June 21, 2017

Dynamics of linear ITG modes with flow shear in ballooning space

A Master thesis by Victor I. Dagnelie¹

Supervised by DR. JONATHAN CITRIN² PROF. DR. CRISTIANE DE MORAIS SMITH¹

¹UTRECHT UNIVERSITY Institute for Theoretical Physics ²DIFFER Dutch Institute for Fundamental Energy Research

Abstract

It is widely recognized [7]-[13] that in a toroidal nuclear fusion reactor (tokamak), unstable plasma modes driven by a radial ion temperature gradient (ITG) can be stabilized by the effect of rotational flow shear. In this study these rotational modes are solved from the linear gyrokinetic equations and scrutinized in ballooning space [14][15], using the GENE code [16]. In ballooning space the effects of flow shear are clearly visible, as well as a difference in stabilization between kinetic and adiabatic electron modes at low magnetic shear \hat{s} (the latter is quenched at lower flow shear). Mode shapes consistently equilibrate in ballooning space, whereas they can still fluctuate highly in time (Floquet modes).

To gain physical insight into the mechanics of stabilization and Floquet fluctuations, a toy model is created. The full rotational ITG solution is decomposed into shearless modes to which flow shear is separately added. The model reproduces general mode structures, Floquet fluctuations and the stabilizing impact of flow shear. However, the difference in stabilization of modes with kinetic and adiabatic electrons at low \hat{s} is not captured.

Besides this main research, a linear quench rule is derived from GENE simulations, approximating the impact of flow shear on ITG modes in a section of 4-dimensional parameter space. This rule can serve as a new dimension in a plasma turbulence neural network based on QuaLiKiz [17] runs (Citrin et al., in progress)

Contents

1	Introduction 4						
	1.1	Nuclear fusion	4				
	1.2	Project motivation and outline	5				
2	General plasma physics 7						
	2.1	Single particle motion and drifts	7				
	2.2	Collisions	11				
	2.3	Vlasov equation	12				
	2.4	Gyrokinetic equations	15				
		2.4.1 Gyrophase averaging	15				
		2.4.2 δ -f splitting	17				
		2.4.3 Gyrokinetic Vlasov equations	18				
3	Tokamaks 19						
	3.1	General setup	19				
	3.2	ITG modes	20				
	3.3	Safety factor q	22				
	3.4	Magnetic shear \hat{s}	22				
	3.5	Flow shear γ_E	23				
	3.6	Adiabatic and kinetic electrons	24				
4	GENE 25						
	4.1	Introduction to GENE	25				
	4.2	Field aligned coordinates	26				
	4.3	Boundary conditions	26				
		4.3.1 Binormal boundary condition	27				
		4.3.2 Parallel boundary condition	27				
	4.4	Flow shear in GENE	29				
	4.5	Solutions of Gene	30				
5	Methodology 32						
	5.1	Problem formulation	32				
	5.2	Instability drive competition	33				
	5.3	Ballooning representation	35				
		5.3.1 The Ballooning transformation	35				
		5.3.2 Ballooning angle	37				
		5.3.3 Shifting $k_{x.center}$	38				
	5.4	Growth rates	39				
		5.4.1 Floquet averaging	39				
		5.4.2 τ_{ac} method	39				
		5.4.3 Growth rates of stable and higher k_x modes	40				
	5.5	Mode convergence	41				

6	Linear ITG quenching rule 42			
	6.1 Influence of R/L_{T_i}	42		
	6.2 Influence of ϵ	43		
	6.3 Influence of q	45		
	6.4 Influence of \hat{s}	45		
7	ITG eigenmode dynamics 4			
	7.1 Problem and methodology summary	48		
	7.2 Rotationless modes ϕ_n	49		
	7.2.1 Rotationless radial growth rates	51		
	7.3 Rotational modes Φ	53		
8	Toy Model	57		
	8.1 Model mechanics	57		
	8.2 Constants and parameters	59		
	8.3 Results	60		
	8.4 Explanation of mechanisms	63		
9	Conclusion and discussion 6			

1 Introduction

In a time where man-made climate change is an ever more pressing global issue, people frantically try to device a fully carbon neutral energy network. Wind, water and solar energy are aready deployed on a large scale and heavily subsidized to make the transition as quickly as possible. However, these sources alone are not enough: they suffer from their intermittent nature since they can not produce power continuously, scaled by the consumer's demand. Several solutions to this large issue are available. First of all surplus energy could be stored after its creation, to be released later when it is needed. However, as of yet no such storage solution is viable [1]. The problem can at least be mitigated by creating an EU-wide power net, which would help to distribute surplus power between countries that obtain energy from uncorrelated sources [2]. This remains only a partial solution however, with the drawback of introducing political dependencies. A more complete solution is to fill the gaps of insufficient energy production by an alternative renewable energy source which can be switched on and off at will. Nuclear fission can fulfil this task, but struggles with its own inherent problems: its hazardous chain reaction nature, long half-life radioactive waste and rare ingredients make it an unpopular solution. The opposite reaction, nuclear fusion, does not possess these negative properties, and may perfectly supplement the gaps left by current renewable energy sources. As such it is a good candidate to reach completely carbon neutral power production in the future [3].

1.1 Nuclear fusion

Nuclear fusion is the process of fusing two atom nuclei into a single one. For elements lighter than iron, the binding energy between the protons and neutrons of the resulting core is less than that of the two separate cores from which we started. This energy (or mass) deficit is released during the fusion reaction and the source of energy that we try to harvest. Two positively charged nuclei have a huge mutually repulsive Coulomb force when they come close to each other. Clasically, this Coulomb barrier must be overcome to fuse the nuclei. Luckily at energies below the barrier fusion already takes place because of quantum tunneling. The optimal temperature needed for D-T fusion turns out to be about 100 keV. This energy can be inserted into a system in several ways, but the most popular approach is by increasing the temperature. This approach is called thermonuclear fusion. The main fusion reaction that is focused on right now is the fusion of deuterium and tritium, according to the reaction

$$_{1}D^{2} + _{1}T^{3} \rightarrow _{2}He^{4} + _{0}n^{1} + 17.59 \text{ MeV.}$$
 (1)

The reason that this reaction is prefered above others is that its cross section is the most favourable. Deuterium is plentiful on Earth, with enough reserves present to fulfil human power consumption for several billions of years. Tritium on the other hand is extemely rare in nature, with current estimates of about 20 kg worldwide. However, it is possible to "breed" tritium in a reactor by colliding the neutron that is released during the fusion reaction with lithium, of which there are large reserves again. Although challenging by itself, it is a well-researched solution that enables fusion fuel to be nearly inexhaustive [4].

1.2 Project motivation and outline

The difficulties of nuclear fusion however prove to be mighty. The most common current design for a nuclear fusion reactor is called a tokamak, where inside a toroidal chamber plasma is confined by strong magnetic fields and heated to about 100 million Kelvin. These temperatures push engineering and materials to their limits, which ramps up the costs needed for research and eventual deployment of reactors. For fusion to be a solution to fill the renewable energy gaps it ought to be be sufficiently simple and cheap to be economically viable. A general approach to reach this is by improving the plasma's confinement, which results in smaller and thus cheaper reactors.

A well confined plasma is in a steady state of fusion power being produced and lost. Ideally, the lost energy is compensated by the fusion process itself, in which case we speak of a *burning* plasma: in first instance, energy is added to the plasma to start the fusion reactor ("ignition"), and after that the reaction will keep itself going. Any additional energy that is produced above this steady state requirement is welcome, since it can be harvested from the reactor and turned into electricity. This requirement for fusion is captured well by the triple product (first introduced by Lawson [5]), which approximately states that for D-T fusion:

$$nT\tau_E \ge 5 * 10^{21} \text{ m}^{-3} \text{s KeV}$$
 (2)

where n is the plasma density, T its temperature and τ_E the confinement time, which measures the time after which the plasma loses energy to its surroundings. There turns out to be an optimal density and temperature operation range for fusion, but a lot can still be gained in plasma confinement. When there is a long confinement time, particles have more time to energetically interact with each other and fuse. Apart from the financial and engineering benefits of confinement as mentioned before, reducing the confinement time is thus also a fundamental objective to increase fusion efficiency.

However, inside a tokamak many modes are driven unstable, leading to turbulence and reduced confinement. Stabilizing these modes is a large field of fusion research, and also the main subject of this work. In particular, we focus on the Ion Temperature Gradient (ITG) instability. Many other drives exist in a tokamak to produce a wealth of different modes, some of which will be outlined later, but ITG modes remain a dominant source for unstable heat and momentum loss [6].

To stabilize these modes, rotational flow shear can be added in a tokamak, which is a gradient in the rotation of the plasma flows. This gradient turns out to shear the unstable modes apart into smaller, more stable fluctuations. The beneficiary effects of flow shear on ITG modes are of great importance and thus widely investigated theoretically, experimentally and by simulations [7]-[13]. Still, not all the details of the mechanics behind them have been worked out.

In this work we investigate the stabilization of ITG modes by flow shear in a nonphysical space called *ballooning space* [14] [15]. For this we use the gyrokinetic code GENE [16]. Visualization in ballooning space may shed light on some of the mechanics of the impact of flow shear on ITG modes that are not clear from usual analysis in real space. To further increase the transparency of these mechanics we then create a toy model. This model attempts to reproduce the effects of flow shear in ballooning space by decomposing the sheared ITG mode into shearless modes. The separate implementation of flow shear in this model then provides clear insight on its effects. In particular we focus on its stabilizing impact, the nature of Floquet fluctuations and the cause of an observed difference in mode stabilization when either kinetic or adiabatic electrons are used (these terms are outlined later in this work). Apart from being academically interesting, this can help to better understand how to optimize confinement in tokamaks. Finally, the model may be useful for reduced simulation purposes.

2 General plasma physics

The temperatures required for thermonuclear fusion are tremendous, in the range of 10 to 100 million Kelvin. All the elements combined in such a fusion reactor are then in the plasma state, with the electrons stripped from the atom's nuclei. Containing such a hot plasma is challenging, since all physical containers would evaporate when coming into contact with the hot plasma. A solution is to use magnetic confinement, by which all the charged plasma particles are held together by an externally applied magnetic field. As such a plasma suited for fusion consists of many hot charged particles, whose movement is affected by the surrounding magnetic and electric fields. However, each charged particles' movement in turn affects the fields. Plasma physics describes the orbits of these charged particles in self-consistent electric and magnetic fields. In this section we introduce some of the plasma physics that is required to understand this work.

The high temperatures of a plasma state result in high particle speed. The basic assumption defining a plasma is that typically, the potential energy of a particle due to its nearest neighbour is much smaller than its kinetic energy [18]. Let's consider a simple plasma, in which there is an equal amount of positively charged (ions) and negatively charged (electrons) particles, and the density per of these particle species is n. Then the average distance between a particle and its nearest neighbour is $r = n^{1/3}$ and the corresponding potential energy of the particle due to this neighbour is

$$\Phi| \propto \frac{e^2}{r} \propto n^{1/3} e^2.$$
(3)

The typical kinetic energy of a particle is $E_{kin} \propto T_s$, where T_s denotes the temperature of the particle species s (here ions and electrons). Our plasma assumption then translates to

$$n^{1/3}e^2 << T_s.$$
 (4)

2.1 Single particle motion and drifts

To give an understanding of how particles behave in a magnetized plasma, we consider here the typical forces that affect a single particle's orbit. In a plasma there are several such forces, which are all coupled to one another by the electric and magnetic field. Emergent structures in a plasma are often caused by a subsequent combination of these forces. One such effect will be outlined in section 3.2, when we give an explanation of the ion temperature gradient instability. For a more in-depth treatment of plasma motion we refer to the textbooks [18] [19] [20]. **Gyromotion** First of all, let's consider the motion of a single charged particle in a magnetic field, which is complicated by the Lorentz force. In a constant magnetic field in the z-direction $(\vec{B} = B_0 \hat{z})$, a particle of charge q_s and mass m_s obeys the equation of motion

$$m_s \vec{x} = \frac{q_s}{c} (\vec{x} \times B_0 \hat{z}). \tag{5}$$

We take an initial velocity $\vec{x_0} = (0, v_{\perp}, v_z)$ since the x and y coordinates can be aligned to the particles'motion and the z-direction to the magnetic field, and the initial position is $\vec{x_0} = (x_0, y_0, z_0)$. The solution of the equation of motion is then

$$\vec{x}(t) = \begin{pmatrix} x_0 + \frac{v_\perp}{\Omega_s} (1 - \cos \Omega_s t) \\ y_0 + \frac{v_\perp}{\Omega_s} \sin \Omega_s t \\ z_0 + v_z t \end{pmatrix}, \tag{6}$$

where $\Omega_s = \frac{q_s B_0}{m_s c}$ is the gyrofrequency of the particle. This name comes from the nature of the motion, as illustrated in Fig. 1: the particle moves with a constant speed in the z-direction and a gyration in the x - y plane about the guiding center with angular frequency $|\Omega_s|$. The gyroradius vector is

$$\vec{\rho} = \frac{v_{\perp}}{\Omega(\vec{x})} (\cos(\theta)\hat{x} + \sin(\theta)\hat{y}).$$
(7)

The position of the guiding center can then be denoted as

$$\vec{X} = \vec{x} - \vec{\rho}.\tag{8}$$



Figure 1: Gyromotion of charges particle in magnetic field and guiding center coordinates

ExB drift The orbit of a plasma particle in a magnetic field follows the described gyromotion; when a force perpendicular to the magnetic field is included the particle starts to drift away from this orbit. The usual force is the electric field and the resulting drift is called ExB drift.

Consider an ion gyrating in a magnetic field and electric field as illustrated in Fig. 2. The electric field will accelerate the ion as it moves upward, and decelerate it when it moves down again. Therefore, the gyroradius (being proportional to v_{\perp}) becomes larger at the top than at the bottom of the orbit. This in turn results in the particle obtaining a drift velocity to the right, instead of just moving in the z-direction. This drift is called ExB drift.



Figure 2: ExB drift of a positively charged particle

When we average the motion over a gyroperiod, the acceleration of the particle is zero. The force downward due to the electric field is thus balanced with the force upward due to the Lorentz force. We thus find

$$q_s \vec{E} + \frac{q_s}{c} (\vec{v}_{\text{ExB}} \times \vec{B}) = 0, \qquad (9)$$

so that the ExB drift velocity becomes

$$\vec{v}_{\rm ExB} = \frac{c}{B_0^2} (\vec{E} \times \vec{B}). \tag{10}$$

 ∇B drift An effect very similar to ExB drift is ∇B drift, which is caused by a gradient in the magnetic field. So lets consider again a single ion gyrating in a magnetic field, but now there is a gradient in this magnetic field in the *y*-direction, as illustrated in Fig. 3. Intuitively, we can guess what happens again. Because the magnetic field is stronger in the *y*-direction, and the gyroradius $\rho \propto 1/\Omega_s \propto 1/B$, the gyroradius will be smaller at the top of the orbit than at the bottom. Similarly to the ExB drift, this results in a drift velocity to the left. Quantitatively, the ∇B drift can be derived [18] to yield

$$\vec{v}_{\nabla B} = \frac{v_0^2}{2\Omega_s B_0^2} (\vec{B} \times \vec{\nabla B}).$$
(11)



Figure 3: ∇B drift of a positively charged particle

Curvature drift As we have seen, a charged particle in a magnetic field will move along a field line, while gyrating about it. Now suppose this field line is curved; then the particle will experience a centrifugal force while it is following this bent line. The force is equal to

$$\vec{F}_{curv} = \frac{m_s v_{\parallel}^2}{R_{curv}} \hat{R}_{curv}, \qquad (12)$$

where R_{curv} is the radius of the circle that fits the bend in the field line. The same derivation as from the ExB drift can be followed, replacing the electric force with F_{curv} . We then find for the curvature drift

$$\vec{v}_{curv} = \frac{v_{\parallel}^2}{\Omega_s R_{curv}} (\hat{R}_{curv} \times \vec{B})$$
(13)



Figure 4: Curvature drift of a positively charged particle

Polarization drift Another drift caused by an effective force perpendicular to the magnetic field is the polarization drift, which is caused by an electric field which varies with time. Imagine a particle in an electric and

magnetic field such as considered in Fig. 2, but now the electric field becomes stronger with time. The ExB drift velocity in the x-direction will then increase with time, and the result is that the particle experiences a force in the -x-direction. This effective force in turn induces its own drift, which is called the polarization drift [18]:

$$\vec{v}_p = \frac{c}{\Omega_s B} \frac{d}{dt} \vec{E}.$$
(14)



Figure 5: Polarization drift of a positively charged particle

Magnetic moment Yet another force is caused by the magnetic moment $\mu \propto IA$, where I is the current and A the area of a loop current. For a charged particle gyrating in a magnetic field we then find

$$\mu = \frac{m_s v_\perp^2}{2B}.\tag{15}$$

If there is a gradient in the magnetic field, a magnetic moment experiences a force $-\mu\nabla B$. Remarkably, for a charged particle in a slowly varying magnetic field (both space and time variation is allowed), the magnetic moment turns out to be constant. It is then an example of an *adiabatic invariant*. Using this, one can for example employ an inhomogeneous magnetic field to steer the particles in a plasma, which is done in mirror confinement fusion devices.

2.2 Collisions

Finally, to understand the motion of plasma particles, collisional effects must be discussed. Here, we shall be very brief about this subject, since it quickly becomes a complicated subject and precise understanding of collisions is not of great importance to this work. For a better explanation we refer to [21]. A plasma particle electrostatically interacts with many other particles that are close. However, from the definition of a plasma follows that these interactions (related to the potential energy of a particle) are small compared to a particle's kinetic energy. So collisional effects are small, but they can be very important. There exist turbulent plasma modes that are greatly impacted by collisions [22]. And indeed, fusion itself is a collisional effect! From the plasma definition we can also deduce that collisions rarely lead to large-angle scattering of a particle, since this would require the potential energy of a particle to be comparable to its kinetic energy. Much more common is the occurrence of small-angle scattering. It can be roughly shown, as e.g. done in [18], that a plasma particle's deflection by collisions is much more due to the effect of many small-angle scattering than by the occasional large-angle scattering.

In the next section we derive the Vlasov equation. This equation is widely used in plasma physics, but it ignores collisional effects. This works well in many circumstances, but not always. In a tokamak for example, certain instabilities exist that are highly impacted by collisions [22]. Therefore in later derivations a collision operator is added again, so that the effects are not neglected.

2.3 Vlasov equation

Thus far we have described the motion of single particles in a plasma. To simulate the whole plasma, a set of equations for many particles is needed. To that aim, we first derive here the Klimontovich equation. Collisions are neglected for the moment, which is exact when $\Lambda_s \to \infty$. Consider a single point particle *i* with orbits $\vec{x}_i(t)$ and $\vec{v}_i(t)$ in six-dimensional phase space (\vec{x}, \vec{v}) . The density of this phase space is then simply

$$N(\vec{x}, \vec{v}, t) = \delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i(t)).$$
(16)

This can quickly be generalized to more particle species, with each species containing M particles. We then obtain the density for species s

$$N_s(\vec{x}, \vec{v}, t) = \sum_{i=1}^M \delta(\vec{x} - \vec{x}_i(t)) \delta(\vec{v} - \vec{v}_i(t)).$$
(17)

The time evolution of this density can be calculated, since the time evolution of the particles' orbits through phase space is known:

$$\dot{\vec{x}}_i(t) = \vec{v}_i(t) \tag{18}$$

 $m_{s}\dot{\vec{v}}_{i}(t) = q_{s}\vec{E}^{m}(\vec{x}_{i},t) + \frac{q_{s}}{c}\vec{v}_{i}(t) \times \vec{B}^{m}(\vec{x}_{i},t)$ (19)

from the Lorentz force equation. \vec{E}^m and \vec{B}^m are the microscopic magnetic and electric fields, which are the background fields plus the field produced by all the point particles in the plasma. The fields themselves satisfy Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E}^{m}(\vec{x},t) = 4\pi\rho^{m}(\vec{x},t)$$

$$\vec{\nabla} \cdot \vec{B}^{m}(\vec{x},t) = 0$$

$$\vec{\nabla} \times \vec{E}^{m}(\vec{x},t) = -\frac{1}{c} \frac{\partial \vec{B}^{m}(\vec{x},t)}{\partial t}$$

$$\vec{\nabla} \times \vec{B}^{m}(\vec{x},t) = -\frac{1}{c} \frac{\partial \vec{E}^{m}(\vec{x},t)}{\partial t} + \frac{4\pi}{c} \vec{J}^{m}(\vec{x},t), \qquad (20)$$

with the microscopic charge density

$$\rho^m(\vec{x},t) = \sum_s q_s \int d\vec{v} N_s(\vec{x},\vec{v},t) \tag{21}$$

and the microscopic current density

$$\vec{J}^{m}(\vec{x},t) = \sum_{s} q_{s} \int d\vec{v} \ \vec{v} N_{s}(\vec{x},\vec{v},t).$$
(22)

We can now take the time derivative of Eq. 17 and use the orbit equations to write the result in terms of \vec{E}^m and \vec{B}^m . After some calculation this results in the *Klimontovich* equation [23]:

$$\left[\partial_t + \vec{v} \cdot \vec{\nabla}_x + \frac{q_s}{m_s} \left(\vec{E}^m + \frac{\vec{v}}{c} \times \vec{B}^m\right) \cdot \vec{\nabla}_v\right] N_s(\vec{x}, \vec{v}, t) = 0$$
(23)

The Klimontovich equation, together with Maxwell's equations, gives a complete description of a plasma, once the initial fields and locations and velocities of all the particles are known.

However, the Klimontovich equation tracks all particles separately. In a magnetic fusion device, this would mean tracking about 10^{23} particles, which is not possible to do on any modern supercomputer in the forseeable future. The Klimontovich equation is therefore not of any practical use to simulate the plasma in fusion devices. We need a more statistical approach to

and

the problem. Instead of tracking the individual particles, we now consider the distribution function $f_s(\vec{x}, \vec{v}, t)$ in six-dimensional phase space. Statistically, this distribution function describes the ensemble averaged number of particles per unit volume of this six-dimensional phase space:

$$f_s(\vec{x}, \vec{v}, t) = \langle N_s(\vec{x}, \vec{v}, t) \rangle, \qquad (24)$$

where $\langle ... \rangle$ denotes an ensemble average. The difference between quantities and their ensemble averages are then

$$\delta N_s(\vec{x}, \vec{v}, t) = N_s(\vec{x}, \vec{v}, t) - f_s(\vec{x}, \vec{v}, t)$$

$$\delta \vec{E}_s^m(\vec{x}, \vec{v}, t) = \vec{E}_s^m(\vec{x}, \vec{v}, t) - \vec{E}_s(\vec{x}, \vec{v}, t)$$

$$\delta \vec{B}_s^m(\vec{x}, \vec{v}, t) = \vec{B}_s^m(\vec{x}, \vec{v}, t) - \vec{B}_s(\vec{x}, \vec{v}, t), \qquad (25)$$

where $\vec{E}_s = \langle \vec{E}_s^m \rangle$, $\vec{B}_s = \langle \vec{B}_s^m \rangle$ and $\langle \delta \vec{E} \rangle = \langle \delta \vec{B} \rangle = \langle \delta N_s \rangle = 0$. Using these relations the Klimontovich equations can be adapted to incorporate the distribution function, which yields

$$\begin{bmatrix} \partial_t + \vec{v} \cdot \vec{\nabla}_x + \frac{q_s}{m_s} \left(\vec{E}^m + \frac{\vec{v}}{c} \times \vec{B}^m \right) \cdot \vec{\nabla}_v \end{bmatrix} f_s(\vec{x}, \vec{v}, t) = -\frac{q_s}{m_s} \langle (\delta \vec{E} + \frac{\vec{v}}{c} \times \delta \vec{B}) \cdot \vec{\nabla}_v \delta N_s \rangle.$$
(26)

This equation is known as the exact *plasma kinetic equation*. The left hand side describes collective plasma effects while the right hand side contains collisional effects. When the collisional effects are neglected the *Vlasov equation* [24] is obtained:

$$\left[\partial_t + \vec{v} \cdot \vec{\nabla}_x + \frac{q_s}{m_s} \left(\vec{E}^m + \frac{\vec{v}}{c} \times \vec{B}^m\right) \cdot \vec{\nabla}_v\right] f_s(\vec{x}, \vec{v}, t) = 0.$$
(27)

The Vlasov equation is possibly the single most important equation in plasma physics. It describes the time evolution of the ensemble averaged distribution function through six-dimensional phase space, while ignoring collisions. Although it is strictly only accurate for a large ensemble of plasmas, in practise the large number of particles in a plasma means that it is usually a good approximation for a single plasma. With the ensemble averaged Maxwell's equations the plasma is then fully described:

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = 4\pi\rho(\vec{x}, t) \tag{28}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0 \tag{29}$$

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) = -\frac{1}{c} \frac{\partial \vec{B}(\vec{x}, t)}{\partial t}$$
(30)

$$\vec{\nabla} \times \vec{b}(\vec{x},t) = -\frac{1}{c} \frac{\partial \vec{E}(\vec{x},t)}{\partial t} + \frac{4\pi}{c} \vec{J}(\vec{x},t), \qquad (31)$$

with

$$\rho(\vec{x},t) = \langle \rho^m \rangle = \sum_s q_s \int d\vec{v} f_s(\vec{x},\vec{v},t)$$
(32)

and

$$\vec{J}(\vec{x},t) = \langle J^m \rangle = \sum_s q_s \int d\vec{v} \ \vec{v} s_s(\vec{x},\vec{v},t).$$
(33)

2.4 Gyrokinetic equations

The Vlasov equation together with Maxwell's equations form a closed system of equations which govern the behaviour of a charged particle in a magnetized plasma. Theoretically, the system is solved. In practise however, the equations are not used directly in fusion plasma simulations, because they are still too complicated for modern supercomputers to solve for a plasma of the right size. Luckily, the Vlasov and field equations can be further reduced by making some clever simplifications, called gyro averaging and δ -f splitting. The resulting gyrokinetic equations are mainly applicable in a tokamak core, away from the reactor walls, and form the basis of a large subfield of plasma physics. The derivations are rather involved, so here we only mention the major steps (gyrophase averaging and δ -f splitting) and present their results. Much of this is taken from [26] [27]. A nice Lagrangian approach to the derivation is given in [28].

2.4.1 Gyrophase averaging

In a magnetic confinement reactor, the magnetic field that is applied is large, and as a result particles will move along the magnetic field lines with a very fast gyration about them. Movement in the plane perpendicular to the magnetic field lines is thus highly restricted, and the gyration has a much shorter timescale than the dynamics of interest such as turbulent transport. It is therefore possible to average out the gyrophase, which reduces the particle distribution function dimensionality from six to five. Next a gyrokinetic ordering parameter $\epsilon = \rho_{ref}/L_{ref}$ is introduced, where $\rho_{ref}, /L_{ref}$ are reference length scales for the gyroradius and turbulent plasma, respectively. Since the typical gyroradius is much smaller than the variations of the magnetic field in the plasmas considered, $\epsilon \ll 1$. This is used to further reduce the equations, which finally brings the solving of the equations within the realm of modern supercomputers.

Since the position of a particle can be viewed as a superposition of a fast gyration and a slow drift of the gyrocenters, instead of tracking the particles position \vec{x} , it is useful to transform to its gyrocenter position \vec{X} , as shown in Fig. 1. and given by Eq. 8.

This gyrocenter position vector can be implemented in the Vlasov equation. This derivation is done in detail in many previous works (see e.g. [26] [27]) and will not be repeated here. We present the resulting five-dimensional Vlasov equation for the gyrocenter particle distribution function $F_j(\vec{X}, v_{\parallel}, \mu)$ of species j:

$$\frac{\partial F_j}{\partial t} + \dot{\vec{X}} \Big(\vec{\nabla} F_j + \frac{1}{m_j v_{\parallel}} \big(q_j \vec{E}_1 - \mu \vec{\nabla} (B_0 + \bar{B}_{1\parallel}) \big) \frac{\partial F_j}{\partial v_{\parallel}} \Big) = \langle C_j(F) \rangle.$$
(34)

Here, bars denote gyroaverages about the gyrocenter position \vec{X} , and brackets $\langle ... \rangle$ denote gyroaverages taken about the particle position \vec{x} . The fields were split to equilibrium and perturbed parts $\vec{E} = \vec{E}_0 + \vec{E}_1$ and $\vec{B} = \vec{B}_0 + \vec{B}_1$, which are governed by Maxwell's equations. The gyrocenter velocity can be expressed as

$$\dot{\vec{X}} = v_{\parallel} \vec{b}_0 + \frac{B_0}{B_{0\parallel}^*} (\vec{v}_{E \times B} + \vec{v}_{\nabla B_0} + \vec{v}_c),$$
(35)

where $B_{0\parallel}^* = \vec{b}_0 \cdot \left(\vec{\nabla} \times (\vec{A}_0 + \frac{m_j c}{q_j} v_{\parallel} \vec{b}_0)\right)$ and the different drift velocities are

$$\vec{v}_{E\times B} = \frac{c}{B_0} \vec{b_0} \times \vec{\nabla} \chi_j,\tag{36}$$

$$\vec{v}_{\nabla B_0} = \frac{\mu}{m_j \Omega_j} \vec{b_0} \times \vec{\nabla} B_0, \tag{37}$$

$$\vec{v}_{c} = \frac{v_{\parallel}^{2}}{\Omega_{j}B_{0}} \Big(\vec{b}_{0} \times (\vec{\nabla}B_{0} + \frac{4\pi}{B_{0}}\vec{\nabla}p_{0}) \Big).$$
(38)

To solve Maxwell's equations, we finally need the expressions for the charge and current desities,

$$\rho(\vec{x}) = \sum_{j} q_{j} \int d^{3}v \ F_{j}^{*}(\vec{x}, \vec{v}), \tag{39}$$

$$j_{\parallel} = \sum_{j} q_{j} \int d^{3}v \quad v_{\parallel} F_{j}^{*}(\vec{x}, \vec{v}),$$
(40)

$$j_{\perp} = \sum_{j} q_{j} \int d^{3}v \ v_{\perp} F_{j}^{*}(\vec{x}, \vec{v}).$$
(41)

In these expressions, the distribution function $F_j^*(\vec{x}, \vec{v})$ is taken at the particle position. To compute these from the new gyrocenter distribution function, one can use a transformation involving the so-called pull-back operator T^* , as outlined in [27].

2.4.2 δ -f splitting

A further approximation that enable further simplification is δ -f splitting: $F_j = F_{0j} + f_j$. Here the distribution function F is split into a MHD equilibrium background distribution function F_0 of order one, plus a perturbed part f (often called δ f) of order ϵ . This assumption works well in a tokamak core, but less near the plasma edges. When the plasma in a tokamak interacts with the walls, large fluctuations occur that break the Maxwellian background assumption.

Applying δ -f splitting to the gyrokinetic equations, and then dividing the equation itself into a part containing terms of order one and a part containing all terms of order ϵ , we find

$$\frac{\partial F_{0j}}{\partial t} + v_{\parallel} \vec{b}_0 \cdot \left(\vec{\nabla} F_{0j} - \frac{\mu}{m_j v_{\parallel}} \frac{\partial F_{0j}}{\partial v_{\parallel}} \vec{\nabla} B_0 \right) = \langle C_j(F_0) \rangle \tag{42}$$

for the background distribution function, and

$$\frac{\partial g_j}{\partial t} + \frac{B_0}{B_{0\parallel}^*} (\vec{v}_{E \times B} + \vec{v}_{\nabla B_0} + \vec{v}_c) \left(\vec{\nabla} F_{0j} - \frac{\mu}{m_j v_{\parallel}} \frac{\partial F_{0j}}{\partial v_{\parallel}} \vec{\nabla} B_0 \right)
+ v_{\parallel} \vec{b}_0 \cdot \vec{\Gamma}_j + \frac{B_0}{B_{0\parallel}^*} (\vec{v}_{E \times B} + \vec{v}_{\nabla B_0} + \vec{v}_c) \cdot \vec{\Gamma}_j - \frac{\mu}{m} \vec{b}_0 \cdot \vec{\nabla} B_0 \frac{\partial f_j}{\partial v_{\parallel}}
= \langle C_j(f) \rangle$$
(43)

for the perturbed distribution function, where we defined

$$g_j = f_j - \frac{q_j}{m_j c} \frac{\partial F_{0j}}{\partial v_{\parallel}} \bar{A}_{1\parallel},\tag{44}$$

$$\vec{\Gamma}_j = \vec{\nabla}g_j - \frac{q_j}{m_j v_{\parallel}} \frac{\partial F_{0j}}{\partial v_{\parallel}} \vec{\nabla}\chi_j + \frac{q_j}{m_j c} \bar{A}_{1\parallel} \vec{\nabla} \frac{\partial F_{0j}}{\partial v_{\parallel}}.$$
(45)

The perturbed distribution function contains all the physics of interest for us, namely all plasma waves on top of the static background, and thus all the turbulence. For the background distribution function we assume a Maxwellian distribution:

$$F_{0j} = \left(\frac{m_j}{2\pi T_{0j}}\right)^{2/3} n_{0j} e^{-\frac{m_j v_{\parallel}^2/2 + \mu B_0}{T_{0j}}}.$$
(46)

Inserting this into Eq. 42 and computing all the derivatives yields

$$\frac{\partial F_{0j}}{\partial t} + v_{\parallel} F_{0j} \vec{b}_0 \cdot \left(\frac{1}{n_{0j}} \vec{\nabla} n_{0j} + \left(\frac{m_j v_{\parallel}^2}{2T_{0j}} + \frac{\mu B_0}{T_{0j}} - \frac{3}{2}\right) \frac{1}{T_{0j}} \vec{\nabla} T_{0j}\right) = \langle C_j(F_0) \rangle.$$
(47)

Collision operator We previously mentioned that collisions were a small effect in plasma, and they are neglected in the Vlasov equation. However, their small effect is often important in turbulent tokamak physics. Therefore, a collision operator $\langle C_j(f) \rangle$ is now added to the gyrokinetic equations. The calculation of this operator is rather complicated and not important to this work. We refer the interested reader to [26].

2.4.3 Gyrokinetic Vlasov equations

The main equation to solve is now the Vlasov equation for the perturbed gyrocenter distribution function g_i . Eq. 43 can be simplified further by the Maxwellian choice for F_{0j} . Also nonlinear terms are neglected. Although GENE is capable of solving the nonlinear gyrokinetic equations, these are not used in this work. Thus, the resulting linear equation is given by

$$\frac{\partial g_j}{\partial t} + \frac{B_0}{B_{0\parallel}^*} F_{0j} (\vec{v}_{E \times B} + \vec{v}_{\nabla B_0} + \vec{v}_c) \Big(\frac{1}{n_{0j}} \vec{\nabla} n_{0j} + (\frac{m_j v_{\parallel}^2}{2T_{0j}} + \frac{\mu B_0}{T_{0j}} - \frac{3}{2}) \frac{1}{T_{0j}} \vec{\nabla} T_{0j} \Big) \\
+ v_{\parallel} \vec{b}_0 \cdot \vec{\Gamma}_j + \frac{B_0}{B_{0\parallel}^*} (\vec{v}_{E \times B} + \vec{v}_{\nabla B_0} + \vec{v}_c) \cdot \vec{\Gamma}_j - \frac{\mu}{m} \vec{b}_0 \cdot \vec{\nabla} B_0 \frac{\partial f_j}{\partial v_{\parallel}} \\
= \langle C_j(f) \rangle.$$
(48)

In this equation, we clearly see the gradients of the background density and temperature (per species). These are the drives of the most important turbulent waves considered in this work, in particular, $\vec{\nabla}T_{0j}$ drives the ion temperature gradient (ITG) mode.

The equations for the charge and current densities are needed to solve Maxwell's equations, which in turn are needed to solve the gyrokinetic Vlasov equation. They can also be reduced by gyroaveraging and $\delta - f$ splitting, but we do not present the resulting equations here.

Finally, many quantities are normalized for numeric purposes, by scaling them with certain reference values. These normalizations can be found in [26].

3 Tokamaks

The basic requirements for nuclear fusion are known and the behaviour of a magnetically confined plasma can be described well by a set of equations. It is therefore possible to design a reactor that can actually contain a fusing plasma and harvest the released energy. For each design there will then exist many specific conditions, which lead to specific challenges that must be overcome. The most common design for a thermonuclear magnetic confinement reactor is a tokamak, and a specific challenge of this design (which will be adressed in this work) is the stabilization of turbulent ion temperature gradient (ITG) modes.

3.1 General setup

The name tokamak comes from the russian $toroidalnaja kamera \ s magnituymi katushkami$; or toroidal chamber with magnetic coils. This is indeed exactly what a tokamak is, as shown in Fig. 6.



Figure 6: Schematic picture of a tokamak

The tokamak consists of a toroidal reactor wall, in which the plasma is contained by magnetic fields. These magnetic fields are produced by different sets of coils. Around the torus are coils that produce a toroidal magnetic field. This is the main magnetic field in a common tokamak. In the center of the torus sit coils which induce a toroidal current. This current then results in a secondary poloidal magnetic field inside the tokamak. As such, the field lines in the plasma have both a toroidal and poloidal component and become helical. The charged plasma particles will in general move along these twisted field lines.

This is just a very rough sketch of a tokamak. The precise design is highly complicated; each component is made to cope with the extreme conditions of the plasma. Although interesting, these clever designs are not necessary for this work. For those interested in the wide world of tokamak engineering, I refer to [29].

In this work we use a circular tokamak with major radius R and minor radius a. We introduce toroidal coordinates (ρ, ϕ, θ) , as illustrated in Fig. 7. Later we shall change these coordinates to be aligned not to the torus' topology, but to the magnetic field lines.



Figure 7: Toroidal coordinates (ρ, ϕ, θ)

3.2 ITG modes

Inside a tokamak, various sources exist that drive plasma modes to instability. To name a few of such instabilities, there are parallel velocity gradient (PVG) modes, the ion/electron temperature gradient (ITG/ETG) modes, and trapped electron modes (TEM). All instabilities cause unwanted loss of particles and heat to other regions of the plasma. The reduction of these instabilities is vital to ensure controlled burning of a fusion reactor, and is one of the main challenges of fusion engineering.

Some of the most damaging instabilities are ITG modes. ITG modes (like ETG modes) limit the radial temperature gradients that can be upheld in a tokamak. This in turn limits the plasma core temperature than can be stably reached, such that temperatures needed for nuclear fusion are difficult to maintain. Since in this work we focuse primarily on ITG modes, we give here an intuitive explanation of its nature using the Rosenbluth-Longmire picture [30]. Consider the outer curve of a tokamak, also known as the 'bad curvature' region (the reason for this name will become apparent soon). If there is a gradient in the ion temperature, the edges of the plasma will

be cooler than the core. This is shown in Fig.8 (a). Because the ∇B and curvature drift velocities scale with temperature, these drift velocities of ions are larger in the core than in the cooler edge region.



Figure 8: Rosenbluth-Longmire picture of the ITG instability

Imagine now some initial perturbation on the boundary between a hotter core and cooler edge region, as shown in Fig.8 (b). Then the larger ion drift velocity in the hot core plasma leads to an inhomogeneous distribution of charge in the plasma. This results in the creation of an electric field (Fig.8 (c)) which is directed such, that the resulting ExB drift increases the amplitude of the mode! It is a clear picture of ITG instability: a small mode grows to large proportions in the bad curvature region of a tokamak. Repeating the same steps on the inside curve of a tokamak, it is seen that the ExB drift resulting from the inhomogeneous charge distribution actually reduces the amplitude of the original mode, so that turbulence is quenched; this is why this region is known as the 'good' curvature region.

A more rigorous way to show the existence if an ITG mode is shown in the book by Goldston and Rutherford [19]. Starting from the drift-kinetic equation, a perturbation is added and the resulting ion density n_{i1} perturbation is calculated. The electron density perturbation n_{e1} is assumed to be adiabatic (which is explained later in this section), and the dispersion relation is obtained by setting $n_{i1} = n_{e1}$ (quasineutrality). To find unstable solutions of this dispersion relation, the Nyquist diagram technique is used. This uses some neat complex analysis tricks to arrive at a condition for ITG instability:

$$\eta_i \equiv \frac{\nabla \ln(T_i)}{\nabla \ln(n_i)} > 2 + \frac{4}{\Lambda} \frac{T_{i0}}{T_{e0}} \left(1 + \frac{T_{i0}}{T_{e0}}\right),\tag{49}$$

where $\Lambda = n_i \left(\frac{k_y v_{di}}{k_z v_{t,i}}\right)^2$, v_{di} is the ion diamagnetic drift velocity and $v_{t,i}$ is the ion thermal velocity. Here, a plasma slab was chosen in cartesian coordinates (x, y, z). From this condition can be deduced the impact of ITG instabilities under different conditions. For example, ∇B and curvature drifts are destabilizing, whereas magnetic shear \hat{s} (which will be explained later) is stabilizing.

To give a measure of an ion temperature gradient, we use the quantity R/L_{Ti} in this work, where L_{Ti} gives the length scale over which the ion temperature gradient is reduced by a predefined factor.

3.3 Safety factor q

As discussed before, a tokamak has both a toroidal and poloidal magnetic field, which causes the field lines to become helical. Remember that ITG modes grew unstable in the bad curvature region of a tokamak, whereas they are quenched at the opposite high field side. The twisted field lines reduce this problem; since the plasma generally moves along these lines, they are constantly rotated through the good and bad curvature regions of the tokamak. As such, perturbations can never grow for long in the same direction. The safety factor q gets it name from this stabilizing property. It describes the "steepness" of the helical field lines, or more precise, the ratio between the poloidal and toroidal field at a certain point:

$$q(\rho) \approx \frac{r \ \partial B_{\phi}}{R \ \partial B_{\theta}},\tag{50}$$

where r and R denote the torus' minor and major radius, respectively. Typically, close to the plasma core q is close to 1 and near the edges it is around 6.

3.4 Magnetic shear \hat{s}

Another variable that plays an important role in this work, is the so-called magnetic shear \hat{s} . This shear is defined as the radial variation of the safety factor q:

$$\hat{s} = \frac{\rho}{q} \frac{\partial q}{\partial \rho}.$$
(51)

Regarding its stablizing effect, let's consider a plasma slab such as used in the ITG instability condition Eq. 49. Then magnetic shear can be introduced by adding a small magnetic field $B_y(x)$ to the main field B_z . This results in a lower bound on the the wave vector k_y (hidden in Λ) in Eq. 49 [19], so that the addition of magnetic shear acts stabilizing on ITG modes. Also,

magnetic shear causes modes to be less correlated, so that unstable ITG modes become more localized in the bad curvature region. This too increases stability.

3.5 Flow shear γ_E

Flow shear is a principal variable in this work, since it is the main trick to subdue ITG instabilities. It is known that flow shear has a large stabilizing effect on ITG modes, by 'shearing apart' the unstable radially elongated ITG modes. The effect has been studied experimentally (e.g. [12] [13]), analytically (e.g. [7]) and using simulations (e.g. [8] [9] [10] [11]).

The plasma in a tokamak flows generally along the magnetic field lines. Its velocity thus has a large toroidal component v_{tor} . Flow shear γ_E is given by the radial variation of this plasma velocity:

$$\gamma_E = \frac{\rho}{q} \frac{\partial v_{tor}}{\partial \rho}.$$
(52)

This is illustrated in Fig. 9 (a).



Figure 9: (a) Flow shear γ_E in a tokamak and (b) its shearing effect on radially elongated unstable ITG modes. Red and blue structures show positive and negative vortices.

In Fig. 9 (b) the effect can be seen that flow shear has on radially elongated ITG instabilities in the $\rho - \theta$ plane (so-called "streamers" because of their

shape). The streamers are sheared apart as flow shear increases, so that radial transport losses are minimized.

The stabilizing effect of flow shear in ITG modes is of great use in a tokamak, but unfortunately it is not possible to reduce all instabilities by simply increasing flow shear to very high levels. Besides the practical difficulties in producing high flow shear, it also produces an unfavourable side effect. The toroidal flow that is the cause of flow shear, necessarily (by the tokamak's geometry) has a parallel gradient as well, and this parallel velocity gradient (PVG) is a well known drive for the so-called PVG instabilities [9]. Thus increasing flow shear is thus only useful up to a certain point, above which the PVG driven modes become more unstable than the remaining ITG modes.

3.6 Adiabatic and kinetic electrons

We conclude this section by mentioning different approaches that are used for the electron distribution function. An aim of this work is to find a clear cause of differences between these approaches.

Assume a tokamak with a Maxwellian electron distribution function, but both the density n_{e0} and the temperature T_{e0} can vary significantly in the radial direction. Then it can be calculated [19] that the perturbed electron distribution is approximately given by

$$n_{e1} \approx \frac{n_{e0}e\phi}{T_{e0}} \Big(1 + i(\frac{\pi}{2})^2 \frac{\omega - k_y v_{de}(1 - \eta_e)/2}{|k_z|v_{t,e}} \Big), \tag{53}$$

where v_{de} is the electron diamagnetic drift velocity, $v_{t,e}$ is the electron thermal velocity and ω the electron drift wave frequency. This relation gives a good approximation of the electron's reaction to its surroundings, and is called the *kinetic electron* approach. On the other hand there is the *adiabatic electron* assumption, which neglects the small imaginary terms on the right such that the electrons relax to a Boltzmann distribution, $n_e \approx n_{e0} \exp(e\phi/T_{e0})$. Another way of thinking about this is that electrons move infinitely fast and thus immediately respond to changes in their surroundings.

Of course, in the adiabatic case we have reduced the physics. Nonetheless, this assumption is widely used because it greatly speeds up computations; much of the work on ITG stabilization is based on simulations run with adiabatic electrons. Often the assumption works well, but not always. In particular we will see later that there is a large difference of ITG stabilization between kinetic and adiabatic electron simulation at low \hat{s} , and we will focus on finding a cause for this difference.

4 GENE

The described gyrokinetic equations must be solved numerically. This is the aim of the gyrokinetic code GENE (Gyrokinetic Electromagnetic Numerical Experiment) [16]. Some concepts of this code and numerically motivated modifications must be introduced, since this work extensively uses GENE to perform tokamak simulations, and the interpretation of the output of these simulations requires some knowledge of GENE. Thus, in this section we give a short introduction to the approach of the code, and present the gyrokinetic equations as they are solved by it. We also reproduce in detail the transformation to the coordinate system implemented in GENE (field-aligned and partially in momentum space). This transformation results in a particular formulation of the parallel boundary condition, a vital ingredient to understanding the rest of this work.

4.1 Introduction to GENE

The GENE code is written by the GENE Development Team, stationed at the Max Plack Institut für Plasmaphysik at Garching, Germany, led by F. Jenko. In the code, all the equations as described in the previous sections are modified to be efficiently solvable by a machine, which requires many algorithms and smart implementations. These adaptations are summarized in [26] [27], and will not be outlined in detail here. GENE is a massively parallel code, written in Fortran 90/95. It uses the Eulerian approach, where phase space is discretized and the time evolution of the distribution function is calculated on this grid. Furthermore, the code uses a radially local assumption, which means that relevant turbulent scales are taken to be much smaller than the typical scales of plasma gradients and profiles. Also known as the flux-tube approximation, it simulates a curved and sheared box around a magnetic field line in a tokamak. Around the central field line equilibrium quantities are taylor expanded, and the first derivatives and metric coefficients are taken to be constant inside the flux tube [26]. Because the boundaries of the flux tube in the directions perpendicular to the magnetic field are not physical boundaries, proper boundary conditions must be used. Conveniently periodic boundary conditions are used in these directions, such that a spectral representation can be applied which is computationally effective. For this work GENE was run on the Edison supercomputer stationed at the National Energy Research Scientific Computing Center (NERSC) at Berkeley, California, which is equipped with 133.824 cores and 357 TB of RAM memory. Only a small portion of this was needed to run linear GENE simulations, which need a few hundred CPU hours to finish.

4.2 Field aligned coordinates

An intuitive and usually effective pick of coordinate system for a torus is the (ρ, ϕ, θ) system as used before (Fig. 7). However, this coordinate system is not aligned to the actual physical phenomena in tokamaks. Particles travel quickly along the magnetic field lines, and by comparison only slowly perpendicular to it. Aligning the coordinate system to the magnetic field allows us to exploit this anisotropic movement, which can save orders of magnitude in computation time. New coordinates are thus defined:

$$x = \rho \tag{54}$$

$$y = C_y(\phi - q\theta) \tag{55}$$

$$z = \theta, \tag{56}$$

where $C_y = \epsilon/q$ is a geometric constant, with $\epsilon = r/R$ the inverse aspect ratio at the radial location r of the chosen flux tube. The coordinates are shown in Fig. 10. Now x and z just trace the familiar ρ and θ direction, respectively, but y is no longer orthogonal to z. Instead it parameterizes the direction perpendicular to the field lines. To see this, imagine keeping y constant. Then $(\phi - q\theta)$ must be kept constant, so that when we increase ϕ by some amount, $q\theta$ must be increased by the same amount. This will always put us back on the same field line from which we started. Another interesting feature of our new coordinates is that since both x and y are perpendicular to the field lines, moving in z, the poloidal angle, is now the only way to travel in the parallel direction.



Figure 10: Field aligned coordinates (x,y,z)

4.3 Boundary conditions

Because of the local assumption of GENE periodic boundary conditions can be chosen for the directions perpendicular to the magnetic field. In the new coordinates these directions are x and y, for which it is then numerically advantageous to use a spectral representation. We now look at the binormal and parallel boundary conditions in this new representation.

4.3.1 Binormal boundary condition

The binormal boundary condition is a direct translation of the toroidal boundary condition in the previous coordinates:

$$f(\rho,\phi,\theta) = f(\rho,\phi+2\pi,\theta) \Longrightarrow f(x,y,z) = (x,y-2\pi C_y,z), \tag{57}$$

so a full toroidal turn corresponds to moving in the y-direction by $L_y = 2\pi C_y$. However, the largest toroidal wavelengths (corresponding to the lowest mode number n_0) are usually smaller than a full toroidal turn (for which we would have $n_0 = 1$, so that the largest wave obeying periodicity spans exactly one full toroidal turn). In that case, since the torus itself is axisymmetric, there is already periodicity after moving $2\pi/n_0$ in the ϕ -direction, and thus

$$f(\rho,\phi,\theta) = f(\rho,\phi + \frac{2\pi}{n_0},\theta) \Longrightarrow f(x,y,z) = (x,y - \frac{2\pi C_y}{n_0},z), \tag{58}$$

and now $L_y = \frac{2\pi C_y}{n_0}$. In GENE, the *y*-direction is also represented in momentum space. The binormal modenumbers are then

$$k_y = jk_{y,min} = \frac{2\pi j}{L_y} = \frac{jn_0}{C_y}$$
(59)

4.3.2 Parallel boundary condition

The poloidal direction in a torus is also subject to periodicity. Since in the field-aligned coordinates both y and z are dependent on the poloidal angle θ , this periodicity is translated to the condition that

$$f(\rho, \phi, \theta) = f(\rho, \phi, \theta + 2\pi) \Longrightarrow f(x, y, z) = f(x, y + 2\pi C_y q(x), z + 2\pi)$$
(60)

To represent the y-direction in momentum space the above equation is Fourier transformed to find

$$\sum_{k_y} f(x, k_y, z) e^{ik_y y} = \sum_{k_y} f(x, k_y, z + 2\pi) e^{ik_y (y + 2\pi C_y q(x))}.$$
 (61)

If we then define $a_{k_y} = f(x, k_y, z)$ and $b_{k_y} = f(x, k_y, z + 2\pi)e^{ik_y * 2\pi C_y q(x))}$, it is seen that $\sum_{k_y} a_{k_y} e^{ik_y y} = \sum_{k_y} b_{k_y} e^{ik_y y}$, which can only be true if $a_{k_y} = b_{k_y}, \forall k_y$, so

$$f(x, k_y, z) = f(x, k_y, z + 2\pi)e^{ik_y * 2\pi C_y q(x))}.$$
(62)

Next the x-direction must be represented in momentum space as well, which is complicated by the x-dependence of q(x). We thus expand q(x) around x_0 :

$$q(x) \approx q_0 + (x - x_0) \frac{\partial q}{\partial x} |_{x = x_0} = q_0 (1 + \frac{x - x_0}{x_0} \hat{s})$$
(63)

where the magnetic shear $\hat{s} = \frac{x_0}{q_0} \frac{\partial q}{\partial x}$ is taken to be constant. Then a Fourier transformation yields

$$\sum_{k_x} f(k_x, k_y, z) e^{ik_x x} = \sum_{k_x} f(x, k_y, z + 2\pi) e^{ik_x x} e^{ik_y * 2\pi C_y q_0 (1 + \frac{x - x_0}{x_0} \hat{s})}$$
(64)

$$=\sum_{k_x} f(x, k_y, z+2\pi) e^{ix(k_x+2\pi\hat{s}k_y)} e^{ik_y*2\pi x_0(1-\hat{s})}, \quad (65)$$

since $C_y = x_0/q_0$. With a similar argument as used above, the sum on both sides of this equation can be dropped. The result can then be interpreted: when moving around z by 2π , the wavenumber k_x "picks up" a factor of $2\pi \hat{s}k_y$, or more accurately: by moving around in the z-direction, all values of k_x that are $2\pi \hat{s}k_y$ apart are coupled to each other, and are part of the same mode structure. As such it must be true that *every* radial wavenumber k'_x can be written in terms of some k_x by the relation

$$k'_x = k_x + 2\pi \hat{s}k_y,\tag{66}$$

which is the final manifestation of the parallel boundary condition. This coupling of k_x 's is one of the core ingredients of this work, and a severe restriction on how the radial grid is chosen in GENE. If the smallest radial wavenumber is chosen to be

$$k_{x,min} = \frac{2\pi \hat{s} k_{y,min}}{N},\tag{67}$$

for N any integer, then surely all other wavenumbers (which must be a multiple of $k_{x,min}$) to obey Eq. 67. Alternatively one can write

$$N = \frac{2\pi \hat{s}k_{y,min}}{k_{x,min}} = \frac{2\pi \hat{s}L_x}{L_y},\tag{68}$$

so in the code the choice of the simulation box dimension in the x- and y-directions define the value of N. We return to the interpretation and consequences of choosing this N in section 5.3.2.

4.4 Flow shear in Gene

In GENE, rotating plasma with some constant radial variation (so constant flow shear) is implemented by making the radial coordinate grid timedependent, and then radially varying this time-dependence. To quantify this idea, we consider the toroidal coordinate in the frame that moves along with the plasma moving with a velocity v_{tor} :

$$\phi(x,t) = \phi_0 - xv_{tor}(x)t = \phi_0 - \frac{\gamma_E x}{q}t,$$
(69)

where ϕ_0 is the toroidal angle in the stationary frame, and the flow shear $\gamma_E = \frac{x}{q} \frac{\partial v_{tor}}{\partial x}$ is taken to be constant. The binormal coordinate in the rotating frame is then

$$y(x,t) = C_y(q(x)\theta - \phi(x,t)) = y_0(x) + \frac{C_y \gamma_E x}{q} t.$$
 (70)

Let's consider now, in this moving frame, the total derivative operator (with respect to x) working on a function $f(x, y, n_1, n_2, ...)$ which depends, besides x and y, on an arbitrary amount of other variables n_i . Then the total derivative operator is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial}{\partial y} + \sum_{i}\frac{\partial n_{i}}{\partial x}\frac{\partial}{\partial n_{i}} = \frac{\partial}{\partial x} + \left(\frac{\partial y_{0}}{\partial x} + \frac{C_{y}\gamma_{E}}{q}t\right)\frac{\partial}{\partial y} + \sum_{i}\frac{\partial n_{i}}{\partial x}\frac{\partial}{\partial n_{i}},$$
(71)

whereas this operator in the stationary frame would have been

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + \frac{\partial y_0}{\partial x}\frac{\partial}{\partial y} + \sum_i \frac{\partial n_i}{\partial x}\frac{\partial}{\partial n_i}.$$
(72)

As such we find that when changing from the stationary to the rotating frame of reference, only the partial derivative to x transformed as

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{C_y \gamma_E}{q} \frac{\partial}{\partial y} t, \tag{73}$$

or, in momentum space:

$$k_x \to k_x + \frac{C_y k_y \gamma_E}{q} t.$$
(74)

Thus, constant flow shear can be implemented by varying the radial spectral grid in time. This trick lies at the core of this work. In GENE, shifting the entire grid is not computationally efficient; instead only the distribution function and the electromagnetic field are shifted over the grid. This is not necessary at every time step, since it takes several time steps in GENE before $C_y k_y \gamma_E \Delta t/q$ becomes of the order of the grid spacing Δk_x , where Δt is the time interval since the last grid shift. Thus, in the code the distribution function and fields are shifted from their values at k_x to $k_x + \Delta k_x$ whenever $C_y k_y \gamma_E \Delta t/q > \Delta k_x/2$. Since only a finite amount of mode numbers is simulated, this means that a part of the distribution function is off the grid, and a new tail is introduced. To ensure this has little impact on the simulations, the radial box size must be chosen large enough so that the distribution function is close to zero at its edges.

4.5 Solutions of Gene

We previously described the gyrokinetic equations, which are solved by GENE to yield the distribution function g. These eigenmodes contain a superposition of all the modes present in our plasma. Once we have obtained g, we can construct the perturbed particle distribution f_1 and through there physically interesting perturbed quantities like density n_1 , average flow velocity \vec{u}_1 and temperature T_1 via the following relations:

$$n_{1} = \int d^{3}v f_{1}$$

$$\vec{u}_{1} = \frac{1}{n_{0}} \int d^{3}v \vec{v}f_{1}$$

$$T_{1} = \frac{m}{2n_{0}} \int d^{3}v (\vec{v} - \vec{u}_{1})^{2}f_{1} - \frac{n_{1}}{n_{0}}T_{0}$$
(75)

Often, to study the stability of our plasma, we are only interested in the most unstable mode in our system. In the initial phase of a simulation the distribution function g contains all eigenmodes as discussed above. But if the plasma evolves for a while, the most unstable mode grows to much larger orders of magnitude than all the others. The distribution function is thus dominated by this most unstable mode, so that $g \approx g_u$, with g_u the distribution function containing only the most unstable mode. The linear gyrokinetic equation then is then reduced to

$$\frac{\partial g_u}{\partial t} = \mathcal{L}g_u = \lambda_u g_u \tag{76}$$

where $\lambda_u = \lambda_u(k_x, k_y, z)$ is the complex eigenvalue of the most unstable mode, containing its growth rate and frequency. This method is known as the initial value solver of GENE, and it is used for all simulations in this work.

5 Methodology

The goal of this work is to investigate some of the dynamics of ITG modes under influence of flow shear, which reduces the growth rate of these unstable modes. In the previous sections we introduced the necessary theory on tokamaks, gyrokinetics, and the code GENE that is used for simulations. In this section we cover more specific subjects that are useful to investigate the ITG mode dynamics. First of all, the motivation and formulation of the actual subject of this thesis can be introduced. When the goal is finally clear, several technical details and methods are presented that are used to tackle the problem. In particular the ballooning representation of ITG modes is explained, the shifting $k_{x,center}$ method is outlined, and techniques to find the growth rate of a certain mode are given.

5.1 Problem formulation

Much study has been done on the effects of flow shear on plasma instability. In this thesis, instead of only quantifying these quenching effects, we focus on the more academic question of what the solution to the linear gyrokinetic equation in the presence of flow shear entails. In particular, we look at the structure of the solutions in so-called ballooning space, which we introduce later in this section. These structures contain a lot of physics that are obscured. The structures are impacted by flow shear, but how exactly this comes to be is not known. This work aims to provide insight into the mechanics of the impact of flow shear on these ballooning structures. This is done by making a toy model that decomposes the full gyrokinetic solution including flow shear into the solutions without flow shear. Then, the mechanism of flow shear must be added separately to these components. As such, we hope to learn more about the precise workings of flow shear. Section 8 is dedicated to the creation of this toy model.

There is another effect that may be captured by our toy model. We introduced earlier the concept of kinetic and adiabatic electrons. In general, electrons behave kinetically. However, the adiabatic assumption is advantageous because it greatly reduces computational efforts of simulations, and therefore it is used extensively in literature. Examples (and one of the prime motives for this work) are the recent papers by Highcock et al. [10] [11], which investigate ITG quench by flow shear. An important result of the work is the apparent very quick quench of ITG modes at low magnetic shear. For despite this quick linear stabilization, it is found that in these cases so-called subcritical turbulence can still arise [7]. I.e., if the linear modes have enough time to grow to significant amplitudes before being quenched by flow shear, they can interact with each other, producing unstable nonlinear modes.

However, it was found by Citrin (unpublished), that at low magnetic shear, the ITG quenching mechanism by flow shear works very differently when kinetic electrons are used. This difference is reproduced and shown in Fig. 11.



Figure 11: Growth rates of ITG modes at various values of \hat{s} , using kinetic (left) and adiabatic (right) electrons. The temperature gradient is chosen different in these two cases such that rotationless growth rates are similar.

It is seen that at high magnetic shear, growth rates of unstable ITG modes are reduced in much the same way when using kinetic or adiabatic electrons in the simulations. This small difference persist through most values of the magnetic shear, until it is dropped as low as $\hat{s} = 0.1$. At these low values, ITG modes are sheared apart much quicker with adiabatic electrons than kinetic electrons. This corresponds to the quick quench reported in the Highcock *et al.* papers, so that their subsequent subcritical turbulence story applies. However, in the kinetic electron case, the linear ITG modes are not stabilized so easily, such that subcritical turbulence does not play a major role (we still have linear instability). The precise nature of these differences between adiabatic and kinetic electrons is not known. The guess is that the mechanics are visible in the structures of the eigenmodes of the linear gyrokinetic operator. As such, it fits well into the investigations of this thesis.

To summarize, we will take a close look at the gyrokinetic eigenmode structures to see what happens under influence of flow shear. In addition, we hope to find a clear cause for the difference between kinetic and adiabatic electrons at low magnetic shear. We aim to build a toy model that reproduces all these effects.

5.2 Instability drive competition

We restrict this work to ITG modes, which means that when performing simulations, we must be confident that the modes were indeed driven by an ion temperature gradient. In a typical initial value simulation in GENE the output is the most unstable eigenmode of the system. But when we scan over some parameter, it will often be the case that in one section of our parameter scan the most unstable mode it of a different physical origin than in another section. Also, sometimes there may be a clear winner, i.e. the most unstable mode has a much higher growth rate than all others, while at other places there may be two or more modes competing to be strongest. It is important to retrace the nature of the most unstable mode at every point of parameter space, so that behaviour can clearly be coupled to the right causes.



Figure 12: Typical growth rate of most unstable mode under influence of flow shear, including different instability drives

A typical example is shown in Fig. 12. A picture of the growth rate of the most unstable mode at varying levels of flow shear and $R/L_T = 6.9$ is given by the red line. In this simulation all instability drives were included. When we set $R/L_{T_e}=4$ (yellow line), we reduce the TEM drive which mainly acts at high flow shear. Indeed, these TEM are quenched so that we no longer see high growth rates at large flow shear: the mode is quenched around $\gamma_E = 2\gamma_{max}$, which is a well known quench rule for ITG modes [8]. Since we focus on ITG modes in this work, we will often use this setting of reduced TEM drive. Next, when our radial flow shear is nonzero, the toroidal plasma rotation also induces parallel flow shear, the drive of PVG instabilities. We can remove this PVG instability drive by setting the parallel velocity gradient to zero. A big hump is then removed from the growth rates, which then was clearly the cause of PVG taking over the dominant instability drive at the respective parameters. The remaining mode is then a sole ITG mode that is nearly linearly reduced by flow shear.

5.3 Ballooning representation

Although GENE solves the gyrokinetic equations numerically to yield the eigenmodes of the distribution function, it is convenient for our investigation to also have an analytical form for these modes. Such a form can be conveniently constructed by using that waves in a tokamak propagates rapidly along the field lines and only slowly varies perpendicular to it. Using toroidal symmetry it can thus be argued that the eigenmodes of the linear distribution function can be written in the form

$$\varphi(\rho,\phi,\theta) = \tilde{\varphi}(\rho,\theta)e^{in(\phi-q(x)(\theta-\theta_0))},\tag{77}$$

with θ_0 the so-called ballooning angle, which is just a constant shift in the direction in which the wave grows. The eigenvalue equation of these plasma modes is then reduced to just two-dimensions:

$$\mathcal{L}(\rho,\theta)\tilde{\varphi}(\rho,\theta) = \lambda\tilde{\varphi}(\rho,\theta),\tag{78}$$

which makes it much easier to solve. In the next section we transform these solutions to a different space, by using the so-called ballooning transformation. The resulting ballooning eigenmodes will be used extensively througout the rest of this work, since they give insight in some behaviour which is not apparent in real space.

5.3.1 The Ballooning transformation

The historic motivation for inventing the ballooning transformation was a problem with solutions of the form of Eq. 77. Physical waves present in the plasma must fulfill the periodic boundary condition $\varphi(\rho, \phi, \theta) = \varphi(\rho, \phi, \theta + 2\pi)$, which results in the requirement for our solutions

$$\tilde{\varphi}(\rho,\theta)e^{in(\phi-q(\rho)(\theta-\theta_0))} = \tilde{\varphi}(\rho,\theta+2\pi)e^{in(\phi-q(\rho)(\theta+2\pi-\theta_0))}.$$
(79)

This can only be satisfied if $nq(\rho)$ has a constant integer value m, in which case a field line winds around the torus onto itself after m toroidal rotations. Clearly, this is not generally the case; $q(\rho)$ is not constant if there is finite magnetic shear in the system $(\hat{s} = \frac{\rho}{q} \frac{\partial q}{\partial \rho} \neq 0)$, and even without shear it can take irrational values so that the field lines never double up on themselves. So our wave representation (Eq.77) is not compatible with periodicity in θ ! This problem can be overcome by using the ballooning transformation [14] [15]:

$$\tilde{\varphi}(\rho,\theta) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta \ e^{im(\theta-\eta)} \varphi_b(\rho,\eta), \tag{80}$$

where $\hat{\varphi}(\rho, \eta)$ itself is a solution of the eigenvalue equation

$$\mathcal{L}(\rho,\eta)\varphi_b(\rho,\eta) = \lambda\varphi_b(\rho,\eta). \tag{81}$$

The ballooning transformation is in essence a map of $\theta \in [-\pi, \pi] \to \eta \in (-\infty, \infty)$, with the crucial difference that $\varphi_b(\rho, \eta)$ is not periodic in η . One can quickly check that a non-periodic solution $\varphi_b(\rho, \eta)$ of Eq. 81 in $\eta \in (-\infty, \infty)$ generates a *periodic* solution $\tilde{\varphi}(\rho, \theta)$ of Eq. 78 with the same eigenvalue:

$$\mathcal{L}(\rho,\eta)\varphi_b(\rho,\eta) = \lambda\varphi_b(\rho,\eta), \text{ so then}$$

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta \ e^{im(\theta-\eta)} \mathcal{L}(\rho,\eta)\varphi_b(\rho,\eta) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta \ e^{im(\theta-\eta)}\lambda\varphi_b(\rho,\eta),$$
thus, $\mathcal{L}(\rho,\theta)\tilde{\varphi}(\rho,\theta) = \lambda\tilde{\varphi}(\rho,\theta),$

since $\sum_{m} e^{im(\theta-\eta)} \propto \delta(\theta-\eta)$. The periodicity of $\tilde{\varphi}(\rho,\theta)$ is easily verified. The eikonal representation exploits the same anisotropic behaviour of the plasma in our system as the (x,y,z) coordinates we introduced in section x, and in these new coordinates the eigenfunctions in real and ballooning space become

$$\varphi(x, y, z) = \tilde{\varphi}(x, z)e^{-in(y-q(x)\theta_0)}$$

= $\tilde{\varphi}(x, z)e^{-iC_y k_y (y-q(x)\theta_0)},$
= $e^{-iC_y k_y (y-q(x)\theta_0)} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\eta \ e^{im(z-\eta)}\varphi_b(x, \eta),$ (82)

where we used $n = jn_0 = C_y k_y$ in the second line and the ballooning transformation in the third. Note that we keep the notation θ_0 for the ballooning angle. As a final step we Fourier transform the x-coordinate, since this is the way it is treated by GENE. Inverting the above relations then finally yields the expression for the ballooning eigenfunction $\varphi_b(k_x, \eta)$:

$$\varphi_b(k_x,\eta) = \sum_{m=-\infty}^{\infty} \int \mathrm{d}z \int \mathrm{d}x \ e^{i\left(m(\eta-z)-k_xx+C_yk_y(y-q(x)\theta_0)\right)}\varphi(x,y,z).$$
(83)

This is the representation of the ballooning eigenfunction we will use throughout the rest of this work. When visualizing, we also average over k_x so that we can plot $|\varphi_b(\eta)| = |\langle \varphi_b(k_x, \eta) \rangle_{k_x}|$. The resulting eigenmodes are the main subject of investigation of this work. A typical picture of $|\varphi_b(\eta)|$ is given in Fig. 13.



Figure 13: Typical eigenfunction structure in ballooning space

In ballooning space we typically see a structure that is peaked around $\theta_b = 0$. This corresponds to a mode that is growing radially outwards in the bad curvature region of a tokamak. This ballooning direction will be discussed in detail in the next section.

5.3.2 Ballooning angle

We introduced a constant shift in the z-direction called θ_0 , the ballooning angle. We can find a very useful representation of θ_0 by considering the gradient of the term $k_y(y - q(x)\theta_0) = S$:

$$\nabla S(x,y) = \frac{\partial q}{\partial x} \frac{\partial S}{\partial q} \nabla x + \frac{\partial S}{\partial y} \nabla y$$
$$= -k_y \hat{s} \theta_0 \nabla x + k_y \nabla y. \tag{84}$$

But ∇S is a wave vector perpendicular to S and thus $\nabla S = k_{\perp} = k_x \nabla x + k_y \nabla y$ [31]. Equating this to the above expression results in

$$k_x = -k_y \hat{s} \theta_0. \tag{85}$$

Thus, if $\theta_0 = 0$, i.e. the mode balloons around the low field side mid plane, $k_x = 0$ exists. Shifting θ_0 by 2π , we end up at $k'_x = k_x + 2\pi k_y \hat{s}$ which clearly balloons in the same direction. This, of course, implicitly assumes the parallel boundary condition, and the conclusion is the same: every k_x in the system is coupled to $k'_x = k_x + 2\pi k_y \hat{s}$.

We also saw that in our simulations we have the freedom to choose $k_{x,min} = 2\pi \hat{s} k_{y_m in}/N$ for N any integer. This has some interesting implications when

viewed in the light of Eq. 85, where k_x was related to the ballooning angle. Let's consider some set $\{k_x\}$:

 $(\dots, -4k_{x,\min}, -3k_{x,\min}, -2k_{x,\min}, -k_{x,\min}, 0, k_{x,\min}, 2k_{x,\min}, 3k_{x,\min}, 4k_{x,\min}, \dots).$

Choosing for example N = 3 means that starting from $k_x = 0$, going around the poloidal direction once gets us to the coupled mode $k'_x = 2\pi k_y \hat{s} = 3k_{x,min}$. Doing the same for the other values of k_x allows us to identify three independent sets of mutually coupled k_x 's, below denoted by their colours:

 $(\dots, -4k_{x,\min}, -3k_{x,\min}, -2k_{x,\min}, -k_{x,\min}, 0, k_{x,\min}, 2k_{x,\min}, 3k_{x,\min}, 4k_{x,\min}, \dots).$

Each set forms its own mode structure. The first set balloons in direction $\theta_0 = 0$, whereas the other sets of mutually coupled k_x 's balloon at an angle $\theta_0 = k_{x,center}/k_y \hat{s}$, where $k_{x,center}$ denotes the value of k_x closest to zero in the respective set.

5.3.3 Shifting $k_{x,center}$

We can now recognize another interesting property of our system. Imagine again some set $\{k_x\}$ which forms mode structures. By shifting all values of k_x by some finite value l, we basically rotate all the present mode structures in θ by an amount $l/k_y \hat{s}$, so that they balloon in a different direction. Since modes are generally most amplified in the direction of bad curvature ($\theta_0 =$ 0), this means we can investigate different amplifications of any mode by controlling the directions in which they balloon. This will be a key feature to investigate the dynamics of the modes.

This is closely related to flow shear as well. Recall that in GENE, flow shear is implemented by shifting the distribution function and electromagnetic field over the radial spectral grid in time. This was equivalent to shifting the grid itself in time, from k_x to $k_x + C_y k_y \gamma_E t/q$. This in turn is equivalent to rotating all the modes' ballooning angles by $\theta_k = \frac{-k_{x,center}}{k_y \hat{s}} = \frac{-C_y \gamma_E t}{q\hat{s}}$.

Note that in following procedures, there is an important distinction between the procedure of investigating the impact of different $k_{x,center}$ on some eigenmodes, and looking at eigenmodes which have effectively shifted $k_{x,center}$ as a result of flow shear. The difference is that flow shear shifts $k_{x,center}$ continuously; the modes under consideration have no time to evolve extensively in that particular direction, since a few time steps later they will be shifted again. The resulting modes are then different from those obtained by shifting $k_{x,center}$ without rotation, as the latter simulations are left to evolve for a long time at a single value of $k_{x,center}$.

5.4 Growth rates

The quantity interesting to fusion scientists is the growth rate of the waves present in the plasma. This quantifies the stability of the mode and governs resulting transport of heat and/or particles. The growth rate γ is determined by the time evolution of the density, which obeys

$$\frac{\partial}{\partial t}n(x, y, z, t) = \gamma(x, y, z)n(x, y, z, t),$$
(86)

which solves to $n(x, y, z, t) = e^{\gamma(x, y, z)t}$. Since GENE outputs the average density over the chosen simulation box at every time step, this can easily be recast to find the corresponding growth rates. However, the resulting signal contains fluctuations of many kinds, and there is ambiguity as to how a single value for the growth rate of a mode can be deduced from this signal, which take different assumptions on what relevant time scales are for the calculation of growth rates. Below we outline two of these methods.

5.4.1 Floquet averaging

An often used method to find a final growth rate of a mode is called Floquet averaging. This simply takes the average over the fluctuating growth rates of the most unstable eigenmode in the system. The local growth rate $\gamma_{k,loc}$ between the starting time t_1 and ending time $t_1 + \Delta t$ is defined by

$$\frac{n(t_1 + \Delta t)}{n(t_1)} = e^{\gamma_{k,loc}\Delta t}.$$
(87)

The local growth rate can thus be calculated at each point in time t_1 for a chosen Δt . When flow shear is included, the plasma density and corresponding local growth rates often look similar to Fig. 14. Floquet averaging ignores the large fluctuations in the growth rates and just computes the average. This averaging is done one a large timescale; at least a few Floquet cycles should be averaged over.

5.4.2 au_{ac} method

The Floquet averaging timescale is very large, so that it may not correspond well to physical growth rates. As such another method to calculate growth rates was proposed by Citrin (unpublished). The method is called the τ_{ac} method and is used in the rest of this work. The underlying assumption is that the relevant timescale for growth rate calculations is the non-linear decorrelation time. This is the timescale for linear modes to saturate and become nonlinear. Next, it is assumed that this timescale at a given spatial scale is given by $1/\gamma_k$, where γ_k is the growth rate of the most unstable mode



Figure 14: (a) Typical growth of plasma density n_1 under influence of flow shear and (b) corresponding local growth rates

at the respective spatial scale. As such, the time scale indeed corresponds to the growth time needed to enter a nonlinear regime, and it is a proper timescale over which the linear modes can be averaged. It was shown that this assumption is works well at low values of k_y such as used in this work [32].

Local growth rates $\gamma_{k,loc}$ are defined as in Eq. 87. The final growth rate γ_k is then obtained by taking the average of the peak maxima of the local growth rates. From the assumptions of this method follows that we wish $\gamma_k \Delta t$ to be close to 1. We thus need to iteratively adapt Δt , then calculate the resulting local growth rates and final growth rates and check whether the product $\gamma_k \Delta t$ is satisfactory close to 1. When precision $|1 - \gamma_k \Delta t| < \delta$ is reached, the resulting value of γ_k is accepted. In this work, we always use $\delta = 0.02$.

5.4.3 Growth rates of stable and higher k_x modes

When a mode is quenched (e.g. by flow shear), it experience a negative growth rate. For simple stability investigations of simulations that are left to run to equilibrium this is not important to us; whenever a mode is stable we are happy and do not need the information of how quickly the modes are stabilized. Therefore, in simple investigations such as the quench rule created in the next chapter, we set any negative growth rate manually to zero.

However, in the more detailed analysis of later sections we will want to know how quickly certain modes are growing or shrinking, and not just the most unstable mode in the system. These more stable modes generally live at higher k_x values (i.e. $|\theta_0| > \pi$) and are coupled to the most unstable modes at that particular ballooning angle by the parallel boundary condition. Investigation of these modes is only possible by GENE's Eigenvalue Solver. However, such analyzation is practically impossible because hundreds of modes exist at each ballooning angle besides the most unstable one. We thus need a different approach.

An idea to quickly find the quenching rate of a mode at any k_x , is to simulate a plasma for N = 1 and then suddenly shift to the desired value of k_x . The modes will need to adjust to their new ballooning angle, and this initial response may well give the quench rate of the mode at the target k_x , before the most unstable coupled mode takes over again. The quench rate of this response should depend solely on the target value of k_x . Instead, we find that it depends on the amount of k_x -shift introduced (i.e. the difference between source and target k_x). This means the response to a " k_x -shock" does not correspond to the quench rate of the target k_x value, and this method therefore does not yield any trustworthy results on negative growth rates of modes at high k_x .

For crude results, an approximate quench rate can be obtained by just taking the average of the growth rate over some long time interval. This was done for example in Fig.11, and is used throughout this work whenever negative growth rates are presented.

5.5 Mode convergence

Of course, the linear GENE runs on which this work is based need to be sufficiently converged in order to yield robust results. To investigate the modes present in a plasma at some chosen point of parameter space, we usually look at the evolution of the density within our simulation box. As such, we now investigate the density and corresponding growth rates to find the parameter values necessary for convergence.

The main parameters affecting convergence in our simulations are Δk_x and $k_{x,max}$. Δk_x must be sufficiently small so that enough modes can live in the simulation box. Recall that $\Delta k_x = k_{x,min} = 2\pi/L_x$ with L_x the simulation box size, so that increasing the simulation box size decreases Δk_x . On the other hand, $k_{x,max}$ is bound to be finite in numerical simulations, and we must make sure that the cutoff is not too low. Excessive values for Δk_x and $k_{x,max}$ are also undesirable, as this increases the computational effort of the simulation. We do some tests to ensure our values of Δk_x and $k_{x,max}$ are computationally cheap, yet consistently produce converged results. The same is done for the z, v_{\parallel} and μ directions. Typical values are $\Delta k_x = 0.1$ and $k_{x,max} = 9$, such that the k_x -direction is sampled over 90 grid points. Typical resolutions in other directions are 24 for z, 32 for v_{\parallel} and 8 for μ . In the k_y -direction only 1 value is considered (usually 0.3), since ITG modes are found to be most unstable there.

6 Linear ITG quenching rule

A side goal of this work is to find a rough estimate of how a linear ITG mode is quenched by flow shear, including several parameters that affect this quench rule. The primary motivation for this quench rule is its usefulness in the construction of a 9D neural network of fusion plasma data (Citrin et al., in progress). This neural network is based on QuaLiKiz runs, a quasilinear plasma code which can compute rotationless growth rates very quickly. It takes a factor 5 longer to calculate this when flow shear is included, which makes it a difficult dimension to include in the neural network. Thus, by finding a simple function that can compute the quench of growth rates by flow shear, the efficiency will go up and a 10th dimension can be added to the neural network. This function will also provide insight into the general behaviour of instability quenching by flow shear, which may be a useful introduction to the rest of this work.

We vary the parameters R/L_{T_i} (the ion temperature gradient), q, ϵ and \hat{s} , since these have a large impact on ITG modes or its stabilization. GENE is used to simulate the plasma evolution while scanning over flow shear and the τ_{ac} method is adopted to compute the growth rates of the most unstable modes. This is repeated for many sets of parameters around the CBC set, which uses $R/L_T = 6.75$, $\epsilon = 0.18$, q = 1.4, $R/L_n = 2.2$. The electron temperature gradient is reduced to $R/L_{T_e} = 4$ such that we find a cleaner ITG mode (see section 5.2) and only $k_y = 0.3$ is considered, since growth rates are largest in that regime. The parameters under consideration are then varied one by one to isolate their impact on the reduction of plasma instability by flow shear.

The quench rule we make is linear, and its variables are assumed to be independent of each other. The desired equation is thus of the form

$$\gamma = \gamma_0 - f(R/L_{T_i}, q, \epsilon, \hat{s})\gamma_E, \qquad (88)$$

where γ is the growth rate of the most unstable mode in the system, γ_0 is the rotationless growth rate and γ_E is the ExB shearing rate. The function $f(R/L_{T_i}, q, \epsilon, \hat{s})$ is to be determined, but we restrict it to the form

$$f(R/L_{T_i}, q, \epsilon, \hat{s}) = c_1 (R/L_{T_i})^{n_1} + c_2 q^{n_2} + c_3 \epsilon^{n_3} + c_4 s^{n_4} + c_5,$$
(89)

where c_i are constants and n_i are integers.

6.1 Influence of R/L_{T_i}

First of all, the impact of the ion termperature gradient itself is looked at. The growth rates are plotted versus flow shear for several values of R/L_{T_i} in



Figure 15: Growth rate vs. flow shear for several values of R/L_{T_i} (solid lines). In dotted lines the resulting linear quench rule is given.

Fig. 15. It can quickly be concluded from Fig. 15 that the slopes of linear lines fitted through these curves are roughly parallel, such that the quench rate of the ITG mode is not impacted by the ion temperature gradient itself. We thus conclude that $f(R/L_{T_i}, q, \epsilon, \hat{s}) = f(q, \epsilon, \hat{s})$. The dotted lines indicate the results of the linear quench rule that is obtained at the end of this section, which show reasonable agreement with the actual curves.

6.2 Influence of ϵ

Next we investigate the influence of the geometrical factor ϵ on ITG stabilization. We plot growth rates versus flow shear again, for several values of ϵ , as seen in Fig. 16 (a).

Here a radical impact on the quench rate by ϵ can be seen. To quantify this impact, we use the following method. First a least-squares method is used to fit lines through all the curves, with γ_0 necessarily on this line. Note that the dotted lines in Fig. 16 (a) are not these linear fits, but instead are the result of our linear function estimate at the end of this section (Eq. 90). Next, the slope of these curves (the flow shear quench rate) is plotted versus ϵ^n , with n an integer. n is chosen such that a line can be fitted well (using again a least-squares method) through the slope values. In this case, as shown in Fig. 16 (b), a line fits well through the quench rates plotted versus ϵ^{-1} . Finally then, the slope of this line is the impact of $1/\epsilon$ on the flow shear quench rate. Here, using this method we find $f(q, \epsilon, \hat{s}) = 0.09\frac{1}{\epsilon} + f(q, \hat{s})$.



Figure 16: (a) Growth rate vs. Flow shear for several values of ϵ (solid lines). In dotted lines the resulting linear quench rule is given. (b) Flow shear quench rate vs. ϵ^{-1} , and the resulting fit yielding $c_3 = 0.09$.



Figure 17: Growth rate vs. flow shear for several values of q (solid lines), (a) at $\hat{s} = 0.8$ and (b) at $\hat{s} = 0.4$. In dotted lines the resulting linear quench rule is given. (c) Flow shear quench rate vs. q, with a fit yielding $c_2 = 0.13$.

6.3 Influence of q

The same method is used to find the impact of q on the flow shear quench rate. Behaviour is not as clean as before, so plots are made of varying qat $\hat{s} = 0.4$ and 0.8, as shown in Fig. 17 (a) and (b). Plotting the quench rates of all these curves versus q^1 (as shown in Fig. 17 (c)) yields $f(q, \epsilon, \hat{s}) = 0.09\frac{1}{\epsilon} + 0.13q + f(\hat{s})$

6.4 Influence of \hat{s}

Finally the impact of the magnetic shear is mapped. Again, behaviour is not very clean. Therefore \hat{s} is varied around three different points in parameter space and the corresponding ITG quench plotted (Fig. 20 (b)). In addition, q and \hat{s} are varied together, as this is physically relevant (see for typical q and q/\hat{s} profiles e.g. [32]). This is done for three different values of \hat{s}/q : 0.2, 0.57 and 0.9 (see Fig. 20 (a)), so that the entire range of possible \hat{s}/q from 0 to 1 is mapped.



Figure 18: Flow shear quench rate vs. \hat{s} for (a) varying q and s while keeping s/q fixed, and (b) varying s while keeping q fixed. The resulting fits are combined to yield $c_4 = 0.41$.

By comparing the curves in which q and \hat{s} were varied together with the curves where only q was varied, the impact of \hat{s} can be isolated. Combining all the quench rates found from the above two methods (from direct \hat{s} variation and derived from \hat{s} with q variation) and plotting this against \hat{s}^1 (Fiq. 18) leads to $f(q, \epsilon, \hat{s}) = 0.09\frac{1}{\epsilon} + 0.13q + 0.41\hat{s}$.

Then, there is the final constant c_5 from Eq. 89, which can be chosen ambiguously such that the function works well at a specific point in parameter space. We like the function to work especially well around CBC, which is for $c_5 = 1.65$. We can thus present our final linear quench rule to be



Figure 19: (a) Growth rate vs. parallel flow shear, while radial flow shear (γ_E) is zero (b) Growth rate vs. radial flow shear (γ_E) , while parallel flow shear is set to zero.

$$\gamma = \gamma_0 - (0.09\frac{1}{\epsilon} + 0.13q + 0.41\hat{s} - 1.65) \ \gamma_E.$$
(90)

This is a larger \hat{s} -dependence than expected. It is interesting to see if \hat{s} mainly has impact on the stabilizing effect of radial flow shear (γ_E) or the destabilizing effect of parallel flow shear (PVG drive) which also comes with increased rotational flows. Therefore two more sets of simulations are carried out, setting $\gamma_E = 0$ while scanning over the parallel flow shear in one, and the other way around in the other. This is shown in Fig. 19.

It is clear that the magnetic shear has no impact on the PVG drive, since all the curves for pure parallel flow shear are parallel. Indeed, the impact of \hat{s} is on radial flow shear; the higher \hat{s} , the less stabilizing radial flow shear is on the ITG modes. This impact of \hat{s} on the modes with pure radial flow shear is quantified, and the quench rate is found to be proportional to 0.22 \hat{s} . This is unexpected, because when parallel flow shear is included again for this single simulation set, a different proportionality of 0.32 \hat{s} is found, even though we just concluded that \hat{s} had no impact on PVG destabilization. The discrepancy between these values is most likely the result of a higher order effect that is not captured by the linear fits used in this section.



Figure 20: Growth rate vs. flow shear. (a) Varying q and s while keeping s/q fixed. (b) Varying s while keeping q fixed. In dotted lines the resulting linear quench rule is given.

7 ITG eigenmode dynamics

We are finally ready to start tackling the main problem of this thesis. We have been looking at a rough quench rule of linear ITG modes by flow shear. In this rule, there was little attention for precise physical content, and many interesting mechanisms have been blurred to produce an easy-to-use result. Although the quench rule is useful in its own right, we shall now focus more on the actual dynamics that go behind this quenching of ITG modes. In particular, we look at the dynamics that are visible in the eigenmodes of the linear gyrokinetic operator, represented in ballooning space.

In this section we first repeat in a quick summary the problem formulation and specific methodology. After that, we present the results that have been found by applying this methodology. In this process, we find all kinds of things that adapt our initial guesses and motivate new questions and methods, which are presented thereafter.

7.1 Problem and methodology summary

The most unstable eigenmode of the linear qyrokinetic operator with flow shear γ_E included can be visualized well in ballooning space. However, many of the mechanics that form this solution are not visible. We therefore attempt to create a toy model that coherently forms the final rotational eigenmode Φ by decomposing it into fixed-angle solutions ϕ_n . As such the initial guess of this toy model was of the form

$$\Phi(\theta_b, t) = \sum_n a_n(t)\phi_n(\theta) \ e^{\gamma_n t},\tag{91}$$

where γ_n denotes the growth rate of the fixed-angle eigenmode ϕ_n , and $a_n(t)$ are their coefficients, which by flow shear will change in time. The exact terms will be constructed such that the toy model reproduces some key physics of the final rotational mode. One of those key mechanisms is the Floquet behaviour we have seen before, and other things are found when we look at the rotational solution in detail throughout the next section. By constructing this toy model we thus hope to uncover some mechanics of flow shear that play beneath the surface of the full rotational solution. In addition to providing insight, we hope that the model is able to illustrate the marked difference that is found between the impact of flow shear on simulations that used kinetic or adiabatic electrons at low magnetic shear \hat{s} (remember, modes with adiabatic electrons are stabilized for much lower γ_E).

Before a toy model can be constructed, we must take a close look at the dynamics of the rotational and fixed-angle eigenmodes. These are always visualized in ballooning space. As such, three equations that have been described are key to understand results:

$$k'_{x} = k_{x} + 2\pi \hat{s}k_{y} \qquad \text{(parallel boundary condition)}$$

$$k'_{x} = k_{x} + \frac{C_{y}k_{y}\gamma_{E}}{q}t \qquad \text{(flow shear consequence)}$$

$$k_{x} = -k_{y}\hat{s}\theta_{0} \qquad \text{(ballooning angle).} \qquad (92)$$

With these relations we can always translate the quantities ballooning angle, time and mode number k_x into each other. This is used throughout the next section.

7.2 Rotationless modes ϕ_n

First of all, we look at the mode structures of the rotationless solutions ϕ_n in ballooning space. Different ballooning angles are obtained from GENE by shifting $k_{x,center}$ as described before. As in the previous section, we restrict ourselves to clean ITG driven modes by setting $R/L_{Ti} = 6.75$ and $R/L_{Te} = 4$. For proper comparison of the stabilization process of modes with kinetic and adiabatic electrons, we ramp up to $R/L_{Ti} = 11$ in the adiabatic case so that rotationless growth rates are equal to the kinetic modes. We visualize modes that evolved at ballooning angles ranging from 0 to π , for $\hat{s} = 0.8$ (Fig. 21) and $\hat{s} = 0.1$ (Fig. 21). By radial symmetry these are equal to the modes from 0 to $-\pi$. Although other modes live at higher values of k_x , these are more stable than their coupled modes at smaller k_x at the same ballooning angle. Because the Initial Value solver of GENE yields only the most unstable mode in the system, we are restricted to these modes in the range of $\theta_0 \in [-\pi, \pi]$.

At $\hat{s} = 0.8$, the mode consists of a single peak at $\theta_b = 0$. As expected, this peak is unstable when it grows in the low field side mid-plane ($\theta_0 = 0$), and at higher ballooning angles it is gradually stabilized. For $|\theta_0| > \frac{\pi}{2}$ no unstable mode lives at all. From the figure, it is clear that there is little difference between the mode structures when kinetic or adiabatic electrons are used. This is in line with the similar ITG growth rates we found at high magnetic shear.

At $\hat{s} = 0.1$, modes are more complicated. First of all they form a wider structure in ballooning space, i.e. they have large wings at higher ballooning angles. When the modes evolve at a nonzero ballooning angle, much of the mode's energy is shifted towards these wings. There is a clear difference visible between kinetic and adiabatic electrons: in the adiabatic case the wings are larger at $\theta_0 = 0$, and the structure shift is very large at only small ballooning angles. At ballooning angles closer to π , the kinetic and adiabatic cases converge again. A final, not unknown observation is that consistently, adiabatic modes drop down to (practically) zero at large ballooning coordinates, whereas kinetic modes remain finite there. This is clearly visible when $|\Phi|$ is shown on a logarithmic plot, as shown in Fig. 22.



Figure 21: Rotationless eigenmode structures (a) at $\hat{s} = 0.8$ and (b) at $\hat{s} = 0.1$, at various ballooning angles. Blue lines are for kinetic electrons and orange lines for adiabatic electrons.



Figure 22: $|\phi|$ on logarithmic scale for kinetic and adiabatic electrons

7.2.1 Rotationless radial growth rates

Next, we need the growth rates γ_n of all the rotationless modes ϕ_n . This is shown in Fig. 23. A decrease radially away from zero ballooning angle is expected. This turns out to be true for all values of the magnetic shear except at $\hat{s} = 0.1$.



Figure 23: Growth rate vs. ballooning angle for several values of \hat{s} .

This growth rate distribution can be compared to Floquet modes of ITG instabilities with small flow shear, since we can translate the time it takes

for a mode to complete a full Floquet cycle by taking $\theta_0 = 2\pi$ in Eq. 92. This is done in Fig. 24 for $\gamma_E = 0.05$, where Floquet cycles were smoothed a bit for better visibility. To understand the similarity between the curves, imagine a full eigenmode that is rotated through all ballooning angles very slowly by flow shear. It then has time to evolve relatively long at all these angles before it is shifted again. Therefore, at each angle it will be similar to a shearless mode which evolved for a long time at that ballooning angle. As we see in the figure, growth rates are indeed close.



Figure 24: Growth rate vs. ballooning angle for fixed-angle modes, compared to growth rates in a Floquet cycle of a dynamic mode with small flow shear, for several values of \hat{s} .

For our toy model we need to describe modes in a larger range than $\theta_b \in [-\pi, \pi]$. However, growth rates at these higher ballooning angles are not accessible by GENE simulations because of the periodic k_x boundary conditions inherent in the local assumption. A wider rotationless radial growth rate relation must therefore be conjectured. We start from the known fact that gyroaveraging leads to a Bessel function of the first kind $J_0(k_{\perp}\rho)$. Mathematically important is that this function drops down to zero at large argument values. Physically this means that when averaging over a large gyroradius, the many fluctuations inside this radius cancel out each other. For our purposes, at large θ_0 (or, eequivalently, k_x) we also have large $k_{\perp}\rho$, such that $J_0\phi \to 0$. In addition to this Bessel function, it is known that at many k_x (especially high values) modes become quenched. Combining this knowledge we conjecture the rotationless radial growh rates to be a Gaussian with a negative offset, as shown in Fig. 25. The precise function that

was chosen is given in Table 1. Other shapes (Bessel function with negative offset, linear decrease away from $\theta_0 = 0$) could also have been chosen. It turned out the choice is not of great importance; these alternatives can be tuned to yield similar results.



Figure 25: Growth rate vs. ballooning angle for $\hat{s} = 0.8$ and $\hat{s} = 0.1$. The grey area is inaccessible by GENE simulations and the rotationless radial growth rates are conjectured in that region (blue lines).

7.3 Rotational modes Φ

To find the impact of flow shear on an ITG mode, the modes are visualized in ballooning space for several values of γ_E at $\hat{s} = 0.8$. The resulting structures are shown in Fig. 26 (a) at some random point in time.

Again, in this high magnetic shear case, there is just a more or less Gaussian structure visible, which is slowly quenched as flow shear increases. The modes show large Floquet fluctuations in time (not visualized here). This fluctuating behaviour also clearly affects the wave structures in real space, which are shown in Fig. 26 (b). There, poloidal plasma slices are visualized at a maximum, medium and minmum value of its mode's fluctuating growth rate. At the lower values the instabilities are clearly more suppressed than at the heights of the Floquet modes.

It is remarkable to note that despite these large Floquet fluctuations, the mode structures equilibrate to some shape in ballooning space. These structures' shapes are not affected by the Floquet fluctuations at all (this observation was clear from the creation of several movies).

The same figures are given for low magnetic shear.



Figure 26: Impact of flow shear on ITG modes at $\hat{s} = 0.8$ (a) in ballooning space, blue lines are for kinetic electrons and orange lines for adiabatic electrons, and (b) in real space, where structures are shown at maximum, medium, and minimum values of the Floquet fluctuating growth rates.



Figure 27: Impact of flow shear on ITG modes at $\hat{s} = 0.1$ (a) in ballooning space, blue lines are for kinetic electrons and orange lines for adiabatic electrons, and (b) in real space.

In ballooning space, again structures equilibrate to some shape in space while growing in time. Most noteworthy is the observation that flow shear causes the structures to be shifted towards negative ballooning space. This is highly interesting: in this shift to the left a large difference is visible between adiabatic and kinetic electrons. Also noteworthy is a smaller mode envelope, especially visible in the kinetic modes at nonzero flow shear. There one sees, besides the main peaked structure, another smaller structure of subpeaks. The nature of this small envelope is not known. In real space, there is a clear difference between the kinetic and adiabatic electron cases: as expected, with adiabatic electrons the impact of flow shear is much quicker. Also interesting are the multiple band structures which do not occur at $\hat{s} = 0.8$. This is likely due to the shifted mode structure in ballooning space, which now peaks at high ballooning angles. Multiple bands can be expected for such modes, which live at high k_x but are constrained to the same radial positions.

8 Toy Model

The basic building blocks for a toy model eigenmode decomposition were obtained in the last section, namely the fixed-angle mode structures ϕ_n and their growth rates γ_n . The full rotational modes were investigated in ballooning space, and their main properties can now be identified. These are

- Floquet fluctuations in time at high \hat{s} , but not at low \hat{s}
- Mode shape equilibration in ballooning space
- Shifted ballooning structures at low \hat{s} , but not at high \hat{s}
- A difference in stabilization between kinetic and adiabatic electron modes at low \hat{s} , but not at high \hat{s}

The goal is to reproduce all this in a toy model, to gain physical insight into the dynamics involved. The next section is dedicated to this model.

8.1 Model mechanics

The form of our intended toy model is a decomposition of the full eigenmode Φ into rotationless modes ϕ_n :

$$\Phi(\theta_b, t) = \sum_{n = -\infty}^{\infty} \phi_n(\theta_b, t).$$
(93)

The modes ϕ_n must grow exponentially, and somehow be adapted by flow shear. In GENE flow shear is implemented by shifting $\Phi(\theta_b, t)$ over the k_x grid constantly in time. In our model, we do something similar. At each time step, the ϕ_n 's are evolved according to the recurrence relation:

$$\phi_n(t + \Delta t) = \left(\phi_n(t) - O_n(t)\phi_n(t) + O_{n+1}(t)\phi_{n+1}(t)\right) e^{\gamma_n \Delta t}, \quad (94)$$

where the θ_b dependence of ϕ_n was omitted. $O_n(t)$ quantifies the overlap between two neighbouring modes.

If we put for now $O_n = 0$, two terms drop out and we are left with a model in the absence of flow shear. This simple rotationless model just describes the exponential growth of each mode ϕ_n . This is shown in Fig 28. We start with ϕ_n structures in packets from $\theta_0 \in [-\pi, \pi]$. These packets are periodically initialized to cover a large enough section of ballooning space. Then, they are all given their respective rotationless radial growth rate. Of course, after a while the modes with the largest growth rates will dominate all others.



Figure 28: Initialization of fixed-angle modes. There are 16 modes in the range $\theta_0 \in [-\pi, \pi]$. These packs are initialized at every multiple of 2π . Then the modes are left to grow as per the rotationless radial growth rates created above.

If we include flow shear, modes ϕ_n are shifted. This transformation scales with the overlap O_n with its neighbouring modes (i.e. with the closest ballooning angles):

$$O_n(t) = \frac{\int d\theta_b \quad \min[\phi_n(\theta_b), \phi_{n-1}(\theta_b)]}{\int d\theta_b \quad \max[\phi_n(\theta_b), \phi_{n-1}(\theta_b)]} \quad \in [0, 1].$$
(95)

When such a shift occurs, we see in Eq. 94 that a mode ϕ_n obtains two additional terms: it gives away a portion of its structure to its right neighbour, and receives a portion of its left neighbour. In essence, modes are thus slowly transformed into their neighbours at negative k_x , and a general shift of the total structure towards negative ballooning space results. The shifts do not occur at each time step, but only every τ_s seconds. Decreasing this shift rate increases flow shear: $\gamma_E \propto 1/\tau_s$. The mechanism is illustrated in Fig. 29.



Figure 29: Implementation of flow shear in toy model: mode ϕ_n receives part of its left neighbour and gives a part of itself to its right neighbour.

The key feature of this model is that the fraction of mode shifted to its neighbour depends on their mutual overlap in ballooning space. When this overlap is large, the transformation by flow shear becomes more efficient. In real space, this overlap corresponds to modes which balloon in different angles having similar amounts of energy stored at the same radial wavelengths, since $\theta_b \propto k_x$. Flow shear then causes energy of modes of all wavenumbers k_x to be rotated to new ballooning angles, and this process is more efficient when there is already a similar amount of energy present at those wavenumbers at the resulting ballooning angle. Intuitively at least, this makes sense.

8.2 Constants and parameters

Thus far we omitted the description of several constants that haven been chosen in this model. Most important to mention is the amplitude of flow shear in this model. By Eq. 92, flow shear can be increased by shifting the modes towards negative k_x quicker. As discussed in Section 4.4, in GENE shifts do not occur at each time step, but only each $t = \frac{q\Delta k_x}{C_y k_y \gamma_E}$ seconds. In this model a similar approach is used; modes are shifted every

$$t_p = \max\left(0, \frac{c_0}{\gamma_E} - c_1\right) \tag{96}$$

seconds, where c_0 and c_1 are constants. Also of importance are the choice of mode numbers n (which determine the simulation box size and mode spacing), the amplitude of the overlap O_n and the function that represents the rotationless radial growth rates. In these final two, the constants are defined by:

$$\gamma_n = c_2 e^{-c_3 n^2} - c_4$$

$$O_n(t) = c_6 \frac{\int d\theta_b \min[\phi_n(\theta_b), \phi_{n-1}(\theta_b)]}{\int d\theta_b \max[\phi_n(\theta_b), \phi_{n-1}(\theta_b)]} + c_5$$
(97)

In the table below, all constants, their values and their relevance are given.

Constant	Values	Relevance
n	[-240,240]	radial simulation box size
	(corresponding to $\theta_0 = \frac{n\pi}{8}$)	and mode spacing
Δt	1/8	time step interval in seconds
c_0	0.4 ($\hat{s} = 0.1$), 0.6 ($\hat{s} = 0.8$)	speed of mode shifts by γ_E
c_1	0.3 ($\hat{s} = 0.1$), 0.7 ($\hat{s} = 0.8$)	offset of γ_E -time relation
<i>c</i> ₂	1 ($\hat{s} = 0.1$), 1.3 ($\hat{s} = 0.8$)	base growth rate
c_3	$0.001 \ (\hat{s} = 0.1), \ 0.3 \ (\hat{s} = 0.8)$	radial width of γ_n
c_4	0.3	γ_n negative saturation
c_5	0.015	offset of O_n
c_6	0.75	mode shift amplitude

Table 1: Declaration of constants as used in toy model

8.3 Results

Let's look at the results. In Fig. 31 pictures are shown of the mode evolution in time and the corresponding growth rates, for several values of flow shear, for both high and low magnetic shear. The constructed toy model is compared to linear GENE simulations. In Fig. 30, some of the final mode structures as produced by our model are shown. These can be compared to the results of Fig. 26 and Fig. 27.

At $\hat{s} = 0.8$, results of the model are qualitatively very similar to GENE simulations. Flow shear has immediate impact on unstable growth rates; even at small values growth rates start to fluctuate wildly. As flow shear increases, the frequency of these Floquet modes becomes higher, and the average growth rate decreases. At some point the mode is completely quenched.



Figure 30: Equilibrated mode structures as produced by toy model. Blue lines indicate kinetic electrons and orange lines adiabatic electrons.

This captures all the goals that were aimed for by this model. Quantitative differences are that our model predicts Floquet fluctuations with a smaller amplitude, and frequencies do not increase as quickly as seen in GENE simulations. The model can likely be tuned to yield slightly more comparative results.

At $\hat{s} = 0.1$, general behaviour is also reproduced well. The model yields no Floquet fluctuations and flow shear quenches the full mode, just as shown by GENE. Again, nearly all goals that were aimed for are reproduced by this model, only one thing is missing. The model shows no difference at all between kinetic and adiabatic electrons at low \hat{s} . With the current implementations, the differences that were visible between some of the ϕ_n 's are clearly not enough to create a large difference in growth rate quench by flow shear.



Figure 31: Time evolution of $\log(n_1)$ and corresponding growth rates, (a) for $\hat{s} = 0.8$ and (b) for $\hat{s} = 0.1$, at several values of flow shear. Our toy model is compared to GENE simulations.

8.4 Explanation of mechanisms

The model reproduces all kinds of complicated mechanisms, of which we wanted more insight. Let's start with Floquet behaviour: the fact that this model produces fluctuating growth rates can be explained by the following. All modes grow and give a portion of their structure to their neighbour. On some location(s), overlap between two modes is very small; let's call this a bottleneck location. Imagine this location between the mode ϕ_n that lives at $\theta_0 = 0$ and its right neighbour ϕ_{n-1} . Mode ϕ_n will receive much structure from the left and give nothing to ϕ_{n-1} , so that it grows fast. Because at high \hat{s} radial growth rates are peaked sharply, the difference between modes at either side of a bottleneck is greatly enhanced. We are at the peak of a Floquet cycle. At this point the mode ϕ_n is so large that its left neighbours hardly contribute any more. At the same time, there exists a lower bound on structure shift: $0.015 * \phi_n$ must always be given to ϕ_{n-1} . Because ϕ_n is so large, this fraction is significant. From this point onwards the overlap between ϕ_n and ϕ_{n-1} will grow rapidly in a snowball effect, and ϕ_n will lose ever more of its size. The structure is given to the modes that have lower growth rates, such that we drop down to the bottom of a Floquet cycle. ϕ_n is depleted, but because its growth rate is large, it will slowly come back up again. Finally it will overtake its neighbours and grow large enough to create a new overlap bottleneck. One Floquet cycle has been described. This is illustrated in Fig. 32.



Figure 32: Schematic view of a Floquet cycle in our model

At low \hat{s} growth rates are similar between neighbours, and mode structures are wider by nature. By these two properties overlap bottlenecks are not as pronounced and no Floquet fluctuations are created.

Next there is the quenching impact of flow shear. This can be understood by thinking about the most unstable modes that live around $\theta_0 = 0$. These will grow largest of all modes, and as such (on average) always give away more to their right neighbour than what they receive from the left. Modes are

thus depleted proportional to their growth rate. The right neighbours live at higher ballooning coordinates and as such are more stable. As flow shear increases, shifts occur more frequently such that this effect is enhanced.

Notes on the creation of our model Finally, we must put the current toy model into some context. It may be thought that the model is very simple and with some small adaptations will yield better results. However, the simplicity of the model is rather a strength than a shortcoming. Initially we attempted to constuct the model from intuitive principles, e.g. by just shifting all ϕ_n 's slowly in their entirety, as done in GENE (in accordance to Eq. 92). However, final growth rates never converged to a fluctuating (Floquet) equilibrium. None of several concepts yielded any results, so the research method was changed. Rather than hoping that a intuitive model yielded good results, a more "engineering"-like approach was used: GENE simulations were recreated by any implementations necessary. After a dozen or so fruitless attempts of ever increasing complexity, finally a model was created that showed Floquet fluctuations. This version was complicated and not intuitive. However, this working model could be trimmed down again: any implementations that were not necessary were removed. In the end, surprisingly, everything could be boiled down to the single mechanism presented above: partial mode shifts dependent on overlap between neighbours. Out of many ideas this is the sole survivor, and it is the first time Floquet fluctuations have been reproduced using a k_x -shift approach.

Although we cannot claim that this mechanism is fundamental, i.e. it may be replaced by a similar alternative (although we found none), it is likely that such an alternative should result in mode shifts becoming "bottlenecked and depleted", reminiscent to the process described before.

9 Conclusion and discussion

In this work, the ITG instability was inspected by visualizing the modes in ballooning space. The impact of flow shear was clearly visible: at high magnetic shear, modes grow and shrink periodically (Floquet fluctuations), with frequencies increasing with flow shear. At low magnetic shear modes live shifted towards negative ballooning space, where they are ever more quenched. Shearless modes were also visualized at ballooning angles $\theta_0 \in$ $[-\pi, \pi]$. In both the sheared and shearless modes, differences are visible between kinetic and adiabatic electron modes at low \hat{s} .

Next, a toy model was created that decomposes the full sheared eigenmodes into the shearless ones. In this model flow shear was added to the components by partially shifting mode structures between neigbouring modes in ballooning space, dependent on their mutual overlap. This causes a general shift of the full eigenmode to negative ballooning space, which is similar to the known flow shear implementation of shifting the full solution over the k_x grid.

The toy model qualitatively reproduces the main properties of sheared ITG eigenmodes: at high magnetic shear they exhibit Floquet fluctuations of increasing frequency and decreasing average growth rate as flow shear increases. At low magnetic shear modes quickly become quenched without fluctuations. All structures equilibrate to some shape in ballooning space. The only thing not captured by our model is a large difference between mode quench between kinetic and adiabatic electron modes at low \hat{s} .

The model makes the mechanics of flow shear stabilization inside an unstable ITG solution more transparent. It is now clear that Floquet fluctuations can be caused by "bottlenecks" in the process of rotating the ballooning angles of modes by flow shear. Mode quenching comes to be because rotation spreads the eigenmodes in ballooning space, which is in particular unfavourable for the most unstable modes. In addition to providing this insight, the model puzzles together the effects of flow shear from shearless components. As such, it could be envisaged that the model is useful for reduced modelling, where one could leave out the explicit evolution of an unstable solution in the presence of flow shear. This model only uses a few shearless solutions as input, which are much quicker to calculate. It is estimated that this model can thus produce rotational growth rates a factor 1-10 quicker than the current implementation in QuaLiKiz, dependent on the values of flow shear and magnetic shear (e.g. at $\hat{s} = 0.8$, only a few modes ϕ_n close to $\theta_0 = 0$ are necessary, whereas for $\hat{s} = 0.1$ and large γ_E modes must be initialized in a large section of ballooning space, making the computation much more expensive).

However, the model is not yet ripe for such quantitative applications. It must first be enhanced and tuned to yield more accurate growth rates and Floquet amplitudes and frequencies. Moreover, it should be tested under a wide variety of circumstances: we created the model at CBC parameters and only varied \hat{s} , whereas it is interesting to test the model's robustness for e.g. varying q, ϵ and R/L_T . Its resulting (averaged) growth rates should at least be similar to the linear quenching rule that was created for reduced modelling purpuses in section 6. Finally, the model would be more powerful if it captured the difference between kinetic and adiabatic modes at low \hat{s} . An idea to achieve these goals is to expand the current coupling between modes. At the moment there is only coupling between neighbouring modes in ballooning space, but other schemes could be imagined, e.g. by considering periodic coupling between modes by the parallel boundary condition. With such more advanced couplings the differences found in rotationless modes ϕ_n between kinetic and adiabatic mode structures could have a larger effect on the final mode structures Φ .

Acknowledgements

I'd like to thank above all Jonathan, who provided this whole research opportunity for me and supervised me well during the many months that I worked on it at DIFFER. Thanks to Cristiane, who helped me to keep a sure footing in Utrecht despite my usual absence there. Also many thanks to the GENE team at IPP Garching: Frank, Tobias, MJ, Daniel and Hauke, for being very hospitable during my visitat there and for providing lots of feedback whenever I asked for it.

References

In order of appearance

- [1] F. Wagner, European Physical Journal Plus 131, 445 (2016)
- [2] F. Wagner, European Physical Journal Plus 129, 219 (2014)
- [3] J. Ongena, G. van Oost, Fusion Science and Technology 57, 2T (2010)
- [4] P. Batistoni et al., Nuclear Fusion **52**, 8 (2012)
- [5] J. D. Lawson, Proceedings of the Physical Society 70, 1 (1957)
- [6] P. N. Guzdar, Liu Chen, W. M. Tang and P. H. Rutherford, Physics of Fluids 26, 673 (1983)
- [7] S.L. Newton, S.C. Cowley and N.F. Loureiro, Plasma Phys. Control. Fusion 52, (2010)
- [8] J.E. Kinsey, R.E. Waltz and J. Candy, Physics of Plasmas 12 (2005)
- [9] M. Barnes et al., Physical Review Letters **106** (2011)
- [10] E.G. Highcock et al., Physical Review Letters **105** (2010)
- [11] E.G. Highcock et al., Physical Review Letters 109 (2012)
- [12] K. H. Burrell, Physics of Plasma 4 (1997)
- [13] G.D. Conway et al., Physical Review Letters 84 (2000)
- [14] J.W. Connor, R.J.Hastie and J.B. Taylor, Physical Review Letters 40, 6 (1978)
- [15] J. Candy, R.E. Waltz and M.N.Rosenbluth, Physics of Plasmas 11, 5 (2004)
- [16] F. Jenko, W. Dorland, M. Kotschenreuter and B.N. Rogers, Physics of Plasmas 7, 1904 (2000) http://genecode.org
- [17] C. Bourdelle et al., Physics of Plasmas 14 (2007)
- [18] D.W. Nicholson, Introduction to Plamsa Theory (John Wiley & Sons, 1983)
- [19] R.J. Goldston and P.H. Rutherford, Introduction to Plasma Physics (IOP Publishing, 1995)

- [20] J Wesson, *Tokamaks* (Oxford: Clarendon press, 2004)
- [21] L. Spitzer, Jr. *Physics of Fully ionized gases* (Wiley-Interscience, 1962)
- [22] T. Vernay et al., Plasma Phys. Control. Fusion 55, (2013)
- [23] Y.L. Klimontovich and D. ter Haar, The Statistical Theory of Non-Equilibrium Processes in a Plasma (Pergamon press, 1967)
- [24] A.A. Vlasov, Journal of Experimental and Theoretical Physics (in Russian) 9, 3 (1938)
- [25] J.B. Taylor and R.J.Hastie, Plasma Physics 10, 5 (1968)
- [26] F. Merz, Gyrokinetic Simulation of Multimode Plasma Turbulence, PhD thesis Universität Münster (2009)
- [27] D. Told, Gyrokinetic Microturbulence in Transport Barriers, PhD thesis Universität Ulm (2012)
- [28] M.J. Pueschel, *Electromagnetic Effects in Gyrokinetic Simulations of Plasma Turbulence*, PhD thesis Universität Münster (2009)
- [29] Y. Song, W. Wu and S. Du, Tokamak Engineering Mechanics (Springer, 2013)
- [30] Snyder et al., Nuclear Fusion **51**, 10 (2011)
- [31] I. Predebon and P. Xanthopoulos, Physics of Plasmas 22 (2015)
- [32] J. Citrin et al., Physics of Plasmas 19 (2012)