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Radiative corrections to the anomalous conductivities in holography

Master Thesis

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Abstract

Quantum anomalies are not only one of the more subtle effects that appear in quantum field theory but it is also known that as a consequence of them the hydrodynamics of anomalous chiral imbalanced media presents $\mathcal{P}$-odd and $\mathcal{T}$-even transport. Macroscopically the axial anomaly allows for the generation of vector and axial currents due to external magnetic fields and rotations. The associated transport coefficients have been studied in chiral theories without dynamical gauge fields and it has been found that in this case they are subject to non-renormalization and universality theorems. In a theory that is also coupled to dynamical color fields, such as QCD, there is an extra gluon contribution to the anomalous Ward identities and it is expected for the anomalous conductivities to renormalize and receive corrections when this term is present. On this thesis corrections to the anomalous transport coefficients due to the axial-gluon anomaly are presented, the calculations are done in the strongly coupled regime using the gauge/gravity duality on a universality class of two derivative holographic models.

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Contents

Introduction

Theories that involve massless fermions that transform on either the right handed or left handed representations of the Lorentz group are known as chiral. Chiral theories have played a significant role in theoretical physics, an example of them is the standard model of particle physics. They not only appear on high energy physics but also show up as effective models in condensed matter and cosmology [1].

The dynamics of chiral theories coupled to gauge fields as described by quantum field theory is very rich. One of the most subtle phenomena that they exhibit is the appearance of quantum anomalies. A quantum anomaly is the breaking of a symmetry that existed at the classical level in the Lagrangian by the incorporation of quantum corrections. In particular for chiral theories the symmetry that is broken is the axial one.

The axial current is directly related with the axial charge that characterizes the differences between the number of right handed and left handed fermions. As a consequence of the anomaly the axial charge will not be conserved and this will imply that the number of right handed fermions will in general be different from the number of left handed fermions, in the case of QCD we know that the non trivial topological configurations of the gluons are directly related to a non zero axial charge.

A non zero axial charge can be described at the Lagrangian level by the inclusion of an axial chemical potential, in analogy to the way finite density is incorporated in QFT. This axial chemical potential comes from a non-zero temporal component of an axial gauge field coupled to the axial current. The inclusion of this term implies the possibility of having parity violating effects on the theory.

The presence of the axial anomaly together with a non zero axial charge has macroscopical consequences in the hydrodynamic description of chiral theories. The combination of this both effects allow for non dissipative $\mathcal{P}$-odd and $\mathcal{T}$-even transport coefficients, this means that the generation of currents due to external magnetic fields or to rotations (characterized by a vorticity) will now be allowed at the first order of the hydrodynamic expansion. These novel transport phenomena is collectively known as anomalous transport and the evaluation of the related transport coefficients will be one of the main goals of this work. It should be mentioned that anomalous transport in general is not limited to Ohm's law like behavior but it also considers collective excitations such as the chiral magnetic wave, an anomalous analogue of sound waves.

Anomalous transport manifest itself on different contexts [2] such as the interaction of the quark gluon plasma generated at heavy ion collisions at RHIC or CERN with the strong magnetic fields generated thereby [3], in cosmology it appears as a possible mechanism for the generation of primordial magnetic fields in the early universe due to the chiral imbalance in the primordial electroweak plasma [4, 5] and it has also been proposed as a possible mechanism of the kick in velocity in neutron stars [6]. With the discovery of Dirac and Weyl semimetals [7] anomalous transport has also gained some attention in the condensed matter community where recently and for the first time its effects where measured in an experiment involving $ZrTe_5$ [8]. Besides all its possible applications in physics phenomenology and to technology due to their non dissipative nature the study of anomalous transport in chiral media is interesting by itself as it is a macroscopic manifestation of one of the more subtle quantum field theory effects.

The anomalous conductivities have been widely studied in the context of hydrodynamics [9], effective field theory [10] and holography[11]. Their values have been found to be closely related to the anomaly coefficients that appear in the anomalous Ward identities. The anomalous terms on these identities come from the coupling to external gauge fields, gravitational anomalies that arise when the theory is defined on a curved background spacetime and from the presence of dynamical gauge fields that couple to the theory via a color interaction with the fermions, in QCD these color fields are identified with the gluons. Up to now the study of these coefficients has mainly been done assuming the gluon contribution is not present and under this assumption it has been found that the conductivities are subject to universality and non-renormalization theorems [12].

Once the gluon contributions are taken into account it is expected that some of these conductivities will renormalize and that their value will differ from the universal result [13]. Corrections to their universal value from the coupling with dynamical gauge fields have only been studied analytically on the weak coupling and large N limit using diagrammatic techniques [13] and numerically in the strong coupling regime using holographic techniques [14]. In lattice calculations of the anomalous conductivities a difference of one order of magnitude from the predicted universal result has been observed [15, 16]. A possible reason for this disparity could come from the gluon contribution of the anomaly that is present in the lattice calculations as in there the full color interaction is taken into account.

On this thesis work we will calculate analytically the corrections to the anomalous conductivities due to the inclusion of gluon contributions to the anomaly in the strong coupling regime by using holographic techniques. We will take a bottom up approach and consider a universality class of two derivative gravitational models to derive a general formula for the corrections in terms of the holographic background information. In particular we will solve the problem for a generic holographic theory at finite density and temperature in terms of a series expansion in a small parameter Δ that characterizes the relevance of the gluon fields in the hydrodynamic limit. Novel analytic results for specific holographic backgrounds will be obtained and the effects of these corrections on the conductivities will be discussed.

The thesis is organized as follows:

- The first two chapters have a review nature of the microscopic and macroscopic physics of chiral theories. Chapter 1 deals with anomalies in chiral quantum field theories from a path integral perspective while chapter 2 deals with the hydrodynamic limit of chiral theories with a particular emphasis on the anomalous sector at first order in the hydrodynamic description.
- Chapter 3 has a dual function: To review the gauge gravity correspondence and to introduce the universality class of holographic models that is going to be considered for the rest of the work.
- Chapter 4 and 5 are the core of the thesis work. Chapter 4 deals with an analysis of the class of holographic backgrounds that will be considered for the holographic calculations. In chapter 5 the corrections to the anomalous conductivities is calculated from a linear response approach.

- The final chapter contains the conclusions of the work together with some outlook and motivation for future work.

It should be emphasized that chapter 5 contains the main result of this work, which is an analytic and generic procedure and formula for calculating gluon radiative corrections to the anomalous conductivities in the strong coupling limit. This result should be taken as valid for a chiral quantum field theory in the large number of colors N_c and large number of flavors N_f limit with a constant $\frac{N_f}{N_c}$ ratio.

Chapter 1

Chiral Anomalies in Quantum Field Theory

Quantum anomalies are one of the more subtle phenomena that appear when quantizing a field theory. Their appearance signals the breaking of a symmetry of the field theory that appeared on the classical level. They exist due to the non-invariance of the path integral measure under the now anomalous classical symmetry and have direct consequences on the theory by modifying the Ward identities of the classically conserved current.

For this work we are particularly interested on chiral theories in four spacetime dimensions, namely quantum field theories with chiral fermions. In particular we will consider models containing N_f right handed and N_f left handed Weyl fermions transforming non-trivially under some N_c color group. To make this more explicit the following action can be considered

$$S = -\int_{\mathcal{M}} dx^4 \sqrt{-g} \left[\frac{\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right)}{2g_{\mathrm{YM}}^2} + \bar{\Psi} \not{D} \Psi \right] + \int_{\mathcal{M}} \frac{\theta}{8\pi^2} \mathrm{Tr}\left(G \wedge G\right) \quad (1.1)$$

Where G is the field strength of the color gauge field, g_{YM} is the coupling constant of the gluons to the fermions, θ is an axion field related to CP violation, Ψ is an array of N_f Dirac fermions and $\not{D}$ represents a covariant derivative which in principle includes not only the dynamical color gauge fields but also the coupling to external currents via non-dynamical gauge fields. We have to note that the quantum field theory is defined over an arbitrary curved fixed background manifold $\mathcal{M}$ with metric g.

As the main future application of this work is aimed towards the study of transport properties in the strongly coupled chirally symmetric deconfined phase of the quark gluon plasma that appears on high energy heavy ion collisions at RHIC,Brookhaven and LHC,CERN it is important to include CP-odd effects, as they reflect the topological character of QCD like theories.

Our generic chiral model described by the action (1.1) exhibits a $SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$ symmetry on the classical level that arise from transforming independently the right and left handed fermions as

$$\begin{aligned} P_+\Psi &\to U P_+\Psi \qquad P_-\Psi \to V P_-\Psi \\ P_+\Psi &\to e^{i\alpha_+} P_+\Psi \qquad P_-\Psi \to e^{i\alpha_-} P_-\Psi \end{aligned} \tag{1.2}$$

Where $P_\pm = \frac{1\pm\gamma_5}{2}\mathbb{I}_f$ is the projector operator that separates the right handed component from the left handed component of the N_f Dirac fermions, U and V are independent $SU(N_f)$ matrices and $\alpha_\pm$ are $U(1)$ phases. It is more convenient to combine the $U(1)_L \times U(1)_R$ symmetry into a $U(1)_A \times U(1)_V$ one instead, which in turn will give rise to a vector and axial classically conserved currents given by

$$\begin{aligned} \mathcal{J}^\mu &= \bar{\Psi}\gamma^\mu\Psi \\ \mathcal{J}^\mu_A &= \bar{\Psi}\gamma^\mu\gamma_5\Psi \end{aligned} \tag{1.3}$$

The $SU(N_f)_L \times SU(N_f)_R$ symmetry is going to be spontaneously broken into $SU(N_F)_V$ while the $U(1)_A$ will become anomalous leaving only a conserved $U(1)_V$ vector current[1].

The non-conservation of the axial current due to quantum corrections can be seen from the (anomalous) Ward identities for the currents, which are given by

$$\begin{aligned} \nabla_\mu\langle\mathcal{J}^\mu\rangle &= 0 \\ \nabla_\mu\langle\mathcal{J}^\mu_A\rangle = \frac{\epsilon^{\mu\nu\rho\sigma}}{4}&\left[a_0 F^V_{\mu\nu}F^V_{\rho\sigma} + a_1 F^A_{\mu\nu}F^A_{\rho\sigma} + a_2 \mathrm{Tr}\left(\mathcal{R}_{\mu\nu}\mathcal{R}_{\rho\sigma}\right)\right] \\ &+ \frac{\epsilon^{\mu\nu\rho\sigma}}{4}\left[a_3 \mathrm{Tr}\left(G_{\mu\nu}G_{\rho\sigma}\right)\right] \end{aligned} \tag{1.4}$$

[1]There exist some arbitrariness on the election of conservation of the currents coming from the choice of regularization of the action. The motivation to keep a conserved vector current and turn the axial into the anomalous one is physical as in principle we will like to be able to gauge this symmetry by relating it to the electrodynamics one.

Where $F^A = dA$ and $F^V = dV$ with A and V external non-dynamical gauge fields that couple to the currents via the terms $A_\mu \mathcal{J}^\mu_A$ and $V_\mu \mathcal{J}^\mu$ through the covariant derivative in the Lagrangian[2], $\mathcal{R}$ is the curvature 2-form of the underlying manifold $\mathcal{M}$ given by $\mathcal{R} = \frac{1}{2} R^\mu_{\nu\rho\sigma} dx^\rho \wedge dx^\sigma$ with $R^\mu_{\nu\rho\sigma}$ the Riemann tensor, and the coefficients a_i are explicitly related to the anomalies and are given by

$$a_0 = \frac{N_c \sum q_f^2}{4\pi^2} \qquad a_1 = \frac{N_c N_f}{12\pi^2} \qquad a_2 = \frac{N_c N_f}{96\pi^2} \qquad a_3 = \frac{N_f}{4\pi^2} \tag{1.5}$$

where q_f represents the strength of the coupling of each flavor of quark with the vector current. The terms associated to the coefficients a_0 and a_1 are related to the breaking of the global chiral axial symmetry, the one associated to a_2 is the known gravitational anomaly related to the breaking of diffeomorphism invariance[3] and the one associated to a_3 is a mixed global-gauge anomaly due to the dynamical gluon gauge fields. The chiral and gravitational anomalies are related to global symmetries and can therefor disappear in the absence of external fields in flat spacetime. On the other hand the mixed global-gauge anomaly can not be turned off and will make the axial current anomalous even in the absence of external fields.

The vector and axial currents can be defined differently according to the desired physical constraints that we want to impose, this freedom will give rise to the concept of covariant currents J^μ and J^μ_A. The form used in (1.4) is known as the consistent one and the covariant ones are related to them via a Bardeen-Zumino polynomial as[4]

$$\begin{aligned} J^\mu &= \mathcal{J}^\mu + \frac{N_c N_f}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} A_\nu F^V_{\rho\sigma} \\ J^\mu_A &= \mathcal{J}^\mu_A + \frac{N_c N_f}{12\pi^2} \epsilon^{\mu\nu\rho\sigma} A_\nu F^A_{\rho\sigma} \end{aligned} \tag{1.6}$$

On the following sections it will be shown how the coefficients a_i are calculated and the concept of chiral imbalance will be revised.

[2] We will relate the vector field V to an external electromagnetic field while the axial field A is included for computational purposes as axial fields do not exist in nature.

[3] Just as we did with the $U(1)_V$ symmetry we have chosen to shift this anomaly into the axial current and keep a conserved energy momentum tensor in the absence of external sources

[4] The consistent definition of the currents is physically meaningful and satisfy the Wess-Zumino integrability condition [17] while the covariant one give us a gauge invariant (but still anomalous) definition of the currents [11] that becomes useful in hydrodynamics.

1.1 Path Integrals and Quantum anomalies

On this section we will sketch how the previously shown anomaly coefficients a_i are calculated using path integral methods. The method here outlined is based on the original work of Fujikawa [18] and follows closely the reviews on anomalies done by Fujikawa and Bilal [19, 17]. For starters we will rewrite the Lagrangian of our chiral model in the following way

$$\mathcal{L}[\Psi, \bar{\Psi}, D_\mu\Psi, D_\mu\bar{\Psi}, \mathcal{A}, \theta] = \mathcal{L}_{\text{matter}}[\Psi, \bar{\Psi}, D_\mu\Psi, D_\mu\bar{\Psi}, \mathcal{A}] + \mathcal{L}_{\mathcal{A}}[\mathcal{A}, \theta] \quad (1.7)$$

Where $\mathcal{L}_{\mathcal{A}}$ contains all the terms in the Lagrangian related exclusively to the dynamical gauge fields $\mathcal{A}$ as well as the θ term, and $\mathcal{L}_{\text{matter}}$ contains the fermion related terms as well as their coupling to dynamical and external gauge fields. The more general form that we consider for $\mathcal{L}_{\text{matter}}$ is[5]

$$\mathcal{L}_{\text{matter}} = -\sqrt{-g}\bar{\Psi}e^\mu_k\gamma^k \left[\partial_\mu - i(V_\mu q_f + A_\mu\gamma^5 + \mathcal{A}^a_\mu t^a + \frac{\omega^{mn}_\mu S_{mn}}{2})\right]\Psi \quad (1.8)$$

Where e^μ_k are the components of the tetrad field, the gluon field has been written on its Lie algebra valued form with t^a the generators of $SU(N_c)$ in the adjoint representation, ω^{mn}_μ is the spin connection, and $S_{mn} = \frac{i}{4}[\gamma_m, \gamma_n]$ are the generators of the Lorentz group on its spinor representation. The greek indices denote spacetime components, latin index a is an adjoint gauge group index and latin indices k, m, n are Minkowsky spacetime indices. The term q_f indicates that the coupling is different for each of the flavors and an implicit multiplication in flavor space in the Lagrangian is assumed.

To see the usefulness of considering separately the matter sector of the theory we can recall that when performing a functional integration to calculate vacuum expectation values of time-order operators $\mathcal{O}_i$ that include matter fields it is possible to separate the integration in two steps: First compute the integral over the matter fields

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)e^{i\int \mathcal{L}_{\text{matter}}[\Psi,\bar{\Psi},D_\mu\Psi,D_\mu\bar{\Psi},\mathcal{A}]} \quad (1.9)$$

and afterwards perform the integral over the gauge fields. The complications of gauge fixing and ghosts only appear on this second step and then to study chiral anomalies it is sufficient to only look at the matter sector.

[5] For a quick review of field theory in curved space time see appendix B of [19]

Performing the first step is enough when the operators $\mathcal{O}_i$ correspond to currents such as $J^{\mu}_{\text{matter}} = \frac{\partial \mathcal{L}_{\text{matter}}}{\partial \mathcal{V}_\mu}$ where $\mathcal{V}$ represents any gauge field. The vacuum expectation value of such current operators can be calculated by taking functional derivatives with respect to the gauge fields $\mathcal{V}$ of the effective action $\mathcal{W}[\mathcal{A}, A, V]$ defined as

$$e^{i\mathcal{W}[\mathcal{A},A,V]} = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{i\int \mathcal{L}_{\text{matter}}} \tag{1.10}$$

From now on our focus will be on (1.10) and how all the terms involved on that expression transform under a local chiral transformation and under a local gauge transformation. From the path integral point of view both of these are change of variables and as such the result should be indepedent of it. We will use that to derive the anomalous ward identities and also to relate the gauge transformation of the effective action with the quantum anomaly.

Chiral transformation

A local chiral transformation of the fields of the following form will be considered

$$\begin{aligned} \Psi' &= U(x)\Psi = e^{i\alpha(x)+i\beta(x)\gamma_5}\Psi \\ \bar{\Psi}' &= \bar{\Psi}\bar{U}(x) = \bar{\Psi}e^{-i\alpha(x)+i\beta(x)\gamma_5} \end{aligned} \tag{1.11}$$

In here the properties of the γ_5 matrix have been used, and $\alpha(x)$ and $\beta(x)$ denote the infinitesimal parameters of the local chiral transformation. Under this transformation the measure of the path integral in (1.10) and the action related to the matter Lagrangian transform like

$$\begin{aligned} S'_{\text{matter}} = S_{\text{matter}} + \int d^4x\sqrt{-g}\,[(i\alpha)\,\nabla_\mu \mathcal{J}^\mu + (i\beta)\,\nabla_\mu \mathcal{J}^\mu_A] \\ D\Psi' D\Psi' = D\Psi D\Psi J \end{aligned} \tag{1.12}$$

Where J is the Jacobian that comes from the change of variables. To see that this Jacobian is not necessarily equal to one it is sufficient to recall that the fermion Grassman measure will transform under (1.11) as

$$D\Psi' = (\text{Det}\,\mathcal{U})^{-1} D\Psi \qquad D\bar{\Psi}' = \left(\text{Det}\,\bar{\mathcal{U}}\right)^{-1} D\bar{\Psi} \tag{1.13}$$

Where the operator $\mathcal{U}$ has matrix components in configuration space given by $\langle x|\mathcal{U}|y\rangle = U(x)\delta^4(x-y)$. From this transformation rule we can see that as $\bar{U}(x) \neq U(x)^{-1}$ then $(\text{Det}\,\mathcal{U})^{-1}\left(\text{Det}\,\bar{\mathcal{U}}\right)^{-1} \neq 1$. On this particular basis the determinant of $\mathcal{U}$ and $\bar{\mathcal{U}}$ look like

$$\begin{aligned}(\text{Det}\,\mathcal{U})^{-1} &= e^{-\text{tr}\ln\mathcal{U}} = e^{-\int d^4x\delta^4(0)\text{Tr}(i\alpha(x)+i\beta(x)\gamma_5)}\\ \left(\text{Det}\,\bar{\mathcal{U}}\right)^{-1} &= e^{-\text{tr}\ln\bar{\mathcal{U}}} = e^{-\int d^4x\delta^4(0)\text{Tr}(-i\alpha(x)+i\beta(x)\gamma_5)}\end{aligned} \tag{1.14}$$

Where tr is a functional and matrix trace while Tr is only a matrix trace with respect to the spinor, flavor and color indices. From these expressions the Jacobian is found to be given by

$$\ln J = -2i \lim_{N\to\infty} \sum_n^N \int d^4x\sqrt{-g}\beta(x)\text{Tr}\left(\phi_n^\dagger(x)\gamma_5\phi_n(x)\right) \tag{1.15}$$

where we have performed a change of basis from configuration space to the basis of eigenstates ϕ_n of the Dirac operator $\not{D}$[6], this operator can be read from (1.8), and is given by

$$\not{D} \equiv e_k^\mu\gamma^k\left[\partial_\mu - i\left(V_\mu q_f + A_\mu\gamma^5 + \mathcal{A}_\mu^a t^a + \frac{1}{2}\omega_\mu^{mn}S_{mn}\right)\right] \tag{1.16}$$

The term $\sum_n \phi_n^\dagger(x)\phi_n(x)$ is a $\delta(0)$ term in disguised and as such the Jacobian is found to be divergent, it is from the regularization of this term that the anomaly will arise. Recalling that ϕ_n are eigenfunctions of a gauge invariant operator with eigenvalues λ_n it is possible to regulate the Jacobian by including a regulator in the previous expression as

$$\ln J = -2i \lim_{\Lambda\to\infty} \sum_n^\infty \int d^4x\sqrt{-g}\beta(x)\text{Tr}\left[\phi_n^\dagger(x)\gamma_5 f\left(\frac{\lambda_n^2}{\Lambda^2}\right)\phi_n(x)\right] \tag{1.17}$$

Where Λ is a high-frequency cut-off and $f(x)$ is a regulator function that should satisfy

$$f(0) = 1 \qquad f(\infty) = 0 \qquad xf'(x)|_{x=0} = xf'(x)|_{x=\infty} = 0 \tag{1.18}$$

[6] In principle we are able to use any orthonormal basis but choosing the Dirac operator as the source of this basis is a natural choice as it is gauge invariant and its eigenvectors diagonalize S_{matter} directly relating the Dirac operator with the effective action through $\mathcal{W} \sim \text{Tr}\log\left(\not{D}\right)$

At this point and before continuing we will introduce the definition

$$\mathcal{A}_5 = \lim_{\Lambda\to\infty} \sum_n^{\infty} \mathrm{Tr}\left[\phi_n^\dagger(x)\gamma_5 f\left(\frac{\not{D}^2}{\Lambda^2}\right)\phi_n(x)\right] \tag{1.19}$$

Where $\mathcal{A}_5$ will be called the anomaly function or simply the anomaly. The calculation of the anomaly will not depend on the regulator function as long as it satisfies the conditions (1.18). Evaluating $\mathcal{A}_5$ in curved spacetime is a somewhat lenghty procedure and it can be found in [19] on full detail, we will limit ourselves here to present the final result given by

$$\begin{aligned}\mathcal{A}_5 = \frac{\epsilon^{\mu\nu\rho\sigma}}{32\pi^2}&\left[N_c N_f F^V_{\mu\nu}F^V_{\rho\sigma} + \frac{N_c\sum q_f^2}{3}F^A_{\mu\nu}F^A_{\rho\sigma} + N_f \mathrm{Tr}\left(G_{\mu\nu}G_{\rho\sigma}\right)\right]\\ &+\frac{N_c N_f}{768\pi^2}\epsilon^{\mu\nu\rho\sigma}R^\alpha_{\beta\mu\nu}R^\beta_{\alpha\rho\sigma}\end{aligned} \tag{1.20}$$

Before continuing one final comment about the anomaly is in order. The anomaly is a one loop effect, due to being a determinant, that scales with the coupling constants as $\mathcal{O}\left(g^2_{\mathrm{YM}}q_f\right)$[7] which then implies that the anomaly manifest itself on one loop triangle diagrams and it is on this type of diagrams that the anomaly was originally discovered [20].

After this comment the calculation of the transformation rule of all the elements present in (1.10) can be considered complete. Consequently under a chiral transformation the path integral transforms like

$$\begin{aligned}\int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_{\mathrm{matter}}} &= \int\mathcal{D}\Psi'\mathcal{D}\Psi' e^{iS'_{\mathrm{matter}}}\\ = \int\mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_{\mathrm{matter}}}&\left[1+\int d^4x\sqrt{-g}i\left[\alpha\left(\nabla_\mu\mathcal{J}^\mu\ \right)+\beta\left(\nabla_\mu\mathcal{J}^\mu_A-2\mathcal{A}_5\right)\right]+...\right]\end{aligned}$$

As α and β are independent we can read off the desired Ward identities from the last expression

$$\nabla_\mu\langle\mathcal{J}^\mu\rangle = 0 \qquad \nabla_\mu\langle\mathcal{J}^\mu_A\rangle = 2\mathcal{A}_5 \tag{1.21}$$

By using (1.20) and (1.21) we can see that the coefficients a_i from (1.4) are the ones presented in (1.5).

[7]The chosen normalization of the color gauge fields does not show this explicitly but once we rescale the gluon fields with a factor of g_{YM} the right scaling will show up

Gauge transformation

In addition to the chiral transformation presented in (1.11) it is possible to perform the full gauge transformation by transforming the external vector and axial gauge fields in the usual way

$$\begin{aligned} \delta V'_\mu &= V_\mu + \partial_\mu \alpha(x) \\ \delta A'_\mu &= A_\mu + \partial_\mu \beta(x) \end{aligned} \tag{1.22}$$

Once the full gauge transformation is taken into account the action is going to remain invariant, and the transformation of the right hand side of (1.10) will only have the Jacobian contribution accounting to

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_{\text{matter}}} = e^{-2i\int d^4x\sqrt{-g}\ \beta\mathcal{A}_5} \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{iS_{\text{matter}}} \tag{1.23}$$

The effective action on the left hand side of (1.10) will now transform, and such transformation will be denote as

$$e^{i\mathcal{W}[\mathcal{A},A,V]} = e^{i\mathcal{W}[\mathcal{A},A',V']} = e^{i\mathcal{W}[\mathcal{A},A,V]+i\delta\mathcal{W}[\mathcal{A},A,V]} \tag{1.24}$$

Comparing (1.23) and (1.24) is possible to see that the gauge variation of the effective action will be determined by the anomaly according to

$$\delta\mathcal{W} = -\int \beta\left[a_1\left(3F^V\wedge F^V + F^A\wedge F^A\right) + a_2\mathrm{Tr}\left(\mathcal{R}\wedge\mathcal{R}\right) + a_3\mathrm{Tr}\left(G\wedge G\right)\right] \tag{1.25}$$

If we consider the effective action together with $S_\mathcal{A}$ the gauge transformation on the full path integral will be

$$\int \mathcal{D}\mathcal{A} e^{iS_\mathcal{A}+i\mathcal{W}+i\delta S_\mathcal{A}+i\delta\mathcal{W}} \tag{1.26}$$

As the vector and axial fields are non dynamical they do not appear on $S_\mathcal{A}$, however it is possible to transform the θ term in the following way

$$\theta' = \theta + 2N_f\beta \tag{1.27}$$

With such tranformation it is possible to cancel out the mixed global gauge anomaly coming from the gauge transformation.The total change in the final effective action will then only be due to the global anomalies.

1.2 Chiral Imbalance

The existence of the axial anomaly and the non conservation of the axial current implies that the associated chiral density n_5 will not be conserved. This density is given by

$$n_5 = n_+ - n_- \tag{1.28}$$

where $n_\pm$ are the number of left handed and right handed fermions respectively. The relationship between the anomaly and the chiral density does not end there as there is a deeper relationship between the two. This can be seen by realizing that the previously discussed anomaly $\mathcal{A}_5$ relates the analytical properties of the Dirac operator $\not{D}$ with the topological properties of the gauge field configuration. To see this we can first realize that the expression (1) can be written in topological terms as

$$\mathcal{A}_5 = 2\pi[\hat{A}\,(\mathcal{M})\,\mathrm{ch}\,(F)]_4 \tag{1.29}$$

where $\hat{A}\,(\mathcal{M})$ is the Dirac genus of the manifold, $\mathrm{ch}\,(F)$ is the Chern character and $[]_4$ denotes that they are evaluated in four dimensions.The next step is to integrate the anomaly using directly the Dirac basis obtaining

$$\int d^4x\sqrt{-g}\mathcal{A}_5 = n_+ - n_- = n_5 \tag{1.30}$$

The axial density n_5 is also known as the analytical index of the Dirac operator, while the quantity on the left hand side of the previous expression is known as its topological index. The relationship between the topological and analytical indices is known as the Atiyah-Singer index theorem [21] and it tell us that the anomaly is closely related to the topology of the quantum field theory and also to the chiral imbalance of the media that it describes. In the case of QCD in the absence of external fields the relationship (1.30) becomes the known relationship between the axial density and the gluons winding number.

It is possible to introduce finite axial density into the theory by the addition of an axial chemical potential μ_5 through a coupling in the Lagrangian such as

$$\mu_5\bar{\Psi}\gamma^0\gamma_5\Psi \tag{1.31}$$

this axial chemical potential then enters the model as the conjugate variable of the axial density. It can be thought of as coming from an axial coupling $\mathcal{J}_5^{\mu} A^{\mu}$ where the axial gauge field has been given a non zero time component. Although this is analogue to the way finite density is introduced in quantum field theory through a vector chemical potential μ it is important to note that contrary to that case the axial density is not conserved. This means that although μ_5 and n_5 are conjugate to each other, the axial chemical potential is not a true chemical potential since those can only be related to conserved charges. There has been some discussion regarding this issue in the literature [22] but a simple solution is to assume that any effect that relies on chiral imbalance should be thought of as occurring in a time scale smaller than the topological rate transition of the theory [23]. Under this philosophy we will think of the axial chemical potential as a true chemical potential in a local and temporal axial equilibrium.

Chapter 2

Hydrodynamics and Anomalous Transport

Hydrodynamics is currently understood as an effective field theory description of out of equilibrium systems on which local thermodynamic equilibrium is assumed. At the core of this description are the relevant currents of an underlying quantum field theory such as the energy momentum tensor and the global vector and axial U(1) currents. The description is based upon two assumptions: The equations of motion for the currents are given by the Ward identities[1] and additionally they can be written in terms of fluid variables such as the fluid velocity u^μ, the temperature T and the chemical potentials μ and μ_5 through the so called constitutive relations.

The equations of motion to be considered are the ones relevant to the $U(1)$ currents defined in accordance to the imposed physical conditions. The hydrodynamic description requires a gauge invariant description and then consequently the use of the covariant current is more appropriate. To relate now the covariant currents with thermodynamical variables it is convenient to first decompose the constitutive relations as

$$\begin{aligned} \langle J^\mu \rangle &= \mathcal{N} u^\mu + \nu^\mu \\ \langle J^\mu_A \rangle &= \mathcal{N}_A u^\mu + \nu^\mu_A \end{aligned} \tag{2.1}$$

[1]It is important to note that the gluons have no associated hydrodynamical variable as they do not represent any global current. Consequently all the discussion that follows on this chapter should be consider under the assumption that there is no gluonic contribution to the anomaly.

This decomposition is given in terms of the scalar, vector, and tensor components with respect to the rotational symmetry of the velocity field. The vectors ν^μ and ν_A^μ are both transverse to the fluid velocity and together with the scalars $\mathcal{N}$ and $\mathcal{N}_A$ are to be written in terms of thermodynamical variables. On the core of this procedure is the assumption that the relevant scales of variation of observables are larger than any microscopical scale of the system. These assumption allows for the currents to be organized in a gradient expansion usually called hydrodynamic expansion.

At zero order in the hydrodynamic expansion the following constitutive relations are the only ones that can be considered[2]

$$
\begin{aligned}
\mathcal{N} = \rho \qquad \mathcal{N}_A = \rho_A \\
\nu^\mu = 0 \qquad \nu_A^\mu = 0
\end{aligned}
\tag{2.2}
$$

where ρ and ρ_A are the vector and axial charge densities respectively [3]. At first order on the gradient expansion the symmetries of the system allow for the following relations[4] [26]

$$
\begin{aligned}
\mathcal{N} = \rho \qquad \mathcal{N}_A = \rho_A \\
\nu^\mu = \sigma_{VV} B^\mu + \sigma_{VA} B_5^\mu + \sigma_{V\omega}\omega^\mu + \sigma_b \nabla^{\mu\nu}\left(E_\nu^b - T\nabla_\nu\left(\frac{\mu^b}{T}\right)\right) \\
\nu_A^\mu = \sigma_{AV} B^\mu + \sigma_{AA} B^{5\mu} + \sigma_{A\omega}\omega^\mu + \sigma_b \nabla^{\mu\nu}\left(E_\nu^b - T\nabla_\nu\left(\frac{\mu^b}{T}\right)\right)
\end{aligned}
\tag{2.3}
$$

Where the index $b = \{V, A\}$ denotes if the chemical potential, electric field or conductivity σ_b are related to the vector or axial symmetry. The magnetic field B^μ, the axial magnetic field B_5^μ, the electric fields $E^{b\mu}$ and the vorticity ω^μ are defined as

$$
\begin{aligned}
B^\mu = \frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}^V \qquad B_5^\mu = \frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma} u_\nu F_{\rho\sigma}^A \\
\omega^\nu = \tilde{\epsilon}^{\mu\nu\rho\sigma} u_\nu \partial_\rho u_\sigma \qquad E_\mu^b = F_{\mu\nu}^b u^\nu
\end{aligned}
\tag{2.4}
$$

[2]See [24] for a review of relativistic hydrodynamics with general anomalous charges

[3]These relations need to be supplemented with the appropriate thermodynamic equations of state for the densities.

[4]At first order in derivatives there exist some ambiguities due to possible redefinitions of the local thermodynamic parameters. This ambiguity can be solved by choosing an specific frame on which to work, and for anomalous transport the most convenient turns out to be the no-drag frame [25]

Where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the metric independent Levi-Civita density with components ± 1 and F^b are the field strength of some external gauge field. The conductivities σ_{VV}, σ_{VA}, σ_{AV}, σ_{AA}, $\sigma_{V\omega}$ and $\sigma_{A\omega}$ are known as anomalous conductivities and will be the subject of discussion of the next section. We will not go further into the derivative expansion as it is out of the scope of the work, see [26] for an example of how this is done in the context of hydrodynamics.

From a microscopical point of view the constitutive relations should be seen as the one point functions for the U(1) current operators in a near equilibrium situation. Calculation of the transport coefficients (conductivities) in terms of the microscopic parameters of the theory can be done via linear response. This means performing a perturbation of an equilibrium state with an external source and then looking at the response of the system to this perturbation. In this work the following philosophy will be consider to calculate the conductivities:

- A static background at finite temperature and densities in the absence of any external electromagnetic fields will be considered.
- This background will be perturbed by external vector and axial gauge fields as well as adding rotation to the system.
- The fluctuation of the one point functions $\langle \delta J \rangle$ and $\langle \delta J_A \rangle$ due to these perturbation in the linear response regime will be calculated. Namely the zero frequency and zero momentum limit will be taken.
- The calculated fluctuation of the one point functions will be then compare to the anomalous sector of the constitute relationships through

$$\begin{aligned} \langle \delta J^\mu \rangle &= \sigma_{VV} B^\mu + \sigma_{VA} B_5^\mu + \sigma_{V\omega} \omega^\mu \\ \langle \delta J_5^\mu \rangle &= \sigma_{AV} B^\mu + \sigma_{AA} B_5^\mu + \sigma_{A\omega} \omega^\mu \end{aligned} \tag{2.5}$$

  and the conductivities will be read off from the comparison[5].

Alternatively the conductivities can be calculated using two point functions through the so called Kubo formulas, see [11, 13] for an example of how this is done in the context of field theory and holography respectively.

[5]The introduction of the coefficients proportional to the axial magnetic field B_5^μ is that of a tool, as σ_{VA} and σ_{AV} can be identified one with each other and it will turn out that σ_{VA} will be easier to calculate consequently giving us access to the σ_{VA} conductivity

2.1 Anomalous conductivities

The conductivities σ_{VV}, σ_{VA}, σ_{AV}, σ_{AA}, $\sigma_{V\omega}$ and $\sigma_{A\omega}$ are called anomalous conductivities as they can only exist when there is an anomalous charge in the underlying quantum field theory. To see that the anomaly is essential it is necessary to calculate the conductivities and see their dependance with respect to the anomaly coefficients a_i, in particular the conductivities will vanish in an anomaly free theory. They are a macroscopical manifestation of a quantum effect and from considerations of $\mathcal{P}$ and $\mathcal{T}$ symmetries it can be seen that they will be related to non-dissipative transport[1][6].

Of all the conductivities we will only be concerned with the ones associated with an external magnetic field B^μ and a vorticity ω^ν. The associated effects are known as: the Chiral Magnetic Effect (CME) for σ_{VV}, Chiral Magnetic Separation Effect (CMSE) for σ_{AV} , Chiral Vortical Effect (CVE) for $\sigma_{V\omega}$ and Chiral Vortical Separation Effect (CVSE) for $\sigma_{A\omega}$[7].

These coefficients have been calculated in the absence of dynamical gluons (setting $a_3 = 0$) using hydrodynamics [9] , quantum field theory at large N_c [13], effective field theory [10] and holography [11]. It has been found that for this case they are non-renormalizable with an universal value[8] given by

$$\begin{aligned} & \sigma_{VV} = 2a_0\mu_5 \qquad \sigma_{V\omega} = 2a_0\mu\mu_5 \\ \sigma_{AV} = 2a_0\mu \qquad & \sigma_{A\omega} = a_0\mu^2 + 3a_1\mu_5^2 + 8\pi^2 a_2 T^2 \end{aligned} \tag{2.6}$$

Once we allow for dynamical color gauge fields the conductivities are expected to receive radiative corrections, see [13] for an example of such corrections on the weak limit. The main result of the current work is the calculation of such correction in the strongly coupled limit using holographic techniques.

[6]See [3] for a particular discussion of the anomalous conductivities in the context Heavy Ion Collisions

[7]The non existance of axial magnetic fields allow us to focus on only these four conductivities. In addition to this the σ_{AA} has a different physical interpretation from the rest of the conductivities[27]

[8]See [12, 28] for a proof of their universality using holography and [13] using QFT at large N_c. Their non renormalizability comes from comparing their universal value at weak coupling with their universal value at strong coupling

Chapter 3

Holography and Anomalous Transport

The study of strongly coupled quantum field theories is not possible using regular perturbative techniques. The AdS/CFT correspondence brought a new set of techniques to do the calculation of correlation functions of strongly coupled quantum field theories by setting up a relationship between a quantum field theory and a string theory. In its original form it relates $\mathcal{N} = 4$ super Yang-Mills theory in 4D with a type IIB string theory on an $AdS_5 \times S^5$ background [29] [1], where the parameters of the different theories are related to each other via

$$g_{YM}^2 = 2\pi g_s \qquad 2g_{YM}^2 N_c = \frac{L^4}{l_s^4} \equiv 2\lambda \tag{3.1}$$

Where g_s is the string coupling constant, g_{YM} is the coupling constant of the super Yang-Mills theory, R is the AdS radius, l_s is the string length and λ is known as the 't Hooft coupling. The correspondence conjectures that these two theories are dynamically equivalent and such statement can be formalized by relating the partition function of type IIB string theory with the generating functional of the CFT correlation functions[30]. This statement can be written as

$$\langle e^{\int d^4 \mathcal{O} \phi_{(0)}} \rangle = Z_{\text{string}}|_{\phi \to \phi_{(0)}} \tag{3.2}$$

[1] This insight can be justify by looking at a stack of N_c D-branes from two different perspectives: an open string perspective and a supergravity perspective

where $\mathcal{O}$ is an operator on the CFT and ϕ is a field from the string theory side with boundary value $\phi_{(0)}$. Both of them transform in the same representation of the symmetry group of both theories and the field ϕ is said to source the operator $\mathcal{O}$. The mapping relates the behaviour of the string theory on the conformal boundary of AdS with the CFT itself, this is the reason why the correspondance is sometimes also known as the holographic correspondance.

The usefulness of AdS/CFT comes when the following limit of the parameters is taken

$$N_c \to \infty \qquad g_{YM} \to 0 \qquad \lambda >> 1 \tag{3.3}$$

This limit is known as the 't Hooft limit and brings many advantages. It allow us to ignore string interactions by making g_s small, it reduces the string theory that effectively contains arbitrary higher derivative terms in its effective action into an effective two derivative super gravity theory by making the curvature small, and it focus on the strongly (effective) coupled regime of the field theory allowing us to study its non-perturbative behavior.

The next well understood cases after the original $\mathcal{N} = 4$ sYM are the ones that can be obtained from it by relevant or marginal deformations with a relationship between the theories parameters along the lines of (3.2) [31, 32]. As the main goal of this work will be to study the behaviour of QCD like theories we will be interested on writing down an holographic dictionary that allow us to relate operators in QCD with fields on some efective supergravity action. There are two approaches one can take: a top down approach and a bottom up approach.

Before discussing the two possible approaches it is important to mention that to study QCD holographically it is necessary to include a flavor sector into the theory. This is usually done by introducing flavor branes and taking into account their effective action[2] together with the supergravity gluon sector that derives from the stack of color branes[3] [33].

[2]On the low curvature limit the effective action for the flavor branes will just be given by the DBI action which in principle should include tachyonic degress of freedom

[3]The color open strings begin and end on color branes and as a consequence they transform in the adjoint representation of $SU(N_c)$ relating them to the gluons, while the flavor open strings have one end in a flavor brane and another one in a color brane making them to transform in the fundamental representation of both $SU(N_c)$ and $SU(N_f)$ allowing us to relate them with the fermion sector of the theory

By introducing flavor into the theory the number of flavors N_f enters as a parameter of the model. A fix N_f together with the 't Hooft limit is known as the probe limit, as the flavor branes do not back react onto the supergravity background. A different limit known as the Veneziano limit can also be taken [34], for this we define the Veneziano parameter x and take a large N_f limit as

$$x = \frac{N_f}{N_c} = \text{constant} \qquad N_f \to \infty \qquad N_c \to \infty \tag{3.4}$$

This limit will introduce a full back reaction of the branes into the geometry . Additionally a coupling between the low energy states of the flavor open string with the background supergravity fields should be taken into account via the inclusion of Wess-Zumino coupling terms [35].

The approach of setting up D-branes in some string theory and then relating the modes of that theory with the operators of a particular QFT is the so called top down approach[4]. Even though this approach provides a precise dictionary the resulting theories are often different from QCD and exhibit extra complications when trying to decouple the KK modes from the Yang-Mills sector[5] [36]. To solve these extra complications it is necessary to consider larger curvatures and as a consequence a full higher derivative string theory is needed to study QCD like theories within a top-down approach[36].

Giving the technical difficulties of finding a dual of QCD it is often more convenient to work in a bottom up approach. The basic idea of this approach is to give up the ambitious goal of finding an exact dictionary for QCD like theories, but to construct an IR effective theory to capture the IR dynamics of relevant and/or marginal operators in the theory. This will be done by writing down effective super gravity actions that capture the symmetries and properties of QCD like theories in the Veneziano limit. On the next section it will be discuss which are the minimal necessary ingredients for an holographic model of QCD like theories.

[4]As there is usually necessary to do a dimensional reduction by performing some compactification there is an appearance of an infinite number of KK modes on the supergravity side which spans an additional sector in the Hilbert space of the QFT

[5]The KK modes also appear on the original correspondence but in there it is possible to have a well defined low energy limit where a subsector of the Hilbert space containing the $\mathcal{N} = 4$ super multiplets of the energy momentum tensor and the flavor currents can be put in direct correspondence with the low-lying gravitational fields. When trying to obtain a dual of QCD conformality needs to be broken, which induces an IR energy scale Λ_{QCD} that makes this decoupling complicated.

3.1 Bottom-up approach to QCD like theories

As it has been argued on the QCD literature [37] a sector of relevant and marginal low-lying operators can be treated separately from the rest of the Hilbert space of operators. In particular for the gluon sector there exist only three of those such operators: the energy momentum tensor $T^{\mu\nu}$, the scalar glueball operator $\mathrm{Tr}G^2$, and the axionic glueball operator $\mathrm{Tr}G \wedge G$. There will be additional operators that arise from the flavor sector, such as the flavor currents J^μ and J^μ_A and the operator $\bar{q}q$ that signals the appearance of a quark-antiquark condensate.

To describe QCD like theories holographically it is necessary to include an extra holographic dimension r that will correspond to the RG scale in the dual gauge theory. Therefore the holographic theory we are looking for should be a solution of a non-critical 5D string theory. There is no much knowledge about non-critical string theory but as we are only interested on the IR physics where QCD is strongly coupled we expect to be able to approximate its holographic model by a two-derivative gravitational action. It is important to note that if we want to apply the rules of the AdS/CFT correspondence we need solutions that approach the AdS_5 spacetime near the conformal boundary. Besides this we do not want to have all the isometries that come from AdS, as we want to break scale invariance with a running coupling constant on the field theory side.

It is possible to build a theory that fulfills all the above mentioned requirements, at least phenomenologically in comparison to QCD. The necessary degrees of freedom on the supergravity side can be achieved with: a metric field $g_{\mu\nu}$ dual to the energy momentum tensor, a dilaton field ϕ related to the operator TrF^2[6], an axion field C_0 dual to the operator $\mathrm{Tr}\,(G \wedge G)$ and sourcing the θ term on the boundary, two gauge fields V and A related to the vector and axial currents respectively[7] and a complex tachyon field $T = \tau e^{i\xi}$ whose module is related to the quark condensate operator while its phase can be (roughly) related to the operator $\bar{\Psi}\gamma_5\Psi$[40].

[6] The dilaton is a crucial ingredient to the theory as it is essential to establish confinement or deconfinement as well as the running of the coupling constant, not to mention its direct relationship with the string coupling constant. [38, 39]

[7] Formally we need to introduce left and right handed gauge fields which we can later combine into the vector and axial ones. This is important to consider as both left and right handed gauge fields appear on their own DBI action, it will not matter to us since we will take an approximation of the DBI action which will easily allow us to write them in terms of the vector and axial symmetry

Once the necessary degrees of freedom are taken into account, the physics phenomenology can be achieved by taking a non-trivial two derivative Maxwell-Einstein dilaton theory with back-reacting flavor branes that includes a dilaton potential $V(\phi)$ that allow us to have the desired IR physics. In addition to this it is necessary to take into account the anomalous behavior of the axial symmetry by including Chern-Simon terms and a CP-odd sector to the action. The desired action will have the following form[8]

$$S = S_g + S_f + S_a + S_{CS} + S_{GH} + S_{CSK} + S_{ct} \tag{3.5}$$

Where S_g is the glue sector of theory given by[9]

$$S_g = M_p^3 N_c^2 \int_{\mathcal{M}} \left[R \star 1 - \frac{1}{2} d\phi \wedge \star d\phi - V(\phi) \star 1 \right] \tag{3.6}$$

where M_p is the planck energy scale of the 5D theory and we made the dependance of N_c explicit on the action. The next term is the flavor sector S_f given by

$$S_f = -\frac{1}{2} M_p^3 N_c^2 x \int_{\mathcal{M}} \left[Z_V(\phi) F^V \wedge \star F^V + Z_A(\phi) F^A \wedge \star F^A \right] \tag{3.7}$$

In here the N_f dependance has also been made explicit, and $Z_A(\phi)$ and $Z_V(\phi)$ denote an arbitrary coupling to the dilaton for the Maxwell terms. It is important to note that we have used an approximation of the DBI action instead of the full DBI, this was done for the sake of simplicity and to have a general action for two derivatives holographic models. The tachyonic degrees of freedom has also been left out from the action, the reason for this is that we are interested on the deconfined region of the quark gluon plasma and it is known that the degrees of freedom related to the chiral condensate do not play any role there.

[8] On the literature of holographic QCD in the Veneziano limit the same structure for the action is used but it is written in terms of phenomenological potentials and a slightly more general flavor sector. The model we are presenting here is a general action that represents any holographic two derivative model with the necessary degrees of freedom that can reproduce the anomalous Ward identities. See [41] for a review of this types of model in the specific context of QCD

[9] On holographic QCD models, motivated by the underlying non-critical string theory, it is common to have an unconventional normalization of the dilaton kinetic term. Nevertheless for the present work the conventional normalization will be used

The next term is the CP-odd sector S_a given by

$$S_a = -\frac{1}{2}M_p^3 N_c^2 \int_{\mathcal{M}} Z_0(\phi)\,(dC_0 - mxA) \wedge \star\,(dC_0 - mxA) \tag{3.8}$$

where $Z_0(\phi)$ is a non trivial coupling with the dilaton and m is a mass to be fixed by matching with the anomaly. This mass term is a crucial ingredient of the model as it is necessary to reproduce the gluon contribution to the anomaly[10]. The next term is a Chern-Simons term that is introduce to reproduce the anomalies due to external sources and is given by[11]

$$S_{CS} = M_p^3 N_c^2 x \int_{\mathcal{M}} A \wedge \left[\kappa F^V \wedge F^V + \frac{\gamma}{3} F^A \wedge F^A + \lambda \mathrm{Tr}\,(\mathcal{R} \wedge \mathcal{R})\right] \tag{3.9}$$

where the coefficients κ, γ and λ are Chern-Simons coefficients that have to be fixed by matching with the anomalous Ward identities. The terms S_{GH} is the Gibbons-Hawking term introduced to make the gravitational variational problem well defined, the term S_{CSK} is introduce to have a well defined mixed gauge gravitational anomaly at an arbitrary cut-off[12] and S_{ct} stands for the possible counter terms that will need to be introduce to regularize the action at the AdS boundary. The explicit form of the terms S_{GH} and S_{CSK} is presented below

$$S_{GH} = 2M_p^3 N_c^2 \int_{\partial\mathcal{M}} d^4x\sqrt{-h}K \tag{3.10}$$

$$S_{CSK} = -8M_p^3 N_c^2 x \int_{\partial\mathcal{M}} d^4x\sqrt{-h}\lambda n_\mu \epsilon^{\mu\nu\rho\sigma\tau} A_\nu K_{\rho\lambda} \nabla_\sigma \left(K^\lambda_\tau\right) \tag{3.11}$$

Where h is the induced metric on the boundary, n^μ is the (outward directed) unit normal to the boundary and K is the extrinsic curvature defined as

$$K_{\mu\nu} = -\nabla_\mu n_\nu = \frac{1}{2} n^\rho \partial_\rho h_{\mu\nu} \qquad K = h^{\mu\nu} K_{\mu\nu} \tag{3.12}$$

[10] This term can be motivated from a top-down approach through a WZ coupling of the three form C_3 with the axial field A and in a more general case it should consider tachyonic contributions. See [42, 43] for a derivation

[11] It is also possible to motivate this term with a WZ coupling of the flavor branes with the RR gravitational fields [43].

[12] It is important to note however that the variational problem is not well defined at an arbitrary cut-off in the presence of the gravitational Chern-Simons term as the associated Hamiltonian have third order derivative terms. For a discussion about this issue see [44]

3.2 Holographic equations of motion

The model discussed above is well defined up to any possible boundary terms coming from the counter terms. On the bulk the dynamics are governed by the equations of motion of the fields, the relevant equations of motion are the following: the equation of motion for the dilaton

$$\nabla_\mu \left[\partial^\mu \phi\right] = \partial_\phi V + \frac{\partial_\phi Z_A}{4}\left(F^A\right)^2 + \frac{\partial_\phi Z_V}{4}\left(F^V\right)^2 + \frac{\partial_\phi Z_0}{2}\left(dC_0 - mxA\right)^2 \tag{3.13}$$

the equation of motion for the axion field

$$\nabla_\mu \left[Z_0 \left(\partial^\mu C_0 - mxA^\mu\right)\right] = 0 \tag{3.14}$$

the vector Maxwell equation

$$\nabla_\mu \left[Z_V F^{V,\mu\nu}\right] - \epsilon^{\mu\nu\alpha\rho\sigma}\partial_\mu \left[\kappa A_\alpha F^V_{\rho\sigma}\right] = 0 \tag{3.15}$$

the axial Maxwell equation

$$\begin{aligned} \nabla_\mu \left[Z_A F^{A,\mu\nu}\right] + m^2 x Z_0 \left[\frac{\partial^\mu C_0}{xm} - A^\mu\right] = \\ = \epsilon^{\mu\nu\alpha\rho\sigma}\partial_\mu \left[\frac{\kappa}{2} V_\alpha F^V_{\rho\sigma} + \frac{\gamma}{2} A_\alpha F^A_{\rho\sigma} + \lambda\left(\Gamma^\tau_{\lambda\alpha}\partial_\rho \Gamma^\lambda_{\tau\sigma} + \frac{2}{3}\Gamma^\tau_{\lambda\alpha}\Gamma^\lambda_{\beta\rho}\Gamma^\beta_{\tau\sigma}\right)\right] \end{aligned} \tag{3.16}$$

where $\Gamma^\rho_{\mu\nu}$ is the Christofell connection.And Einstein's equations are

$$\begin{aligned} R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{V}{3}g_{\mu\nu} \\ x\left[\frac{Z_V}{2}\left(F^V_{\mu\rho}F^{V,\rho}_\nu - \frac{1}{6}g_{\mu\nu}F^V_{\rho\sigma}F^{V,\rho\sigma}\right) + \frac{Z_A}{2}\left(F^A_{\mu\rho}F^{A,\rho}_\nu - \frac{1}{6}F^A_{\rho\sigma}F^{A,\rho\sigma}\right)\right] \\ + \frac{x\lambda}{2}\nabla_\lambda\left[\Sigma^\lambda_{(\mu\nu)} - \frac{1}{3}\nabla_\lambda\Sigma^{\lambda\rho}_\rho g_{\mu\nu}\right] + \frac{Z_0 x^2 m^2}{2}\left(A_\mu - \frac{\partial_\mu C_0}{xm}\right)\left(A_\nu - \frac{\partial_\mu C_0}{xm}\right) \end{aligned} \tag{3.17}$$

Where the spin current $\Sigma^\lambda_{\mu\nu}$ is defined as

$$\Sigma^\lambda_{\mu\nu} = -g_{\tau\mu}\epsilon^{\tau\alpha\beta\rho\sigma}F^A_{\alpha\beta}R^\lambda{}_{\nu\rho\sigma} \tag{3.18}$$

On the next chapter we will set up a background anzats for the previously shown equations.

3.3 Holographic effective action

From the holographic dictionary we know that the boundary gravitational action can be interpret as the effective action for the quantum field theory. From this we know that the one point function for the (consistent) gauge currents can be defined as

$$\mathcal{J}^{\mu} = \lim_{r\to\infty} \frac{\delta S}{\delta V^{\mu}} = \lim_{r\to\infty} M_p^3 N_c^2 x \left[\sqrt{-g} Z_V F^{V,\nu r} + \kappa \tilde{\epsilon}^{r\nu\mu\alpha\beta} A_{\mu} F^{V}_{\alpha\beta}\right] \quad (3.19)$$

$$\mathcal{J}^{\mu}_{A} = \lim_{r\to\infty} \frac{\delta S}{\delta A^{\mu}} = \lim_{r\to\infty} M_p^3 N_c^2 x \left[\sqrt{-g} Z_A F^{A,\nu r} + \frac{\gamma}{3} \tilde{\epsilon}^{r\nu\mu\alpha\beta} A_{\mu} F^{A}_{\alpha\beta}\right] \quad (3.20)$$

where r represents the holographic coordinate and the conformal AdS boundary is set at $r \to \infty$. From the last expressions we can identify the last terms of the consistent currents as Bardeen-Zumino polynomials and identify the covariant currents as

$$J^{\mu} = \lim_{r\to\infty} M_p^3 N_c^2 x \sqrt{-g} Z_V F^{V,\nu r} \quad (3.21)$$

$$J^{\mu}_{A} = \lim_{r\to\infty} M_p^3 N_c^2 x \sqrt{-g} Z_A F^{A,\nu r} \quad (3.22)$$

It will be the linear perturbation of these two last expressions that we will be interested on calculating to evaluate the conductivities. It is important to note that these expressions for the currents is before renormalizition, which means that the renormalized version of them can change and have extra contributions coming from the counter terms.

To finalize the chapter we will evaluate the gauge transformation of the gravitational action and we will relate the boundary value of that transformation to the gauge transformation of the effective action in the field theory side. The gauge transformation that leave the gravitational action invariant up to boundary terms is given by

$$\begin{aligned} V' &= V + d\alpha \\ A' &= A + d\beta \\ C_0' &= C_0 + m x \beta \end{aligned} \quad (3.23)$$

This gauge transformation will change the action like

$$\delta S = M_p^3 N_c^2 x \int_{\partial \mathcal{M}} \beta \left[\kappa F^V \wedge F^V + \frac{\gamma}{3} F^A \wedge F^A + \lambda \mathrm{Tr} \left(\mathcal{R} \wedge \mathcal{R} \right) \right] \tag{3.24}$$

Identifying the previous expression with the gauge transformation of the quantum field theory effective action given by (1.25)[13] it is possible to fix the Chern-Simons coefficients in terms of the anomaly coefficients a_i accordingly

$$\kappa = -\frac{a_0}{M_p^3 N_c^2 x} \qquad \gamma = -\frac{3a_1}{M_p^3 N_c^2 x} \qquad \lambda = -\frac{a_2}{M_p^3 N_c^2 x} \tag{3.25}$$

For this identification to make sense it is necessary that the boundary value of the axion field is identified with $\frac{\theta}{N_c}$ so that the mass m is only a numerical value that can be matched from the gauge transformation of the θ field (1.27) as[14]

$$m = 4\pi a_3 \tag{3.26}$$

On the following chapters the main results of this work will be presented. On them the linear response of an holographic system described by the previously discussed action will be evaluated.

[13]It is important to note that the Chern-Simons term only reproduces the global anomaly. As we have discussed in the first chapter this is true because the gauge transformation of the θ term removes the gluon contribution of the anomaly from the gauge transformation of the effective action

[14]To calculate this value it is possible to think of a probe color brane that couples minimally to the axion through the operator $\mathrm{Tr} G \wedge G$, the mass can then be fixed by asking that the gauged transformed value of the coupling reproduces the gluon anomaly

Chapter 4

Holographic Background

On this chapter we will discuss the holographic background over which the linear response perturbations will be performed. The aim of this work is to study the anomalous conductivities on the strongly interactive and deconfined phase of the quark gluon plasma at finite density and temperature. This means that we need an holographic background that satisfy these physical requirements.

To allow for an energy scale to be dynamically generated we ask for a non trivial dilaton $\phi(r)$ that depends only on the bulk radius. To work in the deconfined phase a black brane solution for the metric is assumed

$$ds^2 = -g_{tt}(r)dt^2 + g_{xx}(r)dx_i dx^i + g_{rr}(r)dr^2 \tag{4.1}$$

The only requirements that the metric should satisfy is to have a non-extremal horizon and to have an asymptotically AdS boundary. Both requirements can be written as

$$\begin{gathered}\lim_{r\to\infty} g_{tt} \sim r^2 + \dots, \qquad \lim_{r\to\infty} g_{tt} \sim r^2 + \dots, \qquad \lim_{r\to\infty} g_{tt} \sim r^{-2} + \dots \\ \lim_{r\to r_h} g_{tt} \sim t_h\,(r-r_h) + \dots, \qquad \lim_{r\to r_h} g_{xx} \sim x_h + \dots, \qquad \lim_{r\to r_h} g_{rr} \sim \frac{\rho_h}{(r-r_h)} + \dots\end{gathered} \tag{4.2}$$

Where t_h, x_h and ρ_h are just constants that characterize the near horizon behavior of the metric, the horizon is taken to be at $r = r_h$ and the AdS boundary is assume to be at $r \to \infty$.

The AdS boundary is required for the holographic dictionary to work in its simplest form while the non-extremal horizon[1] is required to have a finite T on the system, which in terms of the metric components can be written as [33]

$$T = \frac{1}{4\pi}\sqrt{\frac{g'_{tt}g'_{rr}}{g_{rr}^2}}\Bigg|_{r_h} \tag{4.3}$$

where the prime denotes a derivative with respect to the radial component. To be able to set the system at a finite density through a non zero chemical potential we will assume a non-trivial and radially dependent time component[2] for the gauge fields [33], namely

$$V = V_t(r)dt \qquad A = A_t(r)dt \tag{4.4}$$

The chemical potentials will be set through the boundary conditions of these components. On the following section we will discuss some issues regarding the definition of a chemical potential for anomalous symmetries and also what does a chemical potential means when the gauge field is massive. A non-trivial axion field $C_0(r, \mathbf{x})$ will also be assume, the functional dependance of this axion will be defined by consistency of the equations of motion with the rest of our Ansatz.

In general we will use the previously shown Ansatz to write down a schematic solution of all the holographic equations of motion, this means that full back-reaction will in general be assumed. The only case when this is not going to be true is for a calculation of the conductivities on a non backreacting background that will take place on the following chapter.

On the following sections we will show the relevant conserved charges of the system as well as an analysis of the behavior of the solutions near the boundaries. We will finalize the chapter with some brief comments regarding holographic renormalization and by writing down the renormalized one point functions.

[1] The horizon is necessary if we want to be in the deconfined phase. The easiest way to picture this is to take an open string that stretches from the AdS boundary into the horizon and reaches a brane beyond it leaving a single quark on the conformal boundary

[2] It is important to note that we are assuming gauge symmetry and fixing a gauge such that $A_r = V_r = 0$. This means that we no longer have any freedom left after making this choice

4.1 Conserved Charges

The relevant equations of motion that we will analyze are the ones for the axion field (3.14), the two Maxwell equations (3.15) and (3.16), and Einstein's equations (3.17). The equation of motion for the dilaton will not be considered as it will not play any role in any future calculation.

For the analysis of the Maxwell equation it will be convenient to define a vector and axial charges via

$$Q \equiv Z_V \sqrt{\frac{g_{xx}^3}{g_{tt} g_{rr}}} V_t' \qquad Q_5 \equiv Z_A \sqrt{\frac{g_{xx}^3}{g_{tt} g_{rr}}} A_t' \tag{4.5}$$

The vector Maxwell equations only tell us that the vector charge is conserved, namely

$$Q' = 0 \tag{4.6}$$

The axial Maxwell equations contain the following set of equations

$$\partial_c C_0 \qquad \partial_r C_0 = 0 \tag{4.7}$$

$$Q_5' = \frac{\sqrt{-g}}{g_{tt}} Z_0 x m^2 B_t \tag{4.8}$$

Where we have defined $B_t = A_t - \frac{\partial_t C_0}{xm}$ and where the index c denotes the spatial boundary coordinates. Taking into account equations (4.7) the axion equation of motion gets reduced to

$$\partial_t^2 C_0 = 0 \tag{4.9}$$

This last equation can easily be solved to $C_0 = C + xmC_1 t$, which means that there should be a non-trivial time dependance on the axion. The constant C is irrelevant for our analysis.

For our choice of background there are three non-trivial Einstein equations (where R_{xx} stands for the Ricci tensor of any of the spatial coordinates):

$$R_{tt} = -g_{tt}\left[\frac{V}{3} + \frac{x}{6}\left(\frac{Z_V V_t'^2 + Z_A A_t'^2}{g_{rr} g_{tt}}\right)\right] + \frac{x^2 Z_0 m^2}{2}\left[A_t - C_1\right]^2 + \frac{x}{2g_{rr}}\left[Z_V V_t'^2 + Z_A A_t'^2\right] \tag{4.10}$$

$$R_{xx} = g_{xx}\left[\frac{V}{3} + \frac{x}{6}\left(\frac{Z_V V_t'^2 + Z_A A_t'^2}{g_{rr} g_{tt}}\right)\right]s \tag{4.11}$$

$$R_{rr} = \frac{1}{2}\phi'^2 + \frac{\Psi}{2}\chi'^2 + g_{rr}\left[\frac{V}{3} + \frac{x}{6}\left(\frac{Z_V V_t'^2 + Z_A A_t'^2}{g_{rr} g_{tt}}\right)\right] - \frac{x}{2g_{tt}}\left[Z_V V_t'^2 + Z_A A_t'^2\right] \tag{4.12}$$

It is possible to combine all these equations to get a conserved charge K defined as

$$K \equiv \sqrt{\frac{g_{xx}^5}{g_{rr} g_{tt}}}\partial_r\left(\frac{g_{tt}}{g_{xx}}\right) - x\left[V_t Q + B_t Q_5\right] \tag{4.13}$$

To summarize this section we can write down all the relevant conserved charges and solutions for our background

$$\begin{aligned} dC_0 &= C_1 dt \\ \partial_r Q &= 0 \\ \partial_r Q_5 &= \frac{\sqrt{-g}}{g_{tt}} Z_0 x m^2 B_t \\ \partial_r K &= 0 \end{aligned} \tag{4.14}$$

4.2 Asymptotics and boundary conditions

In this section we will analyze the asymptotic and near horizon behavior of the gauge fields and the axion.

Vector field asymptotics

The asymptotic behavior of the vector gauge field can be derived from $Q' = 0$, near the Ads-Boundary it tell us

$$\lim_{r\to\infty} V_t \sim V_t^\infty + \frac{\bar{V}_t^\infty}{r^2} \tag{4.15}$$

where V_t^∞ and $\bar{V}_t^\infty$ are both constants. There is still some residual gauge symmetry on the time component of the vector field allowing us to change its value by a constant and the only condition we will impose is that its radial flux its equal to the chemical potential, namely

$$\mu = \int dr \partial_r V_t = \lim_{r\to\infty} V_t - V_t(r_h) \tag{4.16}$$

This leaves two obvious choices for boundary conditions on the gauge field, as the vector gauge symmetry is fulfilled we will choose to have a vanishing field at the AdS boudnary and to set the chemical potential at the horizon, namely

$$V_t(r_h) = -\mu \qquad \lim_{r\to\infty} V_t(r) = 0 \tag{4.17}$$

As it has been discussed in [11] this option is the more physically satisfactory one.

Axial field asymptotics

The asymptotic behavior of the axial gauge field can be derived from equation (4.8) and near the Ads Boundary it is given by

$$\lim_{r\to} A_t \sim C_1 + A_t^\infty r^\Delta + \frac{\bar{A}_t^\infty}{r^{2+\Delta}} \tag{4.18}$$

with $\Delta = \sqrt{1 + Z_0^\infty x m^2} - 1 > 0$ where Z_0^∞ is the value of the coupling constant of the dilaton with the mass term at the conformal boundary. From the scaling of this field it is possible to note that the conformal dimension of the associated current operator will be given by $[J_A] = 3 + \Delta$, which means that if we want the operator to be either relevant or marginal it is necessary to keep $\Delta < 1$[3]. As we know that there is no hydrodynamical variable that represents the gluon degrees of freedom we will choose to incorporate the anomaly in a weakly way by asking that $\Delta << 1$ [4]. It is important to note that only on the zero mass limit the term C_1 becomes a gauge freedom.

[3]The operator should be relevant or marginal if we want the AdS boundary behavior to be satisfied. If we wanted to consider an irrelevant operator a different procedure on the renormalization and asymptotic behavior should be considered.

[4]This may be the only limit that is in agreement with the philosophy behind the hydrodynamic description

Is possible to notice that defining an axial chemical potential in the same way as we did for the vector field becomes troublesome as

$$\mu_5 \equiv \int dr \partial_r A_t \to \infty \tag{4.19}$$

It will be necessary then to think of the chemical potential as a coupling in the Hamiltonian and define it through the asymptotic behavior of the gauge field in the following form

$$\lim_{r\to} A_t \to C_1 + \mu_5 r^{\Delta} \tag{4.20}$$

As it was discussed in [14] this choice allows for a direct comparison with the massless case. This is the only sensible way to set up the boundary conditions in the AdS boundary as regularity at the horizon demands that

$$A_t(r_h) \sim C_1 + A_t^h(r - r_h) \tag{4.21}$$

This condition arises from the mass term that becomes singular if the invariant combination of the gauge field $(A_t - C_1)$ does not vanishes at least linearly near the horizon.

Boundary conditions of the Axion

It is possible to give an interpretation to the temporal component of dC_0 when the mass term is set to zero. For this we can first consider adding the following coupling to the action

$$S_0 = M_p^3 N_c^2 x \int_{\partial\mathcal{M}} C_0 \left[\kappa F^V \wedge F^V + \frac{\gamma}{3} F^A \wedge F^A + \lambda \mathrm{Tr}\left(\mathcal{R} \wedge \mathcal{R}\right)\right] \tag{4.22}$$

If this action is consider together with the whole holographic action it can be seen that the full action will become invariant. This does not mean that the theory is not anomalous but it only means that if we want to related the theory to another one related via a gauge transformation then it is necessary to consider this term on the calculations [11].If in this case the axion field is non dynamical it can take a value $C_0 = -\mu_5 t$ which will make any two field configurations related through a gauge transformation equivalent. In the dynamical case this term is enforced by the holographic renormalization [14].

Renormalized one point functions

The holographic renormalization for a model with the same asymptotic behavior in the AdS boundary has already been done in [14]. Based on their results we can write down the renormalized one point functions of the model

$$J^{\mu}_{\text{ren}} = \lim_{r\to\infty} M_p^3 N_c^2 x\sqrt{-g}\left[Z_V F^{V,\mu r} + \kappa\tilde{\epsilon}^{\mu\nu\rho\sigma r} B_\nu F^V_{\rho\sigma}\right] \tag{4.23}$$

$$J^{\mu}_{A\text{ren}} = \lim_{r\to\infty} M_p^3 N_c^2 x\sqrt{-g} r^{\delta}\left[Z_V F^{V,\mu r} + r\Delta A^{\mu}\right] \tag{4.24}$$

For the details of the renormalization procedure see the appendices on [14].

Chapter 5

Radiative corrections to the anomalous conductivities

On this chapter we will perform a linear perturbation on the background shown in the previous chapter. The calculation will be done in three different cases and the correction to the conductivities due to the mass term on the limit $\Delta << 1$ will be calculated. The source of the perturbation will be given by some external vector, axial and gravitational gauge fields. These perturbations can be assumed to be wave like and have their momentum oriented in one particular direction, we will keep it general and assume an arbitrary direction. It is known that the only relevant set of perturbations to be considered are the vector symmetry preserving ones given by[1]

$$\begin{aligned} \delta V_c &= v_c(\mathbf{x}) + \beta_c(r,x) \\ \delta A_c &= \alpha_c(r,x) \\ \delta g_{tc} &= g_{xx}\gamma_c(r,x) \end{aligned} \tag{5.1}$$

Where c denotes any spatial component. These perturbations do not couple to the rest of the perturbations and scalar fields, and so they can be studied independently by setting the rest of the fluctuations to zero.

[1]A response of the axion field δC_0 is also considered, but it does not play any role in the final solution.

A zero frequency (no time dependance) has already been assumed and additionally a zero momentum limit is going to be assumed by considering the perturbations to be linear functions of the boundary coordinates. Following [12] it is possible for the vector field to directly include the source of the perturbation as follows

$$B^b = \tilde{\epsilon}^{tbcdr}\partial_c v_d \tag{5.2}$$

Where the indices b, c, d indicate any spatial component and B^b denotes an external magnetic field. Before identifying the sources of the axial and gravitational perturbations a convenient notation will be introduced

$$\nabla \times \boldsymbol{\alpha} = \mathcal{A}(r) \qquad \nabla \times \boldsymbol{\beta} = \mathcal{V}(r) \qquad \nabla \times \boldsymbol{\gamma} = \mathcal{K}(r) \tag{5.3}$$

where a bold symbol denotes a vector in the spatial boundary coordinates while $\nabla\times$ indicate the curl of it on those same coordinates. It is easy to notice that equation (5.2) can easily be written in this notation.

From the introduced variables in (5.3) it is possible to define the external axial and gravitational sources as [2]

$$\begin{aligned} \lim_{r\to\infty} \mathcal{V} &= 0 \\ \lim_{r\to\infty} \mathcal{A} &= B_5 r^{\Delta} \\ \lim_{r\to\infty} \mathcal{K} &= \omega \end{aligned} \tag{5.4}$$

Where B_5 denotes an external axial magnetic field and the boundary conditions has been incorporated in a similar fashion as the axial chemical potential for the axial background gauge field. The source ω can be taken as a vorticity due to a small rotation of the fluid, this can be seen by comparing the metric perturbation with the metric of a rotating fluid with vorticity ω.

An equation for the fluctuations of the unrenormalized covariant currents $j^c = \delta J^c$ and $j_5^c = \delta J_A^c$ can be easily calculated from Maxwell equations

$$j^c(r) = -2M_p^3 N_c^2 x \left[\kappa A_t|_{r_h}^r B^c + \int_{r_h}^r dr\,[\kappa V_t' \mathcal{A} + \kappa A_t' \mathcal{V}]\right] \tag{5.5}$$

$$j^c(r)_5 = -2M_p^3 N_c^2 x \left[\kappa V_t|_{r_h}^r B^c + \int_{r_h}^r dr\,[\kappa V_t' \mathcal{V} + \gamma A_t' \mathcal{A} - \lambda j_g + M^c(r)]\right] \tag{5.6}$$

[2] The fluctuation γ should vanish at the horizon for regularity of the Ricci scalar [12]

where the gravitational Chern-Simons current j_g^c and the dynamic mass term $M^c(r)$ are given by[3]

$$j_G^c = \delta^{cb}\left[-\partial_r\left(\frac{r^4 g f'}{2f}\mathcal{K}'\right) - \frac{r^4 g\left(f'\mathcal{K}'' - f''\mathcal{K}'\right)}{2f}\right] \tag{5.7}$$

$$M^c(r) = Z_0 x m^2\sqrt{-g}\left[\frac{\alpha_b - \frac{\partial_b C_0}{xm}}{g_{xx}} + \frac{\gamma_b B_t}{g_{tt}}\right]\delta^{bc} \tag{5.8}$$

for this last term the following form for the metric was assumed

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 g(r)} + r^2 dx^c dx^d \delta_{cd} \tag{5.9}$$

There are two things that can be noted at this point:

1. The response of the vector current only depends on the variables $\mathcal{A}$ and $\mathcal{V}$. So to calculate the CME,CVE and CSE conductivities it will suffice to solve for these perturbations.

2. The response of the axial current depends on the full perturbations $\mathcal{V}$, α and γ which means that in general it will not be possible to calculate the CVSE conductivity with full generality without solving the entire system. An example will be discuss where some simplifications will make it possible to calculate the CVSE conductivity corrections.

The goal is now to solve only for the perturbations $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{K}$ as this is the minimum set that allows us to calculate three out of the four desired conductivities. By taking the curl of the fluctuation of Maxwell and Einsteins equations it is possible to get a close system of equations [4]

$$\mathcal{K}^c = -\frac{Z_V}{Q}\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\mathcal{V}'^c \tag{5.10}$$

$$\partial_r\left[Q_5\frac{g_{tt}}{g_{xx}}\left(\frac{\mathcal{A}'^c}{A_t'} - \frac{\mathcal{V}'^c}{V_t'}\right)\right] = \frac{g_{tt}}{g_{xx}}\partial_r Q_5\left[\frac{\mathcal{A}^c}{A_t} - \frac{\mathcal{V}'^c}{V_t'}\right] \tag{5.11}$$

$$\partial_r\left(\frac{\mathcal{V}'^c}{V_t'}\right) = x\frac{Q_5 A_t g_{rr}}{\sqrt{-g}}\left[\frac{\mathcal{A}'^c}{A_t'} - \frac{\mathcal{V}'^c}{V_t'}\right] \tag{5.12}$$

On the following sections we will solve this system and calculate the corrections to the conductivities.

[3] The derivation of the fluctuation of Maxwell equations and the derivation of the gravitational Chern-Simons term can be found on the appendix

[4] The derivation of this form of the equations can also be found on the appendix

5.1 A non back-reacting example

As a first example we will work on the same model as the one in [14]. In the work of Jimenez-Alba, Landsteiner and Melgar trivial dilatonic couplings are assumed and a fixed Ads blackhole background is also considered. These means that the background functions for their model are given by

$$Z_A = Z_V = Z_0 = 1 \tag{5.13}$$

$$f(r) = g(r) = 1 - \frac{r_h^4}{r^4} \tag{5.14}$$

Where the blackening factors f and g come from the form of the metric given by (5.9). It is possible to rescale the coordinates and set $r_h = 1$, for this example we will work with this value for r_h from now on.

As the background is fixed there is no back-reaction and the metric fluctuation and the fluctuation of Einstein equation can be disregarded, namely we will set $\mathcal{K} = 0$. Consequently there will be no vorticity induced phenomena and the only conductivities of interest will be σ_{VV} and σ_{AV}. This simplification will also reduce the system of equations described by(A.63), (5.11) and (5.12) into the single equation

$$\partial_r \left[Q_5 \frac{g_{tt}}{g_{xx}} \frac{\mathcal{A}'^c}{A'_t} \right] = \frac{g_{tt}}{g_{xx}} \partial_r Q_5 \frac{\mathcal{A}^c}{A_t} \tag{5.15}$$

Using the specific AdS blackhole background this equation will greatly simplify into

$$\partial_r \left[r^3 f(r) \mathcal{A}' \right] = rxm^2 \mathcal{A} \tag{5.16}$$

that can be solved analytically in terms of hypergeometric functions $F(a, b, c, r^4) \equiv_2 F_1(a, b, c, r^4)$ as

$$\mathcal{A}^c = B_5^c D_1(\Delta) \left[F\left(-\frac{\Delta}{4}, \frac{2+\Delta}{4}, \frac{1}{2}, r^4 \right) + D(\Delta) r^2 F\left(-\frac{\Delta - 2}{4}, \frac{4+\Delta}{4}, \frac{3}{2}, r^4 \right) \right] \tag{5.17}$$

The AdS boundary conditions were imposed together with regularity at the horizon to fix the integration constants D and D_1 as

$$D_1(\Delta) = \frac{(-1)^{\frac{\Delta}{4}}}{\sqrt{\pi}\Gamma\left(\frac{\Delta+1}{2}\right)}\left[\frac{1}{\left[\Gamma\left(\frac{\Delta+2}{4}\right)\right]^2} - \frac{iD(\Delta)}{\left[\Gamma\left(\frac{\Delta+4}{4}\right)\right]^2}\right]^{-1} \tag{5.18}$$

$$D(\Delta) = \frac{\Delta}{2}\frac{\Gamma\left(\frac{2-\Delta}{4}\right)\Gamma\left(\frac{4+\Delta}{4}\right)}{\Gamma\left(\frac{2+\Delta}{4}\right)\Gamma\left(\frac{4-\Delta}{4}\right)} \tag{5.19}$$

Where $\Gamma(\Delta)$ is the Euler gamma function. The only thing missing to calculate the unrenormalized vector one point function is the solution of the background gauge field V_t given by

$$V_t = -\frac{\mu}{r^2} \tag{5.20}$$

Using (5.20) and (5.17) into (5.5) the following expression is found

$$j^c(r) = -2M_p^3 N_c^2 x\kappa\left[A_t|_{r_h}^r B^c + B_5^c \mu D_1 \int_{r_h}^{\infty}\left[\frac{F_1(r)}{r^3} + D\frac{F_2(r)}{r}\right]\right] \tag{5.21}$$

Where F_1 and F_2 stand for the first and second hypergeometric function showed in (5.17) respectively. To calculate the conductivity we first have to evaluate the renormalized one point function at the Ads Boundary, after using (5.21) and the renormalized expression (4.23) we are left with

$$j^c_{\text{ren}} = 2a_0 B_5^c \mu D_1(\Delta)\int_1^{\infty}\left[\frac{F_1(r)}{r^3} + D(\Delta)\frac{F_2(r)}{r}\right] \tag{5.22}$$

where we have used the corresponding value for the Chern-Simons coefficient. The CME and CSE conductivities can be read off from this last expression by identifying it with the constitutive relationship and performing the integral, obtaining

$$\sigma_{\text{CME}} = 0 \tag{5.23}$$

$$\begin{aligned}\sigma_{CSE} = (2a_0\mu)\, D_1(\Delta)\Bigg[&-\frac{{}_3F_2\left[-\frac{1}{2}, \frac{\Delta+2}{4}, -\frac{\Delta}{4}; \frac{1}{2}, \frac{1}{2}; r^4\right]}{2r^2} \\ &+\frac{D(\Delta)\sqrt{\pi}}{4\Gamma\left(\frac{2-\Delta}{4}\right)\Gamma\left(\frac{4+\Delta}{4}\right)} G_{11}^{22}\left(-r^4 \,\middle|\, \begin{matrix} -\frac{\Delta}{4}, \frac{2+\Delta}{4}; 1 \\ 0, 0; -\frac{1}{2}\end{matrix}\right)\Bigg]\Bigg|_1^{\infty}\end{aligned} \tag{5.24}$$

In this last expression ${}_3F_2$ is a Hypergeometric function and G is the Meijer G-function. The obtained expression for the CSE conductivity might not be too instructive but the quantity $\frac{\sigma_{\rm CSE}}{\mu}$ is worth plotting, its plot and comparison with the one obtained numerically in [14] is shown in figure 5.1. The difference between the analytical result here presented and the result of [14] is of around 3%.

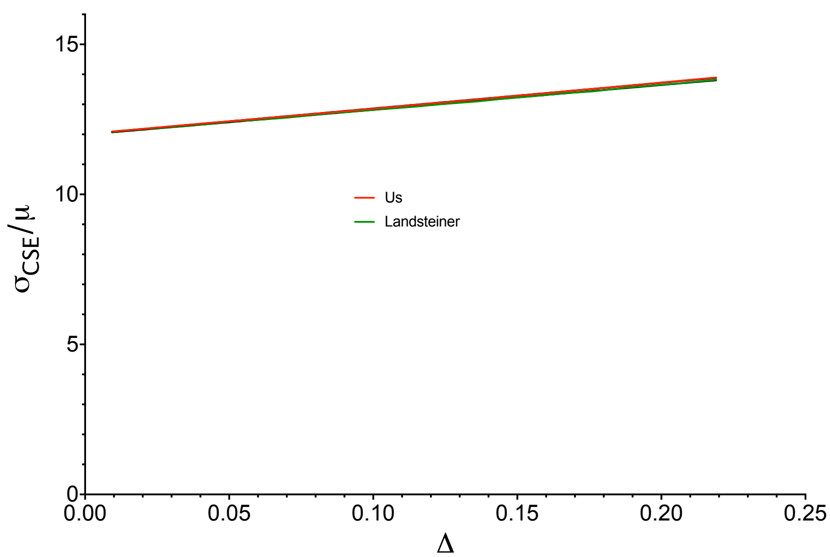

Figure 5.1: CSE conductivity as a function of the current anomalous dimension Δ, the universal result on this numerical units is ≈ 12. A comparison of our analytical result with the numerical results of [14] is also shown.

Before finishing this section some comments are in order:

1. **CME conductivity**: It takes its consistent current universal value (it vanishes). The reason for this is that the Chern-Simons term that originally differentiate between covariant and consistent needs to be taken into account once the renormalization has taken place and the distinction between the two disappears. There are no correction due to the gluons.

2. **CSE conductivity**: In the limit of zero mass ($\Delta \to 0$) the universal value for this conductivity is recovered. The contribution of the gluons enhance the conductivity otherwise, for small Δ the enhancement can be characterize by the simple relation

$$\sigma_{\text{CSE}} = \left(\frac{N_c \sum q_f^2 \mu}{2\pi^2}\right) \left[1 + \ln(2)\Delta + \frac{\pi^2 + 12\ln 2\,(\ln 8 - 2)}{48}\Delta^2\right] \tag{5.25}$$

From this we could conclude that the slope on figure 5.1 is equal to $\ln(2)$. The dependence of the CME conductivity with the chemical potential is linear and as expected it does not depend on the axial chemical potential.

5.2 CME, CSE and CVE (on a Δ expansion)

The previous example was an instructive example on how to proceed with the calculation of anomalous conductivities on the probe limit. But now we will be interested on taking full back-reaction into account. Solving the system of equations for $\mathcal{A}$, $\mathcal{V}$ and $\mathcal{K}$ for a general background configuration is a non-trivial task and it will be necessary to resort to the limit $\Delta << 1$ from the beginning and solve the system of equations perturbatively on the Δ parameter. For this we will expand every field $\mathcal{T}$ as follows

$$\mathcal{T} = \mathcal{T}^{[0]} + \mathcal{T}^{[1]}\Delta + \frac{\mathcal{T}^{[2]}}{2}\Delta^2 + ... \tag{5.26}$$

The Veneziano parameter will be assumed of $\mathcal{O}(1)$ with respect to the Δ parameter. The system of equations (A.63), (5.11), and (5.12) will be solved order by order in the Δ expansion, it is important to note that also the background fields need to be expanded. The boundary conditions for the fields will be obtained from regularity at the horizon and

$$\begin{aligned} \lim_{r\to\infty} \mathcal{V}^c &= 0 \\ \lim_{r\to\infty} \mathcal{A}^c = B_5^c + B_5^c \ln r\Delta + \frac{B_5^c}{2}\left(\ln r\right)^2 \Delta^2 + ... \\ \lim_{r\to\infty} \mathcal{K}^c &= \omega^c \end{aligned} \tag{5.27}$$

On the rest of this section the zeroth and first order equations will be solved and a particular example will be discussed.

5.2.1 $\mathcal{O}(1)$ and universality

The $\mathcal{O}(1)$ equations are given by

$$\partial_r\left[\frac{1}{Q_5^{[0]}}Z_A^{[0]}\sqrt{\frac{g_{xx}^{[0]}g_{tt}^{[0]}}{g_{rr}^{[0]}}}\mathcal{A}'^{[0]}\right]=\partial_r\left[\frac{1}{Q^{[0]}}Z_V^{[0]}\sqrt{\frac{g_{xx}^{[0]}g_{tt}^{[0]}}{g_{rr}^{[0]}}}\mathcal{V}'^{[0]}\right] \tag{5.28}$$

$$\partial_r\left[\frac{\mathcal{V}'^{[0]c}}{V_t'^{[0]}}\right]=x\frac{A_t^{[0]}Z_A^{[0]}}{g_{tt}^{[0]}V_t'^{[0]}}\left[\mathcal{A}'^{[0]}V_t'^{[0]}-A_t'^{[0]}\mathcal{V}'^{[0]}\right] \tag{5.29}$$

Where we used that the non-conservation of the axial charge comes at order $\mathcal{O}(\Delta)$ and where we have omitted the component index for simplicity. After using the background equations of motion at order $\mathcal{O}(1)$ the first equation can be integrated into the following condition

$$\left[\mathcal{A}'^{[0]}V_t'^{[0]}-A_t'^{[0]}\mathcal{V}'^{[0]}\right]=0 \tag{5.30}$$

This allow us to use (5.29) and find solutions for $\mathcal{A}$, $\mathcal{V}$ and $\mathcal{K}$

$$\begin{aligned}\mathcal{V}^{[0]}&=CV_t^{[0]}\\ \mathcal{A}^{[0]}&=\tilde{B}_5-C\mu_5+CA_t^{[0]}\\ \mathcal{K}^{[0]}&=-C\frac{g_{tt}^{[0]}}{g_{xx}^{[0]}}\end{aligned} \tag{5.31}$$

Where C is an integration constant, and to get to this expression we have used the boundary conditions of the background fields plus the AdS boundary conditions for the perturbations. The integration constant C can easily be found from the boundary conditions of $\mathcal{K}$ through $C=-\mathcal{K}_\infty^{[0]}=-\omega$. These leave us with the solution to the system given by

$$\begin{aligned}\mathcal{V}^{[0]}&=-\omega V_t^{[0]}\\ \mathcal{A}^{[0]}&=\tilde{B}_5+\omega\mu_5-\omega A_t^{[0]}\\ \mathcal{K}^{[0]}&=\omega\frac{g_{tt}^{[0]}}{g_{xx}^{[0]}}\end{aligned} \tag{5.32}$$

We will seize this result to show the universality of all the conductivities in the absence of gluon contributions to the anomaly. For this we will substitute the $\mathcal{O}(1)$ solutions into the unrenormalized one point functions (5.5) and (5.6)

$$j^c = -2M_p^3 N_c^2 x \left[\kappa\mu_5 B^c - \omega\kappa \int_{r_h}^{\infty} dr \left[V_t'^{[0]} A_t + A_t'^{[0]} V_t\right] + \kappa\mu(\tilde{B}_5 + \omega\mu_5) + \mathcal{O}(\Delta)\right] \tag{5.33}$$

$$\begin{aligned} j_5^c = -2M_p^3 N_c^2 x \Bigg[\kappa\mu B^c - \omega \int_{r_h}^{\infty} \left[\kappa V_t'^{[0]} V_t^{[0]} + \gamma A_t'^{[0]} A_t^{[0]}\right] + \gamma\mu_5 \left[\tilde{B}_5 + \omega\mu_5\right] \\ -\omega\lambda \int_{r_h}^{\infty} \partial_r \left[\frac{r^4 f'^2 g}{2f}\right]^{[0]} + \mathcal{O}(\Delta)\Bigg] \end{aligned} \tag{5.34}$$

The integrals can easily be done by realizing all of them are total derivatives to finally arrive at

$$j^{c[0]} = 2a_0 \left[\mu_5 B^c + \mu B_5 + \omega\mu_5\mu\right] \tag{5.35}$$

$$j_5^{c[0]} = 2a_0 \left[\mu B^c + \frac{\omega}{2}\mu^2\right] + 6a_1 \left[\mu_5 B_5 + \frac{\omega}{2}\mu_5^2\right] + 8\pi^2 a_2 T^2 \tag{5.36}$$

If there was no mass term, these currents will give the renormalized one point functions and from them we would be able to read the known universal result for the anomalous conductivities. Once the renormalized one point functions at order $\mathcal{O}(1)$ are considered the following conductivities are found

$$\sigma_{\text{CME}} = 0 \tag{5.37}$$

$$\sigma_{\text{CSE}} = 2a_0\mu \tag{5.38}$$

$$\sigma_{\text{CVE}} = 2a_0\mu\mu_5 \tag{5.39}$$

$$\sigma_{\text{CVSE}} = a_0\mu^2 + 3a_1\mu_5^2 + 8\pi^2 a_2 T^2 \tag{5.40}$$

All but the CME conductivity take their known universal value. Due to the contribution of the Chern-Simons term the CME conductivity takes once again its consistent current value.

5.2.2 $\mathcal{O}(\Delta)$ corrections

The $\mathcal{O}(\Delta)$ equations that we need to solve are

$$\partial_r\left[\left(Q_5\frac{g_{tt}}{g_{xx}}\right)^{[0]}\left[\frac{\mathcal{A}'^c}{A'_t}-\frac{\mathcal{V}'^c}{V'_t}\right]^{[1]}\right]=\frac{\mathcal{A}_h^{[0]}}{A_t^{[0]}}\left(\frac{g_{tt}}{g_{xx}}\right)^{[0]}(\partial_r Q_5)^{[1]} \tag{5.41}$$

$$\left[\partial_r\left(\frac{\mathcal{V}'^c}{V'_t}\right)\right]^{[1]}=x\left(\frac{Q_5A_tg_{rr}}{\sqrt{-g}}\right)^{[0]}\left[\frac{\mathcal{A}'^c}{A'_t}-\frac{\mathcal{V}'^c}{V'_t}\right]^{[1]} \tag{5.42}$$

The relevant solutions are (Were we used that $K_\infty^{[1]}=0$)

$$\left[\frac{\mathcal{V}'^c}{V'_t}\right]^{[1]}=-xA_h^{[0]}F(r) \tag{5.43}$$

$$\left[\frac{\mathcal{A}'^c}{A'_t}\right]^{[1]}=\mathcal{A}_h^{[0]}\left[\left(\frac{g_{xx}}{Q_5g_{tt}}\right)^{[0]}D(r)-xF(r)\right] \tag{5.44}$$

where $F(r)$ and $D(r)$ are given by

$$F(r)=\int_r^\infty dr'\left(\frac{g_{xx}g_{rr}}{g_{tt}\sqrt{-g}}A_t\right)^{[0]}D(r') \tag{5.45}$$

$$D(r)=\int_{r_h}^r dr'\left(\frac{g_{tt}}{g_{xx}}\frac{1}{A_t}\right)^{[0]}(\partial_r Q_5)^{[1]} \tag{5.46}$$

The axial one point function becomes inaccessible at this order, but the vector one can still be calculated

$$j^c=2a_0\left[\mu_5B+\mu B_5+\mu_5\mu\omega+\Delta\left(\mu_5\ln rB+\mu\mathcal{A}_h^{[1]}+\mathcal{A}_h^{[0]}\mu G\right)+\mathcal{O}(\Delta^2)\right] \tag{5.47}$$

Where the relevant constants are given by

$$A_h^{[1]}=\lim_{r\to\infty}\left[B_5\ln r+\int_{r_h}^r dr'\left(\mathcal{A}_h^{[0]}A_t'^{[0]}\left[xF(r)-\left(\frac{g_{xx}}{Q_5g_{tt}}\right)^{[0]}D(r)\right]-\omega\frac{A_t'^{2[0]}}{A_t'^{[1]}}\right)\right] \tag{5.48}$$

$$\mu G = \int_{r_h}^{\infty} dr \left[xF(r)\partial_r \left(A_t V_t\right) - A_t' V_t \left(\frac{g_{xx}}{Q_5 g_{tt}}\right) D(r)\right]^{[0]} \tag{5.49}$$

The CME, CSE and CVE conductivities can be read off after taking into account the renormalized one point function to obtain that

$$\sigma_{\text{CME}} = 0 \tag{5.50}$$

$$\sigma_{\text{CSE}} = 2a_0\mu \left[1 + \Delta \left(G + A_{hB}^{[1]}\right)\right] \tag{5.51}$$

$$\sigma_{\text{CVE}} = 2a_0\mu\mu_5 \left[1 + \Delta \left(G + A_{h\omega}^{[1]}\right)\right] \tag{5.52}$$

Where the constants $A_{hB}^{[1]}$ and $A_{h\omega}^{[1]}$ are extracted from $A_h^{[1]}$ and are given by

$$A_{hB}^{[1]} = \lim_{r\to\infty} \left[\ln r + \int_{r_h}^{r} dr' A_t'^{[0]} \left[xF(r) - \left(\frac{g_{xx}}{Q_5 g_{tt}}\right)^{[0]} D(r)\right]\right] \tag{5.53}$$

$$A_{hB}^{[1]} = \lim_{r\to\infty} \left[\int_{r_h}^{r} dr' \left(\mu_5 A_t'^{[0]} \left[xF(r) - \left(\frac{g_{xx}}{Q_5 g_{tt}}\right)^{[0]} D(r)\right] - \frac{A_t'^{2[0]}}{A_t'^{[1]}}\right)\right] \tag{5.54}$$

There are some important things to notice from these results:

1. The CME conductivity takes its known universal consistent value. It seems that there are no corrections to this conductivity due to the gluons. It is possible to notice by looking at the renormalized one point function that this should be true at all orders in Δ.

2. Only the a_0 Chern-Simons coefficient enter the results, this is also clear by noticing that all the other coefficients are closely related to the axial one point function.

3. All the expression are given in terms of definite integrals and background information. And from that we can notice that the difference between the correction of the CSE and CVSE conductivity is given by $A_{hB}^{[1]} - A_{h\omega}^{[1]}$. It will be instructive to calculate this expression for a generic background in a later work.

5.3 CVSE (on a Δ expansion at zero density)

To conclude the chapter we will show how to calculate the CVSE conductivity on a background at zero density. For that we will first take a look at the renormalized axial one point function on this limit:

$$\delta J_5 \propto \lim_{r\to\infty} j_5^{[0]}(r) + \lim_{r\to\infty} \Delta \left[j_5^{[1]}(r) + r^2 \delta A^{[0]} + \ln r\, j_5^{[0]}(r) \right] \tag{5.55}$$

By considering zero chemical potentials all equations decouple and is possible to solve first for the metric fluctuations, obtaining

$$\mathcal{K} = \frac{g_{tt}}{g_{xx}} \omega = f\omega \tag{5.56}$$

The remaining equation is the axial Maxwell equation that can be written in two forms

$$\partial_r \left[Z_A \sqrt{\frac{g_{xx} g_{tt}}{g_{rr}}} \delta A'^c \right] = -\partial_r \left(\frac{r^4 g f'^2}{2f} \right) \omega + \frac{Z_0}{Z_0^\infty} \Delta(2+\Delta) \frac{\sqrt{-g}}{g_{xx}} \delta A^c \tag{5.57}$$

and

$$-\partial_r j_5 = -\partial_r \left(\frac{r^4 g f'^2}{2f} \right) \lambda\omega + \frac{Z_0}{Z_0^\infty} \Delta(2+\Delta) \frac{\sqrt{-g}}{g_{xx}} \delta A^c \tag{5.58}$$

From these equations we can calculate all the necessary ingredients for the renormalized axial one point function in a Δ expansion (we have to realize that the background is independent of Δ, as we switched off the gauge fields), obtaining:

$$j_5^{[0]}(r) = -\lambda\omega 8\pi^2 T^2 \left(1 - \frac{r_h^6}{r^6} \right) \tag{5.59}$$

$$\delta A^{[0]} = \lambda\omega 2\pi T \int_r^\infty d\tilde{r} \left[Z_A^{-1} \frac{\tilde{r}^2}{\tilde{r}_h^3} \left(1 - \frac{r_h^6}{\tilde{r}^6} \right) \partial_r \left(\ln f \right) \right] \tag{5.60}$$

$$j_5^{[1]}(r) = \frac{2}{Z_0^{\infty[0]} \left(4\pi T r_h^3\right)} \int_{r_h}^r d\tilde{r} \left(\tilde{r}^6 f' \delta A^{[0]} \right) \tag{5.61}$$

The CVSE conductivity is then just given by

$$\sigma_{\mathrm{CVSE}} = 8\pi^2 a_2 T^2 \left[1 + \lim_{r\to\infty} \Delta \left(j_5^{[1]}(r) + r^2 \delta A^{[0]} + \ln r\, j_5^{[0]}(r) \right) \right] \tag{5.62}$$

AdS-Blackhole example

It is instructive to see an example on a simple background. For this we will take the same background as in the non back-reacting example, namely and Ads-Blackhole geometry. On that case we get the following relevant solutions

$$\delta A^{[0]} = -\frac{\lambda\omega 8\pi T}{r_h}\left[\frac{r_h^4}{4r^4} + \frac{1}{2}\ln\left(1 + \frac{r_h^2}{r^2}\right)\right] \tag{5.63}$$

$$j_5^{[1]} = (16\pi T r_h)\lambda\omega\left[\frac{1}{8}\left(1 - \frac{r_h^2}{r^2}\right) + \frac{r^2}{4r_h^2}\ln\left(\frac{r_h^2}{r^2} + 1\right) + \frac{1}{2}\ln\left(\frac{r}{r_h}\right) + \frac{1}{4}\ln\left(1 + \frac{r_h^2}{r^2}\right) - \frac{\ln 2}{2}\right] \tag{5.64}$$

Using this the conductivity can easily be calculated, obtaining that

$$\sigma_{A\Omega} = 8\pi^2 a_2 T^2\left[1 + \Delta\left(\ln(2) - \frac{1}{4}\right) + \mathcal{O}(\Delta^2)\right] \tag{5.65}$$

It is possible to notice the following

1. As expected only the gravitational anomaly contributes to the temperature dependent expression.
2. The correction is positive, which means that the mass is enhanced just as in the non back-reacting example.

Chapter 6

Conclusions

A general procedure for calculating gluon corrections to the anomalous conductivities was established. This procedure is given in terms of a perturbative expansion in a small parameter Δ related to the coupling of the gluons in the hydrodynamic limit. The method was implemented on a universality class of two derivative holographic models and was capable of giving analytic solutions to the corrections in terms of holographic background information.

This expansion is used to solved a sector of the field perturbation equations that decouple and enable us to calculate the conductivities without having to solve the full perturbation system of equations. The sector that decouples can consistently be solved on a Δ expansion and comes from a non trivial reduction of the fluctuation of Maxwell and Einstein's equations. Three different cases were studied:

1. A non back-reacting background case. A background on which the back-reaction of the gauge fields into the geometry is disregarded was considered for illustrative purposes. A particular example with an AdS blackhole geometry was considered and an analytical result for the corrections of the CME and CSE conductivities was calculated. Our result is consistent with the numerical study that was done for the same AdS Blackhole model on [14].

2. The CME, CSE and CVE conductivities at finite density and temperature were calculated analytically up to order $\mathcal{O}(\Delta)$. The results at $\mathcal{O}(1)$, including the CVSE conductivity, show the universal behavior of the conductivities when the gluon contribution is ignored [12].

3. The CVSE conductivity at zero density and finite temperature was calculated analytically up to order $\mathcal{O}(\Delta)$. A particular AdS blackhole background was discussed for illustrative purposes.

There are a couple of comments to be made regarding the correction of the conductivities for the Ads Blackhole backgrounds, the non renormalization of the CME conductivity and regarding the Δ expansion method in general.

- The CME conductivity was shown to get no corrections from the gluons. It takes its known consistent universal value, this is in agreement with was already discussed in [14] from a numerical study of these type of corrections.

- The CSE conductivity and the CVSE conductivity (at zero density) for an AdS blackhole background are enhanced from the gluon contributions. A further study of the general properties of the corrections from the derived analytical solutions should be carried on to see if the corrections are expected to be always positive or if there is the possibility of having a suppression of the conductivities due to the gluons.

- The Δ expansion procedure can be easily systematized to higher orders. The first order results shown in this work should be tested with full back-reacting backgrounds and the general behavior of the corrections should be study to see if the gluon can have suppressing and enhancement effects depending on the background.

- The Δ expansion provides a reliable method to get analytical expressions for the corrections.

One last question remains to be discussed before going into the possible outlook and continuations of this work. As it was discussed on the introduction, the lattice calculations of the CVSE conductivity at zero density seem to have $\mathcal{O}(1)$ suppressions from the universal value [16]. This is in disagreement with what, at least, happens in the CVSE conductivity here calculated on an AdS Blackhole background. Even if for a different background, or after summing the full perturbative Δ expansion, the correction give rise to a suppression it is not expected to have a significant effect on the conductivity as we are always working on the hydrodynamic limit $\Delta << 1$.

In general we saw that $\Delta < 1$ for the axial current to be marginal or relevant, this means that to include larger corrections that accounts for the lattice result a different mechanism should be introduced. The inclusion of global gravitational anomalies has been proposed as a mechanism to account for $\mathcal{O}(1)$ corrections of the CVSE conductivity [13], this idea will be follow up on future works.

A possible immediate extension of this work is to consider two full DBI actions instead of the two derivative Maxwell approximation that was taken on this work, this will give us a more realistic behavior of a QCD model. The inclusion of tachyon degrees of freedom might be interesting if the phase near the appearance of the chiral condensate wants to be studied.

Appendix A

Appendix

On the appendix we will show the main steps for the derivation of the fluctuation of the equations of motion. For this we will divide the appendix in the following sections:

- Gravitational Terms: The fluctuation of the relevant gravitational components will be shown with some detail. This includes a gravitational Chern-Simons current and the fluctuation of the spin current.

- Currents: The fluctuation of a generic Maxwell current, Chern-Simons current and gravitational Chern-Simons current will be shown.

- Maxwell and Einstein: The fluctuation of the vector and axial Maxwell equations will be shown as well as the fluctuation of Einstein's equation. The reduction to the relevant equations (5.11), (A.63) and (5.12) will be sketch.

Gravitational Terms

As the calculation will be easier using the Cartan formalism of gravity, we will first show all the relevant background components that are needed. We will first start by recalling the two structure equations:

$$\begin{aligned} de^m + \omega^m_n \wedge e^n &= 0 \\ d\omega^m_n + \omega^m_l \wedge \omega^l_n &= R^m_n \end{aligned} \tag{A.1}$$

where e^m is the tetrad field related to the metric via $ds^2 = e^m e^n \eta_{mn}$, $R^m_n = R^m_{nkl} e^k \wedge e^l$ is the 2 form curvature (we used R for tetrad coordinates and $\mathcal{R}$ for spacetime coordinates) and ω^m_n is the spin connection. For our type of metric we have the following tetrad fields

$$e^t = g_{tt}^{\frac{1}{2}} dt \qquad e^c = g_{xx}^{\frac{1}{2}} dx^c \qquad e^r = g_{rr}^{\frac{1}{2}} dr \tag{A.2}$$

Solving the first structure equation give us the following components for the spin connection

$$\omega^t_r = \frac{1}{g_{rr}^{\frac{1}{2}}} \partial_r \left(g_{tt}^{\frac{1}{2}} \right) dt \qquad \omega^c_r = \frac{1}{g_{rr}^{\frac{1}{2}}} \partial_r \left(g_{xx}^{\frac{1}{2}} \right) dx^c \tag{A.3}$$

Using the second structure equation we get the following non vanishing components of the 2-form curvature

$$\begin{aligned}
R^t_r &= \frac{1}{g_{rr}^{\frac{1}{2}} g_{tt}^{\frac{1}{2}}} \partial_r \left[\frac{1}{g_{rr}^{\frac{1}{2}}} \partial_r \left(g_{tt}^{\frac{1}{2}} \right) \right] e^r \wedge e^t \\
R^t_c &= \frac{1}{g_{tt}^{\frac{1}{2}} g_{xx}^{\frac{1}{2}} g_{rr}} \partial_r \left(g_{tt}^{\frac{1}{2}} \right) \partial_r \left(g_{xx}^{\frac{1}{2}} \right) e^c \wedge e^t \\
R^c_r &= \frac{1}{g_{rr}^{\frac{1}{2}} g_{xx}^{\frac{1}{2}}} \partial_r \left[\frac{1}{g_{rr}^{\frac{1}{2}}} \partial_r \left(g_{xx}^{\frac{1}{2}} \right) \right] e^r \wedge e^c \\
R^c_d &= \frac{1}{g_{xx} g_{rr}} \left[\partial_r \left(g_{xx}^{\frac{1}{2}} \right) \right]^2 e^d \wedge e^c
\end{aligned} \tag{A.4}$$

Variation of the curvature two form

As we will be only interested on calculating the fluctuation of Einstein equation it is possible to see that only the fluctuation of the Ricci tensor is necessary. To calculate this fluctuation we will first take the fluctuation of the structure equations

$$\begin{aligned}
d\delta e^m + \delta\omega^m_n \wedge e^n + \omega^m_n \wedge \delta e^n &= 0 \\
D_\mu \omega_{\nu mn} - D_\nu \omega_{\mu mn} &= \delta R_{\mu\nu mn}
\end{aligned} \tag{A.5}$$

where the tetrad and metric fluctuations can be related by noting that

$$d\delta s^2 = \eta_{ab} \delta \left(e^a e^b \right) \tag{A.6}$$

After using this relation it is possible to write the variation of the tetrad fields in terms of the metric fluctuations

$$\delta e^c = \frac{\delta g_{ct}}{2 g_{xx}^{\frac{1}{2}}} dt \qquad \delta e^t = -\frac{\delta g_{tc}}{2 g_{tt}^{\frac{1}{2}}} dx^c \tag{A.7}$$

Solving the first structure equation give us the following connection coefficients

$$\begin{aligned}
\delta\omega^t_c = \frac{1}{2 g_{xx}^{\frac{1}{2}} g_{tt}^{\frac{1}{2}}} \left[-\partial_c \left(\delta g_{dt}\right) dx^d + \left(g_{tt}^{\frac{1}{2}} \partial_r \left(\delta g_{ct} g_{tt}^{-\frac{1}{2}} \right) - g_{xx}^{\frac{1}{2}} \partial_r \left(\delta g_{ct} g_{xx}^{-\frac{1}{2}} \right) \right) dr \right] \\
\delta\omega^t_r = -\frac{g_{xx}^{\frac{1}{2}}}{2 g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \partial_r \left(\delta g_{ct} g_{xx}^{-\frac{1}{2}} \right) dx^c \\
\delta\omega^c_r = \frac{g_{tt}^{\frac{1}{2}}}{2 g_{xx}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \partial_r \left(\delta g_{ct} g_{tt}^{-\frac{1}{2}} \right) dt \\
\delta\omega^c_d = \frac{\partial_d \left(\delta g_{ct}\right) - \partial_c \left(\delta g_{dt}\right)}{2 g_{xx}} dt
\end{aligned} \tag{A.8}$$

The variation of the Riemann tensor with mixed indices is given by

$$\begin{aligned}
\delta R_{\mu\nu mn} = \left[\partial_\mu \left(\delta\omega_{\nu mn}\right) - \partial_\nu \left(\delta\omega_{\mu mn}\right)\right] + \left[\omega_\mu{}^k{}_n \delta\omega_{\nu km} - \omega_\mu{}^k{}_m \delta\omega_{\nu kn} \right] \\
- \left[\omega_\nu{}^k{}_n \delta\omega_{\mu km} - \omega_\nu{}^k{}_m \delta\omega_{\mu kn} \right]
\end{aligned} \tag{A.9}$$

And for the Ricci tensor in particular we have that

$$\delta\mathcal{R}_{ct} = g_{tt}^{\frac{1}{2}} \left[g_{xx}^{-\frac{1}{2}} \delta R_{dcet} \delta^{de} + g_{rr}^{-\frac{1}{2}} \delta R_{rcrt} \right] \tag{A.10}$$

Which can be used in the expression $\delta(g^{\mu\alpha} e^n_\beta e^m_\alpha R_{\mu\nu mn})$ to get the two non-trivial variations of the Ricci tensor

$$\delta\mathcal{R}_{ct} = -\frac{g_{tt}}{2\sqrt{-g}} \partial_r \left[\sqrt{\frac{g_{xx}^5}{g_{rr} g_{tt}}} \partial_r \left(\frac{\delta g_{ct}}{g_{xx}} \right) \right] + \frac{R_{xx}}{g_{xx}} \delta g_{ct} + \frac{1}{2 g_{xx}} \left[\nabla \times \left(\nabla \times \delta g_{ct} \right) \right]_c \tag{A.11}$$

$$\delta\mathcal{R}_{rt} = -\frac{1}{2 g_{xx} g_{tt}} \left[g_{tt} \partial_r \left[\nabla \cdot \boldsymbol{\delta g} \right] - g'_{tt} \nabla \cdot \boldsymbol{\delta g} \right] \tag{A.12}$$

Gravitational Chern-Simons current fluctuation

The gravitational Chern-Simons current form j^{ν}_{G} is given by

$$j^{\nu}_{G} = \mathrm{Tr} d\left(\omega \wedge d\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) \tag{A.13}$$

Using our previous results for the variation of the spin connection we can calculate the fluctuation of this current by noting first that:

$$\begin{aligned}\delta \mathrm{Tr}\left(\omega \wedge d\omega\right) = 2\left[\omega^{t}{}_{rt}\partial_{d}\left(\delta\omega^{r}{}_{tc}\right) - \omega^{x}{}_{rx}\partial_{d}\left(\delta\omega^{r}{}_{ct}\right)\right] dt \wedge dx^{d} \wedge dx^{c}\\ +2\left[\delta\omega^{r}{}_{tc}\partial_{r}\left(\omega^{t}{}_{rt}\right) - \delta\omega^{r}{}_{ct}\partial_{r}\left(\omega^{x}{}_{rx}\right) - \omega^{t}{}_{rt}\partial_{r}\left(\delta\omega^{r}{}_{tc}\right) + \omega^{x}{}_{rx}\partial_{r}\left(\delta\omega^{r}{}_{ct}\right)\right] dt \wedge dx^{c} \wedge dr\end{aligned} \tag{A.14}$$

$$\begin{aligned}\frac{2}{3}\delta \mathrm{Tr}\left(\omega \wedge \omega \wedge \omega\right) = -2\left[\omega^{x}{}_{rx}\omega^{x}{}_{rx}\delta\omega_{cdt} + 2\omega^{t}{}_{rt}\omega^{x}{}_{rx}\delta\omega^{t}{}_{dc}\right] dt \wedge dx^{d} \wedge dx^{c}\\ -4\left[\omega^{t}{}_{rt}\omega^{x}{}_{rx}\delta\omega^{t}{}_{cr}\right] dt \wedge dx^{c} \wedge dr\end{aligned} \tag{A.15}$$

We can also see that

$$\begin{aligned}\delta \mathrm{Tr}\left(\omega \wedge d\omega\right)_{tcr} = \partial_{r}\left[\frac{\partial_{r}\left(\delta g_{ct}\right)}{g_{rr}^{\frac{1}{2}}}\left(\frac{\omega^{t}{}_{rt}}{g_{tt}^{\frac{1}{2}}} - \frac{\omega^{x}{}_{rx}}{g_{xx}^{\frac{1}{2}}}\right)\right] - 2\partial_{r}\left(\delta g_{ct}\right)\left[\frac{\partial_{r}\left(\omega^{t}{}_{rt}\right)}{g_{rr}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}} - \frac{\partial_{r}\left(\omega^{x}{}_{rx}\right)}{g_{rr}^{\frac{1}{2}}g_{xx}^{\frac{1}{2}}}\right]\\ +2\delta g_{ct}\left[\frac{\partial_{r}\left(\omega^{t}{}_{rt}\right)\omega^{x}{}_{rx}}{g_{xx}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}} - \frac{\partial_{r}\left(\omega^{x}{}_{rx}\right)\omega^{t}{}_{rt}}{g_{xx}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}}\right]\end{aligned} \tag{A.16}$$

$$\frac{2}{3}\delta \mathrm{Tr}\left(\omega \wedge \omega \wedge \omega\right)_{tcr} = 2\delta g_{ct}\left[\frac{g_{rr}^{\frac{1}{2}}\omega^{x}{}_{rx}\left(\omega^{t}{}_{rt}\right)^{2}}{g_{xx}^{\frac{1}{2}}g_{tt}} - \frac{g_{rr}^{\frac{1}{2}}\omega^{t}{}_{rt}\left(\omega^{x}{}_{rx}\right)^{2}}{g_{tt}^{\frac{1}{2}}g_{xx}}\right] \tag{A.17}$$

$$\delta \mathrm{Tr}\left(\omega \wedge d\omega\right)_{tdc} = \frac{\partial_{r}\left(\partial_{d}\delta g_{ct}\right)}{g_{rr}^{\frac{1}{2}}}\left[\frac{\omega^{x}{}_{rx}}{g_{xx}^{\frac{1}{2}}} - \frac{\omega^{t}{}_{rt}}{g_{tt}^{\frac{1}{2}}}\right] \tag{A.18}$$

$$\frac{2}{3}\delta \mathrm{Tr}\left(\omega \wedge \omega \wedge \omega\right)_{tdc} = -2\frac{\left(\partial_{d}\delta g_{ct}\right)\omega^{x}{}_{rx}}{g_{xx}^{\frac{1}{2}}}\left[\frac{\omega^{x}{}_{rx}}{g_{xx}^{\frac{1}{2}}} - \frac{\omega^{t}{}_{rt}}{g_{tt}^{\frac{1}{2}}}\right] \tag{A.19}$$

We can now easily realized that the chern-simon form will have the following structure

$$\begin{aligned}j_{G} = -\Bigg[\partial_{r}\left(\delta \mathrm{Tr}\left(\omega \wedge d\omega\right)_{tdc} + \frac{2}{3}\delta \mathrm{Tr}\left(\omega \wedge \omega \wedge \omega\right)_{tdc}\right)\\ +\partial_{d}\left(\delta \mathrm{Tr}\left(\omega \wedge d\omega\right)_{tcr} + \frac{2}{3}\delta \mathrm{Tr}\left(\omega \wedge \omega \wedge \omega\right)_{tcr}\right)\Bigg] dt \wedge dx^{d} \wedge dx^{c} \wedge dr\end{aligned} \tag{A.20}$$

For this term it will be convenient to work with the general form of the perturbation $\delta g_{ct} = g_{xx}\gamma_c$ and also to use the gauge for the metric

$$ds^2 = -r^2 f(r)dt^2 + \frac{dr^2}{r^2 g(r)} + r^2 dx^c dx^d \delta_{cd} \tag{A.21}$$

By considering this the form of the gravitational Chern-Simons term will be given by

$$j_G^c = \delta^{cb}\left[-\partial_r\left(\frac{r^4 g f'}{2f}\left(\nabla\times\gamma_b'\right)\right) - \frac{r^4 g\left(f'\left(\nabla\times\gamma_b''\right) - f''\left(\nabla\times\gamma_b'\right)\right)}{2f}\right] \tag{A.22}$$

Spin current fluctuation

The spin current is given by

$$\Sigma^\lambda_{\mu\nu} = -g_{\tau\mu}\epsilon^{\tau\alpha\beta\rho\sigma}F_{\alpha\beta}R^\lambda{}_{\nu\rho\sigma} = \frac{g_{\tau\mu}}{\sqrt{-g}}\tilde{\epsilon}^{\tau\alpha\beta\rho\sigma}F_{\alpha\beta}R^\lambda{}_{\nu\rho\sigma} \tag{A.23}$$

And its covariant derivative is

$$\nabla_\lambda \Sigma^\lambda_{\mu\nu} = \partial_\lambda\Sigma^\lambda_{\mu\nu} + \Gamma^\lambda_{\lambda\tau}\Sigma^\tau_{\mu\nu} - \Sigma^\lambda_{\mu\tau}\Gamma^\tau_{\lambda\nu} - \Sigma^\lambda_{\nu\tau}\Gamma^\tau_{\lambda\mu} \tag{A.24}$$

On our background the spin current is identical to zero for all of its components, so the variation of the covariant derivative of the spin current is just given by $\nabla_\lambda\delta\Sigma^\lambda_{\mu\nu}$, in particular we are interested on $\nabla_\lambda\delta\Sigma^\lambda_{ct}$ which is just given by

$$\nabla_\lambda\delta\Sigma^\lambda_{(\mu\nu)} = \partial_\lambda\left[\delta\Sigma^\lambda_{(ct)}\right] + \frac{g_{xx}}{2\sqrt{-g}}\delta\Sigma^r_{tc}\partial_r\left(\frac{\sqrt{-g}}{g_{xx}}\right) + \frac{g_{tt}}{2\sqrt{-g}}\delta\Sigma^r_{ct}\partial_r\left(\frac{\sqrt{-g}}{g_{tt}}\right) - \Gamma^r_{tt}\delta\Sigma^t_{cr} - \Gamma^r_{xx}\delta\Sigma^c_{tr} \tag{A.25}$$

And we know that

$$\begin{aligned}
\delta\Sigma^t_{cr} &= \frac{4g_{xx}}{\sqrt{-g}}\left[\nabla\times\boldsymbol{\delta A}\right]_c R^t_{rrt} + \frac{2g_{xx}A_t'}{\sqrt{-g}}\tilde{\epsilon}^{tcder}\delta R^t_{rde} \\
\delta\Sigma^c_{tr} &= -\frac{4g_{tt}}{\sqrt{-g}}\left[\nabla\times\boldsymbol{\delta A}\right]_c R^x_{rrx} \\
\delta\Sigma^\lambda_{ct} &= -\frac{4g_{xx}}{\sqrt{-g}}\left[\nabla\times\boldsymbol{\delta A}\right]_c R^r_{ttr}\delta^{\lambda r} + \frac{4g_{xx}}{\sqrt{-g}}\tilde{\epsilon}^{tcbdr}\partial_r\delta A_d R^x_{ttx}\delta^{\lambda b} + \frac{2g_{xx}}{\sqrt{-g}}A_t'\delta R^\lambda_{tde}\tilde{\epsilon}^{tcder} \\
\Sigma^\lambda_{tc} &= \frac{4g_{tt}}{\sqrt{-g}}\left[\nabla\times\boldsymbol{\delta A}\right]_c R^r_{xrx}\delta^{\lambda r}
\end{aligned} \tag{A.26}$$

And now we have to realize the following:

$$\begin{aligned}
\delta R^t_{rde}\tilde{\epsilon}^{tcder} &= \frac{g'_{xx}\left[\nabla\times\boldsymbol{\delta g}\right]_c - g_{xx}\partial_r\left[\nabla\times\boldsymbol{\delta g}\right]_c}{g_{tt}g_{xx}} \\
\delta R^r_{tde}\tilde{\epsilon}^{tcder} &= \frac{g'_{xx}\left[\nabla\times\boldsymbol{\delta g}\right]_c - g_{xx}\partial_r\left[\nabla\times\boldsymbol{\delta g}\right]_c}{g_{rr}g_{xx}} \\
\partial_b\delta\Sigma^b_{(ct)} = \frac{4g_{xx}}{\sqrt{-g}}R^x_{ttx}\left[\nabla\times\boldsymbol{\delta A}\right]_c &- \frac{2A'_t\nabla^2\left[\nabla\times\boldsymbol{\delta g}\right]_c}{\sqrt{-g}} - \frac{A'_t g'_{xx}\partial_r\left[\nabla\times\boldsymbol{\delta g}\right]_c}{g_{rr}\sqrt{-g}}
\end{aligned} \tag{A.27}$$

Which then everything can be reduced to

$$\nabla_\lambda\delta\Sigma^\lambda_{(\mu\nu)} = \partial_\lambda\left[\delta\Sigma^\lambda_{(ct)}\right] + \frac{g^2_{xx}}{2\sqrt{-g}}\delta\Sigma^r_{tc}\partial_r\left(\frac{\sqrt{-g}}{g^2_{xx}}\right) + \frac{g^2_{tt}}{2\sqrt{-g}}\delta\Sigma^r_{ct}\partial_r\left(\frac{\sqrt{-g}}{g^2_{tt}}\right) \tag{A.28}$$

From all this we can conclude that

$$\delta\left(\nabla_\lambda\Sigma^\lambda_{(ct)}\right) = \Sigma(B_5,\omega,\nabla\times\boldsymbol{\alpha},\nabla\times\boldsymbol{\gamma},r) \tag{A.29}$$

where $\Sigma(B_5,\omega,\nabla\times\boldsymbol{\alpha},\nabla\times\boldsymbol{\gamma},r)$ is just a function of the curls of the perturbations and the bulk radius.

Currents

In this section we will look at the variation of two different types of currents: A maxwel current $j^{\mu\nu}$ for a generic gauge field D with field strength $H = dD$ and a Chern-Simons current j^ν_{CS} with two gauge fields D_1 and D_2 with field strengths $H_1 = dD_1$ and $H_2 = dD_2$.

The generic gauge fields are assumed of the form $D = D_t dt$ and their variations $\delta D_c = \delta(\mathbf{x}, r)$.

Maxwell Current

A generic Maxwell current is defined as

$$j^{\mu\nu} = Z(\phi)\sqrt{-g}H^{\mu\nu} \tag{A.30}$$

Where Z is some unimportant coupling with the dilaton and the current still needs to be properly normalized by some constant $\tilde{G}$. We are now interested on the variation of this current, namely

$$\delta j^{\mu\nu} = Z\delta\left(\sqrt{-g}H^{\mu\nu}\right) = Z\sqrt{-g}\delta H^{\mu\nu} \tag{A.31}$$

which can easily be computed to be

$$\delta j^{\mu\nu} = \left[\frac{Q_D}{g_{xx}}\delta g_{ct} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\partial_r\left(\delta D_c\right)\right]\left[\delta^{\mu r}\delta^{\nu c} - \delta^{\mu c}\delta^{\nu r}\right] + Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\delta H_{dc}\left[\delta^{\mu d}\delta^{\nu c}\right] \tag{A.32}$$

where we have defined a charge

$$Q_D = Z\sqrt{\frac{g_{xx}^3}{g_{rr}g_{tt}}}D_t' \tag{A.33}$$

We can see that there are two non-trivial components of the anti-symmetric current, namely δj^{rc} and δj^{cd} (We have to note that the fluctuation of the consistent current of the gauge fields is proportional to δj^{rc}). We are also interested on the vector $\delta q^{\nu} = \partial_{\mu}\delta j^{\mu\nu}$, which is going to be given by

$$\begin{aligned}\delta q^{\nu} = &\left[\partial_r\left(\frac{Q_D}{g_{xx}}\delta g_{ct} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\partial_r\left(\delta D_c\right)\right) + Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\left(\delta^{ed}\partial_e H_{dc}\right)\right]\delta^{\nu c}\\ &- \left[\frac{Q_D}{g_{xx}}\left(\delta^{dc}\partial_d\delta g_{ct}\right) + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\partial_r\left(\delta^{dc}\partial_d\delta D_c\right)\right]\delta^{\nu r}\end{aligned} \tag{A.34}$$

or in terms of the source perturbations

$$\begin{aligned}\delta q^{\nu} = &\left[\partial_r\left(Q_D\gamma_c + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\delta_c'\right) + Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\left(\delta^{ed}\partial_e H_{dc}\right)\right]\delta^{\nu c}\\ &- \left[Q_D\left(\delta^{cd}\partial_d\gamma_c\right) + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\partial_r\left(\delta^{dc}\partial_d\delta_c\right)\right]\delta^{\nu r}\end{aligned} \tag{A.35}$$

We can now further study the divergence of H on the boundary manifold (for simplicity we are going to use 4-dimensional Euclidian vector notation, as the time component is not present).

$$\begin{aligned}\delta^{ed}\partial_e\delta H_{dc} = -\tilde{\epsilon}_{tcefr}\partial_e\left(\tilde{\epsilon}_{tfbcr}\partial_b\delta D_c\right) = -\left[\nabla\times\left(\nabla\times\delta D\right)\right]_c\\ \delta^{ed}\partial_e\delta H_{dc} = -\left[\nabla\times\left(\nabla\times\boldsymbol{\delta}\right)\right]_c\end{aligned} \tag{A.36}$$

Using this we are going to rewrite three quantities that are going to be important: δq^r, δq^c and $\tilde{\epsilon}^{tbcdr}\partial_c q_d$ (lowering the indices with an spatial euclidian metric):

$$\delta q^r = -\nabla\cdot\left[Q_D\boldsymbol{\gamma} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\delta}'\right] \tag{A.37}$$

$$\delta q^c = \partial_r \left[Q_d \gamma_b + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\delta'_b \right] \delta^{bc} - Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times \boldsymbol{\delta}\right]^c \tag{A.38}$$

$$\tilde{\epsilon}^{tbcdr}\partial_c q_d = \partial_r \left(Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}} B'^b_D \right) + \partial_r \left[\nabla \times \left(Q_D \boldsymbol{\gamma} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\delta}' \right) \right]^b - Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times (\nabla \times \nabla \times \delta)\right]^b \tag{A.39}$$

Then the last three equations reduce to the form we are going to use to work with them

$$\delta q^r = -\nabla \cdot \left[Q_D \boldsymbol{\gamma} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\delta}' \right] \tag{A.40}$$

$$\delta q^c = \partial_r \left[Q_d \gamma_b + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\delta'_b \right] \delta^{bc} - Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times \boldsymbol{\delta}\right]^c \tag{A.41}$$

$$\tilde{\epsilon}^{tbcdr}\partial_c q_d = \partial_r \left[\nabla \times \left(Q_D \boldsymbol{\gamma} + Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\delta}' \right) \right]^b - Z\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times (\nabla \times \nabla \times \delta)\right]^b \tag{A.42}$$

Chern-Simons Current

A generic Chern-Simons current between two gauge fields D_1 and D_2 is given by

$$j^\nu_{CS-1,2} = \tilde{\epsilon}^{\mu\nu\alpha\rho\sigma}\partial_\mu \left(D_{1,\alpha} H_{2,\rho\sigma}\right) \tag{A.43}$$

Which can easily be rewritten as

$$j^\nu_{CS-1,2} = 2\tilde{\epsilon}^{\mu\nu\alpha\rho\sigma} \left(\partial_\mu D_{1,\alpha}\right)\left(\partial_\rho D_{2,\sigma}\right) \tag{A.44}$$

The perturbation is easy to calculate and is given by

$$\delta j^b_{CS-1,2} = -2\left[D'_{2,t}\left(\tilde{\epsilon}^{tbcdr}\partial_c \delta D_{1,d}\right) + D'_{1,t}\left(\tilde{\epsilon}^{tbcdr}\partial_c \delta D_{2,d}\right)\right] \tag{A.45}$$

or in terms of our fluctuation decomposition

$$\delta j^b_{CS-1,2} = -2\left[D'_{2,t}\nabla \times \boldsymbol{\delta}_1 + D'_{1,t}\nabla \times \boldsymbol{\delta}_2 \right]^b \tag{A.46}$$

It is also convenient to write down an expression for $\tilde{\epsilon}^{tbcdr}\partial_c \delta j^d_{CS-1,2}$

$$\tilde{\epsilon}^{tbcdr}\partial_c \delta j^d_{CS-1,2} = -2\left[D'_{2,t}\nabla \times (\nabla \times \boldsymbol{\delta}_1) + D'_{1,t}\nabla \times (\nabla \times \boldsymbol{\delta}_2) \right]^b \tag{A.47}$$

Maxwell and Einstein

Using the results from the previous section the following fluctuation equations are found (a background with $C_0 = 0$ is assumed):

Vector Maxwell fluctuation

The vector maxwell equation can be written in terms of a Maxwell current and a Chern-Simons current as

$$\partial_\mu j_V^{\mu\nu} = -\kappa j_{CS-AV}^{\nu} \tag{A.48}$$

From the fluctuation of this equation and from our previous results we can write down the two non trivial equalities that arise from it

$$\begin{aligned}\partial_r \left[Q\gamma_b + Z_V \sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}} \beta_b' \right] \delta^{bc} = +Z_V \sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times (\nabla \times \boldsymbol{\beta}) \right]^c \\ +2\kappa \left[A_t' B^c \right] + 2\kappa \left[A_t' \nabla \times \boldsymbol{\beta} + V_t' \nabla \times \boldsymbol{\alpha} \right]^c\end{aligned} \tag{A.49}$$

$$-\nabla \cdot \left[Q\boldsymbol{\gamma} + Z \sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}} \boldsymbol{\beta}' \right] = 0 \tag{A.50}$$

To write down the equations that we are going to work with we first have to recall that $j_V^{rc} = -M_p^3 N_c^2 x j^c$ with j^c the covariant vector current, from this we can rewrite (A.49) as

$$\begin{aligned}\partial_r j^c = -M_p^3 N_c^2 x \kappa \left[\partial_r \left(A_t B^c \right) \right] \\ -M_p^3 N_c^2 x \left[\kappa A_t' \nabla \times \boldsymbol{\beta} + \frac{Z_V}{2} \sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times (\nabla \times \boldsymbol{\beta}) \right] + V_t' \nabla \times \boldsymbol{\alpha} \right]^c\end{aligned} \tag{A.51}$$

As the metric perturbations should vanish at the horizon we can notice that the consistent current at the horizon is equal to zero, from this we can get our main result from this section by integrating our previous equation

$$\begin{aligned}j^c(r) = -M_p^3 N_c^2 x \kappa \left[A_t(r) B^c + \int_{r_h}^{r} dr V_t' B_5^c \right] \\ -M_p^3 N_c^2 x \int_{r_h}^{r} \left[\kappa A_t' \nabla \times \boldsymbol{\beta} + \frac{Z_V}{2} \sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}} \left[\nabla \times (\nabla \times \boldsymbol{\beta}) \right] + V_t' \nabla \times \boldsymbol{\alpha} \right]^c\end{aligned} \tag{A.52}$$

As complement to this equation we will consider the curl of equation (A.49) and equation (A.50) as constraints on the deformation of the fluctuation sources

$$\nabla\cdot\left[Q\boldsymbol{\gamma}+Z_V\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\beta'}\right]=0 \tag{A.53}$$

$$\nabla\times\left[Q\boldsymbol{\gamma}+Z_V\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\beta'}\right]=\int_{r_h}^{r}\left[\left(\kappa A_t'+\frac{Z_V}{2}\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\right)\nabla\times(\nabla\times\nabla\times\boldsymbol{\beta})\right.$$
$$\left.+2\kappa V_t'\nabla\times(\nabla\times\boldsymbol{\beta})+2A_t'\kappa\nabla\times(\nabla\times\boldsymbol{\alpha})\right] \tag{A.54}$$

Axial Maxwell fluctuation

The axial Maxwell equations can be written in terms of a Maxwell current, the gauge invariant vector C_ν, two Chern-Simons currents and a gravitational Chern-Simons current as

$$\partial_\mu j_A^{\mu\nu}=-\frac{1}{2}\left(\gamma j_{AA}^{\nu}+\kappa j_{VV}^{\nu}\right)+\lambda j_G^{\nu}+Z_0xm^2C^\nu \tag{A.55}$$

And with all our previous results we can straightforwardly write down the non-trivial equations that come from the perturbation of this equation

$$\partial_r\left[Q_5\gamma_b+Z_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\alpha_b'\right]\delta^{bc}-Z_A\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\left[\nabla\times(\nabla\times\boldsymbol{\alpha})\right]^c=2\delta^{cb}\left[\gamma A_t'\nabla\times\boldsymbol{\alpha}+\kappa V_t'\nabla\times\boldsymbol{\beta}\right]_b$$
$$+2\kappa\left[V_t'B^c\right]-\lambda j_G^c+Z_0xm^2\sqrt{-g}\left[\frac{\alpha_b-\partial_b\delta C_0\,(xm)^{-1}}{g_{xx}}+\frac{\gamma_bB_t}{g_{tt}}\right]\delta^{bc} \tag{A.56}$$

$$-\nabla\cdot\left[Q_D\boldsymbol{\gamma}+Z\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\boldsymbol{\alpha}'\right]=Z_0xm^2\frac{\partial_r\delta C_0}{g_{rr}N_fm} \tag{A.57}$$

The important equation that we are going to be concerned with is going to be given by the curl of (A.56)

$$\partial_r\left[Z_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\nabla\times\boldsymbol{\alpha}'\right]^c-\frac{Z_0xm^2\sqrt{-g}}{g_{xx}}\left[\nabla\times\boldsymbol{\alpha}\right]^c=2\gamma A_t'\left[\nabla\times(\nabla\times\boldsymbol{\alpha})\right]^c$$
$$+2\kappa V_t'\left[\nabla\times(\nabla\times\boldsymbol{\beta})\right]^c+Z_A\sqrt{\frac{g_{rr}g_{tt}}{g_{xx}}}\left[\nabla\times(\nabla\times\nabla\times\boldsymbol{\alpha})\right]^c$$
$$-\partial_r\left[Q_5\nabla\times\boldsymbol{\gamma}\right]^c-\lambda j_G^c+\frac{Z_0xm^2\sqrt{-g}}{g_{tt}}B_t\left[\nabla\times\boldsymbol{\gamma}\right]^c \tag{A.58}$$

Einstein fluctuation

The Einstein equation we are interested on is the one corresponding the the tc component given by

$$\delta\mathcal{R}_{ct} = \frac{R_{xx}}{g_{xx}}\delta g_{ct} + x\frac{Z_V}{2g_{rr}}V_t'\partial_r\delta V_c + x\frac{Z_A}{2g_{rr}}A_t'\partial_r\delta A_c + x^2\frac{Z_0 m^2}{2}B_t\delta B_c + x\frac{\lambda}{2}\delta\left[\nabla_\lambda\Sigma^\lambda_{(\mu\nu)}\right] \tag{A.59}$$

Using our result for $\delta\mathcal{R}_{ct}$ and multiplying by $\frac{2\sqrt{-g}}{g_{tt}}$ the equation gets reduced to

$$\begin{aligned}-\partial_r\left[\sqrt{\frac{g_{xx}^5}{g_{rr}g_{tt}}}\partial_r\left(\frac{\delta g_{ct}}{g_{xx}}\right)\right] + \frac{\sqrt{-g}}{g_{tt}g_{xx}}\left[\nabla\times(\nabla\times\delta g_{ct})\right]_c &= x\partial_r\left[Q\delta V_c + Q_5\delta B_c\right]\\ &+ xQ_5\partial_r\partial_c\delta C_0 + \frac{x\lambda\sqrt{-g}}{g_{tt}}\delta\left[\nabla_\lambda\Sigma^\lambda_{(ct)}\right]\end{aligned} \tag{A.60}$$

We can take the curl of this equation to get

$$\begin{aligned}&\partial_r\left[\sqrt{\frac{g_{xx}^5}{g_{rr}g_{tt}}}\partial_r\left(\nabla\times\boldsymbol{\gamma}\right)_c + xQ\left[\nabla\times\boldsymbol{\beta}\right]_c + xQ_5\left(\nabla\times\boldsymbol{\alpha} + \boldsymbol{B_5}\right)_c\right]\\ &= \nabla\times\left[\nabla\times(\nabla\times\boldsymbol{\gamma})\right]_c - \frac{x\lambda\sqrt{-g}}{g_{tt}}\left[\nabla\times\boldsymbol{\Sigma(B_5,\omega,\nabla\times\alpha,\nabla\times\gamma,r)}\right]_c\end{aligned} \tag{A.61}$$

Defining form of equations

For the fluctuation of the one point functions to be independent of the boundary coordinates we need to ask that for the curl of the deformations to be given by a constant value on the boundary manifold coordinates, namely we need to ask that

$$\nabla\times\boldsymbol{\alpha} = \mathcal{A}(r) \qquad \nabla\times\boldsymbol{\beta} = \mathcal{V}(r) \qquad \nabla\times\boldsymbol{\gamma} = \mathcal{K}(r) \tag{A.62}$$

where $\mathcal{A}, \mathcal{V}$ and $\mathcal{K}$ are the same functions that were introduced in the main text. After using these assumptions the vector and axial fluctuations of the one point function take the form presented in (5.5) and (5.6).

As it was argued in the main text, the vector one point function is completely determined in terms of the curls of the perturbations. To get a close set of equations for them we will consider the curl of Maxwell and Einstein equations, namely equations (A.54),(A.58) and (A.61), together with the assumption that $\mathcal{A}$, $\mathcal{V}$ and $\mathcal{K}$ are only radially dependent. This will give rise to the following equations

$$\mathcal{K}^c = -\frac{Z_V}{Q}\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\mathcal{V}'^c \tag{A.63}$$

$$\partial_r\left[Z_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\mathcal{A}'^c\right] - \frac{\sqrt{-g}Z_0 x m^2}{g_{xx}}\mathcal{A}^c = -Q_5\mathcal{K}'^c \tag{A.64}$$

$$\sqrt{\frac{g_{xx}^5}{g_{rr}g_{tt}}}\mathcal{K}'^c + xQ\mathcal{V}^c + xQ_5\mathcal{A}^c = S \tag{A.65}$$

Where S is some constant. To eliminate this constant from the equations we will now integrate equation (A.65) by combining it with the background equation of motion

$$\sqrt{\frac{g_{xx}^5}{g_{rr}g_{tt}}}\partial_r\left(\frac{g_{tt}}{g_{xx}}\right) = K + xV_tQ + xA_tQ_5 \tag{A.66}$$

and rewrite it as

$$\mathcal{K}'^c\left(K + xV_tQ + xA_tQ_5\right) + xQ\partial_r\left(\frac{g_{tt}}{g_{xx}}\right)\mathcal{V}^c + xQ_5\partial_r\left(\frac{g_{tt}}{g_{xx}}\right)\mathcal{A}^c = S\partial_r\left(\frac{g_{tt}}{g_{xx}}\right) \tag{A.67}$$

We can also use equations (A.64) and (A.63) to simplify it as follows

$$\begin{aligned} S\partial_r\left(\frac{g_{tt}}{g_{xx}}\right) = {} & \partial_r\left(K\mathcal{K}^c\right) - x\left[\partial_r\left(Z_V\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\mathcal{V}'^c\right)V_t - \partial_r\left(Q\frac{g_{tt}}{g_{xx}}\right)\mathcal{V}^c\right] \\ & -x\left[\partial_r\left(Z_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\mathcal{A}'^c\right)A_t - \partial_r\left(Q_5\frac{g_{tt}}{g_{xx}}\right)\mathcal{A}^c\right] \end{aligned} \tag{A.68}$$

conveniently written as a total radial derivative

$$S\partial_r\left(\frac{g_{tt}}{g_{xx}}\right) = \partial_r\left[K\mathcal{K}^c + xZ_V\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\left(V_t'\mathcal{V}^c - V_t\mathcal{V}'^c\right) + xZ_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\left(A_t'\mathcal{A}^c - A_t\mathcal{A}'^c\right)\right] \tag{A.69}$$

Integrating this last expression from the horizon to some arbitrary r give us that

$$S\frac{g_{tt}}{g_{xx}} = K\mathcal{K}^c + xZ_V\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\left(V_t'\mathcal{V}^c - V_t\mathcal{V}'^c\right) + xZ_A\sqrt{\frac{g_{xx}g_{tt}}{g_{rr}}}\left(A_t'\mathcal{A}^c - A_t\mathcal{A}'^c\right) \tag{A.70}$$

Using (A.65) together with (A.70) is possible to get a new equation that does not involve S, namely

$$\partial_r\left(\frac{\mathcal{V}'^c}{V_t'}\right) = x\frac{Q_5 A_t g_{rr}}{\sqrt{-g}}\left[\frac{\mathcal{A}'^c}{A_t'} - \frac{\mathcal{V}'^c}{V_t'}\right] \tag{A.71}$$

This last equation is an integrated version of the curl of the fluctuation of Einstein equation. Inspire by its form is also possible to rewrite one of the Maxwell equations as

$$\partial_r\left[Q_5\frac{g_{tt}}{g_{xx}}\left(\frac{\mathcal{A}'^c}{A_t'} - \frac{\mathcal{V}'^c}{V_t'}\right)\right] = \frac{g_{tt}}{g_{xx}}\partial_r Q_5\left[\frac{\mathcal{A}^c}{A_t} - \frac{\mathcal{V}'^c}{V_t'}\right] \tag{A.72}$$

These last two expressions are the ones used in the main texted for the derivation of the conductivities.

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