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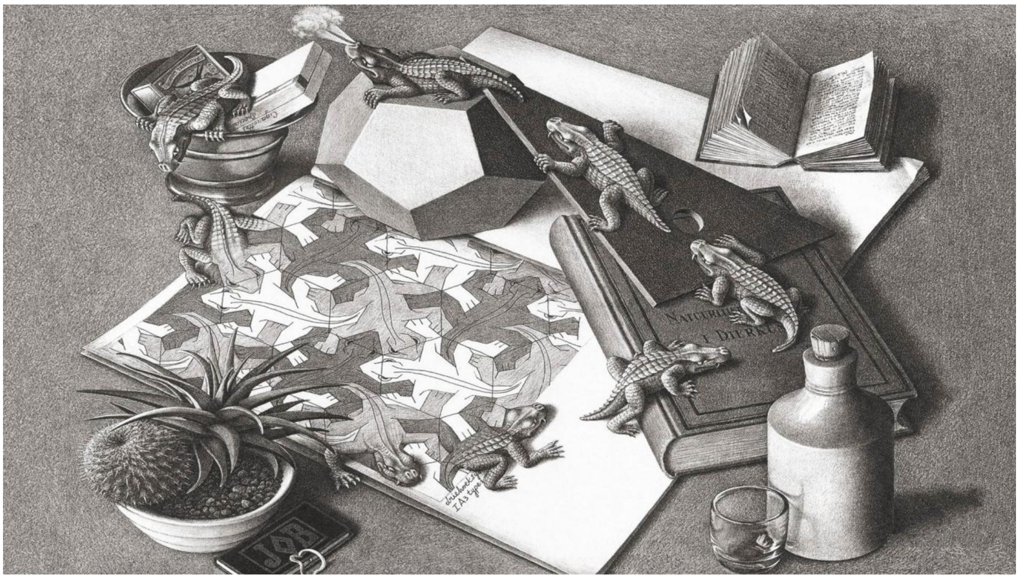
Quantum intuitionistic logic and the Gelfand representation theorem



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About the cover image

This famous drawing of Maurits Cornelis Escher provides a nice metaphor for the quantum mechanical description of the world. The crocodiles can be seen as the states of some system and the drawing on the paper of these crocodiles as the physical act of measurement, which forces the system to take some observable state. As a quantum mechanical principle, we cannot observe the rich and versatile state of the system itself, because the measurement of one observable is not always compatible with the measurement of another. After the measurement, the its states immediately become alive an involved again, escaping from their classical context of compatible observables forced upon that was forced upon them. The metaphysical concept of forcing the state to be in some compatible structure by measurement is one of the key features of the quantum logic studied and developed in this thesis.

ABSTRACT

The subject of this thesis is on the shared horizon of theoretical physics and logic: Quantum logic. In 1936, Birkhofff and von Neumann already formulated a quantum logic. But, this logic has some undesirable properties concerning the interpretation of the logical conjunction and disjunction as "and" and "or". Therefor, Landsman has recently set out to develop a new logic for quantum mechanics, which is set up inside a topos and thus is intuitionistic.

The construction of Landsmans intuitionistic quantum logic relies on the Gelfand representation theorem. However, to apply the theorem a constructive proof is needed. Coquand and Spitters recently published such a proof, but the material that was published is too condensed for our taste. So in this thesis we attempt to expand on this constructive proof of the Gelfand representation theorem and check all details the original authors left out. As a result, more insight is gained in the constructive proof of the Gelfand representation theorem, providing a stronger foundation for intuitionistic quantum logic.

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1

INTRODUCTION

In the academic fields of physics and mathematics, the disciplines of theoretical physics and logic respectively take similar, fundamental place. In physics one often leans on various mathematical disciplines such as (functional) analysis, numerical methods and algebra, but one does not expect logic to be one of them. However, the intersection of theoretical physics and logic is nonempty and contains many interesting subjects.

Most of these subjects have in common that they concern quantum mechanics. For example, in 1977 Davis uses Boolean algebras to obtain a relativistic interpretation of quantum mechanics [1]. The Boolean algebras determine a "frame of reference" that characterizes the real numbers in terms of self-adjoint operators on a Hilbert space. Davis then interprets quantum mechanical principles, such as the Heisenberg uncertainty relations, just as a consequence of operators not being in the same frame of reference.

Another, more recent example is a suggested correspondence between quantum field theory, topology, logic and computation theory [2]. Category theory provides a Rosetta stone analogue for these four disciplines by defining a category with as objects physical systems, manifolds, propositions or data types, and as arrows physical processes, cobordisms, proofs of programs respectively.

The subject of this thesis will provide another example of quantum mechanics being at the shared horizon of theoretical physics and logic. Whereas the previous examples build on and expand from quantum mechanics, this thesis will question and investigate the mathematical foundations of quantum mechanics itself: Quantum logic.

In 1932, von Neumann published his *Mathematische grundlagen der Quantenmechanik*, establishing the mathematical foundations of quantum mechanics [3]. This new formalism needed a new logic, as the logic of classical mechanics was not applicable. So in 1936 Birkhoff and von Neumann defined the classical quantum logic, which is still used today [4]. This logic has some remarkable features as we shall see. Most striking is the physical interpretation of the logical disjunction and conjunction, which cannot be interpreted as "or" and "and" respectively.

Furthermore, the law of excluded middle (stating that a statement is either true or false) holds in this logic. For many purposes this might be convenient, but it goes against the quantum mechanical intuition provided by the thought experiment of Schrödingers cat [5]: The cat does not have to be either alive or dead but can be in a superposition of the two. It is precisely the law of excluded middle, which dominates our classical intuition, that makes it hard to obtain the intuition propagated by this thought experiment. One would not expect the law of excluded middle to hold in the logic of quantum mechanics.

Recently, a new logic for quantum mechanics has been developed, motivated by these objections to the classical quantum logic of Birkhoff and von Neumann [6]. The logical structure is defined within a "quantum mechanical topos" and is therefor intuitionistic. As we shall see, this logic encompasses a similar view on quantum mechanics as Davis proposed, by having its propositions not just contain a statement about the outcome of some measurement, as a proposition must also specify a classical context in which the measurement is done. This prevents us from formulating propositions that are physically incompatible (for example about the measurement of both position and momentum), which would otherwise lead any truth valuation on our propositions to be ill defined.

A crucial ingredient in obtaining this logic is the Gelfand representation theorem, which states that elements of a certain class of Banach algebras can be represented by continuous functions on that Banach algebra. The theorem provides us with a quantum phase space, which in effect lists all possible sets compatible observations. Due to the probabilistic nature of quantum mechanics, this phase space is not determined by the state of the system, contrasting with deterministic classical mechanics. As a consequence, in most states we cannot predict the outcome of a measurement with absolute certainty. The quantum phase space obtained from the Gelfand represtentation theorem, fortunately provides a useful structure that allows us to define a Kripke semantics and an associated truth valuation on our propositions.

The Gelfand representation theorem is well known and originates from functional analysis. However, to apply it internally in the quantum mechanical topos, we need to prove it constructively. Such a proof is given in a very concise article by Coquand and Spitters [7], but it is too concise for out taste. In this thesis we will attempt to work out this proof extensively.

The thesis is structured in the following way. First, we expand on the development of quantum logic from the logic of classical mechanics. We then summarize the objections to this logic for quantum mechanics constructed by Birkhoff and von Neumann, that lead to the development intuitionistic quantum logic. Although the logical structure can be postulated as it is, its development relies on the Gelfand representation theorem. To gain a proper understanding of this theorem, we state and prove its original version in functional analysis after defining the necessary concepts. Then, Stone's version of the Gelfand representation theorem is proved non-constructively. Finally, we set out to give a detailed constructive proof of the localic Gelfand representation theorem following Coquand and Spitters.

The reader is expected to have a good understanding of algebra and category theory. For the relevant algebraic background, we refer to [8, 9, 10]. The reader is assumed to have a little knowledge on functional analysis, however most relevant concepts will be defined and explained in this thesis. A basic knowledge about quantum mechanics is recommended to fully grasp the implications and relevant context of this thesis, but it is not required. A good background is provided for example in [11].

2

QUANTUM LOGIC

One of the core ideas of quantum theory is that systems can be in a superposition of observable states. Referring to the famous thought experiment of Schrödingers cat [5], classically one would say that the cat is either alive or dead, whereas quantum theoretically the cat is in a superposition of both the alive state and the dead state, as long as it is not observed. It takes some time to fully wrap your head around this idea, but it turns out that doing so enables you to better describe nature in terms of physical laws [12].

The logical counterpart of Schrödingers cat is intuitionistic logic. Whereas in classical logic one would say that a statement is either true or false (this is the "law" of excluded middle), intuitionistically no such claim is made; statements do not have to be true or false, as long as no proof is given. Rejecting the law of excluded middle seems to be a natural choice for setting up a logical system for quantum theory. However, the current, mainstream logic for quantum mechanics is in fact obeying the law of excluded middle and, as we shall see, has some interpretation problems concerning the logical connectives.

In 1927 von Neumann formulated the theory of linear operators on Hilbert spaces as the mathematical structure for quantum mechanics. This framework still stands strong today. As this mathematical framework differs from the mathematical framework of classical mechanics, a new logic was needed. In 1936 Birkhoff and von Neumann formulated the quantum logic which was (intentionally) very similar to the classical logical structure of classical mechanics. To properly understand its origin, we will first briefly describe the classical logic of classical mechanics and consequently explain the logical system set up by Birkhoff and von Neumann. Objections to this classical quantum logic will lead to the intuitionistic quantum logic. After explaining the essential connection between this logic and the Gelfand representation theorem, we will conclude this chapter by giving some background on logic in sheaf toposes of which the intuitionistic quantum logic is an example.

2.1 CLASSICAL LOGIC IN CLASSICAL MECHANICS

Given some classical system, we denote X as its configuration space and an observable is a function $f : X \to \mathbb{R}$. Denote $O(X)$ as the set of observables of X. The Gelfand spectrum of X is defined as the set

$$\Sigma(O(X)) = \{\omega : O(X) \to \mathbb{C} | \omega(fg) = \omega(f)\omega(g), \omega \text{ is linear and nonzero } \}. \tag{1}$$

Physically, these maps describe a set such that each element corresponds to a value of an observable that is compatible with the values for the other observables in that set. By defining for each $x \in X$ a map ω_x by

$$\omega_x(f) = f(x), \tag{2}$$

we see that the configuration space X completely determines its Gelfand spectrum. ω_x is called the Gelfand transform of x and it is the Gelfand representation theorem, which we discuss extensively in the next chapter, that ensures that this map is in fact an isomorphism. This is what we would expect: Classical mechanics is deterministic. If you know the state of a system, you can predict the outcome of any measurement of an observable with infinite accuracy.

To set up a logic we need a set Σ_X of atomic propositions, which in our case are given by the expressions of the form $\underline{f \in \Delta}$ where f is an observable and Δ is a subset of $\mathbb{R}$. We will underline propositions, to separate our notation for the logic of the physical system with the physical system itself. This expression has the interpretation of the statment "the value of a measurement of f lies is Δ". Other propositions are constructed inductively along the iterative rules of propositional logic generating the set B_X of propositions.

At this point we can begin to formulate a propositional theory for classical mechanics. For example our theory must contain an axiom stating that if $\Delta \subseteq \Delta'$ then $\underline{f \in \Delta}$ should imply $\underline{f \in \Delta'}$.

One can now define a *valuation at the point* $x \in X$ as a function $V_x : \Sigma_X \to \{0, 1\}$ defined by

$$V_x(\underline{f \in \Delta}) = 1 \text{ if and only if } f(x) \in \Delta. \tag{3}$$

By recursive use of the truth tables of the logical connectives one can extend this map to B_X. One can now interpret a proposition to be true if its valuation has value 1 and false if its valuation has value 0. Since this valuation map is well defined, the law of excluded middle clearly holds.

The semantic entailment relation $\models_X$ on B_X is defined by $\alpha \models_X \beta$ if and only if $V_x(\alpha) = 1$ implies $V_x(\beta) = 1$ for all $x \in X$. The equivalence relation $\sim_X$ is simply defined by $\alpha \sim_X \beta$ if and only if $\alpha \models_X \beta$ and $\beta \models_X \alpha$. Using this equivalence we can show that the Lindenbaum algebra L_X (the algebra of equivalence classes of propositions of B_X) is isomorphic to the power set $\mathcal{P}(X)$ by the map φ, defined by the following inductive system

$$\varphi([f \in \Delta]) = f^{-1}(\Delta), \tag{4}$$
$$\varphi([\neg\alpha]) = \varphi([\alpha])^c, \tag{5}$$
$$\varphi([\alpha \vee \beta]) = \varphi([\alpha]) \cup \varphi([\beta]), \tag{6}$$
$$\varphi([\alpha \wedge \beta]) = \varphi([\alpha]) \cap \varphi([\beta]). \tag{7}$$

Indeed we have

$$(f \in \Delta) \sim_X (1_{f^{-1}(\Delta)} = 1), \tag{8}$$

where 1_U is the indicator function of U and $1_{f^{-1}(\Delta)} = 1$ means $1_{f^{-1}(\Delta)} \in 1$. And by use of the set operations of union, intersection and complement when dealing with conjunction, disjunction and negation respectively, one can show by induction on its complexity that any proposition in B_X is logically equivalent to one of the form $1_U = 1$ for some $U \in \mathcal{P}(X)$.

2.2 QUANTUM LOGIC OF BIRKHOFF AND VON NEUMANN

The crucial point in von Neumanns work in 1927 on the mathematical structure of quantum mechanics was to describe observables by self adjoint operators $a : H \to H$, where H is the Hilbert space $l^2(X)$ (the vector space of functions $\psi : X \to \mathbb{C}$ with a norm defined by its inner product). As the operators not necessarily commute, the Gelfand spectrum is empty which, as we shall see, will cause some problems. To physically comprehend that the Gelfand spectrum is indeed empty, we give the following argument. The Gelfand spectrum expresses the compatible sets of values that observables can take. Due to non-commuting observables, we cannot give such a specification, as is reflected in the Heisenberg uncertainty principle. We cannot specify a value for, say, the position, compatible with a value for momentum.

We proceed as follows. The elementary propositions in quantum logic will be of the form $\underline{a \in \Delta}$. The physical meaning of such a proposition now is "a measurement of the observable a yields a value that lies in Δ". To evaluate the truth of such propositions we need the quantum analogue to the points $x \in X$: The role of configurations $x \in X$ is classical mechanics is assumed by pure states ω_ψ in quantum mechanics: A *state* is a complex-linear map ω on $B(H)$, the set of linear operators on H, to $\mathbb{C}$, satisfying

$$\omega(a^* a) \geq 0, \tag{9}$$

$$\omega(1_H) = 1. \tag{10}$$

Here a^* is the adjoint of a. A state is called *pure* if it is determined by a unit vector $\psi \in H$ in the following way

$$\omega_\psi(a) = \langle \psi, a\psi \rangle. \tag{11}$$

We can now set up a valuation by saying that the elementary proposition $\underline{a \in \Delta}$ is:

- true with respect to ω_ψ if and only if $\psi \in H_{a,\Delta}$ where $H_{a,\Delta}$ is the eigenspace of a for the eigenvalues in Δ;
- false with respect to ω_ψ if and only if $\psi \perp H_{a,\Delta}$.

As in general $H \neq H_{a,\Delta} \cup H^{\perp}_{a,\Delta}$, there can be states which fail to assign a truth value to a proposition.

One would now like to extend this valuation to all propositions, but this is not straightforward anymore since for example a pure state ω_ψ may neither assign a truth value to α nor to β, but it might still make $\alpha \vee \beta$ true. As a consequence we will extend the valuation by the following rules

- The negation $\neg\alpha$ is true with respect to ω_ψ if and only if α is false with respect to ω_ψ.

- The conjunction $\alpha \wedge \beta$ is true with respect to ω_ψ if and only if both α and β are true with respect to ω_ψ.
- The disjunction $\alpha \vee \beta$ is true with respect to ω_ψ if and only if $\neg(\neg\alpha \wedge \neg\beta)$ is true with respect to ω_ψ.

In particular this means that conjunctions behave classically, since $(\underline{a \in \Delta}) \wedge (\underline{b \in \Delta'})$ is true with respect to ω_ψ if and only if $\psi \in H_{a,\Delta} \cap H_{b,\Delta'}$. That means $(\underline{a \in \Delta}) \wedge (\underline{b \in \Delta'})$ is true with respect to ω_ψ if and only if $(\underline{a \in \Delta})$ and $(\underline{b \in \Delta'})$ are both true with respect to ω_ψ . However, the disjunction does not behave classically as $(\underline{a \in \Delta}) \vee (\underline{b \in \Delta'})$ is true with respect to ω_ψ if and only if $\psi \in H_{a,\Delta} + H_{b,\Delta'} \neq H_{a,\Delta} \cup H_{b,\Delta'}$.

We can now define the semantic entailment relation similar to the classical case. The analogue of equation (8) is

$$\underline{a \in \Delta} \sim_H e_{a,\Delta} = 1. \tag{12}$$

Here, $e_{a,\Delta}$ is the projection operator on $H_{a,\Delta}$. Using the properties of these projections one can show that the set of equivalence classes of propositions is isomorphic to the set $\mathcal{P}(H)$ of projections on H by the map φ defined by

$$\varphi([\underline{a \in \Delta}]) = e_{a,\Delta}, \tag{13}$$
$$\varphi([\neg\alpha]) = 1 - \varphi([\alpha]), \tag{14}$$
$$\varphi([\alpha \vee \beta]) = \varphi([\alpha]) + \varphi([\beta]) - \varphi([\alpha])\varphi([\beta]), \tag{15}$$
$$\varphi([\alpha \wedge \beta]) = \varphi([\alpha])\varphi([\beta]). \tag{16}$$

Objections

To the above described quantum logic some conceptual objections can be made. Firstly, the logical conjunction and disjunction are hard to interpret, mainly because they do not distribute over each other. But also because, as already mentioned, there are states in which $\alpha \wedge \beta$ is false whilst neither α nor β is false and there are states in which $\alpha \vee \beta$ is true whilst neither α nor β is true.

Secondly, and most troublesome, the law of excluded middle holds in the quantum logic of Birkhoff and von Neumann. Indeed

$$\varphi([\alpha \vee \neg\alpha]) = \varphi([\alpha]) + 1 - \varphi([\alpha])(1 - \varphi([\alpha])) = 1, \tag{17}$$

as the projections are idempotents $\varphi([\alpha])^2 = \varphi([\alpha])$. This strongly contrasts with the concept of Schrödingers cat.

A final objection to this quantum logic is the inability to assign a truth value to some propositions. The intuitionistic logic of Brouwer and Heyting will overcome these objections and is therefor the appropriate candidate for quantum logic.

2.3 QUANTUM LOGIC OF LANDSMAN

The above objections are the reason for Landsman to develop a quantum logic using topos theory [6]. Toposes have an internal universe to do mathematics in and the internal logic is in most cases intuitionistic. The problems of classical quantum logic mainly arise due to the fact that the algebra of observables is non-commutative and as a result, the Gelfand spectrum is empty. However, in the approach of Landsman this algebra of observables is internalized into a specific topos and this internal algebra of observables is commutative. The details of the internal setup are given in chapter 4. Here, we briefly describe the topos and postulate the resulting logical structure.

The "quantum mechanical" topos is constructed from the algebra of observables $\mathcal{A}$, i.e. the self adjoint operators on the complex Hilbert space of the physical system. In fact, $\mathcal{A}$ is a unital C*-algebra (see definition 14), and let $\mathcal{C}(\mathcal{A})$ be the poset of all unital commutative C*-subalgebras ordered by inclusion. We can regard $\mathcal{C}(\mathcal{A})$ as a category in which $C \in \mathcal{C}(\mathcal{A})$ are objects, and there is a unique arrow $C \to D$ if and only if $C \subseteq D$. The topos $T(\mathcal{A})$ consists of the functors $F : \mathcal{C}(\mathcal{A}) \to Sets$, so

$$T(\mathcal{A}) = [\mathcal{C}(\mathcal{A}), Sets] \cong Sh(\mathcal{C}(\mathcal{A})). \tag{18}$$

The isomorphism concerns $\mathcal{C}(\mathcal{A})$ equipped with the Alexandrov topology and is defined by the map sending a functor $F : \mathcal{C}(\mathcal{A}) \to Sets$ to the sheaf $\tilde{F} : \mathcal{O}(\mathcal{C}(\mathcal{A})) \to Sets$, where the image is defined on an upper set in the Alexandrov topology by $\tilde{F}(\uparrow C) = F(C)$.

The unital commutative C*-subalgebras of $\mathcal{A}$ will be seen as the classical context. An elementary proposition in our quantum logic will be a pair (C, e), with C the classical context and $e \in \mathcal{P}(C)$ a projection in C. Such a pair will define a map $S_{(C,e)} : \mathcal{C}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ by

$$S_{(C,e)}(D) = \begin{cases} e & \text{if } C \subseteq D \\ \perp & \text{otherwise} \end{cases}. \tag{19}$$

Now we define $\mathcal{O}(\Sigma)$ by

$$\mathcal{O}(\Sigma) = \{S : \mathcal{C}(A) \to \mathcal{P}(A) | S(C) \in \mathcal{P}(C) \text{ and if } C \subseteq D, \text{ then } S(C) \leq S(D)\}, \tag{20}$$

where projections e, f are partially ordered by $e \leq f$ if and only if $ef = e$. $\mathcal{O}(\Sigma)$ inherits this partial order pointwise, i.e.

$$S \leq T \text{ if and only if } S(C) \leq T(C) \text{ for all } C \in \mathcal{C}(\mathcal{A}). \tag{21}$$

Since $\mathcal{P}(C)$, the lattice of projections in C, is Boolean, the logical structure described by $\mathcal{O}(\Sigma)$ is defined pointwise by the rules

$$\top(C) = 1 \text{ (the identity projection)}, \tag{22}$$

$$\perp(C) = 0 \text{ (the 0 projection)}, \tag{23}$$

$$(S \wedge T)(C) = S(C) \wedge T(C), \tag{24}$$

$$(S \vee T)(C) = S(C) \vee T(C), \tag{25}$$

$$(S \Rightarrow T)(C) = \bigvee\{e \in \mathcal{P}(C) | \forall D \supseteq C : e \leq S(D)^{\perp} \wedge T(D)\}, \tag{26}$$

$$(\neg S)(C) = \bigvee\{e \in \mathcal{P}(C) | \forall D \supseteq C : e \leq S(D)^{\perp}\}. \tag{27}$$

Boolean lattices are distributive, so our logical connectives $\vee$ and $\wedge$ on $\mathcal{O}(\Sigma)$ are physically meaningful since they are restricted to a context C in which the projections commute. The structure of $\mathcal{O}(\Sigma)$ is that of a Heyting algebra and hence is a model for intuitionistic predicate calculus.

Each $S \in \mathcal{O}(\Sigma)$ is equal to a disjunction of elementary proposition pairs (C, e) by

$$S = \bigvee_{C \in \mathcal{C}(\mathcal{A})} S_{(C, S(C))}. \tag{28}$$

Hence $\mathcal{O}(\Sigma)$ is a good candidate for quantum logic.

For the truth valuation we define a *Kripke semantics* $V_\omega : \mathcal{O}(\Sigma) \to Upper(\mathcal{C}(A)) = \mathcal{O}(\mathcal{C}(\mathcal{A}))$ (because $\mathcal{C}(\mathcal{A})$ carries the Alexandrov topology), given a state ω, by

$$V_\omega(S) = \{C \in \mathcal{C}(A) | \omega(S(C)) = 1\}. \tag{29}$$

Using the order preservation of S we indeed have that $V_\omega(S)$ defines an upper set, for if $C \subseteq D$, then $S(C) \leq S(D)$, so $1 = \omega(S(C)) \leq \omega(S(D)) \leq 1$. This map in fact lists the contexts C in which the classical proposition $S(C)$ (in the logic of Birkhoff and von Neumann) is true. In the new quantum logic, a proposition $S \in \mathcal{O}(\Sigma)$ is:

- true with respect to ω if and only if $V_\omega(S) = \mathcal{C}(A)$.
- false with respect to ω if and only if $V_\omega(S) = \emptyset$.

So a proposition S is true if and only if $\omega(S(C)) = 1$ for all $C \in \mathcal{C}(\mathcal{A})$. Using the linearity of states, we then have that $\neg S$ is true if and only if S is false, $S \wedge T$ is true if and only if both S and T are true and $S \vee T$ is true if and only if either S or T is true.

We wish to highlight an interesting similarity between this logical structure developed by Landsman and the interpretation of quantum mechanics by Davis [1]. Davis uses Boolean valued models of set theory to give a characterization of the real numbers in terms of a commutating set of self adjoint operators using the spectral theorem. When preforming a measurement, one assigns to some observables a real value. Is this way it is impossible to assign a real numbers to non commuting operators simultaneously. So one cannot measure operators that do not commute within one measurement. The specific set of commutating self adjoint operators is then seen as a "reference frame", relative to which a measurement is done. An observer thus chooses one such a frame when doing a measurement, which in turn determines the physics that are observed.

In the logic of Landsman, a similar thing is going on as a proposition cannot consist only of a projection e (which, as in the quantum logic of von Neumann and Birkhoff, corresponds to a statement about the outcome of a measurement) and needs a specified classical context C as well. As the projections have to lie in the Boolean algebra of projections of C, the classical context C takes on the role of a reference frame similar to the Boolean reference frame of Davis.

Derivation of intuitionistic quantum logic

The logical structure for quantum mechanics has been postulated above and provides, on its own, desirable properties to make it a good candidate for a new quantum logic. We expand here on the derivation of this logical structure, to provide a further argument for its soundness.

As in general the algebra $\mathcal{A}$ of observables is non-commutative, its Gelfand spectrum cannot be computed. To overcome this problem, we consider all commutative unital C^*-subalgebras of $\mathcal{A}$ which has a partial order defined by inclusion, thus we can see it as a posetal category. The objects of this catergory do have the desired property of being commutative, so we want to define some map, that contains the relevant structure of A, but is defined pointwise, so it preserves the desirable properties. We can do so by defining a functor A from $\mathcal{C}(\mathcal{A})$ to *Sets* that sends objects to their corresponding set, forgetting the algebra structure, and arrows to the inclusion map. This has thus invited us to define the sheaf topos of $\mathcal{C}(\mathcal{A})$. We show in the first section of chapter 4 that the functor A indeed is a commutative C^*-algebra in terms of the relevant definitions in the topos.

As a result we can obtain the Gelfand spectrum of the internalized algebra of observables A. As we did with classical mechanics, we rely on the Gelfand representation theorem to obtain again a notion of a quantum phase space, i.e. of compatible observations. Due to the pointwise nature of A, we obtain such a notion for each object $C \in \mathcal{C}(\mathcal{A})$. This Gelfand spectrum has an external description, which is provided by $\mathcal{O}(\Sigma)$.

Thus the Gelfand representation theorem plays a crucial role in the process of setting up a logic for quantum (but also classical) mechanics. The theorem is well known and originates from the field of functional analysis. But, to apply its concept within the quantum mechanical topos, a localic version of the theorem is needed with a constructive proof. Johnstone formulated a localic version [13] but with a non-constructive proof. Recently constructive proofs have been given [14, 7]. Because of the crucial role of the Gelfand representation theorem, it will be studied extensively in the following chapters. Before concluding this chapter, we provide some background on logic in sheaf toposes.

2.4 LOGIC IN SHEAF TOPOSES

Sheaf toposes (such as $T(\mathcal{A})$) have a lot of structure that makes them convenient to work with, especially for the construction of a first-order logic. Given some topological space S a sheaf in $Sh(S)$ can be defined in terms of only the topology of S (whose structure characterizes locales), without mentioning its points. Thus we can generalize the sheaf toposes of some topological space to a notion of sheaf toposes, where we only have to specify a locale that does not necessarily correspond to the topology of some space. The sheafification of a topological space to a topos thus factors trough a sheafification of locales. In such a sheaf topos of a locale, the internal locales have a simple external description as a map of locales in *Sets*. We wish to give a brief overview of the relevant concepts and constructs and recommend [10] for a more thorough treatment.

As we have seen $\mathcal{C}(\mathcal{A})$ is a topological space with the Alexandrov topology. In general, for any topological space S we can consider its topology $O(S)$ (the set of all open subsets of S) as a lattice, partially ordered by inclusion, which has finite meets and arbitrary joins. Furthermore, as meets and joins are given by the set theoretic intersection and union respectively, one can show that the infinite distributive law holds, which states that for any $U \in O(S)$ and an arbitrary family $V_i \in O(S)$ we have

$$U \wedge \bigvee_i V_i = \bigvee_i (U \wedge V_i). \tag{30}$$

This specific lattice structure has led to the notion of frames, i.e. a frame is a lattice with finite meets, arbitrary joins and that satisfies the infinite distributive law. It follows that frames have a top and bottom element, by considering the empty meet and join respectively.

A map of frames then is a map of the underlying posets that preserves order, finite meets and arbitrary joins. Then we can construct the category *Frm* of frames and maps of frames. If we have two frames $O(X)$ and $O(Y)$ being the topology of some spaces X and Y, we see that a continuous map $f : X \to Y$ induces a map of frames which is just the inverse image map $f^{-1} : O(Y) \to O(X)$. As this yields an arrow in the opposite direction we thus can define a functor from the category of topological spaces to Frm^{op}. Hence we define the category *Loc* of locales to be Frm^{op}. Hence locales are the same as frames (but for a locale L, the corresponding frame is written as $O(L)$, although there does not need to be a particular topological space L for which the frame $O(L)$ is a topology) and a map of locales $f : X \to Y$ is a map of frames $f^{-1} : O(Y) \to O(X)$. The functor has a right adjoint, but does not give an equivalence between the category of topological spaces and *Loc*, as not all topological spaces can be recovered from their topology. For example, in the trivial topology of a nonempty set S the topology does not give you the points of S.

To give the internal description of frames (or locales) in some sheaf topos $Sh(L)$, with L a locale, we define a category whose objects are maps of frames in *Sets* that have $O(L)$ as domain. An arrow between such objects $\pi_A^{-1} : O(L) \to O(A)$ and $\pi_B^{-1} : O(L) \to O(B)$, is a map of frames $f^{-1} : O(B) \to O(A)$ that satisfies

$$f^{-1} \circ \pi_B^{-1} = \pi_A^{-1}. \tag{31}$$

We claim this category is dual to the category of internal frames in $Sh(L)$.

To prove the claim, let X be a locale in $Sh(L)$ and $O(X)$ the corresponding frame. As $O(X)$ is an object of the sheaf topos of L, we obtain the frame $O(A) := O(X)(L)$ in *Sets* and thereby the external description of $O(X)$ as a map $\pi^{-1} : O(L) \to O(A)$. Conversely, given such a map of frames, we can define a sheaf from it as the functor that sends $U \in O(L)$ to the set $\{V \in O(A) | V \leq \pi^{-1}(U)\}$. Indeed, if $O(A) = O(X)(L)$, then this sheaf is precisely $O(X)$. Furthermore, an arrow from π_A^{-1} to π_B^{-1} (represented by a map of frames $f^{-1} : O(B) \to O(A)$) defines an internal map of frames $\phi^{-1} : O(Y) \to O(X)$ in $Sh(L)$ with $O(Y)(L) = O(B)$ and $O(X)(L) = O(A)$, which thus is a natural transformation $(\phi_U^{-1} : O(Y)(U) \to O(X)(U))_{U \in O(L)}$, whose components are defined by the arrow in *Sets* that sends $y \in O(Y)(U) \subseteq O(B)$ to $f^{-1}(y)$.

As a result, we have a straightforward external description of the internal frames. This is used later on to obtain the sheaf of Dedekind real numbers in a Sheaf topos. Applying this duality to the frame of the internal Gelfand spectrum in the quantum mechanical topos $T(\mathcal{A})$, we thus recover the frame $\mathcal{O}(\Sigma)$ to provide the external description of the Gelfand spectrum as a map of frames $\pi_\Sigma^{-1} : O(\mathcal{C}(\mathcal{A})) \to \mathcal{O}(\Sigma)$, which sends some upper set $\uparrow C$ to the map $S_{(C,1)}$ (see equation (19)).

In general, a locale A is a complete Heyting algebra, as one can define an implication and negation operator for any two elements U and V of A as follows

$$\begin{aligned} U \Rightarrow V &= \bigvee \{W \in A | W \wedge U \leq V\} \\ \neg U &= \bigvee \{W \in A | W \wedge U = \bot\} = (U \Rightarrow \bot). \end{aligned} \tag{32}$$

In this way, the pointwise Heyting algebra structure of $\mathcal{O}(\Sigma)$ was uncovered.

We want to conclude this section with a note on the occurrence of intuitionistic logic. The above result about the structure of $\mathcal{O}(\Sigma)$ is an example of the well known theorem which state that for any object A of a topos, its set of subobjects $Sub(A)$, seen as a poset, is a Heyting algebra [10]. This has a corresponding internal statement. One can define the notion of a Heyting algebra object internally and show that for any object A of the topos, its power object $P(A)$ is an internal Heyting algebra. Moreover, for each object B we have the canonical isomorphism between $Sub(A \times B)$ and $Hom(B, P(A))$ the arrows from B to $P(A)$, yielding a whole family of external Heyting algebra's.

To provide a model for classical propositional calculus, Boole considered the powerset of some set X and defined the natural structure of Boolean algebra's on it. We see now that within the universe of a topos, the powerobject of some object X is in general not a Boolean algebra, but a Heyting algebra due to the above theorems. In that way toposes provide models for intuitionistic propositional caculus.

3

THE GELFAND REPRESENTATION THEOREM

The Gelfand representation theorem originally is a theorem in functional analysis where it is known as the Gelfand-Naimark Theorem. To properly understand the theorem, one first needs to get acquainted with the necessary concepts of functional analysis. For the localic version of the representation theorem, some background on locales is given. Equipped with the necessary background, we will set out to prove both representation theorems.

3.1 BACKGROUND

Functional analysis

We start out with the notion of a Banach spaces and expand from there.

Definition 1. *A* Banach Space *is a complete normed vector space $\mathcal{A}$ over $\mathbb{C}$. i.e. there is a norm $||.||$ on $\mathcal{A}$ such that every Cauchy sequence $(x_n)_n$ in $\mathcal{A}$ has a limit $x = \lim_{n\to\infty} x_n$ in $\mathcal{A}$.*

This notion extends naturally to that of an algebra.

Definition 2. *A* Banach Algebra *is a Banach space $\mathcal{A}$ whose elements form an algebra such that the algebra multiplication satisfies for all $x, y \in \mathcal{A}$*

$$||xy|| \leq ||x||||y||. \tag{33}$$

Definition 3. *A Banach algebra is called* unital *if it has a unit element e with respect to the multiplication. The elements of a unital Banach algebra that have inverses are called* invertible*.*

Note that although not necessary, it is common practice to rescale the definition of the norm such that $||e|| = 1$.

Proposition 4. *The set $\mathcal{G}(\mathcal{A})$ of all invertible elements of $\mathcal{A}$ is open.*

Proof. Let $a \in \mathcal{A}$ be invertible. Let b be in the open ball $B(a, \frac{1}{||a^{-1}||})$. Then $||a^{-1}(b-a)|| \leq ||a^{-1}||\ ||b-a|| < 1$. Now for x with $||x|| < 1$, $e - x$ is invertible with inverse just the geometric series of x: $\sum_{n=0}^{\infty} x^n$. Indeed

$$\begin{aligned}(e-x)\sum_{n=0}^{\infty} x^n &= \sum_{n=0}^{\infty} x^n - \sum_{n=1}^{\infty} x^n \\ &= x^0 = e.\end{aligned} \tag{34}$$

So $e - (-a^{-1}(b-a))$ is invertible. Hence $b = a(e - (-a^{-1}(b-a))$ is a product of invertible elements, hence b itself is invertible. So the entire ball $B(a, \frac{1}{||a^{-1}||})$ is contained in $\mathcal{G}(\mathcal{A})$. So $\mathcal{G}(\mathcal{A})$ is open. □

Definition 5. *The* spectrum of $x \in \mathcal{A}$ *is*

$$\sigma(x) = \{\lambda \in \mathbb{C} | \lambda e - x \text{ is not invertible}\}. \tag{35}$$

For $\lambda \notin \sigma(x)$ *we define the* resolvent of x *to be the operator*

$$R_x(\lambda) = (\lambda e - x)^{-1}. \tag{36}$$

The spectral radius *of* x *is*

$$\rho(x) = sup\{|\lambda|\ |\lambda \in \sigma(x)\}. \tag{37}$$

Proposition 6. *Let* $\mathcal{A}$ *be a Banach algebra. Then* $\sigma(x)$ *is nonempty for any* $x \in \mathcal{A}$.

Proof. Given $x \in \mathcal{A}$ the function $R_x(\lambda)$ is analytic on $\mathbb{C} \setminus \sigma(x)$: Let $\mu, \lambda \in \mathbb{C} \setminus \sigma(x)$, then

$$\begin{aligned}R_x(\lambda)(\mu-\lambda)R_x(\mu) &= R_x(\lambda)((\mu e - x) - (\lambda e - x))R_x(\mu) \\ &= R_x(\lambda) - R_x(\mu).\end{aligned} \tag{38}$$

So

$$\lim_{\mu \to \lambda} \frac{R_x(\lambda) - R_x(\mu)}{\lambda - \mu} = -R_x(\lambda)^2. \tag{39}$$

Now if $\sigma(x)$ would be empty, R_x would be an entire analytic function. By Liouville's theorem it then must be constant. And since $||R_x(\lambda)|| = |\lambda|^{-1}||(e - \lambda^{-1}x)^{-1}|| \to 0$ as $\lambda \to \infty$, R_x is the constant 0 function, which is a contradiction since 0 clearly is not an inverse of any element. □

Notice that, by use of the geometric series, $\sigma(x)$ is a closed subset of the disc $\{\lambda|\ |\lambda| \leq ||x||\}$, so $\rho(x) \leq ||x||$. In fact we can be more precise.

Proposition 7. $\rho(x) = \lim_{n\to\infty} ||x^n||^{1/n}$.

Proof. Observe that

$$\begin{aligned}(\lambda e - x)\sum_{j=0}^{n-1}\lambda^j x^{n-1-j} &= e\left(\sum_{j=1}^{n}\lambda^j x^{n-j} - \sum_{j=0}^{n-1}\lambda^j x^{n-j}\right) \\ &= e\lambda^n - x^n.\end{aligned} \tag{40}$$

So if $e\lambda^n - x^n$ is invertible with inverse a, then $a\sum_{j=0}^{n-1}\lambda^j x^{n-1-j}$ is the inverse of $e\lambda - x$. Hence if $\lambda \in \sigma(x)$, that is, $\lambda e - x$ is not invertible, then $\lambda^n \in \sigma(x^n)$. So $\rho(x)^n \leq \rho(x^n) \leq ||x^n||$ hence $\rho(x) \leq \liminf_n ||x^n||^{1/n}$.
For the other inequality let ϕ be a continuous linear functional on $\mathcal{A}$. In proposition 8 we have shown when R_x is analytic. Thus the composition $\phi \circ R_x(\lambda)$ is analytic for $|\lambda| > \rho(x)$. In this region we can write $\phi \circ R_x(\lambda) = \sum_{n=0}^{\infty}\lambda^{-(n+1)}\phi(x^n)$ by a Laurent series expansion around $\lambda = \infty$. This is thus a convergent sum, hence all terms must be bounded uniformly. That is, there is a bound $C_\phi \in \mathbb{R}$ such that $|\lambda^{-(n+1)}\phi(x^n)| \leq C_\phi$. By the uniform boundedness principle we thus have a bound $C > 0$ such that $|\lambda|^{-n}||x|| \leq C$. So $\limsup_n ||x^n||^{1/n} \leq \rho(x)$. This concludes the proof. □

Banach algebras are determined by their non invertible elements in the following sense.

Proposition 8 (Gelfand Mazur theorem)**.** *A Banach algebra in which every nonzero element is invertible is isomorphic to $\mathbb{C}$.*

Proof. Suppose $\mathcal{A}$ is a Banach algebra in which every nonzero element is invertible. We claim that we can write each $x \in \mathcal{A}$ as $e\lambda$ for $\lambda \in \mathbb{C}$. The natural isomorphism then follows immediately.
Suppose $x \notin e\mathbb{C}$. Then for all $\lambda \in \mathbb{C}$ we have $e\lambda - x \neq 0$. Hence by assumption $e\lambda - x$ is invertible for all $\lambda \in \mathbb{C}$. Thus $\sigma(x)$ is empty, contradicting proposition 6. □

There also is a notion of spectrum for an entire commutative unital Banach algebra.

Definition 9. *Let $\mathcal{A}$ be a commutative unital Banach algebra. The* spectrum of $\mathcal{A}$, *denoted by $\sigma(\mathcal{A})$ is the set of all nonzero homomorphisms from $\mathcal{A}$ to $\mathbb{C}$. $C(\sigma(\mathcal{A}))$ is the space of continuous complex-valued functions on $\sigma(\mathcal{A})$.*

The structure of $\sigma(\mathcal{A})$ will become clear from the following proposition.

Proposition 10. *For any $f \in \sigma A$ we have $f(e) = 1$ and $|f(x)| \leq ||x||$ for all $x \in \mathcal{A}$.*

Proof. Choose an element $x \in \mathcal{A}$ such that $f(x) \neq 0$. Then $f(x) = f(ex) = f(e)f(x)$ so indeed $f(e) = 1$.
For the second statement, first observe that it is equivalent to the implication if $||x|| < |\lambda|$ then $|f(x)| < |\lambda|$. So suppose $||x|| < |\lambda|$. As in the proof of proposition 4 we derive that $\lambda e - x$ is invertible. Now notice that for invertible

elements $a \in \mathcal{A}$ we have $1 = f(e) = f(a^{-1}a) = f(a^{-1})h(a)$, so $f(a) \neq 0$. Thus $f(\lambda e - x) = \lambda - f(x) \neq 0$. Thus $|f(x)| - |\lambda| \leq |\lambda - f(x)| = r > 0$ that is $|f(x)| < |\lambda|$, as desired. □

Now, let $\mathcal{A}^*$ be the space of bounded complex valued linear functionals on $\mathcal{A}$ with norm $||f||_{\mathcal{A}^*} = \sup_{x \in \mathcal{A}-0} \frac{|f(x)|}{||x||}$. Then the above proposition gives that $\sigma(\mathcal{A})$ is a subset of the closed unit ball in $\mathcal{A}^*$. Imposing the weak* topology as a subset of $\mathcal{A}^*$, $\sigma(\mathcal{A})$ is a topological space. Since the defining properties of $\sigma(\mathcal{A})$ are closed under pointwise limits, $\sigma(\mathcal{A})$ is a closed subset of the unit ball in $\mathcal{A}^*$. Hence, we can apply the Banach-Alaoglu theorem, which gives that $\sigma(\mathcal{A})$ is a compact Hausdorff space.

A natural norm on $C(\sigma(\mathcal{A}))$ is the supremum norm $||.||_{sup}$. For $x \in \mathcal{A}$ we can define a function $\widehat{x} \in C(\sigma(\mathcal{A}))$ in a natural way.

Definition 11. *Let $x \in \mathcal{A}$. Define $\widehat{x} : \sigma(\mathcal{A}) \to \mathbb{C}$ by*

$$\widehat{x}(h) = h(x) \tag{41}$$

The map $\Gamma_{\mathcal{A}}$ sending x to $\widehat{x}$ is called the Gelfand transform in $\mathcal{A}$.

One may wonder when the Gelfand transform is isometric, as this is part of the statement of the Gelfand-Naimark theorem. First we demonstrate how the norm $||.||$ on a commutative unital Banach algebra and the sup norm of a Gelfand transform $||\widehat{.}||_{sup}$ are related.

Lemma 12. *Let $\mathcal{A}$ be a commutative unital Banach algebra and let $x \in \mathcal{A}$. Then*

$$range(\widehat{x}) = \sigma(x). \tag{42}$$

This has the immediate consequence

$$||\widehat{x}||_{sup} = \rho(x). \tag{43}$$

Proof. This result will follow from the claim that x is not invertible if and only if $\widehat{x}(h) = 0$ for some $h \in C(\sigma(\mathcal{A}))$. Then we have, given some $\widehat{x}(h) = \lambda \in \mathbb{C}$, that $\widehat{\lambda e - x}(h) = 0$ which, by the claim, holds if and only if $\lambda e - x$ is not invertible. The latter is equivalent to stating $\lambda \in \sigma(x)$.
To prove the claim we will need some algebra and Zorn's lemma. The reader is referred to [8] for the necessary background in algebra. The proof is a concatenation of the following equivalences.

- x is not invertible if and only if the ideal generated by x is proper.
  Indeed if x is not invertible, the ideal generated by x will not contain e. And if x is invertible, then the ideal generated by x will be $\mathcal{A}$ since it will contain $xx^{-1} = e$.

- The ideal generated by x is proper if and only if x is contained in a maximal ideal.
  This follows from Zorn's lemma. Suppose the ideal generated by x is proper. Define a poset of all proper ideals containing x, which is ordered by the inclusion relation. For each chain $I_1 \subseteq I_2 \subseteq ...$ in this poset, define $M = \bigcup_i I_i$. Clearly M is an upper bound for this chain. An M itself is an element of the poset; M is proper because $e \notin I_i$ for all i. M is an ideal and M contains x. By Zorn's lemma the entire poset has a maximal element, which thus is a maximal ideal containing x. The converse is trivial.

- x is contained in a maximal ideal if and only if $\widehat{x}(h) = 0$ for some $h \in C(\sigma(\mathcal{A}))$.
  All maximal ideals are kernels of $h \in C(\sigma(\mathcal{A}))$ and all kernels of $h \in C(\sigma(\mathcal{A}))$ are maximal ideals. To prove the first statement, let $M \subseteq \mathcal{A}$ be a maximal ideal. By the first equivalence M cannot contain invertible elements. So $M \subseteq \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$ and the latter is closed by proposition 4. So $M \subseteq \overline{M} \subseteq A \setminus \mathcal{G}(\mathcal{A})$, hence by maximality of M, $M = \overline{M}$, that is, M is closed.
  Then it is easy to check that $\mathcal{A}/M$ is a Banach algebra with norm $||[x]|| = inf\{||x+m|| \, |m \in M\}$. Let $I \subseteq \mathcal{A}/M$ be an ideal and let $\pi : \mathcal{A} \to \mathcal{A}/M$ be the canonical projection. Then the inverse image $\pi^{-1}(I)$ is an ideal in $\mathcal{A}$. By construction $M \subseteq \pi^{-1}(I) \subseteq \mathcal{A}$ so by maximality of M either $\pi^{-1}(I) = M$, in which case I is just $\{0\}$, or $\pi^{-1}(I) = \mathcal{A}$ in which case $I = \mathcal{A}/M$. So any $y \in \mathcal{A}/M$ has to be invertible, otherwise it would generate a non trivial ideal by the first equivalence above. Thus by proposition 8 we have the isomorphism $\phi : \mathcal{A}/M \to \mathbb{C}$. Now $m \in M$ if and only if $\pi(m) = 0$ if and only if $\phi \circ \pi(m) = 0$, so M is the kernel of $\phi \circ \pi$. Maps in $\sigma(\mathcal{A})$ are uniquely determined by their kernel. Indeed suppose $ker(h) = ker(h')$ then for any $a \in \mathcal{A}$, $a - h(a)e \in ker(h)$. So also $h'(a - h(a)e) = 0$, that is $h'(a) = h(a)$. This proves the first statement.
  To prove the second statement we first note that for any $h \in \sigma(\mathcal{A}$ the kernel of h is a proper ideal since $h(e) = 1 \neq 0$. Then we observe that the kernel of h is a maximal ideal, since adding any element $a \in \mathcal{A} \setminus ker(h)$ to the kernel would give us the entire algebra $\mathcal{A}$. Indeed for $b \in \mathcal{A}$ we have $b = a\frac{h(b)}{h(a)} + (b - a\frac{h(b)}{h(a)})$. And $b - a\frac{h(b)}{h(a)} \in ker(h)$ so indeed $\mathcal{A} = a\mathbb{C} + ker(h)$, that is, $ker(h)$ is a maximal ideal.

□

The following lemma provides a powerful tool the determine whether the Gelfand Transform is isometric.

Lemma 13. *Let $\mathcal{A}$ be a commutative unital Banach algebra and let $x \in \mathcal{A}$. Then $||\widehat{x}||_{sup} = ||x||$ if and only if $||x^{2^k}|| = ||x||^{2^k}$ for all $k \in \mathbb{N}$.*

Proof. Suppose $||\widehat{x}||_{sup} = ||x||$. Then $||x^{2^k}|| \leq ||x||^{2^k} = ||\widehat{x}||_{sup}^{2^k} = ||\widehat{x^{2^k}}||_{sup} \leq ||x^{2^k}||$, where the last inequality is a direct consequence of lemma 12.
Now suppose $||x^{2^k}|| = ||x||^{2^k}$ for all $k \in \mathbb{N}$. We claim $||\widehat{x}||_{sup} = \lim_{k\to\infty} ||x^{2^k}||^{1/2^k} = ||x||$. To prove the claim, we again use lemma 12 to conclude that $||\widehat{x}||_{sup} = \rho(x)$. Then by applying proposition 7 we immediately get the desired result. □

To generalize the concept of complex conjugation to general algebras, we define the involution.

Definition 14. *An* involution *on an algebra $\mathcal{A}$ is an automorphism $(.)^* : \mathcal{A} \to \mathcal{A}$ such that for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$*

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x*, \qquad (xy)^* = y^* x^*, \qquad x^{**} = x. \quad (44)$$

An algebra which has an involution is called a $*$-algebra *and a Banach $*$-algebra in which*

$$||xx^*|| = ||x||^2 \quad (45)$$

holds, is called a C*-algebra.

In C^*-algebras we have $||x^*|| = ||x||$ because $||xx^*|| = ||x||^2 \leq ||x||||x^*||$, so $||x|| \leq ||x^*||$ and thus $||x^*|| \leq ||x^{**}|| = ||x||$.

Definition 15. *Given Banach algebras $\mathcal{A}$ and $\mathcal{B}$. A* homomorphism *ϕ from $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that for all $x, y \in \mathcal{A}$ $\phi(xy) = \phi(x)\phi(y)$.*
A $*$-homomorphism *from $*$-algebra $\mathcal{A}$ to $*$-algebra $\mathcal{B}$ is a homomorphism ϕ such that for all $x \in \mathcal{A}$ $\phi(x^*) = \phi(x)^*$. A $*$-homomorphism that is bijective is called a* $*$-isomorphism.

Lemma 16. *Commutative C^*-algebras are symmetric, i.e. for all x in a commutative C^*-algebra $\widehat{x^*} = \overline{\widehat{x}}$.*

Proof. The proof of this lemma relies on two claims: The first is that a commutative Banach algebra $\mathcal{A}$ is symmetric if and only if for $x \in \mathcal{A}$ we have $x = x^*$ implies $\widehat{x}$ is real valued. Secondly, we claim that the latter indeed holds in commutative C^*-algebras.
To prove the first claim, suppose $\mathcal{A}$ is symmetric and assume $x = x^* \in \mathcal{A}$. Then $\widehat{x} = \widehat{x^*} = \overline{\widehat{x}}$, so $\widehat{x}$ is real. Now suppose $x = x^*$ implies $\widehat{x}$ is real. Given any $y \in \mathcal{A}$, define $u = \frac{1}{2}(y + y^*)$ and $v = \frac{1}{2i}(y - y^*)$. Then $u = u^*$ and $v = v^*$ so both $\widehat{u}$ and $\widehat{v}$ are real. Hence $\widehat{y^*} = \widehat{(u+iv)^*} = \widehat{u} - i\widehat{v} = \overline{\widehat{u} + i\widehat{v}} = \overline{\widehat{y}}$.
To prove the second claim, suppose $x = x^* \in \mathcal{A}$ with $\mathcal{A}$ a commutative C^*-algebra. Let $h \in C(\sigma(\mathcal{A}))$. Then write $\widehat{x}(h) = h(x) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Define $z = x + ite$ with $t \in \mathbb{R}$. Then $h(z) = \alpha + i(\beta + t)$. As a consequence of lemma 12 $|h(z)| \leq ||z||$. So in particular

$$|h(z)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2 \leq ||z||^2 = ||z^* z|| = ||x^2 + t^2 e|| \leq ||x^2|| + t^2, \quad (46)$$

must hold for all t. This forces $\beta = 0$ so $\widehat{x}$ is real valued. □

We are now equipped with everything we need to state and prove the Gelfand-Naimark Theorem.

Stone theory

For the localic version of the Gelfand Duality, we need a slight addition to our terminology.

Definition 17. *A* Stone C* algebra $\mathcal{A}$ *is a* $\mathbb{Q}$*-algebra with multiplicative unit e which has the following properties*

1. *$\mathcal{A}$ has a partial order $\geq$ which satisfies $a^2 \geq 0$ for every $a \in \mathcal{A}$.*
2. *For every $a \in \mathcal{A}$ there exists an integer n such that $ne \geq a$ ($\mathcal{A}$ is Archimedean).*
3. *If $e \geq na$ for all positive integers n, then $0 \geq a$.*
4. *A is complete in the well defined norm*

$$||a|| = \inf\{q \in \mathbb{Q}^+ | qe \geq a \text{ and } qe \geq -a\}. \tag{47}$$

The statement that the norm is well defined follows from $\mathcal{A}$ being Archimedean. One can easily verify that this norm is indeed a proper well defined norm, which satisfies for all $a, b \in \mathcal{A}$ and $q \in \mathbb{Q}$

$$||a + b|| \leq ||a|| + ||b||, \tag{48}$$
$$||ab|| \leq ||a|| \, ||b||, \tag{49}$$
$$||e|| = 1, \tag{50}$$
$$|| - a|| = ||a||, \tag{51}$$
$$||qa|| = q||a||. \tag{52}$$

Note that property (3) makes sure that $||a|| = 0$ implies $a = 0$.

For a maximal ideal M of a Stone C*-algebra $\mathcal{A}$ we can define a function $||.||_M : \mathcal{A} \to \mathbb{R}$ by

$$||a||_M = \inf\{||b|| \, |(a - b) \in M\}, \tag{53}$$

which induces a norm on the quotient ring $\mathcal{A}/M$. Indeed $||1||_M = 1$; $1 - 1 = 0 \in M$, so $||1||_M \leq 1$. Now suppose $||b|| < 1$ and $(1 - b) \in M$. As we have seen $(1 - b)$ is invertible for $||b|| < 1$, so if $(1 - b) \in M$, we must have $M = \mathcal{A}$, contradicting maximality of M. And clearly $||M + a||_M = 0$ if and only if $a \in M$ (here $M + a$ denotes the equivalence class in $\mathcal{A}/M$ represented by a).

Accordingly, we can now define the analogue to the Gelfand transform.

Definition 18. *Let $\mathcal{A}$ be a Stone C*-algebra and $max(\mathcal{A})$ the set of maximal ideals of $\mathcal{A}$. We denote the Hausdorff space of bounded functions from $max(\mathcal{A})$ to $\mathbb{C}$ by $B(max(\mathcal{A}))$. The* Stone Gelfand transform $\gamma_{\mathcal{A}} : \mathcal{A} \to B(max(\mathcal{A}))$ *is defined by*

$$\gamma_{\mathcal{A}}(a) = \hat{a} : max(\mathcal{A}) \to \mathbb{C}, \tag{54}$$

with

$$\hat{a}(M) = M + a. \tag{55}$$

Note that by the Gelfand Mazur theorem (proposition 8), the right hand side of the above equation now is identified with some complex number.

The supnorm $||\hat{a}||_{sup} = \sup\{||a||_M | M \in max(\mathcal{A})\}$ then is bounded by $||a||$ since $0 \in M$ for any $M \in max(\mathcal{A})$, which implies $||a||_M \leq ||a||$. The statement that this bound is attained, i.e. that the Stone Gelfand transform is isometric, is again a consequence of the axiom of choice. But first we need another lemma.

Lemma 19. *Let $\mathcal{A}$ be a Stone C^*-algebra and $a \in \mathcal{A}$ with $||a|| = 1$, then either $e - a$ or $e + a$ is not invertible in A.*

Proof. Suppose both $e + a$ and $e - a$ are invertible, then their product $e - a^2$ is invertible as well. So $a \neq \pm e$. Furthermore, $0 \leq a^2 \leq e||a||^2 = e$ since $||a|| = 1$. So $||a^2|| < 1$, thus $a^2 \leq (1 - \varepsilon)e$ for some real $1 > \varepsilon > 0$. Now the sequence

$$(c)_m = -e \sum_{n=0}^{m} \frac{1 \cdot 3 \cdots (2n-3)}{2^n!} \varepsilon^n, \tag{56}$$

thus forms a Cauchy sequence in $\mathcal{A}$, and its limit equals the square root of $e(1 - \varepsilon)$. As the partial sum is decreasing for increasing m we have $|a| \leq \sqrt{1-\varepsilon}e \leq (1 - \frac{1}{2}\varepsilon)e$ so $||a|| \leq || \, |a| \, || \leq 1 - \frac{1}{2}\varepsilon < 1$, a contradiction. □

3.2 the representation theorems

The Gelfand representation in functional analysis

Theorem 20 (Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then*

$$\mathcal{A} \cong C(\sigma(\mathcal{A})) \tag{57}$$

by the isometric $*$-isomorphism $\Gamma_{\mathcal{A}}$.

Proof. First, we show $\Gamma_{\mathcal{A}}$ is isometric.
Let $x \in \mathcal{A}$, then define $y = x^*x$. We show by induction that $||y^{2^k}|| = ||y||^{2^k}$. Indeed $y = y^*$ which gives $||y^2|| = ||y^*y|| = ||y||^2$. Then $||y^{2^k}|| = ||(y^{2^{k-1}})^*y^{2^{k-1}}|| = ||y^{2^{k-1}}||^2 = ||y||^{2^k}$, using the induction hypothesis in the last equality.
Then using lemma 13 we have $||\widehat{y}||_{sup} = ||y||$. Then the defining property of C^*-algebras, equation (45), gives us that

$$||x^*x|| = ||x||^2 = ||y|| = ||\widehat{y}||_{sup} = ||\widehat{x^*x}||_{sup} = ||(|\widehat{x}|)^2||_{sup} = ||\widehat{x}||^2_{sup}. \tag{58}$$

The second last equality follows from $h(xx^*) = h(x)h(x^*) = h(x)\overline{h(x)} = |h(x)|^2$ for any $h \in \sigma(\mathcal{A})$. Since norms non negative we get $||x|| = ||\widehat{x}||_{sup}$. Hence $\Gamma_{\mathcal{A}}$ is an isometry.

Second, we show that $\Gamma_{\mathcal{A}}$ is a $*$-isomorphism.
By lemma 16 our commutative unital C^*-algebra $\mathcal{A}$ is symmetric. That is $\widehat{x^*} = \overline{\widehat{x}}$, so $\Gamma_{\mathcal{A}}$ is a $*$-homomorphism.
For the injectivity of $\Gamma_{\mathcal{A}}$ suppose $\widehat{x} = \widehat{y}$. Then $||\widehat{x-y}||_{sup} = ||\widehat{x} - \widehat{y}||_{sup} = 0 = ||x - y||$ since the Gelfand transform is isometric. Thus, since $||.||$ is a proper norm, we must have $x - y = 0$, that is $x = y$. Hence $\Gamma_{\mathcal{A}}$ is injective.
Finally, we need to show that $\Gamma_{\mathcal{A}}$ is surjective. This follows from the fact that $Im(\Gamma_{\mathcal{A}})$ is closed, because $\Gamma_{\mathcal{A}}$ is an isometry, and $\Gamma_{\mathcal{A}}(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$. The latter claim about density is proved by application of the Stone-Weierstrass theorem: Indeed, if $x \neq y$ then $\widehat{x} \neq \widehat{y}$, so $\Gamma_{\mathcal{A}}$ separates points; $\Gamma_{\mathcal{A}}(\mathcal{A})$ is closed under complex conjugation because $\mathcal{A}$ is symmetric by lemma 16; And $\Gamma_{\mathcal{A}}(\mathcal{A})$ contains all constants since $\widehat{\lambda e} = \lambda$ for all $\lambda \in \mathbb{C}$. We thus obtain $\Gamma_{\mathcal{A}}(\mathcal{A}) = C(\sigma(\mathcal{A}))$. This concludes the proof. □

Note that the proof relies on the axiom of choice, since in lemma 12 Zorn's lemma is used.

The Gelfand representation in Stone theory

Theorem 21 (Stone-Gelfand-Naimark). *Let $\mathcal{A}$ be a Stone C^*-algebra. Then*

$$\mathcal{A} \cong B(max(\mathcal{A})) \tag{59}$$

by the isometric isomorphism $\gamma_{\mathcal{A}}$.

Proof. Again, we start out by showing that the Stone Gelfand transform $\gamma_{\mathcal{A}}$ is isometric, and injectivity will immediately follow.
Let $a \in \mathcal{A}$ and assume without loss of generality that $||a|| = 1$ (since other cases can be reduced to this one by multiplication with the scalar $1/||a||$). By lemma 19 either $e + a$ or $e - a$ is invertible. By use of Zorn's lemma as in the proof of lemma 12 this noninvertible element generates a proper ideal, which is contained in some maximal ideal M. So $||a||_M = 1$ hence $1 = ||a||_M \leq ||\hat{a}|| \leq ||a|| = 1$. That is $||\hat{a}|| = ||a||$.
Injectivity follows now from the fact that if $\hat{a} = \hat{b}$ then $0 = ||\widehat{a-b}|| = ||a-b||$ and the norm is proper, so $a = b$.
Surjectivity again follows by applying the Stone-Weierstrass theorem: The Stone Gelfand transform separates points and is isometric and $\mathcal{A}$ is complete, so its image is a closed subalgebra of $B(max(\mathcal{A}))$, so it must be the whole of $B(max(\mathcal{A}))$. □

3.3 gelfand duality

These theorems can be formulated as a duality, yielding the well known Gelfand duality. The above results can be extended to a duality by considering the converse of the notion that any continuous map $f : B \to A$ induces a homomorphism $C(A) \to C(B)$ by composition with f.

The duality arises from the fact that any homomorphism $\phi : C(A) \to C(B)$ is induced by a unique continuous map $f : B \to A$: Let $b \in B$ and let M be the maximal ideal containing b. M determines a homomorphism $b^* : C(B) \to \mathbb{R}$ as we have seen. The composition $\beta \circ \phi : C(A) \to \mathbb{R}$ then has the maximal ideal N of $C(A)$ as a kernel, which in turn defines a unique point $a \in A$. So we can define $f : B \to A$ by sending b to $f(b) = a$. Notice that similarly $f(b)$ defines a homomorphism $f(b)^*$. Since for any $g \in C(A)$ we have $g(f(b)) = f(b)^*(g) = b^*(\phi(g)) = \phi(g)(b)$, f indeed defines a homomorphism. Now let $U = g^{-1}(\mathbb{R} - \{0\})$ for $g \in C(A)$. Then $f^{-1}(U) = \phi(g)^{-1}(\mathbb{R} - \{0\})$. So f is continuous, thus establishing the desired result.

4

A CONSTRUCTIVE PROOF

To derive the quantum logic proposed by Landsman, a constructive version of the Gelfand representation theorem is needed, which can be applied internally in the quantum mechanical topos $T(\mathcal{A})$. In the literature, constructive proofs can indeed be found. The two most used proofs are by Banaschewski and Mulvey [14] and by Coquand and Spitters [7]. The proof of Banaschewski and Mulvey is comprehensive, but lengthy, and refers back to numerous articles [15, 16, 17]. In their proof Barr's theorem is used which states that if a geometric statement can be proved from a geometric theory, using classical logic and the axiom of choice, it can also be proved constructively [18]. The proof of Barr's theorem is non-constructive and one might object that this compromises the constructivity of Banaschewski and Mulvey's proof. However, Barr's theorem can be applied outside the topos to some geometric statement, thus providing the existence of a constructive proof of that statement. This constructive proof can then be used internally to prove the statement within the topos.

Regardless, Coquand and Spitters saw the use of Barr's theorem in the proof of Banaschewski and Mulvay as a motivation to give an alternative proof of the Gelfand representation theorem, by reducing the theorem to apply only to real C^*-algebra's as in Stone's version of the Gelfand representation theorem. This reduction allows for a direct and concise proof. However the proof given in [7] is too direct and concise for our taste, so in this thesis we will work out their proof in more detail for the quantum mechanical topos $T(\mathcal{A})$.

We have to start out by defining the necessary concepts such as C^*-algebra's and locales in the topos $T(\mathcal{A})$. From the previous section one can distill the structure of the proof of the representation theorems: The main ingredient is establishing that the Gelfand transform is isometric, this immediately yields injectivity from the norm property. Finally, surjectivity follows from the Stone-Weierstrass theorem. We will use the same structure here.

4.1 INTERNAL SETUP

In the quantum mechanical topos $T(\mathcal{A})$ we have a real numbers object that is isomorphic to the functor R which is defined by sending an object $C \in \mathcal{C}(\mathcal{A})$ to the continuous functions from C to $\mathbb{R}$ because $T(\mathcal{A})$ is a sheaf topos (see theorem 2 of chapter VI in [10]). However, on $\mathcal{C}(\mathcal{A})$ with the Alexandrov topology, any continuous function f to $\mathbb{R}$ is locally constant: Let U be an open of $\mathcal{C}(\mathcal{A})$, that is an upper set of elements of the preorder $\mathcal{C}(\mathcal{A})$. Let $C, D \in U$ with $C \leq D$ and let $V \in \mathbb{R}$ be an open, such that $f(C) \in V$. Since f is continuous, $f^{-1}(V)$ is open and contains C. The smallest open in $\mathcal{C}(\mathcal{A})$ containing C is $\uparrow C = \{C' \in \mathcal{C}(\mathcal{A}) | C \leq C'\}$. Since $C \leq D$ we obtain $f(D) \in V$. For all $\varepsilon > 0$ we can take $V = (f(C) - \varepsilon, f(C) + \varepsilon)$ so that $f(C) - \varepsilon < f(D) < f(C) + \varepsilon$. Thus we conclude $f(D) = f(C)$. Hence, the Dedekind reals object is given by the constant functor $R : C \mapsto \mathbb{R}$.

The terminal object is the constant functor $1 : C \mapsto *$ where $*$ is a singleton. Given an object A of $T(\mathcal{A})$ we can define it to be a commutative *-algebra if there are arrows

$$\begin{aligned} \cdot &: \mathbb{C} \times A \to A \text{ (scalar mulitplication)}, \\ + &: A \times A \to A \text{ (addition)}, \\ \times &: A \times A \to A \text{ (multiplication) and} \\ * &: A \to A \text{ (involution)}, \end{aligned} \tag{60}$$

that satisfy the usual axioms (as in definition 14). The algebra A is unital if it has a unit $1_A : 1 \to A$, with 1 the terminal object, such that $\times \circ (id_A, 1_A) \circ \langle id_A, f \rangle = id_A$, where f is the unique arrow $A \to 1$. 0_A is just the zero constant in A which satisfies a similar condition for addition and multiplication.

Although we are working in a topos, to maintain a clear image of the constructions, from now on we will mostly refer the the contructed (sub)objects as (sub)sets and just denote 1_A as e. To internalize the axioms on the norm and completeness, we start out by defining a seminorm and then formulate a notion of Cauchy sequences and thus obtain a corresponding completeness statement.

We say a *-algebra A is seminormed when we have a map $N : \mathbb{Q}^+ \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the set of subsets of A, which satisfies the conditions for all $p, q \in \mathbb{Q}^+$, $a, b \in A$ and $\lambda \in \mathbb{C}$:

1. $\exists r \in \mathbb{Q}^+ : a \in N(r)$,
2. if $a \in N(p)$ and $b \in N(q)$, then $a + b \in N(p+q)$ and $ab \in N(pq)$,
3. $0 \in N(q)$,
4. if $a \in N(q)$, then $a^* \in N(q)$,
5. if $q > 1$, then $e \in N(q)$,
6. $a \in N(q)$ if and only if $\exists r < q : a \in N(r)$.

Similarly we can define a Cauchy sequence on a seminormed commutative *-algebra A, which is a map $C : \mathbb{N} \to \mathcal{P}(A)$ that satisfies

1. $\forall n \in \mathbb{N} \exists a \in A : a \in C(n)$
2. $\forall k \in \mathbb{N} \exists m \in \mathbb{N} \forall n > m \forall n' > m$, we have $\forall a \in C(n) \forall b \in C(n')$ that $a - b \in C(1/k)$.

This Cauchy sequence is said to converge to an element $b \in A$ if we have

$$\forall k \in \mathbb{N} \exists m \in \mathbb{N} \forall n > m \text{ , any } a \in C(n) \text{ satisfies } a - b \in C(1/k). \tag{61}$$

A seminormed commutative *-algebra A is called complete if every Cauchy sequence converges to a unique element $b \in A$.

The seminorm of such a complete seminormed commutative *-algebra A then is a norm since if for an element $a \in A$ we have $\forall q \in \mathbb{Q}^+ : a \in N(q)$ then by uniqueness of the convergent limits of Cauchy sequences we must have $a = 0$. The norm of an element $a \in A$ is the Dedekind real determined by the upper cut $N^{-1}(\{a\})\{q \in \mathbb{Q}^+ | a \in N(q)\}$ which we shall denote by $||a||$.

We thus obtain the internal definition of commutative C^*-algebra's: a A commutative C^*-algebra in the quantum mechanical topos $T(\mathcal{A})$ is a complete normed commutative *-algebra with the additional condition that $a \in N(q)$ if and only if $a^*a \in N(q^2)$, that is $||a^*a|| = ||a||^2$.

So what object is a commutative C^*-algebra in $T(\mathcal{A})$? The obvious functor to consider is

$$\begin{aligned} A : \mathcal{C}(\mathcal{A}) &\to Sets \\ A(C) &= C \\ A(C \to D) &= C \hookrightarrow D \text{ , the inclusion map.} \end{aligned} \tag{62}$$

Indeed, all operation in A are natural transformations, whose components are defined pointwise by the corresponding operation in each commutative C^*-subalgebra of $\mathcal{A}$. This is also where the commutativity follows from (note that $\mathcal{A}$ in general is not commutative), since all $C \in \mathcal{C}(\mathcal{A})$ are commutative. The unit of A is the natural transformation, whose components are $(1_A)_C = id_C$. The norm is directly inherited from $\mathcal{A}$. And finally, completeness also follows from the completeness of each $C \in \mathcal{A}$: A Cauchy sequence $S : \mathbb{N} \to \mathcal{P}(A)$ describes a sequence $(S(n))_n$ of subobjects of A. As A is a functor, the $S(n)$ are natural transformations with components $C_n := (S(n))_C$ which are just subsets of C. Axiom 2 of the internal definition of Cauchy sequences then gives that any sequence $(c_n)_n$ where c_n is chosen in C_n is a Cauchy sequence in C in the external sense. Externally, we can use the Axiom of countable choice to choose such a sequence, and, since all $C \in \mathcal{C}(\mathcal{A})$ are complete, obtain it's limit $b \in C$. Axiom 2 now insures that this limit is independent on the choice of the sequence and thus provides us with the internal convergence. Thus A is complete.

Let A be a commutative C^* algebra and $A_{sa} = \{a \in A | a^* = a\}$ the set of self adjoint elements of A. So A_{sa} is a complete commutative real Banach algebra. We will prove that A_{sa} is a real C^*-algebra. For this we rely on a theorem by Palmer [19], which states states that a real Banach algebra B is a real C^*-algebra if for all $a, b \in B$ we have $||a||^2 \leq ||a^*a + b^*b||$. We will prove the latter as a proposition, since we will use this result later on as well.

Proposition 22. *For $a, b \in A_{sa}$ we have $||a^2|| \leq ||a^2 + b^2||$.*

Proof. Define $c = a + ib$. Then $c^*c = a^2 + b^2$, so $||a^2 + b^2|| = ||c^*c|| = ||c||^2$, where the last equality is a property of C^*-algebra's. Furthermore, $||a|| = ||\frac{c+c^*}{2}|| \leq \frac{1}{2}(||c|| + ||c^*||) = ||c||$ as $||c^*|| = ||c||$. Finally, $||a^2|| = ||a^*a||$ because a is self adjoint, so $||a^2|| = ||a||^2 \leq ||c||^2 = ||a^2 + b^2||$. □

To conclude this subsection, we will give the internal definition of a locale and introduce the relevant locales for the representation theorem. A Locale L is a lattice such that for each $l \in L$ and $U \subseteq L$

1. L contains $\bigvee U$ and $\bigwedge U$, and
2. $l \wedge \bigvee U = \bigvee\{u \wedge l | u \in U\}$.

A map of locales $f : L \to M$ is a mapping $f^{-1} : M \to L$ that preserves finite meets and arbitrary joins. A point of a locale L is a map of locales $l : 1 \to L$, where 1 has the one point space topology.

A locale is said to be compact provided that for any subset $U \subseteq L$ with $\bigvee U = \top$ there is a finite subset $U' \subseteq U$ such that $\bigvee U' = \top$. On any locale, one can define the binary rather below relation $\lhd$ by writing $k \lhd l$ if and only if there is $m \in L$ such that $k \wedge m = \bot$ and $l \vee m = \top$, with $\bot$ and $\top$ the bottom and top elements of the lattice respectively. The binary completely below relation $\blacktriangleleft$ is defined by writing $k \blacktriangleleft l$ if and only if there is a family m_q indexed by rationals $0 \leq q \leq 1$ such that $m_0 = k$, $m_1 = l$ and $m_p \lhd m_q$ whenever $p < q$.

A locale L is said to be completely regular if for any $l \in L$ we have

$$l = \bigvee\{k \in L | k \blacktriangleleft l\}. \tag{63}$$

The spectrum of a commutative C^*-algebra A is obtained by considering the propositional geometric theory of multiplicative linear functionals [20, 21, 14]. The Lindenbaum locale generated by this theory is said to be the spectrum of the commutative C^*-algebra A and is denoted by A. The elementary propositions of this theory are $a \in (r, s)$ for each $a \in A$ and open rectangle (r, s) of the complex plane. The axioms generating the Lindenbaum locale are

1. $0 \in (r, s)$ if and only if 0, the element of $\mathbb{C}$ lies in (r, s),
2. $1 \in (r, s)$ if and only if 1 lies in (r, s),
3. if $a \in (r, s)$ and $a' \in (r', s')$, then $a + a' \in (r + r', s + s')$,

4. $a \in (r,s)$ implies $\lambda a \in (\lambda r, \lambda s)$ for $\lambda \in \mathbb{C}$,

5. $a \in (r,s)$ implies $a^* \in \overline{(r,s)}$,

6. if $||a|| < 1$ then $a \in (-1,1)$,

7. if $(r,s) \lhd (r',s') \vee (r'',s'')$ and $a \in (r,s)$, then $a \in (r',s')$ or $a \in (r'',s'')$,

8. $a \in (r,s)$ if and only if $\bigvee_{(r',s') \lhd (r,s)} a \in (r',s')$, and

9. if $aa' \in (r,s)$, then $\bigvee_i a \in (r_i,s_i) \wedge a' \in (r'_i,s'_i)$ for every family of pairs $((r_i,s_i),(r'_i,s'_i))_i$ whose product family $((r_i,s_i) \times (r'_i,s'_i))_i$ covers (r,s) in $\mathbb{C}$.

The locale $MFn(A)$ is thus obtained by all finite conjunctions and arbitrary disjunctions of these propositions modulo equivalence, which are ordered by entailment.

The locale that is used to derive the intuitionistic quantum logic [6], is of a similar form, but we will define it from different generators, following the notation of [7]: The locale $MFn(A_{sa})$ generated by symbols $D(a)$ for $a \in A_{sa}$ and relations

1. $D(1) = \top$

2. $D(-a^2) = \bot$

3. $D(a+b) \leq D(a) \vee D(b)$

4. $D(a) \wedge D(-a) = \bot$

5. $D(ab) = (D(a) \wedge D(b)) \vee (D(-a) \wedge D(-b))$

6. $D(a) = \bigvee_{q \in \mathbb{Q}^+} D(a-r)$

for all $a, b \in A_{sa}$. Observe that the multiplicative linear functionals of norm ≤ 1 on A or A_{sa} are models of the above theories, hence they form the points of the corresponding locales. The proposition $D(a)$ can be interpreted as the open set $\{\phi \in MFn | \phi(a) > 0\}$. Notice that since $\mathbb{Q}$ is dense, $MFn(A_{sa})$ is completely regular, by axiom 6.

We wish to show that the locales $MFn(A)$ and $MFn(A_{sa})$ are also compact. To do so, we must first gain more insight in their structure. We interpret the element $a \in (r,s)$ in $MFn(A)$ by the element

$$D(a_1 - r_1) \wedge D(s_1 - a_1) \wedge D(a_2 - r_2) \wedge D(s_2 - a_2) \tag{64}$$

where $a = a_1 + ia_2$ with a_1, a_2 self adjoint, and $(r,s) = (r_1 + ir_2, s_1 + is_2)$. Let F be a model of $MFn(A)$. Then we can assign to this model a multiplicative linear functional f [20], by

$$r < f(a) < s \text{ , whenever } F \models a \in (r,s). \tag{65}$$

Conversely, let f be a multiplicative linear functional. Then we determine a model F by making $a \in (r, s)$ true whenever $r < f(a) < s$. To check whether this correspondence is an isomorphism, we need to observe that axioms 1-5 express that f indeed is a linear functional, axiom 6 demands the norm of the functional to be bounded by 1 and axiom 7 and 8 express the lower and upper cuts are open and have a corresponding Dedekind real. The details are worked out in [21]. As for axiom 9 thus makes $MFn(A)$ a closed sublocale of the dual space A^* of multiplicative linear functionals on A. The above correspondence embeds $MFn(A)$ into the closed unit ball of the dual space A^*. By a constructive version of Alaoglu's [20] theorem, the latter is compact. As $MFn(A)$ is a closed sublocale, it is compact as well.

Now we define the Gelfand transform by sending and element $a \in A_{sa}$ to the map of locales $\widehat{a} : MFn(A_{sa}) \to \mathbb{R}$ defined by

$$\widehat{a}^{-1}(r, s) = D(a - r) \wedge D(s - a). \tag{66}$$

Finally, we need to define a norm on the space $\mathbb{R}(MFn(A_{sa}))$ of maps of locales from $MFn(A_{sa})$ to $\mathbb{R}$. We do so by assigning to $\widehat{a}$ the Dedekind real $||\widehat{a}||$ defined by the upper cut $\{q \in \mathbb{Q} | D(q - a) \wedge D(q + a) = 1\}$. This defines a seminorm. We have to show that this is a proper norm as this is crucial to show that the Gelfand transform is injective and separates points. Fortunately, the bottom element of $\mathbb{R}(MFn(A_{sa})$ is the map $0 : MFn(A_{sa}) \to \mathbb{R}$ defined by 0^{-1} which sends every open interval (r, s) in $\mathbb{R}$ to the bottom element $\perp$ of $MFn(A_{sa})$. Now, we will see later on that for any upper real r, $D(-r) = \perp$ so we have $0 = \widehat{0}$ in $\mathbb{R}(MFn(A_{sa}))$. For any $q \in \mathbb{Q}^+$ we have $D(q - 0) \wedge D(0 + q) = 1$ so indeed $||\widehat{0}|| = 0$. Once we have proven the isometry of the Gelfand transform we have that $||\widehat{a}|| = 0$ implies $||a|| = 0$, which in turn implies $a = 0$. That is all that is needed.

Now we can set out to prove the localic Gelfand respresentation theorem 21 internally with respect to the quantum topos $T(\mathcal{A})$.

4.2 isometry

The approach to showing that the Gelfand transform defined previously is isometric is to define an ordering on A_{sa} with respect to the set of all squares P. Clearly P is closed under multiplication, we show that it is also closed under addition, which we prove together with some other fundamental properties concerning squares.

Proposition 23. *Let $a \in A_{sa}$. If $||1-a|| \leq 1$ then a is a square.*

Proof. We will construct a sequence $(b_n)_n$ in A_{sa} and a sequence $(q_n)_n$ in $\mathbb{Q}$, where $n \in \mathbb{N}$, and prove that the sequence $((b_n)^2)$ converges to a by completeness. The sequence $(b_n)_n$ is constructed by the Taylor series for $\sqrt{a} = \sqrt{1-(1-a)}$ around $1-a=0$. Observe that the Taylor series is given by

$$\sqrt{1-(1-a)} = 1 - \frac{1-a}{2} - \frac{(1-a)^2}{8} - \frac{(1-a)^3}{16} - \frac{5(1-a)^4}{128} - ... \tag{67}$$

So we define $a_0 = 0$ and $a_n = \frac{(1-a)+a_{n-1}^2}{2} = 1 - b_n$, and that we wish to show that the sequence $((1-a_n)^2)_n$ converges to a.

The sequence $(q_n)_n$ in $\mathbb{Q}$ has to estimate the norm of a_n. Observe that we have $||a_n|| \leq (1+||a_{n-1}||^2)/2$. So define $q_0 = 0$ and $q_n = \frac{1+q_{n-1}^2}{2}$. Then by induction we immediately have $||a_n|| \leq q_n$ for all n. Now we have

$$\begin{aligned} (1-a_n)^2 - a &= (1-a+a_n^2) - 2a_n \\ &= 2(a_{n+1}-a_n). \end{aligned} \tag{68}$$

Furthermore, notice that

$$\begin{aligned} 2(a_{n+1}-a_n) &= (1-a) + a_n^2 - (1-a) - a_{n-1}^2 \\ &= a_n^2 - a_{n-1}^2. \end{aligned} \tag{69}$$

So we can show by double induction that $||a_{n+1} \pm a_n|| \leq q_{n+1} \pm q_n$. Indeed for $n=0$ we have $||a_1|| \leq 1/2 = q_1$. Suppose it holds for $n-1$, then we have

$$\begin{aligned} ||a_{n+1}-a_n|| &= ||a_n^2 - a_{n-1}^2||/2 \\ &= ||(a_n+a_{n-1})(a_n-a_{n-1})||/2 \\ &\leq ||a_n+a_{n-1}||\,||a_n-a_{n-1}||/2 \\ &\leq (q_n+q_{n-1})(q_n-q_{n-1})/2 \\ &= (1+q_n^2-(1+q_{n-1}^2)/2 \\ &= q_{n+1}-q_n. \end{aligned} \tag{70}$$

For the first inequality we used the property of Banach algebra's given by equation (33) and the second inequality is the double induction hypothesis.

As a consequence, we have

$$||(1-a_n)^2 - a|| \leq 2(r_{n+1} - r_n) = (1-r_n)^2, \tag{71}$$

which converges to 0 when r_n converges to 1. So we have to show that q_n converges to 1. By induction we have $q_n \leq 1$. So suppose $k \in \mathbb{N}$, then there always are $n \in \mathbb{N}$ such that $q_n \leq 1 - 1/k$. For such n, we have

$$\begin{aligned} 1 - q_{n-1} &= \frac{1-q_n^2}{2} \\ &= (1-q_n)(1+q_n)/2 \\ &\leq (1-q_n)(1-\frac{1}{2k}). \end{aligned} \tag{72}$$

And clearly we have $q_{n-1} \leq q_n$ so we have $1 - qn + 1 \leq (1-\frac{1}{2k})^{n+1}$. So choose $m \in \mathbb{N}$ such that $(1-\frac{1}{2k})^m \leq \frac{1}{k}$, and let $n \geq m$. If $q_n \geq 1 - \frac{1}{k}$ then $1 - q_n \leq \frac{1}{k}$. If $q_n \leq 1 - \frac{1}{k}$, then, by the above $1 - q_n \leq (1-\frac{1}{2k})^n \leq (1-\frac{1}{2k})^m \leq \frac{1}{k}$. Hence q_n indeed converges to 1 and this concludes the proof. □

Proposition 24. *For $a \in A_{sa}$ we have $||a|| \leq 1$ and $||1-a|| \leq 1$ if and only if both x and $1-x$ are squares.*

Proof. Notice that the if part is just proposition 23. For the only if part, suppose a and $1-a$ are squares, say $x = u^2$ and $1 - a = v^2$. Then $u^2 + v^2 = 1$, hence $||a|| = ||u^2|| \leq ||u^2 + v^2|| = 1$ by proposition 22. Similar for $1-a$. □

Proposition 25. *A sum of two squares is a square.*

Proof. Suppose $a, b \in A_{sa}$ are squares. We can assume $||a|| \leq 1$ and $||b|| \leq 1$ by multiplication with a scalar. So by proposition 23 both $1-a$ and $1-b$ are squares. Thus, by proposition 24 $||1-a|| \leq 1$ and $||1-b|| \leq 1$. Hence

$$||1 - \frac{(x+y)}{2}|| \leq \frac{1}{2}(||1-x|| + ||1-y||) \leq 1 \tag{73}$$

Hence, by proposition 23 $\frac{(x+y)}{2}$ is a square. Since $||1-2|| = 1$, 2 is a square as well, so $x+y$ is a square. □

Since squares are closed under multiplication and addition, P is a cone. Observe that for all $q \in \mathbb{Q}^+$ we have $q \in P$: Write $q = m/n$ with $n, m \in \mathbb{N}$. Clearly $1/n^2 \in P$, and since P is closed under addition we can add $1/n^2$ nm times to itself, yielding the element $q \in P$.

The binary relation $\leq$ defined by $a \leq b$ if and only if $b - a \in P$ for $a, b \in A_{sa}$ defines a preorder on A_{sa}. We now define for $q \in \mathbb{Q}$ and $a \in A_{sa}$ that $q \ll a$ if and only if $\exists p > q$ such that $a - p \in P$. Notice that if $0 \ll a$ for some $a \in A_{sa}$, then by multiplication with a scalar we can assume without loss of generality that $1 \leq a$. Furthermore, if $1 \leq a$ then clearly $0 \ll a$. We will now show that this ordering is related to the propositions $D(a)$ in $MFn(A_{sa})$. First, we need a proposition.

Proposition 26. *For $a, b \in A_{sa}$ with $1 \leq ab$ and $0 \leq b$ we have $0 \ll a$*

Proof. Since A is Archimedean, the exists an $N \in \mathbb{N}$, such that $-N \leq a \leq N$ and an $L \in \mathbb{N}$ such that $0 \leq c \leq L$. Define $c = 1 - \frac{b}{L}$, then $0 \leq c \leq 1 - \frac{1}{NL} \leq 1$. So $\frac{1}{L} \leq a\frac{b}{L} = a(1-c)$. As $0 \leq c$ we have $a(1-c) \leq a(1-c)(1+c+c^2+...+c^n) = a(1-c^n)$ for any $n \in \mathbb{N}$. Thus, $\frac{1}{L} - Nc^n \leq \frac{1}{L} + ac^n \leq a$. Now we take n big enough so that $c^n \leq \frac{1}{2NL}$, so $\frac{1}{2L} \leq a$. That is $a - \frac{1}{2L} \in P$, so indeed $0 \ll a$. □

We then have the following proposition

Proposition 27. *Let $a \in A_{sa}$, then $D(a) = \top$ if and only if $0 \ll a$.*

Proof. We start out with the only if part. The proof depends on two claims:

1. If $a \leq b$ for $a, b \in A_{sa}$, then $D(a) \leq D(b)$:
   Indeed, suppose $a \leq b$, that is $b - a \in P$, say $b - a = c^2$. Then $D(a) = D(b - c^2) \leq D(b) \vee D(-c^2)$ by axiom 3 in the axioms generating $MFn(A_{sa})$. So, by axiom 2, $D(a) \leq D(b) \vee \bot = D(b)$.

2. For all $q \in \mathbb{Q}$, if $q > 0$, then $D(q) = \top$:
   Write $q = m/n$ with $m, n \in \mathbb{N}$ we have $\top = D(1) = D(\frac{1}{n} + ... + \frac{1}{n}) \leq D(\frac{1}{n}) \vee ... \vee D(\frac{1}{n}) = D(\frac{1}{n})$ by axiom 3. So $D(\frac{1}{n}) = \top$. By axiom 6 we have $D(m) = \bigvee_{p \in \mathbb{Q}^+} D(m-p) \geq D(1) = \top$ by considering $p = m - 1$, so $D(m) = 1$. We conclude that by axiom 5 we have $D(q) = D(m) \wedge D(\frac{1}{n}) = \top$.

Now suppose $0 \ll a$, so there is a $q \in \mathbb{Q}$ such that $a - q \in P$, which implies $q \leq a$, so by the above $\top = D(q) \leq D(a)$, so $D(a) = \top$.
For the other direction a cut elimination argument is used to prove that $D(b_1) \wedge ... \wedge D(b_n) \leq D(a_1) \vee ... \vee D(a_k)$ if and only if there is a relation $m + p = 0$ where m is in the multiplicative monoid generated by $b_1, ..., b_n$ and p is in $P(-a_1, ..., -a_k)$, that is the mulitplicative and additive closure of $P \cup \{-a_1, ..., -a_k\}$. The proof can be found in [22, 23, 24] but is beyond the scope of this thesis. We only need that $\top = D(a)$ thus gives a relation $m + p = 0$ with $m = 1$ and $p = b - ac$ for $b, c \in P$. Thus $b \geq 0$ and $c \geq 0$, so $ac = 1 + b \geq 1$. Hence by application of proposition 26 we obtain $0 \ll a$. □

Now we can prove

Lemma 28. *For $a \in A_{sa}$ we have $||\widehat{a^2}|| = ||\widehat{a}||^2$.*

Proof. Suppose $||\widehat{a}||^2 < q^2$ for $q \in \mathbb{Q}^+$, so $D(q-a) \wedge D(a+q) = \top$. Then

$$D(q^2 - a^2) = (D(q-a) \wedge D(a+q)) \vee (D(a-q) \wedge D(-q-a)) = \top, \quad (74)$$

by axiom 5. Now, as $q > 0$ we can assume $1 \leq q^2 + a^2$. So $\top = D(1) \leq D(q^2 + a^2)$ for all $q \in \mathbb{Q}^+$. Hence $||\widehat{a^2}|| \leq ||\widehat{a}||^2$.
Now suppose $||\widehat{a^2}|| < q^2$. Then $D(q^2 - a^2) = \top$ which gives $0 \ll q^2 - a^2 =$

$(q-a)(a+q)$. Define $u = a+q$, $v = q-a$, then $\exists s > 0$ such that $uv - s$ is a square. So $uv + u^2 - s$ is a sum of squares, hence also a square, so $0 \ll u(u+v)$ and $u + v = 2q \geq 0$. So by proposition 26 we obtain $0 \ll u$ and similar for v we obtain $0 \ll v$ so by proposition 27 we have $||\widehat{a}||^2 < q^2$, so $||\widehat{a^2}|| \geq ||\widehat{a}||^2$. □

From this result we finally obtain the isometry

Theorem 29. *The Gelfand transform is isometric: For all* $a \in a_{sa}$, $||\widehat{a}|| = ||a||$.

Proof. Suppose $||a|| < q$ for $q \in \mathbb{Q}^+$. Then by proposition 23, $q - a$ is a square unequal to 0 (since $||a|| < q$). Hence $D(q-a) = \top$. Similarly, $||-a|| < q$, so we obtain $D(q-(-a)) = D(q+a) = \top$. We thus have $||\widehat{a}|| < q$. So $||a|| \geq ||\widehat{a}||$. Now suppose $||\widehat{a^2}|| < q$, then $D(q - a^2) = \top$ so by proposition 27, $\exists p > 0$ such that $q - a^2 - p$ is a square. Since the sum of squares is a square, $q - a^2$ is a square, say b^2. Then $a^2 + b^2 = q$, and by proposition 22 we have $||a^2|| \leq ||a^2 + b^2|| = q$. We obtain $||a^2|| \leq ||\widehat{a^2}||$. Since a is self adjoint $||a^2|| = ||a^* a|| = ||a||^2$. Combined with lemma 28 we have $||a||^2 \leq ||\widehat{a}||^2$. □

4.3 STONE-WEIERSTRASS THEOREM

After having obtained the isometry, we immediately obtain injectivity of the Gelfand transform as a consequence of the norm being proper, as we have also seen in the proofs of theorems 20 and 21. What remains to be checked is that the Gelfand transformation is surjective. We follow the previous proofs by applying a version of the Stone-Weierstrass theorem, which admits a constructive proof that can be found in [17]. First we will need a notion of separability.

Let M be some compact, completely regular locale and $\widehat{B}$ be a commutative C^*-subalgebra of $\mathbb{R}(M)$. Then $\widehat{B}$ separates M if each open set U of M can be expressed as

$$U = \bigvee_{\widehat{b} \in \widehat{B} : \widehat{b}^{-1}(0,\infty) \subseteq U} \widehat{b}^{-1}(0,\infty). \tag{75}$$

Theorem 30. *Let M be a compact, completely regular locale. Then any closed C^*-subalgebra of the C^*-algebra $\mathbb{C}(M)$ of maps of locales from M to $\mathbb{C}$, which separates M is equal to $\mathbb{C}(M)$.*

The proof is beyond the scope of this thesis and can be found in [17]. We thus have to show that $\widehat{A_{sa}} := \{\widehat{a} | a \in A_{sa}\}$ separates $MFn(A_{sa})$. Notice that

$$\widehat{a}^{-1}(0,\infty) = \widehat{a}^{-1}\Big(\bigcup_{q \in \mathbb{Q}^+} (0,q)\Big) = \bigvee_{q \in \mathbb{Q}^+} D(a) \wedge D(q-a). \tag{76}$$

Furthermore, by axiom 4,

$$\bigvee_{q \in \mathbb{Q}^+} D(-a) \wedge D(-(q-a)) = D(-a) \wedge D(a) = \bot. \tag{77}$$

So

$$\begin{aligned} \bigvee_{q \in \mathbb{Q}^+} D(a) \wedge D(q-a) &= \bigvee_{q \in \mathbb{Q}^+} (D(a) \wedge D(q-a)) \vee (D(-a) \wedge D(-(q-a))) \\ &= \bigvee_{q \in \mathbb{Q}^+} D(a(q-a)) \\ &\leq \bigvee_{q \in \mathbb{Q}^+} D(aq) \\ &= D(a) \end{aligned} \tag{78}$$

By axiom 5 for the second line, axioms 2 and 3 for the third line and axiom 5 again for the last equality, combined with, which we have already proved, all $q \in \mathbb{Q}^+$ are squares.

If we can argue that the inequality in the third line actually is an equality, we would get the desired result since $MFn(A_{sa})$ is generated by the opens of the form $D(a)$. Then $\widehat{A_{sa}}$ separates $MFn(A_{sa})$. Since the Gelfand transform is isometric, $\widehat{A_{sa}}$ is closed. Thus, we can apply the constructive Stone-Weierstrass theorem to obtain that the Gelfand transform is surjective.

5

DISCUSSION

Before concluding this thesis, we wish to highlight some points of discussion and open problems. Most relevant is the issue concerning the structure of $MFn(A_{sa})$. In the previous section a map is given between the locales $MFn(A)$ and $MFn(A_{sa})$. In [7] it is just stated without proof that this correspondence is an isometry. However, we feel that this statement needs to be checked. The constructive proof of the Gelfand representation theorem uses the isometry to obtain properties of $MFn(A_{sa})$ by relying on known results for $MFn(A)$, for example, the compactness of $MFn(A_{sa})$.

The second point of concern we wish to discuss is one regarding the norm on $\mathbb{R}(MFn(A_{sa}))$. As defined in [7], it is defined only for elements in $\mathbb{R}(MFn(A_{sa}))$ of the form $\widehat{a}$ with $a \in A_{sa}$. At that point, it has not been established that the Gelfand transformation is a surjective mapping, so one might wonder wether the norm is well defined on the whole of $\mathbb{R}(MFn(A_{sa}))$.

Finally, the proof of surjectivity is incomplete in this thesis. There is a usefull version of the Stone-Weierstrass theorem present. But we have not managed to prove that the image of the Gelfand transform $\widehat{A_{sa}}$ indeed separates $MFn(A_{sa})$ and thereby satisfies the conditions of the theorem.

These issues should be further investigated and worked out to provide a complete understanding of the Gelfand representation theorem and its applicability to the internal commutative C^*-algebra of observables.

6

CONCLUSION

In this thesis we have investigated the logical structure of quantum mechanics and have attempted to further develop the understanding of the intuitionistic quantum logic. First, we have set out to study the origin of the quantum logic by Birkhoff and von Neumann in classical mechanics. Inconveniences in this logic, which mainly concern the lacking of a physical interpretation of the logical conjunction and disjunction, have led us to the quantum logic of Landsman.

In the quantum logic of Landsman, elementary propositions now are made up by a pair (C, e), where C is the classical context, and e a projection in C. The logical structure is defined pointwise for each classical context C. Within such a classical context, the projections commute. So, as the lattice of projections in C is Boolean, and thus distributive, the logical disjunction and conjunction are now physically meaningful. Furthermore, this logic is set up inside the quantum mechanical topos $T(\mathcal{A})$, which makes it intuitionistic. Considering the non-deterministic nature of quantum mechanics, one might argue that an intuitionistic quantum logic is actually well suited.

We wish to point out an interesting analogy between this quantum logical structure and the relativistic interpretation of quantum mechanics by Davis [1]. Davis characterizes the real numbers in terms of self adjoint operators on a Hilbert space using Boolean valued models of set theory, which he calls the "Boolean frame of reference". Uncertainty principles are then seen as a consequence of the corresponding operators not being in the same frame. The analogy with the intuitionistic quantum logis is that propositions also have to specify some frame of reference, which in this case comes from the classical context C. Projections in this frame commute and thus can be measured simultaneously.

To derive this logical structure, the Stone-Gelfand-Naimark theorem (better known as the localic Gelfand representation theorem) is used. As we have seen the established proofs of the Gelfand representation theorem and its localic version are non-constructive due to the use of Zorn's lemma. For its internal application, a constructive proof of the localic representation theorem is needed. In 2009, Coquand and Spitters published such a proof, which can indeed be applied in the quantum mechanical topos. As the Gelfand representation theorem is a crucial ingredient in the development of Landsmans quantum logic, we have set out to obtain a thorough understanding of the theorem and its proofs (both constructive and non-construcitve).

In this thesis we have worked out many details of the constructive proof the authors have left out. This has allowed us to come to a better understanding of the proof. However, some issues concerning the proof remain unsolved and are highlighted. We suggest these issues are further investigated in future research. Furthermore, the intiutionistic quantum logic is relatively new and thus leads to numerous interesting topics for future research. For example, one might study the Heisenberg uncertainty relations in this new light, or one can turn to philosophical debates concerning quantum mechanics, where reasonings often use the logic of Birkhoff and von Neumann.

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