Fluctuations in the Holographic Superconductor

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A thesis submitted for the degree of

Master of Science

Utrecht 2017

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Acknowledgements

First, I would like to thank my supervisors, Henk and Nick, for their help and encouragement. With their enthusiasm and knowledge, they guided me through this project making it a really enjoyable experience.

Then, of course, I would also like to thank all the people in the Master room for the moments they made me laugh as well as for all the interesting and useful discussions we had. A special thank to Francisco for letting me use his incredibly fast computer when my laptop could not handle the heavy computations, and to Emma, that, as the deadline was approaching, kept telling me I needed to start writing if I wanted to graduate on time. Without her constant reminder, probably, I would have not started writing yet.

Finally, I would like to thank my family for giving me the opportunity to study here and for always being supportive. This beautiful experience would have not been possible without them.

Abstract

We study fluctuations in the holographic superconductor model, both above and below the critical temperature where the various fields involved are coupled. In particular, we introduce the concept of intrinsic dynamics and explain its relevance to the problem. We then compute spectral functions for the intrinsic dynamics of the scalar and gauge bulk field away from the probe limit, in order to understand the effect of the coupling of the order parameter and the gauge field in the superconducting phase and analyze the low-energy dynamical behavior of the holographic superconductor model.

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Chapter 1

Introduction to gauge/gravity duality

Gauge/gravity duality, often referred to as AdS/CFT or holographic duality, is one of the most remarkable achievements of string theory. First conjectured by Juan Maldacena in 1997 [15], it quickly developed into a useful tool to study strongly interacting quantum field theories (QFT) and, in the last decade, it also made its appearance into the field of condensed matter.

As the name itself suggests, the conjecture links together two seemingly unrelated theories of physics, gauge field theories and the theory of (quantum) gravity in one higher dimension (the quantum field theory can be thought as a projection of the gravitational theory in a lower dimensional space, hence the name *holography*). In particular, it was first discovered as a correspondence between two theories with a particular set of symmetries, **Conformal Field Theories** (CFT) and string theories in the maximally symmetric **anti-de Sitter** (AdS) spacetime. After its first formulation, however, the conjecture has been extended to more general theories and spacetime geometries. Nonetheless, the name AdS/CFT is still commonly used today even to refer to these extended applications of the correspondence.

The usefulness of the relation lies in the fact that it represents a *strong/weak* duality. In informal terms, what this means is that when one side of the correspondence is described by a weakly interacting theory (e.g., classical gravity), the other side is characterized by strong interactions. Weakly interacting problems are easier to treat, as they can be studied starting from a known and well understood free theory and slightly modify it taking into account increasingly smaller corrections due to these interactions until one reaches the desired accuracy (in what is called *perturbation theory*). When the interactions are strong, however, the behavior of the system can quite drastically change with respect to its free counterpart, and it is not possible anymore to understand the underlying physics by means of a simple perturbation expansion. A strong/weak duality represents therefore an amazing tool to study hard to treat, strongly interacting problems by simply shifting the focus on the weak side of the correspondence, where standard perturbation theory techniques can be applied.

In the case of AdS/CFT, the main computational device was provided, shortly after Maldacena's discovery, by Gubser, Klebanov, Polyakov [5] and Witten [18]. This is known as the **GKPW rule** and, as we will explain in details later, it provides a relation between a QFT generating functional and a partition function in terms of fields defined in a higher dimensional gravitational theory.

In the remainder of this chapter, we give a brief introduction to the correspondence, focusing on the limit of classical gravity and strongly interacting quantum theories (the *large-N* limit). This is the limit that is considered in applications of the duality to condensed-matter systems, as for example the holographic superconductor we consider in this work. In such context, string theory does not play a role, and we will therefore not include it in the introduction, for the readers interested in learning more about the role of string theory in AdS/CFT we refer to, for example [3]. In particular, we quickly review the main properties of the AdS spacetime, we introduce the *dictionary* that relates the two sides of the correspondence and we explain the role of symmetries in constructing holographic models.

In chapter 2 we present some detailed calculations of two-point functions for a CFT using the holographic dictionary, in order to provide the reader with a better understanding of the practical use of the correspondence as well as to present some details that will be useful in the remaining chapters.

In chapter 3 we explain how to modify the geometry of the AdS spacetime in order to introduce a temperature and a chemical potential in the quantum boundary theory, and we introduce the holographic superconductor model. We describe how the model gives rise to a second order phase transition characteristic of a superconductor and present some numerical results showing the behavior of the order parameter across the phase transition.

Chapter 4 contains the main results of this thesis. In this chapter we introduce the concept of **intrinsic dynamics** and study fluctuations of the scalar and gauge fields in the holographic superconductor model, providing the theoretical background for the calculations and presenting the numerical results for the spectral functions for various temperatures, both above and below the critical point characterizing the phase transition.

In the conclusive chapter, we shortly review the results obtained and give the outline of possible future developments.

1.1 A brief tour of AdS/CFT

As mentioned in the preamble, the discovery of the correspondence comes from string theory. In particular, Maldacena noticed that type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ (i.e., with 5 of the 10 dimension of the theory compactified on a 5 dimensional sphere, leaving a $(4+1)$-dimensional anti-de Sitter spacetime for the remaining dimensions) can be equivalently formulated as a $\mathcal{N} = 4\, SU(N)$ super-conformal Yang-Mills theory in $3+1$ dimensions. At first sight, one may wonder how this relation can be of any use in condensed matter, especially due to a lack of a perturbative field theoretic formulation of string theory, trying to get insights in a condensed-matter problem by means of string theory does not seem to be a winning approach. However, we know the effective low-energy limit of string theory very well, it is just Einstein theory of general relativity. It would, therefore, be useful to work in this low-energy limit on the gravity side, but what does it imply on the dual quantum field theory? It turns out that the classical limit corresponds to taking the N, in the $SU(N)$ symmetry group of the Super-Yang-Mills theory to infinity, in the so-called *large-N* limit together with the **strong coupling** limit of the CFT. We do not go into details of the derivation, but we sketch a dimensional argument to explain why these limits are related.

On the gravity side, the string theory present three length scales, the AdS radius of curvature L, the Planck length l_P and the string scale l_s. As previously mentioned, in this thesis we do not go into the details of string theory underlying the correspondence, everything we need to know is that the low-energy effective action is given by the Einstein-Maxwell action, plus a matter contribution, plus curvature corrections that are higher order in the string scale l_s:

$$S_{IIB} \sim \frac{1}{G_N} \int \sqrt{g} L + \mathcal{L}_{matter} + l_s^4 L^4 + \cdots \ , \tag{1.1}$$

with G_N the gravitational constant in $d+1$ dimensions (the numerical value of which, of course, is not known except for $d = 3$).

On the CFT side, the $\mathcal{N} = 4$ Super-Yang-Mills theory has two dimensionless parameters, the N of the gauge group $SU(N)$ and the coupling constant of the theory g_{YM}. In 1974 't Hooft realized that the large N limit reorganizes the perturbation expansion in terms of a new effective coupling, that in the case of the Super-Yang-Mills theory is given by

$$\lambda = g_{\mathrm{YM}}^2 N \ , \tag{1.2}$$

called the 't Hooft coupling. The strength of the interaction for large value of N is then controlled by this new parameter instead of g_{YM}.

The correspondence between the parameters of the two theories is

$$\frac{L^{d-1}}{G_N} \sim \left(\frac{L}{l_P}\right)^{d-1} \sim N^2 \tag{1.3}$$

$$\lambda \sim \left(\frac{L}{l_s}\right)^4 \ . \tag{1.4}$$

From this we can see why AdS/CFT is a strong/weak duality. In order to work with classical gravity, both loops and higher curvature correction must be suppressed, on the gravitational side. This implies $\lambda \gg 1$ and $N \gg 1$. On the other hand, if we take a weakly interacting field theory, we have that $l_s \gg L$ and string theory corrections are not suppressed anymore.

Of course what we presented is far from being a proof and it considers only a specific example of conformal field theory. However, the statement is true in general: weakly coupled asymptotic AdS gravitational theories are dual to strongly coupled quantum field theories with a large number of degrees of freedom.

Before continuing to explain the "dictionary rules" of the correspondence, let us quickly review some basics notions about the anti-de Sitter spacetime and conformal field theories.

1.2 AdS spacetime and CFT

As is often the case in physics, in AdS/CFT symmetries play a key role, so let us start by briefly reviewing the concept of symmetry in general relativity.

Mathematically, a d-dimensional spacetime is defined on a d-dimensional differentiable manifold $\mathcal{M}$, and the metric is a symmetric $(0, 2)$ tensor defined on $\mathcal{M}$, encoding the information needed to describe its curvature. In differential geometry, a symmetry of a manifold is a transformation mapping $\mathcal{M}$ to itself that leaves the geometry invariant. Such a transformation is called **isometry**. Isometries of the spacetime are in one to one correspondence with linearly independent *Killing vectors*, that we denote K^μ, defined by

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ , \tag{1.5}$$

where ∇_μ is the covariant derivative on $\mathcal{M}$. To make sense of the above definition in the case of the metric, let us take an infinitesimal coordinate transformation along the Killing vector $x^\mu \to x'^\mu = x^\mu + \epsilon K^\mu$. Under a coordinate transformation the metric transforms as a rank-2 tensor

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} \ , \tag{1.6}$$

that gives[1] to linear order in ϵ

$$g_{\mu\nu} \to g_{\mu\nu} - \epsilon(\nabla_\mu K_\nu + \nabla_\nu K_\mu) + \mathcal{O}(\epsilon^2) \ . \tag{1.7}$$

If we want the metric to be invariant under such a coordinate transformation, we see that equation (1.5) must hold.

A spacetime with the maximum number of independent Killing vectors is called **maximally symmetric**. The simplest example of maximally symmetric spacetime is $\mathbb{R}^d$. We know that translation and rotation are

[1]Details of the calculations: for ϵ small, we have up to linear order in ϵ:

$$\begin{aligned} g_{\mu\nu}(x) =& g'_{\alpha\beta}(x')\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} = g'_{\alpha\beta}(x^\lambda + \epsilon K^\lambda)\frac{\partial(x^\alpha + \epsilon K^\alpha)}{\partial x^\mu}\frac{\partial(x^\beta + \epsilon K^\beta)}{\partial x^\nu} \\ =& \left[g'_{\alpha\beta}(x) + \epsilon\partial_\lambda g'_{\alpha\beta}(x)K^\lambda\right](\delta^\alpha_\mu + \epsilon\partial_\mu K^\alpha)(\delta^\beta_\nu + \epsilon\partial_\nu K^\beta) \\ =& g'_{\alpha\beta}(x) + \epsilon g'_{\alpha\nu}(x)\partial_\mu K^\alpha + \epsilon g'_{\beta\mu}(x)\partial_\nu K^\beta + \epsilon\partial_\lambda g'_{\mu\nu}(x)K^\lambda = g'_{\mu\nu}(x) + \epsilon(\nabla_\mu K_\nu + \nabla_\nu K_\mu) \ , \end{aligned}$$

where the last step can be verified by using the definition of covariant derivative. Inverting this relation by bringing the covariant derivatives on the left-hand side, we obtain (1.7) isometries of the space. We have d independent axes along which we can perform a translation and therefore d total translations. In order to count all the possible rotations, imagine fixing a point in $\mathbb{R}^d$ and put an orthogonal set of axes on it, each independent rotation moves one of the axes into another. Of course, the opposite action represents the same rotation (moving a into b is the same rotation as moving b into a). Since we have d axes, after picking one we can choose one of the remaining $d-1$ axes to rotate into, therefore we have $(d-1)/2$ independent rotations, giving a total of $d(d+1)/2$ symmetries for $\mathbb{R}^d$. Any d-dimensional space with such a number of symmetries (i.e., any space with $d(d+1)/2$ independent Killing vectors) is a maximally symmetric space.

Anti-de Sitter spacetime (AdS in short) is one of only three possible solutions of the vacuum Einstein equations that are maximally symmetric, alongside Minkowski and de Sitter spacetimes. These solutions are characterized by the group formed by their Killing vectors. In particular, the Killing vector for d-dimensional AdS form the group $SO(2, d-1)$. As we will shortly review, this is the same group of a $d-1$-dimensional conformal field theory.

For a maximally symmetric spacetime, the curvature must be the same everywhere and in every direction. What this implies is that the Ricci tensor is constant and it locally completely characterizes the spacetime, that is, knowing the Ricci scalar we can reconstruct the Riemann tensor of the spacetime. Since the curvature is the same in every direction, the Riemann tensor cannot depend on derivatives, hence it takes the form

$$R_{\rho\sigma\mu\nu} = C\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \ , \tag{1.8}$$

with C a constant. Contracting both sides twice we obtain

$$\begin{aligned} R =& g^{\sigma\nu}R_{\sigma\nu} = g^{\sigma\nu}g^{\rho\mu}R_{\rho\sigma\mu\nu} = Cg^{\sigma\nu}(d-1)g_{\sigma\nu} \\ =& Cd(d-1) \ . \end{aligned} \tag{1.9}$$

The three maximally symmetric solutions are characterized by R being positive, negative or zero. The anti de-Sitter solution corresponds to negative values of R. Analogously, it is often said that AdS is the solution with a negative cosmological constant. We can see this from the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = 0 \tag{1.10}$$

by contracting with the metric tensor to obtain, for $d > 2$,

$$R = \frac{2d}{d-2}\Lambda \ , \tag{1.11}$$

and therefore,

$$R_{\rho\sigma\mu\nu} = \frac{2\Lambda}{(d-1)(d-2)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \ . \tag{1.12}$$

A maximally symmetric spacetime is then equivalently locally completely specified by the cosmological constant, and and the three different maximally symmetric solutions correspond to $\Lambda > 0$ (de Sitter), $\Lambda = 0$ (Minkowski) and $\Lambda < 0$ (anti-de Sitter).

These spaces have an analog in Euclidean signature; the solution with positive cosmological constant corresponds to a d-dimensional sphere, Minkowski spacetime corresponds to $\mathbb{R}^d$, while the anti-de Sitter solution is the Lorentzian version of the Euclidean hyperboloid. As is well known, a hyperboloid in d-dimensional Euclidean space is the solution to

$$-X_{-1}^2 + X_0^2 + \cdots + X_{d-1}^2 = -L^2 \ , \tag{1.13}$$

that in Lorentzian signature then becomes

$$-X_{-1}^2 - X_0^2 + X_1^2 \cdots + X_{d-1}^2 = -L^2 \ , \tag{1.14}$$

that defines the AdS$_d$ spacetime. This implies that AdS$_d$ has the same isometries of $\mathbb{R}^{2,d-1}$, and we then see that its Killing vector forms the $SO(2,d-1)$ group. The positive constant L is called the AdS radius, and it is related to the Ricci scalar, and therefore to the cosmological constant, by $R=-2/L^2$.

A solution to (1.14) is given by

$$\begin{aligned}
X_{-1} =& L\cosh\rho\cos\tau \\
X_0 =& L\cosh\rho\sin\tau \\
X_1 =& L\sinh\rho\cos\theta_1 \\
X_2 =& L\sinh\rho\sin\theta_1\cos\theta_2 \\
&\vdots \\
X_{d-2} =& L\sinh\rho\sin\theta_1\sin\theta_2\cdots\cos\theta_{d-2} \\
X_{d-1} =& L\sinh\rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{d-2}\ ,
\end{aligned} \tag{1.15}$$

that gives the metric

$$ds^2 = L^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{d-2}\right)\ , \tag{1.16}$$

with $d\Omega_{d-2}$ the line element of S^{d-2} (a $d-2$-dimensional sphere). This metric with $\rho \in \mathbb{R}^+$ and $\tau \in [0,2\pi]$ covers the Minkowski hyperboloid exactly once, but we have a periodic time coordinate, giving closed timelike curves (that is, if we take all the other coordinates to be constant, as $\tau \to \tau + 2\pi$ we get back to the same spacetime point following a timelike path). This sounds quite peculiar, since an observer standing still would, after a while, get back at the same point in time, giving room to all sort of paradoxes. However, this is not a property of the spacetime itself but merely a consequence of our choice of the solution. The anti-de Sitter spacetime is defined as the *universal cover* of this, where we take $\tau \in \mathbb{R}$, and there are no closed timelike curves. Since these coordinates cover the entire spacetime, we call them *global*.

In order to better understand the topology of AdS$_d$ we can make the coordinate transformation $\rho = \operatorname{arcsinh}\tan\theta$ with $\theta \in [0,\pi/2]$ (for $d>2$). In these coordinates the line element becomes

$$ds^2 = \frac{L^2}{\cos^2\theta}\left(d\theta^2 - d\tau^2 + \sin^2\theta d\Omega^2_{d-2}\right) \tag{1.17}$$

that is topologically equivalent to a cylinder, where θ is the radial direction, τ the longitudinal one, and each point on this cylinder is a S^{d-2} sphere. The metric is multiplied by a θ dependent conformal factor. The interesting feature of this metric is that it shows that at infinity, corresponding to $\theta = \pi/2$ in these coordinates, the AdS spacetime presents a timelike boundary[2] (a surface is called timelike if the vector normal to the surface is everywhere spacelike).

There is another convenient set of coordinates for the anti de-Sitter spacetime, that is the one we will mostly use in the next chapters, given by a $d-1$ Lorentz (co)vector x_μ and a coordinate $r \in \mathbb{R}^+ \cup \infty$:

$$\begin{aligned}
X_0 =& Lrt \\
X_i =& Lrx_i \qquad 1 \leq i < d \\
X_{-1} =& \frac{1}{2r}\left(1 + r^2\left(L^2 + t^2 - \sum_i x_i^2\right)\right) \\
X_{d-1} =& \frac{1}{2r}\left(1 - r^2\left(L^2 + t^2 - \sum_i x_i^2\right)\right)\ ,
\end{aligned} \tag{1.18}$$

which gives a metric

$$ds^2 = L^2\left(\frac{dr^2}{r^2} + r^2(\eta_{\mu\nu}dx^\mu dx^\nu)\right)\ , \tag{1.19}$$

[2]more precisely, it is a **conformal boundary**, i.e. a boundary of the conformally equivalent metric ds^2/r^2, but the adjective conformal is often dropped

called **Poincaré patch**.

What we see from this form of the metric is that spacetime slices of constant r are isomorphic to $(d-1)$-dimensional Minkowski spacetime, making it a suitable choice to describe a boundary field theory living in Minkowski spacetime. It is interesting to notice that the Minkowski metric is multiplied by r^2, that is sometimes called the *warp factor*. This implies that moving along the radial direction, is equivalent to rescaling all the lengths in the Minkowski slice by a factor of r. As we will see in more details later, this property of the AdS spacetime gives an intuition on how to interpret the extra dimension from the point of view of the boundary quantum field theory. Moving from the boundary to the interior, the length scale in the Minkowski spacetime becomes increasingly larger (equivalently the energy scale becomes smaller and smaller), so that one can think of moving through the extra dimension in the bulk as moving through the energy scale of the theory. This intuitive interpretation turns out to be correct, in gauge/gravity duality the radial extra dimension geometrically encodes the renormalization group flow of the boundary theory.

While the boundary is the surface at $r=\infty$, the point at $r=0$ is a horizon, since it does not correspond to a singularity, but to a surface where the Killing vector ∂_t has norm equal to zero. It is important to be aware of the fact that this choice of coordinates covers only half of the hyperboloid. Relating it to the global coordinates (1.15) shows that the Poincaré patch covers the colored region in figure 1.1.

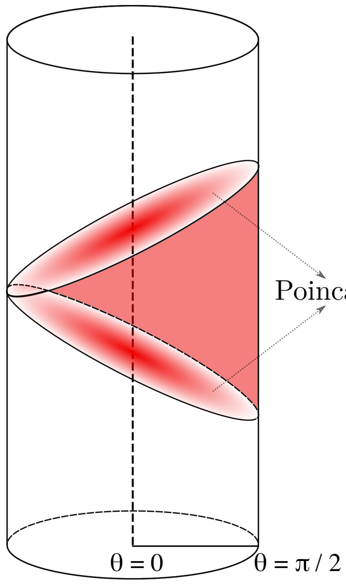


Figure 1.1: Topology of the AdS spacetime. Each point represents a $d-2$ dimensional sphere. The colored region is the Poincaré patch, that covers only portion of the spacetime.

There are alternative forms of the Poincaré patch, related to (1.19) by a redefinition of the radial coordinate. Two commonly used are $r = 1/z$ and $r = e^\rho$ that give respectively

$$ds^2 = L^2 \left(\frac{dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{z^2} \right) \tag{1.20}$$

and

$$ds^2 = L^2 \left(d\rho^2 + e^{2\rho} \eta_{\mu\nu} dx^\mu dx^\nu \right) \tag{1.21}$$

where the boundary is mapped to $z=0$ and $\rho=\infty$ and the horizon at $z=\infty$, $\rho=-\infty$ (the first one in particular is the preferred form for numerical calculations).

The Minkowski boundary of AdS inherits the symmetries of the spacetime, that is the group $SO(2,d-1)$. If we want to define a field theory on this boundary, it has to respect the symmetry of the spacetime, but as we previously mentioned, $SO(2,d-1)$ is the same symmetry group of a $d-1$ conformal field theory. Therefore, the quantum field theory living on the boundary has to be invariant under conformal transformation, in other words, it must be a conformal field theory. This argument of matching the symmetries, although certainly not a proof of the relation of the duality, plays an important role in a bottom-up approach to holography. Requiring for consistency that symmetries of the boundary theory are related to symmetries in the bulk, gives us important hints on how to match operators of the quantum field theory with the corresponding classical fields in the bulk, providing a sort of "dictionary" to translate between the two theories.

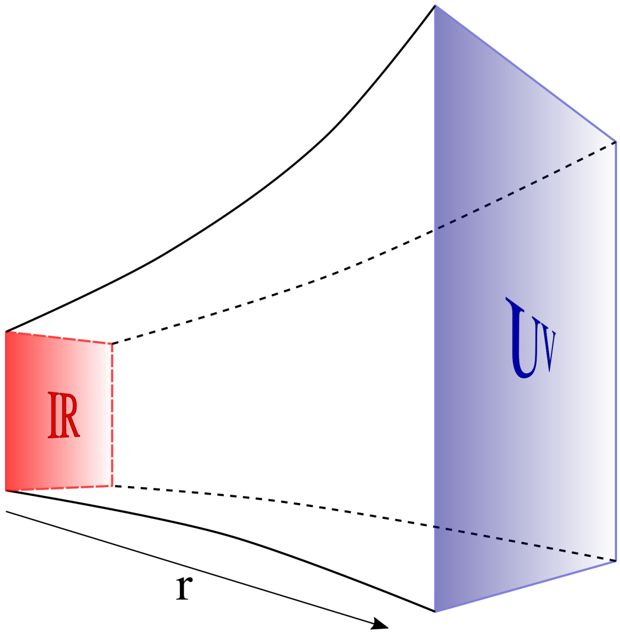


Figure 1.2: Pictorial representation of the extra dimension of the bulk gravitational theory as an energy scale. The deep interior determines the infrared behavior of the dual field theory, while moving towards the boundary at $r = \infty$ is equivalent to probe increasingly higher energies.

1.3 Conformal field theories

A conformal field theory is, of course, a theory that is invariant under **conformal transformations**. In general, a conformal transformation of coordinates is an invertible map, that leaves the metric invariant up to a local rescaling,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu} \ . \tag{1.22}$$

Notice that the set of conformal symmetries contains isometries, corresponding to $\Lambda(x) = 1$, as well as **scale transformations** corresponding to $\Lambda(x) = C$, where C is a generic constant. The field theories we are interested in live in Minkowski spacetime and the metric in (1.22) is then just $\eta_{\mu\nu}$, where the isometries are given by the usual Poincaré group, and the quantum field theory has the peculiar additional property of being scale invariant. The most commonly studied quantum theories that present scale invariance, are also invariant under the full conformal group. However, conformal invariance is a stronger requirement, as it requires invariance under *special conformal transformation*. The full set of conformal transformation is:

- Translation: $x'^{\mu} = x^{\mu} + a^{\mu}$
- Rigid rotations: $x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$
- Dilatations: $x'^{\mu} = C x^{\mu}$
- Special Conformal transformation: $x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2 b_{\mu} x^{\mu} + b^2 x^2}$

From this we can deduce the number of generators in d dimensions. As we noticed before, there are d generators of translation and $d(d-1)/2$ generators of rotations, for dilatations the number of generators is clearly one, while for the special conformal transformation we have again d generators for a total of $(d+1)(d+2)/2$ that is exactly the number of isometries of a AdS_{d+1} spacetime.

In order to see why the boundary of the AdS is invariant under conformal transformations it is useful to introduce the *inversion* I, a transformation defined by

$$I : x^\mu \to x'^\mu = \frac{x^\mu}{x^2} \ . \tag{1.23}$$

The relevance of inversion is due to the fact that a special conformal transformation is equivalent to first performing an inversion, then a translation, followed by another inversion (notice however, that this does not imply that inversion is necessarily a symmetry of the theory). If we look again at the boundary of AdS in Poincaré coordinates, it is clear that it possesses translational and rotational invariance (here we are referring to rotations in Minkowski spacetime by including rotations of the time dimension), as well as scaling symmetry

$$x^\mu \to \lambda x^\mu, \qquad r \to \frac{r}{\lambda} \ . \tag{1.24}$$

The more subtle one to check is invariance under special conformal transformations. However, from the form of the metric (1.20), we can easily verify that Poincaré coordinates are invariant under

$$x^\mu \to \frac{r^2 x^\mu}{1 + r^2 \eta_{\mu\nu} x^\mu x^\nu}, \qquad z \to \frac{r}{1 + r^2 \eta_{\mu\nu} x^\mu x^\nu} \ , \tag{1.25}$$

that on the boundary $r = \infty$ reduces to inversion symmetry $x^\mu \to \frac{x^\mu}{x^2}$. We therefore see that the AdS boundary is indeed invariant under the full conformal group.

All these symmetries impose significant restrictions on the form of the correlation function of the field theory. in particular, in a scale invariant theory an operator $O(x)$ transform under dilatation $x \to x' = \lambda x$ as

$$O(x) \to O'(x') = \lambda^{-\Delta} O(x) \ , \tag{1.26}$$

where Δ is called **scaling** or **conformal dimension**. Using the invariance under dilatations, the two-point function of two scalar conformal operators O_1 and O_2 with scaling dimensions Δ_1 and Δ_2 transform as

$$\langle O_1(\lambda x) O_2(\lambda x) \rangle = \lambda^{-\Delta_1} \lambda^{-\Delta_2} \langle O_1(x) O_2(x) \rangle \ . \tag{1.27}$$

Moreover, due to Poincaré invariance, the two-point function $\langle O_1(x_1) O_2(x_2) \rangle$ can only depend on $(x_1 - x_2)^2$, and we obtain that

$$\langle O_1(x_1) O_2(x_2) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \tag{1.28}$$

The full conformal invariance imposes additional constraints, in particular, by applying an inversion, we see that the correlation function is zero unless both fields have the same scaling dimension.

Having quickly reviewed some of the basics of the anti-de Sitter spacetime and conformal field theories, we now illustrate how the duality relates Green's function on the boundary theory with the bulk gravitational theory.

1.4 GKPW rule

In quantum field theory in the path integral formalism, the most relevant object is the **partition function** Z_{QFT}, that contains all the physical information of interest.

In order to compute an n-point function for a combination of local operators $O_i(x)$, the trick is to perturb the system with a source $J_i(x)$ that couples linearly to the corresponding operator, so that the Lagrangian becomes

$$\mathcal{L} \to \mathcal{L} + \sum_j J_j(x) O_j(x) \ . \tag{1.29}$$

With the inclusion of the source term, the partition function acts as a generating functional for the theory

$$Z_{\text{QFT}}[J] = \left\langle e^{i\int d^dx\, J_i(x)O^i(x)} \right\rangle \ , \tag{1.30}$$

and we can extract all n-point connected correlation functions by successive differentiation of the logarithm of Z_{QFT} and then take the limit of vanishing sources at the end of calculations:

$$\langle O_1(x_1)O_2(x_2)\cdots O_n(x_n)\rangle = \frac{1}{i^n}\frac{\delta^n}{\delta J_1(x_1)\delta J_1(x_2)\cdots\delta J_n(x_n)}\ln Z_{\text{QFT}}\bigg|_{J=0} \ . \tag{1.31}$$

The Gubser-Klebanov-Witten rule explains how the generating functional of the field theory is encoded in the boundary theory. By means of the correspondence, the QFT partition function is equal to the partition function on the gravity side, however, how is a source represented in the bulk?
The key contribution of GKPW was to realize that the source in the field theory should be encoded by a field in the gravitational theory, that reduces to the corresponding source term on the boundary of the (asymptotically) AdS spacetime. Calling $\phi(x)$ the bulk field dual to the operator $O(x)$ sourced by $J(x)$ the GKPW rule states:

$$Z_{\text{QFT}}[J(x)] = \int \mathcal{D}\phi\, e^{iS_{\text{bulk}}[\phi(x,r)]}\Big|_{\phi(x,r=\infty)=J(x)} \ , \tag{1.32}$$

and we see that $J(x)$ acts as a boundary condition for the field $\phi(x,r)$.

The reason we used the subscript QFT and we mention *asymptotically* AdS geometry, is that the validity of the rule is not restricted to pure AdS spacetime dual to CFT, but is valid for spacetime geometries that are asymptotically anti-de Sitter, corresponding to more general field theories where conformal invariance may be broken (an example is the holographic superconductor we present in chapter 3).

The power of (1.32) is that, thanks to the strong-weak nature of the duality in the large-N limit mentioned in section 1.1, it enables us to compute n-point functions for a strongly correlated field theories, by means of simple (at least in principle) functional differentiation of classical fields in a gravitational action. Moreover, in contrast with numerical methods used to study strongly correlated systems such as the quantum Monte Carlo method, that only work with the Euclidean formalism[3], gauge/gravity duality provides a tool for computing physical quantities in the real-time formalism, avoiding possible difficulties in performing a Wick rotation from imaginary time to obtain physically relevant results (remember that the dynamical properties measured in an experiment correspond, of course, to quantities in the real-time formalism). Nonetheless, we have to be a bit more careful with boundary conditions when working in real-time formalism in AdS/CFT, as we will show in more detail in chapter 2, where we give examples of computations in both real and imaginary time.

It is also interesting to notice that the GKPW rule is consistent with the interpretation of the radial dimension as the energy scale of the theory. According to (1.32), the bulk fields on the boundary, corresponding to the UV limit, give rise to the operators $O_i(x)$ that are, in fact, the bare UV operators of the theory.

1.5 Holographic dictionary

Now that we have a mathematical rule explaining how to compute Green's function from holography, we would like to understand how to pair operators we want to consider in our field theory with the corresponding classical field in the bulk. There is no general procedure, and sometimes in a bottom-up approach, it is a matter of (smart) trials and errors. However, symmetries here play a fundamental role, as they give important restrictions on the form of the dual fields.

[3]numerical methods uses the Euclidean formalism because the path integral in the real-time formalism is not a well defined mathematical entity, and for numerical methods one needs a well defined convergent integral. In other words, in real-time formalism the exponential is complex valued and it cannot be used as a real positive definite probability distribution.

The first operator we should consider is the energy-momentum tensor $T_{\mu\nu}$ as it is always present as a generator of translation symmetries. The corresponding conservation law is $\partial_\mu T^{\mu\nu} = 0$. In a conformal field theory, we also have symmetry under scaling. We can easily derive the conservation law related to this symmetry: using the definition of the energy-momentum tensor (in Minkowski spacetime) $T^{\mu\nu} \propto \frac{\delta S}{\delta \eta_{\mu\nu}}$ and for the action to be conserved under a variation $\delta\eta_{\mu\nu}$ we have

$$0 = \delta S = \int \frac{\delta S}{\delta \eta_{\mu\nu}} \delta\eta_{\mu\nu} \ , \tag{1.33}$$

for an infinitesimal scale transformation $\delta\eta_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$, with Ω any constant, (1.34) gives

$$0 = \int \frac{\delta S}{\delta \eta_{\mu\nu}} \Omega^2 \eta_{\mu\nu} \qquad \forall\, \Omega \ . \tag{1.34}$$

Scaling symmetry therefore implies tracelessness of the energy-momentum tensor $\eta_{\mu\nu} T^{\mu\nu} = 0$ (more precisely, at the quantum level, $\eta_{\mu\nu} \langle T^{\mu\nu} \rangle = 0$). In summary, we have that Lorentz invariance projects out the spin-1 part, while scaling invariance project out the spin-0 part, and thus the energy-momentum tensor has the degrees of freedom of a pure spin-2 operator. Since the boundary operator is just the asymptotic behavior of the bulk field, it has to match the symmetries of the operator. Together with the fact that the scaling dimension of the stress tensor $\Delta = d$ implies that the bulk field is massless (more on this in the next chapter), we can deduce the dual field in the bulk. The only consistent spin-2 massless field is the graviton

$$T_{\mu\nu} \longleftrightarrow g_{\mu\nu} \ . \tag{1.35}$$

One apparent subtlety is that that $g_{\mu\nu}$ is defined in on a $d+1$-dimensional manifolds, while the energy-momentum tensor in a spacetime with only d-dimension, so the indices of $g_{\mu\nu}$ do not match with the one of $T_{\mu\nu}$ as the naively written relation (1.35) seems to imply. Here it is important to remember that $g_{\mu\nu}$ has gauge degrees of freedom, these extra degrees of freedom are responsible for the non-matching of the indices, but we can always choose a gauge where $g_{r\mu} = 0$ with $\mu = 0, 1, \cdots, r$ (and in application of AdS/CFT, this is in fact almost always the most convenient choice).

In a similar fashion, if we want to introduce a global conserved current in the field theory $\partial_\mu J^\mu = 0$, that has the degrees of freedom of a pure spin-1 operator, we see that the corresponding field in the bulk is a massless spin-1 vector field A_μ with local gauge symmetry, that we may use to set $A_r = 0$. For a scalar operator in the field theory, it is then easy to predict that the dual field should be a scalar field.

From the examples above we can notice another property that is general to AdS/CFT: global symmetries in the boundary theory are mapped to local gauge symmetries in the bulk. We summarize in the box below the dictionary rule we learned up to now, along with one we will clarify in the next chapter.

CFT partition function	$\longleftrightarrow$	Gravitational partition function
Scalar operator $O(x)$	$\longleftrightarrow$	Scalar field $\phi(x, r)$
Global conserved current $J^\mu(x)$	$\longleftrightarrow$	Maxwell field $A_\mu(x, r)$
Energy-momentum tensor $T^{\mu\nu}(x)$	$\longleftrightarrow$	Metric field $g_{\mu\nu}(x, r)$
Conformal dimension	$\longleftrightarrow$	Mass of the corresponding field

1.6 Remarks on application to condensed matter

The strong/weak nature of the correspondence surely sounds very exciting for condensed-matter physics, where a prescription to study strongly correlated quantum theories is still missing, but we also notice that the quantum field theory side of the duality has a high number of symmetries, and in addition, in the tractable limit of classical gravity, we need to consider a very large number of degrees of freedom N. These are certainly not very common properties of realistic condensed-matter systems. In condensed matter we are usually interested in systems where supersymmetry is already explicitly broken by a finite temperature, a finite chemical potential etc. However, as we will see in chapter 3 this can be achieved by changing the geometry of the spacetime in the deep interior to break conformal invariance in the low-energy limit, while requiring that the spacetime geometry remains asymptotically AdS (i.e., the theory remains supersymmetric in the UV limit).

The reason why it does make sense to study condensed-matter system using the holographic approach is that we are interested in highly phenomenological theories that describe the emergent properties at low energy, that might be independent of the details of the theory in the ultraviolet (UV *independence*) and on the number of degrees of freedom N. How reliable these assumptions are is still an open question, nonetheless, they have been tested on several systems where the infrared behavior of the strongly interacting field theory is known and provided a solid qualitative description.

Chapter 2

Examples of the AdS/CFT correspondence

In this chapter we present example calculations of two-point functions for conformal operators using the GKPW rule of AdS/CFT. We start with a massive scalar field and show the relation between the mass of the bulk field and the conformal dimension of the corresponding operator on the boundary of the AdS spacetime. We give details of the calculations both in Euclidean and Minkowski signature, that allow us to introduce one of the subtleties encountered in computing real-time correlation functions. We then move on to the slightly more complicated example of a massless bulk gauge field. The aim of these examples is to guide the reader through the general steps necessary to calculate Green's function using the GKPW rule, as well as analyse general features of the asymptotic behavior of these two types of fields, that appear in the holographic superconductor model (as we will explain in more detail later, the holographic superconductor consists of a gauge field minimally coupled to a complex scalar field and to gravity, in a different, but asymptotically AdS, background metric).

2.1 Scalar field correlation function

In order to illustrate the procedure of computing the two point function from AdS/CFT we consider the simple toy model of a massive scalar field $\phi(z,x)$ in AdS$_{d+1}$ dual to a scalar operator $O(x)$

$$S = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\left(\partial_{\mathrm{M}}\phi\partial^{\mathrm{M}}\phi + m^2\phi^2\right)\ . \tag{2.1}$$

It is convenient to work in Poincaré coordinates where the metric takes the form

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right)\ , \tag{2.2}$$

with the boundary at $z = 0$ and the Poincaré horizon at infinity.

To avoid confusion, in this chapter we adopt the convention of using capital Latin letter M,N,... as indices of the full AdS$_{d+1}$ spacetime coordinates, while small Greek letters $\mu,\nu,\ldots$ represent coordinates of a d-dimensional spacetime slice transverse to the radial direction z, i.e., $x^{\mathrm{M}} = (z, x^\mu)$. In the next chapters, for convenience, we will abandon this convention as the range of the indices should be clear from the context.

2.1.1 Conformal dimension and mass relation

Upon variation of the action (2.1) up to first order we find

$$\begin{aligned}\delta S^{(1)} &= -\int d^{d+1}x\,\sqrt{-g}\left(\partial_{\mathrm{M}}\delta\phi\partial^{\mathrm{M}}\phi + m^2\phi\delta\phi\right) = \int d^{d+1}x\,\sqrt{-g}\delta\phi(\Box - m^2)\phi - \int d^{d+1}x\partial^{\mathrm{M}}(\sqrt{-g}\delta\phi\partial_{\mathrm{M}}\phi)\\ &= \int d^{d+1}x\,\sqrt{-g}\delta\phi(\Box - m^2)\phi - \int d^dx\,\sqrt{-h}n^{\mathrm{M}}\delta\phi\partial_{\mathrm{M}}\phi\ ,\end{aligned} \tag{2.3}$$

where in the last step we used Stoke's theorem and the box operator is defined by $\Box \equiv \nabla_\mu \nabla^\mu$. From the above equation we can read the equation of motion of the scalar field

$$\left(\Box - m^2\right)\phi = \frac{1}{\sqrt{-g}}\partial_{\rm M}\left(\sqrt{-g}g^{\rm MN}\partial_{\rm N\phi}\right) = 0 \ , \tag{2.4}$$

that using the explicit form of the metric (2.2) reads

$$\frac{z^{d+1}}{L^2}\partial_z\left(z^{-(d-1)}\partial_z\phi(z,k^\mu)\right) + \frac{z^2}{L^2}\eta_{\mu\nu}\partial^\mu\partial^\nu\phi(z,k^\mu) - m^2\phi(z,k^\mu) = 0 \ . \tag{2.5}$$

Fourier transforming the bulk field $\phi(z,x)$ only with respect to the spacetime coordinates of the d-dimensional boundary theory

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d}\,\phi(z,k)e^{ik_\mu x^\mu} \tag{2.6}$$

with $k^\mu = \left(\omega, k^i\right)$ we obtain

$$z^2\partial_z^2\phi(z,k^\mu) - z(d-1)\partial_z\phi(z,k^\mu) - (m^2L^2 + k_\mu k^\mu z^2)\phi(z,k^\mu) = 0 \ . \tag{2.7}$$

This equation can be exactly solved, but let us first analyze its asymptotic behavior. In the limit $z \to 0$, the term $z^2k_\mu k^\mu\phi$ in (2.7) can be neglected and the differential equation simply becomes:

$$z^2\partial_z^2\phi(z,k^\mu) - z(d-1)\partial_z\phi(z,k^\mu) - m^2L^2\phi(z,k^\mu) = 0 \quad \text{for} \quad z \to 0 \tag{2.8}$$

Using the ansatz $\phi(z,k^\mu) = \phi(k^\mu)z^\Delta$ we find an equation for Δ

$$\Delta(\Delta-1) - (d-1)\Delta - m^2L^2 = 0 \tag{2.9}$$

that gives the two roots

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \equiv \frac{d}{2} \pm \nu \ , \tag{2.10}$$

and the asymptotic solution is then

$$\phi(z,k^\mu) = \phi_0(k)\left(\frac{z}{L}\right)^{\Delta_-} + \phi_+(k)\left(\frac{z}{L}\right)^{\Delta_+} + \cdots \ , \tag{2.11}$$

where the dots stands for higher order terms in z. Moreover, since in the limit we dropped the k^2 term in (2.7), this asymptotic solution can be straightforwardly reformulated in position space

$$\phi(z,x^\mu) = \phi_0(x)\left(\frac{z}{L}\right)^{\Delta_-} + \phi_+(x)\left(\frac{z}{L}\right)^{\Delta_+} + \cdots \ . \tag{2.12}$$

Usually, ϕ_0 is the *non-normalizable* term, while $\phi_+$is the *normalizable* one. However this is not always the case so let's clarify a bit what we mean by *normalizable*. The inner product for two scalar field in curved spacetime is defined as

$$\langle\phi_1,\phi_2\rangle = -i\int_{\Sigma_t} dzd^{d-1}x\,\sqrt{-g}g^{tt}\left(\phi_1\partial_t\phi_2 - \phi_2\partial_t\phi_1\right) \ , \tag{2.13}$$

with Σ_t a constant time surface. Neglecting the dependence on the other variables, let us consider a solution of the form $\phi(x) \sim z^\Delta$, then the integrand in (2.13) goes as $z^{2\Delta-d+1}$, and the integral is then convergent for $2\Delta - d + 1 > -1 \to \Delta > d/2 - 1$. The solution $\propto z^{\Delta_+}$ is then always normalizable since $\Delta_+ = d/2 + \nu$ with ν real and positive (we will see shortly why this is the case). On the other hand, there is a small window where the norm of what we called non-normalizable solution is actually finite, that is for $\nu < 1$, so in summary we have:

z^{Δ_+} mode: always normalizable

$$z^{\Delta_-} \text{ mode: } \begin{cases} \text{normalizable} & 0 \le \nu < 1 \\ \text{non-normalizable} & \nu \ge 1 \end{cases}$$

The non-normalizable mode defines the boundary value of the field

$$\phi_0(x) \equiv \lim_{z\to 0} z^{-\Delta_-}\phi(z,x) \ . \tag{2.14}$$

As explained in the previous chapter, the GKPW rule identifies the boundary value of the field with the source term of the associated operator. This implies that the term we add to the lagrangian in the partition function of the conformal field theory is

$$\int d^dx\, \phi_0(x)O(x) = \int d^dx \lim_{z\to 0} \phi(z,x)z^{-\Delta_-}O(x) \ . \tag{2.15}$$

Equation (2.15) identifies Δ_+ as the scaling dimension of the boundary operator $O(x)$. Recalling that the conformal dimension of an operator is implicitly defined by its behavior under a global rescaling

$$x^\mu \to x'^\mu = \lambda x^\mu \qquad O(x) \to O'(x') = \lambda^{-\Delta}O(x) \ , \tag{2.16}$$

we can use the bulk isometry of the AdS spacetime

$$x^\mu \to x'^\mu = \lambda x^\mu \ , \qquad z \to z' = \lambda z \ , \tag{2.17}$$

and the fact that the boundary action should be conformal invariant

$$\int d^dx\, \phi_0(x)O(x) = \int d^dx'\, \phi_0(x')O'(x') \tag{2.18}$$

to see that:

$$\begin{aligned} \int d^dx'\, \phi_0(x')O(x') = \int d^dx' \lim_{z'\to 0} z'^{\Delta_-}\phi'(z',x')O'(x') = \int d^d(\lambda x) \lim_{z\to 0}(\lambda z)^{\Delta_-}\phi(z,x)O'(x') \\ = \lambda^{\Delta_+}\int d^dx\, \phi_0(x)\lambda^{-\Delta}O(x) \ , \end{aligned} \tag{2.19}$$

where we used the fact that $\phi(z,x)$ is a scalar and is therefore invariant under a change of coordinates, and in the last equality $\Delta_+ = d - \Delta_-$. In order for (2.18) to hold the conformal dimension Δ of $O(x)$ must then be

$$\Delta = \Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2L^2} \ . \tag{2.20}$$

Hence, we just showed that the mass of the bulk field determines the scaling behavior of the operator in the conformal field theory. In particular, we have the following three possibilities for a scalar operator:

- $m^2 = 0$, $\Delta = d \Rightarrow O(x)$ is a marginal operator;
- $m^2 > 0$, $\Delta > d \Rightarrow O(x)$ is an irrelevant operator;
- $m^2 < 0$, $\Delta < d \Rightarrow O(x)$ is a relevant operator.

There is an interesting thing to notice from this *mass-dimension* relation. The interpretation of Δ_+ as a conformal dimension implies that it must be a real number. From the definition (2.10) we find that the reality condition implies

$$m^2 \ge -\frac{d^2}{4L^2} \ , \tag{2.21}$$

that is, the mass squared can assume slightly negative values. Not only are negative values allowed, but we have just seen that in order to have a relevant scalar operator in the boundary theory we must choose negative values (relevant operators are the one of interest in the low energy, long wavelength limit).

This may sound strange as negative mass squared are usually associated with tachyonic instabilities. This is true in flat spacetime, where for a scalar QFT

$$\partial^2\phi - m^2\phi = 0$$

and the second derivative of the potential around the vacuum expectation value is $V''(\phi_v) = m^2 < 0$, giving an "upside down" potential, i.e., perturbation around the vacuum solution are unstable. However, this is not true anymore in AdS, where the curvature of the spacetime can compensate for small negative values of m^2. A simple way to see this, it to consider the action (2.1) for a scalar field $\phi = \phi(z)$, that becomes

$$S = -\frac{1}{2}\int dz d^d x \, \frac{1}{z^{d+1}} \left(z^2\partial_z\phi\partial_z\phi + m^2L^2\phi^2\right) \ . \tag{2.22}$$

redefining the field $\phi = z^{d/2}\varphi$ and making a change of variable, $y = \ln z$

$$S = -\frac{1}{2}\int dy d^d x \, \left(\partial_y\varphi\partial_y\varphi + \left(\frac{d^2}{4} + m^2L^2\right)\varphi^2\right) \ , \tag{2.23}$$

with metric

$$ds^2 = L^2\left(dy^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \ . \tag{2.24}$$

We can interpret the action for the scalar field ϕ as an action in flat spacetime with effective potential $V_{\text{eff}} = d^2/4m^2L^2$, that is stable as long as (2.21), known as the *Breitenlohner-Freedman bound*, is satisfied.

It is interesting now to notice that when the Breitenlohner-Freedman bound is saturated, the boundary operator has the lowest allowed conformal dimension $\Delta = d/2$. In conformal field theories however, the *unitary bound*, that is the minimum scaling dimension of the conformal scalar operator, is $\Delta > d/2 - 1$, so how can we describe operator with conformal dimensions in the range $d/2 - 1 < \Delta < d/2$? Well, we have seen this value $\Delta = d/2 - 1$ already, it happens to be the lower limit of the interval where both fields are normalizable. In this range, the association of the non-normalizable mode with the source on the boundary is now a bit ambiguous. In this range, we can indeed choose which term corresponds to the source, and which to the one-point function. Of course the choice has deep implications on the boundary theory we describe, since choosing a different term as the source corresponds to a theory of operators with a different conformal dimension. In particular, the choice of keeping the usual $\propto z^{\Delta_-}$ term as the source is called *standard quantization*. In *alternative quantization*, the $\propto z^{\Delta_+}$ solution act as a source and the conformal dimension of the operator is determined by Δ_-, that is bounded below exactly by the unitary bound.

2.1.2 Expectation value and two-point function

Now that we understand the boundary expansion, we can use the GKPW rule (1.32) to compute the correlations function of the operator $O(x)$. Here we use standard quantization, therefore we identify the source with the coefficient of the leading term in the boundary expansion $\phi_0(x)$, and the GKPW rule then reads

$$Z_{CFT}[\phi_0(x)] = Z_{bulk}\left[\lim_{z\to 0} z^{-\Delta_+}\phi(z,x) = \phi_0(x)\right] \ , \tag{2.25}$$

where Z_{bulk} in the semi-classical limit is defined by the path integral

$$Z_{bulk} = \int_{\lim_{z\to 0} z^{-\Delta_+}\phi(z,x)=\phi_0} D\phi \, e^{iS[\phi]} \ , \tag{2.26}$$

with $S[\phi]$ the scalar action (2.1). In this limit we can compute the leading order partition function using the saddle point approximation:

$$Z_{bulk} = \exp\left(iS[\phi_c]\right) \ , \tag{2.27}$$

with ϕ_c the classical solution to the equation of motion (2.7) that satisfies the boundary conditions. From the expansion of the action (2.1) up to first order we see that the bulk term vanish on the classical solution (obviously, since the classical solution is defined by the vanishing of the first order bulk term) and we are left with the contribution of the boundary term

$$S[\phi(z,x)_c] = -\int d^d x \sqrt{h} n^{\mathrm{M}} \phi_{(c)} \partial_{\mathrm{M}} \phi_{(c)} \Big|_{z=0}^{z=\infty} , \tag{2.28}$$

where the subscript c is a reminder that we are evaluating the action on the classical solution. The metric $h_{\mu\nu} = L^2/z^2 \eta_{\mu\nu}$ is the boundary metric and $n^{\mathrm{M}} = \sqrt{g^{zz}} \delta_z^{\mathrm{M}}$ is the unit vector orthogonal to the boundary pointing outwards, so that the action becomes

$$S[\phi(z,x)_c] = -\int d^d x \left(\frac{L}{z}\right)^d \phi_{(c)} \partial_z \phi_{(c)} \Big|_{z=0}^{z=\infty} , \tag{2.29}$$

and Fourier transforming to momentum space

$$S[\phi(z,k)_c] = -\int \frac{d^d k}{(2\pi)^d} \left(\frac{L}{z}\right)^{d-1} \phi_{(c)}(k) \partial_z \phi_{(c)}(-k) \Big|_{z=0}^{z=\infty} . \tag{2.30}$$

In order for the solution to be regular at infinity, we must impose $\phi(z=\infty, x) = 0$ and the action vanishes at infinity, therefore we are left with evaluating (2.30) at the boundary. However, it seems that we have a problem now, since we know from the boundary expansion (2.12) derived in the previous section that one of the term diverges as $z \to 0$. On second thought, this divergence should not come as a surprise. Approaching the boundary we get closer and closer to the UV regime, and we already know from QFT that in the short wavelength limit we generally encounter divergences, and we need to regularize them. When we presented the GKPW rule in the previous chapter we did not address this issue, so to be precise, a more correct way to write the computational rule for connected n-point function in the large-N limit, for a generic field Φ sourced by $\phi(x)$, is

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \frac{\delta^n \log Z_{bulk}^{(R)}}{\delta\phi(x_1) \dots \delta\phi(x_n)} \Big|_{\phi=0} = \frac{\delta^n S[\Phi_c]^{(R)}}{\delta\phi(x_1) \dots \delta\phi(x_n)} \Big|_{\phi=0} \tag{2.31}$$

where the equality is between the renormalized partition function (identified by the superscript (R)) and the classical action after having applied the regularization procedure.

We now need a way to regularize the boundary action. As the radial direction encoded the energy scale, the standard way to proceed is to set a UV cutoff ϵ, that is, to evaluate the action at an infinitesimal distance from the boundary $z = \epsilon$ and then modify the theory by adding counterterms defined on the d-dimensional spacetime determined by $z = \epsilon$, that cancel the divergences. Since these are only boundary terms, they have no influence on the equation of motion, and at the end of the calculations, after having removed all the divergent terms, we can safely perform the limit $\epsilon \to 0$ to obtain the desired answer.

In the current example, following the procedure we find

$$\begin{aligned}
S[\phi(z,k)_c] &= \frac{1}{2} \lim_{\epsilon\to 0} \int \frac{d^d k}{(2\pi)^d} \left(\frac{L}{z}\right)^{d-1} \phi_{(c)}(k) \partial_z \phi_{(c)}(-k) \Big|_{z=\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \left(\frac{L}{z}\right)^{d-1} \left(\phi_0(k) \left(\frac{z}{L}\right)^{\Delta_-} + \phi_+(k) \left(\frac{z}{L}\right)^{\Delta_+} + \cdots \right) \partial_z \Bigg(\phi_0(-k) \left(\frac{z}{L}\right)^{\Delta_-} \\
&\qquad\qquad + \phi_+(-k) \left(\frac{z}{L}\right)^{\Delta_+} + \cdots \Bigg) \Bigg|_{z=\epsilon} \\
&= \frac{1}{2} \lim_{\epsilon\to 0} \int \frac{d^d k}{(2\pi)^d} L^{-1} \Bigg(\Delta_- \phi_0(k) \phi_0(-k) \left(\frac{z}{L}\right)^{-2\nu} + \Delta_- \phi_+(k) \phi_0(-k) \\
&\qquad\qquad + \Delta_+ \phi_+(-k) \phi_0(k) + \mathcal{O}(z^{2\nu}) \Bigg) \Bigg|_{z=\epsilon} ,
\end{aligned} \tag{2.32}$$

and we can clearly see the divergent term $\propto z^{-2\nu}$. It is not difficult to guess the the correct form of the countertem

$$\begin{aligned} S_{counter} =& - \lim_{\epsilon \to 0} \frac{1}{2L} \int \frac{d^dk}{(2\pi)^d} \, \Delta_- \sqrt{-h} \phi(k)\phi(-k) \Big|_{z=\epsilon} \\ =& - \lim_{\epsilon \to 0} \frac{L^{-1}}{2} \int \frac{d^dk}{(2\pi)^d} \left(\Delta_- \phi_0(k)\phi_0(-k) \left(\frac{z}{L}\right)^{-2\nu} + 2\Delta_- \phi_0(k)\phi_+(-k) + \mathcal{O}(z^{2\nu}) \right) \Big|_{z=\epsilon} . \end{aligned} \tag{2.33}$$

Our regularized boundary term then becomes

$$\begin{aligned} S^{(R)}[\phi(z,k)_c] =& \frac{L^{-1}}{2} \lim_{\epsilon \to 0} \int \frac{d^dk}{(2\pi)^d} \left(2\nu \phi_0(k)\phi_+(-k) + \mathcal{O}(z^{2\nu}) \right) \Big|_{z=\epsilon} \\ =& \frac{L^{-1}}{2} \int \frac{d^dk}{(2\pi)^d} \left(2\nu \phi_0(k)\phi_+(-k) \right) \end{aligned} \tag{2.34}$$

and we can compute the expectation value of $O(k)$:

$$\langle O(k) \rangle = (2\pi)^d \frac{\delta S[\phi(z,k)_c]}{\delta \phi_0(-k)} \Big|_{\phi_0 = 0} = \frac{2\nu \phi_+(k)}{L} . \tag{2.35}$$

The factor of L is not very relevant, as it could have simply be absorbed in a global normalization constant for the action (2.1), an in what follows we will therefore neglect it (or more precisely, we choose to measure z in units of L). From this result, we see that the expectation value of the corresponding operator is indeed determined by the subleading term in the boundary expansion $\langle O \rangle \propto \phi_+$. In general, the coefficient of the non-normalizable mode corresponds to the source of the boundary operator, while the normalizable mode is proportional to the expectation value. If we are in the range of mass values where both solutions are normalizable, which term corresponds to the source on the boundary depends on the quantization scheme chosen. In standard quantization for $\nu < 1$, leading and subleading term switch roles, and the source is then given by the coefficient of the subleading term.

Moreover, the two-point function is given by the linear response of the one-point function to the source[1] ϕ_0:

$$\langle O(k) \rangle = G(k)\phi_0(k) \tag{2.36}$$

and we then have

$$G(k) = \langle O(k) O(-k) \rangle = 2\nu \frac{\phi_+(k)}{\phi_0(k)} . \tag{2.37}$$

To find the explicit expression, now we just have to determine the value of the coefficients of the expansion by explicitly solving equation (2.7). This is usually very difficult or impossible analytically and one have to rely on numerical methods, but this simple toy model is exactly solvable. However, there is still one subtlety that we need to tackle, and is related to the solution in real-time formalism. What we did up to know is equally valid in Euclidean signature, where we send $\omega \to -i\omega_{\mathrm{E}}$, where things are a bit simpler. In the Euclidean formalism, (2.7) becomes

$$z^2 \partial_z^2 \phi(z, k_{\mathrm{E}}) - z(d-1)\partial_z \phi(z, k_{\mathrm{E}}) - (m^2 L^2 + \delta_{\mu\nu} k_{\mathrm{E}}^\mu k_{\mathrm{E}}^\nu z^2)\phi(z, k_{\mathrm{E}}) = 0 \tag{2.38}$$

with k_{E} the momentum in Euclidean signature. This is a Bessel equation with two independent solutions in terms of modified Bessel function (in units of L)

$$\phi(z, k_{\mathrm{E}}) = c^{(1)} z^{d/2} K_\nu(z, k_{\mathrm{E}}) + c^{(2)} z^{d/2} I_\nu(z, k_{\mathrm{E}}) . \tag{2.39}$$

[1] This is the reason why we do not have a factor of 1/2 in (2.35). The regularized boundary action has the form $\sim \frac{1}{2} \int \phi_0 G \phi_0$, and performing a functional derivative with respect to ϕ_0 gives a factor of 2 that cancels the 1/2.

One of the two initial conditions needed, is given by an overall normalization that drops out in the final answer, the other is a boundary condition in the interior. To fix it, we notice that the asymptotic behavior of the Bessel functions $I_\nu(x)$ and $K_\nu(x)$ for $x \to \infty$ is

$$I_\nu(x) \sim \frac{e^x}{\sqrt{x}} , \qquad K_\nu(x) \sim \frac{e^{-x}}{\sqrt{x}} \tag{2.40}$$

therefore, since $I_\nu(kz)$ diverges in the limit $z \to \infty$, we have to set $c^{(2)} = 0$ in order to ensure regularity at infinity, and we then have

$$\phi(z, k_{\mathrm{E}}) = C z^{d/2} K_\nu(k_{\mathrm{E}} z) , \tag{2.41}$$

Using the asymptotic expansion of $K_\nu(x)$ for $x \to 0$

$$K_\nu(x) = \frac{\Gamma(\nu)}{2}\left(\frac{x}{2}\right)^{-\nu}(1+\mathcal{O}(x^2)) + \frac{\Gamma(-\nu)}{2}\left(\frac{x}{2}\right)^{\nu}(1+\mathcal{O}(x^2)) \quad \text{for} \quad x \to 0 \tag{2.42}$$

we obtain

$$\phi(z, k_{\mathrm{E}}) = C z^{d/2}\left(\frac{\Gamma(\nu)}{2}\left(\frac{k_{\mathrm{E}} z}{2}\right)^{-\nu}(1+\mathcal{O}(z^2) + \frac{\Gamma(-\nu)}{2}\left(\frac{k_{\mathrm{E}} z}{2}\right)^{\nu}(1+\mathcal{O}(z^2))\right) \tag{2.43}$$

$$= C\left(\frac{\Gamma(\nu)}{2}\left(\frac{k_{\mathrm{E}}}{2}\right)^{-\nu} z^{\Delta_-} + \frac{\Gamma(-\nu)}{2}\left(\frac{k_{\mathrm{E}}}{2}\right)^{\nu} z^{\Delta_-}\right) \quad \text{for} \quad z \to 0 , \tag{2.44}$$

that is of the form of the boundary expansion (2.12). From (2.44) we extract the coefficients and we find

$$\langle O(k) O(-k) \rangle = 2\nu \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{k_{\mathrm{E}}}{2}\right)^{2\nu} . \tag{2.45}$$

What does change in the real-time formalism? The problem is that in real time the solution of equation (2.7) is given by Hankel functions (for $k^2 = -\omega^2 + k^2 < 0$, that is, for timelike k)

$$\phi(z, k) = z^{d/2} c^{(1)} H_\nu^{(1)}(-ikz) + z^{d/2} c^{(2)} H_\nu^{(2)}(-ikz) \tag{2.46}$$

that behaves for $z \to 0$ as

$$\begin{aligned} &H_\nu^{(1)}(x) \sim e^{-ix} \\ &H_\nu^{(2)}(x) \sim e^{ix} \qquad x \in \mathbb{R} . \end{aligned}$$

The problem is that now both solutions are regular in the interior and we have two possibilities, either choosing the *infalling boundary condition*, corresponding to $c^{(1)} = 0$, or the *outgoing boundary condition*, $c^{(2)} = 0$. The two possibilities reflects the two type of Green's functions in real-time formalism, *retarded Green's functions*, associated with the infalling boundary condition, and *advanced Green's functions* related to the outgoing condition. In real-time formalism we therefore need to be a bit more careful with boundary conditions and make a choice according to the correlation function we want to obtain. Usually we are interested in retarded Green's functions, and after an expansion of the Hankel function $H_\nu^{(2)}$ near the boundary, similar to what we did for the Bessel function we find that the retarded correlator is

$$\langle O(k) O(-k) \rangle_R = 2\nu e^{i\pi\nu \operatorname{sgn}(\omega)} \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{-ik}{2}\right)^{2\nu} . \tag{2.47}$$

2.2 Vector field two-point function

With a similar procedure, we can compute the two-point function for a vector field, this will let us introduce some feature that will be useful when considering the holographic superconductor model. In this section, we only derive the Green's function in Euclidean formalism, nonetheless, as we have seen for the scalar field, the computation in real-time formalism is very similar and just requires a bit of extra care with the boundary conditions.

The metric is then given by (in units where $L = 1$)

$$\frac{1}{z^2}\left(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu\right) \ . \tag{2.48}$$

The action for a vector field $A_{\rm M}$ with Euclidean signature is given by:

$$S_E = -\frac{1}{4}\int \mathrm{d}^{d+1}x\sqrt{g}F_{\rm MN}F^{\rm MN} \tag{2.49}$$

where $F_{\rm MN} = (\partial_{\rm M}A_{\rm N} - \partial_{\rm N}A_{\rm M})$, and we set $\mu_0 = 1$ for the vacuum permeability. The first order variation of the action then reads

$$\begin{aligned}
\delta S_E =& -\frac{1}{2}\int d^{d+1}x\,\sqrt{g}\delta F_{\rm MN}F^{\rm MN} = -\frac{1}{2}\int \mathrm{d}^{d+1}x\,\sqrt{g}(\partial_{\rm M}\delta A_{\rm N} - \partial_{\rm N}\delta A_{\rm M})F^{\rm MN} \\
=& -\int d^{d+1}x\,\sqrt{g}(\partial_{\rm M}\delta A_{\rm N})F^{\rm MN} = \int d^{d+1}x\delta A_{\rm N}\partial_{\rm M}(\sqrt{g}F^{\rm MN}) - \int d^{d+1}x\,\partial_{\rm M}(\sqrt{g}\delta A_{\rm N}F^{\rm MN}) \\
=& \int d^{d+1}x\,\delta A_{\rm N}\partial_{\rm M}(\sqrt{g}F^{\rm MN}) - \int \mathrm{d}^d x\,\sqrt{h}n_{\rm M}F^{\rm MN}\delta A_{\rm N}
\end{aligned} \tag{2.50}$$

where, as usual, in the last line we used Stoke's theorem. From (2.50) we can read off the equation of motion for the gauge field:

$$\partial_{\rm M}(\sqrt{g}F^{\rm MN}) = 0 \tag{2.51}$$

Plugging in the metric (2.48), we obtain:

$$\partial_{\rm M}\left(\left(\frac{1}{z}\right)^{d+1}\frac{z^4}{R^4}\delta^{\rm ML}\delta^{\rm NK}F_{\rm LK}\right) = \partial_z(z^{3-d}F_{z\rm N}) + z^{3-d}\partial_\mu(\delta^{\mu\rm L}F_{\rm LN}) = 0 \tag{2.52}$$

where we used $\sqrt{g} = (1/z^2)^{\frac{d+1}{2}} = z^{-d-1}$. Multiplying everything by z^{d-3} and rewriting the equation in term of the vector field $A_{\rm M}$ we obtain

$$z^{d-3}\partial_z(z^{3-d}(\partial_z A_{\rm N} - \partial_{\rm N}A_z)) + \partial_\mu(\delta^{\mu\rm L}(\partial_{\rm L}A_{\rm N} - \partial_{\rm N}A_{\rm L})) = 0 \tag{2.53}$$

Due to translational symmetry in the x^μ direction (i.e. the direction parallel to the boundary) we can Fourier transform the field

$$A_{\rm M}(z,\boldsymbol{x}) = \int \frac{d^dk}{(2\pi)^d}A_{\rm M}(z,k)e^{ik\cdot\boldsymbol{x}} \tag{2.54}$$

with $k\cdot\boldsymbol{x} = k_\mu x^\mu = \delta_{\mu\nu}k^\mu x^\nu$, and we obtain the set of equations:

$$ik^\nu(ik_\nu A_z(z,k) - \partial_z A_\nu(z,k)) = 0 \tag{2.55}$$

$$z^{d-3}\partial_z(z^{d-3}\partial_z A_\nu(z,k) - ik_\nu A_z(z,k)) - (k^\mu k_\mu A_\nu(z,k) - k^\mu k_\nu A_\mu(z,k)) = 0 \tag{2.56}$$

We now use gauge invariance to set $A_z = 0$ so that (2.55) and (2.56) become:

$$k^\nu\partial_z A_\nu(z,k) = 0 \tag{2.57}$$

$$z^{d-3}\partial_z(z^{d-3}\partial_z A_\nu(z,k)) - (k^\mu k_\mu A_\nu(z,k) - k^\mu k_\nu A_\mu(z,k)) = 0\ . \tag{2.58}$$

Again, we can fix the momentum in one of the spatial directions, let us call it x, so that $k = (\omega_{\rm E}, \tilde{k}, 0, \ldots, 0, 0)$. Calling τ the time component in Euclidean time, from the first equation we have that $A_\tau = -k/\omega A_x + C(\tilde{k}, \omega_{\rm E})$, but after setting $A_z = 0$, we still have freedom to choose a gauge in the k^μ direction, so that we can use this freedom to set $C(\tilde{k},\omega_{\rm E})=0$, that corresponds to making the gauge choice $k^\mu A_\mu(z,k)=0$ (the *Lorentz gauge*). Equation (2.58) then can be rewritten as

$$z^{d-3}\partial_z(z^{d-3}\partial_z A_\nu(z,k))-k^2A_\nu(z,k)=\partial_z^2A_\nu(z,k)+\frac{(3-d)}{z}\partial_zA_\nu(z,k)-k^2A_\nu(z,k)=0 \tag{2.59}$$

Therefore, we can find the classical solution by solving the second order differential equation:

$$zA''_\mu(z,k)+(3-d)A'_\mu(z,k)-zk^2A_\mu(z,k)=0 \tag{2.60}$$

where $A'_\mu(z,k)\equiv\partial_zA_\mu(z,k)$. But this equation looks exactly like the one for the scalar field, (2.7), with $m=0$ and a prefactor of $(d-3)$ instead of $(d-1)$ for the first order term. We then already know what the solutions look like. But let us again first analyze its asymptotic behavior, in order to understand the differences with the scalar case. Neglecting the the last term in (2.60) in the limit $z\to0$, the differential equation becomes:

$$zA''_\mu(z,k)+(3-d)A'_\mu(z,k)=0 \quad \text{for} \quad z\to0\ , \tag{2.61}$$

that again using a ansatz $\propto z^\Delta$ gives the simple solution:

$$A_\mu(z,k)=a_\mu(k)+b_\mu(k)z^{d-2} \quad \text{for} \quad z\to0 \tag{2.62}$$

For $d>2$, $a_\mu(k)$ is the leading term in the limit $z\to0$, and it corresponds to the non-normalizable mode, therefore it represents the source of the boundary current

$$\lim_{z\to0}A_\mu(z,k)=a_\mu(k) \tag{2.63}$$

with no power of z. On the other hand, $b(k)$ is the non-normalizable mode and for the gauge field we therefore expect

$$\langle J^\mu\rangle\sim b^\mu(k) \tag{2.64}$$

As a consistency check, we compute the conformal dimension of the conserved current J^μ to see if it matches with the expected value of $d-1$ (the fact that we expect $d-1$ can be easily seen in the J^0 component, that represent the *charge density.* Since the charge has no dimension, the dimension of J^0 is given by the density, that has dimension $d-1$ by definition). The conformal dimension of the current Δ is implicitly defined by:

$$x^\mu\to x'^\mu=\lambda x^\mu \qquad J^\mu(x)\to J'^\mu(x')=\lambda^{-\Delta}J^\mu(x) \tag{2.65}$$

using the conformal invariance of the boundary action

$$\int d^dx\,a_\mu(x)J^\mu(x)=\int d^dx'\,a'_\mu(x')J'^\mu(x') \tag{2.66}$$

and paying attention that now we are dealing with a vector field, $a_\mu(x)$, that under global scaling behaves as

$$x^\mu\to x'^\mu=\lambda x^\mu\Longrightarrow a_\mu(x)\to a'^\mu(x')=\lambda^{-1}a_\mu(x) \tag{2.67}$$

and from (2.66) we obtain

$$\int d^dx'\,a'_\mu(x')J'^\mu(x')=\int d^d(\lambda x)\,\lambda^{-1}a_\mu(x)\lambda^{-\Delta}J^\mu(x)=\int d^dx\,\lambda^{d-1-\Delta}a_\mu(x)J^\mu(x)=\int d^dx\,a_\mu(x)J^\mu(x) \tag{2.68}$$

that implies $\Delta=d-1$ as expected. From this we can also infer the dimension of the two point function $\langle J^\mu(x_1)J^\nu(x_2)\rangle$, that has to scale as $\sim x^{-2\Delta}=x^{-2(d-1)}$ in position space, where $x=x_1-x_2$ (the correlation function only depends on (x_1-x_2) due to translation invariance in the boundary). In momentum space the k dependence becomes

$$\langle J^\mu(k)J^\nu(-k)\rangle\sim\int d^dx\ \langle J^\mu(x_1)J^\nu(x_2)\rangle\,e^{-ik\cdot x}\sim x^{2-d}\sim k^{d-2}\ . \tag{2.69}$$

2.2.1 Computation in momentum space

We now want to compute $S_E^{(R)}[A_{\rm M}^{(c)}(z,k)]$ where S_E is the action given in (2.49). By partial integration, analogously to what we did for the variation we obtain:

$$\begin{aligned} S_E[A_{\rm M}(z,x)_c] &= \int \mathrm{d}^{d+1}x\, A_{\rm N}^{(c)} \partial_{\rm M}\left(\sqrt{g}F_{(c)}^{\rm MN}\right) - \int \mathrm{d}^{d+1}x \partial_{\rm M}\left(\sqrt{g}A_{\rm N}^{(c)}F_{(c)}^{\rm MN}\right) \\ &= \int \mathrm{d}^{d+1}x\, A_{\rm N}^{(c)} \partial_{\rm M}\left(\sqrt{g}F_{(c)}^{\rm MN}\right) - \int \mathrm{d}^{d}x \sqrt{h} n_{\rm M} F_{(c)}^{\rm MN} A_{\rm N}^{(c)} \\ &= -\lim_{\epsilon\to 0}\int \mathrm{d}^d x\, \sqrt{h} n_{\rm M} F_{(c)}^{\rm MN} A_{\rm N}^{(c)}\Big|_{z=\epsilon}^{z=\infty}, \end{aligned} \tag{2.70}$$

where we inserted the UV cutoff ϵ to account for possible divergences. Plugging in the explicit form of the AdS metric and the expression for the normal vector we obtain

$$\begin{aligned} S_E[A_{\rm M}(z,x)_c] &= -\lim_{\epsilon\to 0}\int \mathrm{d}^d x\, \sqrt{g} g^{zz} h^{\mu\nu} F_{z\mu}^{(c)} A_\nu^{(c)}\Big|_{z=0}^{z=\infty} \\ &= -\int \mathrm{d}^d x\, z^{3-d}\delta^{\mu\nu}\left(\partial_z A_\mu^{(c)}\right) A_\nu^{(c)}\Big|_{z=\epsilon}^{z=\infty} \end{aligned} \tag{2.71}$$

that in Fourier space is

$$S_E[A_{\rm M}(z,k)_c] = +\lim_{\epsilon\to 0}\int \frac{d^d k}{(2\pi)^d} z^{3-d}\delta^{\mu\nu}\left(\partial_z A_\mu^{(c)}(-k)\right) A_\nu^{(c)}(k)\Big|_{z=\epsilon}, \tag{2.72}$$

where the plus sign comes from the fact that we evaluated the action at $z=\infty$ (as before, imposing regularity at the horizon it evaluates to 0). Since the transverse sector completly decouples, we are only intereseted in the terms in A_τ and A_x in the boundary action, in other words, we can set the sources of the transverse compenents to zero without influencing the VEV of the longitutinal degrees of freedom, hence we obtain:

$$S_E[A_{\rm M}(z,k)_c] = +\lim_{\epsilon\to 0}\int \frac{d^d k}{(2\pi)^d} z^{3-d}\left(A'^{(c)}_x(-k)A_x^{(c)}(k) + A'^{(c)}_\tau(-k)A_\tau^{(c)}(k)\right)\Big|_{z=\epsilon}. \tag{2.73}$$

The two terms are related by our gauge condition $k_\mu A^\mu = 0 \Rightarrow \omega_{\rm E} A_\tau + \tilde{k} A_x = 0$, from which we obtain

$$A'^{(c)}_\tau A_\tau^{(c)} = \frac{\tilde{k}^2}{\omega_{\rm E}^2} A'^{(c)}_x A_x^{(c)}$$

and we can rewrite (2.73) as

$$S_E[A_{\rm M}(z,k)_c] = +\lim_{\epsilon\to 0}\int \frac{d^d k}{(2\pi)^d} z^{3-d}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right) A'^{(c)}_x(-k)A_x^{(c)}(k)\Big|_{z=\epsilon}. \tag{2.74}$$

All what is left to do is to study (2.72) for $z\to 0$ and regularize eventual divergences. In the case of the gauge field we have, from the asymptotic behavior (2.62):

$$S_E[A_{\rm M}(z,k)_c] = \lim_{\epsilon\to 0}\int \frac{d^d k}{(2\pi)^d}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right) z^{3-d}\left((d-2)b_x(k)z^{d-3}\right)\left(a_x(-k)+b_x(-k)z^{d-2}\right)\Big|_{z=\epsilon} \tag{2.75}$$

$$= \lim_{\epsilon\to 0}\int \frac{d^d k}{(2\pi)^d}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right)(d-2)a_x(-k)b_x(k) + b_x(k)b_x(-k)z^{d-2}) \tag{2.76}$$

$$= \int \frac{d^d k}{(2\pi)^d}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right)(d-2)a_x(-k)b_x(k) \tag{2.77}$$

for $d>2$, and we see that there seems not to be any UV divergence. This however, is not true for any number of dimensions, and we will get back to this point at the end of the section.

From (2.75), we can read off the one-point function:

$$\langle J^x(k)\rangle = (2\pi)^d \frac{\delta S_E[A_{\rm M}(z,k)_c]}{\delta a_\nu(-k)}\bigg|_{a_\mu=0} = (d-2)\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right) b_x(k) \tag{2.78}$$

that is indeed proportional to the normalizable mode $b_\mu(k)$, as predicted. The Euclidean two point is then given by

$$\langle J^x(k)\rangle = G_E^{xx}(k)a_x(k) + G_E^{x\tau}(k)a_t(k) = 2G_E^{xx}(k)a_x(k) \tag{2.79}$$

where we used the gauge choice and the Ward identities in Euclidean space $\omega_{\rm E}G^{\tau\tau}+\tilde{k}G^{z\tau} = \tilde{k}G^{zz}+\omega_{\rm E}G^{\tau z} = 0$ and we have

$$G_E^{xx}(k) = \langle J^x(k)J^x(-k)\rangle = \frac{(d-2)}{2}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right)\frac{b_x(k)}{a_x(k)} \tag{2.80}$$

The explicit solution of the full equation (2.60), after imposing regularity at the horizon is:

$$A_\mu(k,z) = C_\mu z^{d/2-1} K_{\frac{d}{2}-1}(kz) \tag{2.81}$$

Using the asymptotic expansion (2.42) we obtain

$$\tag{2.82}$$

$$A_\mu(k,z) = C_\mu\left(\frac{\Gamma(d/2-1)}{2}\left(\frac{k}{2}\right)^{-d/2+1} + \frac{\Gamma(1-d/2)}{2}\left(\frac{k}{2}\right)^{d/2-1} z^{d-2}\right) \quad \text{for} \quad z\to 0 \tag{2.83}$$

and we can therefore extract the coefficients $a_\mu(k)$ and $b_\mu(k)$. Using the explicit value of the coefficients in (2.80), we find that the 2-point function is:

$$\langle J^x(k)J^x(-k)\rangle = \frac{d-2}{2^{d-1}}\left(1+\frac{\tilde{k}^2}{\omega_{\rm E}^2}\right)\frac{\Gamma(1-d/2)}{\Gamma(d/2-1)}k^{d-2} \tag{2.84}$$

that shows the predicted k^{d-2} dependence. Similarly we could have computed the $G^{\tau\tau}$ correlation function that is obtained by simply switching $\omega_{\rm E}$ and $\tilde{k}$, and we can obtain $G^{\tau x} = G^{x\tau}$ using the Ward identities (we indeed only have a single longitudinal degrees of freedom, as expected for a photon). The gauge choice we made was particularly convenient because the differential equations decouple. When they are coupled, the leading order for one component $a_{\tilde{\mu}}$ in general source a linear combination of all the operators J^ν. We will learn how to deal with this situation in chapter 4 where we study the Green's function of the superconductor model.

There is one important feature we naively glossed over. The K_ν Bessel function expansion for $z\to 0$ is valid for $\nu\notin\mathbb{Z}$. For the scalar field, the value of ν is a function of a continuous parameter, the mass of the field, and in the cases we consider in the next chapters we always have a mass such that $\nu\notin\mathbb{Z}$. However, for the gauge field $\nu = d/2-1$, that means, for an odd number of dimensions (even number of spatial dimensions plus time) $\nu\notin\mathbb{Z}$, but for d even $(d/2-1)\in\mathbb{Z}$ and the Bessel function present a logarithmic divergence that needs to be regularized. When we expanded the action close to the boundary we used a power law ansatz, but a logarithmic term of the form $z^{d-2}\log(z)$ is also a solution, and it indeed comes out from the asymptotic behavior of the Bessel function for ν integer. For example in $d = 3+1$ dimensions, we have

$$K_1(z) = \frac{1}{z} + \alpha z + \frac{z}{2}\log\left(\frac{z}{2}\right) + \mathcal{O}(z^3)\ . \tag{2.85}$$

Adding the necessary counterterms modify the Green's functions by adding *contact terms* $c^{\mu\nu}$

$$G^{\mu\nu} \propto \frac{b^\nu}{a^\mu} + c^{\mu\nu}\ . \tag{2.86}$$

In chapter 4 we show the precise couterterms needed to regularize the gauge field in the holographic superconductor for $d = 3+1$, and we then derive the explicit form of the contact terms in that example, that turn out to be real and they therefore do not influence the spectral function.

2.2.2 Computation in position space

In position space there is an incredibly elegant trick to find the 2-point function due to Witten [18], that is worth mentioning. As a first step, we want to find the *bulk-to-boundary* propagator, that is the "Green's function" $K(z,x)$, regular in the bulk, that solves the equation of motion $(EOM)K_\mu(z,x)=0$, and reduces to a delta function $\delta(x)$ approaching the boundary (i.e. for $z\to 0$). Instead of solving the full equations of motion as we did in momentum space, Witten proposed to pick a "point at ∞" and then use the isometry of AdS to move the point to the boundary. For $z\to\infty$ we approach the low infrared limit of the theory, and choosing a point at infinity implies that the bulk-to-boundary propagator is x-independent $K_\mu^{(\infty)}(z,x)=K_\mu^{(\infty)}(z)$. To understand this, we can see that in the limit $z\to\infty$ the part of the metric transverse to the radial component goes to 0

$$\frac{dx_\mu dx^\mu}{z^2}\to 0 \quad \text{for} \quad z\to\infty \tag{2.87}$$

therefore the spacetime "shrinks" to a single point and the propagator cannot depends on x^μ. Then the equations of motion in position space (2.53) simplify and become:

$$\partial_z(z^{3-d}(\partial_z K_\mu^{(\infty)}(z))=0 \tag{2.88}$$

Since we are interested in the solutions that vanish for $z\to 0$, we find:

$$K_\mu^{(\infty)}(z)=c_\mu z^{d-2} \tag{2.89}$$

This is the bulk-to-boundary propagator when the "bulk point" is the point at infinity. Witten proposed to use the invariance of the AdS spacetime under inversions, transforming the point $x^{\mathrm{M}}\to x'^{\mathrm{M}}$ with

$$x^{\mathrm{M}}=x'^{\mathrm{M}}/(x'^{\mathrm{N}}x'_{\mathrm{N}}) \tag{2.90}$$

that is:

$$x^\mu\to\frac{x^\mu}{z^2+x^2} \tag{2.91}$$

$$z\to\frac{z}{z^2+x^2} \tag{2.92}$$

that maps the point $z=\infty$ to the boundary origin $z=0$.

The bulk-to-boundary propagator is a vector, and it must transform as such under the diffeomorphism:

$$K'_{\mathrm{M}}(z')=K_{\mathrm{N}}^{(\infty)}(z)\frac{\partial x^{\mathrm{N}}}{\partial x'^{\mathrm{M}}} \tag{2.93}$$

using equation (2.90) we obtain

$$\frac{\partial x'^{\mathrm{N}}}{\partial x'^{\mathrm{M}}}=\frac{\delta^{\mathrm{N}}_{\mathrm{M}}}{x'^{\mathrm{L}}x'_{\mathrm{L}}}-2\frac{x'^{\mathrm{N}}x'_{\mathrm{M}}}{(x'^{\mathrm{L}}x'_{\mathrm{L}})^2} \tag{2.94}$$

remembering that $K_{\mathrm{Z}}^{(\infty)}(z)=0$, N only runs on the indices of the d-dimensional spacetime μ, and we obtain (omitting the primes, i.e., renaming $x\equiv x'$, $z\equiv z'$):

$$K_{\mathrm{M}}(z,x)=-\,2c_\mu\frac{z^{d-2}}{(z^2+x^2)^{d-1}}\left(\delta^\mu_{\mathrm{M}}-\frac{2x^\mu x_{\mathrm{M}}}{z^2+x^2}\right) \tag{2.95}$$

That is the propagator from a point in the bulk to the point x^μ on the boundary. We can see, that $\lim_{z\to 0}K_{\mathrm{M}}=0$ unless $x^2=0$, where the limit diverges, hinting to the fact that the propagator presents the predicted δ-function behavior. Due to translation invariance, we can generalize the equations for two point

x, x' simply by replacing $x \to (x - x')$. We can then write the bulk-to-boundary propagator from a point in the bulk $x^{\mathrm{M}} = (z, x^\mu)$ to an arbitrary point at the boundary $\bar{x}^\nu = (x - x')^\nu$ in terms of its components as:

$$G_{z\nu}(z, x, x') = -\, 2C \frac{z^{d-1}(x - x')_\nu}{(z^2 + (x - x')_\sigma (x - x')^\sigma)^d} \tag{2.96}$$

$$G_{\mu\nu}(z, x, x') = C \frac{z^{d-2}}{(z^2 + (x - x')_\sigma (x - x')^\sigma)^{d-1}} \left(\delta_{\mu\nu} - 2 \frac{(x - x')_\mu (x - x')_\nu}{z^2 + (x - x')_\sigma (x - x')^\sigma} \right) \tag{2.97}$$

where C now is just a multiplicative constant.

The solution of the equation of motion $A^{(c)}_{\mathrm{M}}(z, x)$ can be written in terms of the Green's function as

$$A^{(c)}_{\mathrm{M}}(z, x) = \int d^d x' \, G_{\mathrm{M}\nu}(z, x, x') a^\nu(x') \tag{2.98}$$

where $a_\nu(x) = \lim_{z \to 0} A^{(c)}_\nu(z, x)$ is the source, as before. Plugging this into the Euclidean action (2.71) gives

$$\int \mathrm{d}^d x \int \mathrm{d}^d x' z^{3-d} \delta^{\mu\nu} \partial_z G_{\mu\rho}(z, x, x') a^\rho(x') A^{(c)}_\nu(z, x) \Big|_{z=0}^{z=\infty} \tag{2.99}$$

Therefore the two-point function for the conserved current in position space is given by

$$\left\langle J_\mu(x) J_\nu(x') \right\rangle = \frac{\delta S_E[A_{\mathrm{M}}(z, k)_c]}{\delta a^\mu(x) \delta a^\nu(x')} \bigg|_{a_\mu, a_\nu = 0} = \lim_{z \to 0} \left(z^{3-d} \partial_z G_{\mu\nu}(z, x, x') \right) \tag{2.100}$$

Performing the derivative with respect to z and taking the limit we obtain

$$\left\langle J_\mu(x) J_\nu(x') \right\rangle = C(d - 2) \left(\frac{\delta_{\mu\nu}}{|x - x'|^{2(d-1)}} - 2 \frac{(x - x')_\mu (x - x')_\nu}{|x - x'|^{2d}} \right) \tag{2.101}$$

and again we find that the two-point function presents the expected x-dependence $\sim x^{-2(d-1)}$.

This method avoids to solve the full differential equation, however, it is restricted to AdS spacetime since it is based on its isometries, while the computation in momentum space can be easily generalized to different spacetime geometries.

Chapter 3

Introduction to the holographic superconductor

In this chapter we introduce the necessary ingredients to build a holographic superconductor model. We first present how to set a nonzero temperature and chemical potential in the boundary theory, then we explain how to modify the gravitational background to describe the second-order phase transition characterizing a superconductor.

3.1 Black holes and finite temperature

In condensed-matter physics, we are often interested in systems at a nonzero temperature and entropy, especially if we want to study a to study a phase transition at a nonzero critical temperature T_c as the one describing a superconductor. We then need to understand how the notion of entropy and temperature translate in the bulk gravitational theory. There is one idea that may immediately jump to the mind of the reader with some background in black hole thermodynamics. As Hawking discovered, black holes are thermal objects that radiate and possess a finite entropy S_{BH}, called the *Bekenstein-Hawking entropy.* From statistical physics we know that the entropy scales with the volume of the system, but the entropy of a black hole is quite peculiar in this regard as it scales with the area of the black hole horizon:

$$S_{BH} = k_B \frac{A_H}{4l_P^2} \ . \tag{3.1}$$

This, however, is exactly what we need since the surface of $d+1$-dimensional object has the dimension of a d-dimensional volume and we see that the entropy of a black hole has the correct scaling behavior to describe a thermal field theory on the lower dimensional boundary[1].

AdS supports black holes, in the sense that, as for Minkowski spacetime, black hole solutions exist that modify the geometry of the spacetime but far enough from the black hole, it reduces to AdS. Inserting a black hole, sets a cutoff scale in the radial dimension in the bulk $r_h > 0$ (in Poincaré coordinates (1.19)), and since we identified the radial direction with the energy scale, this idea is consistent with our intuition of the effect of a finite temperature in the boundary theory. Turning on a temperature in a field theory sets an energy scale that breaks conformal invariance, modifying the IR physics. The effects of excitations with an energy lower than the scale set by the temperature are just modified by thermal excitations and only higher energy effects, corresponding in the gravitational dual to scale $r \gtrsim r_h$, can be observed. On the other hand, for energies much higher than this scale, the theory is not sensible to the effects of the finite temperature and we recover the conformal field theory, that corresponds to the spacetime being asymptotically AdS for $r \to \infty$.

[1]This feature, that the number of degrees of freedom of a gravitational system in $d+1$ dimensions are encoded in a field theoretical d-dimensional theory, is believed to be a more general property of quantum gravity called *the holographic principle* and it formed the basis of AdS/CFT.

We then have our first ingredient for constructing a holographic superconductor, a metric that represents a black hole solution.

The temperature of the boundary theory corresponds to the *Hawking temperature* of the bulk black hole. There are several ways for computing this quantity, and Hawking first derived it by quantizing matter fields in a black hole background. Nonetheless, there is a simpler derivation that does not require field quantization. In a general background geometry the Hawking temperature can be computed by performing a Wick rotation to Euclidean gravity and requiring the black hole solution to be smooth.

The relation between a temperature and Euclidean time is known in QFT. To describe a system at finite temperature T, we analytically continue to Euclidean signature, i.e. $t \to -i\tau$, and let τ to be periodic: $\tau \sim \tau + \hbar\beta$, with $\beta = 1/k_B T$. Conversely, if the Euclidean continuation of a QFT is periodic in the time direction, we can conclude that the QFT is at finite temperature. An intuitive way to think about this seemingly weird analogy is by noticing that the partition function for a statistical system in thermal equilibrium (in the grand canonical ensemble) has the form in the Schrödinger picture at a constant time $t = 0$:

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \sum_{\psi} \langle\psi(0)|\, e^{-\beta H}\, |\psi(0)\rangle \ ,$$

with H the Hamiltonian of the system and we are summing over a complete set of states. On the other hand, we know that time evolution of a state is given by $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$, and we can then think of the Boltzmann factor $e^{-\beta H}$ as a time evolution operator in imaginary time and write

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \sum_{\psi} \langle\psi(0)|\psi(-i\beta)\rangle$$

where the left-hand side now represents the vacuum amplitude as we are evolving the state from $\tau = 0$ to $\tau = \beta$, and requiring the final state to be the same as the initial state forces τ to be periodic. This is the argument we are going to use to interpret the temperature of a black hole.

Here we present the computation for a particular black hole solution in asymptotically AdS spacetime, that, as we will see, it will turn out to be the metric that describes the holographic superconductor. Anyhow, the procedure is exactly the same for different geometries.

The metric we consider is

$$ds^2 = -c^2 f(r)e^{-\chi(r)}dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}d\mathbf{x}^2_{d-1} \ . \tag{3.2}$$

We analytically continue the above metric with $t \to -i\tau$

$$ds^2_E = c^2 f(r)e^{-\chi(r)}d\tau^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}d\mathbf{x}^2_{d-1} \tag{3.3}$$

and study the periodicity near the horizon.

At the horizon $f(r_h) = 0$ and the metric (3.2) blows up. So we have to Taylor expand $f(r)$ near the horizon. We obtain:

$$f(r) \approx f(r_h) + f'(r_h)(r - r_h) + \mathcal{O}((r - r_h)^2) = f'(r_h)(r - r_h) + \mathcal{O}((r - r_h)^2) \tag{3.4}$$

and the near-horizon metric becomes:

$$ds^2_E \approx c^2 f'(r_h)(r - r_h)e^{-\chi(r_h)}d\tau^2 + \frac{dr^2}{f'(r_h)(r - r_h)} + \frac{r_h^2}{L^2}d\mathbf{x}^2_{d-1} \tag{3.5}$$

if we now make a change of variable $(\tau, r) \to (\theta, R)$ with

$$\theta = c\frac{f'(r_h)}{2}e^{-\frac{\chi(r_h)}{2}}\tau \equiv K\tau \tag{3.6}$$

$$R = \frac{4(r - r_h)}{f'(r_h)} \tag{3.7}$$

the near-horizon metric can be written as

$$ds_E^2 \approx R^2 d\theta^2 + dR^2 + \frac{r_h^2}{L^2} d\mathbf{x}_{d-1}^2 \tag{3.8}$$

so that the radial and time component are just Euclidean polar coordinates in $\mathbb{R}^2$. This metric presents a *conical singularity* unless θ is periodic with period 2π (a way to visualize a conical singularity is to take a piece of paper and cut off a circle. If the period is $\theta < 2\pi$ we remove a "slice" corresponding to a section with an angle at the centre of the circle of $2\pi - \theta$, as depicted in figure 3.1. We can then glue together the two ends and obtain a cone, that has a sharp tip at the end and cannot represent a smooth black hole horizon). The periodicity of the θ implies that the Euclidean time must have the period

$$\tau \sim \tau + \frac{2\pi}{K} \tag{3.9}$$

and we then get[2]

$$T = \frac{1}{k_B \beta} = \frac{\hbar K}{2\pi k_B} = \frac{\hbar c f'(r_h) e^{-\frac{\chi(r_h)}{2}}}{k_B 4\pi} \ . \tag{3.10}$$

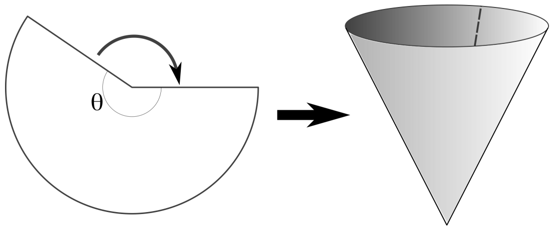


Figure 3.1: Pictorial representation of the origin of a conical singularity for periodicity $\theta < 2\pi$

3.2 Nonzero density and chemical potential

Now that we know how to put the system at a nonzero temperature, the next step to build an interesting condensed matter system is to introduce a nonzero density of particles $\rho_a = N_a/V$, with N_a the number of particles of a species a, associated with a chemical potential μ_a. The reason we need to introduce a density is that in a conformal field theory, in the absence of other scales, all nonzero temperatures are equivalent, and in order to obtain a phase transition we then need to introduce another scale, and the choice of the chemical potential is the natural one as we want to describe the condensation of matter leading to a superconductor phase transition.

Conservation of the particles number in a QFT is described by a $U(1)$ global symmetry associated to a conserved current J_a^μ, with corresponding Noether charge Q_a, so that $N_a = \langle Q_a \rangle$. In statistical physics, a system at fixed temperature and volume, but where the number of particles is allowed to fluctuate is described in the *grand canonical ensemble*, defined by the partition function

$$Z_{\text{grand}} = \text{Tr}\left[e^{-\beta(H - \mu_a Q_a)}\right] \ . \tag{3.11}$$

In a field theory, this corresponds to modify the action by a term

$$\int d^d x \, \mu_a J_a^t \ , \tag{3.12}$$

[2]A caveat, the temperature of a black hole as felt by a stationary observer depends on its position with respect to the black hole (this is a manifestation of the general fact, known as *Ehrenfest–Tolman effect*, that in curved spacetime at thermal equilibrium the temperature is not constant, but depends on the curvature). For observer at some fixed r, the proper time $\tilde{t}$ is given by $d\tilde{t} = \sqrt{f(r)}dt$, and the local temperature is then $\tilde{T} = Tf(r)^{-1/2}$. Notice that the local temperature diverges for $r \to r_h$.

and we already have enough hints to understand how to encode a finite density on the gravity side. The *global* $U(1)$ symmetry in the boundary theory translates to a *local* $U(1)$ symmetry in the bulk, and the boundary term (3.12) implies that, in the large-N limit, we need the time component of a classical gauge field A_μ^a dual to J_a^μ, that for a spatially uniform chemical potential, can only depend on the radial component. For a single particle species, the bulk geometry dual to a finite density field theory is then given by the Einstein-Maxwell theory in AdS

$$S_{EM} = \int d^{d+1}x\, \sqrt{-g}\left(\frac{c^3}{16\pi G}(R-2\Lambda) - \frac{1}{4\mu_0 c}F_{\mu\nu}F^{\mu\nu}\right) \tag{3.13}$$

with boundary conditions

$$\lim_{r\to\infty} A_t(r) = \mu\ . \tag{3.14}$$

If we are interested in a system with both finite temperature and density, we have to merge the presence of a charge with the black-hole solution in the bulk. The first thing that comes to mind is to blend together the two requirements and consider an electrically charged black hole solution of the Einstein-Maxwell equations, the *Reissner-Nordström black hole*. This solution can be derived by variation of the action (3.13) that gives

$$ds^2 = -\, c^2 f(r)dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}d\mathbf{x}_{d-1}^2\ , \tag{3.15}$$

$$A_t(r) = \mu\left(1-\left(\frac{r_h}{r}\right)^{d-2}\right)\ , \tag{3.16}$$

with,

$$f(r) = \frac{r^2}{L^2}\left(1+\frac{Q^2}{r^{2(d-1)}}-\frac{M}{r^d}\right) \tag{3.17}$$

$$\mu = \frac{\mu_0 c Q\sqrt{d-1}}{2\sqrt{2\pi G(d-2)}L^2 r_h^{d-2}} \tag{3.18}$$

where Q and M represent respectively the charge and the mass of the black hole.

The horizon radius is implicitly defined by $f(r_h) = 0$ that in general has two distinct solutions. The Reissner-Nordström black hole then presents two horizons, the *inner* and the *outer* horizon, denoted respectively with r_- and r_+. In the context of gauge/gravity duality, it is the outer horizon that sets the temperature scale on the boundary, while the inner horizon does not have any particular relevance, for this reason, from now on we will always refer to the outer horizon simply as the black hole horizon and we will denote it with $r_h \equiv r_+$.

3.3 Holographic superconductor

We have presented almost all the ingredients necessary to build a model of a holographic superconductor, we know how to define a temperature and a chemical potential, but we still miss the most important characteristic of the superconducting phase transition, the spontaneous breaking of a $U(1)$ symmetry. A spontaneous symmetry breaking is described by the appearance of a nonzero expectation value of an order parameter O without the presence of a source term. In the BCS theory, the order parameter is a bosonic operator (a scalar) that represents the pairing of two fermions. In a holographic superconductor we do not really know what the order parameter represent from a microscopical point of view, but in order for the system to undergo a transition to a superconducting phase we still need a bosonic operator that can represent the condensate.

As we have previously seen, a scalar operator is simply dual to a scalar field in the bulk. In chapter 2 we studied the example of a real scalar field, however, we need our boundary system to have a global $U(1)$ symmetry, dual to a gauge symmetry in the bulk, and we therefore need the scalar field to be invariant under gauge transformations. This requirement means that the classical field in the bulk needs to be a complex charged scalar field. A clever guess then, is to modify the Einstein-Maxwell action by minimally coupling the scalar field to the gauge field and to gravity and we obtain the action (Einstein-Maxwell-Higgs action)

$$S = \int d^{d+1}x \sqrt{-g} \left(\frac{c^3}{16\pi G}(R - 2\Lambda) - \frac{1}{4\mu_0 c} F_{\mu\nu}F^{\mu\nu} - |D_\mu \phi|^2 - V(\phi) \right) \tag{3.19}$$

with $D_\mu \equiv \nabla_\mu - \frac{iq}{\hbar} A_\mu$, q the charge of the scalar field, is the **gauge covariant** derivative that minimally couples the scalar field with the Maxwell field (and the spacetime covariant derivative ∇_μ provides the coupling with the gravity field).

We now have a model with everything we need. Why however does (3.19) represent a superconductor? How is the characteristic phase transition encoded in the action?

First of all, a phase transition in the boundary theory has to be described by some changes in the geometry of the interior. In particular, we are dealing with a second-order transition, where the entropy changes smoothly across the transition. In terms of the geometry of the bulk, this suggests that the transition cannot be described by a sudden change of the black hole area, since the area encodes the entropy of the thermal boundary system. As the horizon area depends on both the charge and mass of the black hole, we do not expect the transition to be described by an abrupt change of neither the metric or the gauge field. On the contrary, in order to describe a second-order phase transition we need to turn on a vacuum expectation value of the scalar operator dual to the complex scalar bulk field, that is nonzero even in absence of a source (this is the concept of *spontaneous* symmetry breaking, a vacuum expectation value generated by a source would break the symmetry explicitly). As we have shown in chapter 2, the leading and subleading order of the boundary expansion of a bulk field are related respectively to the source and vacuum expectation value (sometimes we will just refer to it as VEV) of the corresponding operator, so we need a solution of the equation of motion for the scalar field that allows a boundary expansion with finite subleading term, but with zero leading order.

To be able to discuss the scalar solution, we first have to fix the potential $V(\phi)$ in (3.19). In principle, the exact form of the potential could be derived from consistent truncations and compactification of a theory of quantum gravity in a *top-down* approach. Here, however, as in most of the AdS/CFT applications, we are taking a phenomenological *bottom-up* approach, and we should choose a term that generates the solution we are looking for. The simplest potential that is compatible with the description of a phase transition is just the simple quadratic potential

$$V(\phi) = \frac{m^2c^2}{\hbar^2}\phi\phi^* = \frac{m^2c^2}{\hbar^2}|\phi|^2 \; . \tag{3.20}$$

The model with this form of the potential is called the *minimal holographic superconductor* since it ignores higher order interaction of the fields and represents the simplest framework to understand holographic superconductivity (this minimal model is an especially good approximation close to the critical temperature, where the amplitude of the scalar field is small and higher order terms can be neglected). To summarize, the holographic superconductor action we are going to study is:

$$S = \int d^{d+1}x \sqrt{-g} \left(\frac{c^3}{16\pi G}(R - 2\Lambda) - \frac{1}{4\mu_0 c} F_{\mu\nu}F^{\mu\nu} - |D_\mu \phi|^2 - \frac{m^2c^2}{\hbar^2}|\phi|^2 \right) \; . \tag{3.21}$$

3.3.1 Normal and superconducting phase

In order to set a temperature on the boundary theory, we have to look for a black hole solution of the Einstein equations associated to the action (3.21). Since in the normal phase (i.e., above the critical temperature T_c) the system reduces to a normal finite temperature and density solution, the metric should reduce to the Reissner-Nordström metric (3.15), so we look for a generalization of that metric, that is given by the ansatz

$$ds^2 = -c^2 f(r) e^{-\chi(r)} dt^2 + \frac{1}{f(r)} dr^2 + \frac{r^2}{L^2} d\mathbf{x}_{d-1}^2 \; . \tag{3.22}$$

Moreover, we want a chemical potential and an order parameter that are constant everywhere on the boundary, and we therefore choose for the gauge and the scalar field

$$A_\mu = A_\mu(r)\delta_t^\mu \ , \qquad \phi = \phi(r) \ . \tag{3.23}$$

Varying the action (3.21) we can find the classical equation of motion for the "matter" fields (the gauge and the scalar field)

$$\nabla_\mu F^{\mu\nu} - iq\left(\phi^* D^\nu \phi - \phi (D^\nu \phi)^*\right) = 0 \tag{3.24}$$

$$\left(D_\mu D^\mu - m^2\right)\phi = 0 \tag{3.25}$$

$$\left(D_\mu^* D^{\mu *} - m^2\right)\phi^* = 0 \tag{3.26}$$

and the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \ . \tag{3.27}$$

From the r component of the Maxwell equation (3.24) we obtain (defining $\partial_r \phi \equiv \phi'$)

$$\phi^* \phi' = \phi(\phi^*)' \Rightarrow \text{ the phase } \theta \text{ is constant,} \tag{3.28}$$

and we can therefore, without loss of generality, set the phase to zero and consider a real scalar field $\phi = \phi^*$. Explicitly writing the equations using the ansätze we obtain:

$$f' + \left(\frac{d-2}{r} - \frac{\chi'}{2}\right) f + \frac{16\pi G}{(d-1)c^3} r \left(\frac{e^\chi A_t'^2}{2\mu_0 c^3} + \frac{m^2 c^2}{\hbar^2}\phi^2\right) - \frac{rd}{L^2} = 0 \tag{3.29}$$

$$\chi' + \frac{32\pi G}{(d-1)c^3} r \left(\phi'^2 + \frac{q^2 e^\chi}{\hbar^2 c^2 f} A_t^2 \phi^2\right) = 0 \tag{3.30}$$

$$\phi'' + \left(\frac{f'}{f} + \frac{d-1}{r} - \frac{\chi'}{2}\right)\phi' - \left(\frac{m^2 c^4 - q^2 A_t^2 \frac{e^\chi}{f}}{\hbar^2 c^2 f}\right)\phi = 0 \tag{3.31}$$

$$A_t'' + \left(\frac{d-1}{r} + \frac{\chi'}{2}\right) A_t' - 2\frac{q^2 \mu_0 c \phi^2}{\hbar^2 f} A_t = 0 \ . \tag{3.32}$$

Solutions to these equations of motion define the equilibrium phases of the theory.

The first phase we look for, is the unbroken phase i.e., the phase describing the system above the critical temperature. This corresponds to the solution with $\phi = \chi = 0$. The metric (3.22) reduces to the Reisnner-Nordström metric (RN) (3.15) and the action becomes the Einstein-Maxwell action, with solutions:

$$A_t(r) = \frac{\mu}{q}\left(1 - \left(\frac{r_h}{r}\right)^{d-2}\right) \tag{3.33}$$

and

$$f(r) = \frac{r^2}{L^2} - \left(\frac{r_h}{r}\right)^d \frac{r^2}{L^2} + \frac{8\pi G}{\mu_0 c^6}\frac{d-2}{d-1}\left(\frac{\mu}{q}\right)^2 \left[\left(\frac{r_h}{r}\right)^{2(d-2)} - \left(\frac{r_h}{r}\right)^{d-2}\right] \tag{3.34}$$

that has the form of the RN solution (3.17) and (3.18).

Below the critical temperature the phase is characterized by the full solutions with $\phi, \chi \neq 0$. Often these equation are considered in the probe limit, that is the limit $q \to \infty$, keeping $q\phi$ and qA_t fixed. In this limit the matter terms in the Einstein equations can be neglected and the problem therefore decouples from gravity. In the following, however, we want to consider the full background, and we proceed to integrate the equations numerically.

3.3.2 Boundary conditions and degrees of freedom

To simplify the equations, we first introduce dimensionless fields and coordinates:

$$\begin{cases} \left(\tilde{r}, \tilde{t}, \tilde{\mathbf{x}}\right) = \left(r, ct, \mathbf{x}\right)/L \\ \tilde{m} = \frac{cL}{\hbar} m \\ \tilde{A}_{\tilde{t}} = \sqrt{\frac{16\pi G}{\mu_0 c^6}} A_t \\ \tilde{\phi} = \sqrt{\frac{16\pi G}{c^3}} \phi \\ \tilde{q} = \sqrt{\frac{\mu_0 c^6}{16\pi G}} \frac{L}{\hbar c} q \end{cases} . \tag{3.35}$$

This eliminates G, μ_0 and L from the equation of motions. Under this change of coordinates the action changes as

$$S/\hbar = \frac{c^3 L^{d-1}}{16\pi G \hbar} \tilde{S} \equiv N_G \tilde{S} \ , \tag{3.36}$$

with $\tilde{S}$ the dimensionless action that does not contain any factor of G, μ_0 and L and N_G a dimensionless constant, related to the parameter N of the large-N limit. From now on, we will always use dimensionless quantities, unless stated otherwise, omitting the tilde for convenience. Notice that in this coordinates energies are measured in units of $\hbar c/L$ and length scales are measured in terms of L.

In order to integrate the equations of motion from the boundary to the horizon defined by $f(r_h) = 0$ we need to fix the boundary conditions. Since we have two first order differential equations and two second-order ones, we need in total six initial conditions. First of all, we have that the time component of the metric is null at the horizon $g_{tt}(r_h) = -f(r_h)e^{-\chi(r_h)/2} = 0$ that implies

$$\lim_{r\to r_h} A^2(r) = \lim_{r\to r_h} A_\mu(r)A^\mu(r) = \lim_{r\to r_h} g^{tt}(r)A_t(r)A_t(r) = \infty \tag{3.37}$$

for any nonzero value of $A_t(r_h)$. If we want the norm to be finite we must impose $A_t(r_h) = 0$ (one may argue that the bulk gauge field is not a physical field, so there seems to be no reason why it cannot diverge at the horizon. However, we have to keep in mind that the field is dual to the physical current on the boundary theory). Multiplying the Maxwell equation by f and evaluation at the horizon gives the constraint

$$f'(r_h)\phi'(r_h) = m^2\phi(r_h) \ , \tag{3.38}$$

and we are left with four independent boundary values

$$r_h,\ \chi(r_h),\ A_t'(r_h),\ \phi(r_h) \ . \tag{3.39}$$

However, in order to use the AdS/CFT correspondence we know that our spacetime has to be asymptotically anti-de Sitter, that requires $\chi(\infty) = 0$. Using the isometries of the spacetime, in the numerical calculation we can set the initial condition $\chi(r_h) = 0$ and then rescale the solution using

$$e^\chi \to C^2 e^\chi \ , \qquad t \to Ct \ , \qquad A_t \to A_t/C \ , \tag{3.40}$$

with $C = e^{-\chi(\infty)/2}$. In the same way we can use the symmetry of the equation of motions under a scaling

$$r \to ar \ , \qquad (t, \mathbf{x}) \to (t, \mathbf{x})/a \ , \qquad f \to a^2 f \ , \qquad A_t \to aA_t \ , \tag{3.41}$$

to set $r_h = 1$, and we are left with a two free boundary conditions. On the boundary, there are five parameters that fully describe the theory, the chemical potential μ and charge density ρ, the leading and sub-leading coefficient of the scalar field at the boundary ϕ_0, ϕ_+ and M the mass of the black hole that defines the energy density of the field theory. Integrating the equations of motion then defines a map

$$\left(A_t'(r_h),\ \phi(r_h)\right) \mapsto (\mu,\ \rho,\ \phi_0,\ \phi_+,\ M) \ , \tag{3.42}$$

for each fixed value of the free parameters d, m^2, and q.

However, as we previously discussed, a spontaneous symmetry breaking corresponds to a nonzero expectation value of O in the absence of a source term, that means, in order to study the phase transition we have to impose an extra constraint, the vanishing of the source term dual to the scalar field. In chapter 2 we have seen that there is a range of values for the mass where both modes on the boundary can act as a source, in this range we are then free to choose which term to set to zero. In this thesis, even for values of the mass inside that range, we will always choose the canonical quantization, i.e., we will always consider ϕ_0 as the source term and $\phi_+ \propto \langle O \rangle$, giving the constraint $\phi_0 = 0$ (numerically, we impose the condition using the *shooting method*[3]).

The map (3.42) reduces to a one parameter family of solutions (for a choice of d, m^2 and q), that fix the scale, we can choose it to be the dimensionless quantity T/μ. We can think of it as being the temperature of the theory at a fixed chemical potential (working in the grand canonical ensemble).

3.3.3 Instability and phase transition

In section 3.1 we computed the temperature associated to the metric

$$T = \frac{\hbar c f'(r_h) e^{-\frac{\chi(r_h)}{2}}}{k_B 4\pi} \ . \tag{3.43}$$

Using the same procedure, we can compute the the temperature associated to the RN solution, that gives

$$T = \frac{\hbar c f'(r_h)}{k_B 4\pi} = \frac{d}{L^2} r_h - \frac{8\pi G (d-2)^2}{\mu_0 c^6 (d-1)} \frac{\mu^2}{q^2 r_h} \ , \tag{3.44}$$

that in terms of the dimensionless variables becomes

$$T = d r_h - \frac{(d-2)^2}{2(d-1)} \frac{\mu^2}{q^2 r_h} \ .$$

This solution exists for every $T/\mu \geq 0$, and it therefore exists also for temperatures smaller than the critical temperature, so why do we expect the system to undergo a phase transition? First of all, we can notice that the $T = 0$ solution corresponds to $r_h = r_+ = r_-$, i.e., to the extremal Reissner-Nordström black hole. The term *extremal* derives from the fact that the $T = 0$ solution corresponds to a black hole with all the mass due to the electromagnetic charge of the black hole, as one can see by rewriting the $T = 0$ condition in terms of the mass M and charge Q as in (3.15). It should be now clear why this solution corresponds to zero temperature. The black hole, due to the conservation of its electric charge cannot reduce its mass further via thermal Hawking radiation. The extremal black hole however, has a finite radius, and therefore a finite Bekenstein-Hawking entropy $S \propto A \propto r_h^{d+1}$, that means our field theory on the boundary is characterized by a finite entropy at zero temperature and it is then a very unstable state.

What triggers the instability when lowering the temperature is the violation of the Breitenlohner-Freedman bound. In chapter 2 we derived this BF bound for a scalar field in AdS$_{d+1}$ spacetime

$$m^2 L^2 \geq -\frac{d^2}{4} \ . \tag{3.45}$$

[3]The shooting method is a procedure to reduce a boundary value problem (setting $\phi_0 = 0$ on the boundary) to an initial value problem (giving initial conditions at the horizon). For a second-order ordinary differential equation, we can illustrate it as follows. Defining

$$y''(x) = f(x, y(x), y'(x)), \qquad y(x_0) = y_0, \qquad y(x_1) = y_1$$

the boundary value problem, we denote $y(x; \alpha)$ the solution corresponding to the initial value problem

$$y''(x) = f(x, y(x), y'(x)), \qquad y(x_0) = y_0, \qquad y'(x_0) = \alpha \ .$$

We can then define a function $F(\alpha) \equiv y(x_1; \alpha) - y_1$. Each root $\tilde{\alpha}$ of $F(\alpha)$ gives a initial condition $y'(x_0) = \tilde{\alpha}$ that corresponds to a solution of the boundary value problem

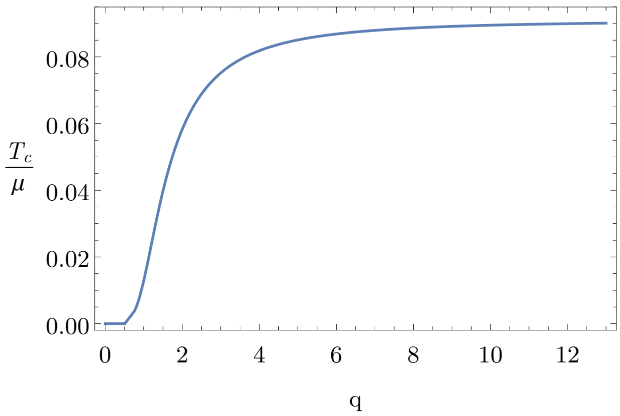


Figure 3.2: Critical temperature as a function of the charge q (scaled by μ to make it dimensionless) for $m^2 = -3.5$. For larger value of q the gauge term provides a smaller (i.e., more negative) contribution to the effective mass giving a larger value of the critical temperature. As we keep increasing q, the contribution of the mass of the scalar field becomes negligible and the critical temperature approaches a constant value. For $q = 0$ T_c is very small, but non zero.

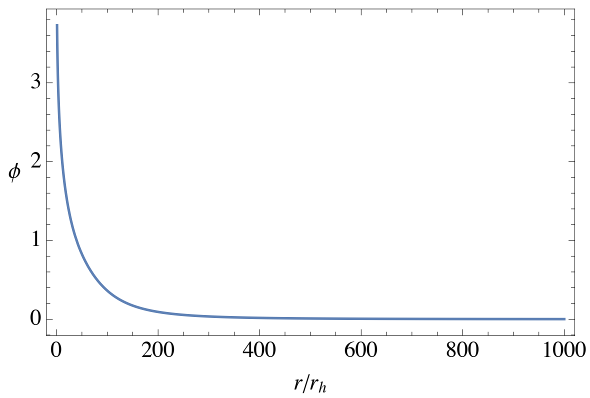
If we now look at the part of the action (3.21) describing the scalar field

$$S_\phi = -\int d^{d+1}x \left(|D_\mu\phi|^2 + m^2|\phi|^2\right) = -\int d^{d+1}x \left(|\partial_\mu\phi|^2 + \left[q^2 A_\mu A^\mu + m^2\right]|\phi|^2\right) \tag{3.46}$$

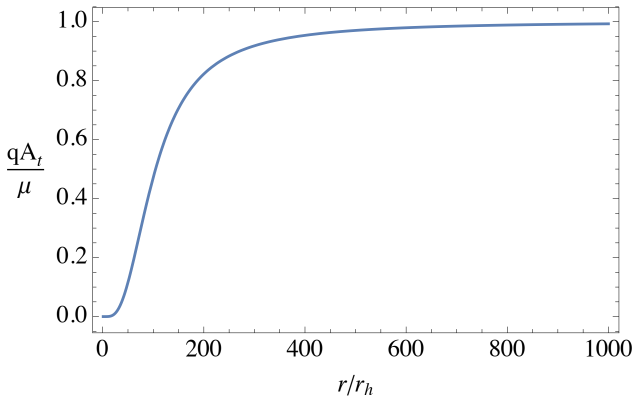
we see that the coupling with the gauge field gives an effective mass term, $m^2_{eff} = q^2 g^{tt} A_t^2 + m^2$. Since $g^{tt} < 0$ and $A_t \in \mathbb{R}$, the extra term act as a negative mass squared. Near the AdS boundary, the extra term is subleading as $g^{tt}A_t^2 \propto \mu^2/(q^2r^2)$ for $r \to \infty$, and the vacuum is stable as long as the mass satisfies the BF bound. Close to the horizon on the other hand, the negative mass squared contribution becomes more important as the temperature is lowered (notice that for the extremal black hole solution $f'(r_h) = 0$ and we have $f(r) \sim f''(r_h)(r - r_h)^2$ for $r \to r_h$ making $g^{tt}A_t^2 = const$), and for large enough value of q it will eventually break the BF bound making the scalar field unstable. Based on this reasoning we expect the critical temperature to be higher for higher values of the charge q.

In figure 3.2 we plotted the critical temperature for a fixed mass as a function of the charge that shows the expected behavior. However, we find a small but nonzero value of the critical temperature even for a neutral operator (i.e., for $q = 0$). The reason is that the $g^{tt}A_t^2$ term is not the only contribution to the instability, especially at very low temperature. The BF bound (3.45) is valid in AdS$_{d+1}$, but the presence of the RN black hole modifies the geometry in the deep interior. First, we should notice that the horizon (hyper)surface in asymptotically AdS spacetime is not spherically symmetric as for Minkowski solution, but it is a d-dimensional membrane, and a more proper term would then be **black brane**. Nonetheless, in the context of gauge/gravity duality, it is common to refer to it as black hole despite the non-spherical symmetry, simply because the emphasis is on the casual structure more than on the actual shape of the horizon. In particular the RN extremal black hole has a geometry $\mathbb{R}^{d-1} \times \text{AdS}_2$, that means that the BF bound near the horizon becomes $m^2 L_2^2 \geq -1/4$ (the relation between curvature radius is $L^2 \equiv L^2_{d+1} = L_2^2 d(d+1)/2$) and we have an interval of values of the mass parameter, $-\frac{d^2}{4L^2} \leq m^2 \leq -\frac{d(d+1)}{8L^2}$, where the field is stable in the AdS$_{d+1}$ geometry close to the boundary, but tachyonic near the horizon. Since the critical temperature for $q = 0$ is very small, the near horizon geometry is close to the $\mathbb{R}^{d-1} \times \text{AdS}_2$ of the extremal solution, and the geometry of the spacetime drives the instability even in the absence of a charge.

When the system becomes unstable, the solution with a nontrivial scalar profile is energetically favorable. This solution is rather complicated and we have to resort to numerical methods to solve it, nonetheless, we can have an idea of what happens qualitatively. From the asymptotic behavior near the boundary we know that the scalar profile (in standard quantization) vanishes as $\phi \sim r^{-\Delta_+}$, on the contrary, on the horizon, where we already set all the other fields to zero $f(r_h) = A_t(r_h) = \chi(r_h) = 0$, we can see from the near horizon behavior of (3.31) that the scalar field goes to a constant at the horizon $\phi(r_h) \propto 1$. This is quite remarkable, the superconducting phase corresponds to a **black hole with scalar hair**. The energy density



(a) Scalar field solution ϕ



(b) Gauge field solution $(qA_t/\mu$

Figure 3.3: Profile of the scalar field and the gauge field solutions for the broken phase with $q = 3$, $m^2 = -3.5$ in $d = 3 + 1$ dimensions. The temperature is fixed at $T = 0.56T_c$. We can see that most of the scalar field energy is concentrated deep in the interior, close to the horizon, and then decreases towards the boundary, as we would expect from the interpretation of the radial direction as an energy scale, since the UV limit is not affected by the condensate in the ground state.
The profile for the gauge solution has been made dimensionless by rescaling it with q/μ. We can see from the plot the asymptotic behavior $A_t \sim \mu/q$ for large r.

of the atmosphere of scalar hairs that form in the broken phase is responsible for the the nonzero order parameter in the boundary theory. This may sound a bit confusing at first because in general relativity we are familiar with the *no-hair theorem* that forbids hairy black hole solutions, however, the conjecture is not true in asymptotically anti-de Sitter geometries where these solutions are allowed.

In figure 3.3 we show the profile of the scalar and gauge field for a fixed value of q, d and m^2. We can see that the scalar field energy is mostly concentrated in the IR regime close to the horizon where it has its maximum, and then decrease towards the boundary, a clear manifestation of the fact that, moving towards the UV regime, the influence on the solution of the presence of a broken phase in the low energy regime is negligible.

In figure 3.4 we plotted different solutions for the scalar field at a temperature just slightly below T_c and for the same value of q and m^2, corresponding to different initial conditions for the scalar field at the horizon. This is due to the fact that in general there is more than one initial condition corresponding to the same boundary value problem, in our example it means that there are several values for $\phi(r_h)$ that give an unsourced solution, $\phi_0 = 0$, on the boundary. The more thermodynamically stable solution is the strictly positive solution that corresponds to the higher critical temperature (and then to the less stable solution against scalar field fluctuations). This is intuitive since the positive solution corresponds to a higher energy of the scalar atmosphere around the black hole that compensates the instability of the RN solution. Numerically integrating all the three scalar profiles we find that the solution with two nodes is the one with the least energy, and we see that it indeed corresponds to the lowest critical temperature. In the results that follow, we will always use the initial condition corresponding to the least stable solution.

Now that we have chosen a solution, we can look at the behavior of the vacuum expectation value of order parameter with $d = 3 + 1$, presented in figure 3.5, and we can see the curve is very similar to the one of BCS theory. Looking closely at temperature close to T_c (figure 3.6) we find the square root dependence $\langle O \rangle \propto (T - T_c)^{1/2}$ typical of the mean-field theory description of a second-order phase transition. This mean-field behavior is present in all dimensions $d \geq 3$, including $d = 2 + 1$. This sounds a bit strange, since we do not expect a mean-field theory description of the phase transition in 2 spatial dimensions[4]. The reason behind this strangeness resides in the large-N limit implicit in the classical gravity dual of the AdS/CFT correspondence. Thermal fluctuations of the order parameter are suppressed in this limit, forcing

[4]In $d \leq 2$ spatial dimensions the **Coleman-Mermin-Wagner** theorem states that a continuous symmetry cannot be broken at finite temperature

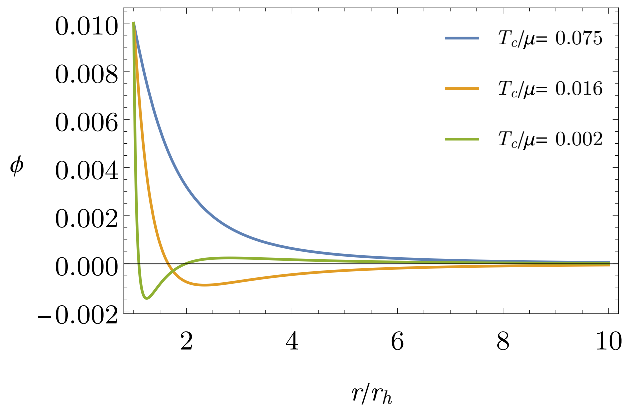


Figure 3.4: Three solution for the scalar field ϕ for $q = 3$, $m^2 = -3.5$ and $d = 3+1$. Each profile corresponds to different initial conditions at the horizon that gives a solution with $\phi_0 = 0$ (i.e. an unsourced expectation value). The temperature is fixed slightly below the critical temperature, so that the scalar field is small. The positive solution (blue line) is the one with the more thermodynamically stable with the higher value for T_c/μ, followed by the solution with only one node (yellow line), and the two nodes solution (green line). This is consistent with the fact that, numerically integrating the profiles, we find the positive solution to be the one with the higher energy density while the two nodes solution has the lower value.

a mean-field behavior even for dimensions where usually fluctuations destroy the long-range order in the low-temperature phase.

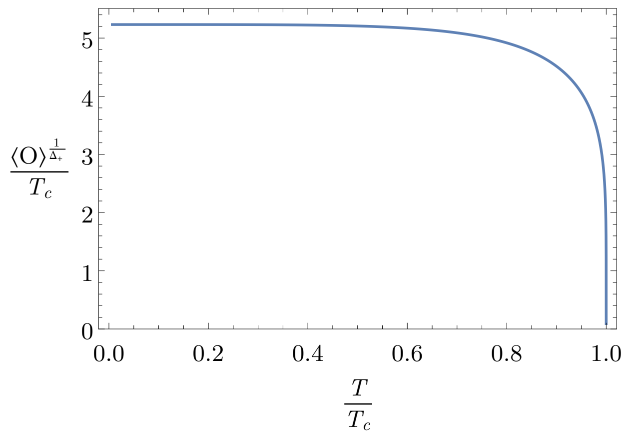


Figure 3.5: Plot of the dimensionless ratio $\frac{\langle O\rangle^{1/\Delta_+}}{T_c}$ as a function of temperature for a fixed chemical potential (i.e., in the grand canonical ensemble). In the plot we used the solution for $q = 3$, $m^2 = -3.5$ in $d = 3+1$ dimension. The profile shows the typical behavior of a second-order phase transition in mean-field theory.

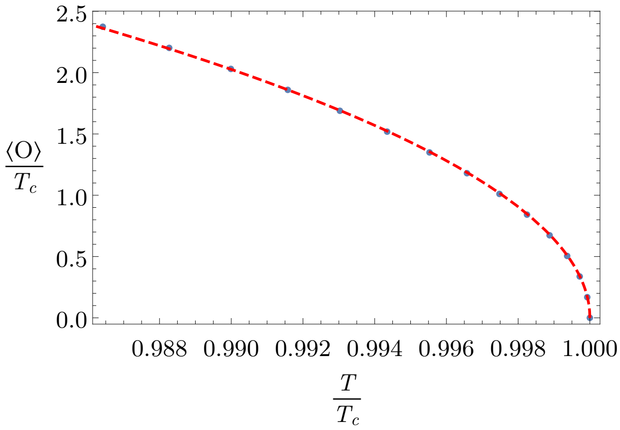


Figure 3.6: Details of the expectation value of the order parameter $\langle O\rangle$ close to T_c for $q = 3$, $m^2 = -3.5$ in $d = 3+1$ dimension. The blue dots repesent the numerical solution while the dashed red line is a fit of the form $(x-1)^{1/2}$ that shows the mean-field behavior $\langle O\rangle \propto (T - T_c)^{1/2}$.

Chapter 4

Fluctuations in the holographic superconductor

In this chapter we study fluctuations of the scalar and gauge fields on top of the superconducting background. We introduce the concept of *intrinsic dynamics* and explain its relevance from a physical point of view. We then present numerical results for the spectral functions describing the intrinsic dynamics of the order parameter and analyze how they differ from what we expect from a standard superconductor. Next, we proceed to study how the spectral function changes as we include the coupling with the fluctuations of the gauge field into the description.

4.1 Holographic superconductor dynamics

The background solution of the gravitational theory that is dual to a superconductor captures the equilibrium properties of the theory. If we want to study dynamical properties we have to consider non-stationary solutions in the bulk, that is, we have to introduce fluctuations of the classical field on top of the background solutions:

$$\phi \to \phi + \delta\phi \ , \quad A_\mu \to A_\mu + \delta A_\mu \ , \quad g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \ . \tag{4.1}$$

Since all the fields are coupled at low energies, introducing fluctuations of any of the fields sources fluctuations in all the others and we have to deal with a coupled problem, this is the holographic equivalent of operator mixing from a field theory point of view.

In chapter 2 we showed how to compute retarded Green's function from the gravitational theory, however, we presented examples where the equations of motion for the fluctuations were not coupled. In the following, we explain how to generalize the procedure for a system of coupled equations.

Let us consider a general set of M fields $\{\Phi^I\}_{I=1,\cdots,M}$, with I denoting the different fields, and a corresponding action

$$S\left[\Phi^I\right] = \int d^{d+1}x\, \mathcal{L}[\Phi^I] \ .$$

As for the uncoupled case, in order to extract the retarded Green's function we have to expand the action up to second order in fluctuations of the fields $\Phi^I \to \Phi^I + \delta\Phi^I$. After partial integration and application of Stokes' theorem, the part of the action containing second order terms, that we denote with $S^{(2)}$, assumes the form of a bulk term plus a boundary contribution[1]

$$S^{(2)} = S^{(2)}_{bulk} + S^{(2)}_{bdy} = -\frac{1}{2}\int d^{d+1}x\, \delta\boldsymbol{\Phi}^\dagger \boldsymbol{G}_B^{-1} \delta\boldsymbol{\Phi} + S^{(2)}_{bdy} \ , \tag{4.2}$$

with $\delta\boldsymbol{\Phi} = (\Phi^1, \cdots, \Phi^M)^T$ the vector containing all the fields Φ^I and the matrix operator $\boldsymbol{G}_B^{-1}$ defining the *linearized equation of motion* (the equations of motion for the fluctuations $\delta\Phi^I$) in the bulk as

$$\boldsymbol{G}_B^{-1}\delta\boldsymbol{\Phi} = 0 \ . \tag{4.3}$$

If the operator is diagonal we have an uncoupled problem, while in a more general case (4.3) defines a system of M coupled differential equations.

The next step to extract the Green's function is to find the asymptotic behavior of the fields by studying the linearized equations of motion in the limit $r \to \infty$. A general boundary expansion takes the form

$$\delta\Phi(r,x)^I \sim \delta\Phi(x)^I_s r^{-\Delta^I_-} + \delta\Phi(x)^I_v r^{-\Delta^I_+} \qquad \text{for} \qquad r \to \infty \tag{4.4}$$

where $\Delta^I_\pm$ depends on the field considered and $\Delta^I_- < \Delta^I_+$, so that $\delta\Phi(x)^I_s$ is the coefficient of the leading term and it represent the change in the source related to the field Φ^I, hence the subscript s, while the coefficient of the subleading order is related to the change of the VEV of the corresponding operator and we denote it as $\delta\Phi(x)^I_v \propto \delta\langle O_I\rangle$. In a theory with translation invariance we can Fourier transform the field to obtain a set of M second order ordinary differential equations in r. We define the Fourier transform as:

$$\delta\Phi(r,x)^I = \int \frac{d^dk}{(2\pi)^d}\, \delta\Phi(r,k)^I e^{ik_\mu x^\mu} \ , \tag{4.5}$$

with $x_\mu k^\mu = \eta_{\mu\nu}x^\mu k^\mu$ and $k^\mu = (\omega, \boldsymbol{k})$. The regularized boundary term $S^{(2)}_{bdy}$ then assumes the form

$$S^{(2)}_{bdy} = \frac{1}{2}\int d^dk \sum_{I=1}^{M} \delta\langle O_I\rangle\, \delta\Phi^I_s \ , \tag{4.6}$$

from which we can extract the fluctuations of the vacuum expectation values $\delta\langle O_I\rangle$ for each operator.

In the uncoupled problem, we know that in linear response theory we could define $G_{OO}\delta\Phi_s = \delta\langle O\rangle$ and extract the retarded Green's function by simply taking the ratio of expectation value obtained from (4.6) and the corresponding source term. In the fully coupled problem, however, a change in any of the operators, for example $\delta\langle O_J\rangle$, is given by a linear combination of changes in all the source terms

$$\delta\langle O_J\rangle(k) = \sum_{I=1}^{M} G^R_{O_JO_I}(k)\delta\Phi^I_s(k) \ , \tag{4.7}$$

and we can then rewrite the boundary term as

$$S^{(2)}_{bdy} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \sum_{I,J}^{M} \delta\Phi^I_s G^R_{O_IO_J}\delta\Phi^J_s \equiv \frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \sum_{I,J}^{M} \delta\boldsymbol{\Phi}_s \boldsymbol{G}^R \delta\boldsymbol{\Phi}_s \ , \tag{4.8}$$

where we defined the retarded Green's function matrix $\boldsymbol{G}^R$ as $\boldsymbol{G}^R_{IJ} = G^R_{O_IO_J}$.

The big difference is due to the fact that when the sources are coupled, a single solution of the linearized equations of motion is not enough. In order to extract the Green's functions, we need M independent solutions. So how do we proceed to extract these independent solutions?

[1]Here when we refer to the boundary term we consider the finite boundary term that we obtain after the renormalization procedure explained in chapter 2. As a brief reminder, when using the Stokes' theorem after partial integration we evaluate the integrand at a finite cutoff $\Lambda \gg 1$, we then introduce boundary terms to cancel the divergent terms in the limit $\Lambda \to \infty$ and we then perform this limit to obtain a regularized boundary action.

The linearized equations of motion (4.3) define a system of M second order linear ordinary differential equations (in general, we can have $N \leq M$ coupled differential equations, and then we would need only N independent solutions), and we therefore need to provide two initial conditions to solve them. As explained in chapter 2, a crucial part in the holographic computation of retarded Green's functions is the choice of the correct boundary conditions at the horizon. In real-time formalism, the solution close the horizon assumes the form

$$\Phi_I \sim \alpha_I(k)(r - r_h)^{\pm\beta_I} \qquad \text{for } r \to r_h \tag{4.9}$$

with $\alpha_I(k)$ a prefactor that cannot depend on r, and β_I a complex number. In many cases the exponent assumes the form $\beta = \frac{i\omega}{4\pi T}$ (however, this is not always true as we will see for the time component of the gauge field in our model). We then have two possibilities, corresponding to the different sign at the exponent, ingoing and outgoing boundary conditions. In order to obtain the retarded time correlators, we have to impose the ingoing wave condition at the horizon. With the Fourier convention we have chosen, this corresponds to picking the solution with the minus sign as it can be seen by restoring the time dependence in the solution[2]. Choosing the ingoing solutions accounts for one of the two degrees of freedom we have in imposing the boundary conditions for the second order differential equations, the other is determined by fixing the prefactors $\alpha_I(k)$. Due to the linearity of the differential equations, of course in the uncoupled problem the prefactor does not play any role in the solution, however, when the equations are coupled, a linearly independent set of prefactors generates independent solutions. A solution is therefore completely determined by all the M prefactors, that we can organize in a vector

$$\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \cdots, \alpha_{\mathrm{M}}) \ , \tag{4.10}$$

and we then have a map between independent solutions to the linearized equations of motion and linearly independent vectors $\boldsymbol{\alpha}$.

In order to generate the necessary solutions to extract the Green's function we then just have to define a set of linearly independent vectors that we can choose for example as:

$$\begin{aligned}
\boldsymbol{\alpha}^{(1)} =&(1, 1, 1, \cdots, 1) \ , \\
\boldsymbol{\alpha}^{(2)} =&(1, -1, 1, \cdots, 1) \ , \\
&\vdots \\
\boldsymbol{\alpha}^{(J)} =&(1, 1, 1, \cdots, \underbrace{-1}_{J\text{th element}}, \cdots, 1) \ , \\
&\vdots \\
\boldsymbol{\alpha}^{(M)} =&(1, 1, 1, \cdots, -1) \ ,
\end{aligned} \tag{4.11}$$

and solve the linearized equations of motion M times, one for each α^I. From each solution, using the boundary expansion (4.4), we can extract the sources of every operator and construct a vector of sources fluctuations

$$\delta\boldsymbol{\Phi}_s^I \equiv (\delta\Phi_{s,1}^I, \delta\Phi_{s,2}^I, \cdots, \delta\Phi_{s,\mathrm{M}}^I) \ , \tag{4.12}$$

corresponding to solution I. Putting all the solutions together, we can construct a matrix $\boldsymbol{A}$, where each line is vector of sources fluctuations corresponding to different solutions, that is:

$$\boldsymbol{A} = \begin{pmatrix} \delta\Phi_{s,1}^1 & \delta\Phi_{s,2}^1 & \cdots & \delta\Phi_{s,\mathrm{M}}^1 \\ \delta\Phi_{s,1}^2 & \delta\Phi_{s,2}^2 & \cdots & \delta\Phi_{s,\mathrm{M}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta\Phi_{s,1}^{\mathrm{M}} & \delta\Phi_{s,2}^{\mathrm{M}} & \cdots & \delta\Phi_{s,\mathrm{M}}^{\mathrm{M}} \end{pmatrix} . \tag{4.13}$$

[2] Restoring the time dependence $e^{-i\omega t}$ at the horizon the solution behaves as $\sim e^{-i\omega\left(t \pm \frac{\log(r-r_h)}{4\pi T}\right)}$. Defining the new coordinate $\tilde{r} = \log(r - r_h)/(4\pi T)$, that sets the horizon at $-\infty$ and the boundary at $+\infty$, the near horizon behavior is $\sim e^{-i\omega(t \pm \tilde{r})}$ and we see that the solution with the minus sign describes a wave moving towards the horizon as t grows.

In a similar way we construct a matrix of the changes in the vacuum expectations value that we obtain from (4.6)

$$\boldsymbol{B} = \begin{pmatrix} \delta\langle O\rangle_1^1 & \delta\langle O\rangle_2^1 & \cdots & \delta\langle O\rangle_{\mathrm{M}}^1 \\ \delta\langle O\rangle_1^2 & \delta\langle O\rangle_2^2 & \cdots & \delta\langle O\rangle_{\mathrm{M}}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta\langle O\rangle_1^{\mathrm{M}} & \delta\langle O\rangle_2^{\mathrm{M}} & \cdots & \delta\langle O\rangle_{\mathrm{M}}^{\mathrm{M}} \end{pmatrix} . \tag{4.14}$$

Now, in matrix form, equation (4.7) reads

$$\boldsymbol{A}\boldsymbol{G}^R = \boldsymbol{B} \tag{4.15}$$

and all that is left to do to obtain the Green's functions is to invert the matrix A and compute $\boldsymbol{G}^R = \boldsymbol{A}^{-1}\boldsymbol{B}$. Since a matrix $\boldsymbol{M}$ is invertible if and only if its determinant is nonzero, we see that the poles are the same for all the Green's functions and they are determined by $\det[\boldsymbol{A}(k)] = 0$, where we have explicitly written the momentum dependence of the matrix $\boldsymbol{A}$ as a reminder that the coefficients are not constant in k.

In order to better understand the procedure, in the remainder of the chapter we will go through the steps of the computation for the gauge and scalar field fluctuations in the holographic superconductor model.

4.2 Fluctuations and intrinsic dynamics

The action describing the minimal model of the holographic superconductor presented in chapter 3 is

$$S = \int d^{d+1}x\,\sqrt{-g}\left(\frac{c^3}{16\pi G}(R-2\Lambda) - \frac{1}{4\mu_0 c}F_{\mu\nu}F^{\mu\nu} - |D_\mu\phi|^2 - \frac{m^2c^2}{\hbar^2}|\phi|^2)\right) , \tag{4.16}$$

that depends on three fields: the metric tensor $g_{\mu\nu}$, the gauge vector field A_μ and the complex scalar field ϕ, coupled to the other two through covariant derivatives. As we have just seen, what this implies is that when we consider fluctuations of one field, we necessarily have to turn on fluctuations of the other two. This problem is in principle quite straightforward, but it involves solving a very large set of coupled ODE that becomes a very challenging task numerically, so we start with a simplification. We study what we call **intrinsic dynamics**. We define it as the dynamics of a *less coupled* problem obtained by neglecting fluctuations of some of the fields involved, that is for example, the intrinsic dynamics of the order parameter can be obtained by letting the complex scalar field fluctuates $\phi \to \phi + \delta\phi$ but considering the other fields static (i.e., simply setting the other fields fluctuations to zero). This is surely a good approximation very close to the critical temperature where the order parameter that regulates the coupling is small, however, as we lower the temperature the strength of the coupling becomes more and more relevant and the contributions due to the interactions with other fields fluctuations completely change the dynamics and they cannot be ignored to obtain the physical solution. So why are we interested in the intrinsic dynamics?

The reason is that looking at the intrinsic dynamics gives us insight into the contribution of each field fluctuations to the fully coupled problem. What we expect, from the Dyson equation, is that even at low temperature, where the coupling can completely change the behavior of the solution, we can still write, at least for the long-wavelength dynamics, the Green's function of the full problem as composed by the Green's function of the intrinsic dynamics, plus a self-energy contribution due to the coupling that will change the behavior of the system. That is, if we call G_0 the Green's function of the intrinsic dynamics, for example of the scalar field, we expect to be able to write the Green's functions of the scalar sector of the the fully coupled problem G as

$$G^{-1} = G_0^{-1} + \Sigma . \tag{4.17}$$

with Σ the self-energy due to the coupling[3].

[3]Remember that the Dyson equation is $G = G_0 + G_0\Sigma G$.

This is important because it gives an idea of how each term in the model contributes to the pole structure, moreover, when the coupling is strong, for example at low temperature for the gauge and scalar field coupling, we must rely on numerical methods to obtain the Green's function as we cannot do a perturbation expansion. One of the shortcomings of numerical solutions to a problem with several variables involved is that it may be extremely difficult to extract a mathematical model that correctly describes the result, as this depends on many variables and the exact contribution of each of them is not known and may be hard to understand from a physical perspective. If (4.17) holds, studying the intrinsic dynamics gives us a procedure to start from a simpler problem, and once we understand it, we can build from it by gradually introducing the contributions of the coupling with other fields until we understand the full problem.

4.3 Fluctuations of the scalar and gauge field

In this thesis we study the intrinsic dynamics of the scalar field and the gauge field, that is, we neglect fluctuations of the metric tensor. Notice that this is different from the *probe limit* where we send the charge q to infinity while keeping the qA_μ and $q\phi$ fixed. In this limit, the metric decouples from the other fields, and we can indeed neglect metric fluctuations. However, in the following we use a fully coupled background solution defined by the equations (3.29)-(3.32) for $q \sim 1$, that is different from the background in the probe limit.

Since the metric is kept fixed, we can focus our attention on the matter part of the action

$$S = -\int d^{d+1}x\,\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^* D_\mu\phi + m^2\phi^*\phi\right) \tag{4.18}$$

Upon variation of the fields, the terms second order in fluctuations give $S^{(2)} = S^{(2)}_{bulk} + S^{(2)}_{bdy}$ with

$$\begin{aligned}\delta S^{(2)}_{bulk} = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\Big[& -\delta\phi^*\left(D_\mu D^\mu - m^2\right)\delta\phi - \delta\phi\left(D_\mu D^\mu - m^2\right)\delta\phi^* \\ & + \delta A_\mu\left(g^{\mu\nu}\left(-\Box + 2q^2\phi^2\right) + \nabla^\mu\nabla^\nu\right)\delta A_\nu \\ & + \delta\phi\left(-iq\phi^*\nabla^\nu - 2iq\partial^\nu\phi^* + 2q^2A^\nu\phi^*\right)\delta A_\nu + \delta A_\mu\left(iq\phi^*\partial^\mu + 2q^2A^\mu\phi^* - iq\partial^\mu\phi^*\right)\delta\phi \\ & + \delta\phi^*\left(iq\phi^*\nabla^\nu + 2iq\partial^\nu\phi + 2q^2A^\nu\phi\right)\delta A_\nu + \delta A_\mu\left(-iq\phi\partial^\mu + 2q^2A^\mu\phi + iq\partial^\mu\phi\right)\delta\phi^*\Big] ,\end{aligned} \tag{4.19}$$

that can be conveniently rewritten in matrix form (notice the change of sign in the prefactor)

$$\delta^2 S_{bulk} = \frac{1}{2}\int d^{d+1}x\,\delta\boldsymbol{\Phi}^\dagger \boldsymbol{G}_B^{-1}\delta\boldsymbol{\Phi}\ , \tag{4.20}$$

with

$$\delta\boldsymbol{\Phi} = \begin{pmatrix}\delta\phi\\ \delta\phi^*\\ \delta A_\nu\end{pmatrix} \qquad\qquad \delta\boldsymbol{\Phi}^\dagger = \begin{pmatrix}\delta\phi^* & \delta\phi & \delta A_\mu\end{pmatrix}\ , \tag{4.21}$$

and

$$\boldsymbol{G}_B^{-1} = \begin{pmatrix} D_\mu D^\mu - m^2 & 0 & -iq\phi\nabla^\nu - 2iq\partial^\nu\phi - 2q^2A^\nu\phi \\ 0 & D^*_\mu D^{\mu *} - m^2 & iq\phi^*\nabla^\nu + 2iq\partial^\nu\phi^* - 2q^2A^\nu\phi^* \\ iq\partial^\mu\phi^* - 2q^2A^\mu\phi^* - iq\phi^*\partial^\mu & -iq\partial^\mu\phi - 2q^2A^\mu\phi + iq\phi\partial^\mu & g^{\mu\nu}\left(\nabla_\sigma\nabla^\sigma - 2q^2\phi^2\right) - \nabla^\nu\nabla^\mu \end{pmatrix} . \tag{4.22}$$

As explained in the previous section, the linearized equations of motions are defined by $\boldsymbol{G}_B^{-1}\delta\boldsymbol{\Phi} = 0$.

The boundary term, before regularization, is

$$\delta S^{(2)}_{bdy} = -\frac{1}{2}\int d^{d+1}x\,\sqrt{-g}\nabla^\mu\Big(iq\phi^*\delta A_\mu\delta\phi - iq\phi\delta A_\mu\delta\phi^* + (\partial_\mu\delta\phi^* + iq\delta\phi A_\mu)\,\delta\phi + (\partial_\mu\delta\phi - iq\delta\phi A_\mu)\,\delta\phi^* \\ - \nabla_\nu\delta A_\mu\delta A^\nu + \nabla_\mu\delta A_\nu\delta A^\nu\Big)\;. \tag{4.23}$$

Using Stokes' theorem and inserting a high energy cutoff $r=\Lambda$ we can rewrite it as

$$\delta S^{(2)}_{bdy} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int d^dx\,\sqrt{-h}n^\mu\left[h^{\nu\sigma}F_{\mu\sigma}\delta A_\nu + \delta\phi(\partial_\mu\phi^* + iqA_\mu\phi^*) + \delta\phi^*(\partial_\mu\phi - iqA_\mu\phi)\right]\Big|_\Lambda\;, \tag{4.24}$$

where we used the fact that the integrand is zero when evaluated at the horizon, due to the initial conditions of the background fields.

To make calculations simpler, we choose a gauge where $\delta A_r = 0$. Notice that this does not completely fix the gauge as we are still free to make a gauge transformation that does not depend on r.

Now the first step consists of Fourier transforming the fields and writing down the linearized equations of motion in terms of the background ansatz presented at the beginning of chapter 3. For the Fourier transform, we choose the convention (4.5) for all the fields. Notice that this means that we treat $\delta\phi$ and $\delta\phi^*$ as two different fields and we use the same sign in the exponent of the Fourier transform: $\delta\phi(x)\propto\int d^dk\,\delta\phi(k)e^{ik_\mu x^\mu}$ and $\delta\phi^*(x)\propto\int d^dk\,\delta\phi^*(k)e^{ik_\mu x^\mu}$. We could just as well choose to use the complex conjugate Fourier transform for $\delta\phi^*$ and in the end we would obtain the same results as long as we stay consistent all along the computation (in particular we have to be careful at which solution corresponds to ingoing boundary condition at the horizon, changing convention requires a change in the sign of the exponent in the ingoing boundary condition). First of all, thanks to rotational invariance we can set the momentum in one direction, without loss of generality let us pick the x direction so that $k^\mu = (\omega, k, 0, \cdots, 0)$, this simplifies the linearized EOM and from (4.22) with the metric ansatz (3.22) we obtain:

$$0 = f\delta\phi'' + \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\phi' + \left(\frac{e^\chi\omega^2}{f} - \frac{k^2}{r^2} + \frac{2qe^\chi\omega A_t}{f} + \frac{q^2e^\chi A_t^2}{f} - m^2\right)\delta\phi \\ + \frac{qk\phi}{r^2}\delta A_x + \left(\frac{qe^\chi\omega\phi}{f} + \frac{2q^2e^\chi\phi A_t}{f}\right)\delta A_t \tag{4.25}$$

$$0 = f\delta\phi^{*\prime\prime} + \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\phi^{*\prime} + \left(\frac{e^\chi\omega^2}{f} - \frac{k^2}{r^2} - \frac{2qe^\chi\omega A_t}{f} + \frac{q^2e^\chi A_t^2}{f} - m^2\right)\delta\phi^* \\ - \frac{qk\phi}{r^2}\delta A_x - \left(\frac{qe^\chi\omega\phi}{f} - \frac{2q^2e^\chi\phi A_t}{f}\right)\delta A_t \tag{4.26}$$

$$0 = \delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_x' + \left(\frac{e^\chi\omega^2}{f^2} - \frac{2q^2\phi^2}{f}\right)\delta A_x + \frac{e^\chi k\omega}{f^2}\delta A_t + \frac{qk\phi}{f}\left(\delta\phi - \delta\phi^*\right) \tag{4.27}$$

$$0 = \delta A_t'' + \left(\frac{\chi'}{2} + \frac{d-1}{r}\right)\delta A_t' - \left(\frac{k^2}{r^2f} + \frac{2q^2\phi^2}{f}\right)\delta A_t - \frac{k\omega}{r^2f}\delta A_x - \frac{q\omega\phi}{f}\left(\delta\phi - \delta\phi^*\right) - \frac{2q^2\phi A_t}{f}\left(\delta\phi + \delta\phi^*\right) \tag{4.28}$$

$$0 = \frac{e^\chi\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' + q\phi\left(\delta\phi' - (\delta\phi^*)'\right) + q\phi'\left(\delta\phi^* - \delta\phi\right) \tag{4.29}$$

$$0 = \delta A_i'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_i' + \left(\frac{e^\chi\omega^2}{f^2} - \frac{k^2}{r^2f} - \frac{2q^2\phi^2}{f}\right)\delta A_i\;, \tag{4.30}$$

where the last line is a set of $d-2$ equations, one for each transverse components $x^i \neq t, x$. These equations are decoupled from all the others and do not play any significant role in the holographic superconductor description (since they are decoupled from the order parameter fluctuations they do not affect the dynamics of the order parameter in the superconducting phase), so we will not consider them further.

What we are interested in is then a set of five coupled equations in four unknowns, so it looks like there is one extra equation. The five equations, however, are compatible and later we will see that the constraint equation (4.29) has a nice interpretation on the boundary, it gives the Ward identities of the theory.

As the next step, we do the same for the boundary term, inserting the expression for the metric in (4.24) and considering the gauge choice we obtain

$$S_{bdy}^{(2)} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int d^dx\, r^{d+1}\left[r^{-2}\eta^{\nu\sigma}\delta A'_\sigma\delta A_\nu + \delta\phi(\delta\phi^*)' + \delta\phi^*\delta\phi'\right]\Big|_{r=\Lambda} , \tag{4.31}$$

that in Fourier space becomes

$$S_{bdy}^{(2)} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\, r^{d+1}\left[r^{-2}\eta^{\nu\sigma}\delta A'_\sigma(r,k^\mu)\delta A_\nu(r,-k^\mu) + \delta\phi(r,k^\mu)\delta\phi^{*\prime}(r,-k^\mu) + \delta\phi^*(r,k^\mu)\delta\phi'(r,-k^\mu)\right]\Big|_{r=\Lambda} . \tag{4.32}$$

We now need to find the boundary behavior of the fields involved. This can be done by studying the equations (4.25)-(4.29) in the limit $r\to\infty$. However, we have done this already. The result is equivalent to the boundary behavior of the zero temperature solution since the spacetime is asymptotically AdS and the the different geometry in the bulk does not influence the expansion for $r\to\infty$. We have already seen in chapter 2 the asymptotic solution for both the scalar and the gauge field in AdS. For a scalar field it is given by

$$\delta\phi = \delta\phi_s(\omega,k)r^{-\Delta_-} + \delta\phi_v(\omega,k)r^{-\Delta_+} + \cdots \tag{4.33}$$

with

$$\Delta_\pm = \frac{d}{2}\pm\sqrt{\frac{d^2}{4}+m^2} . \tag{4.34}$$

Notice that now we are considering fluctuations of the order parameter as a response to a source, so we do not want to put the source term to zero as it was for computing the VEV of the scalar operator.

For the gauge field on the other hand, in chapter 2 we first derived the boundary expansion using a power law ansatz, but as we notice at the end from the full solution in AdS, this ansatz fails to capture a divergent logarithmic term in boundary theories with an even number of dimensions $d\in 2\mathbb{N}$ (as a reminder, the solution close to the boundary is proportional to the Bessel function $\propto K_{d/2-1}$, that for integer indices, i.e., for d even, contains a term $r^{-2}\log(r)$). We postponed the treatment of the correct renormalization procedure until now, but we will explain it in detail below. Up to this point, we decided to treat everything in terms of a general number of dimensions d, in order to be able to quickly adapt our results for systems of different dimensions. The most commonly studied superconductor model is in $d = 2+1$, mostly because several high-temperature superconducting materials are effectively two-dimensional. Nonetheless, in the following, we will work in $d = 3+1$, with the example in mind of a Fermi gas at unitarity. In such a number of dimensions, the asymptotic behavior of the gauge field becomes

$$\delta A_\mu(r,\omega,k) = \delta a_\mu(\omega,k) + \delta b_\mu(\omega,k)r^{-2} + \delta c_\mu(\omega,k)r^{-2}\log(r) + \cdots , \tag{4.35}$$

where the dots denote higher order terms (the argument of the logarithm is, in general, the dimensionless quantity r/r_h, but we used the isometries of AdS to set $r_h = 1$). The three momentum dependent coefficients, are not all independent. Inserting this boundary expansion into the linearized equation of motions and matching coefficients of the same order in r give:

$$\delta c_t = -\frac{k}{2}(\omega\delta a_x + k\delta a_t) \tag{4.36}$$

$$\delta c_x = \frac{\omega}{2}(\omega\delta a_x + k\delta a_t) . \tag{4.37}$$

Substituting the boundary expansions of the field in the boundary action we have

$$S^{(2)}_{bdy} = \lim_{\Lambda\to\infty} \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[\eta^{\mu\sigma} \left(2\delta b_\sigma \delta a_\mu + 2\delta c_\sigma \delta a_\mu \log(r) + \delta c_\sigma \delta a_\mu + \mathcal{O}(r^{-2}\log^2(r))\right) \\ + \left(\Delta_- \delta\phi_s \delta\phi_s^* r^{2\nu} + \Delta_+ \delta\phi_s \phi_v^* + \Delta_- \delta\phi_v \phi_s^* + \mathcal{O}(r^{-2\nu})\right) \\ + \left(\Delta_- \delta\phi_s^* \delta\phi_s r^{2\nu} + \Delta_+ \delta\phi_s^* \phi_v + \Delta_- \delta\phi_v^* \phi_s + \mathcal{O}(r^{-2\nu})\right) \Big]\Big|_{r=\Lambda} , \tag{4.38}$$

and we can now clearly see that there are divergent terms in the action, both for the scalar field components and the gauge field. The counterterms that respect the symmetries of the action and cancel the divergences are

$$S_{c.t.\,\phi} = -\int d^dx\, \sqrt{-h}\Delta_- \phi\phi^* \Big|_\Lambda \tag{4.39}$$

$$S_{c.t.\,A} = \frac{\log(\Lambda)}{4} \int d^dx\, \sqrt{-h} F_{\mu\nu}F^{\mu\nu} \Big|_\Lambda , \tag{4.40}$$

and after expanding in fluctuations of the fields we find

$$S^{(2)}_{c.t.\,\phi} = -\frac{1}{2}\int d^4x\, \sqrt{-h} 2\Delta_- \delta\phi\delta\phi^* \Big|_{r=\Lambda} \\ S^{(2)}_{c.t.\,A} = \frac{1}{2}\int d^4x\, \log(\Lambda)\sqrt{-h}\delta A_\nu \nabla_\mu \delta F^{\mu\nu} \Big|_{r=\Lambda} , \tag{4.41}$$

that in k-space and in the boundary limit become:

$$S^{(2)}_{c.t.\,\phi} = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \left(2\Delta_- \delta\phi_s \delta\phi_s^* r^{2\nu} + 2\Delta_- \delta\phi_s \delta\phi_v^* + 2\Delta_- \delta\phi_s^* \delta\phi_v + \mathcal{O}(r^{-2\nu})\right)\Big|_{r=\Lambda} \tag{4.42}$$

$$\delta^2 S_{c.t.\,A} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \log(\Lambda)\, \eta^{\mu\rho}\eta^{\nu\sigma}\delta a_\nu \left(k_\mu k_\rho \delta a_\sigma - k_\mu k_\sigma \delta a_\rho\right) + \mathcal{O}(r^{-2}\log^2(r))\Big|_{r=\Lambda} \tag{4.43}$$

$$. \tag{4.44}$$

Given the explicit form of c_μ (4.36) and (4.37) the second counterterm can be written as

$$S^{(2)}_{c.t.\,A} = \int \frac{d^4k}{(2\pi)^4} \left[\log(r)\left(-k(k\delta a_t + \omega\delta a_x)\delta a_t - \omega(k\delta a_t + \omega \delta a_x)\delta a_x\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda} \\ = \int \frac{d^4k}{(2\pi)^4} \left[\log(r)\left(2\delta c_t \delta a_t - 2\delta c_x \delta a_x\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda} \\ = -\int \frac{d^4k}{(2\pi)^4} \left[\log(r)\,\eta^{\mu\sigma}\left(2\delta c_\sigma \delta a_\mu\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda} . \tag{4.45}$$

Adding the counterterms to (4.38), we obtain the regularized boundary action and we can perform the limit $\Lambda \to \infty$ and read off the fluctuations in the expectation values[4]

$$S^{(2),R}_{bdy} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \Big[\left(-2\delta b_t(\omega,k) + \delta c_t(\omega,k)\right)\delta a_t(-\omega,-k) + \left(2\delta b_x(\omega,k) - \delta c_x(\omega,k)\right)\delta a_x(-\omega,-k) \\ + 2\nu\delta\phi_v(\omega,k)\delta\phi_s^*(-\omega,-k) + 2\nu\delta\phi_v(\omega,k)\delta\phi_s^*(-\omega,-k) \Big]\Big|_{r=\Lambda} \\ = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \sum_I \delta\langle O_I(\omega,k)\rangle\, \delta\Phi^I_s(-\omega,-k) , \tag{4.46}$$

[4]Remember that the 1/2 in front of the boundary action should not be included in the expectation value. In a Taylor expansion up to second order we have $S[\Phi+\delta\Phi] = S[\Phi] + \frac{\delta S[\Phi]}{\delta\Phi}\delta\Phi + \frac{1}{2}\frac{\delta^2 S[\Phi]}{\delta\Phi\delta\Phi}(\delta\Phi)^2$, and the 1/2 in front of the integral is exactly that factor in front of the second order term in the expansion. However, the expectation value from the GKPW rule is defined by a functional derivative of the action $\delta S[\Phi]/\delta\phi_s$ and it is therefore clear that the right relation is (4.6).

Now we that we know the form of the boundary terms, the only thing left to do is to finally solve the coupled system of differential equations (4.25)-(4.27) and extract the coefficients of the expansion. In order to impose initial conditions at the horizon, we expand the solution for $r \to r_h$, and use the power law ansatz $(r - r_h)^\beta$. For all the fields there are going to be to possible solutions for β, corresponding to the incoming and outgoing wave condition. With the convention we chose for the Fourier transform, the incoming boundary conditions corresponds to the negative solution for β. We then obtain

$$\begin{aligned} \delta A_x &\sim \alpha_x (r - r_h)^{-\frac{i\omega}{4\pi T}} \\ \delta A_t &\sim \alpha_t (r - r_h)^{1-\frac{i\omega}{4\pi T}} \\ \delta\phi &\sim \alpha_\phi (r - r_h)^{-\frac{i\omega}{4\pi T}} \\ \delta\phi^* &\sim \alpha_{\phi^*} (r - r_h)^{-\frac{i\omega}{4\pi T}} \ , \end{aligned} \tag{4.47}$$

where the coefficients are not all independent, but related by the constraint equation, that gives:

$$\alpha_t = i \left(\frac{k}{r_h^2} \alpha_x + q\phi(r_h)(\alpha_\phi - \alpha_{\phi^*}) \right) \frac{e^{-\chi(r_h)/2}}{1 - \frac{i\omega}{4\pi T}} \ . \tag{4.48}$$

This seems to generate a problem, since we now have only three independent coefficient to set for the initial conditions, and we can only generate three different solutions. There is however, a fourth solution that comes from a gauge transformation

$$\delta a_x = -k \ , \quad \delta a_t = \omega \ , \quad \delta\phi = -q\phi(r), \quad \delta\phi^* = q\phi(r), \tag{4.49}$$

and we can then construct the matrix $\boldsymbol{A}$ of sources as

$$\boldsymbol{A} = \begin{pmatrix} \delta a_t^1 & \delta a_x^1 & \delta\phi_s^1 & \delta\phi_s^{*1} \\ \delta a_t^2 & \delta a_x^2 & \delta\phi_s^2 & \delta\phi_s^{*2} \\ \delta a_t^3 & \delta a_x^3 & \delta\phi_s^3 & \delta\phi_s^{*3} \\ -\omega & k & 0 & 0 \end{pmatrix} \ . \tag{4.50}$$

where in the last line we used the fact that $\phi(r)$ is unsourced. Similarly we write the corresponding matrix of expectations value

$$\boldsymbol{B} = \begin{pmatrix} \delta\left\langle J^t \right\rangle_1 & \delta\langle J^x \rangle^1 & \delta\langle O \rangle^1 & \delta\langle O^* \rangle^1 \\ \delta\left\langle J^t \right\rangle_1 & \delta\langle J^x \rangle^2 & \delta\langle O \rangle^2 & \delta\langle O^* \rangle^2 \\ \delta\left\langle J^t \right\rangle_3 & \delta\langle J^x \rangle^3 & \delta\langle O \rangle^3 & \delta\langle O^* \rangle^3 \\ 0 & 0 & -q\langle O \rangle & q\langle O \rangle \end{pmatrix} \tag{4.51}$$

$$= \begin{pmatrix} -2\delta b_t^1 & 2\delta b_x^1 & 2\nu\delta\phi_s^1 & 2\nu\delta{\phi^*}_s^1 \\ -2\delta b_t^2 & 2\delta b_x^2 & 2\nu\delta\phi_s^2 & 2\nu\delta{\phi^*}_s^2 \\ -2\delta b_t^3 & 2\delta b_x^3 & 2\nu\delta\phi_s^3 & 2\nu\delta{\phi^*}_s^3 \\ 0 & 0 & -q\langle O \rangle & q\langle O \rangle \end{pmatrix} + \begin{pmatrix} -\frac{k}{2}(\omega\delta a_x^1 + k\delta a_t^1) & -\frac{\omega}{2}(\omega\delta a_x^1 + k\delta a_t^1) & 0 & 0 \\ -\frac{k}{2}(\omega\delta a_x^2 + k\delta a_t^2) & -\frac{\omega}{2}(\omega\delta a_x^2 + k\delta a_t^2) & 0 & 0 \\ -\frac{k}{2}(\omega\delta a_x^3 + k\delta a_t^3) & -\frac{\omega}{2}(\omega\delta a_x^3 + k\delta a_t^3) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.52}$$

(notice that in the last line $\langle O \rangle$ is the equilibrium VEV and not the $\delta\langle O \rangle$). We can define $\boldsymbol{B} \equiv \tilde{\boldsymbol{B}} + \boldsymbol{C}$ where $\tilde{\boldsymbol{B}}$ is the matrix of the subleading coefficient in the expansion in (4.52), while $\boldsymbol{C}$ is the matrix containing the terms coming from the regularization of the gauge field.

The full set of Green's function

$$\boldsymbol{G}_R = \begin{pmatrix} G^{tt} & G^{xt} & G^{Ot} & G^{O^*t} \\ G^{tx} & G^{xx} & G^{Ox} & G^{O^*x} \\ G^{tO} & G^{xO} & G^{OO} & G^{O^*O} \\ G^{tO^*} & G^{xO^*} & G^{OO^*} & G^{O^*O^*} \end{pmatrix} \tag{4.53}$$

is found by inverting the matrix $\boldsymbol{A}$

$$\boldsymbol{G}_R = \boldsymbol{A}^{-1}\boldsymbol{B} = \boldsymbol{A}^{-1}\tilde{\boldsymbol{B}} + \boldsymbol{A}^{-1}\boldsymbol{C} = \boldsymbol{A}^{-1}\tilde{\boldsymbol{B}} + \boldsymbol{K} \tag{4.54}$$

where we can easily show that

$$\boldsymbol{K} = \begin{pmatrix} -\frac{k^2}{2} & -\frac{k\omega}{2} & 0 & 0 \\ -\frac{k\omega}{2} & -\frac{\omega^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{4.55}$$

Notice then that the terms coming from renormalization of the gauge field only add a real contribution to the gauge spectral functions, therefore they do not influence the spectral function, $\mathcal{A}(\omega, k)$ that is related to the imaginary part of the Green's function:

$$\mathcal{A}(\omega, k) = -\frac{2}{\pi}\mathrm{Im}\left[G_R(\omega, k)\right] . \tag{4.56}$$

In section 4.6 we show the results obtained by numerically solving the linearized equation of motion and extracting the coefficients of the leading and subleading term in order to construct the matrix $\boldsymbol{A}$ and $\tilde{\boldsymbol{B}}$. However, let us first take a look at some simpler cases, namely, the intrinsic dynamics of the scalar field and the optical conductivity.

4.4 Intrinsic dynamics of the scalar field

In this section we present the results for the intrinsic dynamics of the order parameter fluctuations. Remember that this means we have to ignore the coupling of the scalar field fluctuation with both the metric and the gauge field fluctuations. Since we already ignored the metric fluctuations in the previous section, this accounts to setting $\delta A_\mu = 0$ in the $S^{(2)}$ derived before, equation (4.32). The operator $\boldsymbol{G}_B$ in (4.3) reduces to a 2×2 diagonal matrices

$$\boldsymbol{G}^{intr}_{B,O}\delta\boldsymbol{\Phi} = \begin{pmatrix} D_\mu D^\mu - m^2 & 0 \\ 0 & D^*_\mu D^{\mu *} - m^2 \end{pmatrix} \begin{pmatrix} \delta\phi \\ \delta\phi^* \end{pmatrix} \tag{4.57}$$

that defines two uncoupled linearized equations of motion. Since they are related by complex conjugation, we can focus on one of them, for example

$$(D_\mu D^\mu - m^2)\delta\phi = 0 . \tag{4.58}$$

Setting the gauge fluctuations to zero in the regularized boundary term (4.46) leaves us with

$$S^{(2),R}_{bdy} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left[2\nu\delta\phi_v(\omega, k)\delta\phi^*_s(-\omega, -k) + 2\nu\delta\phi_v(\omega, k)\delta\phi^*_s(-\omega, -k)\right] , \tag{4.59}$$

with the difference that now, since the equations are decoupled fluctuations of the expectation values of the order parameter $\delta\langle O\rangle(\omega, k) = 2\nu\delta\phi_v(\omega, k)$ are only sourced by the corresponding source term $\delta\phi_s$ and we can extract the Green's function by simply taking a ratio of the subleading and leading order in the boundary expansion

$$G^{intr}_{OO}(\omega, k) = 2\nu\frac{\delta\phi_v(\omega, k)}{\delta\phi_s(\omega, k)} . \tag{4.60}$$

Before showing the numerical results, we briefly review what we should expect in the low-energy limit from a Ginzburg-Landau theory of standard superfluid.

4.4.1 Ginzburg-Landau theory of superfluidity

As we have seen at the end of chapter 3, we are dealing with a second-order phase transition that resembles the usual phase transition of superfluidity, this suggests that we can model the order parameter dynamics with an effective Ginzburg-Landau theory.

In order to understand the phenomenology of the symmetry breaking, is enough to consider a simple model with only temperature dependence, described in $d = 3 + 1$ by the action

$$S = -\int d^4x \left(\alpha(T)|O|^2 + \frac{\beta(T)}{2}|O|^4 \right) , \tag{4.61}$$

with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. For $\alpha \geq 0$ the action clearly presents a minimum for $\langle O \rangle = 0$, but for $\alpha < 0$, the stationary point at $\langle O \rangle = 0$ becomes a maximum and new minima occur at $\langle O \rangle = \sqrt{-\alpha/\beta} e^{i\theta}$, and we now have an infinitely degenerate ground state. Choosing one of the configurations (i.e., fixing a phase θ) accounts for the spontaneous symmetry breaking for $T < T_c$. Here we pick $\langle O \rangle = \sqrt{-\alpha/\beta}$, in accordance to our choice of a real background field $\phi(r)$ in the holographic theory.

As in mean-field superfluidity, we showed that the holographic results behaves as $\langle O \rangle \propto |T - T_c|^{1/2}$ for $T \approx T_c$ and we can therefore choose $\alpha(T) \approx \alpha_0(T - T_c)$ and $\beta(T) \approx \beta_0$ to obtain the observed temperature dependence near T_c.

In order to study the low-energy dynamics of the order parameter fluctuations, however, we obviously need to introduce a space and time dependence in the model (4.61). The time-dependent low-energy effective Ginzburg-Landau model, consistent with the symmetries of the problem is

$$S = -\int dt \int d^3x \left(iaO^* \partial_t O + \gamma|\nabla O|^2 + \alpha|O|^2 + \frac{\beta}{2}|O|^4 \right) . \tag{4.62}$$

It is a low-energy limit as it only accounts for the lowest order terms in the gradient expansion, and is then valid for small values of ω and k. The coefficient a is complex, and it relates to dissipation of the order parameter due to thermal fluctuations that can break the condensed pair. As we are interested in fluctuations of the order parameter, we let it fluctuates around its equilibrium expectation value $O = \langle O \rangle + \delta O$.

In the unbroken phase we have $\langle O \rangle = 0$, and in Fourier space we get

$$S^{(2)} = \frac{-1}{(2\pi)^4} \int d\omega \int d^3k \, \delta O^*(\omega, k) \left(a\omega + \gamma \left| k^2 \right| + \alpha \right) \delta O(\omega, k) , \tag{4.63}$$

and we obtain the Green's function

$$G_{OO^*}(\omega, k) = \frac{1}{a\omega + \gamma|k|^2 + \alpha} , \tag{4.64}$$

where close to T_c we can write $\alpha = \alpha_0(T - T_c)$. This results predicts a quadratic mode

$$\omega = -\frac{\mathrm{Re}(a)}{|a|^2} \left(|k|^2 + \alpha \right) + i \frac{\mathrm{Im}(a)}{|a|^2} \left(|k|^2 + \alpha \right)$$

and we see that the minimum of this mode moves towards $\omega = 0$ as $T \to T_c$ since $\alpha \to 0$ in this limit.

On the other hand, as we lower the temperature below T_c, the system acquires a non zero expectation value $\langle O \rangle = \sqrt{-\alpha/\beta}$. Using this value in the expansion, the action with terms second order in fluctuations reads

$$\begin{aligned} S^{(2)} = & \frac{-1}{(2\pi)^4} \int d\omega \int d^3k \, \delta O^*(\omega, k) \frac{(a\omega + \gamma \left| k^2 \right| - \alpha)}{2} \delta O(\omega, k) + \delta O(-\omega, -k) \frac{(-a^*\omega + \gamma \left| k^2 \right| - \alpha)}{2} \delta O(-\omega, -k) \\ & - \frac{\alpha}{2} \left(\delta O^*(\omega, k) \delta O^*(-\omega, -k) + \delta O(\omega, k) \delta O(-\omega, -k) \right) \\ = & \frac{-1}{2(2\pi)^4} \int d\omega \int d^3k \begin{pmatrix} \delta O^*(\omega, k) & \delta O(\omega, k) \end{pmatrix} \boldsymbol{G}_R^{-1} \begin{pmatrix} \delta O(-\omega, -k) \\ \delta O^*(-\omega, -k) \end{pmatrix} \end{aligned} \tag{4.65}$$

with

$$G_R^{-1} = \begin{pmatrix} -a^*\omega + \gamma\left|k^2\right| + \alpha & -\alpha \\ -\alpha & a\omega + \gamma\left|k^2\right| - \alpha \end{pmatrix} . \tag{4.66}$$

The pole of G^{OO^*} are given by the zeros of the determinant of G_R^{-1}

$$\det\left[G_R^{-1}\right] = -|a|^2\omega^2 + \gamma|k|^2\left(\gamma|k|^2 - 2\alpha\right) + 2i\mathrm{Im}(a)\omega\left(\gamma|k|^2 - \alpha\right) \tag{4.67}$$

and are then defined by

$$\omega = \frac{1}{|a|^2}\left(i\mathrm{Im}(a)\left(\gamma|k|^2 - \alpha\right) \pm \sqrt{|a|^2\gamma|k|^2(\gamma|k|^2 - 2\alpha) - \mathrm{Im}(a)^2(\gamma|k|^2 - \alpha)^2}\right) . \tag{4.68}$$

Here we see that, at nonzero temperature where $\mathrm{Im}(a) < 0$, for small value of k such that $\gamma|k|^2 \ll |\alpha|$ the poles become purely imaginary. We therefore expect to see strongly overdamped mode for $k \approx 0$, corresponding to broad regions of non zero values in the spectral function. As we lower the temperature $\mathrm{Im}(a) \to 0$, and we should see this overdamped mode becoming increasingly sharp in the spectral function, and when we reach $T = 0$, they become the two sound modes known as Anderson-Bogliubov modes, given by $\omega = \pm\sqrt{2|\gamma|}|k|/|a|$. In figure 4.1 we show qualitatively what the poles looks like in this model, that is, we fix some values for the coefficients γ, α and $\mathrm{Re}(a)$ and plot the spectral function as we send $\mathrm{Im}(a)$ to zero. Of course this is just an oversimplification since we are not taking into account the actual temperature dependence of the parameters other than $\mathrm{Im}(a)$, however it gives us a good qualitative understanding of what to expect when we study the order-parameter dynamics.

4.4.2 High-temperature solution

In the unbroken phase, that is for temperatures greater than T_c, the dynamics of the gauge fluctuations, described by (4.58), is defined on a Reissner-Nordström background and the equation of motion explicitly written in terms of the metric components can be obtained from (4.25) by setting the fields $\phi(r) = 0$ and $\chi(r) = 0$ (and their derivatives):

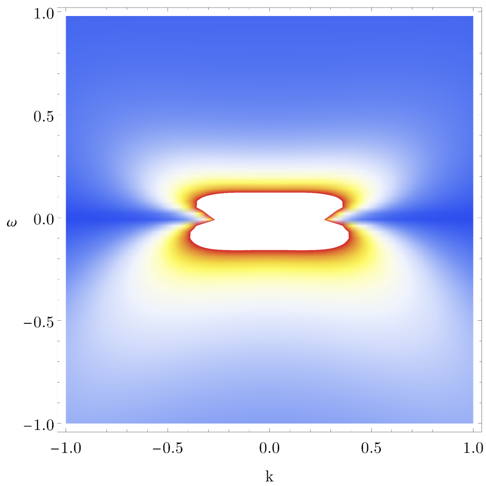
$$f\delta\phi'' + \left(f' + \frac{(d-1)f}{r}\right)\delta\phi' + \left(\frac{\omega^2}{f} - \frac{k^2}{r^2} + \frac{2q\omega A_t}{f} + \frac{q^2A_t^2}{f} - m^2\right)\delta\phi = 0 \tag{4.69}$$

where $f(r)$ and $A_t(r)$ are given by the RN solutions (3.17) and (3.18) respectively.

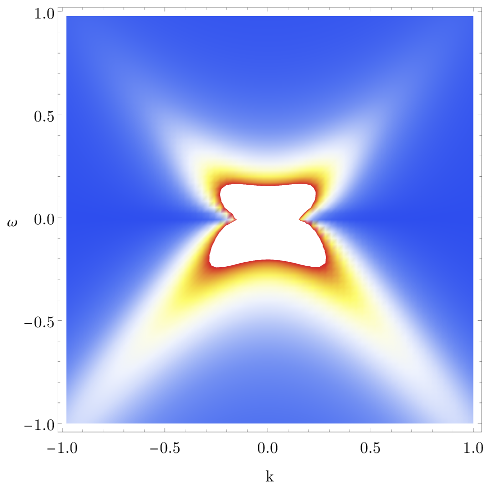
Notice that in this background the fluctuations decouple, and the dynamics of the order parameter is therefore independent of the gauge field fluctuations.

We solved the equation numerically for several values of ω and k, and for each extracted the coefficients of the boundary expansion $\delta\phi_s(\omega, k)$ and $\delta\phi_v(\omega, k)$ to obtain the Green's function from (4.60). In particular we are interested in the spectral function (4.56), as it gives the density of states. In figure 4.2a and 4.2b we show the numerical results for $T = 3.5T_c$ and $T = 1.5T_c$. The color gradient shows the absolute value of the spectral function, rescaled by $\mu^{2\nu}$ to make it dimensionless.

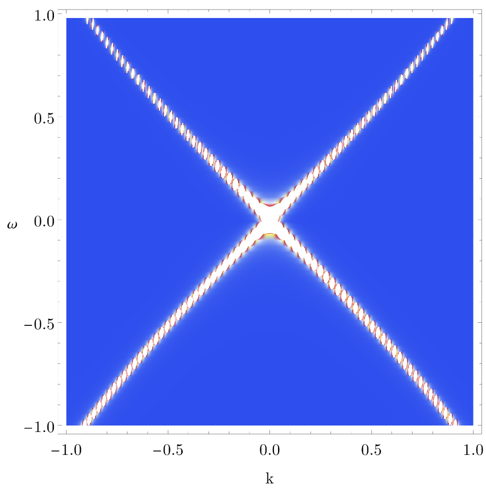
In the short-wavelength limit, when the energy scale becomes larger than the scale set by the chemical potential and the temperature, the system does not feel the presence of this low-energy scale and we should, therefore, recover the AdS result we obtained in chapter 2. In figure 4.3 we plotted the spectral function for this AdS result for comparison (the absence of a color scale is due to the fact that in the AdS solution there is not a scale and only relative values matter), and we can notice that as we move to larger ω in the holographic superconductor spectral function we indeed recover the AdS result. However there are some important differences, first of all, in the AdS solution the spectral function is symmetric and nonzero everywhere inside the lightcone $|\omega| = |k|$, the spectral function for the superconductor model is shifted by the presence of the chemical potential that moves the lightcone to $|\omega + \mu| = |k|$ and generates an asymmetry in the spectral weight for positive and negative values of ω. Furthermore, for the holographic superconductor model, the



(a) $\mathrm{Im}(a) = -1$



(b) $\mathrm{Im}(a) = -1/4$

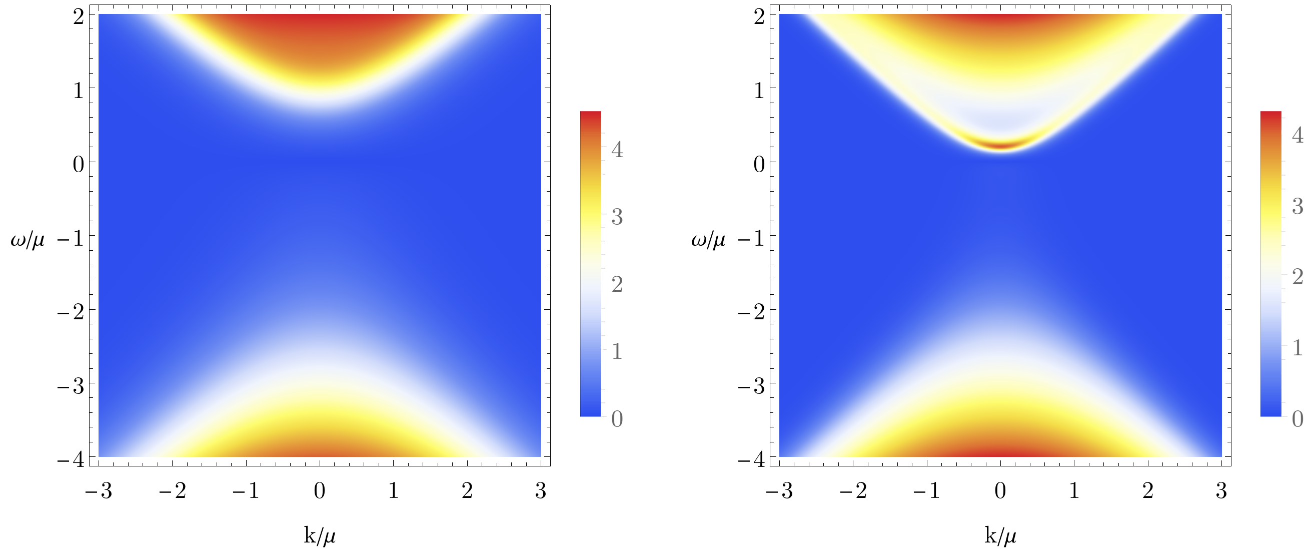


(c) $\mathrm{Im}(a) = 10^{-6}$

Figure 4.1: Qualitative behavior of the spectral function in the G-L model for fixed $\alpha = -1$, $\gamma = 1/2$, and $\mathrm{Re}(a) = 1$ as we decrease the value of $|\mathrm{Im}(a)|$. Notice that this does not precisely represent the actual behavior of the model as we are neglecting the temperature dependece of all the parameters but $\mathrm{Im}(a)$, however it gives a qualitative idea of the shape of the spectral function. As $\mathrm{Im}(a) \to 0$, we see the appereance of the Anderson-Bogoliubov modes.

lightcone is not filled by the spectral weight but we see a gap opening up, that becomes increasingly evident as we raise the temperature as we can see in figure 4.2a. The reason is that in the AdS conformal solution there is no notion of temperature and even low-energy levels are accessible. As we turn on a temperature, we see a gap opening, consistent with the order parameter being massive.

As we move towards the critical temperature, we can see the large-wavelength behavior of the system, described by the effective Ginzburg-Landau theory. In particular, in figure 4.2b we see the predicted quadratic dispersion relation shifted up by the value of $\alpha > 0$.



(a) $T = 3.5T_c$

(b) $T = 1.5T_c$

Figure 4.2: Absolute value of the spectral function for the intrisic dynamics of the order parameter for two values of $T > T_c$. In this plots $q = 3$, $m^2 = -3.5$ and the spectral function has been rescaled by $\mu^{2\nu}$ to make it dimensionless.

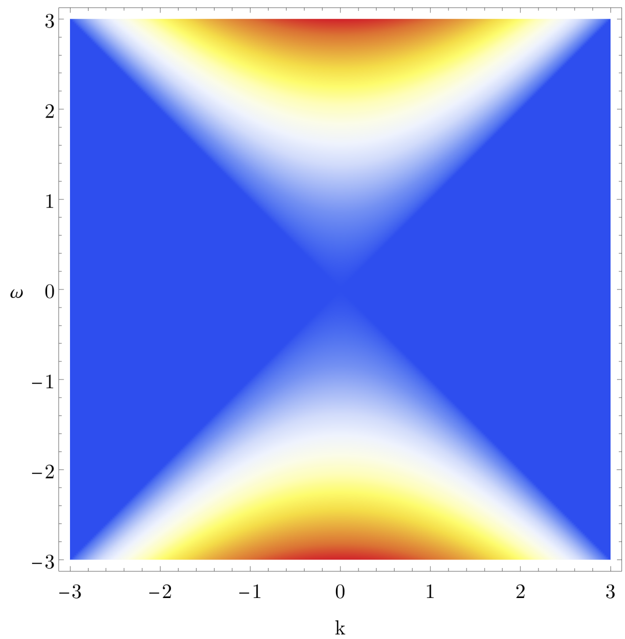


Figure 4.3: Absolute value of the spectral function for the AdS scalar field solution.

4.4.3 Superfluid phase solution

In the superfluid case, we have to restore the background fields $\phi(r)$ and $\chi(r)$, while setting to zero δA_μ, $\mu = x, t$ in (4.25), in order to consider the intrinsic dynamics. The defining equation then becomes

$$f\delta\phi'' + \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\phi' + \left(\frac{e^\chi\omega^2}{f} - \frac{k^2}{r^2} + \frac{2qe^\chi\omega A_t}{f} + \frac{q^2e^\chi A_t^2}{f} - m^2\right)\delta\phi = 0 \tag{4.70}$$

with $f(r)$ now determined by the full background (3.29), that we solved numerically.

The results for different values of the temperature are shown in figure 4.4.

We can notice that for $T = 0.99T_c$ where we are very close to the phase transition the spectral function resembles the one for the solution in the unbroken phase, where the quadratic mode is clearly visible, and it has a minimum positioned at $\omega = 0$, consistent with the fact that $\alpha \approx 0$ in the Ginzburg-Landau model for $T \approx T_c$.

As we lower the temperature, we see the quadratic peak getting sharper, until it becomes a delta function peak as $T \to 0$, and in order for the peak to be visible in 4.4 for $T = 0.03T_c$ we needed to add a small imaginary part $\omega \to \omega + i\epsilon$. We can also notice that the spectral weight for lower temperature moves inside a cone $|\omega| < c_s(k)|k|$, where the speed c_s is a function of mometum. In particular we have that at low temeperature the $c_s \approx 0.85$, while it grows for higher momenta and asymptically approach $c_s = 1$. This coincide with the effective speed of light in the deep interior of the bulk (see figure 4.5). From the metric (3.22), we see that the slice of constant r is described by

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + r^2 d\mathbf{x}_{d-1}^2 \ . \tag{4.71}$$

For a fixed r, massless particle have an effective speed defined by $ds^2 = 0$:

$$c(r) = \sqrt{\frac{f(r)e^{-\chi(r)}}{r^2}} \tag{4.72}$$

that approaches the speed of light in the limit $r \to \infty$ as $f(r) \sim r^2$ and $\chi(r) \sim 0$ in this limit.

None of the observed results coincide with the prediction of the standard Ginzburg-Landau model. In [17], Plantz, Stoof and Vandoren propose a large-N Ginzburg-Landau model to describe the qualitative behavior of the order parameter intrinsic dynamics.

They propose that the boundary field theory should be described by an effective theory of N complex order parameter $O_n\big|_{n=1,\cdots,N}$. The gravitational dual only contains one complex scalar field with a $U(1)$ symmetry, therefore the holographic model only describes the breaking of one of the order parameter on the boundary.

With N order parameters the Ginzburg-Landau model becomes

$$S = -\int dt \int d^3x \sum_{n=1}^{N}\left(iaO_n^*\partial_t O_n + \gamma|\nabla O_n|^2 + \alpha|O_n|^2 + \frac{\beta}{2N}\sum_{m=1}^{N}|O_n|^2|O_m|^2\right) \tag{4.73}$$

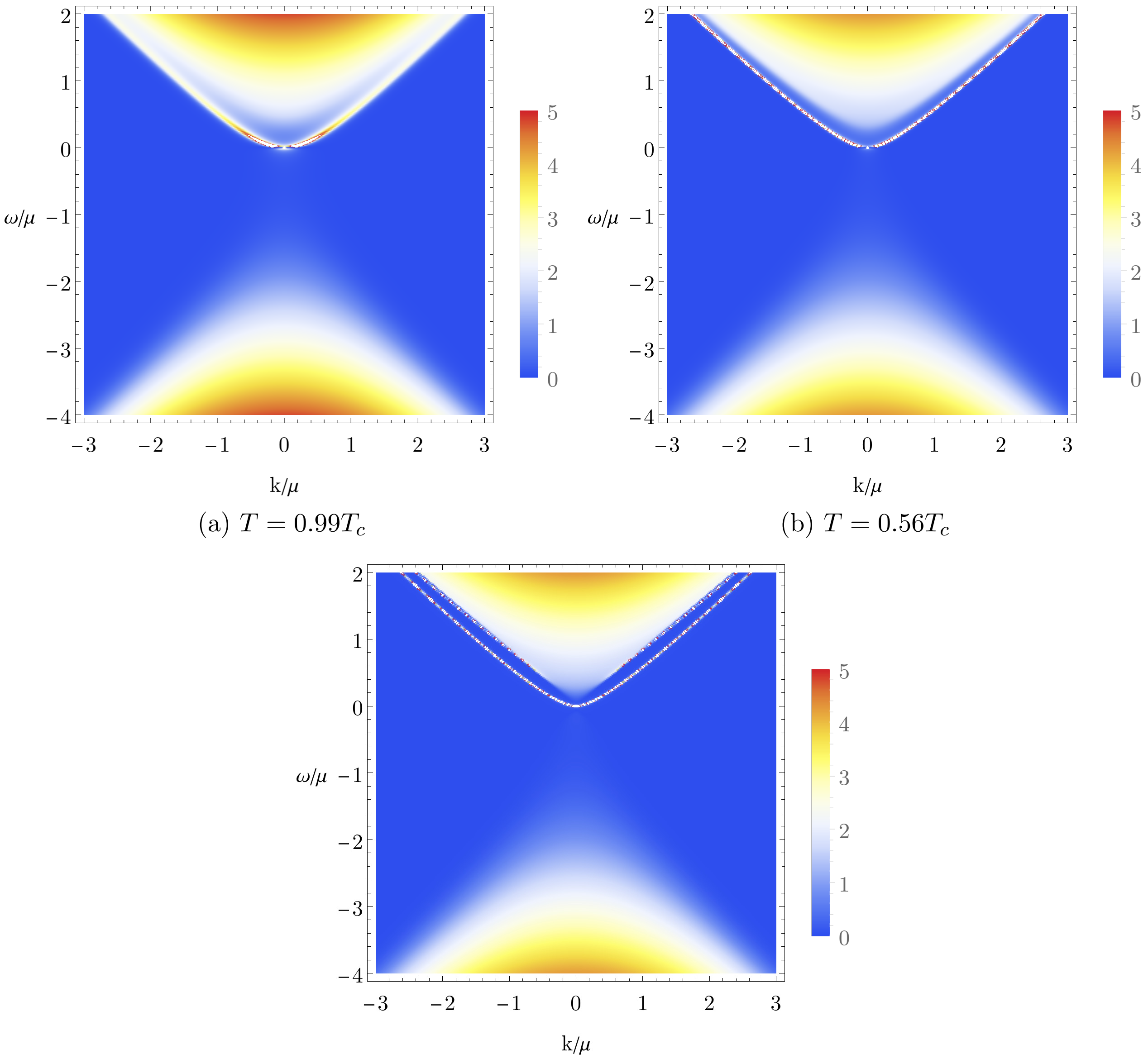
where we do not have mixed terms of the form $O_iO_j^*$ with $i \neq j$ as we want to respect the $U(1)$ symmetry for each operator. We call O_1 the operator that condensed, that is, $O_1 \in \mathbb{R}$ and

$$\langle O_1\rangle = \sqrt{-\frac{\alpha N}{\beta}} \tag{4.74}$$

in the superfluid phase, while $\langle O_i\rangle = 0$ for $i \neq 1$.

Above the critical temperature, the terms second order in fluctuations are (in momentum space)

$$S = -\frac{1}{(2\pi)^4}\int d\omega \int d^3k \sum_{n=1}^{N}\left[\delta O_n^*\left(a\omega + \gamma|k|^2 + \alpha\right)\delta O_n'\right] \tag{4.75}$$



(a) $T = 0.99T_c$

(b) $T = 0.56T_c$

(c) $T = 0.03T_c$

Figure 4.4: Absolute value of the spectral function for the intrinsic dynamics of the order parameter for different values of the dimensionless temperature, with $q = 3$ and $m^2 = -3.5$. For $T = 0.03T_c$ a small imaginary part was added to the frequency in order to observe the quadratic dispersion.

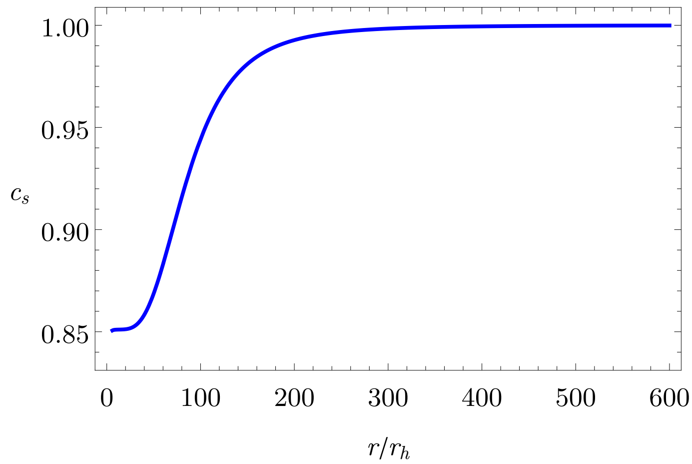


Figure 4.5: Effective speed of light in the bulk as a function of r for $T = 0.03T_c$.

that, defining a vector $\delta\boldsymbol{O} = (\delta O_1\, \delta O_1^*, \cdots, \delta O_N\, \delta O_N^*)$, gives the Green's function as

$$S = -\frac{1}{2}\frac{1}{(2\pi)^4}\int d\omega \int d^3k\, \delta\boldsymbol{O}^\dagger \boldsymbol{G}_N^{-1}\delta\boldsymbol{O} \tag{4.76}$$

and we obtain a $2N \times 2N$ matrix, with 2×2 submatrices on the diagonal of the form

$$\begin{pmatrix} \frac{1}{a\omega+\gamma|k|^2+\alpha} & 0 \\ 0 & \frac{1}{-a^*\omega+\gamma|k|^2+\alpha} \end{pmatrix} \tag{4.77}$$

and we then have

$$G_{O_i^* O_i} = \frac{1}{a\omega + \gamma|k|^2 + \alpha}\ , \tag{4.78}$$

reproducing the correct results of the standard Ginzburg-Landau model above T_c if we define the order parameter described by the boundary theory as

$$O = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} O_i \tag{4.79}$$

and take the large-N limit. Repeating the procedure below the phase transition, however, gives a Green's function

$$\boldsymbol{G}_N = \begin{pmatrix} \boldsymbol{G}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_{N-1} \end{pmatrix} \tag{4.80}$$

with

$$\boldsymbol{G}_{N-1} = \boldsymbol{I}_{N-1} \otimes \begin{pmatrix} \frac{1}{a\omega+\gamma|k^2|} & 0 \\ 0 & \frac{1}{-a^*\omega+\gamma|k^2|} \end{pmatrix} \tag{4.81}$$

and $\boldsymbol{G}_1$ reproducing the normal Ginzburg-Landau results

$$\boldsymbol{G}_1 = \frac{1}{|a|^2\omega^2 - \gamma|k|^2\left(\gamma|k|^2 - 2\alpha\right) - 2i\mathrm{Im}(a)\omega\left(\gamma|k|^2 - \alpha\right)}\begin{pmatrix} a^*\omega - \gamma|k|^2 + \alpha & \alpha \\ \alpha & -a\omega - \gamma|k|^2 + \alpha \end{pmatrix}. \tag{4.82}$$

However, we see that in this large-N model, since $\langle O^* O\rangle = \frac{1}{N}\sum_i \langle O_i^* O_i\rangle$ this last term is suppressed by a factor of $1/N$, and when taking the large-N limit we obtain

$$G_{O^*O} = \frac{1}{a\omega + \gamma|k|^2} \tag{4.83}$$

that describes the observed quadratic mode. Notice that as we move towards zero temperature, we expect the imaginary part of a to vanish and the peak to get a delta function peak, as observed in the numerical results. Within this model, we can think of the parameter N as different species of fermions. The cone structure with the effective speed of light may be interpreted as free fermions of species that are not gapped by $\langle O_1 \rangle \neq 0$, with the effective speed due to strong interactions among them.

4.5 Optical Conductivity

One of the transport properties we would like to study for a superconductor is the *optical conductivity*, that is, the linear response to a small external electric field $J(\omega) = \sigma(\omega)E(\omega)$. This is a simpler version of the coupled problem defined by (4.3), since here we are interested in a fixed values of the momentum, and in particular we can choose $k = 0$ as it is the easiest to probe experimentally (in a system in a laboratory typically one has to deal with particles with momentum way less than the speed of light $k \ll c$). Setting $k = 0$ the equations decouple and we can focus on the electric current J^x dual to fluctuations of A_x, without worrying about the coupling with fluctuations of the other fields (actually, the A_x fluctuations are coupled to fluctuations of the metric element g_{tx}, but we are ignoring them in this chapter)

$$\delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2\phi^2}{f}\right)\delta A_x = 0 \; . \tag{4.84}$$

What we can see is that the effect of the condensate is to introduce a mass term proportional to $q^2\phi^2$. Since the equation decouples, the response of the current expectation value to a small perturbation is simply

$$\delta\langle J^x\rangle = G^{xx}\delta a_x \; . \tag{4.85}$$

However, we want to express the source of the perturbation δa_x in terms of the electric field. Remember that the x component of an electric field is $\delta E_x(\omega, k) = \delta F_{xt}(\omega, k) = ik\delta A_t + i\omega\delta A_x$. Then the relation (4.85) in terms of a small applied electric field becomes

$$\delta\langle J^x\rangle = \frac{G^{xx}}{i\omega}\delta E_x \; . \tag{4.86}$$

and we find the conductivity

$$\sigma(\omega) = \frac{\delta\langle J^x\rangle}{\delta E_x} = \frac{G^{xx}}{i\omega} \; . \tag{4.87}$$

For $d = 3+1$ we can read off the conductivity from the boundary expansion (4.46)

$$\sigma(\omega) = \frac{2\delta b_x - \delta c_x}{i\omega\delta a_x} = 2\frac{\delta b_x}{i\omega\delta a_x} + i\frac{\omega}{2} \; . \tag{4.88}$$

We solved equation (4.84) numerically and extracted the coefficients from the asymptotic behavior. The plot of the real part of the conductivity for different values of the temperature is shown in figure 4.6. The real part is the inverse resistivity of the system. It is the physical part, in the sense that it is directly measurable and is related to the amount of work needed to run a current at a given frequency. The imaginary part is the reactive part and it is related to the real part by the *Kramers-Kronig relation* for retarded Green's function

$$\mathrm{Im}[G^R(\omega)] = -\mathcal{P}\int_{-\infty}^{+\infty}\frac{d\omega'}{\pi}\,\frac{\mathrm{Re}[G^R(\omega)]}{\omega'-\omega} \; , \tag{4.89}$$

$$\mathrm{Re}[G^R(\omega)] = \mathcal{P}\int_{-\infty}^{+\infty}\frac{d\omega'}{\pi}\,\frac{\mathrm{Im}[G^R(\omega)]}{\omega'-\omega} \; . \tag{4.90}$$

Let us first make some observation about the expected behavior of the conductivity. First of all, in chapter 2, we showed that the conformal dimension of the current $\Delta = d-1$ implies that the current-current Green's function for the conformal field theory scale as $G^{xx}(\omega, k=0) \sim \omega^{d-2}$. Therefore, we expect the conductivity at large frequencies (high energies), where the theory forgets about the breaking of conformal symmetries in the IR, to scale as $\sigma(\omega) \sim G^{xx}/\omega \sim \omega^{d-3}$. This asymptotic behavior is clearly seen in figure 4.6 where $d = 3+1$ and the conductivity becomes linear for large ω as well as in the $d = 2+1$ case in figure 4.9, where the conductivity asymptotes to a constant value.

For high temperature $T \gg \mu$ the presence of the chemical potential is less important (from the gravitational point of view, as we raise the temperature we recover a Schwarzschild black hole solution) and real part of the conductivity quickly get to its linear (or constant in $d = 2+1$) behavior. As the temperature is lowered, $\mathrm{Re}[\sigma(\omega)]$ develops a minimum for $\omega \to 0$. As we lower the temperature further below its critical value, $\mathrm{Re}[\sigma(\omega)]$ start developing a gap where $\mathrm{Re}[\sigma(\omega)] \approx 0$ but still finite. For $T \to 0$ the gap appears to become a *hard gap* in the sense that the real $\mathrm{Re}[\sigma(\omega)]$ is exactly zero in the gap[5].

This is remarkably similar to the standard weakly interacting superconductor described by BSC theory, where a gap opens up in at $T = 0$ due to a macroscopic condensation of Cooper pairs. However, there are important differences. The most important one is that BCS theory describes a system characterized by weak interaction, while we are considering a strongly interacting system thanks to the weak-strong nature of the duality. The size of the gap in the holographic superconductor is related to the VEV of the order parameter, however, contrary to the weakly interacting theory where it is determined by the energy necessary to break a Cooper pair $2\Delta = E_{Cooper}$, there seems not to be such a simple relation in the holographic theory (notice that in general $\langle O \rangle$ does not even have dimensions of an energy, but a meaningful relation cannot be established even after matching the dimensions). Unfortunately, the holographic approach only provides phenomenological results, and it does not shed lights on the microscopic interpretation of the order parameter.

One of the most peculiar features of superconductivity is that the DC conductivity (i.e., conductivity at $\omega = 0$) diverges (and then the resistivity is zero). In $\mathrm{Re}[\sigma(\omega)]$ this is represented as a delta function $\delta(\omega)$, but it is not possible to see it from the numerics. Nonetheless, from the Kramers-Kronig relation (4.89) we see that a delta function at $\omega = 0$ corresponds to a pole $1/\omega$ in the imaginary part, that is clearly visible in figure 4.7, where the red line is the numerical result of $\mathrm{Im}[\sigma(\omega)]$ for $T \approx 0.03 T_c$.

One, however, can argue that the peak in the conductivity in not a feature of superconductivity, but it is a simple consequence of translation invariance of the theory. Every system with translation invariance does indeed present an infinite DC conductivity, as the momentum is conserved and it cannot relax, so even for a small applied electric field, the charges in the conductor keeps accelerating. In our model, however, we fixed the background metric by neglecting metric fluctuations and this means that we do not take into account the conservation law $\partial_\mu \delta \langle T^{\mu\nu} \rangle = 0$ associated to metric fluctuations. Neglecting these Ward identities is the reason why we do not see a pole in the imaginary part of the conductivity for $T \gtrsim T_c$, as shown by the blue line in figure 4.7. It is similar to the analysis in the probe limit in [7], where they argue that fixing the metric in the probe limit breaks translation invariance. Here we are not considering the probe limit, but we fix the background by neglecting fluctuations, breaking the abovementioned conservation law, that has a similar effect.

This is analogous to the standard computation of conductivity in fluids with a conserved current $\partial_\mu J^\mu = 0 = \partial_t \rho + \nabla \cdot J$, but neglecting conservation of energy and momentum (in real materials this is in general a good assumption, as momentum conservation degrades quickly due to scattering with impurities, and, since diffusion is the slowest process, on a large enough timescale momentum conservation can be neglected). In the low-energy limit, from a derivative expansion we can obtain the **constitutive relation** $J = -D\nabla\rho$

$$J = -D\nabla\rho = -D\frac{\partial\rho}{\partial\mu}\nabla\mu = D\chi E = \sigma E \ , \tag{4.91}$$

with both D and χ real constants for $\omega = 0$. Based on this reasoning, we see that by neglecting the metric fluctuations we should find a conductivity $\mathrm{Re}[\sigma(\omega = 0)] = D\chi$ as we observe in the numerical results. Notice, moreover, that we expect $\mathrm{Im}[\sigma(\omega = 0)] = 0$, as we obtained in figure 4.7.

Including metric fluctuations for $k = 0$ does not alter the qualitative behavior of the results presented above, as all it does is to renormalize the mass term in equation (4.84) by adding a term $\propto {A_t'}^2$ (see for example [7]),

[5] With numerical methods there are always going to be numerical precision issues, that is why we cannot really prove that $\mathrm{Re}[\sigma(\omega)] = 0$ exactly in for $T = 0$.

restoring the neglected Ward identities. As a result, the metal described in this case is a perfect conductor and a pole in the imaginary part can be observed even in the unbroken phase, as we show in figure 4.8. In the superconducting phase, however, the pole at $\omega = 0$ is a combination of translation invariance and the results of the system becoming a superconductor, and the delta function due to the superconducting nature would remain even if we would explicitly break translation invariance.

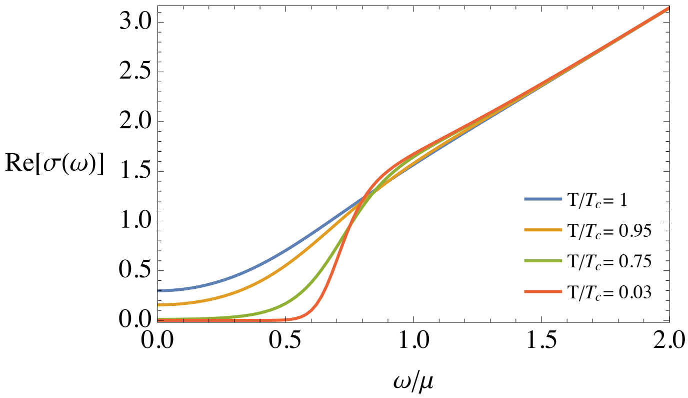


Figure 4.6: Real part of the conductivity for several value of T. The value of $\text{Re}[\sigma(\omega)]$ has been rescaled by a factor μ^2 to make it dimensionless. We can see that as we lower the temperature below T_c we observe a gap opening, that eventually becomes a hard gap at $T = 0$.

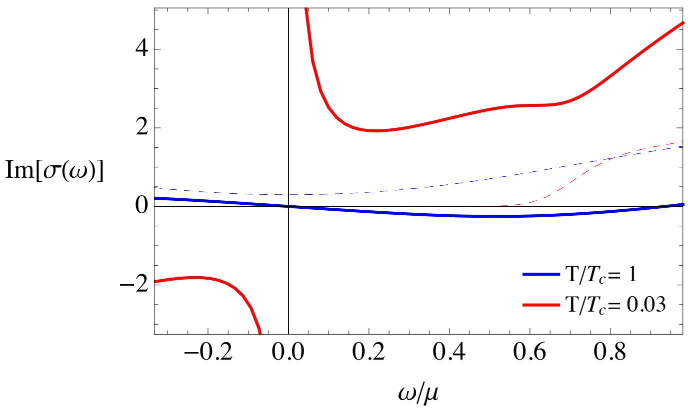


Figure 4.7: Imaginary part of the conductivity for $T \to T_c^+$ (blue line) and $T = 0.03T_c$, rescaled by a factor of μ^2 to make them dimensionless. We can clearly see the pole $\sim 1/\omega$ for the low-temperature solution, corresponding to the superconducting delta function in $\mathrm{Re}[\sigma(\omega = 0)]$. On the other hand $\mathrm{Im}[\sigma(\omega)]$ is smooth across $\omega = 0$ in the unbroken phase, as the conservation law $\partial_\mu \delta\langle T^{\mu\nu}\rangle = 0$ is implicitely broken by neglecting metric fluctuations. The dashed line represent the respective real part of the conductivity.

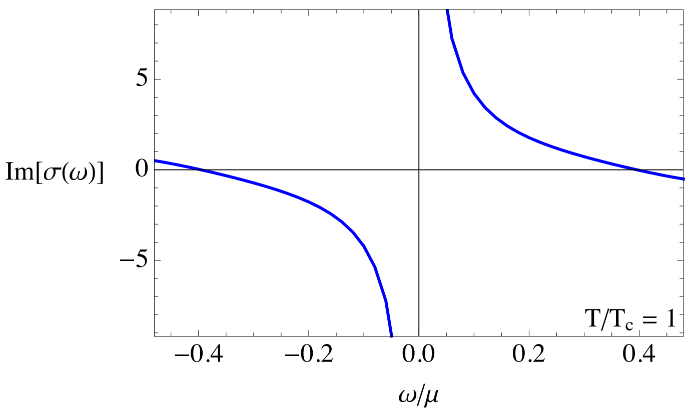


Figure 4.8: Imaginary part of the conductivity including metric fluctuations for $T \to T_c^+$, rescaled by a factor of μ^2 to make it dimensionless. We can see that including metric fluctuations, and hence restoring the conservation law $\partial_\mu \delta\langle T^{\mu\nu}\rangle = 0$, gives a pole $\sim 1/\omega$ even for T above T_c, as expected for a conductor without disorder (i.e., with translation invariance).

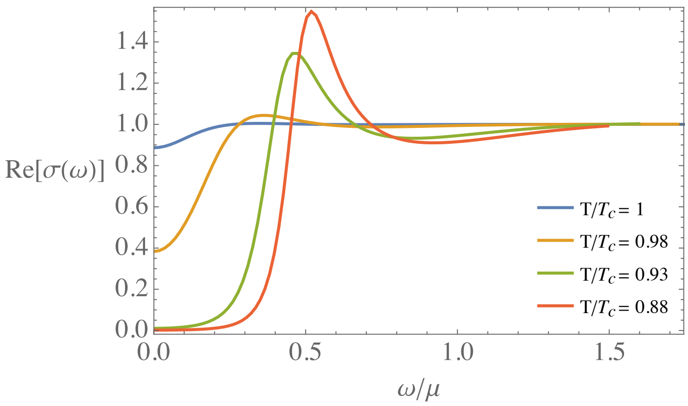


Figure 4.9: Real part of the conductivity for several value of T in $d = 2+1$ dimensions. The value of Re[$\sigma(\omega)$] has been rescaled by a factor μ^2 to make it dimensionless. The long-wavelength behavior is determined by the scaling dimension of the gauge field, that gives $\sigma \sim \omega^{d-3} \sim 1$ for $d = 2 + 1$.

4.6 Gauge field fluctuations

In this section we finally consider the more general case of gauge fluctuations for different values of the momentum k and compute the spectral function arising for different values of the temperature.

First we present the high-temperature solution where the scalar field $\phi = 0$ and there is therefore no coupling with the scalar field fluctuations (see equations (4.27) and (4.28) where the coupling with the scalar fluctuations is determined by ϕ^2). Nonetheless, the time component A_t and the spatial component A_x of the gauge field are coupled together, and in order to extract the Green's functions we still need to perform the procedure for coupled fields we explained in the introduction. We will see that this gives rise to the usual gauge invariant form in terms of a single scalar Green's function $\Pi(\omega, k)$.

4.6.1 High-temperature solution

As we did for the scalar field, we can find the linearized equation of motion for the gauge field in the unbroken phase by simply setting $\phi = \chi = 0$ in equations (4.27) and (4.28), giving:

$$0 = \delta A_x'' + \left(\frac{f'}{f} + \frac{d-3}{r}\right)\delta A_x' + \frac{\omega^2}{f^2}\delta A_x + \frac{k\omega}{f^2}\delta A_t \ , \tag{4.92}$$

$$0 = \delta A_t'' + \frac{d-1}{r}\delta A_t' - \frac{k^2}{r^2 f}\delta A_t - \frac{k\omega}{r^2 f}\delta A_x \ . \tag{4.93}$$

We can also notice that in the high-temperature solution the constraint (4.29) becomes

$$\frac{\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' = 0 \ . \tag{4.94}$$

Remember that in the high-temperature case $f(r)$ is given by the Reissner-Nordström solution (3.34).

As always, we first make use of the fact that the boundary behavior is independent of the geometry in the deep interior of the bulk to realize that the boundary expansion for the Reissner-Nordström solution is the same as the one we already computed given by (4.46). Since scalar fluctuations and transverse components of the gauge field are decoupled we only focus on longitudinal terms

$$\begin{aligned} S_{bdy}^{(2),R} =& \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\left(-2\delta b_t(\omega,k) + \delta c_t(\omega,k)\right)\delta a_t(-\omega,-k) + \left(2\delta b_x(\omega,k) - \delta c_x(\omega,k)\right)\delta a_x(-\omega,-k)\Big] \\ =& \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[\delta\langle J^x\rangle\,\delta a_x + \delta\langle J^t\rangle\,\delta a_t\Big] \ , \end{aligned} \tag{4.95}$$

and we can then read off the fluctuations in the density and current expectation values

$$\begin{aligned}\delta\langle J^x\rangle(\omega,k) =&2\delta b_x(\omega,k)-\frac{\omega}{2}(\omega\delta a_x(\omega,k)+k\delta a_t(\omega,k))\\ \delta\left\langle J^t\right\rangle(\omega,k) =&-2\delta b_t(\omega,k)-\frac{k}{2}(\omega\delta a_x(\omega,k)+k\delta a_t(\omega,k))\ ,\end{aligned}\tag{4.96}$$

where we used the expressions for the c_i coefficients (4.36) and (4.37).

In the high-temperature phase, linear response theory gives

$$\delta\langle J^x\rangle(\omega,k) =G^{xx}(\omega,k)\delta a_x(\omega,k)+G^{xt}(\omega,k)\delta a_t(\omega,k)\tag{4.97}$$

$$\delta\left\langle J^t\right\rangle(\omega,k) =G^{tt}(\omega,k)\delta a_t(\omega,k)+G^{tx}(\omega,k)\delta a_x(\omega,k)\tag{4.98}$$

$$\tag{4.99}$$

and given two independent solutions, we can solve the linear system

$$\begin{pmatrix}\delta a_t^1 & \delta a_x^1\\ \delta a_t^2 & \delta a_x^2\end{pmatrix}\begin{pmatrix}G^{tt} & G^{xt}\\ G^{tx} & G^{xx}\end{pmatrix}=\begin{pmatrix}\delta\left\langle J^t\right\rangle_1 & \delta\langle J^x\rangle_1\\ \delta\left\langle J^t\right\rangle_2 & \delta\langle J^x\rangle_2\end{pmatrix}\ .\tag{4.100}$$

As we did before, we now need to solve the equation of motion for the fluctutions imposing incoming-waves conditions at the horizon. These are simply given by (4.47)

$$\delta A_x\sim c_x(r-r_h)^{-\frac{i\omega}{4\pi T}}\tag{4.101}$$

$$\delta A_t\sim c_t(r-r_h)^{1-\frac{i\omega}{4\pi T}}\ ,\tag{4.102}$$

with the constraint obtained from (4.48) after setting $c_\phi=c_{\phi^*}=0$:

$$c_t=i\left(\frac{k}{r_h^2}c_x\right)\frac{e^{-\chi(r_h)/2}}{1-\frac{i\omega}{4\pi T}}\ .\tag{4.103}$$

As for the full problem, we then see that the two coefficients are not independent and we can only generate one solution for the linearized equation of motion. The second solution we need, is a pure gauge solution due to the residual gauge freedom remained after fixing the gauge $A_r=\delta A_r=0$. It is easy to verify that the ansatz

$$\delta a_t=\lambda\omega,\qquad \delta a_x=-\lambda k\tag{4.104}$$

with λ a constant, does indeed solve the linearized equation of motion for the high-temperature solution, (4.92) and (4.93). The gauge solution does not source any response, and after numerically computing the non-trivial solution, extracting the sources δa_i and the responses $\delta\left\langle J^i\right\rangle$, we can write (4.100) as

$$\begin{pmatrix}G^{tt} & G^{xt}\\ G^{tx} & G^{xx}\end{pmatrix}=\begin{pmatrix}\delta a_t & \delta a_x\\ \lambda\omega & -\lambda k\end{pmatrix}^{-1}\begin{pmatrix}\delta\left\langle J^t\right\rangle & \delta\langle J^x\rangle\\ 0 & 0\end{pmatrix}\ ,\tag{4.105}$$

that gives the Green's functions in terms of the coefficients of the boundary expansion (using (4.96))

$$\begin{aligned}G^{tt}(\omega,k)=&-\frac{2k\delta b_t}{\omega\delta a_x+k\delta a_t}-\frac{k^2}{2}\\ G^{xt}(\omega,k)=&\frac{2k\delta b_x}{\omega\delta a_x+k\delta a_t}-\frac{\omega k}{2}\\ G^{tx}(\omega,k)=&-\frac{2\omega\delta b_t}{\omega\delta a_x+k\delta a_t}-\frac{\omega k}{2}\\ G^{xx}(\omega,k)=&\frac{2\omega\delta b_x}{\omega\delta a_x+k\delta a_t}-\frac{\omega^2}{2}\ .\end{aligned}\tag{4.106}$$

These solutions may at first seem a bit strange, as for example, it is not apparent that $G^{xt} = G^{tx}$ as required by time inversion symmetry. However, let us take a look at the constraint equation in the boundary limit

$$\frac{\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' = 0 \ . \tag{4.107}$$

Inserting the expansion for the gauge field and using $f(r) \sim r^2$ for $r \to \infty$ we can rewrite it as

$$0 = -\omega(-2\delta b_t + \delta c_t) + k(2\delta b_x - \delta c_x) = 2\omega\delta b_t + 2k\delta b_x = -\omega\delta\langle J^t\rangle + k\delta\langle J^x\rangle \ , \tag{4.108}$$

and we see that the constraint equation is simply the Fourier transform of current conservation for the fluctuations of the expectation values $\partial_\mu\delta\langle J^\mu\rangle = 0$. From this, we find that $\delta b_x = -\frac{\omega}{k}\delta b_t$ and we can rewrite (4.106) as (multiplying and dividing $G^{\mu\nu}$ by $k_\mu/(k^2-\omega^2)$)

$$\begin{aligned}
G^{tt}(\omega,k) =& \frac{k^2}{k^2-\omega^2}\left[\frac{2(\omega\delta b_x + k\delta b_t)}{\omega\delta a_x + k\delta a_t} - \frac{k^2-\omega^2}{2}\right] && (4.109)\\
G^{xt}(\omega,k) =& \frac{k\omega}{k^2-\omega^2}\left[\frac{2(\omega\delta b_x + k\delta b_t)}{\omega\delta a_x + k\delta a_t} - \frac{k^2-\omega^2}{2}\right] && (4.110)\\
G^{tx}(\omega,k) =& \frac{\omega k}{k^2-\omega^2}\left[\frac{2(\omega\delta b_x + k\delta b_t)}{\omega\delta a_x + k\delta a_t} - \frac{k^2-\omega^2}{2}\right] && (4.111)\\
G^{xx}(\omega,k) =& \frac{\omega^2}{k^2-\omega^2}\left[\frac{2(\omega\delta b_x + k\delta b_t)}{\omega\delta a_x + k\delta a_t} - \frac{k^2-\omega^2}{2}\right] && (4.112)\\
&, && (4.113)
\end{aligned}$$

where the numerators are in terms of the coefficients of the expansion of the gauge invariant electric field $E_x = F_{xt} = ikA_t + i\omega A_x$, that defines the scalar Green's function $\Pi(\omega,k)$ containing all the information of the gauge field Green's functions.

We could have indeed computed the Green's functions by realizing that the two equations (4.92) and (4.93) can be combined in a single equation for the variation of the electric field δE_x. Summing the two and using the constraint relation we obtain the equation (in $d = 3+1$)

$$\delta E_x'' + \frac{r^2\omega}{r^2\omega^2 - k^2 f}\left(\frac{\omega f'}{f} - \frac{3k^2 f}{r^3\omega} + \frac{\omega}{r}\right)\delta E_x' + \left(\frac{\omega^2}{f^2} - \frac{k^2}{r^2 f}\right)\delta E_x = 0 \ , \tag{4.114}$$

where the electric field has a boundary expansion

$$\delta E_x = \delta E_{x,0} + \delta E_{x,1} r^{-2} + \frac{k^2-\omega^2}{2} r^{-2}\log(r) + \cdots \tag{4.115}$$

The response to the electric field perturbation δE_x is related to the scalar Green's function $\Pi(\omega,k)$, that we can extract following the usual procedure. From it, we can then recover the gauge field Green's function using the Ward identities, that give

$$G^{\mu\nu}(\omega,k) = \frac{k^\mu k^\nu}{k^2-\omega^2}\Pi(\omega,k) \ . \tag{4.116}$$

Notice that the Ward identities can be derived from the boundary limit of the constraint $-\omega\delta\left\langle J^t\right\rangle + k\delta\left\langle J^x\right\rangle = 0$ by using the expression for the linear response (4.97) giving

$$\left(-\omega G^{tt} + kG^{xt}\right)\delta a_t + \left(-\omega G^{tx} + kG^{xx}\right)\delta a_x = 0 \tag{4.117}$$

that has to be valid for every value of the sources, that implies:

$$\begin{aligned}
-\omega G^{tt} + kG^{xt} =& 0\\
-\omega G^{tx} + kG^{xx} =& 0 \ .
\end{aligned} \tag{4.118}$$

The approach in terms of the gauge invariant field can be better in this situation as it requires to solve a single equation instead of a coupled system that usually requires a higher computational time and gives a lower precision when solving numerically. However, the first procedure is more general and can be used to derived the Green's function of any set of coupled operators, moreover, even for a theory with a gauge field it may often be highly non trivial how to reduce the equations for A_t and A_x to one single equation for the electric field.

In figure 4.10 we show the resulting spectral function for $T = 1.5T_c$. We can see that the spectral function fills the light cone, and there is no gap in the high-temperature solution, as we already noticed in the conductivity, whose real part is related to the spectral function $\mathrm{Re}[\sigma(\omega)] = \frac{\pi}{2\omega}\mathcal{A}(\omega, k = 0)$.

In the next section we study how the spectral function changes in the broken phase due to the coupling with the scalar field.

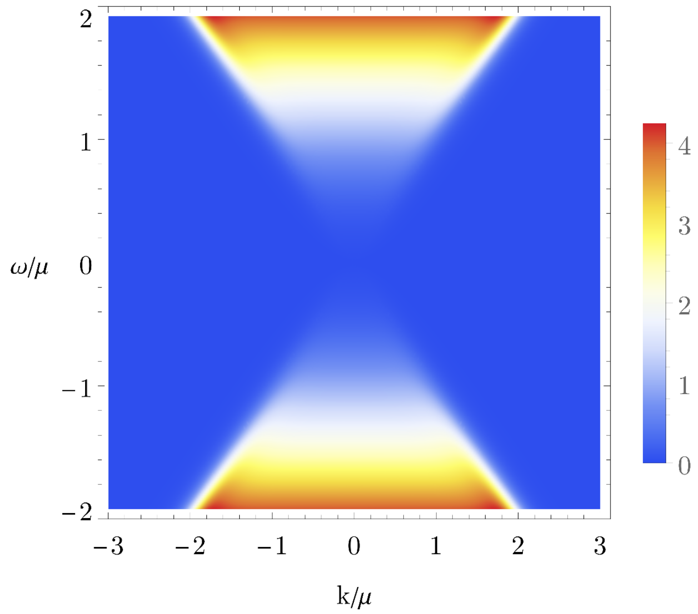


Figure 4.10: Absolute value of the spectral function for G^{xx} in the normal phase, rescaled by a factor of μ^2 to make it dimensionless. There are no quasinormal modes, as we only have a diffusive pole $\omega = -iDk^2$.

4.6.2 Broken phase solution

Below the critical temperature we turn on the scalar field in the bulk and we now have to deal with the full set of differential equations (4.25) to (4.28) presented in the introduction to this section (remember that we are studying the intrinsic dynamics and ignoring the fluctuations of the metric).

Before presenting the solutions let us take a look at how the constraint equation is modified in the superconducting phase. For $\phi \neq 0$, that is given by (4.29), and we rewrite here for convenience:

$$\frac{e^{\chi}\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' + q\phi\left(\delta\phi' - (\delta\phi^*)'\right) + q\phi'\left(\delta\phi^* - \delta\phi\right) = 0\ .$$

Taking the boundary limit of this equation and substituting the boundary behavior of the fields we find

$$-\omega\delta\langle J^t\rangle + k\delta\langle J^x\rangle = q\,\langle O\rangle\,(\delta\phi_s - \delta\phi_s^*) = 2iq\,\langle O\rangle\,\delta\eta_s\ , \tag{4.119}$$

where we defined $\delta\eta$ as the imaginary component of the scalar field fluctuations: $\delta\eta = \mathrm{Im}[\delta\phi]$. We can then see that the right hand side only depends on the source of phase fluctuations of the operator O. To see that the imaginary part corresponds to phase fluctuations, we can write

$$\delta\phi = \delta\left(\rho e^{i\theta_0}\right) = \delta\rho e^{i\theta_0} + i\delta\theta\rho e^{i\theta_0}$$

where θ_0 is the phase of the condensate, and we set $\theta_0 = 0$ by imposing O to be real. We therefore obtained

$$\delta\sigma \equiv \mathrm{Re}[\delta\phi] = \delta\rho \ , \quad \delta\eta \equiv \mathrm{Im}[\delta\phi] = \rho\delta\theta \ . \tag{4.120}$$

In real space, the relation assumes the form

$$\partial_\mu \delta\langle J^\mu\rangle = -2q \langle O\rangle \eta_s \tag{4.121}$$

that describes particles flowing into and out of the condensed phase.

From this, we can also obtain the Ward identity in the superconducting phase

$$\begin{aligned} -\omega G^{tt} + kG^{xt} =& 0 \\ -\omega G^{tx} + kG^{xx} =& 0 \\ -\omega G^{tO} + kG^{xO} =& 2q \langle O\rangle \\ -\omega G^{tO^*} + kG^{xO^*} =& -2q \langle O\rangle \end{aligned} \tag{4.122}$$

and we can see that among all the Green's functions we obtain, we only have 4 independent ones. As we noticed above, since the poles are given by the zeros of the determinant of the matrix of sources, all the Green's functions present the same pole structure. Below we show the results of the absolute value of the spectral functions we are mostly interested in, the one related to G^{xx} (that determines all the Green's function in the gauge sector) and the G^{OO^*}, to understand how the dynamics of the order parameter is changed by the effect of the coupling.

What we can see is that as $T \to T_c^-$ the spectral function of the scalar field is the same as the one obtained from the intrinsic dynamics in 4.4 as we would expect because the coupling with the gauge field is regulated by the order parameter and as we approach T_c the effect of the coupling becomes negligible.

Lowering the temperature, a gap is opening in the spectral function of the gauge field for low frequency, as we already observed when we studied the conductivity. The main feature we notice in the spectral function coming from G_{O^*O}, is that the quadratic mode seen in the intrinsic dynamics disappeared and it has been replaced by a linear mode, particularly evident at positive frequencies. The spectral function at higher energies resembles the one seen in 4.4, but combined with a quadratic peak that is symmetric with respect to the origin, implying it is a contribution coming from the gauge field.

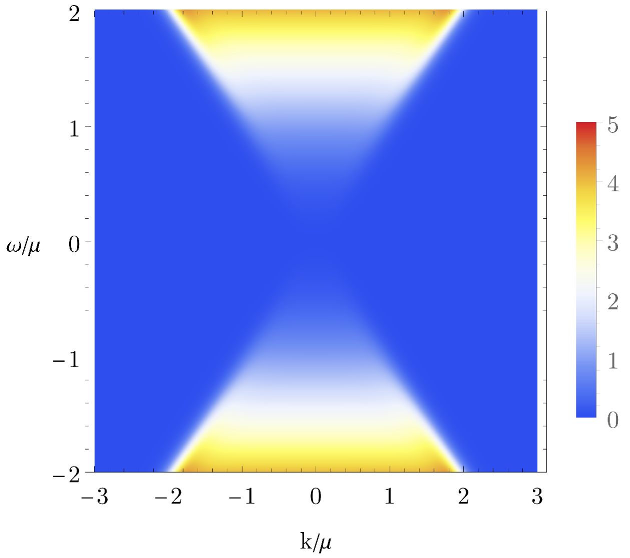
It is interesting to notice, that this quadratic peak does not sharpen as we lower the temperature, and it therefore represent states with a finite lifetime even at zero temperature. On the other hand, the peak corresponding to the linear modes that reside outside the lightcone is always narrow and it does not smoothen out as we increase the temperature, corresponding to long-living modes even at nonzero temperature.

Below we include some more plots, showing the variation of the speed of the linear mode with the temperature, as well as some 3-dimensional plot in which the linear mode is more easily observed.

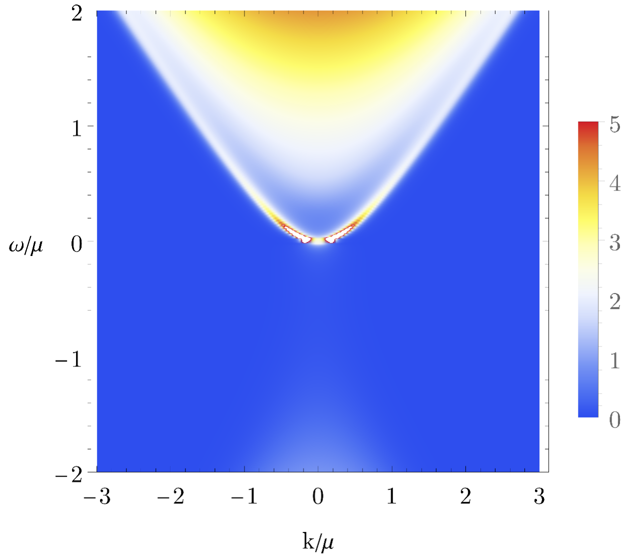
4.6.3 Linear modes formation and speed of sound

As the temperature approaches its critical value from below, the coupling with the gauge field becomes smaller and we observed that we recover the result obtained by studying the intrinsic dynamics of the scalar field, that is, a quadratic mode at low energies. As we lower the temperature, this quadratic mode transforms into two linear modes by the effect of the coupling, that therefore have a nonzero speed of sound. We then expect the speed of sound to increase as we move from T_c, where the speed of sound is $c_s = 0$, towards lower temperatures. In figure 4.16, we show the linear mode appearing as we move away from the critical temperature.

In figure 4.17, we analyze the speed of sound as a function of T/T_c. As expected, $c_s \to 0$ as $T/T_c \to 1$, however, we see that it asymptotes to a constant value as $T \to 0$.



(a) Spectral function from G^{xx} at $T = 0.99T_c$.



(b) Spectral function from G^{O^*O} at $T = 0.99T_c$.

Figure 4.11: Absolute value for the spectral function with the coupling of the gauge and scalar field, and for $q = 3$, $m^2 = -3.5$ and $T = 0.99T_c$. The spectral function from the G^{xx} component, here as in the plots below has been rescaled by μ^2 to make it dimensionles, while the one from the scalar sector by $\mu^{2\nu}$. We can see that for temperature close to T_c, where the coupling between the gauge and the scalar field is weak, the solution for G^{O^*O} is similar to the one obtained from the intrinsic dynamics.

4.6.4 Details of the spectral function for the scalar component

Here we check if the structure of the coupled dynamics of the scalar field is related to the intrinsic dynamics behavior. To do so, we take the 2×2 lower matrix of $\boldsymbol{G}_R$, corresponding to the scalar sector, and we invert it. The idea is to check if we can write this inverse matrix G_s^{-1} as

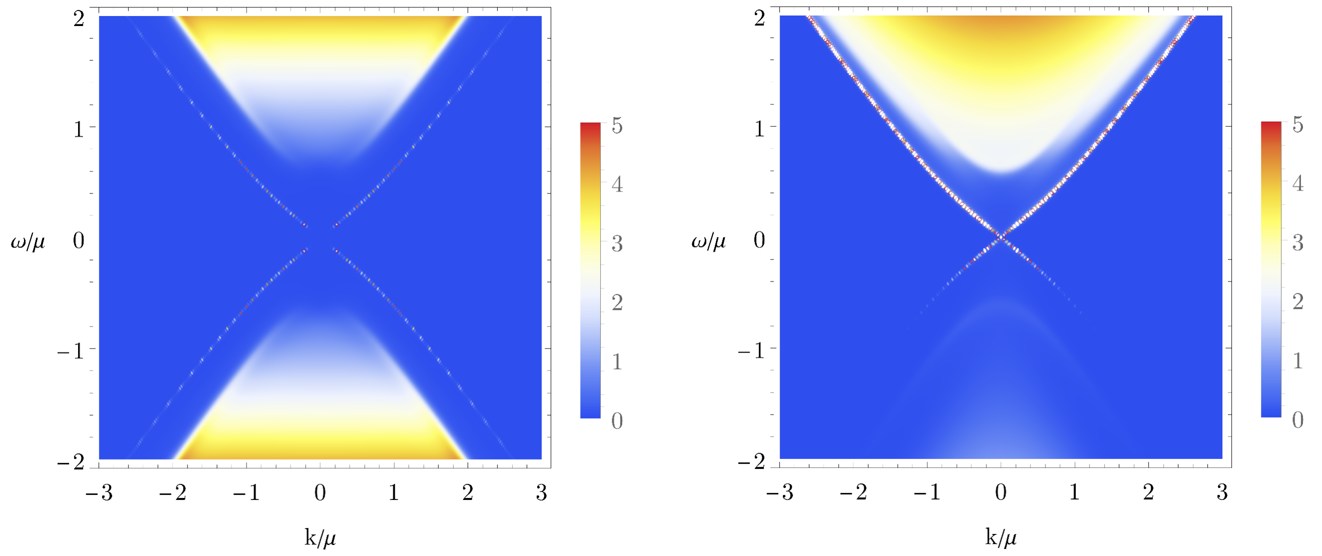
$$G_s^{-1} = G_0^{-1} + \Sigma$$

with G_0^{-1} the matrix from the intrinsic dynamics of the scalar field, as explained in 4.2

Preliminary results seems to indicate that for $(\omega, k) \to 0$, G_s^{-1} assumes the form

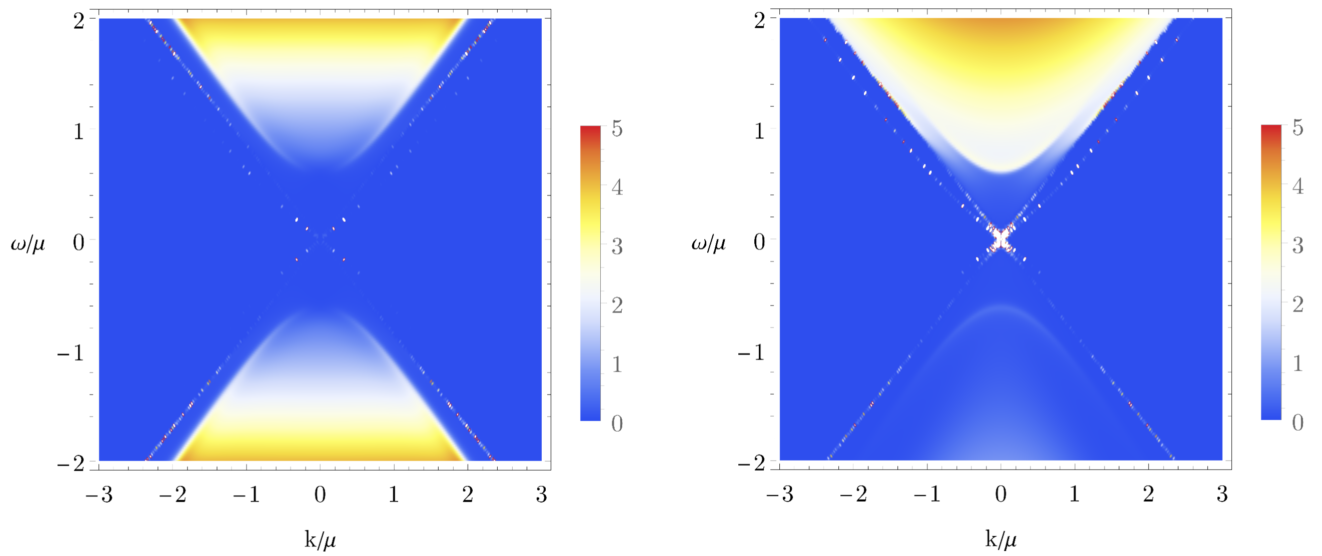
$$G_s^{-1} = G_0^{-1} + C \tag{4.123}$$

with C a 2×2 constant matrix, hinting at a coupling of the current with the phase of the order parameter, represented in the low-energy effective action by a term $\propto (O + O^*)\partial_\mu J^\mu$. This behavior is already enough to generate linear modes at low energies, however further results are needed for a better understanding of the form of this effective low-energy coupling to see if it correctly fits the numerical results.



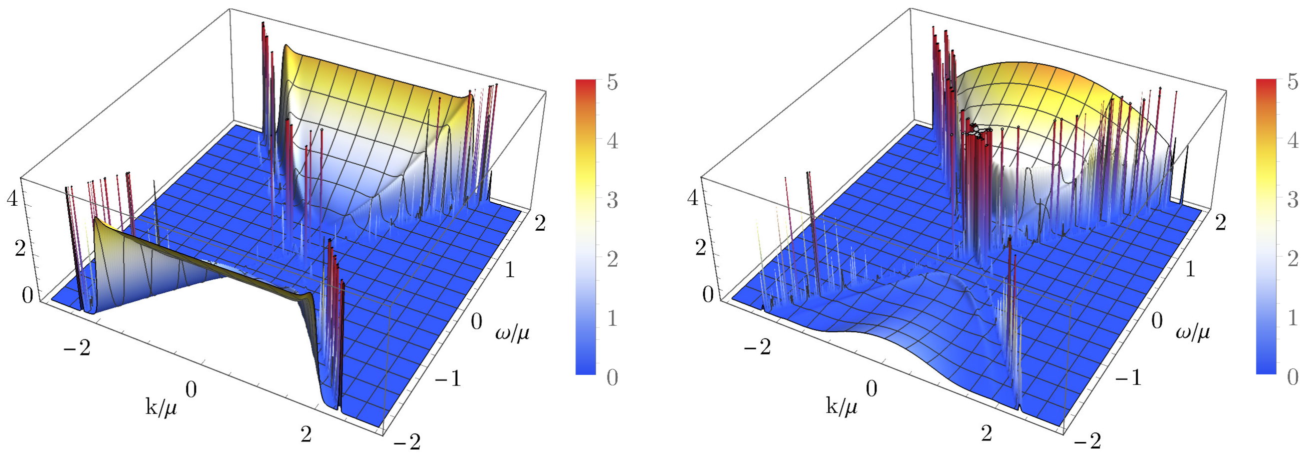
(a) Spectral function from G^{xx} at $T = 0.56T_c$ (b) Spectral function from G^{O^*O} at $T = 0.56T_c$

Figure 4.12: Absolute value for the spectral function with the coupling of the gauge and scalar field, and for $q = 3$, $m^2 = -3.5$ and $T = 0.56T_c$. We can see a long-lived linear mode in both figures. The plot for the scalar function resembles the one for the intrinsic dynamics with the additional contribution of 2 symmetic quadratic peak from the coupling with the gauge field.



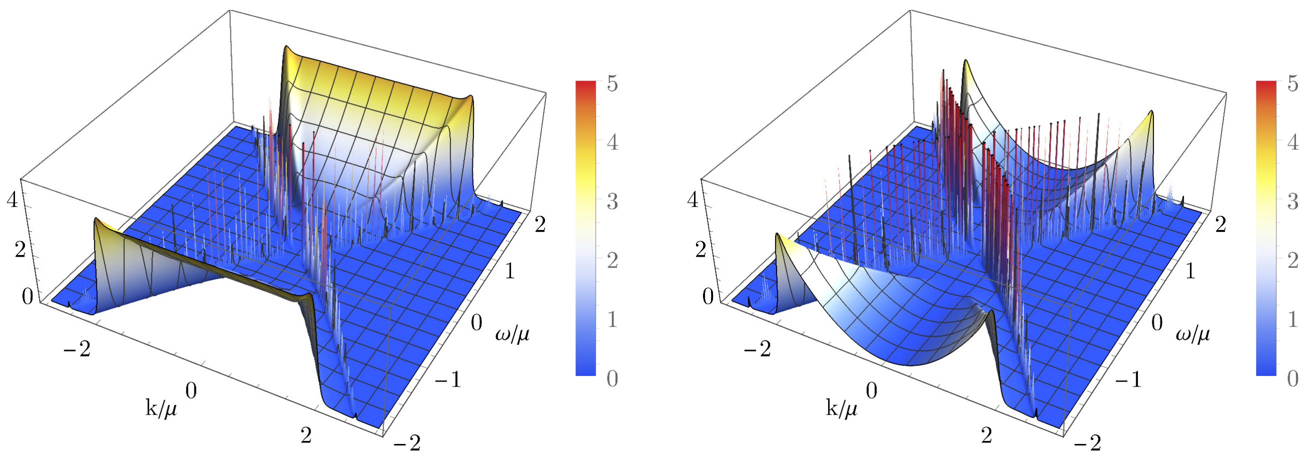
(a) Spectral function from G^{xx} at $T = 0.03T_c$ (b) Spectral function from G^{O^*O} at $T = 0.03T_c$

Figure 4.13: Absolute value for the spectral function with the coupling of the gauge and scalar field, and for $q = 3$, $m^2 = -3.5$ and $T = 0.03T_c$. In the spectral function for the scalar component we can see two linear modes, the one that we observe in the plot above at higher temperature, and one defining the cone structure we already saw in the intrinsic dynamics.

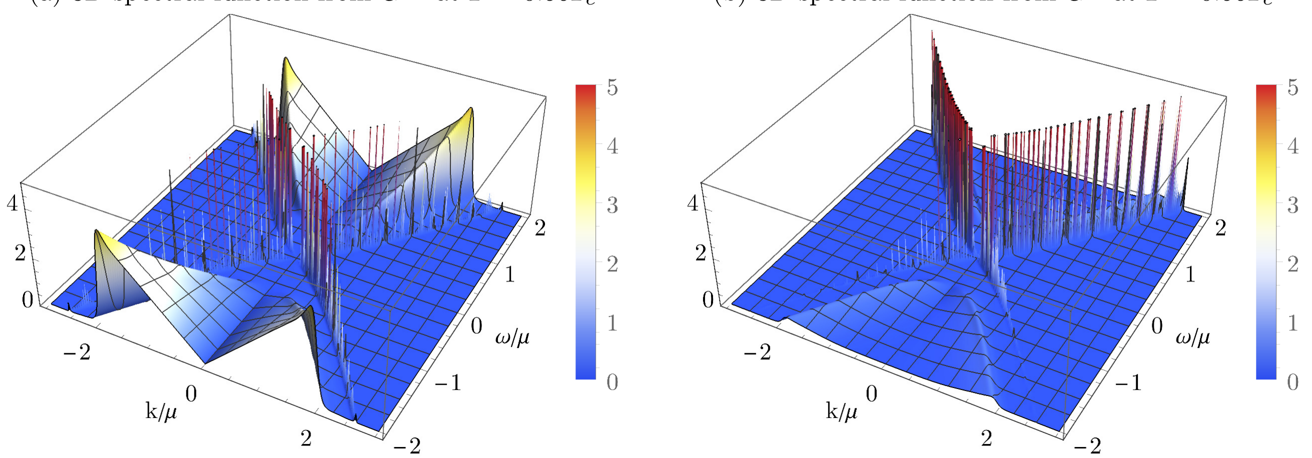


(a) 3D spectral function from G^{xx} at $T = 0.03T_c$ (b) 3D spectral function from G^{O^*O} at $T = 0.03T_c$

Figure 4.14: 3D version of the plot for the absolute value of the spectral function for $T = 0.03T_c$. Here it is easier to observe the linear mode as it is clearly visible from the high peaks. The discretness of the peaks is due to numerical discretization.

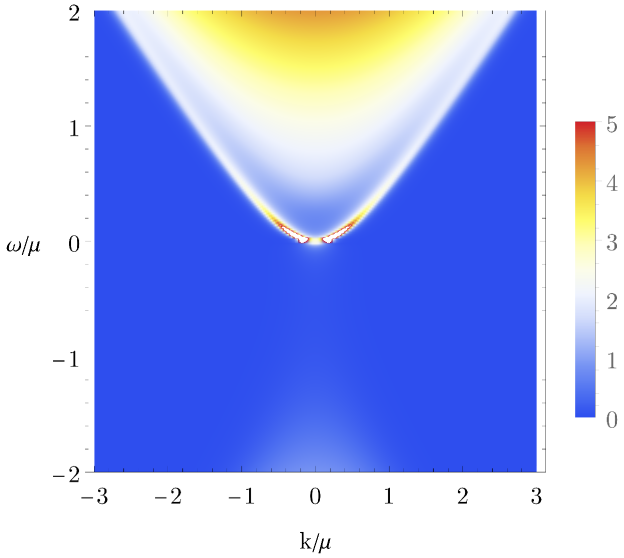


(a) 3D spectral function from G^{xx} at $T = 0.56T_c$ (b) 3D spectral function from G^{tt} at $T = 0.56T_c$

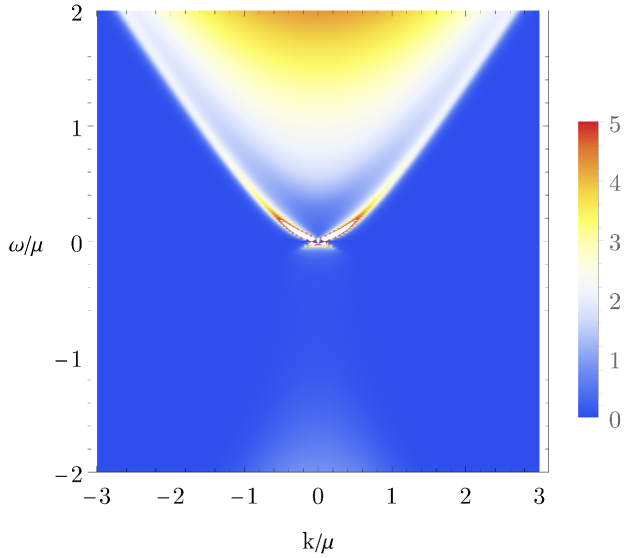


(c) 3D spectral function from G^{tx} at $T = 0.56T_c$ (d) 3D spectral function from G^{xO} at $T = 0.56T_c$

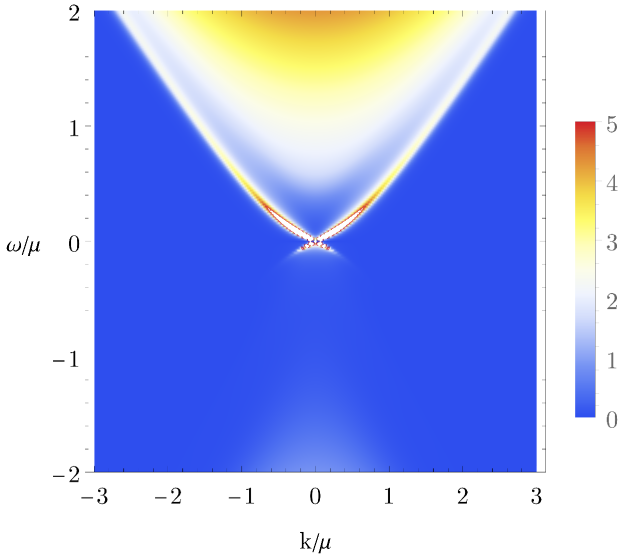
Figure 4.15: Plots of the 3 gauge spectral functions and the one associated to G^{xO} for $T = 0.56T_c$, for $q = 3$ and $m^2 = -3.5$. In the gauge spectral functions we can clearly see behavior related to the Ward identities, that imposes $G^{\mu\nu} \sim k^\mu k^\nu \Pi(\omega, k)$.



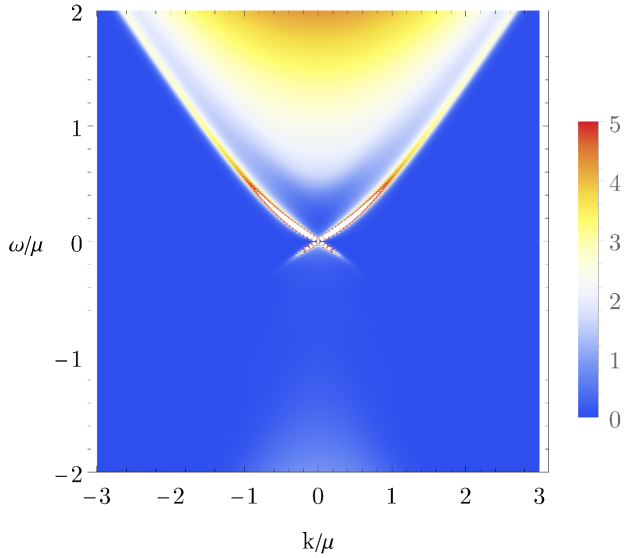
(a) $T = 0.999T_c$



(b) $T = 0.975T_c$



(c) $T = 0.944T_c$



(d) $T = 0.902T_c$

Figure 4.16: Absolute value of the spectral function associated to G^{O^*O} for different temperatures close to T_c. As we lower the temperature, we can see the quadratic mode from the intrinsic dynamics turning into two linear modes.

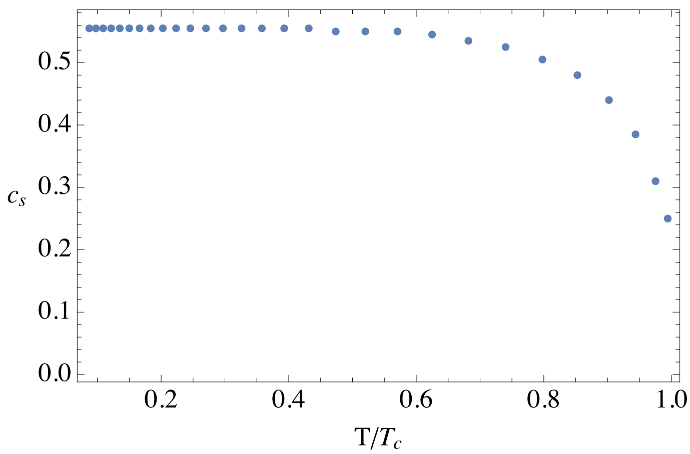


Figure 4.17: Speed of sound as a function of T/T_c. We can notice that it stabilizes to a constant value as we lower the temperature.

4.7 Intrinsic dynamics without amplitude fluctuations

In order to better understand the origin of the linear mode in the spectral function, we studied the intrinsic dynamics of the field after setting fluctuations of the amplitude of the order parameter to zero. In this section we briefly present the equations and the results obtained.

First of all we rewrite equations (4.25) to (4.29) in terms of the imaginary and the real part of the scalar fluctuations. As we explained above, for our choice of O real, the real part of the fluctuations is associated with amplitude fluctuations, while the imaginary part to phase fluctuations. We will, therefore, set the real part to zero to study the intrinsic dynamics.

Remember that we called $\delta\sigma = \mathrm{Re}[\delta\phi]$ and $\delta\eta = \mathrm{Im}[\delta\phi]$. The linearized equation of motions are already written in terms of real and imaginary part, though not explicitely, except for the first two equation (4.25) and (4.26). Combining them together, we obtain

$$0 = f\delta\sigma'' + \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\sigma' + \left(\frac{e^{\chi}\omega^2}{f} - \frac{k^2}{r^2} + \frac{q^2 e^{\chi} A_t^2}{f} - m^2\right)\delta\sigma + \frac{2iqe^{\chi}\omega A_t}{f}\delta\eta + \frac{2q^2 e^{\chi}\phi A_t}{f}\delta A_t \tag{4.124}$$

$$0 = f\delta\eta'' + \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\eta' + \left(\frac{e^{\chi}\omega^2}{f} - \frac{k^2}{r^2} + \frac{q^2 e^{\chi} A_t^2}{f} - m^2\right)\delta\eta - \frac{iqk\phi}{r^2}\delta A_x - \frac{iqe^{\chi}\omega\phi}{f}\delta A_t - \frac{2iqe^{\chi}\omega A_t}{f}\delta\sigma \tag{4.125}$$

$$0 = \delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2\phi^2}{f}\right)\delta A_x + \frac{e^{\chi}k\omega}{f^2}\delta A_t + 2i\frac{qk\phi}{f}\delta\eta \tag{4.126}$$

$$0 = \delta A_t'' + \left(\frac{\chi'}{2} + \frac{d-1}{r}\right)\delta A_t' - \left(\frac{k^2}{r^2 f} + \frac{2q^2\phi^2}{f}\right)\delta A_t - \frac{k\omega}{r^2 f}\delta A_x - 2i\frac{q\omega\phi}{f}\delta\eta - \frac{4q^2\phi A_t}{f}\delta\sigma \tag{4.127}$$

$$0 = \frac{e^{\chi}\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' + 2iq\left(\phi\delta\eta' - \phi'\delta\eta\right) \ . \tag{4.128}$$

The procedure to obtain the Green's function is then exactly the same we used for the problem in terms of the real and complex field, where now the coefficient of the initial condition at the horizon $\alpha_I(r-r_h)^{-\beta_I}$ are related by

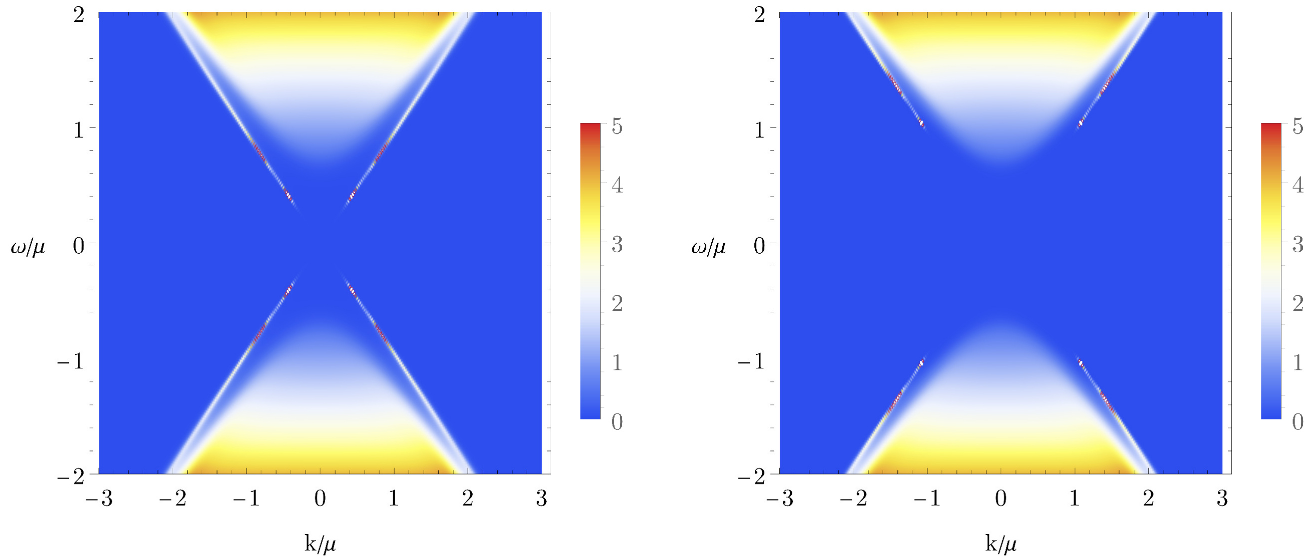
$$c_t = i\left(\frac{k}{r_h^2}c_x + 2iq\phi(r_h)c_\eta\right)\frac{e^{-\chi(r_h)/2}}{1-\frac{i\omega}{4\pi T}} \ . \tag{4.129}$$

If we now neglect (4.124) and set $\delta\sigma = 0$ in all the other equations, we reduce the problem to a 3×3 matrix computation that will give us

$$\boldsymbol{G}^R_{\delta\sigma=0} = \begin{pmatrix} G^{tt} & G^{xt} & G^{\eta t} \\ G^{tx} & G^{xx} & G^{\eta x} \\ G^{t\eta} & G^{x\eta} & G^{\eta\eta} \end{pmatrix} \tag{4.130}$$

The results for the spectral function of the G_{xx} and the $G_{\eta\eta}$ Green's function for different value of the temperature are shown below.

What we observe is that the results are qualitatively similar for the gauge component, and we still obtain a linear mode. However, the speed of sound of this mode is close to the speed of light and seems to be temperature independent.



(a) $T = 0.056T_c$

(b) $T = 0.003T_c$

Figure 4.18: Absolute value of the intrinsic spectral function associated to G^{xx}, obtained by neglecting amplitude fluctuations, for two values of T below the critical temperature. We can see two linear modes with speed close to the speed of light even at low energies.

Chapter 5

Conclusions

In this thesis we reviewed the holographic superconductor model and its thermodynamics properties that show a resemblance with known superconducting systems. In particular, we explained how to encode in the dual gravitational theory the instability that leads to a superconducting phase transition of the boundary system, and we observed the second-order phase transition that presents the typical mean-field behavior $\langle O \rangle \sim |T - T_c|^{1/2}$ for $T \approx T_c$.

We then studied the dynamics of the holographic superconductor model, with the intent of understanding its low energy behavior in order to write down a hydrodynamical model to observe how the dynamical properties compare to the ones observed in known materials. In doing so, we computed the optical conductivity that shows the expected properties for a superconductor model, namely the infinite conductivity in the DC regime ($\omega = 0$), and the formation of a gap in the low-energy limit for temperature below the critical value. This is reminiscent of BCS theory of superconductivity, even though, contrary to BCS, the holographic model represents systems with strong interactions, and a simple relation between the value of the order parameter and the energy necessary to break a pair of particles cannot be established. Here we also observed one of the effects of considering the intrinsic dynamics by neglecting metric fluctuations. Similarly to what happens when we neglect momentum conservation in the computation of the conductivity of a fluid, fixing the background metric implies a finite conductivity at $\omega = 0$ in the normal phase.

We then reviewed the results of a recent paper [17] about the intrinsic dynamics of the order parameter. We recovered the same results, that, contrary to the expectations from a time-dependent Ginzburg-Landau theory, do not present a linear mode in the superconducting phase. We then extended those results by adding fluctuations of the gauge field into the description. In the superconducting phase, the longitudinal components of the gauge field couple to the order parameter modifying its dynamics. We have shown that the coupling turns the quadratic mode observed in the intrinsic dynamics, into linear modes, hinting at the fact that the observed linear modes are not standard second-sound modes expected from superfluid models, as these arise from the phase fluctuations of the order parameter. Moreover, we found that the speed of sound does not always increase as we lower the temperature, but it quickly reaches a constant value.

Further work is still necessary to understand the low-energy dynamics of this coupled system and write down an effective model, and it is the aim of the future development of this project.

In a follow-up of this work, it would be interesting to include metric fluctuations in order to obtain the full spectral functions. This is particularly important not only to understand how the coupling with the metric affects the dynamics of the order parameter, but also because the full spectral functions provide physical information that is in principle comparable with results from experiments.

An alternative problem would be the study of the holographic superconductor dynamics with a different form of the scalar potential $V(\phi)$. For example adding a self-interaction term for the scalar field $V(\phi) = m^2|\phi|^2 + |\phi|^4$, that generates a coupling between the scalar field fluctuations and their complex conjugate already at the level of the intrinsic dynamics for the scalar field. This may change the intrinsic dynamics and it would be of great interest to see if the quadratic mode in the superconducting phase survives with this different form of the potential.

Appendix A

Different calculations for the intrinsic dynamics of the gauge field

In this appendix we explain why it is not possible to study the intrinsic dynamics of the gauge field alone and we present calculations in a different gauge.

A.1 About the intrinsic dynamics for the gauge field

As we have studied the intrinsic dynamics of the scalar fluctuations, one may wonder why we did not consider the intrinsic dynamics of vector field fluctuations as well. At first, we indeed tried to extract the intrinsic dynamics for the gauge field, however, we found that it is not possible. The equations for the intrinsic dynamics of the gauge field are given by (4.27) to (4.29) by setting $\delta\phi = \delta\phi^* = 0$, that gives

$$0 = \delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2\phi^2}{f}\right)\delta A_x + \frac{e^{\chi}k\omega}{f^2}\delta A_t \tag{A.1}$$

$$0 = \delta A_t'' + \left(\frac{\chi'}{2} + \frac{d-1}{r}\right)\delta A_t' - \left(\frac{k^2}{r^2 f} + \frac{2q^2\phi^2}{f}\right)\delta A_t - \frac{k\omega}{r^2 f}\delta A_x \tag{A.2}$$

$$0 = \frac{e^{\chi}\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' \tag{A.3}$$

The first two equation in the superconducting phase are not compatible with the constraint equation from the intrinsic dynamics (A.3). We can see a possible explanation by comparing this constraint equation with the one in the coupled problem

$$0 = \frac{e^{\chi}\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' + q\phi\left(\delta\phi' - (\delta\phi^*)'\right) + q\phi'\left(\delta\phi^* - \delta\phi\right) \tag{A.4}$$

that, as we showed, on the boundary becomes the conservation equation

$$\partial_\mu \delta\langle J^\mu\rangle = -2q\langle O\rangle\,\delta\eta_s \tag{A.5}$$

with the last term describing particles going into or getting out of the superfluid phase. Setting the scalar fluctuations to zero the conservation equation simply becomes

$$\partial_\mu \delta\langle J^\mu\rangle = 0 \tag{A.6}$$

as in the normal phase, without the term representing condensation of particles.

In other words, we explained in chapter 3 that the condensate forms because the Reissner-Nordström background is unstable under scalar fluctuations below a critical temperature. If we kill the scalar fluctuations, then there is nothing driving the field tachyonic generating the superconductiong phase transition.

A.2 Calculation in a different gauge

Here we present the calculation we tried for the intrinsic dynamics of the gauge field coupled only to phase fluctuations, in a gauge where we the phase dependence is not explicit, but $\delta A_r \neq 0$. It turns out this is just an inconvenient gauge choice and the same results could be obtained in the conventional gauge with $\delta A_r = 0$.

The matter part of the action describing our theory, in properly rescaled units, is:

$$S = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left((D^\mu\phi)^* D_\mu\phi + m^2\phi^*\phi\right)\right) \tag{A.7}$$

with a complex scalar field $\phi = |\phi|e^{i\theta}$. Rewriting the action (A.7) explicitly in terms of the amplitude and phase of the scalar fields we obtain:

$$S = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu|\phi|\partial^\mu|\phi| + |\phi|^2\partial_\mu\theta\partial^\mu\theta + q^2|\phi|^2A_\mu A^\mu - 2q|\phi|^2A_\mu\partial^\mu\theta + m|\phi|^2\right) \ . \tag{A.8}$$

The action is invariant under gauge transformations of the form:

$$\begin{aligned} A_\mu &\longrightarrow A'_\mu = A_\mu - \partial_\mu\Lambda(r,t,\boldsymbol{x}) \\ \phi &\longrightarrow \phi' = \phi e^{-iq\Lambda(r,t,\boldsymbol{x})} \end{aligned} \tag{A.9}$$

we can therefore use this invariance to get rid of the phase degree of freedom, by making the gauge choice:

$$\Lambda = -\frac{\theta}{q} \tag{A.10}$$

so that the gauge field transforms as:

$$A_\mu \longrightarrow A_\mu + \partial_\mu\frac{\theta}{q} \tag{A.11}$$

and the action becomes:

$$S = -\int d^{d+1}x\sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu|\phi|\partial^\mu|\phi| + q^2|\phi|^2A_\mu A^\mu + m^2|\phi|^2\right) \ . \tag{A.12}$$

The equations of motion for the gauge field then are:

$$\nabla_\mu F^{\mu\nu} = 2q^2|\phi|^2A^\nu \tag{A.13}$$

and we see that in order to ensure conservation of the current $J^\nu = 2q^2|\phi|^2A^\nu$ we have to impose the condition:

$$\nabla_\nu\left(|\phi|^2A^\nu\right) = 0 \tag{A.14}$$

We can verify that this constraint is compatible with the gauge choice by using the equation of motion for θ derived from the action (A.8):

$$\nabla_\mu(|\phi|^2\partial^\mu\theta) = \nabla_\mu(q|\phi|^2A^\mu) \ , \tag{A.15}$$

if we now make a gauge transformation of the current conservation equation we obtain:

$$\nabla_\nu\left(|\phi|^2A^\nu\right) \longrightarrow \nabla_\nu\left(|\phi|^2\left(A^\nu + \frac{\partial^\nu\theta}{q}\right)\right) = \nabla_\nu\left(|\phi|^2A^\nu\right) + \nabla_\nu\left(|\phi|^2\frac{\partial^\nu\theta}{q}\right) = 2\nabla_\nu\left(|\phi|^2A^\nu\right) = 0 \tag{A.16}$$

as required for consistency.

Summarizing, from the action (A.12) we obtain the three equations:

$$\nabla_\mu F^{\mu\nu} - 2q^2|\phi|^2 A^\nu = 0 \tag{A.17}$$
$$\left(\Box - m^2 - q^2 A_\nu A^\nu\right)|\phi| = 0 \tag{A.18}$$
$$\nabla_\nu\left(|\phi|^2 A^\nu\right) = 0 \tag{A.19}$$

Varying this set of equations we find the linearized equation of motions:

$$\nabla_\mu \delta F^{\mu\nu} - 2q^2|\phi|^2\delta A^\nu - 4q^2|\phi|\delta|\phi|A^\nu = 0 \tag{A.20}$$
$$\left(\Box - m^2 - q^2 A_\nu A^\nu\right)\delta|\phi| - 2q^2|\phi|A_\mu\delta A^\mu = 0 \tag{A.21}$$
$$\nabla_\nu\left(|\phi|^2\delta A^\nu\right) + 2\nabla_\nu\left(|\phi|\delta|\phi|A^\nu\right) = 0 \tag{A.22}$$

Since we are now interested in the decoupled problem, we set $\delta|\phi| = 0$ and consider:

$$\nabla_\mu \delta F^{\mu\nu} - 2q^2|\phi|^2\delta A^\nu = 0 \tag{A.23}$$
$$\nabla_\nu\left(|\phi|^2\delta A^\nu\right) = 0 \tag{A.24}$$

Using the metric in $d = 4$ dimensions (corresponding to a $3+1$ boundary theory)

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\boldsymbol{x}^2 \ , \tag{A.25}$$

Fourier transforming the field

$$\delta A_\nu(r, x^\mu) = \int \frac{d^4k}{(2\pi)^4}\delta A_\nu(r, k^\mu)e^{ik\cdot x} \tag{A.26}$$

and choosing the momentum along the x direction $k^\mu = (\omega, k, 0, 0)$ we obtain:

$$0 = \delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta A_x' + \left(\frac{e^\chi\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f}\right)\delta A_x + \frac{e^\chi k\omega}{f^2}\delta A_t - ik\left[\delta A_r' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta A_r\right] \tag{A.27}$$

$$0 = \delta A_t'' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta A_t' - \left(\frac{k^2}{r^2 f} + \frac{2q^2|\phi|^2}{f}\right)\delta A_t - \frac{k\omega}{r^2 f}\delta A_x + i\omega\left[\delta A_r' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta A_r\right] \tag{A.28}$$

$$0 = \frac{e^\chi\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' - i\delta A_r\left(2q^2|\phi|^2 + \frac{k^2}{r^2} - \frac{e^\chi\omega^2}{f}\right) \tag{A.29}$$

$$0 = \delta A_r' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{3}{r} + 2\frac{|\phi|'}{|\phi|}\right)\delta A_r + i\left(\frac{\omega e^\chi}{f^2}\delta A_t + \frac{k}{r^2 f}\delta A_x\right) \tag{A.30}$$

From (A.30):

$$\delta A_r' = -\left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{3}{r} + 2\frac{|\phi|'}{|\phi|}\right)\delta A_r - i\left(\frac{\omega e^\chi}{f^2}\delta A_t + \frac{k}{r^2 f}\delta A_x\right) \tag{A.31}$$

and substituting this into (A.27) and (A.28) we obtain:

$$\delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta A_x' + \left(\frac{e^\chi\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta A_x + ik\left(\frac{2}{r} + 2\frac{|\phi|'}{|\phi|}\right)\delta A_r = 0 \tag{A.32}$$

$$\delta A_t'' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta A_t' + \left(\frac{e^\chi\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta A_t - i\omega\left(\frac{f'}{f} - \chi' + 2\frac{|\phi|'}{|\phi|}\right)\delta A_r = 0 \tag{A.33}$$

We can now reduce the set of equations to a system of first order differential equations by defining the variables $\delta B_t = \delta A_t'$ and $\delta B_x = \delta A_x'$.

Defining the vector

$$\delta \boldsymbol{F}(r) = (\delta B_x(r), \delta B_t(r), \delta A_x(r), \delta A_t(r), \delta A_r(r))^T \tag{A.34}$$

the system of differential equations can be written as:

$$\delta \boldsymbol{F}'(r) = M(r)\delta \boldsymbol{F}(r) \tag{A.35}$$

with $M(r)$ a 5×5 matrix given by:

$$\begin{pmatrix} -\frac{f'}{f} + \frac{\chi'}{2} - \frac{1}{r} & 0 & \frac{k^2}{r^2 f} + \frac{2q^2|\phi|^2}{f} - \frac{e^\chi \omega}{f^2} & 0 & -ik\left(\frac{2|\phi|'}{|\phi|} + \frac{2}{r}\right) \\ 0 & -\frac{\chi'}{2} - \frac{3}{r} & 0 & \frac{k^2}{r^2 f} + \frac{2q^2|\phi|^2}{f} - \frac{e^\chi \omega}{f^2} & i\omega\left(\frac{f'}{f} + \frac{2|\phi|'}{|\phi|} - \chi'\right) \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{ikf}{r^2} & -\frac{ie^\chi \omega}{f^2} & -\frac{f'}{f} - \frac{2|\phi|'}{|\phi|} + \frac{\chi'}{2} - \frac{3}{r} \end{pmatrix} \tag{A.36}$$

If we look at the high temperature solution, i.e. setting $\phi = \chi = \delta A_r = 0$ in equations (A.27)-(A.29) (we can set δA_r to zero because in the high temperature solution the scalar field is not present, and we therefore have the freedom to choose a gauge in which $\delta A_r = 0$), the equations become:

$$\delta A_x'' + \left(\frac{f'}{f} + \frac{1}{r}\right)\delta A_x' + \frac{\omega^2}{f^2}\delta A_x + \frac{k\omega}{f^2}\delta A_t = 0 \tag{A.37}$$

$$\delta A_t'' + \frac{3}{r}\delta A_t' - \frac{k^2}{r^2 f}\delta A_t - \frac{k\omega}{r^2 f}\delta A_x = 0 \tag{A.38}$$

$$\frac{\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' = 0 \tag{A.39}$$

$$\tag{A.40}$$

and defining δB_x and δB_t as above, and

$$\delta \boldsymbol{F}(r) = (\delta B_x(r), \delta B_t(r), \delta A_x(r), \delta A_t(r))^T, \tag{A.41}$$

the matrix now reads

$$M(r) = \begin{pmatrix} -\frac{f'}{f} - \frac{1}{r} & 0 & -\frac{\omega^2}{f^2} & -\frac{k\omega}{f^2} \\ 0 & -\frac{3}{r} & \frac{k\omega}{r^2 f} & \frac{k^2}{r^2 f} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{A.42}$$

This matrix, as we expect, clearly presents a zero-mode, as it easy to see that $\mathrm{rank}(M) = 3$. Using this fact, we can combine the system to obtain the equation:

$$B_x' + \left(\frac{f'(r)}{f(r)} + \frac{1}{r}\right)B_x + 3\frac{\omega r}{kf}B_t + \frac{\omega r^2}{kf}B_t' = 0 \tag{A.43}$$

A.2.1 Decoupling the δA_r component

The equations shown above can be combined to get rid of the δA_r term and obtain the following two equations in δA_x and δA_t only:

$$\begin{aligned} &\delta A_x'' + \left[\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r} + \left(\frac{2}{r} + 2\frac{|\phi|'}{|\phi|}\right)\frac{k^2 f}{2q^2|\phi|^2 f r^2 + k^2 f - r^2 e^\chi \omega^2}\right]\delta A_x' \\ &+ r^2 \omega k e^\chi \frac{\frac{2}{r} + 2\frac{|\phi|'}{|\phi|}}{2q^2|\phi|^2 f r^2 + k^2 f - r^2 e^\chi \omega^2}\delta A_t' + \left(\frac{e^\chi \omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta A_x = 0 \end{aligned} \tag{A.44}$$

$$\begin{aligned}\delta A_t'' + \left[\frac{\chi'}{2} + \frac{3}{r} - \left(\frac{f'}{f} - \chi' + 2\frac{|\phi|'}{|\phi|}\right)\frac{r^2\omega^2 e^{\chi}}{2q^2|\phi|^2 f r^2 + k^2 f - r^2 e^{\chi}\omega^2}\right]\delta A_t' \\ - k\omega f \frac{\frac{f'}{f} - \chi' + 2\frac{|\phi|'}{|\phi|}}{2q^2|\phi|^2 f r^2 + k^2 f - r^2 e^{\chi}\omega^2}\delta A_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta A_t = 0\end{aligned} \tag{A.45}$$

A.2.2 Equations in terms of gauge invariant fields

Equation (A.27) to (A.29) can be also written in terms of gauge invariant field only, as:

$$-\delta R_x' - \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta R_x + \frac{e^{\chi}\omega}{f^2}\frac{\delta E_x}{i} = \frac{2q^2|\phi|^2}{f}\delta A_x \tag{A.46}$$

$$-\delta R_t' - \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta R_t - \frac{k}{r^2 f}\frac{\delta}{i}E_x = \frac{2q^2|\phi|^2}{f}\delta A_t \tag{A.47}$$

$$-\frac{e^{\chi}\omega}{f^2}\delta R_t - \frac{k}{r^2 f}\delta R_x = i\frac{2q^2|\phi|^2}{f}\delta A_r \tag{A.48}$$

Defining the electric field $E_x = F_{xt} = ikA_t + i\omega A_x$ and the R field $R_x = F_{xr} = ikA_r - A_x'$. The choice of the name variable R is arbitrary, as R_x is actually a component of the magnetic field tensor, (the magnetic field in spatial dimensions different then 3, it is not a vector) but we decided not to call it B_x to avoid confusion with the vector magnetic field and we do not use the proper tensor notation to avoid having double indices. Combining (A.32) and (A.33) we obtain:

$$\delta E_x'' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta E_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta E_x - i\omega\left(\frac{f'}{f} - \chi' - \frac{2}{r}\right)\delta R_x = 0 \tag{A.49}$$

For the constraint equation (A.29) we instead find:

$$\frac{\omega e^{\chi}}{fk}\delta E_x' + i\left(\frac{\omega^2 e^{\chi}}{fk} - \frac{k}{r^2}\right)\delta R_x + 2q^2|\phi|^2\delta A_r = 0 \tag{A.50}$$

and from the current conservation (A.30)

$$\delta R_x' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta R_x + i\frac{e^{\chi}\omega}{f^2}\delta E_x + \frac{2q^2|\phi|^2}{f}\delta A_x = 0\ . \tag{A.51}$$

Now we have equations for δA_r and δA_x in terms of gauge invariant field, and we can therefore construct the second gauge invariant equation using $\delta R_x = ik\delta A_r - \delta A_x'$:

$$\delta R_x = \frac{k}{2q^2|\phi|^2}\left(\left(\frac{\omega^2 e^{\chi}}{fk} - \frac{k}{r^2}\right)\delta R_x - i\frac{\omega e^{\chi}}{fk}\delta E_x'\right) + \frac{d}{dr}\left[\frac{f}{2q^2|\phi|^2}\left(\delta R_x' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta R_x + i\frac{e^{\chi}\omega}{f^2}\delta E_x\right)\right]\ . \tag{A.52}$$

Writing the derivative explicitly we obtain:

$$\begin{aligned}\delta R_x = \frac{f}{2q^2|\phi|^2}\Bigg[\Bigg(\left(\frac{\omega^2 e^{\chi}}{f^2} - \frac{k^2}{fr^2} + \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 - \frac{\chi''}{2} - \frac{1}{r^2}\right)\delta R_x + \delta R_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta R_x' + i\frac{\omega e^{\chi}}{f^2}\left(\chi' - 2\frac{f'}{f}\right)\delta E_x \\ + \left(\delta R_x' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta R_x + i\frac{e^{\chi}\omega}{f^2}\delta E_x\right)\left(\frac{f'}{2q^2|\phi|^2} - 2\frac{f|\phi|'}{2q^2|\phi|^3}\right)\end{aligned} \tag{A.53}$$

That can be written as:

$$\begin{aligned}\delta R_x'' + \left(\frac{\omega^2 e^{\chi}}{f^2} - \frac{k^2}{fr^2} + \frac{f''}{f} - \frac{\chi''}{2} - \frac{1}{r^2} - 2\frac{|\phi|' f'}{|\phi| f} - \frac{\chi' f'}{2f} + \frac{\chi'|\phi|'}{|\phi|} + \frac{f'}{fr} - 2\frac{|\phi|'}{|\phi| r} - \frac{2q^2|\phi|^2}{f}\right)\delta R_x \\ + \left(2\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r} - 2\frac{|\phi|'}{|\phi|}\right)\delta R_x' + i\frac{\omega e^{\chi}}{f^2}\left(\chi' - \frac{f'}{f} - 2\frac{|\phi|'}{|\phi|}\right)\delta E_x = 0\end{aligned} \tag{A.54}$$

and now we have a system of two equations in the gauge independent variables δE_x and δR_x.

We can alternatively solve the system of equations in terms of the electric field and the gauge invariant variable $R_t \equiv -E_r = -i\omega A_r - A_t'$. Written in term of this field, equation (A.49) becomes:

$$\delta E_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{1}{r}\right)\delta E_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2|\phi|^2}{f} - \frac{k^2}{fr^2}\right)\delta E_x + ik\left(\frac{f'}{f} - \chi' - \frac{2}{r}\right)\delta R_t = 0 \tag{A.55}$$

While we can derive the second equation from (A.50) and (A.28), that written in terms of gauge invariant fields reads:

$$\delta R_t' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta R_t - i\frac{k}{r^2 f}E_x + \frac{2q^2|\phi|^2}{f}\delta A_t = 0 \tag{A.56}$$

And we have:

$$\delta R_t = \frac{1}{2q^2|\phi|^2}\left(\frac{e^{\chi}\omega^2}{f}\delta R_t - \frac{k^2}{r^2}\delta R_t + i\frac{k}{r^2}\delta E_x'\right) + \frac{d}{dr}\left[\frac{f}{2q^2|\phi|^2}\left(\delta R_t' + \left(\frac{\chi'}{2} + \frac{3}{r}\right)\delta R_t - i\frac{k}{r^2 f}\delta E_x\right)\right] . \tag{A.57}$$

Computing the derivative we find:

$$\delta R_t'' + \left(\frac{\omega^2 e^{\chi}}{f^2} - \frac{k^2}{fr^2} + \frac{\chi''}{2} - \frac{3}{r^2} + \frac{\chi' f'}{2f} - \frac{\chi'|\phi|'}{|\phi|} + 3\frac{f'}{fr} - 6\frac{|\phi|'}{|\phi| r} - \frac{2q^2|\phi|^2}{f}\right)\delta R_t \tag{A.58}$$

$$+\left(\frac{f'}{f} + \frac{\chi'}{2} + \frac{3}{r} - 2\frac{|\phi|'}{|\phi|}\right)\delta R_t' + i\frac{k}{fr^2}\left(\frac{2}{r} + 2\frac{|\phi|'}{|\phi|}\right)\delta E_x = 0 \tag{A.59}$$

Even though we where able to decouple the A_r component and to find nice expression in terms of gauge invariant terms, this gauge choice is not optimal to perform computations, and to obtain the results presented in the main chapters we used the more conventional choice of setting $\delta A_r = 0$.

Appendix B

Detailed calculations

Here we present the details of the calculations that lead to the results in chapter 3 and chapter 4. We expand the action up to second order in fluctuations of the scalar and gauge fields while keeping the metric fixed. As previously discussed, fluctuations of the metric are coupled to fluctuations of the “matter” fields, therefore, in order to solve the full problem we would need to include the metric fluctuations as well.

B.1 Equations for the background

B.1.1 Einstein equations

Here we derive the equation of motion for the metric tensor.

The metric ansatz is

$$ds^2 = -c^2 f(r) e^{-\chi(r)} dt^2 + \frac{1}{f(r)} dr^2 + \frac{r^2}{L^2} d\mathbf{x}_{d-1}^2 \tag{B.1}$$

so that we have:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 f(r) e^{-\chi(r)} & 0 & \mathbf{0}_{d-1} \\ 0 & \frac{1}{f(r)} & \mathbf{0}_{d-1} \\ \mathbf{0}_{d-1}^T & \mathbf{0}_{d-1}^T & \frac{r^2}{L^2} \mathbf{1}_{d-1,d-1} \end{pmatrix} \tag{B.2}$$

with $\mathbf{0}_{d-1}$ the $d-1$ dimensional null vector and $\mathbf{1}_{d-1,d-1}$ the $(d-1) \times (d-1)$ identity matrix. Since the metric is diagonal the inverse metric is trivially

$$g^{\mu\nu} = \begin{pmatrix} -\frac{e^{\chi(r)}}{c^2 f(r)} & 0 & \mathbf{0}_{d-1} \\ 0 & f(r) & \mathbf{0}_{d-1} \\ \mathbf{0}_{d-1}^T & \mathbf{0}_{d-1}^T & \frac{L^2}{r^2} \mathbf{1}_{d-1,d-1} \end{pmatrix} \tag{B.3}$$

The Christoffel symbol is defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \tag{B.4}$$

The Riemann tensor is given in terms of Christoffel symbols by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{B.5}$$

Since the metric is diagonal $\Gamma^{\lambda}_{\mu\nu} = 0 \qquad \forall \lambda \neq \mu \neq \nu$ and since $g_{\mu\nu} = g_{\mu\nu}(r)$, we have that $\Gamma^{t}_{\alpha\alpha} = 0$, $\Gamma^{x_i}_{\alpha\alpha} = 0$ for $\alpha = r, t, x_1, \ldots, x_{d-1}$. Therefore the only non zero Christoffel symbols are:

$$\Gamma^{r}_{rr} = -\frac{1}{2}\frac{f'(r)}{f(r)} \tag{B.6}$$

$$\Gamma^{r}_{x_i x_i} = -\frac{f(r) r}{L^2} \tag{B.7}$$

$$\Gamma^{r}_{tt} = \frac{c^2}{2} f(r) e^{-\chi(r)} \left(f'(r) - f(r)\chi'(r)\right) \tag{B.8}$$

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2}\frac{f'(r) - f(r)\chi'(r)}{f(r)} \tag{B.9}$$

$$\Gamma^{x_i}_{x_i r} = \Gamma^{x_i}_{r x_i} = \frac{1}{r} \tag{B.10}$$

The relevant Riemann tensor then are:

$$R^{r}{}_{trt} = \frac{c^2}{4} e^{-\chi(r)} f(r) \left(-3f'(r)\chi'(r) + f(r)(\chi'(r)^2 - 2\chi''(r)) + 2f''(r)\right) \tag{B.11}$$

$$R^{r}{}_{xrx} = -\frac{r}{2L^2} f'(r) \tag{B.12}$$

$$R^{t}{}_{rtr} = \frac{1}{4f(r)} \left(3f'(r)\chi'(r) - f(r)\chi'(r)^2 - 2f''(r) + 2f(r)\chi''(r)\right) \tag{B.13}$$

$$R^{t}{}_{xtx} = \frac{r}{2L^2} \left(f(r)\chi'(r) - f'(r)\right) \tag{B.14}$$

$$R^{x}{}_{txt} = \frac{c^2}{2r} e^{-\chi(r)} f(r) \left(f'(r) - f(r)\right) \tag{B.15}$$

$$R^{x}{}_{rxr} = -\frac{f'(r)}{2r f(r)} \tag{B.16}$$

$$R^{x_i}{}_{x_j x_i x_j} = -\frac{f(r)}{L^2} \tag{B.17}$$

Remember that the Ricci tensor is defined by $R_{\mu\nu} \equiv R^{\sigma}{}_{\mu\sigma\nu}$ and the Ricci scalar as $g^{\mu\nu} R_{\mu\nu}$.

The holographic superconductor action, defined in chapter 3 is

$$S = \int d^{d+1}x \sqrt{-g} \left(\frac{c^3}{16\pi G}(R - 2\Lambda) - \frac{1}{4\mu_0 c} F_{\mu\nu}F^{\mu\nu} - |D_\mu \phi|^2 - \frac{m^2 c^2}{\hbar^2}|\phi|^2 \right) \tag{B.18}$$

with $D_\mu \equiv \nabla_\mu - \frac{iq}{\hbar} A_\mu$.

Varying it with respect to the metric field, we obtain the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^3} \left(-4F_{\mu\rho}F_{\nu}{}^{\rho} + F_{\rho\sigma}F^{\rho\sigma} g_{\mu\nu} + g_{\mu\nu} \left(|D\phi|^2 + \frac{m^2 c^2}{\hbar^2}|\phi|^2 \right) - 2D_{[\mu}\phi (D_{\nu]})^* \right) \tag{B.19}$$

where the last term can be dropped since we choose ϕ to be real. The right-hand side of the equation represent the energy momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{B.20}$$

Using the metric ansatz (B.1) and the expressions for the nonzero Ricci tensors and for Ricci scalar we obtain the two equations:

$$\left(4f'(d-1)+8r\Lambda-\frac{(d-2)f}{2r}\right)=-\frac{64\pi Gr}{c^3}\left(\frac{2fA_t'^2e^{\chi}}{c^2f4\mu_0}+f\phi'^2+\frac{q^2A_t^2\phi^2}{c^2f\hbar^2}+\frac{m^2c^2}{\hbar^2}\phi^2\right) \tag{B.21}$$

$$\left(-4f\chi'(d-1)+4f'(d-1)+8r\Lambda-\frac{(d-2)f}{2r}\right)=-\frac{64\pi Gr}{c^3}\left(\frac{2fA_t'^2e^{\chi}}{c^2f4\mu_0}-f\phi'^2-\frac{q^2A_t^2\phi^2}{c^2f\hbar^2}+\frac{m^2c^2}{\hbar^2}\phi^2\right) \tag{B.22}$$

Subtracting (B.22) from (B.21) we obtain

$$\chi'+\frac{32\pi G}{(d-1)c^3}r\left(\phi'^2+\frac{q^2e^{\chi}}{\hbar^2c^2f}A_t^2\phi^2\right)=0 \tag{B.23}$$

while summing them we obtain

$$f'+\left(\frac{d-2}{r}-\frac{\chi'}{2}\right)f+\frac{16\pi G}{(d-1)c^3}r\left(\frac{e^{\chi}A_t'^2}{2\mu_0c^3}+\frac{m^2c^2}{\hbar^2}\phi^2\right)-\frac{rd}{L^2}=0 \tag{B.24}$$

where we used

$$2\Lambda=-\frac{d(d-1)}{L^2}\ . \tag{B.25}$$

Equations (B.23) and (B.24) are two equations of the equations for the background presented in chapter 3, below we derive the remaining two equations.

B.1.2 Equation of motion for the scalar field

The equation of motion for the scalar field ϕ can be found by varying part of the action containing the scalar field

$$S_\phi=-\int d^{d+1}x\sqrt{-g}\left((D^\mu\phi)^*D_\mu\phi+\frac{m^2c^2}{\hbar^2}\phi^*\phi\right) \tag{B.26}$$

that expanding the covariant derivative becomes:

$$S_\phi=-\int d^{d+1}x\sqrt{-g}\left(\partial^\mu\phi^*\partial_\mu\phi-i\frac{q}{\hbar}A_\mu\phi\partial^\mu\phi^*+i\frac{q}{\hbar}A^\mu\phi\partial_\mu\phi+\frac{q^2}{\hbar^2}A_\mu A^\mu\phi^*\phi+\frac{m^2c^2}{\hbar^2}\phi^*\phi\right)\ . \tag{B.27}$$

Varying this action with respect to ϕ^* we obtain, after integrating by parts

$$\delta S_{\phi^*}=boundaryterm+\int d^{d+1}x\sqrt{-g}\left[\left(D_\mu D^\mu-\frac{m^2c^2}{\hbar^2}\right)\phi\right]\delta\phi^*\ , \tag{B.28}$$

so the equation of motion for ϕ is:

$$\left(D_\mu D^\mu-\frac{m^2c^2}{\hbar^2}\right)\phi=0\ . \tag{B.29}$$

and inserting the expression for the metric

$$\phi''+\left(\frac{f'}{f}+\frac{d-1}{r}-\frac{\chi'}{2}\right)\phi'-\left(\frac{m^2c^4-q^2A_t^2\frac{e^{\chi}}{f}}{\hbar^2c^2f}\right)\phi=0\ . \tag{B.30}$$

B.1.3 Equations of motion for the vector field

Similarly we can vary the matter part of the action containing the gauge field

$$S = -\int d^{d+1}x \sqrt{-g}\left(\frac{1}{4\mu_0 c}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^* D_\mu\phi\right) \tag{B.31}$$

to obtain

$$\delta_{A_\mu} S = boundaryterm - \int d^{d+1}x \sqrt{-g}\left(\frac{1}{\mu_0 c}\left(\nabla_\mu\nabla^\mu A^\nu - \nabla_\mu\nabla^\nu A^\mu\right) + \frac{iq}{hbar}\phi^*\partial^\nu\phi - \frac{iq}{hbar}\phi\partial^\nu\phi^* - \frac{2q^2}{hbar^2}\phi A^\nu\phi^*\right) . \tag{B.32}$$

Rewriting (B.32) in terms of $F^{\mu\nu}$ and gauge covariant derivatives we obtain

$$\frac{1}{\mu_0 c}\nabla_\mu F^{\mu\nu} - \frac{iq}{\hbar}\left(\phi^* D^\nu\phi - \phi(D^\nu\phi)^*\right) = 0 \ , \tag{B.33}$$

that in terms of the background fields become the equation of motion for A_t

$$A_t'' + \left(\frac{d-1}{r} + \frac{\chi'}{2})A_t' - 2\frac{q^2\mu_0 c\phi^2}{\hbar^2 f}\right) A_t = 0 \ . \tag{B.34}$$

B.2 Variation of the matter action

In this section we work in rescaled units where all the constants are set to one.

The matter part of the action is then given by

$$S = -\int d^{d+1}x \sqrt{-g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^* D_\mu\phi + m^2\phi^*\phi\right) \tag{B.35}$$

expanding the action in fluctuations of the fields

$$\begin{aligned} \phi &\to \phi + \delta\phi \ , \\ A_\mu &\to A_\mu + \delta A_\mu \ , \end{aligned} \tag{B.36}$$

we have:

$$\begin{aligned} S[\mathbf{\Phi} + \delta\mathbf{\Phi}] = -\int d^{d+1}x \sqrt{-g}\Big(&\frac{1}{4}\left[\partial_\mu(A_\nu + \delta A_\nu) - \partial_\nu(A_\mu + \delta A_\mu)\right]^2 + \left(D^\mu(\phi+\delta\phi)\right)^* D_\mu(\phi+\delta\phi) \\ &+ m^2(\phi^* + \delta\phi^*)(\phi+\delta\phi)\Big) \\ = -\int d^{d+1}x \sqrt{-g}\Big(&\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi^* + iq\partial_\mu\phi A^\mu\phi^* - iqA_\mu\phi\partial^\mu\phi^* + q^2A_\mu A^\mu\phi\phi^* + m^2\phi\phi^* + \\ &+ \nabla_\mu\delta A_\nu F^{\mu\nu} + \left[iq\partial_\mu\phi\phi^* - iq\phi\partial_\mu\phi^* + 2q^2A_\mu\phi\phi^*\right]\delta A^\mu + \\ &+ \left[\partial_\mu\delta\phi\partial^\mu\phi^* + iqA^\mu\phi^*\partial_\mu\delta\phi - iqA^\mu\delta\phi\partial_\mu\phi^* + m^2\delta\phi\phi^* + q^2A_\mu A^\mu\delta\phi\phi^* + h.c\right] + \\ &+ \frac{1}{2}\left(\nabla_\mu\delta A_\nu\nabla^\mu\delta A^\nu - \nabla_\mu\delta A_\nu\nabla^\nu\delta A^\mu\right) + \delta A_\mu q^2\phi\phi^*\delta A^\mu + \\ &+ \left[iq\partial_\mu\delta\phi\delta A^\mu\phi^* - iq\delta A^\mu\phi\partial_\mu\phi\delta\phi + 2q^2\delta A^\mu A_\mu\phi\delta\phi + h.c\right] + \\ &+ \partial_\mu\delta\phi\partial^\mu\delta\phi^* + iq\partial_\mu\delta\phi A^\mu\delta\phi^* - iq\delta\phi A^\mu\partial_\mu\delta\phi^* + q^2\delta\phi A_\mu A^\mu\delta\phi^* + \\ &+ iq\partial_\mu\delta\phi\delta A^\mu\delta\phi^* - iq\delta\phi\delta A^\mu\partial_\mu\delta\phi^* + 2q^2A_\mu\delta\phi\delta A^\mu\delta\phi^* + q^2\delta A_\mu\delta\phi\delta A^\mu\phi^* \\ &+ q^2\delta A_\mu\delta\phi^*\delta A^\mu\phi + +q^2\delta A_\mu\delta\phi\delta A^\mu\delta\phi^* \ , \end{aligned} \tag{B.37}$$

where *h.c* stands for *hermitian conjugate* and $\mathbf{\Phi} + \delta\mathbf{\Phi}$ is a shorthand notation that stands for variations of all the matter fields in the lagrangian.

B.2.1 First order variation and equations of motion

Here we check that the terms first order in fluctuations of the fields give us the equation of motion that we derived before. Focusing first on terms that are first order in fluctuations in (B.37), we perform partial integration and use the definition of the gauge covariant derivative $D_\mu = \nabla_\mu - iqA_\mu$ to obtain

$$
\begin{aligned}
\delta S^{(1)} =& -\int d^{d+1}x\, \nabla_\mu \left(\sqrt{-g}\left[\delta A_\nu F^{\mu\nu} + \delta\phi \left(D^\mu \phi\right)^* + \delta\phi^* D^\mu \phi\right]\right) \\
&+ \int d^{d+1}x \sqrt{-g} \left(\delta A_\nu \left[\nabla_\mu F^{\mu\nu} - iq\left(\phi^* D^\nu \phi - \phi (D^\nu\phi)^*\right)\right] + \left[\delta\phi\left(D^*_\mu D^{\mu *} - m^2\right)\phi^* + h.c.\right]\right) \\
=& \delta S^{(1)}_{bdy} + \delta S^{(1)}_{bulk} \ .
\end{aligned}
\tag{B.38}
$$

From the bulk term we can read off the classical equation of motions

$$\nabla_\mu F^{\mu\nu} = iq\left(\phi^* D^\nu \phi - \phi (D^\nu \phi)^*\right) \tag{B.39}$$

$$\left(D_\mu D^\mu - m^2\right)\phi = 0 \tag{B.40}$$

$$\left(D^*_\mu D^{\mu *} - m^2\right)\phi^* = 0 \tag{B.41}$$

That are the same as we found in the previous section, except for the fact that we are now working in rescaled units.

We denoted the first term (B.38) as $\delta S^{(1)}_{bdy}$ because we can apply Stokes' theorem, and reduce it to a boundary term. However we have to be careful, since the integrand diverges for $r \to \infty$. The general procedure consist in introducing a cutoff Λ at large r that we will send to infinity at the end of the calculations, after adding the proper counterterms to regularize the action if necessary. This gives:

$$
\begin{aligned}
\delta S_{bdy} =& -\int d^{d+1}x\, \nabla_\mu \left(\sqrt{-g}\left[\delta A_\nu F^{\mu\nu} + \delta\phi \left(D^\mu \phi\right)^* + \delta\phi^* D^\mu \phi\right]\right) \\
=& -\lim_{\Lambda\to\infty} \int d^d x \sqrt{-h} n^\mu \left[h^{\nu\sigma} F_{\mu\sigma}\delta A_\nu + \delta\phi(\partial_\mu \phi^* + iqA_\mu \phi^*) + \delta\phi^*(\partial_\mu\phi - iqA_\mu\phi)\right]\Big|_{r_h}^{\Lambda} \ .
\end{aligned}
\tag{B.42}
$$

Where h is the determinant of the metric of the d-dimensional spacetime slice orthogonal to r:

$$h^{\mu\nu} = r^{-2}\eta^{\mu\nu} \tag{B.43}$$

and n^μ is the orthogonal unit vector to the surface of constant r, pointing in the direction of increasing r. It then must have the form $n^\mu = c\delta^\mu_r$, where we can determine the constant c by imposing the normalization condition $n_\mu n^\mu = 1$ that gives:

$$g_{\mu\nu} n^\mu n^\nu = 1 \Rightarrow g_{rr} c^2 = 1 \Rightarrow c = \pm\sqrt{f(r)} \ , \tag{B.44}$$

for fixed r. The two solutions represents the two possible orientation of the vector normal to the surface. Since we want the one pointing outward, we choose the positive solution. Notice that in the boundary limit the vector behaves as

$$n^\mu = r\delta^\mu_r \qquad \text{for} \qquad r \to \infty \ . \tag{B.45}$$

The integrand of (B.42) is zero when evaluated at the horizon due to the initial conditions, therefore we can rewrite it as:

$$\delta S_{bdy} = -\lim_{\Lambda\to\infty} \int d^d x\, \sqrt{-h} n^\mu \left[h^{\nu\sigma} F_{\mu\sigma}\delta A_\nu + \delta\phi(\partial_\mu \phi^* + iqA_\mu \phi^*) + \delta\phi^*(\partial_\mu\phi - iqA_\mu\phi)\right]\Big|_{\Lambda} \ . \tag{B.46}$$

Recalling that the in our gauge the only non zero component of A_μ is $A_t(r)$, from

Since we have divergences in both fields, we need to add two boundary counterterms to regularize the action:

$$S_{c.t.\,\phi} = \int d^d x\,\sqrt{-h}\Delta_- \phi\phi^* \Big|_\Lambda \tag{B.47}$$

$$S_{c.t.\,A} = -\frac{\log(\Lambda)}{4} \int d^d x\,\sqrt{-h} F_{\mu\nu} F^{\mu\nu} \Big|_\Lambda \,. \tag{B.48}$$

Variation of this counterterms give:

$$\delta S_{c.t.\,\phi} = \int d^d x\,\sqrt{-h}\Delta_- (\delta\phi\phi^* + \phi\delta\phi^*) \Big|_\Lambda \tag{B.49}$$

$$\delta S_{c.t.\,A} = \log(\Lambda) \int d^d x \sqrt{-h}\delta A_\nu \nabla_\mu F^{\mu\nu} \Big|_\Lambda \,, \tag{B.50}$$

where we used partial integration in the second integral.

B.2.2 Second order variation and two point functions

We now consider the terms that are second order in fluctuations in (B.37). After partial integration we obtain

$$\begin{aligned}\delta S^{(2)}_{bdy} = -\frac{1}{2} \int d^{d+1} x\,\sqrt{-g} \nabla^\mu \Big(& iq\phi^* \delta A_\mu \delta\phi - iq\phi\delta A_\mu \delta\phi^* + (\partial_\mu \phi^* + iq\delta\phi A_\mu)\,\delta\phi + (\partial_\mu\phi - iq\delta\phi A_\mu)\,\delta\phi^* + \\ & - \nabla_\nu \delta A_\mu \delta A^\nu + \nabla_\mu \delta A_\nu \delta A^\nu \Big)\end{aligned} \tag{B.51}$$

and the bulk term

$$\begin{aligned}\delta S^{(2)}_{bulk} = -\frac{1}{2} \int d^{d+1} x\,\sqrt{-g} \Big[& -\delta\phi^* \left(D_\mu D^\mu - m^2\right)\delta\phi - \delta\phi\left(D_\mu D^\mu - m^2\right)\delta\phi^* \\ & + \delta A_\mu \left(g^{\mu\nu}\left(-\Box + 2q^2\phi^2\right) + \nabla^\mu\nabla^\nu\right)\delta A_\nu \\ & + \delta\phi\left(-iq\phi^*\nabla^\nu - 2iq\partial^\nu\phi^* + 2q^2 A^\nu\phi^*\right)\delta A_\nu + \delta A_\mu\left(iq\phi^*\partial^\mu + 2q^2 A^\mu\phi^* - iq\partial^\mu\phi^*\right)\delta\phi \\ & + \delta\phi^*\left(iq\phi^*\nabla^\nu + 2iq\partial^\nu\phi + 2q^2 A^\nu\phi\right)\delta A_\nu + \delta A_\mu\left(-iq\phi\partial^\mu + 2q^2 A^\mu\phi + iq\partial^\mu\phi\right)\delta\phi^* \Big] \,,\end{aligned} \tag{B.52}$$

that can be conveniently rewritten in matrix form (notice the change of sign in the prefactor)

$$\delta S^{(2)}_{bulk} = \frac{1}{2} \int d^{d+1} x\, \delta\boldsymbol{\Phi} \boldsymbol{G}_B^{-1} \delta\boldsymbol{\Phi} \,, \tag{B.53}$$

with

$$\delta\boldsymbol{\Phi} = \begin{pmatrix} \delta\phi \\ \delta\phi^* \\ \delta A_\nu \end{pmatrix} \qquad\qquad \delta\boldsymbol{\Phi}^\dagger = \begin{pmatrix} \delta\phi^* & \delta\phi & \delta A_\mu \end{pmatrix} \,, \tag{B.54}$$

and

$$G_B^{-1} = \begin{pmatrix} D_\mu D^\mu - m^2 & 0 & -iq\phi\nabla^\nu - 2iq\partial^\nu\phi - 2q^2 A^\nu\phi \\ 0 & D^*_\mu D^{\mu *} - m^2 & iq\phi^*\nabla^\nu + 2iq\partial^\nu\phi^* - 2q^2 A^\nu\phi^* \\ iq\partial^\mu\phi^* - 2q^2 A^\mu\phi^* - iq\phi^*\partial^\mu & -iq\partial^\mu\phi - 2q^2 A^\mu\phi + iq\phi\partial^\mu & g^{\mu\nu}\left(\nabla_\sigma\nabla^\sigma - 2q^2\phi^2\right) - \nabla^\nu\nabla^\mu \end{pmatrix} \,. \tag{B.55}$$

The operator $\boldsymbol{G}_B^{-1}$ defines the *linearized equation of motion* in the bulk as

$$\boldsymbol{G}_B^{-1}\delta\boldsymbol{\Phi} = 0 \,. \tag{B.56}$$

Since $\boldsymbol{G}_B^{-1}$ is not diagonal, we see that the linearized equation of motion are a set of coupled differential equation.

Writing this system of linearized equations of motion explicitly in terms of the background fields we obtain

$$\begin{aligned}0 = f\delta\phi'' &+ \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\phi' + \left(\frac{e^{\chi}\omega^2}{f} - \frac{k^2}{r^2} + \frac{2qe^{\chi}\omega A_t}{f} + \frac{q^2e^{\chi}A_t^2}{f} - m^2\right)\delta\phi \\ &+ \frac{qk\phi}{r^2}\delta A_x + \left(\frac{qe^{\chi}\omega\phi}{f} + \frac{2q^2e^{\chi}\phi A_t}{f}\right)\delta A_t\end{aligned} \tag{B.57}$$

$$\begin{aligned}0 = f\delta\phi^{*\prime\prime} &+ \left(f' + \frac{(d-1)f}{r} - \frac{f\chi'}{2}\right)\delta\phi^{*\prime} + \left(\frac{e^{\chi}\omega^2}{f} - \frac{k^2}{r^2} - \frac{2qe^{\chi}\omega A_t}{f} + \frac{q^2e^{\chi}A_t^2}{f} - m^2\right)\delta\phi^* \\ &- \frac{qk\phi}{r^2}\delta A_x - \left(\frac{qe^{\chi}\omega\phi}{f} - \frac{2q^2e^{\chi}\phi A_t}{f}\right)\delta A_t\end{aligned} \tag{B.58}$$

$$0 = \delta A_x'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_x' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{2q^2\phi^2}{f}\right)\delta A_x + \frac{e^{\chi}k\omega}{f^2}\delta A_t + \frac{qk\phi}{f}\left(\delta\phi - \delta\phi^*\right) \tag{B.59}$$

$$0 = \delta A_t'' + \left(\frac{\chi'}{2} + \frac{d-1}{r}\right)\delta A_t' - \left(\frac{k^2}{r^2f} + \frac{2q^2\phi^2}{f}\right)\delta A_t - \frac{k\omega}{r^2f}\delta A_x - \frac{q\omega\phi}{f}\left(\delta\phi - \delta\phi^*\right) - \frac{2q^2\phi A_t}{f}\left(\delta\phi + \delta\phi^*\right) \tag{B.60}$$

$$0 = \frac{e^{\chi}\omega}{f}\delta A_t' + \frac{k}{r^2}\delta A_x' + q\phi\left(\delta\phi' - (\delta\phi^*)'\right) + q\phi'\left(\delta\phi^* - \delta\phi\right) \tag{B.61}$$

$$0 = \delta A_i'' + \left(\frac{f'}{f} - \frac{\chi'}{2} + \frac{d-3}{r}\right)\delta A_i' + \left(\frac{e^{\chi}\omega^2}{f^2} - \frac{k^2}{r^2f} - \frac{2q^2\phi^2}{f}\right)\delta A_i\ , \tag{B.62}$$

For the boundary term we obtain:

$$\delta S^{(2)}_{bdy} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int d^dx\sqrt{-h}n^{\mu}\left[h^{\nu\sigma}\delta F_{\mu\sigma}\delta A_{\nu} + \Big(\delta\phi(\partial_{\mu}\delta\phi^* + iq\phi^*\delta A_{\mu}) + h.c\Big)\right]\Bigg|_{\Lambda}\ . \tag{B.63}$$

Using the expression for the metric and the gauge choice the boundary action reduce to:

$$\delta S^{(2)}_{bdy} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int d^dxr^{d+1}\left[r^{-2}\eta^{\nu\sigma}\delta A_{\sigma}'\delta A_{\nu} + \delta\phi(\delta\phi^*)' + \delta\phi^*\delta\phi'\right]\Bigg|_{r=\Lambda}\ . \tag{B.64}$$

From now on we focus on the special case $d = 3+1$. The reason is that the boundary action (B.64) is divergent and need to be regularized, however, contrary to the divergence in the scalar field, where the form of the counterterms can be easily generalized to any number of dimensions, the gauge field present a logarithmic divergence only for a even number of dimensions and the form of the counterterms therefore depends on the dimensions of the boundary theory considered. For a $3+1$-dimensional boundary theory, the counterterms we need to add to the boundary action are:

$$\begin{aligned}\delta S^{(2)}_{c.t.\,\phi} &= -\frac{1}{2}\int d^4x\sqrt{-h}2\Delta_-\delta\phi\delta\phi^*\Bigg|_{r=\Lambda} \\ \delta S^{(2)}_{c.t.\,A} &= \frac{1}{2}\int d^4x\log(r)\sqrt{-h}\delta A_{\nu}\nabla_{\mu}\delta F^{\mu\nu}\Bigg|_{r=\Lambda}\ .\end{aligned} \tag{B.65}$$

Fourier transforming the action we obtain

$$\begin{aligned}\delta S^{(2)} = \frac{1}{2}\int d^4x\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}r^5\big[&r^{-2}\eta^{\nu\sigma}\delta A_{\sigma}'(r,k^{\mu})\delta A_{\nu}(r,k^{\mu\prime}) + \delta\phi(r,k^{\mu})(\delta\phi^*(r,k^{\mu\prime}))' \\ &+ \delta\phi^*(r,k^{\mu})\delta\phi'(r,k^{\mu\prime})\big]e^{-i(k+k')_{\mu}x^{\mu}}\Bigg|_{r=\Lambda}\ ,\end{aligned} \tag{B.66}$$

where, due to rotational invariance we can set the four momentum k^{μ} along the x direction, i.e. $k^{\mu} = (\omega, k, 0, 0)$. Integrating over x gives a delta function $(2\pi)^4\delta(k^{\mu} + k^{\mu\prime})$, and integrating this out we get:

$$\delta S^{(2)} = \lim_{\Lambda\to\infty}\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}r^5\left[r^{-2}\eta^{\nu\sigma}\delta A_{\sigma}'(r,k^{\mu})\delta A_{\nu}(r,-k^{\mu}) + \delta\phi(r,k^{\mu})(\delta\phi^*(r,-k^{\mu}))' + \delta\phi^*(r,k^{\mu})\delta\phi'(r,-k^{\mu})\right]\Bigg|_{r=\Lambda} \tag{B.67}$$

In four dimensions the asymptotic behavior of the gauge field is:

$$\delta A_\mu(r,\omega,k) = \delta a_\mu(\omega,k) + \delta b_\mu(\omega,k) r^{-2} + \delta c_\mu(\omega,k) r^{-2}\log\left(\frac{r}{r_h}\right) \ , \tag{B.68}$$

therefore deriving with respect to r we obtain

$$\begin{aligned}\delta A'_\mu(r,\omega,k) =& -2\delta b_\mu(\omega,k) r^{-3} - 2\delta c_\mu(\omega,k) r^{-3}\log\left(\frac{r}{r_h}\right) + \delta c_\mu r^{-3} &\text{(B.69)}\\ \delta A''_\mu(r,\omega,k) =& 6\delta b_\mu(\omega,k) r^{-4} + 6\delta c_\mu(\omega,k) r^{-4}\log\left(\frac{r}{r_h}\right) - 5\delta c_\mu r^{-4} \ . &\text{(B.70)}\end{aligned}$$

The three momentum dependent coefficients, are not all independents, inserting this boundary expansion into the equation of motions and matching coefficients of the same order gives:

$$\begin{aligned}\delta c_t =& -\frac{k}{2}(\omega\delta a_x + k\delta a_t) &\text{(B.71)}\\ \delta c_t =& \frac{\omega}{2}(\omega\delta a_x + k\delta a_t) \ . &\text{(B.72)}\end{aligned}$$

For the boundary expansion of the scalar field we obtain

$$\delta\phi_s r^{-\Delta_-} + \delta\phi_v r^{-\Delta_+} + \cdots \tag{B.73}$$

Substituting this expansions in the boundary action

$$\begin{aligned}\delta S^{(2)} = -\lim_{\Lambda\to\infty}\frac{1}{2}\int\frac{d^4k}{(2\pi)^4} r^5\Bigg[& r^{-2}\eta^{\nu\sigma}\left(-2\delta b_\sigma r^{-3} - 2\delta c_\sigma r^{-3}\log\left(\frac{r}{r_h}\right) + \delta c_\sigma r^{-3}\right)\left(\delta a_\nu + \delta b_\nu r^{-2} + \delta c_\nu r^{-2}\log\left(\frac{r}{r_h}\right)\right)\\ &+ \left(\delta\phi_s r^{-\Delta_-} + \delta\phi_v r^{-\Delta_+}\right)\left(-\Delta_-\delta\phi_s^* r^{-\Delta_- -1} - \Delta_+\delta\phi_v^* r^{-\Delta_+ -1}\right)\\ &+ \left(\delta\phi_s^* r^{-\Delta_-} + \delta\phi_v^* r^{-\Delta_+}\right)\left(-\Delta_-\delta\phi_s r^{-\Delta_- -1} - \Delta_+\delta\phi_v r^{-\Delta_+ -1}\right) + \mathcal{O}(r^{-4}\log r)\Bigg]\Bigg|_{r=\Lambda} \ ,\end{aligned} \tag{B.74}$$

that gives, after performing the products:

$$\begin{aligned}\delta S^{(2)} = \lim_{\Lambda\to\infty}\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\Bigg[& \eta^{\mu\sigma}\left(2\delta b_\sigma\delta a_\mu + 2\delta c_\sigma\delta a_\mu\log\left(\frac{r}{r_h}\right) + \delta c_\sigma\delta a_\mu + \mathcal{O}(r^{-2}\log^2(r))\right)\\ &+ \left(\Delta_-\delta\phi_s\delta\phi_s^* r^{2\nu} + \Delta_+\delta\phi_s\phi_v^* + \Delta_-\delta\phi_v\phi_s^* + \mathcal{O}(r^{-2\nu})\right)\\ &+ \left(\Delta_-\delta\phi_s^*\delta\phi_s r^{2\nu} + \Delta_+\delta\phi_s^*\phi_v + \Delta_-\delta\phi_v^*\phi_s + \mathcal{O}(r^{-2\nu})\right)\Bigg]\Bigg|_{r=\Lambda}\end{aligned} \tag{B.75}$$

and we can now clearly see that there are divergent terms in the action, both for the scalar field components and the gauge field. We therefore need the counterterms (B.65), that in k-space and in the boundary limit become:

$$\begin{aligned}\delta S^{(2)}_{c.t.\,\phi} =& -\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left(2\Delta_-\delta\phi_s\delta\phi_s^* r^{2\nu} + 2\Delta_-\delta\phi_s\delta\phi_v^* + 2\Delta_-\delta\phi_s^*\delta\phi_v + \mathcal{O}(r^{-2\nu})\right)\Big|_{r=\Lambda} &\text{(B.76)}\\ \delta S^{(2)}_{c.t.\,A} =& \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(\frac{r}{r_h}\right)\eta^{\mu\rho}\eta^{\nu\sigma}\delta a_\nu\left(k_\mu k_\rho\delta a_\sigma - k_\mu k_\sigma\delta a_\rho\right) + \mathcal{O}(r^{-2}\log^2(r))\Big|_{r=\Lambda} \ , &\text{(B.77)}\end{aligned}$$

given the explicit form of c_μ (B.71) and (B.72) the second counterterm can be written as

$$\begin{aligned}\delta S^{(2)}_{c.t.\,A} =& \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left[\log\left(\frac{r}{r_h}\right)\left(-k(k\delta a_t + \omega\delta a_x)\delta a_t - \omega(k\delta a_t + \omega\delta a_x)\delta a_x\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda}\\ =& \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left[\log\left(\frac{r}{r_h}\right)\left(2\delta c_t\delta a_t - 2\delta c_x\delta a_x\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda}\\ =& -\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\left[\log\left(\frac{r}{r_h}\right)\eta^{\mu\sigma}\left(2\delta c_\sigma\delta a_\mu\right) + \mathcal{O}(r^{-2}\log^2(r))\right]\Big|_{r=\Lambda} \ .\end{aligned} \tag{B.78}$$

Adding the counterterms to the boundary action we can finally safely perform the limit $\Lambda \to \infty$, to obtain

$$S_{bdy}^{(2),R} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Bigg[(-2\delta b_t(\omega,k) + \delta c_t(\omega,k))\, \delta a_t(-\omega,-k) + (2\delta b_x(\omega,k) - \delta c_x(\omega,k))\, \delta a_x(-\omega,-k) \\ + 2\nu\delta\phi_v(\omega,k)\delta\phi_s^*(-\omega,-k) + 2\nu\delta\phi_v(\omega,k)\delta\phi_s^*(-\omega,-k) \Bigg] \Bigg|_{r=\Lambda} , \tag{B.79}$$

from which we can read off the variation in the expectation values of the operators and determine the Green's functions, as explained in chapter 4.

B.3 Near horizon boundary condition

Here we show how to derive the initial conditions at the horizon using the example of intrinsic dynamics of the scalar field, however the calculations including the gague field are analogous. The equation of motion for scalar field fluctuations is given by

$$\delta\phi'' + \left(\frac{f'(r)}{f(r)} + \frac{d-1}{r} - \frac{\chi'(r)}{2} \right) \delta\phi' - \frac{m^2 - (qA_t + \omega)^2 \frac{e^{\chi(r)}}{f(r)} + \frac{k^2}{r^2}}{f(r)} \delta\phi = 0 \; . \tag{B.80}$$

Expanding near the horizon (here we neglect gauge fluctuations because we are considering the uncoupled problem), with $f(r) = f'(r_h)(r - r_h) + \mathcal{O}((r - r_h)^2)$ for $r \to r_h$ and $A_t(r_h) = 0$ we have:

$$\delta\phi'' + \left(\frac{1}{r - r_h} + \frac{d-1}{r_h} - \frac{\chi'(r_h)}{2} \right) \delta\phi' - \frac{m^2 - \omega^2 \frac{e^{\chi(r_h)}}{f'(r_h)(r-r_h)} + \frac{k^2}{r_h^2}}{f'(r_h)(r - r_h)} \delta\phi = 0 \quad \text{for} \quad r \to r_h \; . \tag{B.81}$$

Using the ansatz $\delta\phi = (r - r_h)^\alpha$ we obtain:

$$(r - r_h)^{\alpha - 2} \left(\left[\alpha^2 + \omega^2 \frac{e^{\chi(r_h)}}{f'^2(r_h)} \right] + (r - r_h) \left[\frac{d-1}{r_h} - \frac{\chi'(r_h)}{2} - \frac{m^2}{f'(r_h) - \frac{k^2}{f'(r_h) r_h}} \right] \right) , \tag{B.82}$$

neglecting higher order terms, we then find

$$\alpha^2 = -\omega^2 \frac{e^{\chi(r_h)}}{f'^2(r_h)} \tag{B.83}$$

that, using the formula for the black hole temperature (3.10) gives

$$\alpha = \pm \frac{i\omega}{4\pi T} \tag{B.84}$$

and then near the black hole horizon

$$\delta\phi(r) \sim (r - r_h)^{\pm \frac{i\omega}{4\pi T}} \; . \tag{B.85}$$

The two solutions corresponds to waves *infalling* into black hole or *outgoing* from the black hole horizon. We want to choose the physical solution of infalling waves, computing correlation functions with this boundary conditions gives the retarded Green's functions. In order to see which sign corresponds to infalling boundary conditions, we have to restore the time dependence in (B.85):

$$e^{-i\omega t}(r - r_h)^{\pm \frac{i\omega}{4\pi T}} \tag{B.86}$$

if we now make the change of variable $(r - r_h) \to \bar{r} = \log r - r_h/(4\pi T)$ than (B.85) behaves as

$$\delta\phi(\bar{r}) \sim e^{-i\omega(t \mp \bar{r})} \tag{B.87}$$

and we see that the solution with the minus sign gives a wave $e^{-i\omega(t+\bar{r})}$ moving towards the horizon, therefore we set the boundary condition at the horizon to be:

$$\delta\phi(r) = (r - r_h)^{-\frac{i\omega}{4\pi T}} \quad \text{for} \quad r \to r_h \; . \tag{B.88}$$

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