} + \frac{1}{1-b+j} - \frac{2}{j+1}$, from which the result follows. $\square$

From lemma 3.3 follows that

$$G_1(x) = -\frac{\sqrt{\pi}}{2\Gamma(7/12)\Gamma(11/12)}G_0(x) + \frac{\sqrt{\pi}(-2\pi+\log(1728))}{2\Gamma(7/12)\Gamma(11/12)}F_0(x), \tag{3.18}$$

where we use that $\Gamma(3/2) = \sqrt{\pi}/2$. The vector space of solutions of the hypergeometric differential equation is two-dimensional, and has bases $B_0 = \{F_0, G_0\}$ and $B_1 = \{F_1, G_1\}$, see definition 2.1. From equation (3.18) and equation (3.15) we know that the transformation matrix from B_1 to B_0 is given by

$$M = \begin{pmatrix} \frac{\sqrt{\pi}(2\pi+\log(1728)}{\Gamma(1/12)\Gamma(5/12)} & \frac{\sqrt{\pi}(-2\pi+\log(1728)}{2\Gamma(7/12)\Gamma(11/12)} \\ -\frac{\sqrt{\pi}}{\Gamma(1/12)\Gamma(5/12)} & -\frac{\sqrt{\pi}}{2\Gamma(7/12)\Gamma(11/12)} \end{pmatrix}, \tag{3.19}$$

this means that $(F_1, G_1) = (F_0, G_0) \cdot M$. The monodromy matrix of one counterclockwise loop around $x = 0$ is given by $M_0 = \left(\begin{smallmatrix} 1 & 2\pi i \\ 0 & 1 \end{smallmatrix}\right)$ with respect to the basis B_0.

Now we know the transformation formulas and monodromy matrices, we can calculate the function ${}_2F_1\left(1/12, 5/12; 1/2 \,|x\right)$ after one clockwise loop around $x = 1$: using the matrix M from equation (3.19) we write the function ${}_2F_1\left(1/12, 5/12; 1/2 \,|x\right)$ as a linear combination of $F_0(1-x)$ and $G_0(1-x)$, then we calculate the monodromy of this function, and then we transform back to a linear combination of $F_1(1-x)$ and $G_1(1-x)$. We have

$$ {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; \frac{1}{2} \,|x\right) = F_1(1-x) = -\frac{\sqrt{\pi}}{\Gamma(1/12)\Gamma(5/12)} G_0(1-x) + \frac{\sqrt{\pi}(2\pi + \log(1728))}{\Gamma(1/12)\Gamma(5/12)} F_0(1-x). \tag{3.20} $$

If x starts at 0 and goes one clockwise loop around 1, it follows that $1-x$ starts at 1 and goes one clockwise loop around 0. From this follows that equation (3.20), after one clockwise loop of x around 1, is equal to

$$ -\frac{\sqrt{\pi}}{\Gamma(1/12)\Gamma(5/12)}(G_0(1-x) - 2\pi i F_0(1-x)) + \frac{\sqrt{\pi}(2\pi + \log(1728))}{\Gamma(1/12)\Gamma(5/12)} F_0(1-x). \tag{3.21} $$

Now we transform equation (3.21) to the basis B_1 using the matrix M from equation (3.19), from which follows that the analytic continuation of ${}_2F_1\left(1/12, 5/12; 1/2 \,|x\right)$ after one clockwise loop around 1 is equal to

$$ M^{-1}\begin{pmatrix} 1 & -2\pi i \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + i/2 \\ -i\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} \end{pmatrix} = \left(1 + \frac{i}{2}\right) F_1(1-x) - i\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} G_1(1-x), \tag{3.22} $$

where we identified $F_1(1-x)$ with $(1,0)^t$, and $G_1(1-x)$ with $(0,1)^t$. The calculation in equation (3.22) means that we started with the function $F_1(1-x)$, which is identified with the vector $(1,0)^t$. Then the function $F_1(1-x)$ is sent to a linear combination of $F_0(1-x)$ and $G_0(1-x)$, which is identified with the vector $M \cdot (0,1)^t$. Because $1-x$ goes one clockwise loop around 0, the monodromy matrix $\left(\begin{smallmatrix} 1 & -2\pi i \\ 0 & 1 \end{smallmatrix}\right)$ is used, because $M \cdot (1,0)^t$ is written with respect to the basis B_0: this gives a new linear combination of $F_0(1-x)$ and $G_0(1-x)$. After calculating the monodromy, the linear combination of $F_0(1-x)$ and $G_0(1-x)$ is sent back to the basis B_1, which gives a linear combination of $F_1(1-x)$ and $G_1(1-x)$.

From this follows that the left-hand side of equation (3.11), after analytic continuation along γ, is equal to

$$ \left(1 + \frac{i}{2}\right) F_1(1-0) - i\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} G_1(1-0) = 1 + \frac{i}{2}, \tag{3.23} $$

because $F_1(1) = 1$ and $G_1(1) = 0$. The right-hand side of equation (3.11), after analytic continuation along γ, is equal to

$$ e^{-\pi i/4}\sqrt[4]{\frac{4}{11}}\left(1 + \frac{i}{2}\right) F_1\left(1 - \frac{1323}{1331}\right) - ie^{-\pi i/4}\sqrt[4]{\frac{4}{11}}\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} G_1\left(1 - \frac{1323}{1331}\right) = 1 + \frac{i}{2}, \tag{3.24} $$

where the factor $e^{-\pi i/4}(4/11)^{1/4}$ comes from the analytic continuation of $\left(\frac{3-4z}{z+3}\right)^{1/4}$ along $z = \gamma(t)$. Here we use that in equation (3.11) an equality of power series is mentioned, and the analytic continuation of the left-hand side of equation (3.11) is equal to the analytic continuation of the right-hand side of equation (3.11). From equation (3.24) we have one linear equation in $F_1(8/1331)$ and $G_1(8/1331)$, and if we can produce another linear equation, we can calculate $F_1(8/1331)$ and $G_1(8/1331)$.

Because $F_1(8/1331)$ and $G_1(8/1331)$ are real, we can take the complex conjugate of equation (3.24), which gives another linear equation between $F_1(8/1331)$ and $G_1(8/1331)$. From this follows that

$$\begin{pmatrix} 1+i/2 \\ 1-i/2 \end{pmatrix} = \begin{pmatrix} e^{-\pi i/4}\sqrt[4]{\frac{4}{11}}\left(1+\frac{i}{2}\right) & -e^{\pi i/4}\sqrt[4]{\frac{4}{11}}\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} \\ e^{\pi i/4}\sqrt[4]{\frac{4}{11}}\left(1-\frac{i}{2}\right) & -e^{-\pi i/4}\sqrt[4]{\frac{4}{11}}\frac{\Gamma(7/12)\Gamma(11/12)}{\Gamma(1/12)\Gamma(5/12)} \end{pmatrix} \begin{pmatrix} F_1(8/1331) \\ G_1(8/1331) \end{pmatrix}, \qquad (3.25)$$

which is a system of two linear equations in two unknowns, and we can solve this: we have

$$F_1\left(\frac{8}{1331}\right) = {}_2F_1\left(\frac{1}{12},\frac{5}{12};\frac{1}{2}\middle|\frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11} \qquad (3.26)$$

and

$$\sqrt{\frac{1331}{1323}}G_1\left(\frac{8}{1331}\right) = {}_2F_1\left(\frac{7}{12},\frac{11}{12};\frac{3}{2}\middle|\frac{1323}{1331}\right) = \frac{11^{7/4}}{168\sqrt{3}}\frac{\Gamma(1/12)\Gamma(5/12)}{\Gamma(7/12)\Gamma(11/12)}. \qquad (3.27)$$

3.2 Theoretical motivation

In section 3.1 the function $P(z) = (z+256)^3/z^2$ is used, from which follows a special value of the hypergeometric function. Now we will give the theoretical motivation why we used the function $P(z)$ and the parameters $1/12, 5/12$. Later we want to find new special values of the hypergeometric function using this method, but with other parameters and another rational function. Looking at the example in section 3.1, we formulate some conditions which are necessary to find a new special value of the hypergeometric function.

The function j is an isomorphism from $X(\mathrm{SL}_2(\mathbb{Z}))$ to $\mathbb{P}^1(\mathbb{C})$. On the other hand, we have that $h(z) = \Delta(z)/\Delta(2z)$ is an isomorphism from $X(\Gamma_0(2))$ to $\mathbb{P}^1(\mathbb{C})$. From these isomorphisms follows that the inclusion map $\iota : \Gamma_0(2)\backslash\mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ extends to a holomorphic function $P : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, but we already found this function in table 1.3. Moreover, from lemma 1.8 follows that P is a Belyi function, because in the notation of lemma 1.8 we have $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})$ and $h_2(z) = j(z)$, from which follows that 0 and 1728 are the only values of $h_2(w_0)$, if w_0 is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i or $e^{\pi i/3}$. The map $z \mapsto -\frac{1}{2z}$ is an involution on $\Gamma_0(2)\backslash\mathbb{H}$, and we have $h\left(-\frac{1}{2z}\right) = 2^{12}/h(z)$. From the isomorphism $h : X(\Gamma_0(2)) \to \mathbb{P}^1(\mathbb{C})$ follows an involution on $\mathbb{P}^1(\mathbb{C})$ with $z \mapsto 2^{12}/z$. For a picture of all mentioned maps, see figure 3.2.

$$\begin{array}{ccc} z\mapsto -\frac{1}{2z} \circlearrowleft & & z\mapsto 2^{12}/z \circlearrowleft \\ \Gamma_0(2)\backslash\mathbb{H} & \xrightarrow[h]{\sim} & \mathbb{P}^1(\mathbb{C}) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle P} \\ \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} & \xrightarrow[j]{\sim} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

Figure 3.2: Picture of maps between $\Gamma_0(2)\backslash\mathbb{H}$ and $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

The functions $P_1(z) := P(z)$ and $P_2(z) := P(2^{12}/z)$ are two different functions from $\mathbb{P}^1(\mathbb{C})$ to itself. The hypergeometric differential equation with parameters $1/12, 5/12, 1/2$ has local exponents $0, 1/2$ at $z = 0$, and $0, 0$ at $z = 1$, and $1/12, 5/12$ at $z = \infty$. We have

$$\begin{array}{lll} P_1^{-1}(0) = \{-256,-256,-256\} & P_1^{-1}(1728) = \{-64,512,512\} & P_1^{-1}(\infty) = \{0,0,\infty\} \\ P_2^{-1}(0) = \{-16,-16,-16\} & P_2^{-1}(1728) = \{-64,8,8\} & P_2^{-1}(\infty) = \{\infty,\infty,0\}, \end{array} \qquad (3.28)$$

counting multiplicities. Consider the functions $Q_1(z) = 1 - 1728/P_1(64z - 64)$ and $Q_2(z) = 1 - 1728/P_2(64z - 64)$, then it follows that $Q_2(z) = Q_1\left(\frac{z}{z-1}\right)$ and

$$\begin{aligned} &Q_1^{-1}(0) = \{0, 9, 9\} && Q_1^{-1}(1) = \{1, 1, \infty\} && Q_1^{-1}(\infty) = \{-3, -3, -3\} \\ &Q_2^{-1}(0) = \left\{0, \frac{9}{8}, \frac{9}{8}\right\} && Q_2^{-1}(1) = \{\infty, \infty, 1\} && Q_2^{-1}(\infty) = \left\{\frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right\}, \end{aligned} \tag{3.29}$$

where $0, 1, \infty$ are the only singular points: the extra singular points become regular, because their local exponents are equal to $0, 1$, which follows from the choice of parameters. The points $0, 1, \infty$ in equation (3.29) come from the points $0, -64, \infty$ in equation (3.28), and the set $\{0, -64, \infty\}$ is invariant under the involution $z \mapsto 2^{12}/z$: the point $z = -64$ is invariant, and the points $z = 0, \infty$ get interchanged. In general the set of singular points is not invariant under the involution, from which follows that in some situations an extra Möbius transformation should be applied. Note that because we have only three singular points with given local exponents, the differential equation following from this Riemann scheme is unique.

Because the inverse images in equation (3.29) give the same Riemann scheme, and the functions P_1 and P_2 are different, this gives an algebraic relation, see equation (3.11). Using the algebraic relation equation (3.11), a special value of the hypergeometric function can be calculated, because in this situation we have $Q_1(0) = Q_2(0) = 0$ and $Q_2(9/8) = 0$ and $Q_1(9/8) = 1323/1331 \neq 0$. Note that in equation (3.29) the points $0, \infty$ have different multiplicities, but because the local exponents are equal to $0, 0$, the local exponents at $z = 0$ and $z \to \infty$ are in both cases equal to $0, 0$, which gives the same Riemann schemes for ${}_2F_1\left(1/12, 5/12; 1/2 \,|Q_1(z)\right)$ and ${}_2F_1\left(1/12, 5/12; 1/2 \,|Q_2(z)\right)$.

Now we want to give a motivation why we chose the function P arising from the group inclusion $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z})$, because in principle we can choose every function in $\mathbb{C}(z)$. The first property of P we used was that P is a Belyi function, because then we can use proposition 3.2:

Definition 3.4. A Belyi function is a holomorphic map from a compact Riemann surface to $\mathbb{P}^1(\mathbb{C})$ which ramifies only over three points.

The fact that P is a Belyi function follows from lemma 1.8, because we know that $X(\Gamma_0(2)) \cong \mathbb{P}^1(\mathbb{C})$, and we also know the Hauptmodul of $X(\Gamma_0(2))$. A list of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ which modular curves have genus 0 is given in [13].

Another property we used in the situation in section 3.1, is that in the Riemann scheme of ${}_2F_1\left(1/12, 5/12; 1/2 \,|P(z)\right)$ there are two points with the same exponent difference. If all three points in a Riemann scheme have a pairwise different exponent difference, it is not possible to produce an involution which gives the same Riemann scheme.

3.3 First calculation

We want to find a new special value of the hypergeometric function, using the method from section 3.1, but in this case we consider the congruence subgroup $\Gamma_0(7) \subset \mathrm{SL}_2(\mathbb{Z})$, which has index 8. Our first goal is to prove the identity in equation (3.38). Note that in this calculation we first choose the group $\Gamma_0(7)$, and later in this section we will choose the parameters of the hypergeometric function. A Hauptmodul for $X(\Gamma_0(7))$ is given by $h(z) = \frac{\eta^4(z)}{\eta^4(7z)}$, and a set of coset representatives of $\Gamma_0(7)\backslash\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\left\{z \mapsto z, z \mapsto -1/z, z \mapsto \frac{-1}{z+1}, z \mapsto \frac{-1}{z+2}, z \mapsto \frac{-1}{z+3}, z \mapsto \frac{-1}{z+4}, z \mapsto \frac{-1}{z+5}, z \mapsto \frac{-1}{z+6}\right\}. \tag{3.30}$$

From the inclusion map $\Gamma_0(7)\backslash\mathbb{H} \to \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ follows a map $P : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with

$$\begin{aligned} P^{-1}(0) &= \{h(\gamma e^{\pi i/3}) | \gamma \in \Gamma_0(7)\backslash \mathrm{SL}_2(\mathbb{Z})\} &&= \left\{z \in \mathbb{C} | (z^2+13z+49)(z^2+245z+2401)^3 = 0\right\} \\ P^{-1}(1728) &= \{h(\gamma i) | \gamma \in \Gamma_0(7)\backslash \mathrm{SL}_2(\mathbb{Z})\} &&= \left\{z \in \mathbb{C} | (z^4-490z^3-21609z^2-235298z-823543)^2 = 0\right\} \\ P^{-1}(\infty) &= \{h(\gamma\infty) | \gamma \in \Gamma_0(7)\backslash \mathrm{SL}_2(\mathbb{Z})\} &&= \{0,0,0,0,0,0,0,\infty\}, \end{aligned} \tag{3.31}$$

counting multiplicities. If we choose parameters a, b, c such that $1-c = 1/3$ and $c-a-b = 1/2$ and $b-a = 1/7$, the points in $P^{-1}(0)$ with multiplicity 3 and the points in $P^{-1}(1728)$ have local exponents $0, 1$, and are therefore regular points. The point $0 \in P^{-1}(\infty)$ has local exponent difference 1, and therefore can be made a regular point. From this follows that there are only three singular points, and then we can use that a Fuchsian differential equation with three singular points is unique, given its local exponents. Note that with these parameters the corresponding triangle group is compact, contrary to the non-compact triangle groups we used until here.

Remark 3.5. In this calculation we know the Hauptmodul of $X(\Gamma_0(7))$, from which we could calculate the inverse images of $0, 1728, \infty$. If the Hauptmodul is not known, we can still consider the formulas in equation (3.31), but then we only know the multiplicities of the inverse images of $0, 1728, \infty$. Here we use that for $z_0, z_1 \in \mathbb{H}$ we have $h(z_0) = h(z_1)$ if and only if there exists a $\gamma \in \Gamma_0(7)$ such that $z_0 = \gamma z_1$. The question whether there exists a $\gamma \in \Gamma_0(7)$ such that $z_0 = \gamma z_1$ can be answered using only the group $\Gamma_0(7)$, without knowing its Hauptmodul.

The Atkin–Lehner involution on $\Gamma_0(7)$ is given by $z \mapsto -\frac{1}{7z}$ and we have $h\left(-\frac{1}{7z}\right) = 49/h(z)$. The set of points in equation (3.31) does not contain a subset of three elements which is invariant under $z \mapsto 49/z$, so we cannot use the method from section 3.1: here we use that an involution on a set with an odd number of elements has a fixpoint, but the points ± 7 are not mentioned in equation (3.31). Define

$$P(z) = \frac{(49+13z+z^2)(2401+245z+z^2)^3}{z^7} \tag{3.32}$$

and consider the following function and its Riemann scheme:

$$z^{-1/12}{}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3} \middle| \frac{P(z)}{1728}\right); \qquad \begin{array}{ccc} -\frac{13}{2}+\frac{3i}{2}\sqrt{3} & -\frac{13}{2}-\frac{3i}{2}\sqrt{3} & \infty \\ \hline 0 & 0 & 2/21 \\ 1/3 & 1/3 & 5/21 \end{array}; \tag{3.33}$$

here we chose the function P and parameters $1/84, 13/84, 2/3$ and the factor $z^{-1/12}$, because then equation (3.33) only has three singular points. However, the singular points of equation (3.33) are not equal to $0, 1, \infty$, so we will make a substitution: insert $3i\sqrt{3}z - \frac{13}{2} - \frac{3i\sqrt{3}}{2}$ into equation (3.31); define

$$Q(z) = \frac{P\left(3i\sqrt{3}z - \frac{13}{2} - \frac{3i\sqrt{3}}{2}\right)}{1728} = \frac{2(z-1)z\left(-27iz^2 + \left(-696\sqrt{3}+27i\right)z + 348\sqrt{3} + 844i\right)^3}{\left(6\sqrt{3}z - 3\sqrt{3} + 13i\right)^7}, \tag{3.34}$$

from which follow the following function and Riemann scheme:

$$\left(3i\sqrt{3}z - \frac{13}{2} - \frac{3i\sqrt{3}}{2}\right)^{-1/12} {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3} \middle| Q(z)\right); \qquad \begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & 2/21 \\ 1/3 & 1/3 & 5/21 \end{array}. \tag{3.35}$$

The function in equation (3.35) has singular points $0, 1, \infty$ and has the same Riemann scheme as ${}_2F_1\left(2/21, 5/21; 2/3 \,|z\right)$, so after dividing by $-13/2 - 3i\sqrt{3}/2$ it follows that

$$\left(1 - \frac{27+39\sqrt{3}i}{98}z\right)^{-1/12} {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3}\,|Q(z)\right) = {}_2F_1\left(\frac{2}{21}, \frac{5}{21}; \frac{2}{3}\,|z\right), \tag{3.36}$$

as an equality of power series around $z = 0$; here we use that both sides of equation (3.36) have the same Riemann scheme, evaluate to 1 at $z = 0$ and are holomorphic at $z = 0$. Because $z = 0, 1$ have the same local exponents, it follows for equation (3.36) that inserting $1 - z$ instead of z gives the same Riemann scheme. From this follows that

$$\left(1 - \frac{27+39\sqrt{3}i}{98}z\right)^{-1/12} {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3}\,|Q(z)\right) = \left(1 - \frac{27-39\sqrt{3}i}{98}z\right)^{-1/12} {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3}\,|Q(1-z)\right), \tag{3.37}$$

because both sides of this equality are holomorphic at $z = 0$, and evaluate to 1 at $z = 0$. From this follows that

$${}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3}\,|Q(z)\right) = \left(\frac{98 - 27z - 39\sqrt{3}iz}{98 - 27z + 39\sqrt{3}iz}\right)^{1/12} {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3}\,|Q(1-z)\right). \tag{3.38}$$

Now we know the identity in equation (3.38), we want to construct a curve $\gamma : [0, 1] \to \mathbb{C}$ such that $Q(\gamma(0)) = Q(\gamma(1)) = 0$, because then we can use that ${}_2F_1\left(1/84, 13/84; 2/3 \,|0\right) = 1$, and we can find a linearly independent solution $G_0(x)$ for $|x| < 1$ of the hypergeometric differential equation with $G_0(0) = 0$; the function G_0 will be defined in equation (3.41). If we have $Q(1 - \gamma(1)) \neq 0$, we can find a special value of the hypergeometric function. Now we calculate the analytic continuation of equation (3.38) from $z = 0$ to

$$z = y := \frac{1}{3i\sqrt{3}}\left(\frac{49}{2}\left(-5 - \sqrt{21}\right) + \frac{13}{2} + \frac{3}{2}i\sqrt{3}\right) = \frac{1}{2} + \frac{116}{9}\sqrt{3}i + \frac{49}{6}\sqrt{7}i \approx \frac{1}{2} + 43.9i, \tag{3.39}$$

note that $Q(y) = 0$. Let $\gamma : [0, 1] \to \mathbb{C}$ be a halfcircle going clockwise from $z = 0$ to $z = y$, then $Q(\gamma(t)) \cong \gamma_0^{-1}\gamma_1^{-1}\gamma_0^{-1}\gamma_1^{-1}$ written from right to left, where γ_0 is a counterclockwise loop around $z = 0$, and γ_1 is a counterclockwise loop around $z = 1$. We also have $Q(1 - \gamma(t)) \cong \gamma_1\gamma_0\gamma_1\gamma_0$, followed by a path from 0 to

$$Q(1-y) = -38241952 \cdot \frac{21307687138103583 - 4649976434760203\sqrt{21}}{33275593513886484375} \approx 1.358, \tag{3.40}$$

through the lower half plane. In section 3.1 we could use the same formula in equation (3.23) and equation (3.24), because the two loops were homotopic. However, in this calculation the two loops are not homotopic, so we should calculate two different analytic continuation along two different loops.

Now we want to calculate the monodromy of the functions in equation (3.38), along the curves $Q(\gamma(t))$ and $Q(1 - \gamma(t))$. To do this, we first define local bases of the hypergeometric differential equation with parameters $a = 1/84$ and $b = 13/84$ and $c = 2/3$ around $x = 0$ and $x = 1$. If we know these functions and the transformation matrix between these two bases, we can calculate the monodromy of the functions in equation (3.38), where every time we calculate the monodromy of one loop around $x = 0$ or $x = 1$, we transform to the local basis around that point. Define

$$\begin{aligned} F_0(x) &:= {}_2F_1\left(a,b;c\,|x\right) && |x|<1 \\ G_0(x) &: x^{1-c}{}_2F_1\left(a+1-c,b+1-c;2-c\,|x\right) && |x|<1 \\ F_1(x) &:= {}_2F_1\left(a,b;a+b+1-c\,|1-x\right) && |x-1|<1 \\ G_1(x) &:= (1-x)^{c-a-b}{}_2F_1\left(c-b,c-a;c-a-b+1\,|1-x\right) && |x-1|<1. \end{aligned} \tag{3.41}$$

With $a=1/84$ and $b=13/84$ and $c=2/3$, it follows that the monodromy matrix of the loop γ_0 is given by $M_0=\left(\begin{smallmatrix}1&0\\0&e^{2\pi i/3}\end{smallmatrix}\right)$, with respect to the basis $\{F_0,G_0\}$. The monodromy matrix of the loop γ_1 is given by $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Without proof we state that

$$M=\begin{pmatrix}\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)}\\ \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)} & \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{\pi}\Gamma(2/3)}{\Gamma(43/84)\Gamma(55/84)} & \frac{\sqrt{\pi}\Gamma(4/3)}{\Gamma(71/84)\Gamma(83/84)}\\ -\frac{2\sqrt{\pi}\Gamma(2/3)}{\Gamma(1/84)\Gamma(13/84)} & -\frac{2\sqrt{\pi}\Gamma(4/3)}{\Gamma(29/84)\Gamma(41/84)}\end{pmatrix} \tag{3.42}$$

is the transformation matrix from the basis $\{F_0,G_0\}$ to $\{F_1,G_1\}$, this means that $(F_0,G_0)=(F_1,G_1)\cdot M$.

The analytic continuation of the left-hand slide of equation (3.38) is equal to

$$M_0^{-1}M^{-1}M_1^{-1}MM_0^{-1}M^{-1}M_1^{-1}M\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\alpha\\ \cdot\end{pmatrix}, \tag{3.43}$$

where $27\alpha^6-54\alpha^5+36\alpha^4-36\alpha^3+42\alpha^2-21\alpha+7=0$ and $\alpha\approx1.123+0.071i$. In this calculation, we started with the vector $(1,0)^t$, which we identify with $F_0(x)$, then we calculate the monodromy of the loop $Q(\gamma(t))\cong\gamma_0^{-1}\gamma_1^{-1}\gamma_0^{-1}\gamma_1^{-1}$, transforming to the appropriate basis B_0 or B_1, depending on whether the loop is around $x=0$ or $x=1$. Because $G_0(0)=0$, it follows that the analytic continuation of equation (3.38) along $\gamma(t)$ is equal to α, so we omit the second component of equation (3.43).

To calculate the analytic continuation of the right-hand side of equation (3.38), we use that $\frac{98-27\gamma(t)-39\sqrt{3}i\gamma(t)}{98-27\gamma(t)+39\sqrt{3}i\gamma(t)}$ is a function from $[0,1]$ to $\mathbb{C}\backslash(-\infty,0)$, so its square root can be defined. From this follows that right-hand side of equation (3.38) is equal to

$$\begin{aligned} A &:= \left(\frac{98-27y-39\sqrt{3}iy}{98-27y+39\sqrt{3}iy}\right)^{1/12} MM_0M^{-1}M_1MM_0M^{-1}M_1M\begin{pmatrix}1\\0\end{pmatrix} \\ &= \left(\frac{98-27y-39\sqrt{3}iy}{98-27y+39\sqrt{3}iy}\right)^{1/12} \frac{\sqrt{\pi}\Gamma(2/3)}{(\csc(\pi/84)\csc(13\pi/84)-\sec(\pi/84)\sec(13\pi/84))^2} \\ &\quad\cdot\begin{pmatrix}\frac{\csc^2(\pi/84)\csc^2(13\pi/84)+(1-2i\sqrt{3})\sec^2(\pi/84)\sec^2(13\pi/84)+16(1-i\sqrt{3})\csc(\pi/42)\sec(4\pi/21)}{\Gamma(43/84)\Gamma(55/84)}\\ 2\frac{(-1+2i\sqrt{3})\csc^2(\pi/84)\csc^2(13\pi/84)-\sec^2(\pi/84)\sec^2(13\pi/84)+16(i\sqrt{3}-1)\csc(\pi/42)\sec(4\pi/21)}{\Gamma(1/84)\Gamma(13/84)}\end{pmatrix}, \end{aligned} \tag{3.44}$$

written with respect to the basis $\{F_1,G_1\}$ where we identified $(1,0)^t$ with $F_1(x)$ and $(0,1)^t$ with $G_1(x)$. Here we choose to write equation (3.44) with respect to the basis $\{F_1,G_1\}$, and not with respect to the basis $\{F_0,G_0\}$, later this will be convenient to write down equation (3.46). In this calculation we also started with the function $F_0(x)$, and calculate the monodromy of the loop $Q(1-\gamma(t))\cong\gamma_1\gamma_0\gamma_1\gamma_0$, transforming to the appropriate basis B_0 or B_1, depending on whether the loop is around $x=0$ or $x=1$.

Now we know the analytic continuation of the left-hand side and the right-hand side ofequation (3.38), it follows that equation (3.43) and equation (3.44) are equal to each other, where we in equation (3.43) have to insert $x = 0$ and in equation (3.44) we have to insert $x = Q(1-y)$. From this follows that

$$\alpha = A_1 \cdot F_1(Q(1-y)) + A_2 \cdot G_1(Q(1-y)), \tag{3.45}$$

where α is as in equation (3.43) and A is defined as in equation (3.44). This gives one linear equation in $F_1(Q(1-y))$ and $G_1(Q(1-y))$, but we want another linear equation. In section 3.1 we found one equation, but we also could take the complex conjugate of that equation, from which followed a new linear equation. Because we cannot say something about the complex argument of $F_0(Q(1-y))$ and $G_0(Q(1-y))$, we have written equation (3.44) with respect to the basis $B_1 = \{F_1, G_1\}$. We have $F_1(Q(1-y)) \in \mathbb{R}$ and $G_1(Q(1-y)) \in i\mathbb{R}$, from which follows that $\overline{F_1(Q(1-y))} = F_1(Q(1-y))$ and $\overline{G_1(Q(1-y))} = -G_1(Q(1-y))$. From this follows the equation $\overline{\alpha} = (F_1(Q(1-y)), -G_1(Q(1-y))) \cdot \overline{A}$, from which follows that

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_1} & -\overline{A_2} \end{pmatrix} \begin{pmatrix} F_1(Q(1-y)) \\ G_1(Q(1-y)) \end{pmatrix} = \begin{pmatrix} \alpha \\ \overline{\alpha} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_1 & A_2 \\ \overline{A_1} & -\overline{A_2} \end{pmatrix} (M^t)^{-1} \begin{pmatrix} F_0(Q(1-y)) \\ G_0(Q(1-y)) \end{pmatrix} = \begin{pmatrix} \alpha \\ \overline{\alpha} \end{pmatrix}. \tag{3.46}$$

Solving the right-hand equation in equation (3.46) for $F_0(Q(1-y))$ and $G_0(Q(1-y))$ gives that

$$F_0(Q(1-y)) = (\csc(\pi/84)\csc(13\pi/84) - \sec(\pi/84)\sec(13\pi/84))^2 \left(\frac{205}{1546 - 286\sqrt{21}}\right)^{1/12} \frac{B_1 - B_2}{\sqrt{3}\sqrt[3]{7}B_3} \tag{3.47}$$

with

$$\begin{aligned} B_1 =& (\sqrt{3} + 3i)\alpha \sqrt[12]{2343 + 1287i\sqrt{3} - 1521i\sqrt{7} - 923\sqrt{21}} \\ & \times \left(\csc^2(\pi/84)\csc^2(13\pi/84) + \sec^2(\pi/84)\sec^2(13\pi/84) + 16\csc(\pi/42)\sec(4\pi/21)\right) \end{aligned} \tag{3.48}$$

and

$$B_2 = 3i \sqrt[12]{2343 - 1287i\sqrt{3} + 1521i\sqrt{7} - 923\sqrt{21}} \left(\csc^2(\pi/84)\csc^2(13\pi/84) - \sec^2(\pi/84)\sec^2(13\pi/84)\right) \overline{\alpha} \tag{3.49}$$

and

$$\begin{aligned} B_3 =& \csc^4(\pi/84)\csc^4(13\pi/84) + \sec^4(\pi/84)\sec^4(13\pi/84) + 1024\csc^2(\pi/42)\sec^2(4\pi/21) \\ & + 2\csc^2(\pi/84)\csc^2(13\pi/84)\left(7\sec^2(\pi/84)\sec^2(13\pi/84) + 64\csc(\pi/42)\sec(4\pi/21)\right) \\ & + 128\csc(\pi/42)\sec^2(\pi/84)\sec^2(13\pi/84)\sec(4\pi/21). \end{aligned} \tag{3.50}$$

From this follows that $F_0(Q(1-y))$ is algebraic, but it is a very complicated expression. Note that we have $|Q(1-y)| > 1$, so this argument does not lie in the convergence region of F_0.

3.4 Second calculation

In equation (3.39) we defined a value of y, for another calculation we choose another value:

$$y := \frac{1}{3i\sqrt{3}} \left(\frac{49}{2}\left(-5 + \sqrt{21}\right) + \frac{13}{2} + \frac{3}{2}i\sqrt{3}\right) = \frac{1}{2} + \frac{116}{9}\sqrt{3}i - \frac{49}{6}\sqrt{7}i \approx \frac{1}{2} + 0.717i, \tag{3.51}$$

we have $Q(y) = 0$. We use the same method as in section 3.3, in particular we use the same definitions of F_0, G_0, F_1, G_1. Define $\gamma(t) : [0,1] \to \mathbb{C}$ as the half-circle from $x = 0$ to $x = y$ through the upper half plane, then it follows that $Q(\gamma(t)) \cong \gamma_1^{-1}$ and $Q(1-\gamma(t))$ is homotopic to a clockwise loop around $x = 1$, followed by a path through the lower half plane from $x = 0$ to

$$Q(1-y) = \frac{38241952\left(-21307687138103583 - 4649976434760203\sqrt{21}\right)}{33275593513886484375} \approx -48977.05, \qquad (3.52)$$

see figure 3.3 for a picture.

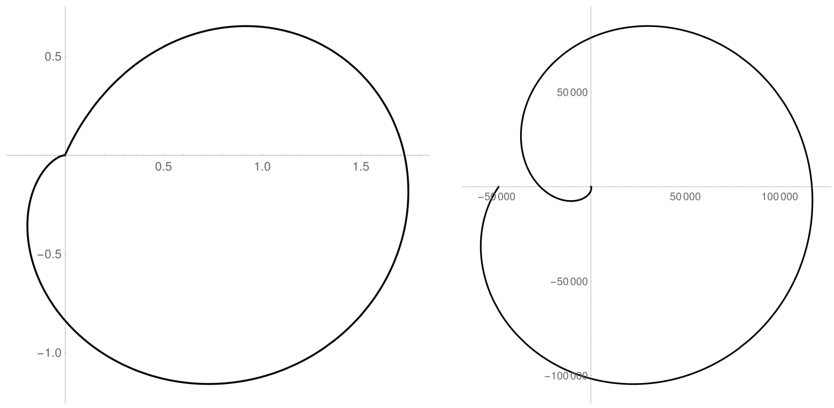

Figure 3.3: Left: picture of the loop $Q(\gamma(t))$; this loop starts at $x = 0$, goes clockwise around $x = 1$, goes clockwise around $x = 0$, and ends at $x = 0$. Right: picture of the loop $Q(1-\gamma(t))$; this loop starts at $x = 0$, goes clockwise around $x = 1$, and goes through the lower half plane to the number in equation (3.52). Here $\gamma : [0,1] \to \mathbb{C}$ is a clockwise half-circle from $z = 0$ to the number in equation (3.51).

From this follows that the analytic continuation of the left-hand side of equation (3.38) along $\gamma(t)$ is equal to

$$M^{-1}M_1^{-1}M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{2\cos(\pi/7)}{\sqrt{3}}\\ \frac{2\Gamma(29/42)\Gamma(41/42)\cdot\pi^{-1/2}\Gamma(7/6)^{-1}}{\sec(\pi/84)\sec(13\pi/84)-\csc(\pi/84)\csc(13\pi/84)}\end{pmatrix} \qquad (3.53)$$

with respect to the basis $\{F_0, G_0\}$; here we can ignore the second component, because $G_0(0) = 0$. The analytic continuation of the right-hand side of equation (3.38) is equal to

$$\begin{aligned} A &:= \sqrt[12]{\frac{1}{820}\left(781 + 923\sqrt{\frac{7}{3}} - 39i\left(11\sqrt{3}+13\sqrt{7}\right)\right)}M^{-1}M_1^{-1}M\begin{pmatrix}1\\0\end{pmatrix} \\ &= \sqrt[12]{\frac{1}{820}\left(781 + 923\sqrt{\frac{7}{3}} - 39i\left(11\sqrt{3}+13\sqrt{7}\right)\right)}\begin{pmatrix}2\cos(\pi/7)/\sqrt{3}\\ \frac{2\Gamma(29/42)\Gamma(41/42)\cdot\pi^{-1/2}\Gamma(7/6)^{-1}}{\sec(\pi/84)\sec(13\pi/84)-\csc(\pi/84)\csc(13\pi/84)}\end{pmatrix}, \end{aligned} \qquad (3.54)$$

with respect to the basis $B_0 = \{F_0, G_0\}$. Note that it is no coincidence that the vectors apparing in equation (3.53) and equation (3.54) are the same, because the loops $Q(\gamma(t))$ and $Q(1-\gamma(t))$ both are homotopic to one clockwise loop around $x = 1$.

Now we know how the right-hand side of equation (3.38) behaves under monodromy, we want to insert $x = Q(1-y)$. However, the functions F_0, G_0 are not defined in $Q(1-y)$, but their analytic continuation is defined, if we specify the root in the definition of G_0. From this follows that equation (3.53) and equation (3.54) are equal to each other, when $x = 0$ is inserted in equation (3.53) and $x = Q(1-y)$ is inserted in equation (3.54); here we use that equation (3.53) and equation (3.54) are both analytic continuations of equation (3.38) along the same curve γ. This gives one linear equation in $F_0(Q(1-y))$ and $G_0(Q(1-y))$:

$$\frac{2\cos(\pi/7)}{\sqrt{3}} = A_1 \cdot F_0(Q(1-y)) + A_2 \cdot G_0(Q(1-y)). \tag{3.55}$$

Now we want to find another linear equation, using complex conjugates. The Euler integral reads

$${}_2F_1\left(a, b; c \,\middle|\, z\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \mathrm{d}t \tag{3.56}$$

for $0 < \Re(b) < \Re(c)$, what in particular proves that ${}_2F_1\left(a, b; c \,\middle|\, z\right)$ is real for $z \in \mathbb{R}_{<0}$. From this follows that $F_0(Q(1-y)) \in \mathbb{R}$ and $G_0(Q(1-y)) \in e^{-\pi i/3}\mathbb{R}_{>0}$, where we use that the path from $x = 0$ to the number in equation (3.52) goes through the lower half plane. From this follows that

$$\begin{pmatrix} A_1 & A_2 \\ \overline{A_1} & e^{-2\pi i/3}\overline{A_2} \end{pmatrix} \begin{pmatrix} F_0(Q(1-y)) \\ G_0(Q(1-y)) \end{pmatrix} = \begin{pmatrix} 2\cos(\pi/7)/\sqrt{3} \\ 2\cos(\pi/7)/\sqrt{3} \end{pmatrix}. \tag{3.57}$$

Solving this equation gives that

$$\begin{aligned} F_0(Q(1-y)) &= {}_2F_1\left(\frac{1}{84}, \frac{13}{84}; \frac{2}{3} \,\middle|\, \frac{38241952\left(-21307687138103583 - 4649976434760203\sqrt{21}\right)}{33275593513886484375}\right) \\ &= \sqrt[12]{\frac{512\left(773 - 143\sqrt{21}\right)}{358817445}} \cdot \Re\left(e^{-\pi i/6}\sqrt[12]{2343 + 1287i\sqrt{3} + 1521i\sqrt{7} + 923\sqrt{21}}\right), \end{aligned} \tag{3.58}$$

however because $|Q(1-y)| > 1$, see equation (3.52), this result is on the analytic continuation of ${}_2F_1\left(1/84, 13/84; 2/3 \,\middle|\, x\right)$, and not about its region of convergence. Fortunately, using an identity we can find a special value which has an argument inside the region of convergence of F_0, which we will show now.

From the identity

$${}_2F_1\left(a, b; c \,\middle|\, z\right) = (1-z)^{-a}{}_2F_1\left(a, c-b; c \,\middle|\, \frac{z}{z-1}\right) \tag{3.59}$$

follows that

$$\begin{aligned} &{}_2F_1\left(\frac{1}{84}, \frac{43}{84}; \frac{2}{3} \,\middle|\, \frac{38241952\left(5289411798647305 - 672452454064707\sqrt{21}\right)}{8443412305470285118248 1}\right) \\ &= \sqrt[84]{\frac{297568118001823519028019 2\left(3224592092541346723\sqrt{21} - 14673095170014395553\right)}{1674802610123026678739408499666232174237347823394822911603359375}} \\ &\qquad \cdot \Re\left(e^{-\pi i/6}\sqrt[12]{2343 + 1287i\sqrt{3} + 1521i\sqrt{7} + 923\sqrt{21}}\right), \end{aligned} \tag{3.60}$$

which is a special value of the hypergeometric function inside its region of convergence: we have

$$\frac{38241952\left(5289411798647305 - 672452454064707\sqrt{21}\right)}{8443412305470285118248 1} \approx 0.99997958. \tag{3.61}$$

Appendix A

Details about Schwarz' theorem

In this appendix, we give a proof of theorem 1.1.

A.1 Definition of the hypergeometric function

Recall from the introduction that the hypergeometric differential equation is given by

$$\left(z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-(a+b+1)z)\frac{\mathrm{d}}{\mathrm{d}z}-ab\right)f(z)=0, \tag{A.1}$$

written with z as variable.

If $z_0 \neq 0, 1, \infty$, the hypergeometric differential equation has a holomorphic local solution around z_0. If $z_0 = 0, 1, \infty$, the solution of the hypergeometric differential equation is not always holomorphic. To see which power series can satisfy the hypergeometric equation, we use the Ansatz $f = z^\rho + \cdots$ around $z = 0$, where the dots indicate higher powers of z. Calculating the coefficient of $z^{\rho-1}$ gives $\rho(\rho-1)+c\rho = 0$, from which follows that $\rho = 0$ and $\rho = 1-c$ are the only possibilities of the starting power of the solution of the hypergeometric differential equation around $z = 0$, which we call *local exponents*. Around $z = 1$ we calculate the coefficient of $(z-1)^{\rho-1}$ in $f = (z-1)^{\rho-1}+\cdots$, which gives $-\rho(\rho-1)+\rho(c-a-b-1) = 0$, from which follows that $\rho = 0$ or $\rho = c-a-b$. To calculate the local exponents around $z = \infty$, we consider functions of $1/z$ around $z = 0$, so we first rewrite the hypergeometric differential equation in the variable $1/z$ instead of z.

We have $\frac{\partial}{\partial z} = \frac{\partial 1/z}{\partial z}\frac{\partial}{\partial 1/z} = -\frac{1}{z^2}\frac{\partial}{\partial 1/z}$ and

$$\frac{\partial^2}{\partial z^2} = -\frac{\partial}{\partial z}\left(\frac{1}{z^2}\frac{\partial}{\partial 1/z}\right) = -\frac{\partial}{\partial z}\left(\frac{1}{z^2}\right)\cdot\frac{\partial}{\partial 1/z} - \frac{1}{z^2}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial 1/z}\right) = \frac{2}{z^3}\frac{\partial}{\partial 1/z}+\frac{1}{z^4}\frac{\partial^2}{\partial (1/z)^2}. \tag{A.2}$$

From this follows that the hypergeometric differential equation is given by

$$z(1-z)\frac{2}{z^3}\frac{\partial f}{\partial 1/z}+z(1-z)\frac{1}{z^4}\frac{\partial^2 f}{\partial(1/z)^2}-(c-(a+b+1)z)\frac{1}{z^2}\frac{\partial f}{\partial 1/z}-abf=0. \tag{A.3}$$

Now we use the Ansatz $f = (1/z)^\rho + \cdots$, and calculate the coefficient of $(1/z)^\rho$ which gives $-2\rho-\rho(\rho-1)+\rho(a+b+1)-ab = -(\rho-a)(\rho-b) = 0$, which gives $\rho = a$ and $\rho = b$ as local exponents around $z = \infty$. Now we define a *Riemann scheme* as a table in which the local exponents are written down, see table A.1.

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

Table A.1: Riemann scheme of the hypergeometric differential equation with parameters a, b, c

In general, consider the equation

$$y'' + p(z)y' + q(z)y = 0 \tag{A.4}$$

where p, q are meromorphic functions. A point $z \in \mathbb{C}$ is called singular if not both p, q are holomorphic at z, and z is called regular singular if p has at most a pole of order 1 at z and q has at most a pole of order 2 at z. The point ∞ is regular (and maybe singular) if p has a root of at least order 1 at ∞ and q has a root of order at least 2 at ∞. It can be shown that if there are at most three singular points, then the singular points and their local exponents determine equation (A.4) uniquely. Suppose that $y(z)$ satisfies equation (A.4). Then we have $(z^a y)' = az^{a-1}y + z^a y'$ and $(z^a y)'' = a(a-1)z^{a-2}y + 2az^{a-1}y' + z^a y''$. From this follows that

$$\begin{aligned}(z^a y)'' + p(z)(z^a y)' + q(z)z^a y &= a(a-1)z^{a-2}y + 2az^{a-1}y' + z^a y'' + ap(z)z^{a-1}y + p(z)z^a y' + q(z)z^a y \\ &= \frac{a(a-1)}{z^2}z^a y + \frac{2a}{z}\left((z^a y)' - \frac{a}{z}z^a y\right) + \frac{ap(z)}{z}z^a y, \end{aligned} \tag{A.5}$$

from which follows that

$$(z^a y)'' + \left(p(z) - \frac{2a}{z}\right)(z^a y)' + \left(q(z) - \frac{a(a-1)}{z^2} + \frac{2a^2}{z^2} - \frac{ap(z)}{z}\right)z^a y = 0, \tag{A.6}$$

which is again a Fuchsian equation, but the local exponents at $z = 0$ are increased by a, and the local exponents at $z = \infty$ are decreased by a. Analogously it can be shown that if y is the solution of a Fuchsian equation, then $(z-1)^a y$ is the solution of a Fuchsian equation, but the local exponents at $z = 1$ are increased by a, and the local exponents at $z = \infty$ are decreased by a.

A.2 Schwarzian derivative

Definition A.1. The *Schwarzian derivative* of a meromorphic function w is defined as

$$S(w) = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2. \tag{A.7}$$

One important property of the Schwarzian derivative is that the Schwarzian derivative of a Möbius transformation is equal to the zero function, which we will prove in corollary A.3. It is also true that every function with a Schwarzian derivative equal to the zero function is a Möbius transformation, this will be proven in corollary A.7.

Proposition A.2. *Let $A, B, C, D \in \mathbb{C}$ such that $AD - BC \neq 0$ and let w be a meromorphic function. Let $v = \frac{Aw+B}{Cw+D}$, then it follows that $S(v) = S(w)$.*

Proof. Suppose that $v = \frac{Aw+B}{Cw+D}$ with $A, B, C, D \in \mathbb{C}$ and $AD - BC \neq 0$. Without loss of generality we can assume that $AD - BC = 1$. We have that

$$\left(\frac{Aw+B}{Cw+D}\right)' = \frac{Aw'(Cw+D) - Cw'(Aw+B)}{(Cw+D)^2} = \frac{w'}{(Cw+D)^2} \tag{A.8}$$

and

$$\left(\frac{Aw+B}{Cw+D}\right)'' = \frac{w''(Cw+D)^2}{(Cw+D)^4} - \frac{2(Cw+D)Cw'^2}{(Cw+D)^4} = \frac{w''}{(Cw+D)^2} - \frac{2Cw'^2}{(Cw+D)^3} \tag{A.9}$$

and

$$\left(\frac{Aw+B}{Cw+d}\right)'' \Bigg/ \left(\frac{Aw+B}{Cw+d}\right)' = \frac{(Cw+D)^2}{w'}\left(\frac{w''}{(Cw+D)^2} - \frac{2Cw'^2}{(Cw+D)^3}\right) = \frac{w''}{w'} - \frac{2Cw'}{Cw+D}. \tag{A.10}$$

From this follows that

$$S\left(\frac{Aw+B}{Cw+d}\right) - S(w) \tag{A.11}$$

is equal to

$$\begin{aligned}
&\left(\frac{w''}{w'} - 2C\frac{w'}{Cw+D}\right)' - \frac{1}{2}\left(\frac{w''}{w'} - 2C\frac{w'}{Cw+D}\right)^2 - \left(\frac{w''}{w'}\right)' + \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \\
&= -2C\left(\frac{w'}{Cw+D}\right)' - \frac{1}{2}\left(\frac{w''}{w'} - 2C\frac{w'}{Cw+D}\right)^2 + \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \\
&= -2C\left(\frac{w'}{Cw+D}\right)' - \frac{1}{2}\left(\left(\frac{w''}{w'}\right)^2 - 4C\frac{w''}{w'}\frac{w'}{Cw+D} + 4C^2\left(\frac{w'}{Cw+D}\right)^2\right) + \frac{1}{2}\left(\frac{w''}{w'}\right)^2 \\
&= -2C\left(\frac{w'}{Cw+D}\right)' + 2C\frac{w''}{Cw+D} - 2C^2\left(\frac{w'}{Cw+D}\right)^2 = 0,
\end{aligned} \tag{A.12}$$

where the last equality follows from working out the quotient derivative. From this follows that $S(v) = S(w)$. ☐

Corollary A.3. *Let v be a Möbius transformation, then $S(v)$ is the zero function.*

Proof. By proposition A.2 we have that $S(v)(z) = S(z)(z)$, because $w(z) = \frac{Az+B}{Cz+D}$ for some $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$. Furthermore we have $\frac{z''}{z'} = \frac{0}{1}$, where the prime denotes differentation with respect to z. From this follows that $S(z) = 0' - \frac{1}{2}0^2 = 0$. ☐

Proposition A.4. *Let v, w be nonconstant meromorphic functions and assume that the image of w is contained in the domain of v. Then it follows that $S(v \circ w)(z) = S(w)(z) + w'(z)^2 \cdot (S(v) \circ w)(z)$.*

Proof. In this proposition we prove an equality of functions. To avoid confusion with composition of functions, we evaluate these functions in some arbitrary z in the domain of w. First note that $(v \circ w)'(z) = v'(w(z)) \cdot w'(z)$ and $(v \circ w)'' = v''(w(z)) \cdot w'(z)^2 + v'(w(z)) \cdot w''(z)$ and

$$\frac{(v \circ w)''(z)}{(v \circ w)'(z)} = \frac{v''(w(z)) \cdot w'(z)^2 + v'(w(z)) \cdot w''(z)}{v'(w(z)) \cdot w'(z)} = \frac{v''(w(z)) \cdot w'(z)}{v'(w(z))} + \frac{w''(z)}{w'(z)}. \tag{A.13}$$

We have that

$$\left(\frac{v''(w(z)) \cdot w'(z)}{v'(w(z))} + \frac{w''(z)}{w'(z)}\right)' \tag{A.14}$$

is equal to

$$\frac{v'''(w(z))\cdot w'(z)^2 + v''(w(z))\cdot w''(z)}{v'(w(z))} - \frac{v''(w(z))w'(z)\cdot v''(w(z))w'(z)}{v'(w(z))^2} + \left(\frac{w''(z)}{w'(z)}\right)'$$
$$= w'(z)^2\cdot\frac{v'''(w(z))}{v'(w(z))} + \frac{v''(w(z))\cdot w''(z)}{v'(w(z))} - w'(z)^2\cdot\left(\frac{v''(w(z))}{v'(w(z))}\right)^2 + \left(\frac{w''(z)}{w'(z)}\right)' \tag{A.15}$$

and

$$-\frac{1}{2}\left(\frac{v''(w(z))\cdot w'(z)}{v'(w(z))} + \frac{w''(z)}{w'(z)}\right)^2 = -\frac{1}{2}\left(\frac{v''(w(z))\cdot w'(z)}{v'(w(z))}\right)^2 - \frac{v''(w(z))\cdot w'(z)}{v'(w(z))}\frac{w''(z)}{w'(z)} - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2$$
$$= -\frac{1}{2}w'(z)^2\cdot\left(\frac{v''(w(z))}{v'(w(z))}\right)^2 - \frac{v''(w(z))\cdot w''(z)}{v'(w(z))} - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2. \tag{A.16}$$

From this follows that

$$S(v\circ w)(z) = w'(z)^2\cdot\frac{v'''(w(z))}{v'(w(z))} - \frac{3}{2}w'(z)^2\cdot\left(\frac{v''(w(z))}{v'(w(z))}\right)^2 + \left(\frac{w''(z)}{w'(z)}\right)' - \frac{1}{2}\left(\frac{w''(z)}{w'(z)}\right)^2, \tag{A.17}$$

which is equal to $w'(z)^2\cdot(S(v)\circ w)(z) + S(w)(z)$, which finishes the proof. □

Proposition A.5 ([11, chapter 8, exercise 19])**.** *Let w be a nonconstant meromorphic function such that $S(w) = 0$. Then $w(z) = \frac{Az+B}{Cz+D}$ for some $A, B, C, D \in \mathbb{C}$ and $AD - BC \neq 0$, so w is a Möbius transformation.*

Proof. First note that if w'' is the zero function, then w is a linear function, so w is a Möbius transformation. Now assume that w'' is not the zero function and define $f = w''/w'$. Because $S(w) = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = 0$, it follows that $2f' = f^2$ and $(1/f)' = -f'/f^2 = -\frac{1}{2}$. From this follows that $1/f(z) = -(z+a)/2$ for some $a \in \mathbb{C}$. From this follows that $(\log w'(z))' = \frac{w''(z)}{w'(z)} = f(z) = -\frac{2}{z+a}$. From this follows that $\log w'(z) = -2\log(z+a) + b$ and $w'(z) = \frac{b}{(z+a)^2}$ for some other $b \in \mathbb{C}$. From this follows that $w(z) = -\frac{b}{z+a} + c = \frac{cz+ca-b}{z+a}$ for some $c \in \mathbb{C}$, so w is a Möbius transformation. Furthermore, if $AD - BC = 0$, it follows that w is a constant function. □

Proposition A.6 ([8, section II.1])**.** *Let v, w be nonconstant meromorphic functions with the same domain $U \subset \mathbb{C}$ such that $S(v)(z) = S(w)(z)$ and v is not a constant function. Then it follows that $v = \frac{Aw+B}{Cw+D}$ for some $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$.*

Proof. First note that v' and w' cannot be both the zero function, because then v and w would be constant functions. Without loss of generality assume that there exists a point $z_0 \in U$ such that $w'(z_0) \neq 0$. From this follows that w is locally invertible around z_0. From this follows that

$$S(v\circ w^{-1})(z) = S(w^{-1})(z) + {w^{-1}}'(z)^2\cdot(S(v)\circ w^{-1})(z)$$
$$= S(w^{-1})(z) + {w^{-1}}'(z)^2\cdot(S(w)\circ w^{-1})(z) = S(w\circ w^{-1})(z) = 0. \tag{A.18}$$

In the second equality we used that $S(v) = S(w)$, and in the last equality we used that the Schwarzian derivative of the identity function $w\circ w^{-1}$ is the zero function, because $\left(\frac{z''}{z'}\right) = \frac{0}{1} = 0$.

Because locally around z_0 holds that $(v\circ w^{-1})(z) = \frac{Az+B}{Cz+D}$, it follows locally that $v(z) = \frac{Aw(z)+B}{Cw(z)+D}$, from which follows that $v(z) = \frac{Aw(z)+B}{Cw(z)+D}$ globally on U. Furthermore, if $AD - BC = 0$, it follows that v is a constant function. □

Corollary A.7. *Let v be a meromorphic function such that $S(v)$ is the zero function. Then it follows that $v = \frac{Aw+B}{Cw+D}$ for some $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$.*

Proof. We use proposition A.6 with $w(z) = z$, then it follows that $S(w)$ is the zero function. Because $S(v)$ is also the zero function, there exist $A, B, C, D \in \mathbb{C}$ with $AD - BC \neq 0$ such that $v(z) = \frac{Az+B}{Cz+D}$. ☐

If we want to solve the equation $S(f) = Q$ for some given meromorphic function Q, then proposition A.6 and corollary A.7 give a sort of uniqueness statement: if we find one solution for f, we have all solutions. Now we want to find one solution of $S(f) = Q$, which will be done in corollary A.9.

Proposition A.8. *Consider the differential equation $y'' + p(z)y' + q(z)y = 0$ with p, q meromorphic functions and y_1, y_2 linearly independent local solutions. Then it follows that $S(y_1/y_2) = 2q - p' - p^2/2$.*

Proof. We have $(y_1/y_2)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2}$ and

$$\begin{aligned} \left(\frac{y_1}{y_2}\right)'' &= \frac{y_1''y_2 - y_1y_2''}{y_2^2} - \frac{2y_2y_2'(y_1'y_2 - y_1y_2')}{y_2^4} \\ &= \frac{(-py_1' - qy_1)y_2 - y_1(-py_2' - qy_2)}{y_2^2} - \frac{2y_2'(y_1'y_2 - y_1y_2')}{y_2^3} \\ &= p \cdot \frac{-y_1'y_2 + y_1y_2'}{y_2^2} - \frac{2y_2'}{y_2} \cdot \frac{y_1'y_2 - y_1y_2'}{y_2^2} = \left(-p - \frac{2y_2'}{y_2}\right)\left(\frac{y_1}{y_2}\right)'. \end{aligned} \tag{A.19}$$

From this follows that

$$\left(\frac{y_1}{y_2}\right)'' \Big/ \left(\frac{y_1}{y_2}\right)' = -p - 2\frac{y_2'}{y_2}. \tag{A.20}$$

From this follows that

$$\begin{aligned} S(y_1/y_2) &= \left(-p - 2\frac{y_2'}{y_2}\right)' - \frac{1}{2}\left(-p - 2\frac{y_2'}{y_2}\right)^2 = -p' - 2\frac{y_2''y_2 - y_2'^2}{y_2^2} - \frac{1}{2}\left(p^2 + 4p\frac{y_2'}{y_2} + 4\frac{y_2'^2}{y_2^2}\right) \\ &= -p' - 2\frac{y_2''}{y_2} - \frac{1}{2}p^2 - 2p\frac{y_2'}{y_2} = -p' + 2\frac{py_2' + qy_2}{y_2} - \frac{1}{2}p^2 - 2p\frac{y_2'}{y_2} = -p' + 2q - \frac{1}{2}p^2, \end{aligned} \tag{A.21}$$

which finishes the proof. ☐

Corollary A.9. *Let p, q be meromorphic functions and define $Q = 2q - p' - p^2/2$. If Q is not the zero function, then the solutions of the differential equation $S(w) = Q$ are given by y_1/y_2 where y_1, y_2 are linearly independent solutions of $y'' + py' + qy = 0$.*

Proof. Let y_1, y_2 be two linearly independent solutions of $y'' + py' + qy = 0$. From proposition A.8 we know that $S(y_1/y_2) = Q$. Let v be a meromorphic function such that $S(v) = Q$, then it follows from proposition A.6 that $v = \frac{Ay_1/y_2+B}{Cy_1/y_2+D}$ for some $A, B, C, D \in \mathbb{C}$. If $AD - BC = 0$, it follows that v is constant and $S(v)$ is the zero function. Furthermore we have $v = \frac{Ay_1+By_2}{Cy_1+Dy_2}$, which shows the connection between Möbius transformations and linear maps. Because $AD - BC \neq 0$, it follows that $Ay_1 + By_2$ and $Cy_1 + Dy_2$ are linearly independent: if $\lambda(Ay_1 + By_2) + \mu(Cy_1 + Dy_2) = 0$, it follows that $A\lambda + C\mu = 0$ and $B\lambda + D\lambda$, so $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ from which follows that $\lambda = \mu = 0$. ☐

Using corollary A.9 we now can solve the equation $S(f) = Q$ for a given meromorphic function Q, which we will need to prove theorem A.24. We end this section about the Schwarzian derivative by a calculation we will need later.

Proposition A.10. *Let $w(z) = (z - z_0)^a u(z)$ with $u(z)$ holomorphic and $u(z_0) \neq 0$ for some $z_0 \in \mathbb{C}$. If $a = -1, 0, 1$, then $S(w)$ is holomorphic at z_0. If $a \neq -1, 0, 1$, it follows that $S(w)(z) = \frac{1-a^2}{2(z-z_0)^2} + \cdots$, so $S(w)$ has a pole of order 2.*

Proof. If $a = 0$, we consider the function $w - w(z_0)$ instead of the function w, note that $S(w) = S(w - w(z_0))$. We have

$$w'(z) = a(z - z_0)^{a-1} u(z) + (z - z_0)^a u'(z) \tag{A.22}$$

and

$$w''(z) = a(a-1)(z - z_0)^{a-2} u(z) + 2a(z - z_0)^{a-1} u'(z) + (z - z_0)^a u''(z) \tag{A.23}$$

and

$$w'''(z) = a(a-1)(a-2)(z-z_0)^{a-3} u(z) + 3a(a-1)(z-z_0)^{a-2} u'(z) + 3a(z-z_0)^{a-1} u''(z) + (z-z_0)^a u'''(z). \tag{A.24}$$

From this follows that

$$\frac{w''(z)}{w'(z)} = \frac{a(a-1)(z - z_0)^{-1} u(z) + 2au'(z) + (z - z_0)u''(z)}{au(z) + (z - z_0)u'(z)} \tag{A.25}$$

and

$$\frac{w'''(z)}{w'(z)} = \frac{a(a-1)(a-2)(z - z_0)^{-2} u(z) + 3a(a-1)(z - z_0)^{-1} u'(z) + 3au''(z) + (z - z_0)u'''(z)}{au(z) + (z - z_0)u'(z)}, \tag{A.26}$$

where we multiplied numerator and denominator with $(z - z_0)^{1-a}$.

If $a = 1$, we have

$$\frac{w''(z)}{w'(z)} = \frac{2u'(z) + (z - z_0)u''(z)}{u(z) + (z - z_0)u'(z)} \quad \text{and} \quad \frac{w'''(z)}{w'(z)} = \frac{3u''(z) + (z - z_0)u'''(z)}{u(z) + (z - z_0)u'(z)}. \tag{A.27}$$

Because $u(z_0) \neq 0$, it follows that $1/(u(z)+(z-z_0)u'(z))$ is holomorphic around z_0, so $w''(z)/w'(z)$ and $w'''(z)/w'(z)$ are both holomorphic around z_0, so $S(w)$ is holomorphic around z_0.

If $a = -1$, we have

$$\frac{w''(z)}{w'(z)} = \frac{2(z - z_0)^{-1} u(z) - 2u'(z) + (z - z_0)u''(z)}{-u(z) + (z - z_0)u'(z)} = -\frac{2}{z - z_0} + 2\frac{u'(z_0)}{u(z_0)} + O(z - z_0) \tag{A.28}$$

and

$$\begin{aligned} \frac{w'''(z)}{w'(z)} &= \frac{-6(z - z_0)^{-2} u(z) + 6(z - z_0)^{-1} u'(z) - 3u''(z) + (z - z_0)u'''(z)}{-u(z) + (z - z_0)u'(z)} \\ &= \frac{6}{(z - z_0)^2} - \frac{6u'(z_0)}{u(z_0)(z - z_0)} + O(1) \end{aligned} \tag{A.29}$$

and

$$-\frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = -\frac{6}{(z - z_0)^2} + \frac{6u'(z_0)}{u(z_0)(z - z_0)} + O(1), \tag{A.30}$$

where in the Laurent expansions $O(z-z_0)$ denotes a holomorphic function around z_0 which is equal to 0 at z_0, and $O(1)$ denotes a function which is holomorphic around z_0. From this follows that $\frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2 = 0 + O(1)$, from which follows that $S(w)$ is holomorphic around z_0.

If $a \neq -1, 0, 1$, we have that

$$\lim_{z\to z_0}(z-z_0)^2\left(\frac{w''(z)}{w'(z)}\right)^2 = (a-1)^2 \qquad \text{and} \qquad \lim_{z\to z_0}(z-z_0)^2\frac{w'''(z)}{w'(z)} = (a-1)(a-2), \quad \text{(A.31)}$$

so $\lim_{z\to z_0}(z-z_0)^2 S(w)(z) = (a-1)(a-2) - \frac{3}{2}(a-1)^2 = \frac{1}{2} - \frac{1}{2}a^2$.

To sum up, for $a \neq 0$, it follows that $S(w)(z) = \frac{1-a^2}{2z^2} + \cdots$, where the dots indicate z^{-1} and higher powers of z. From this follows $S(w)(1/z) = (1-a^2)z^2/2 + \cdots$, where the dots indiate z and lower powers of z. □

A.3 Riemann mapping theorem and curvilinear triangles

Theorem A.11 (Riemann mapping theorem, [7, paragraph X.1])**.** *Let $U \subset \mathbb{C}$ be a simply connected open subset of $\mathbb{C}$, not equal to $\mathbb{C}$. Then there exists a biholomorphic function f from G to $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$. Moreover, let $z_0 \in U$, from the requirements $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}_{>0}$ follows that f is uniquely determined.*

Definition A.12. We call $\gamma = f([a,b]) \subset \mathbb{C}$ an *analytic arc* if there exists a function $f : [a,b] \to \mathbb{C}$ where $[a,b]$ is a real interval, such that for every $t_0 \in [a,b]$ there exists a convergent power series such that $f(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^n$ on some open interval around t_0. Furthermore we require f to be injective and $f'(t) \neq 0$ for all $t \in [a,b]$.

Lemma A.13 ([7, lemma IX.2.4])**.** *Let $f : U \to V$ be an analytic function such that for every compact subset $K \subset V$ the set $f^{-1}(K)$ is compact. If $\{z_n\}$ is a sequence in U approaching the boundary of U, then $\{f(z_n)\}$ approaches the boundary of V.*

Theorem A.14 ([7, theorem IX.2.5])**.** *Let $f : U \to \mathbb{D}$ be a biholomorphic function. Let γ be an analytic arc contained in the boundary of U. Suppose that U lies on one side of γ, then f extends to a function on $U \cup \gamma$, holomorphic on U and continuous on $U \cup \gamma$.*

Definition A.15. A *curvilinear triangle* is a connected open subset of $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$, which has a boundary consisting of three points, and three open line segments or open segments of a circle. It can be shown that a curvilinear triangle with given angles is unique, up to a Möbius transformation.

Lemma A.16. *Let $T \subset \mathbb{C}$ be a curvilinear triangle with angles $\lambda\pi, \mu\pi, \nu\pi$ in counterclockwise order and $0 < \lambda, \mu, \nu < 1$. Let f be the function from theorem A.11: a biholomorphic function from $\mathbb{D}$ to T. Then it follows that f can be extended to a continuous bijective function from $\mathbb{D} \cup S^1$ to $T \cup \partial T$, where S^1 and ∂T are the boundaries of $\mathbb{D}$ and T. Moreover, we have $f(S^1) = \partial T$.*

Proof. We cannot use theorem A.14 to give a continuous extension of f to $D \cup S^1$, with $S^1 = \{z \in \mathbb{C}; |z| = 1\}$, because S^1 is not an analytic arc: there does not exist a continuous bijection from some closed interval in $\mathbb{R}$ to S^1; removing the two endpoints of a closed interval gives a connected set, but removing two distinct points from S^1 gives a disconnected set. To produce a continuous extension of f to $D \cup S^1$, define $S^1_+ = \{z \in S^1; \Im(z) \geq 0\}$ and $S^1_- = \{z \in S^1; \Im(z) \leq 0\}$, then there exist continuous extensions of f to $\mathbb{D} \cup S^1_-$ and to $\mathbb{D} \cup S^1_+$ which agree on $S^1_- \cap S^1_+ = \{-1, 1\}$, because the continuous extension of f is continuous on $\mathbb{D} \cup S^1$. Furthermore, because f is continuous, it follows that the continuous extension of f to $\mathbb{D} \cup S^1$ takes values in $\overline{T}$: if $\{z_n\}$ is a sequence in $\mathbb{D}$ which converges to $z \in \mathbb{D} \cup S^1$, it follows that $f(z) = \lim_{n\to\infty} f(z_n)$, and the sequence $\{f(z_n)\}$ in T converges to a point in $\overline{T}$. Analogously f^{-1} can be continued to a function on $\overline{T}$, using the three circle segments as analytic arcs, and the analytic continuation of f^{-1} to $\overline{T}$ takes values in $\mathbb{D} \cup S^1$.

Now we want to prove that the continuous extensions of f and f^{-1} are each others inverses. From lemma A.13 we know that $f(S^1) \subset \partial T$ and $f^{-1}(\partial T) \subset S^1$, here we use that f and f^{-1} are both holomorphic, so in particular continuous, so the inverse image by f or f^{-1} of a compact set is a compact set. To prove that f is surjective, let $z \in \partial T$, and let $\{z_n\}$ be a sequence in T converging to z. Because f^{-1} is continuous, the sequence $\{f^{-1}(z_n)\}$ in $\mathbb{D}$ converges to a point $y \in S^1$, where we used lemma A.13. Because f is continous, it follows that $f(y) = \lim_{n\to\infty} f(f^{-1}(z_n)) = \lim_{n\to\infty} z_n = z$, which proves that the analytic continuation of f is surjective. Analogously it follows that f^{-1} as a function from ∂T to S^1 is surjective.

Now we want to prove that the continuous extension of f is injective. Suppose that there exists $x \neq y \in S^1$ such that $f(x) = f(y) = z \in \partial T$, from this follows that there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{D}$ which converge to a different limit in S^1, but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ in T converge to the same limit in ∂T. To see this, we use some methods from analysis: we already know that the function f^{-1} can be analytically continued to $\overline{T}$, which is a compact set. From this follows that the analytic continuation of f^{-1} is uniformly continuous on $\overline{T}$, because a continuous function on a compact set is uniformly continuous. Furthermore uniform continuity implies Cauchy continuity, so for two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in T with $\lim_{n\to\infty} |a_n - b_n| \to 0$, it follows that $\lim_{n\to\infty} |f^{-1}(a_n) - f^{-1}(b_n)| \to 0$. Note that f^{-1} is uniformly continuous on T, because f^{-1} is uniformly continuous on $\overline{T}$. From the definition of Cauchy continuity applied to the sequences $\{f(x_n)\}$ and $\{f(y_n)$ follows that $\lim_{n\to\infty} x_n = \lim_{n\to} y_n$, so there do not exist $x \neq y \in S^1$ such that $f(x) = f(y) = z \in \partial T$, so the continuous extension of f is injective. Analogously it follows that f^{-1} as a function from ∂T to S^1 is injective.

Now we want to prove that the continuous extensions of f and f^{-1} are each others inverse. Let $x \in S^1$ and $\{x_n\}$ a sequence in $\mathbb{D}$ which converges to x, then $f^{-1}(f(x)) = \lim_{n\to\infty} f^{-1}(f(x_n)) = \lim_{n\to\infty} x_n = x$. Let $x \in \partial T$ and $\{x_n\}$ a sequence in T which converges to x, then $f(f^{-1}(x)) = \lim_{n\to\infty} f(f^{-1}(x_n)) = \lim_{n\to\infty} x_n = x$. This proves that the continuous extensions of f and f^{-1} are each others inverse. □

Lemma A.17. *Let a curvilinear triangle $T \subset \mathbb{C}$ be given with angles $\lambda\pi, \mu\pi, \nu\pi$ in counterclockwise order. Then there exists a continuous bijection $f : \mathbb{H} \cup \mathbb{R} \cup \{\infty\} \to \overline{T}$, where $\overline{T}$ is the closure of T. Moreover, f is biholomorphic on $\mathbb{H}$ and $f(\mathbb{H}) = T$, and $f(\mathbb{R} \cup \{\infty\}) = \partial T$, where ∂T is the boundary of T, and $f(0)$ is equal to the vertex of T with angle $\lambda\pi$ and $f(1)$ is equal to the vertex of T with angle $\mu\pi$ and $f(\infty)$ is equal to the vertex of T with angle $\nu\pi$.*

Proof. Consider the function $g : \mathbb{H} \cup \mathbb{R} \cup \{\infty\} \to \mathbb{D}$ with $z \mapsto \frac{z-i}{z+i}$. We have that g is a biholomorphic function $\mathbb{H} \to \mathbb{D}$, because $-1 \mapsto i$ and $0 \to -1$ and $1 \to -i$, and h is continuous on $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$, and bijective as function $\mathbb{R} \cup \{\infty\} \to S^1$. From this follows that the function $g \circ f$ is a homeomorphism from $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ to $\overline{T}$ which is also a biholomorphic function from $\mathbb{H}$ to T.

Let h be a Möbius transformation which sends $0, 1, \infty$ to the images of $(g \circ f)^{-1}$ of the vertices of T with angles $\lambda\pi, \mu\pi, \nu\pi$. Because of the conservation of orientation, it follows that h sends $\mathbb{H}$ to itself, so h is a homeomorphism from $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ to itself and also a biholomorphic function from $\mathbb{H}$ to itself. From this follows that $h \circ g \circ f$ is a homeomorphism from $\mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ to $\overline{T}$, is a biholomorpic function from $\mathbb{H}$ to T and sends $0, 1, \infty$ to the vertices of T with angles $\lambda\pi, \mu\pi, \nu\pi$. □

A.4 Schwarz reflection principle and Schwarz triangle map

Theorem A.18 ([7, theorem IX.1.1(ii)]). *Let $U \subset \mathbb{H}$ be open such that the boundary of U contains an interval $I \subset \mathbb{R}$. Let f be a function on $U \cup I$, analytic on U and continous on I. If f is realvalued on I, then f has a unique analytic continuation to $U \cup I \cup \overline{U}$ which satisfies $f(\overline{z}) = \overline{f(z)}$, where the set $\overline{U}$ is the complex conjugate of the set U.*

Corollary A.19 (Schwarz reflection principle). *Let $U \subset \mathbb{H}$ be open such that the boundary of U contains an interval $I \subset \mathbb{R}$. Let f be a function on $U \cup I$, analytic on U and continuous on I. If $f(I)$ lies on a line $L = \{z_0 + \lambda z_1 | \lambda \in \mathbb{R}\}$ with $z_0 \in \mathbb{C}$ and $0 \neq z_1 \in \mathbb{C}$, then f has a unique analytic continuation to $U \cup I \cup \overline{U}$ which satisfies $f(\overline{z}) = \delta(\overline{f(z)})$ with $\delta(z) = z_1/\overline{z_1} \cdot z + z_0 - z_1/\overline{z_1} \cdot \overline{z_0}$.*

If $f(I)$ lies on a circle C with midpoint $z_0 \in \mathbb{C}$ and radius $r_0 \in \mathbb{R}_{>0}$, then f has a unique analytic continuation to $U \cup I \cup \overline{U}$ which satisfies $f(\overline{z}) = \delta(\overline{f(z)})$ with $\delta(z) = \frac{z_0 z + r_0^2 - |z_0|^2}{z - \overline{z_0}}$.

Proof. Suppose that $f(I)$ lies on a line $z_0 + \lambda z_1$ and let $\gamma(z) = (z - z_0)/z_1$, then γ sends the line L to the real line. From this follows that $\gamma \circ f$ sends the real interval I to the real line. From theorem A.18 follows that there exists a unique analytic continuation of $\gamma \circ f$ with $(\gamma \circ f)(\overline{z}) = \overline{(\gamma \circ f)(z)}$. From this follows that $\gamma^{-1}(\gamma \circ f)(\overline{z}) = f(\overline{z}) = \gamma^{-1}\overline{(\gamma \circ f)(z)} = (\gamma^{-1} \circ \overline{\gamma} \circ \overline{f})(z)$, which gives an analytic continuation of f. Note that $\gamma^{-1}(z) = z_0 + z_1 z$, from which follows that $(\gamma^{-1} \circ \overline{\gamma})(\overline{f(z)}) = z_0 + z_1(\overline{f(z)} - \overline{z_0})/\overline{z_1} = z_1/\overline{z_1} \cdot \overline{f(z)} + z_0 - z_1/\overline{z_1} \cdot \overline{z_0}$.

Suppose that $f(I)$ lies on a circle C with midpoint z_0 and radius r_0. Let $\gamma(z) = i\frac{z-(z_0-r_0)}{z-(z_0+r_0)}$, then γ sends the circle C to the real line, because $\gamma(z_0 - r_0) = 0$ and $\gamma(z_0 + r_0) = \infty$ and and $\gamma(z_0 + ir_0) = i\frac{ir_0+r_0}{ir_0-r_0} = i\frac{i+1}{i-1} = 1$. From this follows that $\gamma \circ f$ sends the real interval I to the real line. From theorem A.18 follows that there exists a unique analytic continuation of $\gamma \circ f$ with $(\gamma \circ f)(\overline{z}) = \overline{(\gamma \circ f)(z)}$. From this follows that $\gamma^{-1}(\gamma \circ f)(\overline{z}) = f(\overline{z}) = \gamma^{-1}\overline{(\gamma \circ f)(z)} = (\gamma^{-1} \circ \overline{\gamma} \circ \overline{f})(z)$, which gives an analytic continuation of f. Note that $\gamma^{-1}(z) = \frac{i(z_0+r_0)z + z_0 - r_0}{iz+1}$, because $\gamma^{-1}(0) = z_0 - r_0$ and $\gamma^{-1}(\infty) = z_0 + r_0$ and $\gamma^{-1}(1) = \frac{i(z_0+r_0)+z_0-r_0}{i+1} = z_0 + \frac{ir_0 - r_0}{i+1} = z_0 + ir_0$. Furthermore we have

$$\begin{aligned}\begin{pmatrix} i(z_0+r_0) & z_0 - r_0 \\ i & 1 \end{pmatrix} \overline{\begin{pmatrix} i & -i(z_0 - r_0) \\ 1 & -z_0 - r_0 \end{pmatrix}} &= \begin{pmatrix} i(z_0+r_0) & z_0 - r_0 \\ i & 1 \end{pmatrix} \begin{pmatrix} -i & i(\overline{z_0} - r_0) \\ 1 & -\overline{z_0} - r_0 \end{pmatrix} \\ &= \begin{pmatrix} -2z_0 & 2|z_0|^2 - 2r_0^2 \\ -2 & 2\overline{z_0} \end{pmatrix}, \end{aligned} \tag{A.32}$$

and we define the Möbius transformation δ using the matrix (we multiply all elements in the matrix with $-1/2$) in the previous equation. From this follows for the analytic continuation of f that $f(\overline{z}) = \delta(\overline{f(z)})$. ☐

Remark A.20. To give a concrete description of the function δ from corollary A.19, note that for $r, \theta \in \mathbb{R}$ with $r > 0$ we have

$$\delta(\overline{z_0 + re^{i\theta}}) = \frac{z_0(\overline{z_0 + re^{i\theta}}) + r_0^2 - |z_0|^2}{1(\overline{z_0 + re^{i\theta}}) - \overline{z_0}} = \frac{z_0\overline{re^{i\theta}} + r_0^2}{\overline{re^{i\theta}}} = z_0 + \frac{r_0^2}{\overline{re^{i\theta}}} = z_0 + \frac{r_0^2}{r}e^{i\theta}, \tag{A.33}$$

so $\delta(\overline{z})$ lies on the line segment between z_0 and z, and $|\delta(\overline{z}) - z_0| \cdot |z - z_0| = r_0^2$. This geometric construction is called the *Schwarz reflection principle.*

Lemma A.21. *Let f be the function from lemma A.17. Then $S(f)$ is holomorphic and one-valed on $\mathbb{C}\backslash\{0, 1\}$.*

Proof. We will construct an analytic continuation of f, along a curve which goes from $z \in \mathbb{H}$, through the interval $(1,\infty)$ to the lower half plane, and through the interval $(0,1)$ to the upper half plane, back to z. From corollary A.19 with $I = (1,\infty)$ follows for $z \in -\mathbb{H}$ that $f(z) = \gamma(\overline{f(\overline{z})})$ for some Möbius transformation γ. Now we again use the Schwarz reflection principle, but with the analytic continuation of f which is defined on $-\mathbb{H}$, and the interval $(0,1)$. From this follows that we should define f on $\mathbb{H}$ as $f(z) = \delta(\overline{\gamma(\overline{f(z)})}) = (\delta\overline{\gamma})(f(z))$ for some Möbius transformation δ. These two analytic continuations give that the analytic continuation of f along a curve around the point 1 is equal to a Möbius transformation of f. From proposition A.2 follows that $S(f)$ is invariant under analytic continuation along the curve around 1. Analogously it follows that $S(f)$ is invariant under analytic contination around the point 0. From this follows that $S(f)$ can be defined globally on $\mathbb{C}\backslash\{0,1\}$. Furthermore we have that $S(f)$ is holomorphic on $\mathbb{C}\backslash\{0,1\}$ because f is holomorphic at $\mathbb{H}$, see proposition A.10. From this follows that $S(f)$ is a holomorphic function on $\mathbb{C}\backslash\{0,1\}$ which is one-valued. ☐

Lemma A.22. *Let f be the function from lemma A.17. Then $f = y_1/y_2$, where y_1, y_2 are linearly independent solutions of $y'' + Q(z)y = 0$, where $Q(z)$ has double poles at $z = 0, 1$ and has a double root at $z \to \infty$.*

Proof. Because f sends $0 \in \mathbb{H}\cup\mathbb{R}$ with angle π to $f(0)$ with angle $\lambda\pi$, it follows that $f(z) = z^\lambda u(z)$ for some holomorphic function u around $z = 0$, so $S(f)(z) = \frac{1-\lambda^2}{2z^2} + \cdots$. Analogously it follows that $f(z-1) = (z-1)^\mu u(z)$ for some other holomorphic function u around $z = 1$, so $S(f)(z-1) = \frac{1-\mu^2}{2(z-1)^2} + \cdots$. Furthermore we have $f(1/z) = 1/z^\nu u(1/z)$ for $z = \infty$, from which follows that $S(f)(1/z) = (1-\nu^2)z^2/2 + \cdots$. From this follows that $S(w) = 2Q$ for some rational function Q with double poles in $z = 0$ and $z = 1$ and a double root at $z \to \infty$. Applying corollary A.9 gives that $f = y_1/y_2$ with y_1, y_2 linearly independent solutions of the equation $y'' + Qy = 0$. ☐

Lemma A.23. *The functions y_1, y_2 from lemma A.22 satisfy a Fuchsian second order differential equation with local exponents $1/2 \pm \lambda/2$ at $z = 0$ and $1/2 \pm \mu/2$ at $z = 1$ and $-1/2 \pm \nu/2$ at $z = \infty$.*

Proof. The equation $y'' + Q(z)y = 0$ is a Fuchsian equation, with $0, 1, \infty$ as regular singular points: this follows from the orders of poles and roots at $z = 0, 1, \infty$, see lemma A.22. Because the rigidity of Fuchsian equations with three regular singular points, the differential equation $y'' + Q(z)y = 0$ follows from the local exponents at $z = 0, 1, \infty$. To find the local exponents of y_1, y_2 at $z = 0$, we plug in $y = z^\rho + \cdots$ in $y'' + Qy = 0$ and consider the coefficient of $z^{\rho-2}$, which gives $\rho(\rho-1) + \frac{1-\lambda^2}{4} = 0$, from which follows that $\rho = \frac{1}{2} \pm \frac{\lambda}{2}$: here we use that around $z = 0$ we have $S(f)(z) = 2Q(z) \approx \frac{1-\lambda^2}{2z^2}$. Analogously it follows that around $z = 1$ we have $\rho = \frac{1}{2} \pm \frac{\mu}{2}$. Around $z = \infty$ have to rewrite $y'' + Qy = 0$ in the variable $1/z$ instead of z, which we did in equation (A.3). From this follows that $y''/z^4 + 2y'/z^3 + Qy = 0$, where the prime denotes differentiation with respect to $1/z$. Now we plug in $y = (1/z)^\rho + \cdots$ and consider the coefficient of $(1/z)^{\rho+2}$ which gives $\rho(\rho-1) + 2\rho + (1-\nu^2)/4 = 0$, from which follows that $\rho = -\frac{1}{2} \pm \frac{\nu}{2}$. ☐

Theorem A.24 (Schwarz). *Let $a, b, c \in \mathbb{R}$ and define $\lambda = |1-c|$ and $\mu = |c-a-b|$ and $\nu = |a-b|$. Suppose that $0 \leq \lambda, \mu, \nu \leq 1$, then the function $f = y_1/y_2$ maps $\mathbb{H}$ one-to-one and conformal onto the interior of a curvilinear triangle, where y_1, y_2 are linearly independent solutions of the hypergeometric differential equation equation* (2) *with parameters a, b, c. The vertices of the triangle correspond to the points $f(0), f(1), f(\infty)$ with corresponding angles $\lambda\pi, \mu\pi, \nu\pi$.*

Proof. From lemma A.23 we know that y_1, y_2 satisfy a second order Fuchsian equation with local exponents expressed in λ, μ, ν. We also want to express these local exponents variables a, b, c, but in the statement of this theorem absolute signs are used, so λ, μ, ν are known up to sign. Because the statement of lemma A.23 is invariant under choice of sign of λ, μ, ν, this gives in all cases the same Riemann scheme, see table A.2.

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \frac{1}{2}+\frac{\lambda}{2} & \frac{1}{2}+\frac{\mu}{2} & -\frac{1}{2}+\frac{\nu}{2} \\ \frac{1}{2}-\frac{\lambda}{2} & \frac{1}{2}-\frac{\mu}{2} & -\frac{1}{2}-\frac{\nu}{2} \end{array} \quad = \quad \begin{array}{ccc} 0 & 1 & \infty \\ \hline 1-c/2 & (1-a-b+c)/2 & (-1+a-b)/2 \\ c/2 & (1+a+b-c)/2 & (-1-a+b)/2 \end{array}.$$

Table A.2: Riemann scheme of the differential equation $y'' + Qy = 0$

The function $z^{-c/2}(z-1)^{-1/2-a/2-b/2+c/2}f$ has local exponents $0, 1-c$ at $z = 0$, and $0, c-a-b$ at $z = 1$, and a, b at $z = \infty$, and therefore must be a solution of the hypergeometric differential equation with parameters a, b, c. From this follows that the quotient of two linearly independent solutions of $y''+Qy = 0$ is equal to the quotient of two linearly independent solutions of the hypergeometric differential equation, see equation (2), because the factors $z^{-c/2}(z-1)^{-1/2-a/2-b/2+c/2}$ in the denominator and numerator cancel each other. To sum up, in lemma A.17 we started with a biholomorphic fuction from $\mathbb{H}$ to a curvilinear triangle with angles $\lambda\pi, \mu\pi, \nu\pi$, and then proved after some steps that this function is the quotient of two linearly independent solutions of the hypergeometric differential equation with parameters a, b, c. Note that every choice of y_1, y_2 gives a curvilinear triangle with given angles, because $\frac{Ay_1+By_2}{Cy_1+Dy_2} = \frac{Ay_1/y_2+B}{Cy_1/y_2+D}$ is a Möbius transformation of y_1/y_2. □

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