Utrecht University

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Bachelor Thesis

Schur–Weyl Duality

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Abstract

The subject of Schur–Weyl Duality relates the fields of differential geometry and algebra as it proposes a correspondence between irreducible representations of both the Lie group $\mathrm{GL}(n, \mathbb{C})$ and the finite symmetric group S_k. The intent of this thesis is to amass the theory needed to understand this result.

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1 Introduction

Within the field of representation theory a representation is a way to identify the elements of any group with the endomorphisms on a linear space in such a way that important properties of the group are preserved. Under a suitable choice for the representation and basis this reduces the study of the group to a problem expressable in terms of matrices, thus allowing for the tools in the well understood field of linear algebra to be used. Under certain conditions the representations of a group can be decomposed into components, the smallest of which are called the irreducible representations. These conditions are met for representations of the classical groups and the symmetric group. Schur–Weyl Duality, named after Issai Schur and Hermann Weyl, then says that the irreducible components of a representation of $\mathrm{GL}(n, \mathbb{C})$ on the k-tensor space $\bigotimes^k \mathbb{C}^n$ are uniquely related to the irreducible representations of S_k on this same space. The goal of this thesis is to precisely state these definitions, the Schur–Weyl Duality Theorem, and then to prove the theorem.

The material in this thesis is roughly divided over three sections. The first two are dedicated to introducing the prerequisites and establishing the results needed in order to prove the theorem, which will be done in the fourth section. The first section can be seen as an independent introduction to the essentials of Lie group representations, assuming prior knowledge of topology and group theory; the others however are intimately related and can not be read seperately.

2 Prerequisites

As Lie groups lie at the heart of all theoretical results presented in this thesis, it is natural to start by working towards defining them. In order to do this, acquaintance with the field of differential geometry is required.

2.1 Differential Geometry

One of the main objects studied in differential geometry are smooth manifolds, which are topological manifolds equipped with a smooth structure. These structures allow diffeomorphisms between manifolds to be defined. Diffeomorphisms fill the same role homeomorphisms do in topology, meaning we can treat diffeomorphic objects as the same. As Lie groups are smooth manifolds with an additional group structure the theory in this subsection is essential for understanding them.

Definition 2.1. Let $(M, \mathcal{T})$ be a second-countable Hausdorff space. The space M is said to be an n-dimensional topological manifold if for every $p \in M$ there exists an open subset $U \in \mathcal{T}$ such that $p \in U$ and U is homeomorphic to $\mathbb{R}^n$.

Note that the last condition is equal to the requirement that there exists a homeomorphism $\varphi : U \to \mathbb{R}^n$. Every such pair (U, φ) is called a coordinate chart on M. These coordinate charts, or rather the yet to be introduced transition maps between them, can be classified based on the properties of their derivatives.

Definition 2.2. A function F between Euclidean spaces is of the class C^k for $k \in \mathbb{N}$ if every component function of F has continuous partial derivatives up to the k-th order. If this condition is met, this property is expressed as $F \in C^k$.

Continuous function are said to be of the class C^0. Another special instance is the class C^∞, which is defined as $C^\infty = \cap_{k \in \mathbb{N}} C^k$. If a function belongs to this class it is called smooth. The functions in this class give rise to the concept of a diffeomorphism.

Definition 2.3. A diffeomorphism is a bijective function $F \in C^\infty$ between open subsets of Euclidean spaces that admits an inverse $F^{-1} \in C^\infty$.

This definition encapsulates homeomorphisms. Every diffeomorphism is also a homeomorphism per definition, meaning that for spaces (equipped with a suitable structure) to be diffeomorphic is a stronger condition than them being homeomorphic. It is of importance to realise that at this point only smooth maps and diffeomorphisms with real domain and co-domain have been defined. These definitions will later be extended to their corresponding maps from and to manifolds.

The transition maps mentioned before can now be introduced and classified.

Definition 2.4. Let (U, φ) and (V, ψ) be two coordinate charts on an n-dimensional topological manifold. If the intersection $W = U \cap V$ is non-empty, the composition $\psi \circ \varphi^{-1} : \ \varphi(W) \to \psi(W)$ is called the transition map from φ to ψ.

Because any topological manifold has the assumption that all coordinate charts are homeomorphisms, all transition maps are automatically C^0-maps. In the same vein as the diffeomorphism, a stronger condition would be to require all transition maps to be C^∞-maps. This requirement defines the final key component for the definition of a smooth manifold.

Definition 2.5. A collection $\mathcal{A}$ of coordinate charts is called an atlas for the topological manifold M if for every $p \in M$ there exists a coordinate chart $(U, \varphi) \in \mathcal{A}$ so that p is in the domain of φ. The atlas is referred to as a smooth atlas if all transition maps between the coordinate charts are of the class C^∞.

Equipping a topological manifold with a smooth atlas is the final step to defining a smooth manifold.

Definition 2.6. A smooth manifold is a triple $(M, \mathcal{T}, \mathcal{A})$, consisting of a topological manifold $(M, \mathcal{T})$ extended to include a smooth atlas $\mathcal{A}$.

Although this is the only named type of manifold that will be treated extensively in this section, there are others. They come forth from alternative restrictions on the atlas the manifold is equipped with. Examples include C^k-manifolds defined in the obvious way, or analytic manifolds brought on by restricting the transition maps to C^ω. In this last case ω refers to all maps being real analytic, rather than being k-times continuously differentiable as before. A complex manifold requires a complex analytic structure in the same way, after identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$.

The first example of a smooth manifold, and in fact a Lie group, will be the matrix group $\mathrm{GL}(n,\mathbb{R})$. To show that it is a smooth manifold requires the following definition of an open submanifold.

Definition 2.7. An open submanifold N of a smooth manifold M is an open subset of M equipped with the atlas $\mathcal{A}_N = \{(U,\varphi) \in \mathcal{A} \mid U \cap N \neq \varnothing\}$. Per restriction it is itself a topological manifold and a smooth manifold.

Lemma 2.8. *The group* $\mathrm{GL}(n,\mathbb{R})$ *possesses the structure of a smooth manifold.*

Proof. Define $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n,\mathbb{R}) \mid \det^{-1}(A) \neq 0\}$. As the determinant function is continuous as a map from $\mathrm{M}(n,\mathbb{R})$ to $\mathbb{R}$, $\mathrm{GL}(n,\mathbb{R})$ is defined as the pre-image of an open set in $\mathbb{R}$ and is therefore itself open in $\mathrm{M}(n,\mathbb{R})$. Because there is a natural diffeomorphism between $\mathrm{M}(n,\mathbb{R})$ and $\mathbb{R}^{n^2}$, $\mathrm{GL}(n,\mathbb{R})$ can be seen as an open subset of the latter, thus proving it is an n^2-dimensional smooth submanifold. □

This proof draws from the concept of a diffeomorphism between manifolds. This usage is justified and will follow from the extension of diffeomorphisms to manifolds later in this subsection. As a last remark on this lemma, $\mathrm{GL}(n,\mathbb{C})$ can be proven to be a (complex) manifold in the same manner.

Next, through the following definitions the domains and co-domains of C^k-maps (and therefore diffeomorphisms) will be extended to allow for manifolds.

Definition 2.9. A map $F : M \to N$ between two arbitrary manifolds is called smooth if for all $p \in M$ there exist smooth coordinate charts $(U,\varphi) \in \mathcal{A}_M$ containing p, and $(V,\psi) \in \mathcal{A}_N$ containing $F(p)$, such that $F(U) \subseteq V$ and the composite map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth as a map between open subsets of Euclidean spaces.

Definition 2.10. A diffeomorphism between two manifolds M and N is a bijective map $F : M \to N$ so that both F and its inverse F^{-1} are C^∞-maps. If such a map exists M and N are called diffeomorphic, written $M \cong N$.

Parallel to submanifolds in $\mathbb{R}^n$, smooth manifolds admit tangent spaces. These tangent spaces will be defined abstractly through maps called derivations and will naturally lead to an analogous notion of the derivative on manifolds, called the differential. The reason for this construction is that unlike the aforementioned objects, manifolds need not be embedded in an ambient space.

Definition 2.11. For a smooth manifold M and an arbitrary point $p \in M$, a linear map $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at the point p if it satisfies $v(fg) = f(p)v(g) + g(p)v(f)$ for all $f, g \in C^\infty(M)$. The vector space of all such derivations is called the tangent space to M at the point p, denoted T_pM. The derivations in this set are also referred to as tangent vectors at p.

The derivations in a tangent space can be identified with tangent vectors as it can be shown they correspond to directional derivatives. Such an approach is outlined in ([4], ch. 3). The definition of the differential of a smooth map sprouts from the above definition. Although the definition is abstract, it can be thought of as a coordinate-free generalisation of the known Jacobian matrix.

Definition 2.12. If F is a map between two smooth manifolds M and N, the differential of F at p is defined as the map $dF_p : T_pM \to T_{F(p)}N$ that, given a tangent vector $v \in T_pM$, acts on a function $f \in C^\infty(N)$ by $dF_p(v)(f) = v(f \circ F)$.

The differential has the following properties.

Proposition 2.13. *Let M, N and P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear.*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.*
3. *$d(Id_M)_p = Id_{T_pM} : T_pM \to T_pM$.*
4. *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

The above properties, and the following two propositions, were taken (save for the omission of manifolds with boundary) verbatim from ([4], pg. 55-57). Their proofs can be found on these same pages.

Proposition 2.14. *Let M be a smooth manifold, let $U \subseteq M$ be an open subset, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.*

Proposition 2.15. *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is n-dimensional.*

In the above definitions all derivations were taken at points, therefore leading to tangent spaces at different points. These tangent spaces can be grouped together in the tangent bundle to the smooth manifold M, defined as a disjoint union as follows:

$$TM = \coprod_{p \in M} T_pM.$$

It will be stated without proof that if M is an n-dimensional smooth manifold, the tangent bundle TM can be equipped with a smooth structure so that it becomes a $2n$-dimensional smooth manifold itself. The differential on each tangent space extends naturally to the global differential on the tangent bundle. The global differential, within the same context as before, is defined to be the map $dF : TM \to TN$ so that $dF|_{T_pM} = dF_p$. It behaves in the same way as the points (2)-(4) in the above proposition for differentials prescribe.

In order to be able to properly define Lie subgroups in the next subsection, immersed and embedded submanifolds need to be defined. The following definitions will effectuate this.

Definition 2.16. A smooth map $F : M \to N$ between two smooth manifolds M and N is called a smooth immersion if its differential is injective at each point. If F is also a topological embedding, F is said to be a smooth embedding of M into N.

Definition 2.17. A subset S of a smooth manifold M is called an immersed submanifold if it is equipped with a topology and a smooth atlas with respect to which S is a topological manifold and the inclusion map $\iota_S : S \hookrightarrow M$ is a smooth immersion. If the topology coincides with the subspace topology and ι is a smooth embedding, S is called an embedded submanifold.

In the following, final stage of this subsection the necessary knowledge and tools for defining Lie algebras and the exponential map are given.

Definition 2.18. For $I \subseteq \mathbb{R}$ an interval, a curve on a manifold M is a continuous map $\gamma : I \to M$.

In the following definition $\frac{d}{dt}|_s$ is the basis element that spans $T_s\mathbb{R}$.

Definition 2.19. Let γ be a smooth curve on M. The velocity of γ at the point s is given by $\gamma'(s) = d\gamma(\frac{d}{dt}|_s) \in T_{\gamma(s)}M$.

As $\gamma'(s)$ is a tangent vector, it is a derivation, and it acts on a function $f \in C^\infty(M)$ by:

$$\gamma'(s)f = d\gamma(\tfrac{d}{dt}|_s)f = \tfrac{d}{dt}|_s(f \circ \gamma) = (f \circ \gamma)'(s),$$

following from the definition of the differential. Considering the right hand expression, the action of this derivation is seen as taking the derivative of the function f along the curve γ.

It is possible to reformulate the definition of the tangent bundle to a manifold in terms of curves, and it is sometimes preferrable to make use of this construction. This subsection will refrain from this formulation, but it can be found in ([4], pg. 72).

Definition 2.20. Let M be a smooth manifold. A vector field on M is a continuous map $X : M \to TM,\ X : p \mapsto X_p$ so that $X_p \in T_pM$ for every $p \in M$. If X is smooth as a map from M to TM, it is called a smooth vector field.

The set of all smooth vector fields on a smooth manifold M is denoted $\mathfrak{X}(M)$, and forms a vector space under pointwise addition and scalar multiplication.

If S is an immersed or embedded submanifold of M, and p is a point on S, a vector field X on M is tangent to S at p if $X_p \in T_pS$, where T_pS is identified with a subspace of T_pM by the isomorphism described in Proposition 2.14. The vector field X is said to be tangent to S if it is tangent to S at every point of S. These definitions guarantee that if X is tangent to S, its restriction to S is a vector field on S.

Definition 2.21. For X a vector field on a smooth manifold M, an integral curve of X is a differentiable curve $\gamma : I \to \mathbb{R}$ so that its velocity at each point $s \in I$ equals the value of X at $\gamma(s)$, in formula: $\gamma'(s) = X_{\gamma(s)}$.

This definition is the final concept needed from the field of differential geometry.

2.2 Lie Groups and Lie Theory

Every Lie group is in particular a topological group. Like the previous subsection started with topological manifolds, this subsection will start with the definition of a topological group.

Definition 2.22. A topological group $(H, \mathcal{T})$ is a group H endowed with a topology $\mathcal{T}$ so that both the group operation and inverse group operation are continuous with respect to $\mathcal{T}$.

The definition of a Lie group is a rephrasing of this definition with stronger conditions. It requires H to be a smooth manifold on top of being a group, and both operations to be smooth.

Definition 2.23. A Lie group is a smooth manifold with a group structure so that the group operation and inverse group operation are smooth.

Lemma 2.24. *If G is a smooth manifold with a group operation, and the map $\chi : G \times G \to G,\ \chi : (g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Proof. Let $\chi^*(g) = \chi(e, g) = g^{-1}$ denote the restriction of χ to $\{e\} \times G$. It is smooth as a restriction of a smooth map, therefore the inverse group operation is smooth. Then as a composition $\chi(g, \chi^*(h)) = \chi(g, h^{-1}) = g(h^{-1})^{-1} = gh$ is smooth as well. Hence G is a Lie group. □

For G a Lie group, and g an element of G, two smooth maps can be defined naturally. These maps are the left and right translations, both diffeomorphisms as maps from G to itself:

$$L_g(h) = gh,\ R_g(h) = hg.$$

As before, an example will be given by $\mathrm{GL}(n, \mathbb{R})$.

Lemma 2.25. *The group* $\mathrm{GL}(n, \mathbb{R})$ *is a Lie group.*

Proof. It is known that $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold. This leaves to be proven that the inversion and multiplication of the matrices in this group are smooth operations. Because all matrices $A, B \in \mathrm{GL}(n, \mathbb{R})$ have a finite number of entries, each entry in the matrix product AB is a polynomial with a finite number of terms depending on the entries of A and B. As a polynomial, it is smooth. The same argument holds for the inverse of A, by Cramer's rule. □

Corollary 2.26. *The group* $\mathrm{GL}(n, \mathbb{C})$ *is a Lie group.*

Proof. Every complex matrix can be split into a real and imaginary component. It then follows from the above reasoning applied to each component that $\mathrm{GL}(n, \mathbb{C})$ is a Lie group. $\square$

In the line of expectation, Lie groups can admit Lie subgroups. Furthermore there are analogues to the group homomorphisms and isomorphims known from finite group theory.

Definition 2.27. If G is a Lie group, a subgroup H of G is called a Lie subgroup of G if H allows the structure of a Lie group such that the inclusion map $\iota : H \to G$ is an immersion.

Definition 2.28. If G and H are Lie groups, a Lie group homomorphism from G to H is a smooth map $F : G \to H$ that is also a group homomorphism. If in addition F is a diffeomorphism, it is called a Lie group isomorphism.

Lie groups are often used to model symmetry. For instance, the circle group in $\mathbb{C}$ can be seen as a Lie group that encodes rotations in the plane. These symmetries are applied to sets and spaces through a Lie group action.

Definition 2.29. A left group action of a group G on a set K is a map $G \times K \to K$, $(g, k) \mapsto g \cdot k$ so that for all $g, h \in G$ and $k \in K$, $g \cdot (h \cdot k) = (gh) \cdot k$ and $e \cdot k = k$. A right action is defined in a similar way by a map $K \times G \to K$, $(k, g) \mapsto k \cdot g$.

If G is a topological group and K is a topological manifold, an action is called continuous if the defining map is continuous. If G is a Lie group and K is a smooth manifold, an action is called smooth if the defining map is smooth. Under either of these circumstances K is called a left or right G-space.

Example 2.30. The vector space $\mathbb{C}^n$ is a natural $\mathrm{GL}(n, \mathbb{C})$-space through matrix multiplication.

Next assume M is a smooth manifold, U is an open subset of M and $f : U \to \mathbb{R}$ is a smooth function. Every smooth vector field $X \in \mathfrak{X}(M)$ defines a new function $Xf : U \to \mathbb{R}$ by $(Xf)(p) = X_p f$ which is again smooth. In the same way constructing $YXf = Y(Xf)$ yields another smooth function. This allows the following definition to be stated.

Definition 2.31. Let M be a smooth manifold, and $X, Y \in \mathfrak{X}(M)$. The Lie bracket of X and Y is the unique vector field $[X, Y]$ whose associated derivation is the commutator.

The Lie bracket has the following properties for $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$:

1. $[X, Y]_p f = X_p(Yf) - Y_p(Xf)$.
2. It is bilinear.
3. It satisfies Jacobi's identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
4. $[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X$.

Restricted to only the left-invariant vector fields on a Lie group, the Lie bracket defines its Lie algebra. The definitions and propositions needed are as follows.

Definition 2.32. Let L_g denote left translation on G, for every $g \in G$. Then a vector field X on G is left-invariant if for all $h \in G$, $d(L_g)_h(X_h) = X_{gh}$.

Proposition 2.33. *If G is a Lie group, and X and Y are smooth left-invariant vector fields on G, then $[X, Y]$ is also left-invariant.*

The proof of this proposition can be found in ([4], pg. 189).

Definition 2.34. A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, antisymmetric and satisfies Jacobi's identity.

Definition 2.35. A Lie subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ that is closed under the bracket. Then $\mathfrak{h}$ with the restricted bracket is itself a Lie algebra.

Definition 2.36. If G is a Lie group the set $\mathfrak{L}(G)$ of all smooth left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$ and therefore a Lie algebra. It is called the Lie algebra of G.

The Lie algebra $\mathfrak{g}$ of a Lie group G is tied to the tangent space at the unit element of G, and has the same dimension as G. This follows from the following theorem of which a proof can be found in ([4], pg. 191).

Theorem 2.37. *Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. The evaluation map $\epsilon : \mathfrak{g} \to T_eG$, given by $\epsilon(X) = X_e$, is a vector space isomorphism. Thus $\mathfrak{g}$ is finite-dimensional, with dimension equal to* $\dim(G)$.

There is one more important result involving Lie subalgebras. Although the exponential map $\exp(\cdot)$ is not yet defined, it is useful to bundle these two characterisations.

Theorem 2.38. *Let G be a Lie group, and H a Lie subgroup of G. Then the Lie algebra $\mathfrak{h}$ of H can be characterised as a subset of the Lie algebra $\mathfrak{g}$ of G in the following two ways.*

1. $\mathfrak{h} = \{X \in \mathfrak{g} \mid X_e \in T_eH\}$.
2. $\mathfrak{h} = \{X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H\}$.

The proof follows from ([4], pg. 197, 521). Before moving on to defining the exponential map the treatment of general Lie algebras is concluded with the following definition.

Like for Lie groups, there are homomorphisms and isomorphisms between Lie algebras. They are aptly called Lie algebra homomorphisms and isomorphisms.

Definition 2.39. A Lie algebra homomorphism between the Lie algebras $\mathfrak{g}, \mathfrak{h}$ is a linear map $A : \mathfrak{g} \to \mathfrak{h}$ that preserves the bracket in the sense that $A[X, Y] = [AX, AY]$. If a Lie algebra homomorphism is invertible, it is called a Lie algebra isomorphism.

The exponential map is a map from the Lie algebra of a Lie group into the Lie group itself. It maps smooth vector fields in the Lie algebra to the one-parameter subgroups generated by these vector fields, or equivalently, specific integral curves on G. These concepts will be made precise in the last part of this subsection.

Definition 2.40. A one-parameter subgroup of a Lie group G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$ where $\mathbb{R}$ is considered a Lie group under addition.

It can be shown that the one-parameter subgroups of G are exactly the maximal integral curves with initial point e of the vector fields in $\mathfrak{L}(G)$. This is done in ([4], ch. 20). The one-parameter subgroup brought forth from an $X \in \mathfrak{L}(G)$ under the correspondence described is called the one-parameter subgroup generated by X.

Definition 2.41. Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. The exponential map is defined as $\exp : \mathfrak{g} \to G$, $X \mapsto \gamma(1)$, where γ is the one-parameter subgroup generated by X.

The name of this map is derived from the fact that the over $\mathfrak{gl}(n, \mathbb{R})$ the exponential map is given by the matrix exponential.

Proposition 2.42. *For any* $A \in \mathfrak{gl}(n, \mathbb{R})$ *let:*

$$e^A = \textstyle\sum_{k=1}^{\infty} \frac{1}{k!} A^k.$$

This series converges to an invertible matrix $e^A \in \mathrm{GL}(n, \mathbb{R})$*, and the one-parameter subgroup of* $\mathrm{GL}(n, \mathbb{R})$ *generated by* A *is* $\gamma(t) = e^{tA}$*.*

The proof of this proposition is given in ([4], pg. 517).

Let $X \in \mathfrak{g}$, $s, t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then the exponential map has the following properties:

1. The exponential map is smooth.
2. $\exp{(s+t)X} = \exp{(sX)}\exp{(tX)}$.
3. $\exp{(X)}^n = \exp{(nX)}$.
4. $\exp{(X)}^{-1} = \exp{(-X)}$.

The final result presented in this subsection is the Closed Subgroup Theorem, which grants a relatively easy way to determine if a subgroup of a Lie group is a Lie subgroup. For a proof the reader is referred to ([4], pg. 523).

Theorem 2.43. *Let* G *be a Lie group, and let* H *be a subgroup of* G *that is also a closed subset of* G*. Then* H *is an embedded Lie subgroup of* G*.*

2.3 Representation Theory

Representations of Lie groups are yet one step closer to the theory discussed in the following sections of this thesis. In this subsection they will be defined and briefly discussed, concluding the prerequisites section.

A preliminary result that will later be needed to extend the study of representations to characters will be given before giving the definition of a representation.

Proposition 2.44. *Let G be a Lie group, and $C_c(G)$ the set of compactly supported functions from G to $\mathbb{C}$. There exists a complex linear map $I : C_c(G) \to \mathbb{C}$ which satisfies the following properties:*

1. *If f is real valued then so is $I(f)$; if $f \geq 0$ then $I(f) \geq 0$.*
2. *If $f \geq 0$ and $I(f) = 0$ then $f = 0$.*
3. *For every $y \in G$: $I(f(y\cdot)) = I(f)$.*

If J is a linear map satisfying these same properties, there exists a unique scalar $c > 0$ so that $J = cI$. The map $I(f)$ is written $\int_G f(x)dx$. The measure used is the left Haar measure, which is not treated in this thesis.

The proof of this result can be found in ([1], ch. 19).

Definition 2.45. Let V be a vector space. A continuous representation of a Lie group G is a pair (π, V) of a continuous left action $\pi : G \times V \to V$ and a G-space V such that for every $g \in G$, $\pi(g) \in \mathrm{End}(V)$. If V is a finite dimensional space the representation is called finite dimensional.

The space V is called a G-module. If the group G can be seen as a subset of $\mathrm{End}(V)$ the defining representation of G acts by itself.

Representations are not restricted to Lie groups. They can also be defined on Lie algebras. In particular, a representation of a Lie group gives rise to a representation of its Lie algebra.

Definition 2.46. Let (π, V) be a continuous representation of the Lie group G. Then $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$, the tangent map of π at e, defines a representation of $\mathfrak{g}$.

There is one lemma regarding the representations of Lie algebras that will be useful for this thesis. Its proof can be found in ([1], pg. 100).

Lemma 2.47. *If G is a connected Lie group, V is a G-module and W is a linear subspace of G, then W is G-invariant if and only if it is $\mathfrak{g}$-invariant.*

There are specific representations that are often of interest called irreducible representations. They are defined by their invariance under the left action defined by the representation. Irreducible in this context means they cannot be decomposed into smaller components. The precise definition is as follows.

Definition 2.48. Let (π, V) be a representation of G. An invariant subspace W of V is a linear subspace of V such that for all $g \in G$, $\pi(g)W \subseteq W$. The representation is called irreducible if the only invariant subspaces of V are $\{0\}$ and V itself.

Representations can be further specialised.

Definition 2.49. If (π, V) is a representation of a Lie group G, and an inner product on V is given, then π is called unitary if $\pi(g)$ is unitary with respect to this inner product for all $g \in G$. If no such inner product is given, but it is known it must exist, then π is called unitarisable.

The following proposition and lemma will be given without proof. Instead, a reference to the syllabus on Lie Groups by Erik van den Ban that does these proofs justice will be given.

Proposition 2.50. *If G is a compact Lie group and (π, V) is a continuous finite dimensional representation of G, then π is unitarisable.* ([1], pg. 72)

Lemma 2.51. *Let (π, V) be a unitary representation of the Lie group G. If W is an invariant subspace for π, then its orthocomplement $W^\perp$ is a closed invariant subspace for π. If W is closed, then $V = W \oplus W^\perp$.* ([1], pg. 73)

Corollary 2.52. *Let (π, V) be a continuous finite dimensional representation of a Lie group G. If π is unitarisable, the representation decomposes as a finite direct sum of irreducibles. In other words, there exists a direct sum decomposition $\oplus_{i=1}^n V_j$ of V into invariant subspaces V_j such that the representation $\pi|_{V_j}$ defined by the restriction of π to V_j is irreducible.*

Proof. Let $V_1 \subset V$ be a G-invariant subspace of V. Then by Lemma 2.51, $V = V_1 \oplus V_1^\perp$. Repeat this procedure for $V_2 \subset V_1^\perp$, and keep repeating it for any following V_j until no invariant strict subspaces other than $\{0\}$ exist in any of the V_j. Then $V = \oplus_{i=1}^n V_i$ for an $n \in \mathbb{N}$. □

Corollary 2.53. *Let (π, V) be a continuous finite dimensional representation of a compact Lie group. Then π admits a decomposition as a finite direct sum of irreducible representations.*

Proof. Because G is compact, the representation π is unitarisable. Because all conditions for application of the above corollary have now been met, the representation can be decomposed as described above. □

It is worthy of note that the action of a continuous finite dimensional representation π on a space can be expressed as a matrix. This will often be done from this point onwards.

Definition 2.54. If (π_j, V_j) for $j = 1, 2$ are two continuous representations of G, and the V_j are locally convex spaces, then a continuous linear map $T : V_1 \to V_2$ is said to be equivariant or intertwining if for every $g \in G$, $T \circ \pi_1(g) = \pi_2 \circ T(g)$. The two representations are said to be equivalent if there exists a topological linear isomorphism from V_1 onto V_2 which is equivariant. This is denoted as $\pi_1 \sim \pi_2$.

The space of all G-intertwining maps from V_1 to V_2, both of finite dimension, will be denoted by $\mathrm{Hom}_G(V_1, V_2)$. If $V_1 = V_2$, it is denoted $\mathrm{End}_G(V_1)$. If the maps are not required to be G-intertwining the subscript G is omitted in both cases. If a map in $\mathrm{Hom}_G(V_1, V_2)$ is also an isomorphism, it is called a G-module isomorphism and V_1 and V_2 are isomorphic as G-modules. Furthermore all invertible linear maps in the space of endomorphisms of V_1 will be referred to as $\mathrm{GL}(V_1)$.

The following lemma is a major result in representation theory. It is named Schur's Lemma.

Lemma 2.55. *Let (π, V) be a finite dimensional representation of a group G. Then the following holds.*

a. *If π is irreducible, then every G-intertwining endomorphism of V acts on V by scalar multiplication. In other words,* $\mathrm{End}_G(V) = \mathbb{C}I_V$.

b. *Conversely, if π is unitarisable and* $\mathrm{End}_G(V) = \mathbb{C}I_V$, *then π is irreducible.*

Its proof can be found in any text on representation theory. One such source is again ([1], pg. 75). Using the following lemma it can be rephrased in a different form.

Lemma 2.56. *Let V be a finite dimensional complex linear space and let $A, B \in \mathrm{End}(V)$ be two commuting elements. Then A leaves* $\ker(B), \mathrm{Im}(B)$ *and all eigenspaces of B invariant.*

Proof. The proof follows by applying elementary knowledge of linear algebra. If $x \in \ker(B)$, $Bx = 0$. Then $B(Ax) = BAx = ABx = A0 = 0$ so $Ax \in$

$\ker(B)$. If $x \in \mathrm{Im}(B)$ there exists a vector y in the domain of B so that $By = x$. Then $Ax = A(By) = ABy = BAy = B(Ay) \in \mathrm{Im}(B)$. Finally, if x is an eigenvector of B for the eigenvalue λ, $Bx = \lambda x$. Then $B(Ax) = BAx = ABx = A\lambda x = \lambda(Ax)$ so Ax is an element of the same eigenspace. $\square$

Lemma 2.57. *Let (ρ, V) and (π, W) be two irreducible finite dimensional representations of a group G. If $\rho \sim \pi$,* $\dim \mathrm{Hom}_G(V, W) = 1$ *and if $\rho \nsim \pi$,* $\dim \mathrm{Hom}_G(V, W) = 0$.

Proof. First assume $\rho \sim \pi$. Then per definition of equivalence there must exist a G-module isomorphism $T : V \to W$ in $\mathrm{Hom}_G(V, W)$. Let $T_0 \in \mathrm{Hom}_G(V, W)$ be any map. Then $T_0^{-1}T \in \mathrm{End}_G(V)$ and by Schur's Lemma $T_0^{-1}T = cI$ for a $c \in \mathbb{C}$. Then $T = cT_0$, thus $\dim \mathrm{Hom}_G(V, W) = 1$.
Next assume $\rho \nsim \pi$, and let $T \in \mathrm{Hom}_G(V, W)$. Per the lemma above both $\mathrm{Ker}(T) \subset V$ and $\mathrm{Im}(T) \subset W$ are G-invariant. Because of the assumed irreducibility of ρ and π, $\mathrm{Ker}(T) = \{0\}$ or V and $\mathrm{Im}(T) = \{0\}$ or W. If $\mathrm{Ker}(T) = U$, $T = 0$. If $\mathrm{Ker}(T) = \{0\}$, T is injective. Then $\mathrm{Im}(T) \neq \{0\}$ so $\mathrm{Im}(T) = V$. Then it must follow that T is a G-module isomorphism and $\rho \sim \pi$ which is a contradiction, so $\dim \mathrm{Hom}_G(V, W) = 0$. $\square$

The following definition is a powerful characterisation of a representation. The definition is possible because as stated before, every action of π can be described by a matrix.

Definition 2.58. Let (V, π) be a finite dimensional representation of a group G. The function $\chi_\pi : G \to \mathbb{C}$, $\chi_\pi : g \mapsto \mathrm{tr}\ \pi(g)$ is called the character of π.

The following lemmas will bare some of the important properties of these characters.

Lemma 2.59. *If two finite dimensional representations π, ρ of a group G are equivalent, they have the same character.*

Proof. Recall that the two representations are equivalent if there exists a topological linear isomorphism $T : V_\pi \to V_\rho$ so that $T \circ \pi = \rho \circ T$. Because T is an isomorphism its inverse is defined and it follows that $\rho = T \circ \pi \circ T^{-1}$. Then $\chi_\rho(g) = \mathrm{tr}\ \rho(g) = \mathrm{tr}\ (T \circ \pi(g) \circ T^{-1}) = \mathrm{tr}\ \pi(g) = \chi_\pi(g)$ as the trace function is conjugacy invariant. $\square$

Lemma 2.60. *Every character χ of a representation π of any group is constant on the conjugacy classes of the group.*

Proof. As $\chi(g) = \text{tr } \pi(g)$ for $g \in G$, for any $h \in G$, $\chi(hgh^{-1}) = \chi(hh^{-1}g) = \chi(g)$ since $\text{tr }(xy) = \text{tr }(yx)$ for all endomorphisms x, y. $\square$

A function that is constant on conjugacy classes is often called a class function.

New constructions using representations can now be introduced, and their characters can be determined. If (π_1, V_1) and (π_2, V_2) are two representations of a group G then the direct sum representation of the two representations is given by $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$. This representation acts on the G-space by $\pi(g)(v_1, v_2) = (\pi(g)v_1, \pi(g)v_2)$ for $g \in G, v_1 \in V_1$ and $v_2 \in V_2$. The character of a representation of this form is given by $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$. If the π_i are finite dimensional, their tensor product defines a representation $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ with accompanying character $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1}\chi_{\pi_2}$. Its action is defined by:

$$(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2$$

for $v_1 \in V_1, v_2 \in V_2$ and $g \in G$. Alternatively the exterior tensor representation can be defined as above, but defined by the action:

$$(\pi_1 \otimes \pi_2)(g, h)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(h)v_2$$

for $v_1 \in V_1, v_2 \in V_2$ and $g, h \in G$.

Lemma 2.61. *Let π and ρ be finite dimensional irreducible representations of a compact Lie group G. Then the following statements hold.*

- *a. If $\pi \sim \rho$ then $\langle \chi_\pi, \chi_\rho \rangle = 1$.*
- *b. If $\pi \nsim \rho$ then $\langle \chi_\pi, \chi_\rho \rangle = 0$.*

For $\langle \cdot, \cdot \rangle$ the L^2 inner product defined as $\langle f, g \rangle = I(fg) = \int_G f(x)\overline{g(x)}dx$ for I as in Proposition 2.44.

The proof of this lemma relies on the Schur orthogonality relations and can be found in ([1], pg. 83).

Definition 2.62. Assume a representation π can be decomposed into $\oplus_{i=1}^{n}\delta_i$ for an $n \in \mathbb{N}$, for all δ_i irreducible. Then the multiplicity of an irreducible representation δ in π is defined to be $m(\delta, \pi) = \#\{i \mid \delta_i \sim \delta\}$.

Lemma 2.63. *The multiplicity $m(\delta_j, \pi)$ can be expressed in terms of the characters of the representations δ_j and π.*

Proof. By Lemma 2.61:

$$\langle \chi_{\delta_j}, \chi_\pi \rangle = \langle \chi_{\delta_j}, \textstyle\sum_{k=1}^n \chi_{\delta_k} \rangle = \sum_{k=1}^n \langle \chi_{\delta_j}, \chi_{\delta_k} \rangle = \sum_{i=1}^n \delta_{jk} = m(\delta_j, \pi),$$

where δ_{jk} denotes the Kronecker delta. □

Definition 2.64. Let $\widehat{G}$ denote the set of equivalence classes of irreducible representations of a group G. Then any equivalence class $[\delta] \in \widehat{G}$ is identified with one representative δ, and δ will henceforth be used to refer to any element of this class.

As a sidenote, the shorthand $n\delta$ is often used to signify $\oplus_{i=1}^n \delta$.

The following lemma gives an expression for a representation in terms of its irreducible components, unique up to equivalence.

Lemma 2.65. *Let π be a finite dimensional representation of a compact group G. Then $\pi \sim \oplus_{\delta \in \widehat{G}} m(\delta, \pi)\delta$ and any decomposition of π into irreducibles is equivalent to the above one.*

Proof. By Proposition 2.50 the representation π is unitarisable, and by Corollary 2.52 it is equivalent to a direct sum $\oplus_{i=1}^n \delta_i$ of irreducible representations. Choosing one representative δ for each class in $\widehat{G}$, this expression is equivalent to a direct sum of these classes. Adopting the above notation of writing $n\delta$ for $\oplus_{i=1}^n \delta$, and noting that in this context n is the multiplicity of each δ in π, the result is $\pi \sim \oplus_{\delta \in \widehat{G}} m(\delta, \pi)\delta$. □

Corollary 2.66. *If π, ρ are two finite dimensional continuous representations of the compact group G then $\pi \sim \rho$ if and only if $\chi_\pi = \chi_\rho$.*

Proof. The first implication follows directly from Lemma 2.59. For the reverse implication, assume that $\chi_\pi = \chi_\rho$. Then for every irreducible representation δ of G, $m(\delta, \pi) = \langle \chi_\pi, \chi_\rho \rangle = \langle \chi_\rho, \chi_\pi \rangle = m(\delta, \rho)$ and therefore:

$$\pi \sim \bigoplus_{\delta \in \widehat{G}} m(\delta, \pi)\delta \sim \bigoplus_{\delta \in \widehat{G}} m(\delta, \rho)\delta \sim \rho$$

And thus $\pi \sim \rho$ per transitivity. □

This section is finished with the following definition and lemma.

Definition 2.67. Let G be a compact and commutative Lie group. A multiplicative character of G is a continuous group homomorphism $\xi : G \to (\mathbb{C}^*, \cdot)$.

Lemma 2.68. *If ξ is a multiplicative character, then $|\xi| = 1$.*

Proof. It will be proven that any compact subgroup H of $\mathbb{C}^*$ must be a subset of the unit circle. Then for all $g \in G$, $|\xi(g)| = 1$ as $\xi(g)$ is an element of one such compact subgroup. By compactness and the Heine-Borel Theorem for $\mathbb{C}$ there exists a constant $c > 0$ so that for every $h \in H$, $c^{-1} < |h| < c$. Setting $k = h^n$ for a natural number n it must follow that $c^{-1/n} < |k| < c^{1/n}$. Taking the limit as $n \to \infty$ it becomes apparent that $|k| = 1$. $\square$

3 Preparation

Some preparations need to be made to bridge the gap between the prerequisites of representation theory and the proof of the Schur–Weyl Duality Theorem. This is done in this section.

3.1 Matrix Groups

This subsection will briefly recap the definitions of the three matrix groups relevant to this thesis. Throughout, the space of all square complex $n \times n$ matrices will be denoted by $\mathrm{M}(n, \mathbb{C})$, and it will be seen as the same as $\mathbb{C}^{n^2}$ through the natural diffeomorphism between these spaces. The operation of every matrix group will be composition.

The most general matrix group is the subset $\mathrm{GL}(n, \mathbb{C})$ of $\mathrm{M}(n, \mathbb{C})$ that contains all invertible matrices. Although its definition was presumed to be known from linear algebra it will be stated again, because the other two relevant matrix groups can be described as subgroups of the general linear group. Because a matrix is invertible if and only if its determinant does not equal 0, $\mathrm{GL}(n, \mathbb{C})$ can be characterised using the determinant:

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in \mathrm{M}(n, \mathbb{C}) \mid \det(A) \neq 0\}.$$

The unitary group then consists of all matrices in $\mathrm{GL}(n, \mathbb{C})$ that are unitary. For a matrix this means that its conjugate transpose equals the inverse of the matrix:

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid AA^* = A^*A = I_n\}.$$

Within the unitary group further distinction can be made. A matrix with determinant 1 is sometimes called special. Using this attribute the special unitary group can be defined as follows:

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det(A) = 1\}.$$

These matrix groups are Lie groups (by the Closed Subgroup Theorem - that they are closed will be shown in the next subsection) and therefore possess topological properties. This opens many possibilities for their analysis and will be used extensively in the next subsections.

3.2 Complete Reducibility of the General Linear Group

For the proof of the Schur–Weyl Duality Theorem the assumption that every finite dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ is completely reducible is needed. That is to say that every finite dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ can be decomposed into a direct sum of irreducible representations. This subsection sets out to prove this fact. To do this, some lemmas will be needed: it needs to be proven that $\mathrm{U}(n)$ is compact and connected, and that $\mathrm{GL}(n, \mathbb{C})$ is connected. Then it will be shown through the Lie algebras of $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$ that every finite dimensional representation of $\mathrm{GL}(n, \mathbb{C})$ is completely reducible.

Lemma 3.1. *The Lie group* $\mathrm{U}(n)$ *is compact.*

Proof. Recall that the Heine-Borel Theorem for finite dimensional vector spaces says that as a subset of $\mathbb{C}^{n^2}$, $\mathrm{U}(n)$ is compact if and only if it is closed and bounded. These properties follow exactly from the restriction on the matrices that belong to $\mathrm{U}(n)$. The operator norm of any unitary matrix A is equal to 1:

$$\|A\| = \sup_{v \neq 0} \tfrac{\|Av\|}{\|v\|} = \sup_{v \neq 0} \tfrac{\|v\|}{\|v\|} = 1.$$

This follows from the unitarity of A since:

$$\|Av\| = \sqrt{\langle Av, Av \rangle} = \sqrt{\langle v, A^* Av \rangle} = \sqrt{\langle v, v \rangle} = \|v\|.$$

So $\mathrm{U}(n)$ is bounded in $\mathbb{C}^{n^2}$. Furthermore $\mathrm{U}(n)$ is the intersection of the pre-images of $\{I_n\}$ under the continuous maps $X \mapsto X^* X$ and $Y \mapsto YY^*$, both seen as maps from $\mathrm{M}(n, \mathbb{C})$ to itself. As pre-images of a singleton these pre-images are closed, and as an intersection of two closed subsets the intersection of these pre-images is closed as well. Hence $\mathrm{U}(n)$ is compact. $\square$

It follows from Corollary 2.53 that all representations of $\mathrm{U}(n)$ are completely reducible, which opens a path to proving the complete reducibility of $\mathrm{GL}(n, \mathbb{C})$. The course of action for this is to show that $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$ are connected, which allows the use of Lemma 2.47.

Lemma 3.2. *The Lie group* $\mathrm{U}(n)$ *is connected.*

Proof. It will be proven that there exists a continuous path between every matrix in $\mathrm{U}(n)$ and the identity element I. Then for any two matrices in $\mathrm{U}(n)$ there exist paths to the identity, and joining these paths together forms a path between the matrices. It is known from topology that path-connectedness implies connectedness, from which the statement follows.

Let $A \in \mathrm{U}(n)$. From the Spectral Theorem for normal operators it follows there exist unitary matrices V, D so that:

$$A = VDV^* \text{ with } D = \operatorname{diag}(\lambda_1, ..., \lambda_n).$$

The matrix D is taken with respect to an orthonormal basis $\{f_j \mid 1 \leq j \leq n\}$, and for every j, $Df_j = \lambda_j f_j$ as D is a diagonal matrix. Because it is unitary, for all j:

$$\begin{aligned} \langle Df_j, Df_j \rangle &= \langle f_j, D^* D f_j \rangle = \langle f_j, f_j \rangle = 1 \text{ and} \\ \langle Df_j, Df_j \rangle &= \langle \lambda f_j, \lambda f_j \rangle = \lambda_j \overline{\lambda_j} \langle f_j, f_j \rangle = |\lambda_j|, \end{aligned}$$

from which it follows that for all j, $|\lambda_j| = 1$. For each j there exists a $\theta_j \in \mathbb{R}$ so that $\lambda_j = e^{i\theta_j}$. Let:

$$D(t) = \operatorname{diag}\left(e^{i(1-t)\theta_1}, ..., e^{i(1-t)\theta_n}\right) \text{ and } A(t) = VD(t)V^* \text{ for } 0 \leq t \leq 1.$$

Then $A(t)$ describes a path from A to I. In $t = 0$, $A(0) = A$. In $t = 1$, $A(1) = VIV^* = I$. The entire path is contained in $\mathrm{U}(n)$ as for every t:

$$D(t)^* = \operatorname{diag}\left(e^{-i(1-t)\theta_1}, ..., e^{-i(1-t)\theta_n}\right) = D(t)^{-1},$$

and therefore $A(t) \in \mathrm{U}(n)$ as a product of unitary matrices. $\square$

Next it is proven that $\mathrm{GL}(n, \mathbb{C})$ is connected.

Lemma 3.3. *The Lie group* $\mathrm{GL}(n, \mathbb{C})$ *is connected.*

The proof is a rephrasing of the same proposition found in ([3], ch. 1).

Proof. As for $\mathrm{U}(n)$ this follows from the fact that there exists a path between every matrix in $\mathrm{GL}(n, \mathbb{C})$ and the identity I. It is known that $\mathrm{GL}(1, \mathbb{C})$ is exactly $\mathbb{C}^*$, and that it is path-connected. Now let $n \geq 2$ and let $A \in \mathrm{GL}(n, \mathbb{C})$. Then by the Jordan Normal Form Theorem there exists an upper triangular matrix L with the eigenvalues λ_i $(1 \leq i \leq n)$ of A on its diagonal, and a transformation matrix T so that:

$$A = TLT^{-1}.$$

As $\det(A) \neq 0$, there are no eigenvalues equal to zero, since the determinant of a matrix equals the product of its eigenvalues. Let $L(t)$ be the matrix L with all of the elements above the diagonal parametrised through multiplication by a factor $(1-t)$, and define:

$$A(t) = TL(t)T^{-1} \text{ for } 0 \leq t \leq 1.$$

Next let $\lambda_i(t)$ for $1 \leq t \leq 2$ denote the path from λ_i to 1 in $\mathbb{C}^*$, which exists by the path-connectedness of $\mathbb{C}^*$. Then define:

$$A(t) = T\text{diag}\left(\lambda_1(t), ..., \lambda_n(t)\right) T^{-1} \text{ for } 1 \leq t \leq 2.$$

It is claimed that $A(t), 0 \leq t \leq 2$ is a path from A to I in $\mathrm{GL}(n, \mathbb{C})$. First, $A(0) = A$. As t increases over the interval $[0,1]$ the value $T\text{diag}\left(\lambda_1, ..., \lambda_n\right) T^{-1}$ is reached in $t = 1$. Then as t further increases over the interval $[1,2]$, the value:

$$T\text{diag}\left(1, ..., 1\right) T^{-1} = TIT^{-1} = I$$

is reached in $t = 2$. For $0 \leq t \leq 1$, $A(t) \in \mathrm{GL}(n, \mathbb{C})$ as the diagonal and hence the determinant is left unchanged; for $1 \leq t \leq 2$, $A(t) \in \mathrm{GL}(n, \mathbb{C})$ as each path $\lambda_i(t)$ lies in $\mathbb{C}^*$ and therefore never takes on the value 0. The determinant is then always non-zero, and $A(t) \in \mathrm{GL}(n, \mathbb{C})$. □

By the connectedness of $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$, Lemma 2.47 is now within reach. Before it can be applied to $\mathrm{GL}(n, \mathbb{C})$ to obtain the desired result a connection between the representations of $\mathrm{GL}(n, \mathbb{C})$ and the representations of the completely reducible group $\mathrm{U}(n)$ must be established. This is done through their Lie algebras in the following two lemmas.

Lemma 3.4. *If $\mathfrak{gl}(n, \mathbb{C})$ denotes the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$, the Lie algebra $\mathfrak{u}(n)$ of $\mathrm{U}(n)$ is given by $\{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}$.*

Proof. Let $\mathcal{U} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}$. It will be proven by mutual inclusion that $\mathcal{U} = \mathfrak{u}(n)$. First let $X \in \mathfrak{u}(n)$. By Theorem 2.38, $\exp(tX) \in \mathrm{U}(n)$ for all $t \in \mathbb{R}$. It is known that $(e^{tX})^*(e^{tX}) = (e^{tX^*})(e^{tX}) = e^{-tX}e^{tX} = I$. Differentiating both sides to t and evaluating the result in $t = 0$ yields:

$$\tfrac{d}{dt}|_{t=0}\, (e^{tX^*})(e^{tX}) = [X^*e^{tX^*}e^{tX} + e^{tX^*}Xe^{tX}]_{t=0} = X^* + X = 0.$$

Hence $X^* = -X$, and therefore $X \in \mathcal{U}$. Next let $X \in \mathcal{U}$, then $X^* = -X$. It follows directly that $X \in \mathfrak{u}(n)$ as for all $t \in \mathbb{R}$:

$$(e^{tX})^* = e^{tX^*} = e^{-tX} = (e^{tX})^{-1}.$$

Therefore $\mathcal{U} \subset \mathfrak{u}(n)$, and $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = -X\}$. □

Moreover, as an even stronger result, the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ is the complexification of the Lie algebra of $\mathrm{U}(n)$.

Lemma 3.5. *The Lie algebra* $\mathfrak{gl}(n, \mathbb{C})$ *is the complexification* $\mathfrak{u}(n)_{\mathbb{C}}$ *of* $\mathfrak{u}(n)$.

Proof. It follows from Lemma 3.4 that $i\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* = X\}$, and as both $\mathfrak{u}(n)$ and $i\mathfrak{u}(n)$ are subsets of $\mathfrak{gl}(n, \mathbb{C})$ it suffices to show that $\mathfrak{gl}(n, \mathbb{C}) \subset \mathfrak{u}(n) + i\mathfrak{u}(n)$ to conclude equality. Let $X \in \mathfrak{gl}(n, \mathbb{C})$. Then X can be expressed as:

$$X = \tfrac{X - X^*}{2} + \tfrac{X + X^*}{2}$$

where $\frac{X-X^*}{2} \in \mathfrak{u}(n)$ and $\frac{X+X^*}{2} \in i\mathfrak{u}(n)$. In the case of $\mathfrak{u}(n)$ this follows as:

$$\left(\tfrac{X-X^*}{2}\right)^* = \tfrac{X^*-X^{**}}{2} = \tfrac{X^*-X}{2} = -\left(\tfrac{X-X^*}{2}\right)$$

The argument for $i\mathfrak{u}(n)$ is analogous. Therefore $X \in \mathfrak{u}(n) + i\mathfrak{u}(n)$, $\mathfrak{gl}(n, \mathbb{C})$ is a subset of $\mathfrak{u}(n) + i\mathfrak{u}(n)$ and $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n)_{\mathbb{C}}$. □

And finally the complete reducibility of $\mathrm{GL}(n, \mathbb{C})$ can be proven.

Proposition 3.6. *Any finite dimensional representation of* $\mathrm{GL}(n, \mathbb{C})$ *is completely reducible.*

Proof. Let (π, V) be any finite dimensional representation of $\mathrm{GL}(n, \mathbb{C})$. As $\mathrm{U}(n)$ is a subgroup of $\mathrm{GL}(n, \mathbb{C})$, the restriction $(\pi|_{U(n)}, V)$ defines a representation of $\mathrm{U}(n)$. By Lemma 3.1, $\mathrm{U}(n)$ is compact and hence completely reducible, so there exists a decomposition into irreducible invariant subspaces:

$$V = V_1 \oplus \cdots \oplus V_m$$

for an $m \in \mathbb{N}$. All V_j in this decomposition are $\mathrm{U}(n)$-invariant, and as $\mathrm{U}(n)$ is connected, $\mathfrak{u}(n)$-invariant by Lemma 2.47. Because $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$ it follows that the V_j are $\mathfrak{gl}(n, \mathbb{C})$-invariant and by the same lemma as before, $\mathrm{GL}(n, \mathbb{C})$-invariant. Thus the arbitrary representation π was decomposed into irreducible submodules. □

As the compactness of $\mathrm{SU}(n)$ is a direct consequence of the compactness of $\mathrm{U}(n)$ proven in Lemma 3.1, it will be stated as a corollary here.

Corollary 3.7. *The Lie group* $\mathrm{SU}(n)$ *is compact.*

Proof. Let det be the determinant function from $\mathrm{M}(n, \mathbb{C})$ to $\mathbb{C}$. Then characterise $\mathrm{SU}(n)$ by:

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \det^{-1}(\{1\}).$$

This definition coincides with the definition given in the previous subsection per mutual inclusion. As sets $\mathrm{U}(n)$ and $\det^{-1}(\{1\})$ are both closed so $\mathrm{SU}(n)$ is closed as a subset of $\mathrm{U}(n)$. As a closed subset of a compactum, $\mathrm{SU}(n)$ is compact. $\square$

3.3 Isotypic Decomposition

The decomposition of a representation into its irreducible components is only unique up to equivalence. Because the Schur–Weyl Duality Theorem makes claims about uniqueness of the components of a representation, more constraints will be needed. The answer to these constraints will manifest as the isotypic components (also called isotypic subspaces) of a representation, and the isotypic decomposition of a G-module is unique. This subsection will culminate in an important rephrasing of the isotypic decomposition.

Let G be a completely reducible Lie group and let $[\lambda]$ be a class in $\widehat{G}$. Then for every $\delta \in [\lambda]$ the associated G-module is denoted U_δ. These assumptions will be used throughout this subsection.

Let (ρ, V) be any representation of G.

Definition 3.8. The λ-isotypic subspace of V is defined as the following linear sum of vector spaces:

$$V_{(\lambda)} = \sum_{U \subset V,\ \rho|_U \sim \lambda} U.$$

The next proposition shows that V can be written as a direct sum of these isotypic components, called the isotypic decomposition.

Proposition 3.9. *The G-module V can be written as the direct sum*

$$V = \bigoplus_{[\lambda]\in\widehat{G}} V_{(\lambda)}.$$

The proof of this proposition is based on the proof given in ([2], par. 4.1.5).

Proof. Because every irreducible component is represented in the sum it is straightforward to see that V is contained in the linear sum of the $V_{(\lambda)}$. For this reason it only needs to be shown that this sum is direct. Note that because the representation is finite dimensional there can be at most a finite number of irreducible components in a decomposition into irreducibles, and they can be indexed by natural numbers. It will be shown by means of induction that the sum of any finite collection of isotypic components is direct. To this end, let $\{\lambda_1, ..., \lambda_n\}$ be an arbitrary set of representatives for n different equivalence classes of $\widehat{G}$ such that $V_{(\lambda_i)} \neq \{0\}$. The base case of $W_1 = \sum_{i=1}^{1} V_{(\lambda_i)} = V_{(\lambda_1)}$ is trivially a direct sum. Now assume that $n > 1$ and the hypothesis holds for $W_{n-1} = \sum_{i=1}^{n-1} V_{(\lambda_i)}$. Then $W_{n-1} = \oplus_{i=1}^{n-1} V_{(\lambda_i)}$ and $W_n = W_{n-1} + V_{(\lambda_n)}$. For $1 \leq i < n$ let $P_i : W_{n-1} \to V_{(\lambda_i)}$ be the projection map onto the corresponding component of W_{n-1}, and recall that for two vector subspaces A and B:

$$A + B = A \oplus B \iff A \cap B = \{0\}.$$

Arguing by contradiction, assume that there exists a non-zero element v in $W_{n-1} \cap V_{(\lambda_n)}$. Then the linear span $\mathcal{S}$ of the orbit $G \cdot v$ of v under the action of G is an invariant subspace of V and therefore a G-module. By the assumptions on G it is finite dimensional, reducible and it is contained in $V_{(\lambda_n)}$ as the latter is a G-module, meaning it is invariant. For this reason it can be decomposed in the following way:

$$\mathcal{S} = Y_1 \oplus \cdots \oplus Y_r \text{ for } (\alpha_k, Y_k) \text{ representations so that } \alpha_k \in [\lambda_n].$$

But at the same time $\mathcal{S} \subset W_{n-1}$ so there must exist an $i < n$ so that $P_i(\mathcal{S})$ is not empty. Therefore in the same way:

$$\mathcal{S} = Z_1 \oplus \cdots \oplus Z_q \text{ for } (\beta_j, Z_j) \text{ representations so that } \beta_j \in [\lambda_i].$$

This is a contradiction as $\lambda_n \nsim \lambda_i$. Thus $W_{n-1} \cap V_{(\lambda_n)} = \varnothing$ and $W_n = \oplus_{i=1}^{n} V_{(\lambda_i)}$. □

There is a small amount of additional information needed in order to start rephrasing this primary decomposition in the way needed for the later sections.

For every class in $\widehat{G}$, fix one representation (λ, U_λ). Then the tensor product $\mathrm{Hom}_G(U_\lambda, V) \otimes U_\lambda$ is a G-module under the action $g \cdot (u \otimes w) = u \otimes (g \cdot w)$ for $g \in G$. This defines an action as w is an element taken from the G-module U_λ. It will be shown the G-module $\mathrm{Hom}_G(U_\lambda, V) \otimes U_\lambda$ is isomorphic to $V_{(\lambda)}$.

Proposition 3.10. *For every $[\lambda] \in \widehat{G}$, $V_{(\lambda)} \cong Hom_G(U_\lambda, V) \otimes U_\lambda$ by:*

$$S_\lambda : Hom_G(U_\lambda, V) \otimes U_\lambda \to V, \;\; S_\lambda(u \otimes w) = u(w).$$

Proof. It needs to be shown that S_λ is a G-module isomorphism. By the discussion preceding this proposition it is known that the domain of S_λ is a G-module. Under the defining action the map is G-intertwining as $u \in \mathrm{Hom}_G(U_\lambda, V)$ is. For $g \in G$:

$$g \cdot S_\lambda(u \otimes w) = g \cdot u(w) = u(g \cdot w) = S_\lambda(u \otimes (g \cdot w)) = S_\lambda(g \cdot (u \otimes w)).$$

It is shown by mutual inclusion of $\mathrm{Im}(S_\lambda)$ and $V_{(\lambda)}$ that S_λ is surjective.

Let $T \in \mathrm{Hom}_G(V, W)$, then in order to prove $\mathrm{Im}(S_\lambda) \subset V_{(\lambda)}$ it suffices to show that $T(U_\lambda) \subset V_{(\lambda)}$. For $T = 0$ this is clear. Assume that $T \neq 0$, then $\mathrm{Ker}(T) \subsetneq U_\lambda$ is G-invariant and must therefore be $\{0\}$. Then T is injective and from $\lambda \sim \rho|_{T(U_\lambda)}$ it follows that $T(U_\lambda) \subset V_{(\lambda)}$, and $\mathrm{Im}(S_\lambda) \subset V_{(\lambda)}$.

To prove $V_{(\lambda)} \subset \mathrm{Im}(S_\lambda)$, let $v \in V_{(\lambda)}$. Then v decomposes as $\oplus_{i=1}^{k} v_i$ for a natural number k and each v_i taken from an irreducible subspace $V_i \subset V$ in such a way that $\rho|_{V_i} \sim \lambda$. Hence for every i there exists a G-module isomorphism $T_i : U_\lambda \to V_i \subset V$. For every component V_i of V, $v_i \in V_i = T_i(U_\lambda) = S_\lambda(\mathbb{C}T_i \otimes U_\lambda)$ so in particular $v \in \mathrm{Im}(S_\lambda)$. Therefore $V_{(\lambda)} \subset \mathrm{Im}(S_\lambda)$.

The proof is then completed by showing S_λ is injective. This will be done by showing the kernel is trivial.

Let $V = \oplus_{i=1}^{n} V_i$ for an $n \in \mathbb{N}$ be a decomposition of V into G-invariant subspaces. Then $\mathrm{Hom}_G(U_\lambda, V) \cong \oplus_{i=1}^{n} \mathrm{Hom}_G(U_\lambda, V_i)$. If $\rho|_{V_i}$ is irreducible and $\rho|_{V_i} \nsim \lambda$ then $\mathrm{Hom}_G(U_\lambda, V_i) = 0$ by Lemma 2.57, otherwise $\mathrm{Hom}_G(U_\lambda, V_i) = 1$ by the same lemma. For every i, let $T_i \in \mathrm{Hom}_G(U_\lambda, V_i)$ be non-trivial. Then:

$$\mathrm{Hom}_G(U_\lambda, V) \cong \bigoplus_{i:\ \rho|_{V_i} \sim \lambda} \mathbb{C}T_i.$$

Hence:

$$\mathrm{Hom}_G(U_\lambda, V) \otimes U_\lambda \cong \bigoplus_{i:\ \rho|_{V_i} \sim \lambda} \mathbb{C}T_i \otimes U_\lambda.$$

Let $w \in \ker(S_\lambda)$. It will then be shown that w must be trivial. Write:

$$w = \sum_{i:\ \rho|_{V_i} \sim \lambda} z_i T_i \otimes u_i \text{ with } z_i \in \mathbb{C} \text{ and } u_i \in U_i.$$

Then:

$$S_\lambda(w) = \sum_{i:\ \rho|_{V_i} \sim \lambda} z_i T_i(u_i) = \sum_{i:\ \rho|_{V_i} \sim \lambda} T_i(z_i u_i),$$

since S_λ is linear by the multilinearity of tensor products. Then for every i, $T_i(z_i u_i) \in V_i$ as the T_i were chosen from $\mathrm{Hom}_G(U_\lambda, V_i)$. The V_i are components of the direct sum $V = \oplus_{i=1}^n V_i$ so for all i, $T_i(z_i u_i) = 0$. Because the T_i are non-trivial for all i it follows from $z_i u_i = 0$ that either $z_i = 0$, $u_i = 0$ or both equal 0. It follows that:

$$w = \sum_{i:\ \rho|_{V_i} \sim \lambda} z_i T_i \otimes u_i = 0.$$

As S_λ is proven to be an isomorphism, the modules are isomorphic. □

Recall the definition (Definition 2.62) of the multiplicity of an irreducible representation in a representation. Using this definition the spectrum of the representation (ρ, V) is defined as:

$$\mathrm{Spec}(\rho) = \mathrm{Spec}(V) = \{\lambda \mid [\lambda] \in \widehat{G} \text{ and } m(\lambda, \rho) \neq 0\}.$$

The combination of Propositions 3.9 and 3.10 then results in the following rephrasement of the decomposition:

$$V \cong \bigoplus_{\lambda \in Spec(\rho)} \mathrm{Hom}_G(U_\lambda, V) \otimes U_\lambda.$$

For an element $g \in G$, the action of $\rho(g)$ on V translates to the action $I \otimes \lambda(g)$ on the summand of type λ. This is the rephrasing that was promised in the opening of this subsection.

3.4 The Group Algebra

The name of this subsection implies it will concern itself with algebras. As this is indeed the case, the reader is reminded of the definition of an associative algebra.

Definition 3.11. An associative algebra $\mathcal{A}$ over a field $\mathbb{F}$ is a vector space A equipped with an $\mathbb{F}$-bilinear and associative operation $A \times A \to A$, $(a, b) \mapsto a \cdot b$. It is called unital if there exists a unital element in the algebra.

The definition of a group representation extends to these algebras.

Definition 3.12. A representation (ρ, V) of an algebra $\mathcal{A}$ is an algebra homomorphism $\rho : \mathcal{A} \to \mathrm{End}(V)$. The space V is called an $\mathcal{A}$-module.

The main notion of this subsection is that every representation of a group extends to a representation of an algebra, specifically of the group algebra of the group.

Definition 3.13. The group algebra $\mathcal{A}[G]$ of G is defined to be the linear space of all finitely supported functions from G to $\mathbb{C}$ equipped with the following (convolution) operation:

$$(\varphi * \psi)(g) = \sum_{h \in G} \varphi(gh^{-1})\psi(h)$$

for $\varphi, \psi \in \mathcal{A}[G]$ and $g \in G$.

It is left to the reader to verify that this is indeed a unital associative algebra.

The group algebra will play a pivotal role in the proof of the Schur–Weyl Duality Theorem. In this subsection the bare essentials will be presented. A more thorough explanation can be found in ([2], par. 4.1.1).

As a vector space the group algebra $\mathcal{A}[G]$ has a basis of functions $\{\delta_g \mid g \in G\}$ so that for $x \in G$, $\delta_g(x) = 1$ if $x = g$ and $\delta_g(x) = 0$ otherwise. Each element $g \in G$ is identified with one such basis element δ_g, and each element $a \in \mathcal{A}[G]$ can be uniquely expressed as the sum:

$$\sum_{g \in G} a(g)\delta_g$$

for only a finite number of non-zero coefficients $a(g)$.

Lemma 3.14. *For all group elements $g_1, g_2 \in G$ the convolution $\delta_{g_1} * \delta_{g_2}$ of the basis elements δ_{g_1} and δ_{g_2} of $\mathcal{A}[G]$ equals $\delta_{g_1 g_2}$.*

Proof. This follows directly as for $g \in G$:

$$(\delta_{g_1} * \delta_{g_2})(g) = \sum_{h \in G} \delta_{g_1}(gh^{-1})\delta_{g_2}(h) = \delta_{g_1}(gg_2^{-1}) = \delta_{g_1 g_2}(g),$$

since $\delta_{g_1}(gg_2^{-1}) = 1$ if $g = g_1 g_2$, and $\delta_{g_1}(gg_2^{-1}) = 0$ otherwise. $\square$

The correspondence between representations of G and $\mathcal{A}[G]$ is explained next. If (ρ, V) is a representation of $\mathcal{A}[G]$, then a representation (π, V) of G is constructed from ρ by:

$$\pi : G \to \mathrm{GL}(V), \quad \pi : g \mapsto \rho(\delta_g).$$

Then by the above lemma for any two elements $g, h \in G$ it follows that $\pi(gh) = \rho(\delta_{gh}) = \rho(\delta_g * \delta_h) = \rho(\delta_g)\rho(\delta_h) = \pi(g)\pi(h)$. Conversely if (π, V) is a representation of G then π extends uniquely to a representation (ρ, V) of $\mathcal{A}[G]$ defined in the following way:

$$\rho : \mathcal{A}[G] \to \mathrm{End}(V), \quad \rho : f \mapsto \sum_{g \in G} f(g)\pi(g).$$

Showing that ρ defined in this way is a homomorphism takes more steps than it did for π, and it is clearer to work backwards from $\rho(\varphi)\rho(\psi)$ to $\rho(\varphi * \psi)$.

$$\begin{aligned}
\rho(\varphi)\rho(\psi) &= \left(\sum_{x \in G} \varphi(x)\pi(x) \right) \cdot \left(\sum_{y \in G} \psi(y)\pi(y) \right) \\
&= \sum_{x, y \in G} \varphi(x)\psi(y)\pi(x)\pi(y) \\
&= \sum_{x, y \in G} \varphi(x)\psi(y)\pi(xy) \\
&= \sum_{g, y \in G} \varphi(gy^{-1})\psi(y)\pi(gy^{-1}y) \\
&= \sum_{g, y \in G} \varphi(gy^{-1})\psi(y)\pi(g)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{g \in G} \left[\sum_{y \in G} \varphi(gy^{-1})\psi(y) \right] \pi(g) \\
&= \sum_{g \in G} (\varphi * \psi)(g)\pi(g) \\
&= \rho(\varphi * \psi)
\end{aligned}
$$

Outside of this explanation this correspondence will be assumed to be understood, and the corresponding representations of G and $\mathcal{A}[G]$ will be denoted by the same symbol. There is one other algebra construction that is needed for the next section, and one that is in fact closely related to the group algebra.

Definition 3.15. Let (π, V) be a representation of a group G. Then the span of π is the unital and associative algebra:

$$\mathrm{Span}(\pi) = \{c_1\pi(g_1) + \cdots + c_k\pi(g_k) \mid c_1, ..., c_k \in \mathbb{C} \text{ and } g_1, ..., g_k \in G\}.$$

This definition finds its use in the next section, where it is used to construct an algebra from a group representation where the structure of an algebra is needed. The relation between the group algebra and the span of a representation will be clarified by the final lemma of this section.

Lemma 3.16. *If (π, V) is a representation of a group G and $\mathcal{A}[G]$ is the group algebra of G, then* $\mathrm{Span}(\pi) = \pi(\mathcal{A}[G])$.

Proof. From the previous discussion on the group algebra it is known that $\mathcal{A}[G] = \mathrm{Span}\{\delta_g \mid g \in G\}$. Using this definition the derivation of the equality is straightforward:

$$\mathrm{Span}(\pi) = \mathrm{Span}\{\pi(g) \mid g \in G\} = \mathrm{Span}\{\pi(\delta_g) \mid g \in G\} = \pi(\mathcal{A}[G]).$$

$\square$

4 Schur–Weyl Duality

In this main section of the thesis the proof of the Schur–Weyl Duality Theorem will be given by means of the proofs of the General Duality Theorem and the Double Commutant Theorem. The third subsection will tie these theorems together in the proof of the Schur–Weyl Duality Theorem.

4.1 The Double Commutant Theorem

The Double Commutant Theorem fulfills a vital role in showing the General Duality Theorem presented in the next subsection extends to include the specific incarnation of duality described by Schur–Weyl Duality.

Let V be any finite dimensional complex vector space, and let $\mathcal{S}$ be a subset of $\mathrm{End}(V)$.

Definition 4.1. The commutant of $\mathcal{S}$ in $\mathrm{End}(V)$ is the unital and associative algebra defined as:

$$\mathrm{Comm}(\mathcal{S}) = \{x \in \mathrm{End}(V) \mid xs = sx \text{ for all } s \in \mathcal{S}\}.$$

The Double Commutant Theorem says that for specific algebras, taking the commutant twice results in the algebra itself.

Theorem 4.2. *Let $\mathcal{A} \subset \mathrm{End}(V)$ be a unital associative subalgebra, and let $\mathcal{B} = \mathrm{Comm}(\mathcal{A})$. If V is a completely reducible $\mathcal{A}$-module, $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$.*

This proof is based on the proof given in ([2], pg. 184).

Proof. By definition $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$ so it suffices to prove $\mathrm{Comm}(\mathcal{B}) \subset \mathcal{A}$, and the theorem will follow by mutual inclusion. Choose $\{v_1, ..., v_n\}$ to be a basis of V and let $T \in \mathrm{Comm}(\mathcal{B})$. It will be shown there exists an element $S \in \mathcal{A}$ so that $Sv_i = Tv_i$ for all basis elements. Because S and T then act on the basis in the same way, they must be the same endomorphism. Let $w = \oplus_{i=1}^n v_i \in V^{(n)} = \oplus_{i=1}^n V$. Because V was assumed to be a completely reducible $\mathcal{A}$-module, $V^{(n)}$ is too, and the cyclic submodule $\mathcal{O} = \mathcal{A} \cdot w$ has an $\mathcal{A}$-invariant complement $\mathcal{O}^{\perp}$ by the irreducibility of $\mathcal{O}$. It follows that there is a projection $P : V^{(n)} \to \mathcal{O}$ that commutes with the elements in $\mathcal{A}$. To see this write:

$$V^{(n)} = \mathcal{O} \oplus \mathcal{O}^{\perp},$$

where both components are $\mathcal{A}$-invariant. On $\mathcal{O}$, it is evident that $P = I$. Let $a \in \mathcal{A}$. If $v \in \mathcal{O}$ then $av \in \mathcal{O}$ so $P(av) = av = aP(v)$. If on the other hand $\hat{v} \in \mathcal{O}^{\perp}$, then $a\hat{v} \in \mathcal{O}^{\perp}$ and $P(a\hat{v}) = 0 = a \cdot 0 = aP(\hat{v})$. Therefore $P \circ a = a \circ P$ on both $\mathcal{O}$ and $\mathcal{O}^{\perp}$, hence on $V^{(n)}$. Because the projection commutes with $\mathcal{A}$ its action is given by a matrix in $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$. Furthermore it is known that $Pw = w$, and because T was chosen from $\mathrm{Comm}(\mathcal{B})$ it commutes with P. Now:

$$P \circ T(w) = T \circ P(w) = T(w) = \oplus_{i=1}^{n} Tv_i \in \mathcal{O}.$$

Because $\mathcal{O}$ was defined to be $\mathcal{A} \cdot w$ there must exist an $S \in \mathcal{A}$ such that:

$$S \cdot w = \oplus_{i=1}^{n} Sv_i = \oplus_{i=1}^{n} Tv_i.$$

Thus S has been found, $T = S$ and $T \in \mathcal{A}$. $\square$

4.2 The General Duality Theorem

This subsection is started with a definition.

Definition 4.3. Let V be a (possibily infinite dimensional) complex linear space. A representation (ρ, V) is said to be locally regular if for all $v \in V$ the space $W = \mathrm{Span}\{\rho(g)v \mid g \in G\}$ is finite dimensional and $\rho|_W : G \to \mathrm{GL}(W)$ is continuous.

Let (ρ, L) be any finite dimensional locally regular representation of a completely reducible group $G \subset \mathrm{GL}(n, \mathbb{C})$. Next fix a representation (λ, U_λ) for every $[\lambda] \in \widehat{G}$. From the results in Subsection 3.3 it is known that the G-module L can be decomposed as:

$$L \cong \bigoplus_{\lambda \in Spec(\rho)} \mathrm{Hom}_G(U_\lambda, L) \otimes U_\lambda,$$

with the action of $\rho(g)$ defined by $I \otimes \lambda(g)$ on the summand of type λ. Now assume that $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra satisfying the following three conditions:

1. $\mathcal{R}$ induces an irreducible (defining) representation ϕ on L.

2. If $g \in G$ and $T \in \mathcal{R}$ then $\rho(g)T\rho(g)^{-1} \in \mathcal{R}$. Accordingly, $\mathcal{R}$ is a G-module for the action $(g, T) \mapsto \rho(g)T\rho(g)^{-1}$.

3. The representation of G on $\mathcal{R}$ induced by the action described in the second condition is locally regular.

Define the commutant $\mathcal{R}^G$ of $\rho(G)$ in $\mathcal{R}$ to be:

$$\mathcal{R}^G = \{T \in \mathcal{R} \mid \rho(g)T = T\rho(g) \text{ for all } g \in G\}.$$

Then since ρ extends to an irreducible representation of the group algebra $\mathcal{A}[G]$, the tensor representation $\phi \otimes \rho$ defines a representation of $\mathcal{R}^G \otimes \mathcal{A}[G]$ on L. The action of $\mathcal{R}^G$ by ϕ on $\mathrm{Hom}_G(U_\lambda, L)$ is given by left multiplication, and the above decomposition holds true for L as an $\mathcal{R}^G \otimes \mathcal{A}[G]$-module as well. The action of $\phi \otimes \rho$ is defined by $\phi \otimes \lambda$ on each summand of type λ. The General Duality Theorem states there is a duality between the representations of $\mathcal{R}^G$ and $\mathcal{A}[G]$ through this decomposition. Before making this more precise through the statement of the theorem, a supporting lemma will be given without proof. This lemma is a consequence of Burnside's Theorem, which is itself a corollary to the Jacobson Density Theorem. The proof of the lemma can be found in ([2], pg. 196).

Lemma 4.4. *Let $X \subset L$ be a G-invariant subspace. Then the restriction map $r \mapsto r|_X$ induces a surjective linear map $\mathcal{R}^G \to \mathrm{Hom}_G(X, L)$.*

The General Duality Theorem will now be stated.

Theorem 4.5. *For every $[\lambda] \in \widehat{G}$, the space $Hom_G(U_\lambda, L)$ is an irreducible $\mathcal{R}^G$-module. Furthermore for any pair $\lambda, \mu \in Spec(\rho)$, if $Hom_G(U_\lambda, L) \cong Hom_G(U_\mu, L)$ then $\lambda = \mu$.*

This theorem says that in the decomposition above every $\mathrm{Hom}_G(U_\lambda, L)$ must be an irreducible $\mathcal{R}^G$-module. Moreover this $\mathcal{R}^G$-module can only appear once in the decomposition, and therefore it corresponds uniquely to U_λ. The proof is a rephrasing of the proof given in ([2], pg. 196).

Proof. First it is proven that $\mathrm{Hom}_G(U_\lambda, L)$ is an irreducible $\mathcal{R}^G$-module. Let S and T be non-zero elements of $\mathrm{Hom}_G(U_\lambda, L)$. It will be shown an element $r \in \mathcal{R}^G$ can be found so that $rT = S$. Then the only non-zero invariant subspace of $\mathrm{Hom}_G(U_\lambda, L)$ must be $\mathrm{Hom}_G(U_\lambda, L)$ itself, demonstrating it is an irreducible module. Let $X = TU_\lambda$, $Y = SU_\lambda$. Then X and Y are isomorphic through a G-module isomorphism φ. To see this note that as $\ker(T)$ is an invariant subspace of the irreducible module U_λ and T is non-trivial, the kernel of T must be $\{0\}$ and therefore T must be injective. It is automatically surjective onto its image $X = T(U_\lambda)$, and G-intertwining per definition. Hence $X \cong U_\lambda$. This same reasoning holds for S and Y, and therefore $X \cong U_\lambda \cong Y$. By Lemma 4.4 there exists an element $u \in \mathcal{R}^G$ so that $\varphi = u|_X$. The composition $uT : U_\lambda \to SU_\lambda$ is a G-module isomorphism. By Lemma 2.57, there exists a scalar $c \in \mathbb{C}$ so that $cuT = S$. Take $r = cu$.

Next, assume that $\lambda \neq \mu$. It will be shown that $\mathrm{Hom}_G(U_\lambda, L)$ and $\mathrm{Hom}_G(U_\mu, L)$ are inequivalent $\mathcal{R}^G$-modules. It follows from this that the modules are equivalent if and only if $\lambda = \mu$: the implication $\lambda = \mu \Rightarrow \mathrm{Hom}_G(U_\lambda, L) \cong \mathrm{Hom}_G(U_\mu, L)$ is apparent. The reverse implication then follows from that which is to be proven, by contraposition. Recall the definition of equivalence from Definition 2.54, and assume that:

$$\psi : \mathrm{Hom}_G(U_\lambda, L) \to \mathrm{Hom}_G(U_\mu, L)$$

is a map that meets the requirements for equivalence. Then it must be trivial. To see this, let $T \in \mathrm{Hom}_G(U_\lambda, L)$ be non-zero and set $S = \psi(T)$. Set $U = TU_\lambda + SU_\mu = TU_\lambda \oplus SU_\mu$, the second equality following from the irreducibility of λ, μ and the assumption that $\lambda \neq \mu$. Let $p : U \to SU_\mu$ be the projection map. Lemma 4.4 implies there exists an element $r \in \mathcal{R}^G$ so that $r|_U = p$. Because $pT = 0$ it follows that $rT = 0$, hence:

$$0 = \psi(0) = \psi(rT) = r\psi(T) = rS = pS = S.$$

This shows that $S = 0$, and therefore that $\psi = 0$. □

The Schur–Weyl Duality Theorem will combine this theorem for the specific case of $\mathrm{GL}(n, \mathbb{C})$ with the Double Commutant Theorem.

4.3 Schur–Weyl Duality

In this subsection $\mathcal{C}^k$ will sometimes be used to denote $\bigotimes^k \mathbb{C}^n$ in order to improve readability. Let $(\rho, \mathbb{C})$ be the defining representation of $G = \mathrm{GL}(n, \mathbb{C})$. Then the representation $(\rho_k, \mathcal{C}^k)$ for $k \geq 0$ is the representation $\otimes_{i=1}^k \rho$ that acts on $\mathcal{C}^k$ by:

$$\rho_k(g)(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k \text{ for } g \in \mathrm{GL}(n, \mathbb{C}).$$

Furthermore define the representation $(\sigma_k, \mathcal{C}^k)$ of S_k by:

$$\sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)} \text{ for } s \in S_k.$$

Let E_λ denote $\mathrm{Hom}_G(U_\lambda, \mathcal{C}^k)$, known from the preceding subsections. Within this context the Schur–Weyl Duality Theorem is as follows.

Theorem 4.6. *For every $\lambda \in \mathrm{Spec}(\rho_k)$ there are irreducible, mutually inequivalent S_k-modules E_λ and irreducible, mutually inequivalent $\mathrm{GL}(n, \mathbb{C})$-modules U_λ such that*

$$\bigotimes^k \mathbb{C}^n \cong \bigoplus_{\lambda \in Spec(\rho_k)} E_\lambda \otimes U_\lambda$$

as a representation of $S_k \times GL(n, \mathbb{C})$. The module E_λ uniquely determines U_λ and vice versa.

The proof relies on both the General Duality Theorem and the Double Commutant Theorem. In order for the latter to be applicable, the spans $\mathcal{A} = \mathrm{Span}(\rho_k)$ and $\mathcal{B} = \mathrm{Span}(\sigma_k)$, which in contrast to the images of the representations are unital and associative algebras, will need to be used. The proof is an extended rephrasing of the one given in ([2], pg. 200).

Proof. It will be shown that $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ and $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$. Because these algebras are then each other's commutants, the theory from Subsection 4.2 and in particular the General Duality Theorem (Theorem 4.5) becomes applicable. Within this context, $\mathrm{Span}(\sigma_k)$ will fill the role of $\mathcal{R}^G$, for $G = \mathrm{GL}(n, \mathbb{C})$.

By Proposition 3.6, and because S_k is finite, both ρ_k and σ_k are completely reducible, and $\mathcal{C}^k$ is a completely reducible $\mathcal{A}$-module. Because σ_k commutes with ρ_k, $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$. If it is shown that $\mathrm{Comm}(\mathcal{B}) \subset \mathcal{A}$, then $\mathrm{Comm}(\mathcal{B}) = \mathcal{A}$ and $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$ will follow. This is implicated by the Double Commutant Theorem: by mutual inclusion $\mathcal{A} = \mathrm{Comm}(\mathcal{B})$, and because the conditions for the applications of the Double Commutant Theorem have been met, $\mathrm{Comm}(\mathcal{A}) = \mathcal{B}$. It will now be proven that $\mathrm{Comm}(\mathcal{B}) \subset \mathcal{A}$.

Let $\{e_1, ..., e_k\}$ be the standard basis for $\mathbb{C}^n$. For an ordered k-tuple $(i_1, ..., i_k)$ with $1 \leq i_j \leq n$ for all j, define $|I| = k$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$. The tensors $\{e_I\}$ for all such I then form a basis of $\mathcal{C}^k$. The representation σ_k acts on these basis elements as given in the preceding discussion:

$$\sigma_k(s)e_I = e_{s \cdot I} \text{ where } s \cdot I = s \cdot (i_1, ..., i_k) = (i_{s^{-1}(1)}, ..., i_{s^{-1}(k)}) \text{ for } s \in S_k.$$

Now let $T \in \mathrm{End}(\mathcal{C}^k)$, and let its action relative to the basis $\{e_I\}$ per basis element be given by the matrix $[t_{I,J}]$:

$$Te_J = \sum_I t_{I,J} e_I.$$

Setting $T(\sigma_k(s)(e_J)) = \sigma_k(s)(Te_J)$ for $s \in S_k$, it follows that T commutes with elements of $\mathcal{B}$ if and only if:

$$t_{s \cdot I, s \cdot J} = t_{I,J} \tag{1}$$

for all I, J and all $s \in S_k$. To see this, write:

$$T(\sigma_k(s)e_J) = T(e_{s \cdot J}) = \sum_I t_{I, s \cdot J} e_I = \sum_I t_{s \cdot I, s \cdot J} e_{s \cdot I}$$

$$\sigma_k(s)(Te_J) = \sum_I t_{I,J} e_{s \cdot I}.$$

Let $(\cdot, \cdot) : \mathrm{End}(\mathcal{C}^k) \times \mathrm{End}(\mathcal{C}^k) \to \mathbb{C}$ denote the non-degenerate bilinear form $\mathrm{tr}(\cdot, \cdot)$. The restriction to $\mathrm{Comm}(\mathcal{B})$ of this form is non-degenerate, as will be demonstrated using the following projection:

$$P : \mathrm{End}(\mathcal{C}^k) \to \mathrm{Comm}(\mathcal{B}),\ P : X \mapsto \tfrac{1}{k!} \sum_{s \in S_k} \sigma_k(s) X \sigma_k(s)^{-1}.$$

Its image $P(\mathrm{End}(\mathcal{C}^k))$ is indeed a subset of $\mathrm{Comm}(\mathcal{B})$ as for all elements in $\mathcal{B}$, their action on $P_X = P(X)$ merely rearranges the terms of the sum. For all elements P_X in its image, and $T \in \mathrm{Comm}(\mathcal{B})$:

$$\begin{aligned}
(P_X, T) &= \frac{1}{k!} \sum_{s \in S_k} \mathrm{tr}(\sigma_k(s) X \sigma_k(s)^{-1} T) \\
&= \frac{1}{k!} \sum_{s \in S_k} \mathrm{tr}(\sigma_k(s) X T \sigma_k(s)^{-1}) \\
&= \frac{1}{k!}\, k!\ \mathrm{tr}(X, T) \\
&= (X, T),
\end{aligned}$$

due to the facts that T and σ_k commute and the trace function is conjugation invariant. Hence $(\mathrm{Comm}(\mathcal{B}), T) = 0$ implies $(X, T) = 0$ for all $X \in \mathrm{End}(\mathcal{C}^k)$ and by the non-degeneracy of $(\cdot, \cdot)$ this must imply $T = 0$. Therefore the trace form is non-degenerate on $\mathrm{Comm}(\mathcal{B})$. It is known that $\mathcal{A} \subset \mathrm{Comm}(\mathcal{B})$. Thus for the reverse inclusion it suffices to show that the orthocomplement of $\mathcal{A}$ in $\mathrm{Comm}(\mathcal{B})$ is trivial. Let $T \in \mathrm{Comm}(\mathcal{B})$ be orthogonal to $\mathcal{A}$. If $g = [g_{i,j}] \in \mathrm{GL}(n, \mathbb{C})$ then $\rho_k(g)$ has matrix $g_{I,J} = g_{i_1 j_1} \cdots g_{i_k j_k}$ relative to the basis $\{e_I\}$ due to the multilinearity of the tensor product. Therefore the orthogonality assumption is expressed as:

$$(T, \rho_k(g)) = \sum_{I,J} t_{I,J} g_{i_1 j_1} \cdots g_{i_k j_k} = 0 \tag{2}$$

for all $g \in \mathrm{GL}(n, \mathbb{C})$. Let $f_T : \mathrm{M}(n, \mathbb{C}) \to \mathbb{C}$ define the following polynomial function:

$$f_T(X) = \sum_{I,J} t_{I,J} x_{i_1 j_1} \cdots x_{i_k j_k} \text{ for } X = [x_{i,j}] \in \mathrm{M}(n, \mathbb{C}).$$

For $\det : \mathrm{M}(n, \mathbb{C}) \to \mathbb{C}$ it follows from (2) that $\det(X) f_T(X) = 0$ for all $X \in \mathrm{M}(n, \mathbb{C})$ by continuity of f_T. Hence f_T must be identically zero, and for all $[x_{i,j}] \in \mathrm{M}(n, \mathbb{C})$:

$$\sum_{I,J} t_{I,J} x_{i_1 j_1} \cdots x_{i_k j_k} = 0. \tag{3}$$

It will be shown that it follows from (1) and (3) that $t_{I,J} = 0$ for every pair I, J, and therefore $T = 0$, concluding the proof. To achieve this, (3) will be rewritten as a sum over representatives of equivalence classes rather than over pairs of multi-indices.

Denote $x_{i_1 j_1} \cdots x_{i_k j_k}$ by $x_{I,J}$ and note that they can be treated as monomials $x_{I,J} : \mathrm{M}(n, \mathbb{C}) \to \mathbb{C}$. Let Ξ be the set of all ordered pairs (I, J) of multi-indices of length k. Then S_k acts naturally on Ξ by $s \cdot (I, J) = (s \cdot I, s \cdot J)$ for $s \in S_k$, and this action defines an equivalence relation $\sim$ on Ξ where $(I, J) \sim (I', J')$ if and only if there exists an $s \in S_k$ so that $s \cdot (I, J) = (I', J')$. Choose Γ to be a set of representatives of the equivalence classes in $\Xi / \sim$. Let (I, J) be any pair of multi-indices in Ξ. Then there exists a $\gamma \in \Gamma$ so that $(I, J) \in [\gamma]$. Since all factors in the monomial x_γ commute, it is deduced for all $s \in S_k$ that $x_\gamma = x_{s \cdot \gamma}$, hence the value of x_γ is not dependent on the choice of the representative γ for the class. Furthermore it follows that the value of the sum of the $x_{I,J}$ over the multi-indices (I,J) in a class $[\gamma]$ equals x_γ multiplied by the number of elements in the class $[\gamma]$.

If $x_{I,J} = x_{I',J'}$ for a second pair of multi-indices $(I',J') \in \Xi$ then $x_{I,J}$ and $x_{I',J'}$ are equal as monomials over $\mathrm{M}(n,\mathbb{C})$ and must therefore consist of the same factors $x_{i.,j.}$. These factors are not necessarily in the same order, but there must exist a permutation $s \in S_k$ that orders them in the same way so that $(I,J) = s \cdot (I',J')$. It follows that if $x_{I,J} = x_\gamma$ and $x_{I,J} = x_{I',J'}$ then $(I',J') \in [\gamma]$. This shows that $[\gamma]$ is uniquely determined by x_γ.

For $\gamma \in \Gamma$ let $n_\gamma = |S_k \cdot \gamma|$ denote the cardinality of the orbit, and therefore the size of the equivalence class. If the coefficients $t_{I,J}$ of T satisfy (1) and (3) then as $t_{I,J} = t_\gamma$ for all $(I,J) \in S_k \cdot \gamma$:

$$\sum_{I,J} t_{I,J} x_{i_1 j_1} \cdots x_{i_k j_k} = \sum_{\gamma \in \Gamma} n_\gamma t_\gamma x_\gamma = 0.$$

The linear independence of the set of monomials $\{x_\gamma \mid \gamma \in \Gamma\}$ then implies that for all $\gamma \in \Gamma$, $n_\gamma t_\gamma = 0$. But per definition $n_\gamma \geq 1$ so this must mean that $t_\gamma = 0$ and therefore $t_{I,J} = 0$ for all $(I,J) \in \Xi$. Hence it is proven that $T = 0$.

$\square$

With the proof of this theorem, Schur–Weyl Duality has been established.

5 Outlook

It has been established that there is a correspondence between the S_k-modules and $\mathrm{GL}(n, \mathbb{C})$-modules described in the statement of the Schur–Weyl Duality Theorem, but not if one can be explicitly determined given the other. As it turns out, this is possible. It is achieved through the study of the irreducible characters of S_k and $\mathrm{GL}(n, \mathbb{C})$, the latter of which can be determined through the Weyl Character Formula. A thorough treatment of this theory and possible applications of Schur–Weyl Duality can be found in ([2], ch. 9).

Literature

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