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Master's Thesis

Vortices in p-wave superconductors

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June, 2017

Abstract

In this thesis, we study and compare two approaches to describe vortex lattices for a number of physical systems. The first approach consists of imposing an extra constraint equation on the order parameter, that leads to a Liouville-like partial differential equation for the particle density. The second approach is a new generalisation of a method originally developed by Abrikosov to the case of a certain p-wave superconductors.

With the first approach, we find an infinite number of energetically degenerate solutions. The second approach leads - under suitable conditions - to a phase transition between different vortex lattices.

Contents

1 Introduction

In this Master's thesis, we study vortex lattices in a number of condensed matter systems, primarily in superconductors. The reader is likely to be familiar with vortices from his or her everyday life. They occur as a result of the conservation of angular momentum when draining a bathtub or as tornadoes in our atmosphere.

When quantum mechanics became better understood throughout the Twentieth Century, a number of new physical systems became known that exhibited vortex-like solutions. Among the first of these systems was liquid helium, where Bogolyubov in 1947 [1] and later Feynman [2] predicted their existence. In 1957 Abrikosov [3] predicted the formation of vortices in a certain type of superconductors, a result for which he won the Nobel Prize and that will later be studied in this thesis as well. Another four years later Gross [4] and Piteavskii [5] first described vortices in a Bose-Einstein condensate, a gas of atoms that is cooled till a point very near the absolute zero. Even nowadays, there is a lot of interest in the study of vortices in Bose-Einstein condensates [6].

These quantum systems are described by a complex-valued order parameter ψ whose absolute value in a point $(x, y) \in \mathbb{R}^2$ describes a density. In the case of superfluid helium this density corresponds to that of the super-fluid atoms, while in the case of superconductors it corresponds to that of the Cooper pairs of superconducting electrons and in the case of the Bose-Einstein condensates it corresponds to that of ultracold atoms. One may write this order parameter as

$$\psi = \sqrt{\rho}e^{i\phi}, \tag{1.1}$$

where $\rho : \mathbb{R}^2 \to \mathbb{R}$ is the relevant density and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is the phase of the order parameter.

An important property of a vortex is its winding number $n \in \mathbb{Z}$, that counts how often the order parameter rotates around the central point of the vortex. Because this winding number leads to topological protection, systems with a non-zero winding number tend to be stable.

We study two methods to describe vortex solutions in a number of two-dimensional physical systems in order to gain a better understanding of the relation between them.

The first method was developed by Olesen [9] who originally applied it to find vortex solutions to the so-called Chern-Simons theories in the standard model. It consists of imposing an extra constraint equation on the order parameter and look for zeros of the free energy that satisfy this equation. Let A_i be the magnetic vector potential and $D_i = \partial_i - 2ieA_i$ the covariant derivative. If we set $D_\pm = \frac{1}{2}\sqrt{2}(D_1 + iD_2)$, then the constraint equation is:

$$D_\pm\psi = 0. \tag{1.2}$$

This constraint does not have a clear-cut physical interpretation. Rather, it provides one with a convenient method to factorise the equations of motion and obtain gauge-invariant

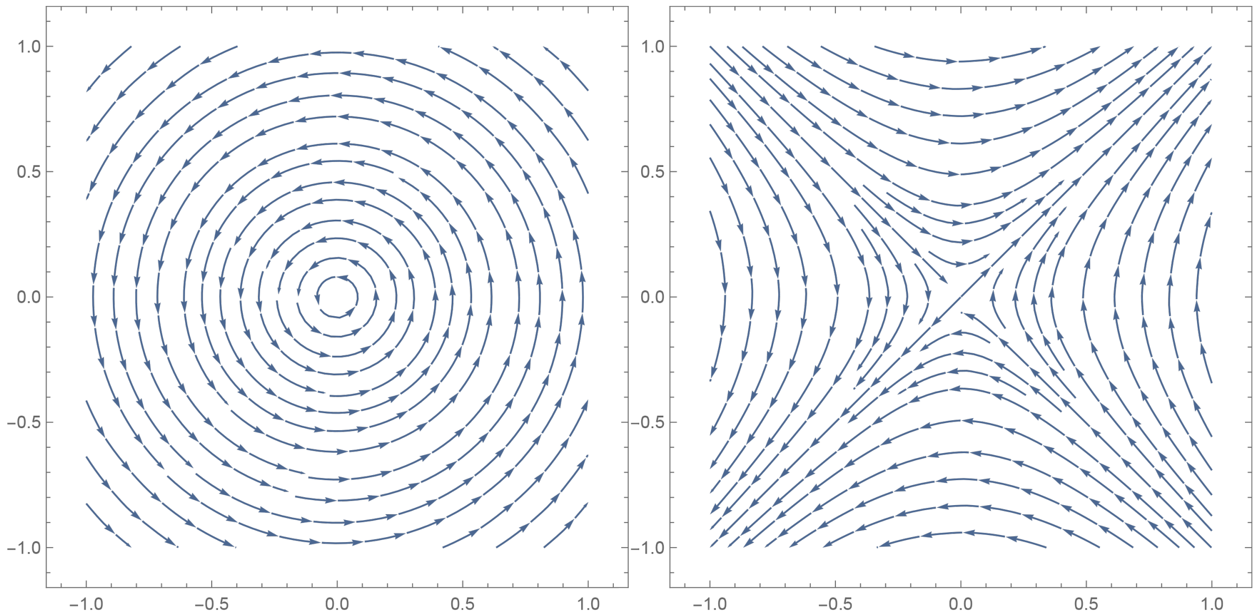

Figure 1: Left: a vortex with winding number $n = 1$; right: an anti-vortex with winding number $n = -1$.

equations. It is possible to rewrite this constraint and obtain one of the form

$$\Delta \log(\sqrt{\rho}) \pm q h_3 = 0, \tag{1.3}$$

where q is the charge of the particles in the system and h_3 the external magnetic field. Then, if h_3 is not fixed, we use the equations of motion to eliminate it and obtain

$$\Delta \log(\sqrt{\rho}) \pm \rho = a, \tag{1.4}$$

where a is a constant that can be expressed in terms of the physical parameters. When $a = 0$, this equation is known as the Liouville equation. A curious feature of this theory lies in the fact that any ρ that we find by this method is a zero of the free energy, therefore all solutions to this equations are energetically degenerate. In particular, there is an infinite degeneracy in the lattice structure according to which the vortices are ordered. Because finding zeros of the free energy is not necessarily the same as solving the equations of motion, some criteria have to be found so that these zeros also correspond to solutions to the equations of motion. These criteria come in the form of restrictions on the physical parameters of the model.

The second method is the one that was originally developed by Abrikosov [3] to describe a superconductor inside a strong magnetic field. In this thesis, we present new work that extends this method to incorporate ferromagnetic p-wave superconductors. It this method, one first neglects the interactions in the model to obtain an approximate periodic solution to the equations of motion. This solution can be written in dimensionless coordinates as

$$\psi = \sum_n C_n H_r \left(\sqrt{K} \left[x - \frac{kn}{K^2} \right] \right) \exp \left(ikny - \frac{K^2}{2} \left[x - \frac{kn}{K^2} \right]^2 \right), \tag{1.5}$$

where K is a constant that can be expressed in terms of the physical parameters of the model and H_r is the r^{th} Hermite polynomial. The C_n must be taken periodically, so that one obtains a doubly periodic order parameter. Then, the solution that allows for the strongest magnetic field is taken and the energy is minimised as a function of K, in order to find the energetically optimal lattice.

1.1 Structure of this thesis

In the first chapter after this introduction, we will review the Ginzburg-Landau model for superconductivity. This is a phenomenological model, based on the model for liquid helium. We will show that there exist two basic types of superconductors: the first type are materials that are either completely superconducting or completely normal when placed in a magnetic field and the second type are materials that exhibit a lattice structures of vortices in which the magnetic field penetrates and the material is in its normal phase, while it is superconducting outside those vortices. We will also give a criterion that determines the type of superconductivity. Finally, we will introduce a model for ferromagnetic superconductors where the order parameter is a three-component wave function, corresponding to a Cooper pair in one of the triplet states.

In the second chapter, we will describe the first method to derive a lattice of vortex solutions to both a regular superconductor and a ferromagnetic p-wave superconductor. This method is valid when certain constraints are placed on the physical parameters of the model. For a regular superconductor, these constraints correspond to the transition point between the two types of regular superconductors described above. We will also show that this leads equation (1.4). When $c = 0$ this equation reduces to the Liouville equation, that is solved by

$$\rho(x, y) = \frac{|f'(x + iy)|^2}{(1 \pm |f(x + iy)|^2)^2}, \tag{1.6}$$

where f is *any* analytic function. We will study this equation in the fourth chapter and determine the set of analytic functions that yield both doubly-periodic and non-periodic vortex solutions. Then, we will proceed with perturbation theory, to determine approximate solutions to the Liouville-like equation when $c \neq 0$. We will conjecture a radius of convergence for this perturbative solution, but are forced to leave the proof as an open problem.

Before this, however, we will briefly study the same approach to find vortex solutions to the Ginzburg-Landau model in the context of rotating Bose-Einstein condensates. The main difference between both systems is the gauge field A that describes the magnetic field in a superconductor is now fixed by the rotation speed of the condensate.

Finally, we will study Abrikosov's approximate solution to the Ginzburg-Landau model for superconductors of the second type and then apply it to ferromagnetic superconductors. As opposed to regular superconductors, the so-called Lowest Landau Level solution is then no longer expected to produce the best approximation and as a result, other solutions have to be taken into account as well. When specialising to the second-lowest level, this will lead to a phase transition between a rectangular lattice and a triangular lattice.

1.2 A brief note on units

Throughout this thesis, we will use so-called natural units. In other words, we will set

$$c = \hbar = k_B = 1, \tag{1.7}$$

where c is the speed of light, $\hbar$ is the reduced Planck's constant and k_B is Bolzmann's constant. Even though the author is painfully familiar with the task of converting a physical quantity given in terms of these units to one that is given in terms of those units that experimentalists actually use to perform measurements on nature, he still estimates the total cost of these hypothetical conversions to be less than the effort it would take to actually write down these constants every time they would appear in the equations to follow.

2 Physical systems

In this chapter, we review important elements in the theories of superconductors and Bose-Einstein condensates. Both systems will be of interest in later chapters, when we apply our methods to describe vortex lattices.

2.1 Conventional superconductivity

We will start by reviewing the Ginzburg-Landau model for superconductivity. This treatment will be based on [7] and [8].

Some materials exhibit a phase transition, where the electrical resistance of the material drops to zero. This takes place when electrons form so-called Cooper pairs: when the electrons form pairs, a gap appears in their energy spectrum. This removes the possibility for the electrons to scatter, because that would mean a small fluctuation in their energy state. Because electron scattering is a primary cause of electrical resistance, the formation of these Cooper-pairs means an increase in the conductivity.

From experimental observations, we know that the phase transition to superconductivity is of second order. Making an educated guess, Ginzburg and Landau expected the free energy density to contain terms of the form

$$\alpha|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{2.1}$$

where α and g are numerical parameters and ψ is a complex-valued order parameter describing the density of electron pairs in a superconducting state. From this expression, we can already say something about α and g. When g is negative, we see that it becomes energetically favourable for the supercurrent to become arbitrarily large in magnitude. Because this scenario does not generally strike one as overly likely, we take $g > 0$. Then, we see that 2.1 attains a minimum at a point $|\psi|^2 \neq 0$ if and only if $\alpha < 0$. Experimentally, we know that this point is exists below some critical temperature T_c, so we assume that $\alpha \approx a(T - T_c)$ for temperatures T near T_c.

Additionally, Ginzburg and Landau added a term proportional to $|\partial_i \psi|^2$, so that large spatial variations in ψ become energetically unfavourable. Analogously to the Schrödinger equation, this term is written as

$$\frac{1}{2m}|\partial_i \psi|^2, \tag{2.2}$$

where m is the effective mass of a Cooper pair. When magnetic fields are taken into account, this term becomes

$$\frac{1}{2m}|D_i \psi|^2, \tag{2.3}$$

where $D_i = \partial_i - 2ieA_i$ is the covariant derivative and A is the magnetic potential satisfying $h_i = \epsilon_{ijk}\partial_j A_k$, where h is the microscopic magnetic field. We note the usage of '$2e$', rather than 'e', because we are describing pairs, rather than single electrons. The final form of the Ginzburg-Landau free energy then becomes

$$V \cdot F = \int_V d^3x \Big\{ \frac{1}{2m} |D_i \psi|^2 + \alpha |\psi|^2 + \frac{g}{2} |\psi|^4 + \frac{1}{8\pi} h_i^2 \Big\}, \tag{2.4}$$

where V is the volume of the superconductor, which we will set to 1 in the rest of this section. The last term in the integral takes the energy of the magnetic field into account.

Performing a variational calculation, we derive the Ginzburg-Landau equations of motion

$$\frac{-1}{2m} D_i^2 \psi + \alpha \psi + g |\psi|^2 \psi = 0 \tag{2.5}$$

$$\frac{1}{4\pi} \epsilon_{ijk} \partial_j h_k = \frac{ie}{m} \Big(\psi (D_i \psi)^* - \psi^* (D_i \psi) \Big). \tag{2.6}$$

We can give solutions to these equations in some limiting cases. In the absence of a magnetic field, we have $A_i = 0$ for $i = 1, 2, 3$. We see then that 2.5 is solved by

$$\psi = 0 \qquad\qquad |\psi|^2 = \frac{-\alpha}{g}. \tag{2.7}$$

The first solutions corresponds to a normal, i.e. non-superconducting state, whereas the second solution exists only if $\alpha \leq 0$ and corresponds in that case also to a lower free energy than the first one.

If we take a new order parameter f so that

$$f = \sqrt{\frac{g}{|\alpha|}} \psi, \tag{2.8}$$

we may rewrite 2.5 in the form

$$\frac{1}{2m\alpha} \partial_i^2 f - f + |f|^2 f = 0. \tag{2.9}$$

The factor in front of the Laplacian defines a natural length scale ξ on which f varies:

$$\xi^2 := \frac{1}{2m|\alpha|} \approx \frac{1}{2ma|T_c - T|}. \tag{2.10}$$

This number is known as the Ginzburg-Landau coherence length.

We may also consider the case where the electron density is approximately constant, that is

$$|\psi|^2 = \rho := \frac{|\alpha|}{g}. \tag{2.11}$$

Then, equation 2.6 reduces to

$$\frac{1}{4\pi}\epsilon_{ijk}\partial_j h_k = \frac{-4e^2\rho}{m}A_i, \tag{2.12}$$

which is known as the London equation and we may extract a second characteristic length scale from it. We consider an infinite superconductor in the region

$$\{(x, y, z) \in \mathbb{R}^3 : x > 0\} \tag{2.13}$$

while we apply a constant magnetic field in the z-direction. Then, because the system remains invariant under translations in the y- and z-direction, A must be independent of y and z. Furthermore, because $h_1 = h_2 = 0$, this observation implies that $A_3 =$ constant and because the magnetic field depends only on derivatives of the A_i, we might as well set $A_3 = 0$. Under these conditions, we can verify that 2.12 is solved by

$$A_1 = A_3 = 0 \qquad\qquad A_2 = C_1 e^{-x/\lambda} + C_2 e^{x/\lambda}, \tag{2.14}$$

where

$$\lambda^2 := \frac{m}{16\pi e^2\rho} \tag{2.15}$$

is known as the London penetration length. Because the magnetic field must be bounded, we set $C_2 = 0$ and let C_1 be determined by the value of the magnetic field outside the superconductor. We conclude that magnetic fields exponentially decrease inside a superconductor, as long as the material is in a superconducting state. This effect is known as the Meissner effect.

A remark about the last condition is in place here. From equation 2.4, it becomes clear that in the presence of magnetic fields, the superconducting state may become energetically unfavourable: when the magnitude of A increases, the term $|D_i\psi|^2$ in the free energy also increases when $\psi \neq 0$. At some critical field h_{crit}, this contribution will cancel out the negative contribution from $\alpha|\psi|^2 + \frac{g}{2}|\psi|^4$ and the material will cease to be superconducting.

From thermodynamic considerations, it is possible to determine the critical field. Suppose we have a superconductor inside an external magnetic field h in the z-direction. We want to vary the magnetic field strength, which means that the relevant free energy is the Gibbs free energy, given by

$$G(T, H) = F - \frac{1}{4\pi}\int d^3x bh, \tag{2.16}$$

where b is the magnetic induction (the combination of the external magnetic field and the magnetisation of the superconductor). Then, a variation in G is given by

$$dG = -SdT - \frac{1}{4\pi}\int d^3x bdh. \tag{2.17}$$

If we increase h from 0 to some spatially constant value $\tilde{h}$ at constant T, the corresponding change in the Gibbs free energy is given by

$$G(T, \tilde{h}) - G(T, 0) = -\frac{1}{4\pi} \int d^3x \int_0^{\tilde{h}} b(h) dh. \tag{2.18}$$

In most materials in the normal, non-superconducting state $b \approx h$ and therefore the change in the normal Gibbs free energy G_n is given by

$$G_n(T, \tilde{h}) - G_n(T, 0) = -\frac{1}{8\pi} \int d^3x \tilde{h}^2 = -\frac{h_{\text{crit}}^2}{8\pi}, \tag{2.19}$$

where we used that the integral over the total volume is one. On the other hand, inside a superconducting material b decays exponentially, so

$$G_s(T, \tilde{h}) \approx G_s(T, 0). \tag{2.20}$$

At the critical value h_{crit} of h, both phases must be in equilibrium, which means that

$$G_s(T, h_{\text{crit}}) = G_n(T, h_{\text{crit}}). \tag{2.21}$$

Combining the above with equations (2.16), (2.19) and (2.20), we find that

$$G_s(T, 0) = G_n(T, 0) - \frac{h_{\text{crit}}^2}{8\pi} \tag{2.22}$$

$$F_s(T, 0) = F_n(T, 0) - \frac{h_{\text{crit}}^2}{8\pi}. \tag{2.23}$$

Comparing our expression for F with 2.4 and 2.7 (recall that we set $V = 1$), we find that

$$\frac{1}{8\pi} h_{\text{crit}}^2 = \frac{\alpha^2}{2g}. \tag{2.24}$$

As it turns out we can distinguish between two types of superconductors: one in which ψ is more or less constant and all of the magnetic flux is expelled from the material and one in which some regions of the material allow magnetic flux to pass through, while others exhibit a Meissner effect. Which type of superconductivity is favoured depends on the Ginzburg-Landau parameter

$$\kappa := \frac{\lambda}{\xi} = \sqrt{\frac{2m^2|\alpha|}{16\pi e^2 \rho}} = \frac{m}{2e}\sqrt{\frac{g}{2\pi}} \tag{2.25}$$

We note that even though λ and ξ both implicitly depend on the temperature T (through ρ and α), κ does not.

We will proceed to demonstrate the role κ has to play in the existence of two types of superconductors. We consider an infinite superconductor in a uniform external magnetic field h in the z direction (so $A_1 = A_3 = 0$ and $A_2 = h_0 x$ with h_0 the magnetic field strength). When $h_0 \gg h_{\text{crit}}$, the material is in the normal state and $\psi = 0$. As we decrease the field strength, some superconducting regions will begin to form inside the material at some critical value of the magnetic field strength $h_{\text{crit},2}$. Because this process takes place when the superelectron density is still small, we may approximate the Ginzburg-Landau free energy by neglecting the quartic term:

$$F = \int d^3x \Big\{ \frac{1}{2m} |D_i \psi|^2 + \alpha |\psi|^2 + \frac{1}{8\pi} h^2 \Big\}, \tag{2.26}$$

which gives rise to the following equations of motion

$$\frac{-1}{2m} D_i^2 \psi + \alpha \psi = 0 \tag{2.27}$$

$$\frac{1}{4\pi} \epsilon_{ijk} \partial_j h_k = \frac{ie}{m} \Big(\psi (D_i \psi)^* - \psi^* (D_i \psi) \Big). \tag{2.28}$$

Equation 2.27 is identical to the Schrödinger equation for a particle in a constant magnetic field with energy $-\alpha = \frac{1}{2} m v_z^2 + (\frac{1}{2} + n)\omega$, where v_z is the velocity in the z-direction, $\omega = \frac{2e h_{\text{crit},2}}{m}$ is the cyclotron orbit frequency and $n \in \mathbb{N}_0$. The ground state is given by $v_z = n = 0$, where $\alpha = \frac{e h_{\text{crit},2}}{m}$. By using 2.24, we find that

$$h_{\text{crit},2} = \sqrt{2} \kappa h_{\text{crit}}. \tag{2.29}$$

In other words, when $\kappa > \frac{1}{2}\sqrt{2}$, there exists a phase in which the material contains superconducting regions, but is not completely superconducting. Perhaps somewhat surprisingly, this state will sometimes remain energetically preferable to one that is completely superconducting. In principle, this can be shown by calculating the energy σ_{ns} of the surface between the superconducting regions and the normal ones. We state the results of this calculation here:

$$\sigma_{ns} = \begin{cases} \frac{\sqrt{2}}{6\pi} h_{\text{crit}}^2 \xi & \kappa \ll 1 \\ 0 & \kappa = \frac{1}{2}\sqrt{2} \\ -\frac{1}{3\pi}(\sqrt{2} - 1) h_{\text{crit}}^2 \lambda & \kappa \gg 1. \end{cases} \tag{2.30}$$

This indicates that the transition to a negative surface energy occurs when $\kappa = \frac{1}{2}\sqrt{2}$ and that it is positive for smaller values of κ. Together with equation (2.29), this motivates the distinction between two types of superconductors:

- Type I ($h_{\text{crit}} > h_{\text{crit},2}$, $\kappa < \frac{1}{2}\sqrt{2}$): the material becomes completely superconducting before nuclei can form and will remain so for all values of $h < h_{\text{crit}}$.

- Type II ($h_{\text{crit}} < h_{\text{crit},2}$, $\kappa > \frac{1}{2}\sqrt{2}$): before the material becomes completely superconducting, there exists a phase with $h_{\text{crit}} < h < h_{\text{crit},2}$ where some parts of the material are superconducting, while others remain in the normal phase.

2.2 Beyond conventional superconductors

In the model from the previous section, it is implicitly assumed that the paired electrons form a singlet state, in which the wave function can be written as

$$\phi(x_1, x_2; \sigma_1, \sigma_2) = \frac{1}{2}\sqrt{2}\phi(x_1, x_2)\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \tag{2.31}$$

where σ_i denotes the projection of the spin along a chosen axis and $(\uparrow\downarrow - \downarrow\uparrow)$ denotes the spin part of the wave function, that is now taken to be the singlet state. Because the complete wave function should be antisymmetric under the exchange of the two electrons, we must have

$$\frac{1}{2}\sqrt{2}\phi(x_1, x_2)\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) = -\frac{1}{2}\sqrt{2}\phi(x_2, x_1)\left(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right) = \frac{1}{2}\sqrt{2}\phi(x_2, x_1)\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \tag{2.32}$$

and therefore

$$\phi(x_1, x_2) = \phi(x_2, x_1). \tag{2.33}$$

In other words, when we take the spin part of the total wave function to be antisymmetric, we find that the orbital part must be symmetric.

In principle, the electrons could also be in one of the triplet states. This type of superconductors is referred to as a *p-wave superconductor* as opposed to an *s-wave superconductor*, where the spins are in the singlet state. Then

$$\phi(x_1, x_2; \sigma_1, \sigma_2) = \phi_{\uparrow\uparrow}(x_1, x_2)|\uparrow\uparrow\rangle + \phi_{\downarrow\downarrow}(x_1, x_2)|\downarrow\downarrow\rangle + \frac{1}{2}\sqrt{2}\phi_{\uparrow\downarrow}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right). \tag{2.34}$$

Because the spin part of the wave function is now symmetric, we must take the orbital part antisymmetric. Now, we can interpret each of the three triplet states as a basis vector and consider

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{2.35}$$

as a three-component complex vector. One can then generalise the Ginzburg-Landau free energy to one that supports a three-component order parameter. This means that, for example, the term $|D_i\psi|^2$ should be replaced by a term of the form $M_{ijkl}(D_i\psi_j)^*(D_k\psi_l)$. However, not all terms like this respect the symmetries that one would like to have present. For example, the free energy should still be gauge-invariant and it must be real-valued.

Here, we are interested in a system described in [9], where the system is assumed to be a square lattice. Then, the free energy should also respect the symmetries of the lattice. This can be arranged by working out the structure of the order parameter and the way the point-group of the crystal lattice, the gauge group and the time-reversal symmetry operate on it. Then, one can analyse which terms are left invariant by the symmetry group (or the subgroup that is left over after some symmetries are broken) to obtain a phenomenological free energy of the desired form. This becomes quite involved, so we just mention the result [10] here, which is the same as was used in [9]:

$$F = F_{\text{grad}} + F_{\text{mixed}} + F_{\text{pot}} + F_{\text{mag}}, \tag{2.36}$$

where

$$F_{\text{grad}} = \int d^3x \Big\{ K_1 (D_i\psi_j)^*(D_i\psi_j) + K_2[(D_i\psi_j)^*(D_j\psi_i) + (D_i\psi_i)^*(D_j\psi_j)] + K_3|D_i\psi_i|^2 \Big\}, \tag{2.37}$$

$$F_{\text{mixed}} = \int d^3x \Big\{ 4\pi i J M_i \epsilon_{ijk} \psi_j^* \psi_k \Big\}, \tag{2.38}$$

$$F_{\text{pot}} = \int d^3x \Big\{ \alpha|\psi_i|^2 + \beta(\psi_i^*\psi_i)^2 + \beta_2|\psi_i^2|^2 \Big\}, \tag{2.39}$$

$$F_{\text{mag}} = \int d^3x \frac{1}{8\pi} h_i^2. \tag{2.40}$$

In addition to the terms in F_{grad}, F_{pot} and F_{mag}, which we recognise as a generalisation from terms in 2.4, we note the presence of F_{mixed}, which describes a coupling between the superelectrons and the magnetisation M_i of the material.

According to [9], there exists the possibility of an unstable mode of which the magnitude grows in time when $\alpha \geq 0$. This can be thought of as the superelectrons condensing into a single state, that dominates the other states. As we will demonstrate in the next chapter, the superelectron density profile is described by a differential equation that reduces to the Liouville equation when we set $\alpha = 0$.

We consider magnitisation in the z-direction. This instability can appear in the ψ_i when

$$\psi := \psi_1 = -i\psi_2 \qquad \text{and} \qquad \psi_3 = 0. \tag{2.41}$$

Moreover, we assume that the system is uniform in the z-direction, so that there is no real z dependence and the integral in that direction merely gives us a numerical factor, which we will set to 1. Thus, the free energy simplifies to

$$F = \int d^2x \Big\{ 2K_1|D_i\psi|^2 + 2K_2|D_i\psi|^2 + K_3 \left(|D_i\psi|^2 - i(D_2\psi)^*(D_1\psi) + i(D_1\psi)^*(D_2\psi) \right) \\ - 8\pi J M_3|\psi|^2 + 2\alpha|\psi|^2 + 4\beta|\psi|^4 + \frac{1}{8\pi} h_i^2 \Big\}. \tag{2.42}$$

If we define $2C_1 := 2K_1 + 2K_2 + K_3$, $C_2 := K_3$ and assume that the magnetisation changes proportionally to the external field $M_3 = \chi h_3$, then the free energy becomes

$$\begin{aligned} F &= \int d^2x \Big\{ 2C_1|D_i\psi|^2 + iC_2\left((D_1\psi)^*(D_2\psi) - (D_2\psi)^*(D_1\psi)\right) - 8\pi\chi J h_3|\psi|^2 - 2\alpha|\psi|^2 + \\ &\qquad\qquad + 4\beta|\psi|^4 + \frac{1}{8\pi}h_i^2 \Big\} \\ &= \int d^2x \Big\{ -2C_1(D_i^2\psi)\psi^* + iC_2\left(\psi^*(D_2D_1\psi) - \psi^*(D_1D_2\psi)\right) - 8\pi\chi J h_3|\psi|^2 + 2\alpha|\psi|^2 + \\ &\qquad\qquad + 4\beta|\psi|^4 + \frac{1}{8\pi}h_i^2 \Big\} + \text{ boundary terms.} \end{aligned} \tag{2.43}$$

Since $[D_2, D_1] = 2ie(\partial_1 A_2 - \partial_2 A_1) = 2ieh_3$, the above expression further simplifies to

$$F = \int d^2x \Big\{ -2C_1(D_i^2\psi)\psi^* - 2\left(4\pi\chi J + eC_2\right)h_3|\psi|^2 + 2\alpha|\psi|^2 + 4\beta|\psi|^4 + \frac{1}{8\pi}h_i^2 \Big\} + \text{ boundary terms.} \tag{2.44}$$

From this, we can read off the equation of motion for ψ

$$0 = -C_1 D_i^2\psi - \left(4\pi\chi J + eC_2\right)B_3\psi + \alpha\psi + 4\beta|\psi|^2\psi, \tag{2.45}$$

while the equation of motion for A becomes

$$\frac{1}{4\pi}\epsilon_{ijk}\partial_j h_k = -4ieC_1\Big(\psi(D_i\psi)^* - \psi^*(D_i\psi)\Big) - 2(4\pi\chi J + eC_2)d_i^\times|\psi|^2, \tag{2.46}$$

with

$$d^\times = \begin{pmatrix} \partial_2 \\ -\partial_1 \\ 0 \end{pmatrix}. \tag{2.47}$$

2.2.1 Possible materials

In [10], it is mentioned that this model is meant to describe $ZrZn_2$ in the ferromagnetic phase. According to the same article, a similar phenomenon occurs in UGe_2. However, the latter material has a different crystalline structure and is therefore described by a different free energy. Other possible materials include UCoGe [11] and URhGe [12].

2.3 A rotating Bose-Einstein condensate

Another case in which one may try to find vortex solutions by requiring $D_\pm\psi = 0$ for some chiral derivative is a rotating Bose-Einstein condensate, in which a gauge field A is *artificially* introduced - hence the term *artificial gauge fields*, that is used in some of the literature on this subject.

We will give a brief introduction to Bose-Einstein condensates, or more briefly Bose-gases. This introduction will follow the lecture notes [13] by Walraven and the book [14] by Pethick and Smith on the matter.

For a Bose-gas, the average total number of atoms in a state s is given by

$$\overline{n}_s = f_{BE}(\epsilon_s) = \frac{1}{e^{(\epsilon_s-\mu)/T} - 1}, \tag{2.48}$$

where ϵ_s is the energy associated to the state s, μ is the chemical potential (or the energetic cost of adding a particle to the condensate) and T is the temperature. The total number of particles N then has to be given by

$$N = \sum_s \overline{n_s}. \tag{2.49}$$

This in turn determines the value of μ.

At high temperatures, the chemical potential μ is generally much less than ϵ_0, the energy of the ground state. This means that all energy levels will contain more or less the same number of atoms. When the temperature decreases, μ increases until it reaches ϵ_0. At this point, $\overline{n_0}$ will become arbitrarily large and we can say that all particles must be in the ground state. This is the point at which a condensate is formed.

We can write down the Schrödinger equation for a particle with mass m in a condensate as

$$-\frac{d}{dt}\psi(x,y,t) = \left(\frac{-1}{2m}\Delta + V(x,y) + g|\psi|^2\right)\psi(x,y,t). \tag{2.50}$$

In the above description $\frac{g}{2}$ describes the interaction strength between two particles. The potential $V(x,y)$ is used to trap the atoms: it is usually realised by means of a spatially dependent magnetic field or by means of a laser trap.

The magnetic field is set up in such a way that it has an extremal value at the place where the atoms are supposed to be trapped. If the extremal value is a maximum, then that place will attract so-called high-field seeking particles and if the extremal value is a minimum, it will attract so-called low-field seeking particles. Whether a particles is attracted to minima or maxima is determined by its hyperfine state.

The laser trap works in such a way that it forces outward-moving particles to emit a photon. This causes the particle to lose some of its kinetic energy, which it experiences as an inward-directed force.

In most experiments, this trap can be described as a harmonic potential with frequency ω. In other words

$$V(x,y) = \frac{m\omega^2}{2}\left(x^2+y^2\right). \tag{2.51}$$

The relevant thermodynamic energy is now the grand potential $K = E - \mu N$. For our Schrödinger equation, this becomes

$$K = \int_V \left\{ \frac{1}{2m} |\nabla\psi|^2 + V(x,y)|\psi|^2 - \mu|\psi|^2 + \frac{g}{2}|\psi|^4 \right\} d^2x \tag{2.52}$$

from which we can read off the equation of motion

$$\frac{-1}{2m}\Delta\psi + V(x,y)\psi + |\psi|^2\psi = \mu\psi. \tag{2.53}$$

2.3.1 Feshbach resonances

One of the primary reasons for the experimental interest in Bose-gases is the ability to tune many of the parameters of the physical model. One of these parameters is the interaction strength, which can be altered by means of Feshbach resonances. We will describe the general formalism behind these resonances and indicate how this can be used to alter the interaction strength. The space of possible two-particle quantum states my be divided into two channels. One channel P contains the states that are energetically accessible for the two particles: this channel is referred to as the *open channel*. The other channel Q describes those states that are energetically forbidden for the two particles and is referred to as the *closed channel*.

Define projection operators $\mathcal{P}$ and $\mathcal{Q}$ that satisfy

$$\mathcal{P}^2 = \mathcal{P}, \qquad \mathcal{Q}^2 = \mathcal{Q}, \qquad \mathcal{P} + \mathcal{Q} = 1, \qquad \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0. \tag{2.54}$$

The operator $\mathcal{P}$ projects a quantum state $|\psi\rangle$ onto the open channel and the operator $\mathcal{Q}$ projects a quantum state $|\psi\rangle$ onto the closed channel. In general, one may write

$$|\psi\rangle = \mathcal{P}|\psi\rangle + \mathcal{Q}|\psi\rangle = |\psi_P\rangle + |\psi_Q\rangle \tag{2.55}$$

If we now consider the Schrödinger equation for both substates and act on it with $\mathcal{P}$, we find

$$0 = \mathcal{P}\left(H - E\right)\left(|\psi_P\rangle + |\psi_Q\rangle\right) = \mathcal{P}H\mathcal{P}|\psi_P\rangle - E|\psi_P\rangle + \mathcal{P}H\mathcal{Q}|\psi_Q\rangle, \tag{2.56}$$

where we used that $\mathcal{P}|\psi_P\rangle = |\psi_P\rangle$ and $\mathcal{Q}|\psi_Q\rangle = |\psi_Q\rangle$ and that $\mathcal{P}\mathcal{Q} = 0$. Similarly, we can derive that

$$0 = \mathcal{Q}H\mathcal{P}|\psi_Q\rangle - E|\psi_Q\rangle + \mathcal{Q}H\mathcal{P}|\psi_Q\rangle, \tag{2.57}$$

If we now define

$$H_{PP} = \mathcal{P}H\mathcal{P}, \qquad H_{PQ} = \mathcal{P}H\mathcal{Q}, \qquad H_{QP} = \mathcal{Q}H\mathcal{P}, \qquad H_{QQ} = \mathcal{Q}H\mathcal{Q}, \tag{2.58}$$

then we obtain two coupled Schrödinger equations

$$\left(E - H_{PP}\right)|\psi_P\rangle = H_{PQ}|\psi_Q\rangle \tag{2.59}$$
$$\left(E - H_{QQ}\right)|\psi_Q\rangle = H_{QP}|\psi_P\rangle \tag{2.60}$$

The formal solution to the second equation is

$$|\psi_Q\rangle = (E - H_{QQ} + i\delta I)^{-1} H_{QP}|\psi_P\rangle, \tag{2.61}$$

with I the identity operator and δ a small parameter that will ensure the inverse exists. Inserting this into equation (2.59), we find

$$(E - H_{PP} - H'_{PP})\,|\psi_P\rangle = 0, \tag{2.62}$$

for

$$H_{PQ}\,(E - H_{QQ} + i\delta I)\,H_{QP}. \tag{2.63}$$

The term H'_{PP} has an interesting interpretation: it describes the process of the two particles in the open channel temporarily entering the closed channel before decaying back to a state in the closed channel.

Going into the precise details of the computations involved to compute H'_{PP} would require more effort than what seems justifiable for this brief detour in the theory of Bose-gases. For more details, the reader can consult [14]. The important thing is that one can generally write down the contribution to the total interaction strength due to H'_{PP} in the form of

$$\frac{C}{E - E_{\text{res}}} \tag{2.64}$$

where E_{res} is the energy of a quantum state in the closed channel and E is the energy of a state in the open channel. By modifying parameters such as the external magnetic field strength, the difference $E - E_{\text{res}}$ can be tuned, because two quantum states typically respond differently to these modifications. Therefore, an experimentalist can set the interaction strength in a Bose-gas almost at will.

This freedom will become important later, when we apply the methods that lead to the Liouville-like equation in superconductors to a rotating Bose-gas and look for solutions with a degeneration in the energy.

2.3.2 Rotating condensates

We will introduce rotating coordinates

$$x' = \cos(\Omega t)x + \sin(\Omega t)y \qquad y' = \cos(\Omega t)y - \sin(\Omega t)x \tag{2.65}$$

and rewrite the Hamiltonian. For this, we will need to compute the Laplace operator in terms of these new coordinates. We have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \cos(\Omega t)\frac{\partial}{\partial x'} - \sin(\Omega t)\frac{\partial}{\partial y'} \tag{2.66}$$

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \sin(\Omega t)\frac{\partial}{\partial x'} + \cos(\Omega t)\frac{\partial}{\partial y'}, \tag{2.67}$$

so a simple computation shows that the Laplacian becomes

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} \tag{2.68}$$

If we rewrite the wave function in terms of these new coordinates, we will also need to consider their time dependency we evaluating the left hand side of the Schrödinger equation. This becomes

$$\begin{aligned} \frac{d}{dt}\psi(x',y',t) =& \frac{\partial}{\partial t}\psi(x,y,t) + \frac{dx'}{dt}\frac{\partial}{\partial x'}\psi(x,y,t) + \frac{dy'}{dt}\frac{\partial}{\partial y'}\psi(x,y,t) \qquad (2.69)\\ =& \frac{\partial}{\partial t}\psi(x,y,t) + \Omega(\cos(\Omega t)y - \sin(\Omega t)x)\frac{\partial}{\partial x'}\psi(x,y,t) \\ & - \Omega(\cos(\Omega t)x + \sin(\Omega t)y)\frac{\partial}{\partial y'}\psi(x,y,t) \\ =& \frac{\partial}{\partial t}\psi(x,y,t) + \Omega y'\frac{\partial}{\partial x'}\psi(x,y,t) - \Omega x'\frac{\partial}{\partial y'}\psi(x,y,t), \end{aligned} \tag{2.70}$$

so that we can rewrite the Hamiltonian as

$$H = \frac{-1}{2m}\Delta + V(x',y') - \Omega L_3 \tag{2.71}$$

with

$$L_3 = -i\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right). \tag{2.72}$$

We can now introduce an artificial gauge field as follows. Let

$$A = \frac{m\Omega}{q}\begin{pmatrix} -y' \\ x' \\ 0 \end{pmatrix}, \qquad B = \nabla \times A = \frac{2m\Omega}{q}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.73}$$

for some arbitrary constant q. With this notation, the Hamiltonian becomes

$$H = \frac{-1}{2m}\left(\nabla - iqA\right)^2 + V(x',y') + V_\Omega(x',y') + \frac{g}{2}|\psi|^2 \tag{2.74}$$

for

$$V_\Omega(x',y') = -\frac{m\Omega^2}{2}(x'^2 + y'^2). \tag{2.75}$$

and the (static) grand potential is given by

$$K = \int_V \left\{\frac{1}{2m}|D\psi|^2 + V(x',y')|\psi|^2 + V_\Omega(x',y')|\psi|^2 + \mu|\psi|^2 + \frac{g}{2}|\psi|^4\right\} d^2x \tag{2.76}$$

with

$$D = \nabla - iqA. \tag{2.77}$$

We thus see that the grand potential of a rotating Bose-gas bears some resemblance to the free energy of a superconductor in the Ginzburg-Landau model. The most important differences are the presence of an effective trapping potential

$$V_{\text{eff}}(x', y') = V(x', y') + V_{\Omega}(x', y') = \frac{m}{2}(\omega^2 - \Omega^2)(x'^2 + y'^2) \tag{2.78}$$

of which the strength can be determined by changing the angular velocity Ω. The second - and arguably more important - difference is the fact that the gauge field A is no longer dynamic: instead it is now fixed by the angular velocity of the condensate. Despite these differences, we can and will still apply the self-dual method of the next chapter to rotating condensates.

3 The self-dual method

In each of the systems that we described in the previous chapter, it is possible to look for a special type of solutions. Using the covariant derivative that we defined for each of those systems, we can define the *chiral derivative*

$$D_\pm = \frac{1}{2}\sqrt{2}(D_1 \pm iD_2). \tag{3.1}$$

And look for solutions ψ that also satisfy

$$D_\pm\psi = 0 \tag{3.2}$$

These are the so-called *self-dual* solutions. These are often degenerate, which makes them a curious aspect of the theory. It is not clear from the start that the equations of motion allow for solutions that satisfy this additional constraint. In general, this is not the case, but solutions do exist for a suitable choice of the physical parameters in the model. In this chapter, we will derive solutions to this constraint equation and we will work out the choice of physical parameters that allows for these solutions to exist.

The method that we use to derive a solution to the constraint equation, results in a differential equation for the gauge-invariant quantity $|\psi|$. This is convenient, because equations of motion that have gauge freedom are generally impossible to solve, unless one specifies a gauge. This method is also in [9] for the p-wave superconductor that is described in the previous chapter and in [20] in the context of the non-relativistic Jackiw-Pi model.

3.1 Regular superconductors and the Liouville equation

We will now consider a two-dimensional superconducting material that is described by the Ginzburg-Landau model (2.4). The magnetic field is assumed to be directed along the z-axis, orthogonal to the material. We can rewrite the free energy (2.4) by introducing the following derivatives:

$$D_\pm := \frac{1}{2}\sqrt{2}(D_1 \pm iD_2). \tag{3.3}$$

We note that

$$\frac{1}{2}D_i^2 = D_-D_+ - eh_3 = D_+D_- + eh_3 \tag{3.4}$$

and therefore the free energy becomes

$$F = \int_A d^2x \left\{ \frac{1}{m}|D_\pm\psi|^2 \pm \frac{e}{m}h_3|\psi|^2 + \alpha|\psi|^2 + \frac{g}{2}|\psi|^4 + \frac{1}{8\pi}h_3^2 \right\}. \tag{3.5}$$

This free energy will become zero whenever

$$D_\pm\psi = 0, \qquad \pm\frac{e}{m}h_3|\psi|^2 + \alpha|\psi|^2 + \frac{g}{2}|\psi|^4 + \frac{1}{8\pi}h_3^2 = 0 \tag{3.6}$$

From the first condition, it follows that

$$0 = D_{\mp} D_{\pm} \psi = \frac{1}{2} D_i^2 \psi \pm e h_3 \psi \tag{3.7}$$

Substituting this expression in the equation of motion (2.5) for ψ, we obtain

$$\pm \frac{e}{m} h_3 + \alpha + g|\psi|^2 = 0. \tag{3.8}$$

Our next step is to eliminate h_3 from this expression. To do this, we write

$$\psi = e^{i\omega} \sqrt{\rho}, \tag{3.9}$$

where ρ is a positive-valued function of the spatial coordinates and ω is a real-valued function of the spatial and (possibly) temporal coordinates that describes the phase of ψ. Then the first condition in equation (3.6) boils down to

$$\begin{aligned} 0 &= (D_1 \pm i D_2) e^{i\omega} \sqrt{\rho} = (\partial_1 - 2ieA_1 \pm i\partial_2 \pm 2eA_2) e^{i\omega} \sqrt{\rho} \\ &= e^{i\omega} \sqrt{\rho} (\partial_1 \log(\sqrt{\rho}) + i\partial_1 \omega - 2ieA_1 \pm i\partial_2 \log(\sqrt{\rho}) \mp \partial_2 \omega \pm 2eA_2). \end{aligned} \tag{3.10}$$

The real and imaginary part of this equation have to be satisfied independently, so

$$0 = \partial_1 \log(\sqrt{\rho}) \mp \partial_2 \omega \pm 2eA_2 \tag{3.11}$$

and

$$0 = \partial_2 \log(\sqrt{\rho}) \pm \partial_1 \omega \mp 2eA_1. \tag{3.12}$$

We can differentiate the first of these equations with respect to the first spatial coordinate and the second equation with respect to the second spatial coordinate to obtain

$$\frac{1}{m} \Delta \log(\sqrt{\rho}) \pm 2 \frac{e}{m} h_3 = 0. \tag{3.13}$$

Together with equation (3.8), this means that

$$\frac{1}{2m} \Delta \log(\sqrt{\rho}) - g\rho = \alpha, \tag{3.14}$$

which reduces to the Liouville equation when $\alpha = 0$.

It is not clear whether these zeroes of the energy correspond to actual solutions to the equations of motion. We will derive a condition that is both necessary and sufficient for this correspondence to exist. We consider the equation of motion (2.6) for the A-field and work out the j^{th} component right hand side:

$$\begin{aligned} \frac{ie}{m} \left(\psi (D_j \psi)^* - \psi^* D_j \psi \right) &= \frac{ie}{m} \left(-2i\rho \partial_j \omega + 4ieA_j \rho \right) \\ &= \frac{-2e}{m} \rho (2eA_j - \partial_j \omega) \\ &= \mp \frac{e}{m} \epsilon_{jk} \partial_k \rho, \end{aligned} \tag{3.15}$$

where the last equality follows from the equations (3.11) en (3.12). We can also work out the left hand side, using equation (3.8). This leads to

$$\frac{1}{4\pi}\epsilon_{jk}\partial_k h_3 = \mp\frac{mg}{4\pi e}\epsilon_{jk}\partial_k \rho. \tag{3.16}$$

Comparing this with equation (3.15), we find that

$$\frac{gm^2}{4\pi e^2} = 1 \qquad \Leftrightarrow \qquad \kappa^2 = \frac{1}{2}. \tag{3.17}$$

When this condition is fulfilled, we can also rewrite the free energy (3.5) up to a total derivative as

$$F = \int_A d^2x \left\{ \frac{1}{m}|D_\pm \psi|^2 + \frac{1}{2g}\left(\frac{\pm e}{m}h_3 + g|\psi|^2 + \alpha\right)^2 \right\}, \tag{3.18}$$

so that the second condition from equation (3.6) becomes identical to equation (3.8).

3.2 p-wave superconductors and the Liouville equation

According to [9], the ψ_i fields of which the components satisfy equation (2.41) become unstable when they satisfy the self-dual equation

$$D_+\psi := (D_1 + iD_2)\psi = 0 \tag{3.19}$$

From the above equation, we find that

$$0 = (D_1 - iD_2)(D_1 + iD_2)\psi = D_i^2\psi \pm i[D_1, D_2]\psi = D_i^2\psi \pm 2eB_3\psi. \tag{3.20}$$

Inserting this into equation 2.45, we find

$$0 = (2eC_1 - eC_2 - 4\pi\chi J)\, B_3 + \alpha + 4\beta|\psi|^2. \tag{3.21}$$

Furthermore, by an argument similar to the one used in the previous section, we find from equation 3.19 that

$$\begin{aligned}
0 &= (\partial_1 + i\partial_2 - 2ieA_1 + eA_2)\, e^{i\omega}|\psi| \\
&= e^{i\omega}\left(\partial_1 + i(\partial_1\omega) + i\partial_2 - (\partial_2\omega) - 2ieA_1 + 2eA_2\right)|\psi| \\
&= e^{i\omega}|\psi|\left(\frac{1}{|\psi|}\partial_1|\psi| + i(\partial_1\omega) + i\frac{1}{|\psi|}\partial_2|\psi| - (\partial_2\omega) - 2ieA_1 + 2eA_2\right) \\
&= e^{i\omega}|\psi|\left(\partial_1 \log(|\psi|) + i(\partial_1\omega) + i\partial_2\log(|\psi|) - (\partial_2\omega) - 2ieA_1 + 2eA_2\right),
\end{aligned}$$

or equivalently

$$\partial_1 \log(|\psi|) + i(\partial_1\omega) + i\partial_2\log(|\psi|) - (\partial_2\omega) - ieA_1 + eA_2 = 0. \tag{3.22}$$

Because the real and imaginary parts of this equation have to be satisfied independently, this means that

$$\partial_i \log(|\psi|) + \epsilon_{ij}(2eA_j - \partial_j\omega) = 0, \tag{3.23}$$

where ω is the time-dependent phase of ψ. After acting with ∂_i, we may substitute this expression in equation 3.21 to obtain

$$0 = \frac{1}{2e}\left(4\pi\chi J + eC_2 - 2eC_1\right)\partial_i\partial_i \log(|\psi|) + \alpha + 4\beta|\psi|^2, \tag{3.24}$$

which reduces to Liouville's equation when $\alpha = 0$.

Like before, we have to verify under what conditions solutions to the self-dual equation exist. Rather than go through the same derivation as in the previous section, we will state the results. When

$$2\sqrt{\frac{\beta}{8\pi}} = \pm\left(2C_1e - 4\pi\chi J - eC_2\right), \tag{3.25}$$

we can rewrite our expression for the free energy that we obtained in equation (2.44) as

$$F = \int d^2\left\{2C_1|D_+\psi|^2 + \frac{1}{4\beta}\left[\left(2C_1e - 4\pi\chi J - eC_2\right)h_3 + 4\beta|\psi|^2 + \alpha\right]^2\right\}, \tag{3.26}$$

and we see upon comparison with equation (3.21) that zeros of this free energy correspond to solutions to the equation of motion for ψ. Verifying that the equation of motion for A is also satisfied is straightforward and can be done analogously to the case for the regular superconductor, where we obtain equations (3.15) and (3.16).

3.3 The self-dual method for a rotating Bose-gas

In this section, we will apply the self-dual method to the rotating bose-gas from the previous section. As before, we may introduce chiral derivatives

$$D_\pm = \frac{1}{2}\sqrt{2}(D_1 \pm iD_2) \tag{3.27}$$

and look for solutions to the equation

$$D_\pm\psi = 0. \tag{3.28}$$

Then, by a process completely analogous to the one that we followed to obtain equation (3.13) for regular a superconductor we find that

$$\Delta\log(\rho) \pm qB_3 = 0, \tag{3.29}$$

with $\rho = |\psi|$. This time, though, there is no need to eliminate B from the equation, because it is fixed. Instead, write

$$\rho = e^f \tag{3.30}$$

so the equation becomes

$$\Delta f \pm qB_3 = 0. \tag{3.31}$$

The solutions to this equation are

$$f(x', y') = \mp\frac{qB_3}{4}(x'^2 + y'^2) + h(x', y') = \mp\frac{m\Omega}{2}(x'^2 + y'^2) + g(x', y'), \tag{3.32}$$

where h is an arbitrary harmonic function. Substituting this back into equation (3.30), we find

$$\rho = \exp\left(\mp\frac{m\Omega}{2}(x'^2 + y'^2) + h(x', y')\right). \tag{3.33}$$

Because one generally wants the wave function to be normalisable, there are some restrictions on h. If we require $D_+\psi = 0$ so that a negative sign arises in ρ, h must satisfy

$$\frac{2h(x', y')}{m\Omega\,(x'^2 + y'^2)} < 1, \qquad \text{as } x'^2 + y'^2 \to \infty. \tag{3.34}$$

On the other hand, if $D_-\psi = 0$ and there is a positive sign in ρ, then we find the requirement

$$\frac{m\Omega}{2}(x'^2 + y'^2) + h(x', y') \leq -\epsilon \log(x'^2 + y'^2), \qquad \text{as } x'^2 + y'^2 \to \infty \text{ for all } \epsilon > 0. \tag{3.35}$$

When we use the condition $D_\pm\psi = 0$ on the equation of motion (2.53) we find as before that

$$\pm\frac{q}{2m}B_3 - \frac{m\Omega^2}{2} + V(x', y') + g|\psi|^2 = \mu \tag{3.36}$$

There are two ways to make sure that this equation is satisfied. When the potential V is that of a harmonic trap, so $V(x', y') = \frac{m\omega^2}{2}(x'^2 + y'^2)$ one can adjust the rotation speed of the condensate so that $\omega = \Omega$ and require that

$$\mu = \pm\frac{q}{2m}B_3 = \pm\Omega. \tag{3.37}$$

This can be arranged by adjusting the temperature and the number of particles in the condensate. Then, by using a Feshbach resonance, one can ensure that $g = 0$.

Alternatively, one could add an additional magnetic field, thereby altering V. For example, if an experimentalist could construct a magnetic field such that

$$V(x', y') = \frac{m\omega^2}{2} - g|\psi|^2 \tag{3.38}$$

and additionally have $\mu = \pm\Omega = \pm\omega$ then there would be no need to use a Feshbach resonance. This would, however, break the degeneracy in the solutions to the equation of motion, because only one choice of the harmonic function h would suit the trapping potential. Additionally, this approach would be impractical in most experiments, because the part where the magnetic coils are located is generally hard to reach and the task of finding the correct coil to produce a suitable magnetic field is not a trivial one.

4 The Liouville equation

As we have seen in the previous chapter, one encounters a Liouville-like equation when applying the self-dual methods to superconductors. The idea is now to obtain numerical approximations to the solutions of this equation by doing perturbation theory. However, before we can do perturbation theory, we need to have exact solutions to the Liouville equations. We derive these solutions in this chapter.

4.1 The general solution to Liouville's equation

The Liouville equation is a partial differential equation in $\mathbb{R}^2$ given by

$$\Delta\psi = \tilde{c}e^{d\psi}, \tag{4.1}$$

for a real-valued function ψ and $\Delta = \partial_1^2 + \partial_2^2$. In the remainder of this section, we will use complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, which allow us to write the equation as

$$\psi_{z\bar{z}} = ce^{d\psi}, \tag{4.2}$$

where $c = \tilde{c}/4$.

By setting $\rho := \log(d\psi)$, the equation can be written in a different form that will become important later:

$$\Delta \log(\rho) = \tilde{c}d\rho. \tag{4.3}$$

We will derive the general solution to equation (4.2). Doing so, we first follow the approach an article by Crowdy [15] and then show that his 'most general' solution is in fact equivalent to the solution discovered by Liouville [16] many years ago.

Equation (4.2) can be cast in a different form:

Lemma 4.1. *Any function $\psi : \mathbb{C} \to \mathbb{R}$ that is a (local) solution to equation (4.2) is also a solution to*

$$\psi_{zz}(z,\bar{z}) - \frac{d}{2}\left(\psi_z(z,\bar{z})\right)^2 = E(z) \tag{4.4}$$

for some complex analytic function E.

Remark 4.1. *To keep the notation compact, we will henceforth write ψ rather than $\psi(z,\bar{z})$.*

Proof. We first show that any solution to (4.2) satisfies (4.4) for some E. By integration of (4.2) with respect to $\bar{z}$, we find that

$$\psi_z = c\int_{\bar{z}_0}^{\bar{z}} e^{c\psi}d\bar{z} + F(z) \tag{4.5}$$

for some analytic complex function F. Differentiating this expression and using (4.2), we obtain

$$\begin{aligned}\psi_{zz} &= cd\int_{\bar{z}_0}^{\bar{z}} \psi_z e^{d\psi} d\bar{z} + F_z(z) = d\int_{\bar{z}_0}^{\bar{z}} \psi_z \psi_{z\bar{z}} d\bar{z} + F_z(z) \\ &= \frac{d}{2}\int_{\bar{z}_0}^{\bar{z}} \frac{\partial}{\partial \bar{z}} (\psi_z)^2 \, d\bar{z} + F_z(z) = \frac{d}{2}\psi_z^2 + F_z(z).\end{aligned}$$

Identifying $F_z(z)$ with $E(z)$, we have proved our claim. $\square$

We can solve equation (4.4) and thereby obtain solutions to Liouville's equation.

Theorem 4.1. *The solutions to (4.2) are given by*

$$\psi = -\frac{2}{d}\log\left(y^\dagger(\bar{z})My(z)\right), \tag{4.6}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ *with the* y_i *linearly independent solutions to a differential equation*

$$y_{zz} + \frac{d}{2}E(z)y = 0 \tag{4.7}$$

for some analytic E *and* M *a Hermitian* 2×2*-matrix satisfying*

$$cd = -2\det(M)W_y W_{\bar{y}}, \tag{4.8}$$

with W_y *the (constant) Wronskian determinant for* y_1 *and* y_2*.*

Remark 4.2. *In our notation, we set* $y^\dagger(\bar{z}) = \begin{pmatrix} \overline{y_1(z)} & \overline{y_2(z)} \end{pmatrix}$.

Proof. By lemma (4.1), the solutions to (4.2) are also solutions to (4.4) for some E. We may transform (4.4) to a linear equation by substituting $\psi = -\frac{2}{d}\log(\tilde{y})$. Indeed, we have

$$\psi_z^2 = \left(-\frac{2}{d}\frac{\partial}{\partial z}\log(\tilde{y})\right) = \frac{4\tilde{y}_z^2}{d^2\tilde{y}^2}$$

and

$$\psi_{zz} = -\frac{2}{d}\frac{\partial}{\partial z}\frac{\tilde{y}_z}{\tilde{y}} = \frac{2(\tilde{y}_z^2 - \tilde{y}_{zz}\tilde{y})}{d\tilde{y}^2}.$$

By inserting these equalities into equation (4.4) and multiplying by $\tilde{y}$ we obtain equation (4.7). We know from the theory of linear differential equations, that there exist two linearly independent solutions y_1, y_2 to equation (4.7), so that any solution $\tilde{y}$ of (4.7) may be written as

$$\tilde{y}(z,\bar{z}) = M_1(\bar{z})y_1(z) + M_2(\bar{z})y_2(z). \tag{4.9}$$

Because ψ and consequently $\tilde{y}$ are real-valued, we may take the complex conjugate of (4.9), to obtain

$$\tilde{y}(z,\bar{z}) = \bar{M}_1(z)\bar{y}_1(\bar{z}) + \bar{M}_2(z)\bar{y}_2(\bar{z}). \tag{4.10}$$

It follows that

$$\tilde{y}(z,\bar{z}) = m_1|y_1(z)|^2 + m_2 y_1(z)\bar{y}_2(\bar{z}) + m_3 y_2(z)\bar{y}_1(\bar{z}) + m_4|y_2(z)|^2 = y^\dagger(\bar{z})My(z), \tag{4.11}$$

for $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$. Again by demanding that $\tilde{y}$ be real, we find that $m_1, m_4 \in \mathbb{R}$ and $m_3 = \bar{m}_2$, so M is indeed a Hermitian matrix and we find that

$$\psi = -\frac{2}{d}\log\left(y^\dagger(\bar{z})My(z)\right),$$

as desired.

It remains to be shown that any real-valued solution to (4.4) thus obtained with the additional constraint on the determinant of M, is also a solution to (4.2). A straightforward computation shows that, for this choice of ψ, the left hand side of equation (4.2) evaluates to

$$\begin{aligned}
\psi_{z\bar{z}} &= -\frac{2}{d}\frac{\left(y^\dagger_{\bar{z}}(\bar{z})My_z(z)\right)\left(y^\dagger(\bar{z})My(z)\right) - \left(y^\dagger_{\bar{z}}(\bar{z})My(z)\right)\left(y^\dagger(\bar{z})My_z(z)\right)}{\left(y^\dagger(\bar{z})My(z)\right)^2} \\
&= -\frac{2}{d}\frac{\det(M)W_y(z)W_{y^\dagger}(\bar{z})}{\left(y^\dagger(\bar{z})My(z)\right)^2},
\end{aligned}$$

where $W_y = y_1y_2' - y_2y_1'$ is the Wronskian determinant corresponding to y_1 and y_2. Evaluating

$$\frac{\partial}{\partial z}W_y(z) = y_1(z)y_2''(z) - y_2(z)y_1''(z) = \frac{1}{2}E(z)\left(y_1(z)y_2(z) - y_2(z)y_1(z)\right) = 0,$$

we conclude that W_y and $W_{\bar{y}}$ - by a similar argument - are constant.

On the other hand, for this ψ, the right hand side of equation (4.2) evaluates to

$$\frac{c}{\left(y^\dagger(\bar{z})My(z)\right)^2},$$

which means that ψ is a solution, precisely when $cd = -2\det(M)W_yW_{\bar{y}}$. □

In practise, some choices for the y_i and the matix M yield the same solution ψ. In particular, because of the condition $cd = -2\det(M)W_yW_{\bar{y}}$ and $c, d \neq 0$, we know that M is diagonalisable and we may write $M = P^\dagger DP$ for some diagonal matrix $D \in M_2(\mathbb{R})$. Then $y^\dagger My = (Py)^\dagger D\,(Py)$ and because the y_i are two linearly independent solutions to a second order linear differential equation, we know that the elements of Py are two independent solutions to the same equation. Consequently, we may (and will) henceforth assume that M is a diagonal matrix. In fact when $\det(M) > 0$ (and thus $cd < 0$), by the transformation

$$
\begin{aligned}
y_i &\mapsto \sqrt{\frac{m_1}{\sqrt{\det(M)}}}\, y_i \\
M &\mapsto \begin{pmatrix} \sqrt{\det(M)} & 0 \\ 0 & \sqrt{\det(M)} \end{pmatrix}
\end{aligned}
$$

we can assure that M is a multiple of the identity matrix, without changing the solution. Finally, in the light of condition (4.8) we observe that for $cd < 0$

$$
\begin{aligned}
y_i &\mapsto \frac{y_i}{\sqrt{W_y}} \\
M &\mapsto \begin{pmatrix} \sqrt{\frac{-cd}{2}} & 0 \\ 0 & \sqrt{\frac{-cd}{2}} \end{pmatrix}
\end{aligned}
$$

also leaves ψ fixed, while allowing us to assume that the Wronskian determinant of our chosen y_i equals 1. Summarising, we have the following result

Corollary 4.1. *Let $cd < 0$, then the solutions to (4.2) are given by*

$$
\psi = -\frac{1}{d} \log\left((|y_1|^2 + |y_2|^2)^2 \right) - \frac{1}{d} \log\left(-\frac{cd}{2} \right), \tag{4.12}
$$

where the y_i are linearly independent solutions to the equation

$$
y_{zz} + \frac{d}{2} E(z) y = 0
$$

with Wronskian determinant $W_y = 1$.

In the case where $cd > 0$ we can through similar transformations take

$$
M = \begin{pmatrix} \sqrt{\frac{cd}{2}} & 0 \\ 0 & -\sqrt{\frac{cd}{2}} \end{pmatrix}
$$

and obtain the following

Corollary 4.2. *Let $cd > 0$, then the solutions to (4.2) are given by*

$$
\psi = -\frac{1}{d} \log\left((|y_1|^2 - |y_2|^2)^2 \right) - \frac{1}{d} \log\left(\frac{cd}{2} \right), \tag{4.13}
$$

where the y_i are linearly independent solutions to the equation

$$
y_{zz} + \frac{d}{2} E(z) y = 0
$$

with Wronskian determinant $W_y = 1$.

Now let Y_1, Y_2 be two arbitrary linearly independent functions and define $y_i := \frac{Y_i}{\sqrt{W_Y}}$ for $i = 1, 2$. It is clear that the y_i are linearly independent as well. Furthermore, if we compute their Wronskian, we find that

$$W_y = \frac{Y_2'\sqrt{W_Y} - \frac{1}{2}Y_2 W_Y^{3/2}}{W_Y}\frac{Y_1}{\sqrt{W_Y}} - \frac{Y_1'\sqrt{W_Y} - \frac{1}{2}Y_1 W_Y^{3/2}}{W_Y}\frac{Y_2}{\sqrt{W_Y}} = \frac{Y_1 Y_2' - Y_2 Y_1'}{W} = 1. \quad (4.14)$$

Therefore

$$0 = \frac{d}{dz}W_y = y_1 y_2'' - y_1'' y_2, \quad (4.15)$$

from which we conclude that the y_i are solutions to the equation $y_{zz} + \frac{d}{2}E(z)y = 0$ for $\frac{d}{2}E(z) = -\frac{y_1''(z)}{y_1(z)} = -\frac{y_2''(z)}{y_2(z)}$. This means that we can use them to cook up a solution to the Liouville equation (4.2), given by

$$\psi = -\frac{2}{d}\log\left(Y^\dagger(\bar{z})MY(z)\right) + \frac{1}{d}\log\left(W_Y(z)W_{\bar{Y}}(\bar{z})\right). \quad (4.16)$$

On the other hand, any solution of the form (4.6) can be obtained from te above equation by taking linearly independent solutions to the equation $y_{zz} + \frac{d}{2}y = 0$ with unit Wronskian for the Y_i. Combining these observations with Corollaries (4.1) and (4.2), we find that

Corollary 4.3. *The solutions to the Liouville equation (4.2) are given by*

$$\psi = -\frac{1}{d}\log\left((|Y_1|^2 \pm |Y_2|^2)^2\right) + \frac{1}{d}\log\left(W_Y(z)W_{\bar{Y}}(\bar{z})\right) - \frac{1}{d}\log\left(\frac{|cd|}{2}\right), \quad (4.17)$$

where the Y_i are arbitrary analytic functions and where we take the sign positive whenever $cd < 0$ and the negative sign whenever $cd > 0$ in the above expression.

We can apply one more refinement to the solution above. With the notation from corollary (4.3), we define $X_1 = \frac{Y_1}{Y_2}$ and $X_2 = 1$. Then the solution to (4.2) corresponding to these functions is given by

$$\begin{aligned}\psi &= -\frac{1}{d}\log\left((|X_1|^2 \pm 1)^2\right) + \frac{1}{d}\log\left(|X_1'(z)|^2\right) - \frac{1}{d}\log\left(\frac{|cd|}{2}\right)\\ &= -\frac{1}{d}\log\left((|\frac{Y_1}{Y_2}|^2 \pm 1)^2\right) + \frac{1}{d}\log\left(\frac{W_Y(z)W_{\bar{Y}}(z)}{|Y_2(z)|^4}\right) - \frac{1}{d}\log\left(\frac{|cd|}{2}\right)\\ &= -\frac{1}{d}\log\left((|Y_1|^2 \pm |Y_2|^2)^2\right) + \frac{1}{d}\log\left(W_Y(z)W_{\bar{Y}}(\bar{z})\right) - \frac{1}{d}\log\left(\frac{|cd|}{2}\right)\end{aligned}$$

Therefore, we really only need one function to give an arbitrary solution to the Liouville equation. Combining the logarithms, we conclude

Corollary 4.4. *The solutions to the Liouville equation (4.2) are given by*

$$\psi_f = \frac{1}{d}\log\left(\frac{2}{|cd|}\frac{|f'(z)|^2}{(|f(z)|^2 \pm 1)^2}\right), \quad (4.18)$$

where f is an arbitrary analytic function and where we take positive signs whenever $cd < 0$ and negative signs whenever $cd > 0$ in the above expression.

Furthermore, the solutions to the logarithmic of the Liouville equation (4.3) are given by

$$\rho_f = \frac{2}{|cd|} \frac{|f'(z)|^2}{(|f(z)|^2 \pm 1)^2}, \tag{4.19}$$

with the same conventions on the sign as before.

This is the solution discovered by Liouville [16], more than a century ago.

4.1.1 Geometric interpretation

Before we proceed to look at special solutions to the Liouville equation, we pause to have a look at a geometric setting in which the Liouville equation occurs. Let M be a two-dimensional, smooth manifold with metric $g(x,y) = \begin{pmatrix} e^{f(x,y)} & 0 \\ 0 & e^{f(x,y)} \end{pmatrix}$ for some smooth function $f : \mathbb{R}^2 \to \mathbb{R}$. Computing the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$, we find

$$\begin{aligned} \Gamma^0_{00} = -\Gamma^0_{11} = \Gamma^1_{10} = \Gamma^0_{01} &= \frac{1}{2}\partial_1 f \\ \Gamma^0_{10} = \Gamma^0_{01} = -\Gamma^1_{00} = \Gamma^1_{11} &= \frac{1}{2}\partial_2 f \end{aligned}$$

.

Using these, we can compute the Ricci tensor $R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\rho}$

$$\begin{aligned} R_{00} = R_{11} &= -\frac{1}{2}\Delta f \\ R_{01} = R_{10} &= 0 \end{aligned}$$

and the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu} = -e^{-f}\Delta f.$$

On a manifold with constant Gaussian curvature $K = R/2$, we see that our function f satisfies the Liouville equation (4.2). The metric on M can therefore be written as

$$dS^2 = \frac{4}{|K|} \frac{|f'(x+iy)|^2}{(|f(x+iy)|^2 \pm 1)^2} \left(dx^2 + dy^2\right), \tag{4.20}$$

where the sign should be take positive in case of positive curvature and negative in the case of negative curvature.

4.2 Special solutions in the plane

As will be shown in other sections of this thesis, the Liouville equation occurs in a number of problems in physics, where the solution of (4.3) (the logarithmic version) equals the density of some physical quantity. For this reason, we want our solution ρ_f to be bounded. By Liouville's theorem, every non-constant entire function is unbounded on open sets and therefore the solutions in (4.19) with a negative sign, will result in density functions with singularities, because the function f inside the solution will attain norm 1 somewhere. The case where we encounter a negative sign will thus indicate a shortcoming in the physical theory.

4.2.1 Solutions with a positive sign

First, we will consider solutions to the Liouville equation on the plane with a positive sign, which ensures that ρ_f is bounded. In this case, it is clear that we want

$$\rho(r) = \mathcal{O}(\frac{1}{r^{2+\epsilon}}) \qquad \text{as } r \to \infty \tag{4.21}$$

for some $\epsilon > 0$, because the integral of ρ over the plane should be finite. Furthermore, we will assume that the function f in (4.19) has at most isolated singularities. Each singularity corresponds to a vortex in the density function, so this assumption boils down to studying solutions with at most isolated vortices. The solutions (4.19) were studied under these assumptions in [17] and we will repeat their results here.

We start with a lemma that relates a complex function to its corresponding solution of the Liouville equation:

Lemma 4.2. *Let $f(z)$ be a complex function with at most isolated singularities and let*

$$\gamma : [0,1] \to \mathbb{C}$$

be a curve in the complex plain that avoids the singularities of f. Furthermore, set $z_0 := \gamma(0)$ and $z_1 := \gamma(1)$; then

$$\frac{|f(z_0) - f(z_1)|}{\sqrt{(1+|f(z_0)|^2)(1+|f(z_1)|^2)}} \leq \int_\gamma \frac{|f'(z)|}{1+|f(z)|^2}|dz|. \tag{4.22}$$

Proof. Via stereographic projection, the metric on the Riemann sphere is given by

$$ds^2 = \frac{4}{(1+|w|^2)^2}|dw|^2, \tag{4.23}$$

where the point w is the image in $\widehat{\mathbb{C}}_w$ imbued with the above metric of a point (x, y, z) on the sphere after stereographic projection. Therefore, the length of a curve $\Gamma : [0,1] \to \widehat{\mathbb{C}}_w$ on the Riemann sphere is given by

$$L(\Gamma) = \int_\Gamma \frac{2}{1+|w|^2}|dw| = \int_0^1 \frac{2|\Gamma'(t)|}{1+|\Gamma(t)|^2}dt \tag{4.24}$$

and the distance between two points $w_0, w_1 \in \widehat{\mathbb{C}}_w$ is given by

$$d_w(w_0, w_1) = \inf_{\substack{\Gamma:[0,1]\to\widehat{\mathbb{C}}_w \\ \Gamma(0)=z_0,\ \Gamma(1)=z_1}} L(\Gamma). \tag{4.25}$$

We can view f as a mapping from $\mathbb{C}$ to $\widehat{\mathbb{C}}_w$; then $f \circ \gamma$ becomes a curve in $\widehat{\mathbb{C}}_w$ and we obtain the following inequality

$$d_w(f(z_0), f(z_1)) \leq \int_{f\circ\gamma} \frac{2}{1+|w|^2}|dw| = \int_\gamma \frac{2|f'(z)|}{1+|f(z)|^2}|dz|. \tag{4.26}$$

But the distance in the left hand side of the above expression is the same as the distance on the Riemann sphere between the points $f(z_0)$ and $f(z_1)$ before stereographic projection and this latter distance is greater than or equal to the natural distance in $\mathbb{R}^3$, which we will now calculate.

Let $\pi : S^2 \to \widehat{\mathbb{C}}_w$ be the stereographic projection map from the sphere to the plane. Then its inverse in a point $w = x + iy$ is given by

$$\pi^{-1}(w) = \left(\frac{2x}{1+|w|^2}, \frac{2y}{1+|w|^2}, \frac{1-|w|^2}{1+|w|^2}\right) \tag{4.27}$$

Then the distance in $\mathbb{R}^3$ between $f(z_0)$ and $f(z_1)$ is given by

$$d(f(z_0)), f(z_1))^2 = (\pi^{-1}(f(z_0) - \pi^{-1}(f(z_1))^2 = \frac{4|f(z_0) - f(z_1)|^2}{(1+|f(z_0)|^2)(1+|f(z_1)|^2)}, \tag{4.28}$$

where we used a computer algebra package to verify the last step. Combining this with (4.26), we finally obtain

$$\frac{|f(z_0) - f(z_1)|}{\sqrt{(1+|f(z_0)|^2)(1+|f(z_1)|^2)}} \leq \int_\gamma \frac{|f'(z)|}{1+|f(z)|^2}|dz|, \tag{4.29}$$

as desired. □

With this lemma, we will be able to derive some properties of the function f that we found in (4.19).

Lemma 4.3. *The function f in (4.19) cannot have an isolated essential singularity in $\mathbb{C}$.*

Proof. Suppose to the contrary that f has an isolated essential singularity in a point z_0. Then f is analytic inside some punctured disc with radius ϵ: $D := D(z_0, \epsilon) \setminus \{z_0\}$. Then, by Picard's Theorem [19] the image $f(D)$ is either all of $\mathbb{C}$ or $\mathbb{C}$ minus a single point. Let $z_0 \in D$ be a point such that $f(z_0) = 0$ (or, if such a point does not exist, $f(z_0)$ arbitrarily close to 0). Furthermore, because ρ is a regular function that goes to 0 at infinity, we there exists an $M \in \mathbb{R}_{\geq 0}$ such that

$$\frac{|f'(z)|}{1+|f(z)|^2} \leq M \qquad\qquad \forall z \in \mathbb{C}. \tag{4.30}$$

Then, let $z_1 \in D$ be arbitrary. By lemma (4.2) we find that

$$\frac{|f(z_1)|}{\sqrt{1+|f(z_1)|^2}} \leq M \int_\gamma |dz| \tag{4.31}$$

for all curves $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Because z_0 and z_1 both lie in D, we may choose γ such that its length equals 2ϵ. Then, if we choose ϵ such that $2\epsilon M \leq \frac{1}{2}$, we find that

$$|f(z_1)|^2 \leq \frac{1}{2}(1+|f(z_1)|^2), \tag{4.32}$$

so $|f(z_1)| \leq 1$. Because z_1 was chosen arbitrarily in D, this is in contradiction with Picard's Theorem. □

The same result can be obtained for isolated essential singularities at infinity.

Lemma 4.4. *The function f in (4.19) cannot have an isolated singularity at infinity.*

Proof. Again, suppose to the contrary that f does have an isolated singularity at infinity and let $D = \{z \in \mathbb{C} : |z| > r\}$ for some $r > 0$ that we will specify later. By Picard's Theorem, we can find a point $z_0 \in D$ such that $f(z_0) = 0$ (or arbitrarily close to 0).

As in the previous proof, we will derive that the presence of the singularity leads to an upper bound on the norm of f inside the region D. We note that our growth condition on ρ implies that

$$|\rho_f(z)| \leq M|z|^{-2-\epsilon} \qquad\qquad \forall z \in D \tag{4.33}$$

for some $M \in \mathbb{R}_{\geq 0}$. Now consider the circle C with radius $r' > r$, a point $z \in C$ and z_1, the point of intersection of C with the half-line from the origin through z_0. By the triangle inequality, we have

$$d_w(0, f(z)) \leq d_w(0, f(z_1)) + d_w(f(z_1), f(z)) \tag{4.34}$$

We can estimate both terms on the right hand side of this expression with the help of lemma (4.2). For the first term, we use the inequalities (4.26) and (4.33) for $\gamma(t) = z_0 + t(z_1 - z_0)$ to find

$$d_w(0, f(z_1)) \leq 2\int_\gamma \sqrt{\rho_f(z)}|dz| \leq \int_\gamma \frac{2M}{|z|^{1+\epsilon/2}}|dz| \leq \frac{4M}{\epsilon r^{\epsilon/2}}, \tag{4.35}$$

where we used the fact that the image of γ is contained in D, which means that the integration variable takes values larger than r.

For the second term, we again use the inequalities (4.26) and (4.33) and integrate over the circular arc with radius r' from z_1 to z. Then

$$d_w(f(z_1), f(z)) \leq 2\int_{\text{arc}} \sqrt{\rho_f(z)}|dz| \leq \int_{\text{arc}} \frac{2M}{|z|^{1+\epsilon/2}}|dz| \leq \frac{4\pi M}{(r')^{\epsilon/2}} \leq \frac{4M}{r^{\epsilon/2}}. \tag{4.36}$$

By lemma (4.2) and the estimates we obtained, we thus find that

$$\frac{|f(z)|}{\sqrt{1+|f(z)|^2}} \leq d_w(0, f(z)) \leq 4\left(\pi + \frac{1}{\epsilon}\right)\frac{M}{r^{\epsilon/2}}. \tag{4.37}$$

Now, we choose r such that $4\left(\pi + \frac{1}{\epsilon}\right)\frac{M}{r^{\epsilon/2}} = \frac{1}{2}$ holds. Then the above inequality implies that $|f(z)| \leq 1$ for all $z \in D$, which is in contradiction with Picard's Theorem. We conclude that f cannot have an essential singularity at infinity. □

Lemmas (4.3) and (4.4) allow us to give a general formula for f:

Corollary 4.5. *The function f in (4.19) is a rational function.*

Proof. By lemmas (4.3) and (4.4), f cannot have an essential singularity anywhere on the Riemann sphere. Theorem 5.64 of Whittaker and Watson [18] then gives us the desired result. □

Not all choices of f yield unique density functions ρ_f. We will prove and use a lemma from [20] to eliminate the unnecessary degrees of freedom.

Lemma 4.5. *Let $f_i : \mathbb{C} \to \mathbb{C}$ for $i = 1, 2$ be two non-constant meromorphic functions. If their associated densities ρ_{f_i} are equal, then there exists a unique matrix $\alpha \in PSU(2)$ such that its lift $\tilde{\alpha} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ satisfies*

$$f_1 = T(\alpha)(f_2) := \tilde{\alpha} \cdot f_2 := \frac{af_2 + c}{cf_2 + d}. \tag{4.38}$$

Conversely, whenever $f_1 = \beta \cdot f_2$ for some $\beta \in U(2)$, we have $\rho_{f_1} = \rho_{f_2}$.

Proof. For the first implication, we note that $\rho_{f_1} = \rho_{f_2}$ implies that

$$d_w(f_1(z_1), f_1(z_2)) = d_w(f_2(z_1), f_2(z_2)) \qquad \text{for all } z_1, z_2 \in \mathbb{C}. \tag{4.39}$$

We now define the map $\iota : \widehat{\mathbb{C}}_w \to \widehat{\mathbb{C}}_w$

$$\iota(w) := f_2(f_1^{-1}(w)). \tag{4.40}$$

This map is well-defined, because whenever $f_1(z) = f_2(z')$ for some $z, z' \in \mathbb{C}$, we have

$$0 = d_w(f_1(z), f_1(z')) = d_w(f_2(z), f_2(z')) \tag{4.41}$$

and hence $f_2(z) = f_2(z')$. We will show that ι is an orientation-preserving isometry of $\widehat{\mathbb{C}}_w$. It is clear that ι is orientation-preserving, because f_1 and f_2 are meromorphic. Furthermore, it is surjective, because f_1 and f_2 are not constant. Finally, it is preserves distances, because

$$d_w(\iota(w_1), \iota(w_2)) = d_w(f_2(f_1^{-1}(w_1)), f_2(f_1^{-1}(w_2)) = d_w(f_1(f_1^{-1}(w_1)), f_1(f_1^{-1}(w_2)) = d_w(w_1, w_2), \tag{4.42}$$

for any $w_1, w_2 \in \widehat{\mathbb{C}}_w$.

Therefore, $f_2 = T(\alpha)(f_1)$, where $T(\alpha)$ is an orientation-preserving isometry of $\widehat{\mathbb{C}}_w$. These isometries are given by matrices $\alpha \in \mathrm{PSU}(2)$ such that their lifts $\tilde{\alpha} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$ act on meromorphic functions f in a point $z \in \mathbb{C}$ as

$$\alpha \cdot f(z) = \frac{af(z) + b}{cf(z) + d}. \tag{4.43}$$

Furthermore, when z is a pole of f, then $\alpha \cdot f(z) = \frac{a}{c}$ if $c \neq 0$ and $\alpha \cdot f(z) = \infty$ is $c = 0$.

For the second implication, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \in \mathrm{U}(2)$ and suppose that $f_1 = \beta \cdot f_2$. Because $\beta \in \mathrm{U}(2)$, we may write $c = -\bar{b}e^{i\theta}$ and $d = \bar{a}e^{i\theta}$ with $|a|^2 + |b|^2 = 1$. Then

$$\rho_{f_1}(z) = \rho_{\beta \cdot f_2}(z) = \frac{|\frac{d}{dz}\beta \cdot f_2(z)|^2}{(1 + |\beta \cdot f_2(z)|^2)^2}, \tag{4.44}$$

where

$$\begin{aligned}
\left|\frac{d}{dz}\beta \cdot f_2\right| &= \left|\frac{d}{dz}\frac{af_2(z) + b}{cf_2(z) + d}\right| \\
&= \left|\frac{af_2'(z)(cf_2(z) + d) - cf_2'(z)(af_2(z) + b)}{(cf_2(z) + d)^2}\right| \\
&= \left|\frac{(ad - bc)f_2'(z)}{(cf_2(z) + d)^2}\right| \\
&= \left|\frac{(|a|^2 + |b|^2)e^{i\theta}f_2'(z)}{(-\bar{b}e^{i\theta}f_2(z) + \bar{a}e^{i\theta})^2}\right| \\
&= \frac{|f_2'(z)|}{|\bar{a} - \bar{b}f_2(z)|^2}.
\end{aligned} \tag{4.45}$$

We now see that equation (4.44) simplifies to

$$\begin{aligned}
\rho_{f_1}(z) &= \frac{|f_2'(z)|^2}{(|\bar{a} - \bar{b}f_2(z)|^2 + |af_2(z) + b|^2)^2} \\
&= \frac{|f_2'(z)|^2}{|a|^2 + |b|^2 - \bar{a}b\bar{f}_2(\bar{z}) - a\bar{b}f_2(z) + a\bar{b}f_2(z) + \bar{a}b\bar{f}_2(\bar{z}) + (|a|^2 + |b|^2)|f_2(z)|^2} \\
&= \frac{|f_2'(z)|^2}{(1 + |f_2(z)|^2)^2} = \rho_{f_2}(z),
\end{aligned} \tag{4.46}$$

as we set out to prove. $\square$

Theorem 4.2. *The solutions*

$$\rho_f(z) = \frac{2}{|cd|}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}$$

to (4.3) with

$$\rho_f(r) = \mathcal{O}(\frac{1}{r^{2+\epsilon}})$$

for some $\epsilon > 0$ as $r \to \infty$ are given by

$$f = \frac{P(z)}{Q(z)}$$

with P and Q polynomials satisfying $\deg(P) < \deg(Q)$*. Also, the coefficient of the highest-order term in Q can taken to be 1, while the coefficient of the highest-order term in P can be taken in $\mathbb{R}_{>0}$.*

Proof. Write $f(z) = \frac{P(z)}{Q(z)}$. First suppose that $\deg(P) > \deg(Q)$. Let $\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{U}(2)$. By lemma 4.5, we know that $\rho_f = \rho_{\alpha\cdot f} = \rho_{1/f}$, where $\frac{1}{f}$ is a rational function with denominator of higher degree than the numerator.

Next, suppose that $\deg(P) = \deg(Q)$. We may rewrite this as $f(z) = f_0 + \frac{\tilde{P}(z)}{Q(z)} = \frac{f_0 Q(z) + \tilde{P}(z)}{Q(z)}$, where $\deg(\tilde{P}) < \deg(Q)$. Then, let $\alpha = \begin{pmatrix} 1 & -f_0 \\ -\bar{f}_0 & 1+|f_0|^2 \end{pmatrix} \in \mathrm{U}(2)$ and compute

$$\begin{aligned} \alpha \cdot f &= \frac{(f_0 + \tilde{P}/Q) - f_0}{-\bar{f}_0(f_0 + \tilde{P}/Q) + (1+|f_0|^2)} \\ &= \frac{\tilde{P}}{Q - \bar{f}_0\tilde{P}}. \end{aligned} \tag{4.47}$$

By lemma 4.5, we know that $\rho_f = \rho_{\alpha\cdot f}$, where $\alpha \cdot f$ is a rational function with denominator of degree strictly higher than the numerator.

For the conditions on the leading-term coefficients, we point out that we can simply rescale P and Q by the same factor, so that the leading-term coefficient of Q becomes one and that we may act on f by a matrix of the form $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{U}(2)$ to ensure that the leading-term coefficient of P lies in $\mathbb{R}_{>0}$. $\square$

4.2.2 Solutions with a negative sign

Even though the solutions to the Liouville equation with a negative sign are unlikely to give a complete description of any physical system one may hope to encounter in the real world, they might still be of interest as approximations to actual physical phenomena. We will therefore study these solutions in this section, deriving results similar to the ones that we obtained before. In contrast to the case with a positive sign, we can no longer exclude functions with essential singularities on the ground that they produce densities ρ_f that are singular in one or more points. Indeed, by the fundamental theorem of algebra, even taking f to be a polynomial will produce singularities, because there must a point $z \in \mathbb{C}$ in which $|f(z)| = 1$. To still obtain results that are comparable to the situation with a positive sign, we are forced to exclude essential singularities by hand.

To determine which choices of f will yield the same density ρ_f, we will need an equivalent to lemma 4.5. To obtain this lemma, we will first prove the following

Lemma 4.6. *After stereographic projection from the two-dimensional hyperbola $\pi : H^2 \to \widehat{\mathbb{C}}_w$ the group $SO(2,1)$ of orientation preserving isometries of H^2 acts as a Möbius transformation on $\widehat{\mathbb{C}}_w$*

$$z \mapsto \alpha \cdot z, \tag{4.48}$$

where $\alpha \in PSU(1,1)$.

Proof. To determine the action of an isometry $T \in \mathrm{SO}(2,1)$ on a point $w = u + iv \in \widehat{\mathbb{C}}_w$, we can compute

$$\pi(T(\pi^{-1}(w))). \tag{4.49}$$

Because $\mathrm{SO}(2,1)$ is generated by matrices of the form

$$R(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.50}$$

$$B_x(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix} \qquad B_y(t) = \begin{pmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & 1 & 0 \\ \sinh(t) & 0 & \cosh(t) \end{pmatrix}$$

it suffices to determine their action on w. A straightforward calculation shows that

$$\pi(R(t)(\pi^{-1}(w))) = \cos(t)u - \sin(t)v + i\sin(t)u + i\cos(t)v = e^{it}w = \frac{e^{it/2}w + 0}{0w + e^{-it/2}}, \tag{4.51}$$

which is a Möbius transformation. For $B_x(t)$, we have

$$\pi(B_x(t)(\pi^{-1}(w))) = \frac{-2iv - 2u\cosh(t) + (1 + u^2 + v^2)\sinh(t)}{-1 + u^2 + v^2 - (1 + u^2 + v^2)\cosh(t) + 2u\sinh(t)}. \tag{4.52}$$

We want to write this as a Möbius transformation as well. To this end, observe that for $a, b \in \mathbb{C}$ and $c, d \in \mathbb{R}$ we have

$$\frac{a(u+iv)+b}{ic(u+iv)+id} = \frac{-iac(u^2+v^2) - i(ad+bc)u + (ad-bc)v - ibd}{c^2(u^2+v^2) + 2cdu + d^2}, \tag{4.53}$$

by multiplication with $\frac{-ic(u-iv)-id}{-ic(u-iv)-id}$. Bearing in mind that

$$2\cosh^2(t/2) = \cosh(t) + 1 \qquad\qquad 2\sinh^2(t/2) = \cosh(t) - 1 \tag{4.54}$$

and

$$2\sinh(t/2)\cosh(t/2) = \sinh(t) \tag{4.55}$$

we can verify that for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mp i\sqrt{2}\cosh(t/2) & \pm i\sqrt{2}\sinh(t/2) \\ \pm\sqrt{2}\sinh(t/2) & \mp\sqrt{2}\cosh(t/2) \end{pmatrix} \tag{4.56}$$

the right hand side of equation (4.52) becomes equal to the right hand side of (4.51). Therefore, after dividing the denominator and the numerator by $i\sqrt{2}$ we find that

$$\pi(B_x(t)(\pi^{-1}(w))) = \frac{\cosh(t/2)(u+iv) - \sinh(t/2)}{-\sinh(t/2)(u+iv) + \cosh(t/2)}. \tag{4.57}$$

Using a similar approach, we also find that

$$\pi(B_y(t)(\pi^{-1}(w))) = \frac{\cosh(t/2)(u+iv) - i\sinh(t/2)}{i\sinh(t/2)(u+iv) + \cosh(t/2)}. \tag{4.58}$$

We conclude that the generators $R(t)$, $B_x(t)$ and $B_y(t)$ of $\mathrm{SO}(2,1)$ correspond to the matrices

$$\begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \begin{pmatrix} \cosh(t/2) & -\sinh(t/2) \\ -\sinh(t/2) & \cosh(t/2) \end{pmatrix} \text{ and } \begin{pmatrix} \cosh(t/2) & -i\sinh(t/2) \\ i\sinh(t/2) & \cosh(t/2) \end{pmatrix} \tag{4.59}$$

respectively, which all lie in $\mathrm{PSU}(1,1)$. □

This lemma allows us to proof a result similar to lemma 4.5:

Lemma 4.7. *Let $f_i : \mathbb{C} \to \mathbb{C}$ for $i = 1, 2$ be two non-constant meromorphic functions. If their associated densities ρ_{f_i} are equal, then there exists a unique matrix $\alpha \in PSU(1,1)$ such that its lift $\tilde{\alpha} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$ satisfies*

$$f_1 = T(\alpha)(f_2) := \tilde{\alpha} \cdot f_2 := \frac{af_2 + c}{cf_2 + d}. \tag{4.60}$$

Conversely, whenever $f_1 = \beta \cdot f_2$ for some $\beta \in U(1,1)$, we have $\rho_{f_1} = \rho_{f_2}$.

Proof. Like before, we will need a distance function; this time for the hyperbolic plane. Via stereographic projection, the metric on the hyperbolic plane is given by

$$ds^2 = \frac{4}{(1-w^2))^2}|dw|^2. \tag{4.61}$$

Therefore, the length of a curve $\Gamma : [0,1] \to \widehat{\mathbb{C}}_w$ on the projected hyperbolic plane is given by

$$L(\Gamma) = \left| \int_\Gamma \frac{2}{1-|w|^2}|dw| \right| = \left| \int_0^1 \frac{2|\Gamma'(t)|}{1-|\Gamma(t)|^2} dt \right| \tag{4.62}$$

and the distance between two points $w_0, w_1 \in \widehat{\mathbb{C}}_w$ is given by

$$d_w(w_0, w_1) = \inf_{\substack{\Gamma:[0,1]\to\widehat{\mathbb{C}}_w \\ \Gamma(0)=z_0,\ \Gamma(1)=z_1}} L(t). \tag{4.63}$$

For the first implication, we note that $\rho_{f_1} = \rho_{f_2}$ implies that

$$d_w(f_1(z_1), f_1(z_2)) = d_w(f_2(z_1), f_2(z_2)) \qquad \text{for all } z_1, z_2 \in \mathbb{C}. \tag{4.64}$$

We now define the map $\iota : \widehat{\mathbb{C}}_w \to \widehat{\mathbb{C}}_w$

$$\iota(w) := f_2(f_1^{-1}(w)). \tag{4.65}$$

This map is well-defined, because whenever $f_1(z) = f_2(z')$ for some $z, z' \in \mathbb{C}$, we have

$$0 = d_w(f_1(z), f_1(z')) = d_w(f_2(z), f_2(z')) \tag{4.66}$$

and hence $f_2(z) = f_2(z')$. We will show that ι is an orientation-preserving isometry of $\widehat{\mathbb{C}}_w$. It is clear that ι is orientation-preserving, because f_1 and f_2 are meromorphic. Furthermore, it is surjective, because f_1 and f_2 are not constant. Finally, it is preserves distances, because

$$d_w(\iota(w_1), \iota(w_2)) = d_w(f_2(f_1^{-1}(w_1)), f_2(f_1^{-1}(w_2)) = d_w(f_1(f_1^{-1}(w_1)), f_1(f_1^{-1}(w_2)) = d_w(w_1, w_2), \tag{4.67}$$

for any $w_1, w_2 \in \widehat{\mathbb{C}}_w$.

Therefore, $f_2 = T(\alpha)(f_1)$, where $T(\alpha)$ is an orientation-preserving isometry of $\widehat{\mathbb{C}}_w$. These isometries are given by matrices $\alpha \in \mathrm{PSU}(1,1)$ such that their lifts $\tilde{\alpha} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2)$ act on meromorphic functions f in a point $z \in \mathbb{C}$ as

$$\alpha \cdot f(z) = \frac{af(z)+b}{cf(z)+d}. \tag{4.68}$$

Furthermore, when z is a pole of f, then $\alpha \cdot f(z) = \frac{a}{c}$ if $c \neq 0$ and $\alpha \cdot f(z) = \infty$ is $c = 0$.

For the second implication, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \in \mathrm{U}(2)$ and suppose that $f_1 = \beta \cdot f_2$. Because $\beta \in \mathrm{U}(1,1)$, we may write $c = \bar{b}e^{i\theta}$ and $d = \bar{a}e^{i\theta}$ with $|a|^2 - |b|^2 = 1$. Then

$$\rho_{f_1}(z) = \rho_{\beta \cdot f_2}(z) = \frac{|\frac{d}{dz}\beta \cdot f_2(z)|^2}{(1 + |\beta \cdot f_2(z)|^2)^2}, \tag{4.69}$$

where

$$\begin{aligned} \left|\frac{d}{dz}\beta \cdot f_2\right| &= \left|\frac{d}{dz}\frac{af_2(z) + b}{cf_2(z) + d}\right| \\ &= \left|\frac{af_2'(z)(cf_2(z) + d) - cf_2'(z)(af_2(z) + b)}{(cf_2(z) + d)^2}\right| \\ &= \left|\frac{(ad - bc)f_2'(z)}{(cf_2(z) + d)^2}\right| \\ &= \left|\frac{(|a|^2 - |b|^2)e^{i\theta}f_2'(z)}{(\bar{b}e^{i\theta}f_2(z) + \bar{a}e^{i\theta})^2}\right| \\ &= \frac{|f_2'(z)|}{|\bar{a} + \bar{b}f_2(z)|^2}. \end{aligned} \tag{4.70}$$

We now see that equation (4.69) simplifies to

$$\begin{aligned} \rho_{f_1}(z) &= \frac{|f_2'(z)|^2}{(|\bar{a} + \bar{b}f_2(z)|^2 - |af_2(z) + b|^2)^2} \\ &= \frac{|f_2'(z)|^2}{|a|^2 - |b|^2 + \bar{a}b\bar{f}_2(\bar{z}) + a\bar{b}f_2(z) - a\bar{b}f_2(z) - \bar{a}b\bar{f}_2(\bar{z}) - (|a|^2 - |b|^2)|f_2(z)|^2} \\ &= \frac{|f_2'(z)|^2}{(1 - |f_2(z)|^2)^2} = \rho_{f_2}(z), \end{aligned} \tag{4.71}$$

as we set out to prove. □

4.3 Special solutions on the torus

In this section, we will have a look at solutions ρ_f on the torus. Identifying the torus with $\mathbb{C}/\Omega$, where $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ for some $\omega_1, \omega_2 \in \mathbb{C}$, we find that these solutions correspond to periodic functions:

$$\rho_f(z) = \rho_f(z + \omega_i) \qquad \text{for all } z \in \mathbb{C}, i = 1, 2 \tag{4.72}$$

The statements about the absence of essential singularities of f in $\mathbb{C}$ carry over from the previous paragraphs. However, because f is completely determined by its behaviour inside a unit cell, there is no analogous statement about singularities at infinity. Therefore, we now only find f to be a function that is meromorphic on the plane.

One obvious choice for f that satisfies the periodicity condition (4.72) would be any elliptic function on $\mathbb{C}/\Omega$. However, it turns out that one can do more than this. Let $\omega \in \Omega$ and define

$$g(z) := f(z + \omega). \tag{4.73}$$

Then, by the lemmas 4.5 and 4.7, we find that $\rho_f = \rho_g$ precisely when

$$g = \gamma_\omega \cdot f \qquad \gamma_\omega \in \mathrm{U}(2), \mathrm{U}(1,1) \tag{4.74}$$

or equivalently when

$$g = T(\gamma_\omega)(f) \tag{4.75}$$

for some $T(\gamma_\omega) \in \mathrm{PSU}(2)$ when ρ_f is a solution as written down in 4.19 with a positive sign and $T(\gamma_\omega) \in \mathrm{PSU}(1,1)$ when ρ_f has a negative sign. Furthermore, ρ_f is periodic when the above holds for all $\omega \in \Omega$. Meromorphic functions f that satisfy condition (4.74) are called $\Omega-$*quasi-elliptic.*

Because

$$T(\gamma_{\omega+\omega'}) = T(\gamma_\omega)T(\gamma_{\omega'}) = T(\gamma_{\omega'})T(\gamma_\omega), \tag{4.76}$$

it suffices to determine γ_{ω_1} and γ_{ω_2} that satisfy equation (4.74). Moreover, these transformations are then the generators of some abelian subgroup G of $\mathrm{PSU}(2)$ or $\mathrm{PSU}(1,1)$, depending on the sign in our solution. This subgroup then acts on our function f as a Möbius transformation.

Because we can take the lift γ_ω of $T(\gamma_\omega)$ in $\mathrm{U}(2)$ or $\mathrm{U}(1,1)$, depending on the sign in our density, the actual matrix γ acting on f is only determined up to a scalar multiplication by a unit $\mu \in \mathbb{C}^*$. Thus, when $T(\gamma_\omega)T(\gamma_{\omega'}) = T(\gamma_{\omega'})T(\gamma_\omega)$, we find that

$$\gamma_\omega \gamma_{\omega'} = \mu \gamma_{\omega'} \gamma_\omega. \tag{4.77}$$

For reasons that will become clear later, we will choose our lifts γ_{ω_1} and γ_{ω_2} such that

$$\gamma_{\omega_j} = \begin{pmatrix} ia_j & \mp b_j e^{-i\phi_j} \\ b_j e^{i\phi_j} & -ia_j \end{pmatrix} \tag{4.78}$$

with $a_j, b_j, \phi \in \mathbb{R}$ satisfying $a_j^2 \pm b_j^2 = 1$, where the sign is chosen in accordance with the sign in our density ρ_f. We will now distinguish between two cases, namely the case where $\mu = 1$ and the matrices commute and the case where $\mu \neq 1$ where we will show that this case is in fact limited to $\mu = -1$ and the matrices anti-commute.

4.3.1 Case I: $\mu = 1$

Because the γ_{ω_j} commute, we can diagonalise them simultaneously: there exists a matrix $U \in \mathrm{SU}(2)$ such that

$$\gamma_{\omega_j} = U^\dagger \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} U \qquad j = 1, 2, \tag{4.79}$$

where we used that any matrix of the form (4.78) has eigenvalues $\pm i$. We thus conclude that our special choice for these matrices has forced them to be identical, so we will use a, b, ϕ from now on, leaving out the subscript in equation (4.78). We now define

$$g := U \cdot f, \tag{4.80}$$

which satisfies

$$g(z + \omega_j) = -g(z) \qquad j = 1, 2. \tag{4.81}$$

So g is an elliptic function of the second kind with multiplier -1. A complete classification of such functions will be given in Appendix A. This means that we can write

$$f = U^\dagger g, \tag{4.82}$$

where g is an elliptic function of the second kind with multiplier -1.

When the density has a positive sign, it is clear that $\rho_f = \rho_g$, because $U \in \mathrm{U}(2)$. This is not as clear when the density has a negative sign: then we need to show that we can take U in $\mathrm{U}(1, 1)$. A straightforward computation shows that the vectors

$$\begin{pmatrix} ie^{-i\phi}\sqrt{\frac{a+1}{2}} \\ \frac{b}{\sqrt{2a+2}} \end{pmatrix}, \begin{pmatrix} \frac{b}{\sqrt{2a+2}} \\ -ie^{i\phi}\sqrt{\frac{a+1}{2}} \end{pmatrix} \tag{4.83}$$

are eigenvectors of the γ_{ω_j} and that U does indeed lie in $\mathrm{U}(1, 1)$ (in fact, it even lies in $\mathrm{SU}(1, 1)$) if we take these vectors as its columns. Therefore, we also get $\rho_f = \rho_g$ in this case.

Conversely, any elliptic function g of the second kind with multiplier -1 yields a periodic associated density ρ_g, because it satisfies

$$g(z + \omega_j) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} /cdot g(z), \qquad j = 1, 2 \tag{4.84}$$

and the claim follows from the lemmas 4.5 and 4.7.

4.3.2 Case II: $\mu \neq 1$

It turns out that this case is in fact restricted to $\mu = -1$:

Lemma 4.8. *Let $\gamma, \gamma' \in M_2(\mathbb{C})$ satisfy*

$$\gamma\gamma' = \mu\gamma'\gamma \tag{4.85}$$

for some $1 \neq \mu \in \mathbb{C}$. Then $\mu = -1$.

Proof. Let $v_1, v_2 \in \mathbb{C}^2$ be the eigenvectors of γ with corresponding eigenvalues λ_1, λ_2. Then

$$\gamma(\gamma' v_i) = \mu\gamma'\gamma v_i = \lambda_i \mu(\gamma' v_i). \tag{4.86}$$

Because $\mu \neq 1$, it is clear that γ' acts as a permutation on the eigenvectors of γ, exchanging v_1 and v_2. Also, we see from this equation that

$$\lambda_1 = \mu\lambda_2, \qquad \lambda_2 = \mu\lambda_1. \tag{4.87}$$

Or, in other words $\mu = \frac{1}{\mu}$. This is only possible when $\mu = \pm 1$ and because $\mu = 1$ was excluded, we find that $\mu = -1$. □

From the proof of the above lemma, it also becomes clear that we can diagonalise γ_{ω_1} while simultaneously anti-diagonalising γ_{ω_2}:

$$\gamma_{\omega_1} = U^\dagger \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} U, \qquad \gamma_{\omega_2} = U^\dagger \begin{pmatrix} 0 & -\lambda \\ \lambda^{-1} & 0 \end{pmatrix} U \tag{4.88}$$

where λ is some complex number with unit modulus[1]. It is worth considering for a moment what this means when the γ_{ω_i} lie in $\mathrm{U}(1,1)$. Because of our choice of the lift of the $T(\gamma_{\omega_i})$ to $\mathrm{U}(1,1)$ in equation (4.78), we know that $U, U^\dagger \in \mathrm{U}(1,1)$. But this means that $\begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} \in \mathrm{U}(1,1)$, which is clearly impossible. We can thus conclude that this case only occurs in $\mathrm{PSU}(2)$.

Now define

$$M := \begin{pmatrix} 1 & 0 \\ 0 & i\lambda \end{pmatrix}. \tag{4.89}$$

Then

$$\begin{aligned} \gamma_{\omega_1} &= U^\dagger M^\dagger \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} MU = V^\dagger \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} V \\ \gamma_{\omega_2} &= U^\dagger M^\dagger \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} MU = V^\dagger \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} V \end{aligned} \tag{4.90}$$

for $V = MU \in \mathrm{U}(2)$. It follows that any Ω-quasi-elliptic function f can be written as

$$f = V^\dagger \cdot g, \tag{4.91}$$

where g satisfies

$$g(z + \omega_1) = -g(z), \qquad g(z + \omega_2) = \frac{1}{g(z)} \tag{4.92}$$

[1] We point out that we do not make use of the special form in equation (4.78), because it doesn't simplify the argument that follows in any way.

Conversely, it also follows by lemma 4.5 that the density ρ_f associated with any such f is indeed periodic.

We will now classify all meromorphic functions that satisfy equation (4.92). Suppose g and g_0 are two such functions and define

$$\phi(z) := \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \frac{g(z)}{g_0(z)} \tag{4.93}$$

Then ϕ satisfies

$$\phi(z+\omega_1) = f(z), \qquad \phi(z+\omega_2) = -\phi(z), \tag{4.94}$$

so ϕ is a multiplicative quasi-elliptic function with multipliers $\mu_1 = 1$ and $\mu_2 = -1$. This means that there are constants

$$a_0, \ldots, a_n \in \mathbb{C} \tag{4.95}$$

and parameters

$$z_1, \ldots, z_n \in \{t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\} \tag{4.96}$$

in the fundamental domain of Ω such that

$$\phi(z) = \left[a_0 + \sum_{k=1}^{n} a_k \frac{d^k\zeta}{dz^k}(z - z_0)\right] \frac{\sigma(z-z_0)^n}{\prod_{j=1}^{n}\sigma(z-z_j)} e^{\zeta(\omega_1/2)z}, \tag{4.97}$$

where

$$z_0 = \frac{\omega_1}{2n} + \frac{1}{n}\sum_{k=1}^{n} z_k. \tag{4.98}$$

Thus g is of the form

$$g(z) = -\frac{\phi(z)-1}{\phi(z)+1} g_0(z), \tag{4.99}$$

while in fact any g of this form satisfies equation (4.92).

What remains is to find a g_0 satisfying equation (4.92). We may try a function of the form

$$g_0(z) = \frac{\wp_{2\omega_1,2\omega_2}(z) + b}{c\wp_{2\omega_1,2\omega_2}(z) + d} \tag{4.100}$$

The reason for this is the fact that there exist 'half-period' formulas for the Weierstrass p-functions:

Lemma 4.9. *Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and define*

$$\wp(z) := \wp_{2\omega_1,2\omega_2}(z), \qquad e_1 := \wp(\omega_1), \qquad e_2 := \wp(\omega_2), \qquad e_3 := -e_1 - e_2. \tag{4.101}$$

Then we have

$$\wp(z+\omega_1) = e_1 + \frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1}, \qquad \wp(z+\omega_2) = e_2 + \frac{(e_2-e_1)(e_2-e_3)}{\wp(z)-e_2}. \tag{4.102}$$

Proof. We know the addition formula for the Weierstrass p-functions:

$$\wp(z_1+z_2)+\wp(z_1)+\wp(z_2)=\frac{1}{4}\left(\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right)^2 \tag{4.103}$$

where $\wp'(z)$ satisfies

$$\wp'(z)^2=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3). \tag{4.104}$$

In particular, we note that the latter equation implies that $\wp'(\omega_i)=0$ for $i=1,2$. Substituting $z_1=z$ and $z_2=\omega_1$ in equation (4.103), we thus find that

$$\begin{aligned}\wp(z+\omega_1)&=-e_1-\wp(z)+\frac{(\wp(z)-e_2)(\wp(z)-e_3)}{\wp(z)-e_1}\\&=e_1+\frac{-(2e_1+\wp(z))(\wp(z)-e_1)+(\wp(z)-e_2)(\wp(z)-e_3)}{\wp(z)-e_1}\\&=e_1+\frac{2e_1^2+e_2e_3-(e_1+e_2+e_3)\wp(z)}{\wp(z)-e_1}\\&=e_1+\frac{(e_1-e_2)(e_1-e_3)}{\wp(z)-e_1},\end{aligned} \tag{4.105}$$

where we used that $e_1+e_2+e_3=0$. The derivation for the other formula works along the same lines. □

With these formulas, we may express $\wp(z+\omega_i)$ in terms of the e_i and $\wp(z)$. Imposing the conditions in equation (4.92) then amounts to simultaneously solving two systems of equations - i.e. one for each condition - and then looking for common solutions. This was carried out by the author using a computer algebra package (Mathematica) and it turns out that one common solution is given by

$$\begin{aligned}b&=-e_1-\sqrt{(e_1-e_2)(e_1-e_3)}\\c&=\frac{e_1-e_2+\sqrt{(e_1-e_2)(e_1-e_3)}}{\sqrt{(-e_1+e_2)(e_2-e_3)}}\\d&=c(-e_1+\sqrt{(e_1-e_2)(e_1-e_3)}).\end{aligned} \tag{4.106}$$

We note that this solution differs from the one given in [20]. The author has checked both solutions numerically and found that the solution as stated in this thesis does indeed yield a density with periods ω_1 and ω_2, whereas the other solution seems to yield a density with exactly twice these periods. After a correspondence with Akerblom, who is one of the authors of the original paper, it there turned out to by a typographical error. Their corrected values are

$$b = \frac{-e_2 + c^2(-2e_1 + e_2)}{1 + c^2} \tag{4.107}$$

$$c = \sqrt{\frac{-3e_1 + 2\sqrt{(e_1 - e_2)(2e_1 + e_2)}}{e_1 + 2e_2}} \tag{4.108}$$

$$d = \frac{c(-2e_1 + e_2 - c^2 e_2)}{1 + c^2}, \tag{4.109}$$

the difference being a $-e_2$ in the numerator of b, rather than a $-e_2^2$.

4.3.3 A general solution on the torus

From the discussion in the above two cases, it becomes clear that we can classify all solutions to the Liouville equation on the torus:

Theorem 4.3. *Let ρ_f be a doubly-periodic solution to the Liouville equation with periods ω_1 and ω_2. It follows that f is a meromorphic function that falls in one of the two categories:*

1. *The function f satisfies*

$$f(z + \omega_j) = -f(z), \tag{4.110}$$

that is, f is an elliptic function of the second kind with multiplier -1 (cf. Appendix A).

2. *There are constants*

$$a_0, \ldots, a_n \in \mathbb{C} \tag{4.111}$$

and parameters

$$z_1, \ldots, z_n \in \{t_1\omega_1 + t_2\omega_2 : 0 \le t_1, t_2 < 1\} \tag{4.112}$$

in the fundamental domain of Ω such that g is of the form

$$g(z) = -\frac{\phi(z) - 1}{\phi(z) + 1} g_0(z), \tag{4.113}$$

where

$$\phi(z) = \left[a_0 + \sum_{k=1}^{n} a_k \frac{d^k\zeta}{dz^k}(z - z_0)\right] \frac{\sigma(z - z_0)^n}{\prod_{j=1}^{n} \sigma(z - z_j)} e^{\zeta(\omega_1/2)z}, \tag{4.114}$$

with

$$z_0 = \frac{\omega_1}{2n} + \frac{1}{n}\sum_{k=1}^{n} z_k \tag{4.115}$$

and

$$g_0(z) = \frac{\wp_{2\omega_1, 2\omega_2}(z) + b}{c\wp_{2\omega_1, 2\omega_2}(z) + d}, \tag{4.116}$$

with

$$\begin{aligned} b &= -e_1 - \sqrt{(e_1 - e_2)(e_1 - e_3)} \\ c &= \frac{e_1 - e_2 + \sqrt{(e_1 - e_2)(e_1 - e_3)}}{\sqrt{(-e_1 + e_2)(e_2 - e_3)}} \\ d &= c(-e_1 + \sqrt{(e_1 - e_2)(e_1 - e_3)}). \end{aligned} \tag{4.117}$$

and

$$e_1 := \wp(\omega_1), \qquad e_2 := \wp(\omega_2), \qquad e_3 := -e_1 - e_2. \tag{4.118}$$

The functions ζ, σ and $\wp$ refer to Weierstrass' elliptic functions with periods ω_1 and ω_2.

4.3.4 Integration

When a density ρ_f is a solution to the Liouville equation with a positive sign on $D = \mathbb{C}/\Omega$, one may be interested in the integral

$$\int_D \rho_f d^2x = \int_D \frac{|f'(x+iy)|^2}{(1+|f(x+iy)|^2)^2} d^2x \tag{4.119}$$

either to find the number of particles per unit cell, or because it is related to some other physical quantity, such as the magnetic flux through one cell[2]. It is often straightforward to do this, without using heavy machinery such as an index theorem or resorting to numerical approximations. We will sketch an approach to this.

If we set $z = x + iy$ and $\bar{z} = x - iy$ and use these as integration variables, we pick a factor $\frac{1}{2}$ from the Jacobian and find

$$\int_D \rho_f d^2x = \frac{1}{2} \int_D \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} d\bar{z}dz. \tag{4.120}$$

We will make another substitution, setting $f(z) = w = u+iv$, and integrate over w. Then we recognise the term $|f'(z)|^2$ as the inverse of the Jacobian belonging to this transformation, so we would like to write

$$\int_D \rho_f d^2x = \frac{1}{2} \int_{f^{-1}(D)} \frac{1}{(1+|w|^2)^2} d\bar{w}dw = \int_{f^{-1}(D)} \frac{1}{(1+u^2+v^2)^2} dudv. \tag{4.121}$$

This is not entirely true, however, because f can attain some values more than once. For example, let g be an elliptic function on D with n poles, counted with multiplicity and let $z_0 \in \mathbb{C}$. Then $g - z_0$ is also an elliptic function with n poles and therefore

$$\sum_{\omega \in D} \mathrm{ord}_\omega(g - z_0) = 0, \tag{4.122}$$

[2] This is the case in the non-relativistic Jackiw-Pi model on a torus [20].

which means that f attains each value in $\mathbb{C}$ n times. Therefore, we now have

$$\int_D \rho_g d^2x = n \int_{\mathbb{R}^2} \frac{1}{(1+u^2+v^2)^2} du dv = n\pi \tag{4.123}$$

5 Perturbation theory

In this section, we study a perturbative approach to solving the Liouville-like equation

$$\Delta f \pm e^f = a. \tag{5.1}$$

We know the exact solutions to this equation when $a = 0$ and one may hope that solutions change smoothly when a takes on different values. Therefore, we write a solution f to equation (5.1) as

$$f(x, y) = \sum_{n=0}^{\infty} f_n(x, y) a^n. \tag{5.2}$$

Next, we write the exponent as a power series in a:

$$\begin{aligned}
\exp(\sum_{n=0}^{\infty} f_n a^n) &= e^{f_0} \left(\sum_{m=0}^{\infty} \frac{a^m}{m!} \left(\sum_{n=1}^{\infty} f_n a^n \right)^m \right) \\
&= e^{f_0} \left(1 + \sum_{n=1}^{\infty} a^n \sum_{\substack{m_1 j_1 + \ldots + m_k j_k = n \\ j_1, \ldots, j_k > 0}} \frac{1}{m_1! \ldots m_k!} f_{j_1}^{m_1} \ldots f_{j_k}^{m_k} \right),
\end{aligned}$$

where we used that

$$\left(\sum_{n=1}^{\infty} f_n a^n \right)^m = m! \sum_{m_1 + \ldots + m_k = m} \frac{a^{m_1 j_1 + \ldots + m_k j_k}}{m_1! \ldots m_k!} f_{j_1}^{m_1} \ldots f_{j_k}^{m_k}. \tag{5.3}$$

Substituting these expressions into equation (5.1) and separating the different powers of a, we find a series of differential equations:

$$\Delta f_0 \pm e^{f_0} = 0 \tag{5.4}$$

$$\Delta f_1 \pm e^{f_0} f_1 = 1 \tag{5.5}$$

$$\Delta f_n \pm e^{f_0} f_n = \mp e^{f_0} \sum_{\substack{m_1 j_1 + \ldots + m_k j_k = n \\ 0 < j_1, \ldots, j_k < n}} \frac{1}{m_1! \ldots m_k!} f_{j_1}^{m_1} \ldots f_{j_k}^{m_k} \text{ for } n \geq 2 \tag{5.6}$$

We know the solution to the first equation and the solution to the n^{th} can be obtained numerically from the solutions to the $n - 1$ previous equations. In general these equations all share the form

$$\Delta f_n \pm e^{f_0} f_n = g_n, \tag{5.7}$$

where g_n is a combination of solutions to the previous equations.

We will now describe a numerical method to find doubly periodic solutions to the differential equations (5.5) and (5.6) on a fundamental domain $D \subseteq \mathbb{R}^2$. The main idea is to divide the fundamental domain D in a rectangular grid and then find an approximation for the solution f_n on each of the grid points.

This is most conveniently done when D itself is also rectangular. This can be arranged at the cost of adding some extra derivatives to the differential equations: suppose f_n is a solution to the differential equation

$$\Delta f_n \pm e^{f_0} f_n = g_n \tag{5.8}$$

and that D is given by

$$D = \{\lambda_1 \hat{e}_1 + \lambda_2 r(\cos(\phi)\hat{e}_1 + \sin(\phi)\hat{e}_2) \in \mathbb{R}^2 : 0 \leq \lambda_1, \lambda_2 \leq 1\} \tag{5.9}$$

for some $r > 0$, $0 < \phi \leq \pi$. Then the function $\tilde{f}_n : \mathbb{R}^2 \to \mathbb{R}$ given by

$$(x, y) \mapsto f_n(x + \cos(\phi)y, \sin(\phi)y) \tag{5.10}$$

is periodic on $\tilde{D} = [0, 1] \times [0, r]$ and it is a solution to the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tilde{f}_n(x - \frac{y}{\tan(\phi)}, \frac{y}{\sin(\phi)}) \pm e^{f_0} \tilde{f}_n(x - \frac{y}{\tan(\phi)}, \frac{y}{\sin(\phi)}) = g_n \tag{5.11}$$

If we define

$$\tilde{x} = x - \frac{y}{\tan(\phi)}, \qquad \tilde{y} = \frac{y}{\sin(\phi)} \tag{5.12}$$

then a straightforward calculation shows that the equation can be rewritten as

$$\frac{\Delta}{\sin^2(\phi)} \tilde{f}_n - \frac{2\cos(\phi)}{\sin^2(\phi)} \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} \tilde{f}_n \pm e^{\tilde{f}_0} \tilde{f}_n = \tilde{g}_n, \tag{5.13}$$

where

$$\tilde{f}_0(x, y) = f_0(x + \cos(\phi)y, \sin(\phi)y), \qquad \tilde{g}_n(x, y) = g_n(x + \cos(\phi)y, \sin(\phi)y) \tag{5.14}$$

Our next step is to choose a grid G inside $\tilde{D}$. Let $N_x, N_y \in \mathbb{N}$ be the number of points on the grid along the x and y directions and let $d_x := \frac{1}{N_x}, d_y := \frac{r}{N_y}$ be the corresponding grid spacings. Then G is given by

$$G = (\mathbb{Z}d_x \times \mathbb{Z}d_y) \cap ([0, 1) \times [0, r)) \tag{5.15}$$

Then, for $0 \leq m < N_x$, $0 \leq n < N_y$, we define the point $(x_m, y_n) \in G$

$$(x_m, y_n) := (md_x, nd_y) \tag{5.16}$$

We can use a Taylor approximation in the point $(x_i, y_j) \in G$ to find

$$\tilde{f}_n(x_i + d_x, y_j) = \tilde{f}_n(x_i, y_j) + d_x \frac{\partial}{\partial x_i} \tilde{f}_n(x_i, y_j) + \frac{d_x^2}{2} \frac{\partial^2}{\partial x_i^2} \tilde{f}_n(x_i, y_j) + \mathcal{O}(d_x^2)$$

$$\tilde{f}_n(x_i - d_x, y_j) = \tilde{f}_n(x_i, y_j) - d_x \frac{\partial}{\partial x_i} \tilde{f}_n(x_i, y_j) + \frac{d_x^2}{2} \frac{\partial^2}{\partial x_i^2} \tilde{f}_n(x_i, y_j) + \mathcal{O}(d_x^2)$$

Adding and substracting these equations, we find

$$\frac{\partial}{\partial x_i}\tilde{f}_n(x_i, y_j) = \frac{\tilde{f}_n(x_{i+1}, y_j) - \tilde{f}_n(x_{i-1}, y_j)}{2d_x} + \mathcal{O}(d_x^3) \tag{5.17}$$

$$\frac{\partial^2}{\partial x_i^2}\tilde{f}_n(x_i, y_j) = \frac{\tilde{f}_n(x_{i+1}, y_j) + \tilde{f}_n(x_{i-1}, y_j) - 2\tilde{f}_n(x_i, y_j)}{d_x^2} + \mathcal{O}(d_x^3), \tag{5.18}$$

where we used that $x_{i\pm1} = x_i \pm d_x$. Using an identical approach, we can of course find similar expressions for the derivatives with respect to y_i.

We may view $\tilde{f}_n(G)$ as an element of $\mathbb{R}^{N_x} \otimes \mathbb{R}^{N_y}$, where $\otimes$ denotes the Kronecker tensor product, given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{1m}B & \dots & a_{nm}B \end{pmatrix} \tag{5.19}$$

for $n \times m$ matrix $A = (a_{ij})$ and $p \times q$ matrix B. Then the first derivatives of $\tilde{f}_n$ on the points of G are given up to $\mathcal{O}(d_x^3)$ by

$$\partial_1 \tilde{f}_n(G) = (D_1 \otimes I_{N_y})\tilde{f}_n(G), \qquad \partial_2 \tilde{f}_n(G) = (I_{N_x} \otimes D_1)\tilde{f}_n(G) \tag{5.20}$$

with

$$D_1 = \frac{1}{2d_x}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & & & 0 \\ 0 & \vdots & \ddots & & & 1 \\ 1 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}. \tag{5.21}$$

We note that the entries in the upper right and lower left corners are to ensure the periodic boundary conditions, ensuring that

$$\partial_1 \tilde{f}_n(x_0, y_j) = \frac{\tilde{f}_n(x_1, y_j) - \tilde{f}_n(x_{N_x-1}, y_j)}{2d_x} \tag{5.22}$$

Meanwhile, the second order derivatives ∂_1^2, ∂_2^2 and $\partial_1\partial_2$ are given by

$$\partial_1^2 \tilde{f}_n(G) = (D_2 \otimes I_{N_y})\tilde{f}_n(G), \qquad \partial_2^2 \tilde{f}_n(G) = (I_{N_x} \otimes D_2)\tilde{f}_n(G) \tag{5.23}$$

and

$$\partial_1\partial_2 \tilde{f}_n(G) = (D_1 \otimes D_1)\tilde{f}_n(G) \tag{5.24}$$

with

$$D_2 = \frac{1}{d_x^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & & & 0 \\ 0 & \vdots & \ddots & & & 1 \\ 1 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}. \tag{5.25}$$

Going back to equation (5.13), we see that we still need to write the multiplication by $e^{\tilde{f}_0}$ in matrix form. This can be arranged by the matrix

$$F_0 = \begin{pmatrix} E_0 & 0 & \dots & 0 \\ 0 & E_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & E_{N_x-1} \end{pmatrix} \tag{5.26}$$

with

$$E_i = \begin{pmatrix} e^{\tilde{f}_n(x_i,y_0)} & 0 & \dots & 0 \\ 0 & e^{\tilde{f}_n(x_i,y_1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\tilde{f}_n(x_i,y_{N_y-1})} \end{pmatrix}. \tag{5.27}$$

Now we can rewrite equation (5.13) on all of the points on G as

$$H\tilde{f}_n(G) = \tilde{g}_n(G) \tag{5.28}$$

with

$$H = \frac{1}{\sin^2(\phi)}\big(D_2 \otimes I_{N_y} + I_{N_x} \otimes D_2\big) - \frac{2\cos(\phi)}{\sin^2(\phi)} D_1 \otimes D_2 \pm F_0 \tag{5.29}$$

so that

$$\tilde{f}_n(G) = H^{-1} g_n(G) \tag{5.30}$$

provided that $\det(H) \neq 0$.

We can summarise the above in the following theorem:

Theorem 5.1. *Consider the equation for f*

$$\Delta f \pm e^{f_0} f = g \tag{5.31}$$

where f_0 and g are a doubly periodic functions with fundamental domain

$$D = \{\lambda_1 \hat{e}_1 + \lambda_2 r(\cos(\phi)\hat{e}_1 + \sin(\phi)\hat{e}_2) \in \mathbb{R}^2 : 0 \leq \lambda_1, \lambda_2 \leq 1\} \tag{5.32}$$

Furthermore, consider $N_x, N_y \in \mathbb{N}$*,* $d_x = \frac{1}{N_x}$*,* $d_y = \frac{r}{N_y}$ *and a rectangular grid*

$$G = (\mathbb{Z}d_x \times \mathbb{Z}d_y) \cap ([0,1) \times [0,r)) \tag{5.33}$$

Suppose that f *is a solution to equation (5.31) that is doubly periodic with fundamental domain* D*. Then* f *is given up to* $\mathcal{O}(\max(d_x^3, d_y^3))$ *on* $TG := \begin{pmatrix} 1 & \cos(\phi) \\ 0 & \sin(\phi) \end{pmatrix} G$ *by*

$$f(TG) = H^{-1} g(TG), \tag{5.34}$$

where H *is defined in equation (5.29).*

In the figures at the end of this section, one can see the results of this procedure for a number of choices of f_0. One may wonder whether (5.2) actually converges, as the numerical results

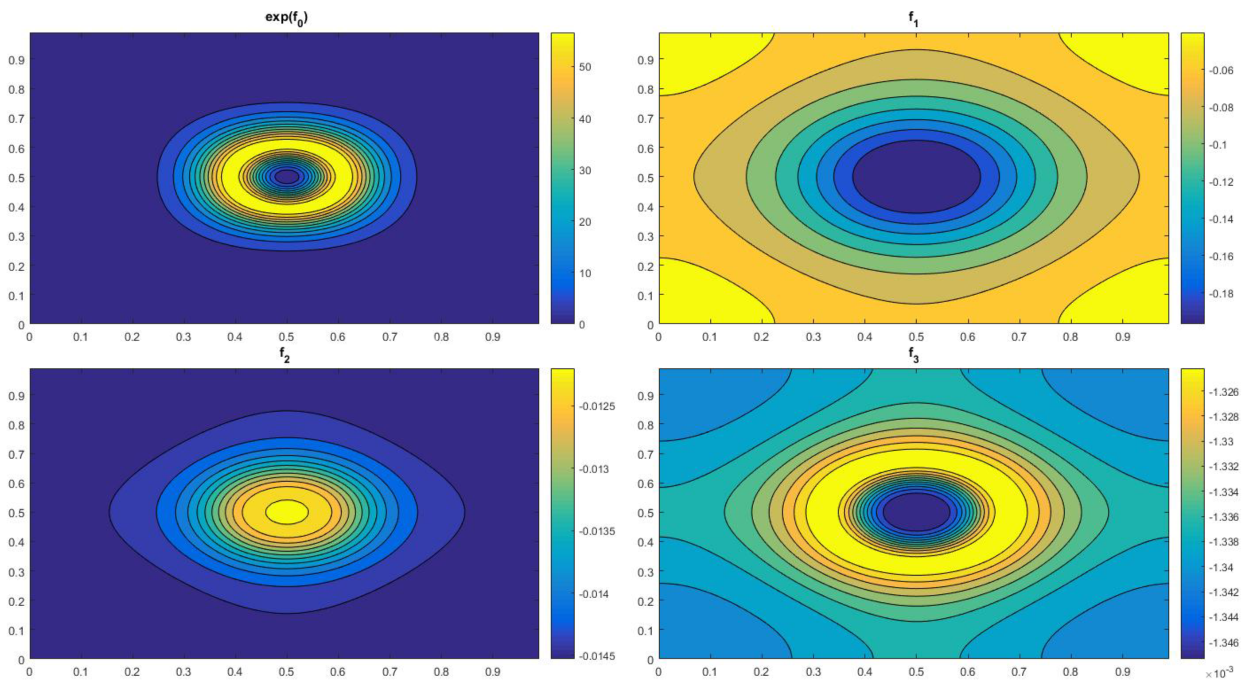

Figure 2: Solutions to equation (5.7) on the torus $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ for $f_0 = \rho_\wp$ and $n = 0, 1, 2, 3$.

seem to indicate. That is, if we can show that $\lim_{n\to\infty} \|f_{n+1}\|_\infty = 0$. Let $P = (x, y) \in D$ be a point in the fundamental domain of f_n. Using the numerical approach that we described above, we can compute f in a point (x_i, y_j) arbitrarily close to P up to an arbitrary precision, provided that f is sufficiently smooth, so using the notation of theorem (5.1) we can say

$$\|f_n\|_\infty = \max(|f_n(TG)|) + \mathcal{O}(\max(d_x^2, d_y^2)) \leq \left\|H^{-1}\right\| \max(|g_n(TG)|) + \mathcal{O}(\max(d_x^2, d_y^2)). \tag{5.35}$$

where

$$\left\|H^{-1}\right\| = \sup\{\left\|H^{-1}v\right\|_\infty : v \in (\mathbb{R}^{N_x} \otimes \mathbb{R}^{N_y}) \text{ with } \|v\|_\infty = 1\} \tag{5.36}$$

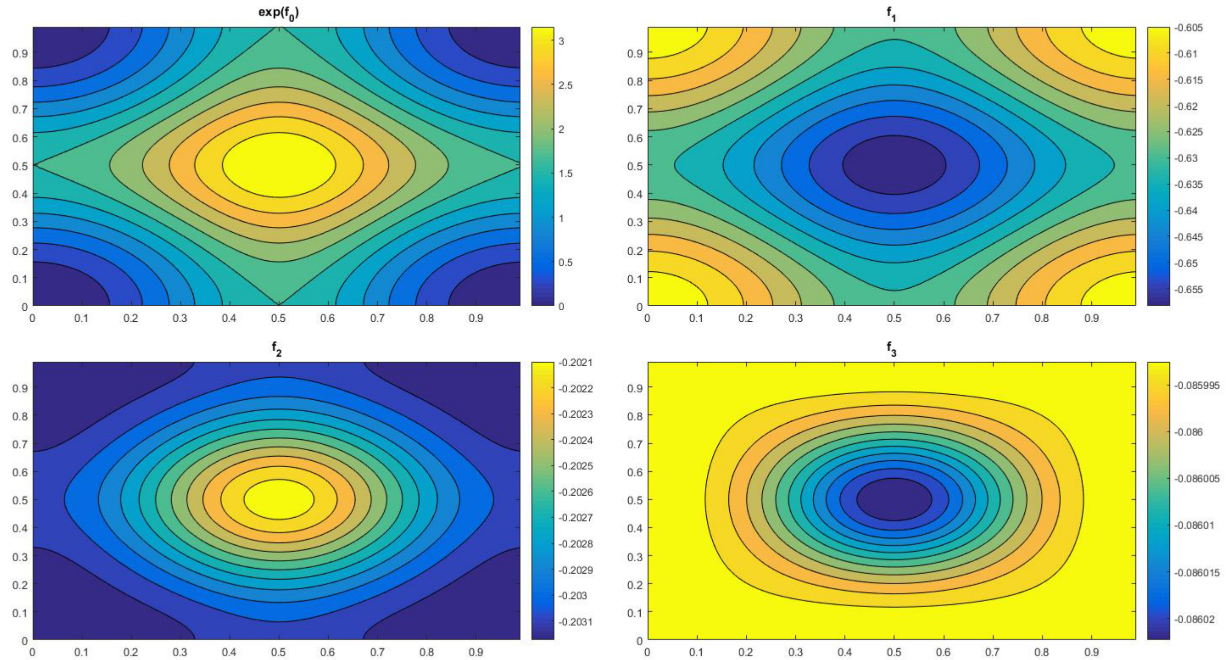

Figure 3: Solutions to equation (5.7) on the torus $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ for f_0 Olesen's solution and $n = 0, 1, 2, 3$.

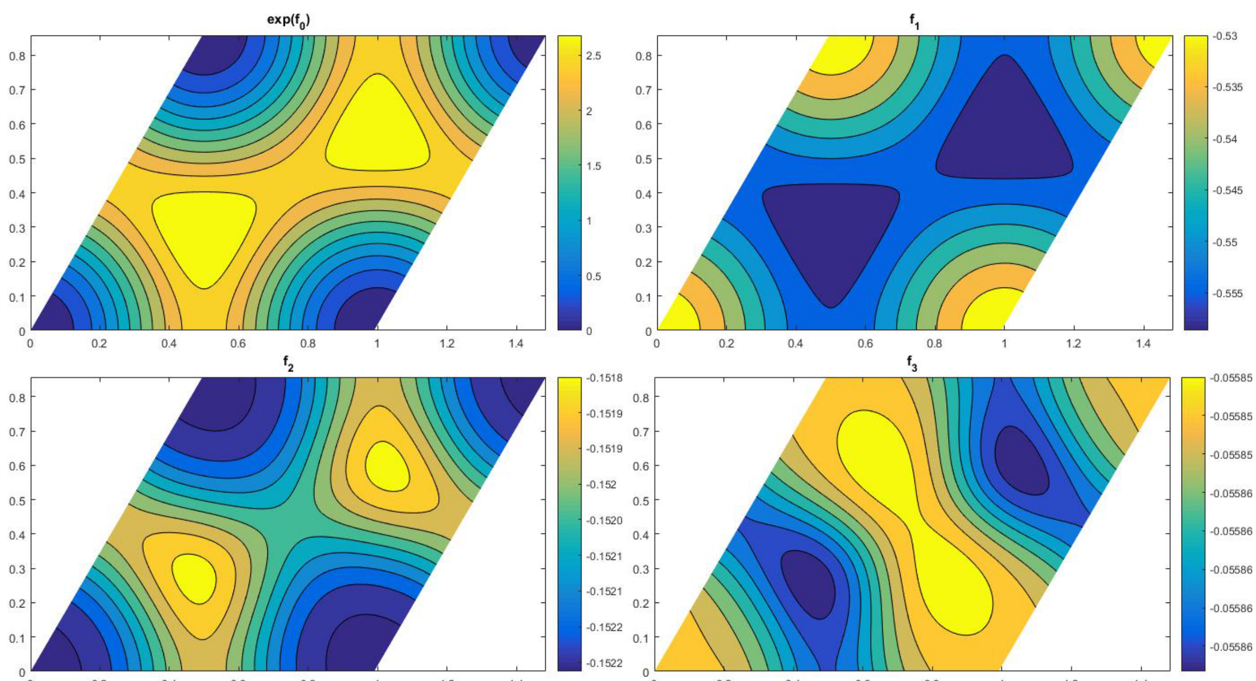

Figure 4: Solutions to equation (5.7) on the torus $\mathbb{C}/(\mathbb{Z}+e^{\pi i/3}\mathbb{Z})$ for f_0 Olesen's solution and $n = 0, 1, 2, 3$.

This means that

$$\|f_n\|_\infty \leq \left\|H^{-1}\right\| \max(|g_n(TG)|) + \mathcal{O}(\max(d_x^3, d_y^3)), \tag{5.37}$$

so the way that the norm of our solution changes as $n \to \infty$ is determined by the norm of g_n. We would like to be able to determine an upper bound for this, but that turns out to be hard. We have done some numerical experiments, trying to determine an upper bound. This led to a peculiar conjecture:

Conjecture 5.1. *Let $n \in \mathbb{N}_{\leq 2}$, then*

$$f_n \approx (-1)^{n+1} \frac{f_1^n}{n}, \tag{5.38}$$

where the margin of error is lower than ∼5%.

This is easy to check with the scales on the figures that we included (note the minus sign on the scales). In fact, we also verified it up to f_8 and the pattern seems to hold.

Because the norm of f_n is strictly smaller than one for each of the cases that we checked, this would mean that our perturbative approach does indeed converge.

The reason that this conjecture is peculiar, is that we can actually 'prove' this equality by induction if we neglect the Laplacian in our differential equation. Indeed, upon inspection of equation (5.6), we see that

$$f_2 = -\frac{f_1^2}{2}, \tag{5.39}$$

when we set $\Delta f_2 = 0$. Suppose that we have verified the equality up to some $n = k$. This means that

$$\begin{aligned} f &= f_0 + af_1 - \frac{a^2 f_1^2}{2} + \frac{a^3 f_1^3}{3} \mp \dots & (5.40) \\ &= f_0 + \log(1 + af_1) + \sum_{j=k+1}^{\infty} \left(f_j - (-1)^{j+1} \frac{f_1^j}{j} \right) a^j, & (5.41) \end{aligned}$$

where we used that $\log(1 + x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}$ for $|x| < 1$. If we substitute this into equation (5.1) and look for the terms of order a^{k+1}, we find that

$$\Delta f_{k+1} + e^{f_0} \left(f_{k+1} - (-1)^{k+2} \frac{f_1^{k+1}}{k+1} \right). \tag{5.42}$$

So if we again neglect the Laplacian, if follows that

$$f_{k+1} = (-1)^{k+2} \frac{f_1^{k+1}}{k+2}, \tag{5.43}$$

as we claimed.

One could imagine extending this procedure to include error terms that allow for

$$f_n = (1+\epsilon)(-1)^{n+1}\frac{f_1^n}{n} \tag{5.44}$$

for some small ϵ. However, this approach seems to lead to some sort of cumulative error, that is in practice hard to bound.

6 Hermite polynomials

In this section we will derive the results that will be used for the computations in chapter 7. These results concern Hermite polynomials and ways to write products of these polynomials as a sum of some new Hermite polynomials. These product expansions are then used to compute a number of integrals and antiderivatives. The first two lemmas and the first corollary are taken from [23].

First recall the definition:

Definition 6.1. *For $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, we define the* $\mathbf{n^{th}}$ ***Hermite polynomial*** *as*

$$H_n(z) := (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \tag{6.1}$$

For the integrals that we will be computing, it is convenient to use the generating function for Hermite polynomials:

Lemma 6.1. *Consider the exponential generating function*

$$\mathcal{H}(t) := \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n. \tag{6.2}$$

We have the equality

$$\mathcal{H}(t) = e^{-t^2+2zt} \tag{6.3}$$

Proof. Notice that

$$\begin{aligned} H_n(z) &= (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} \\ &= e^{z^2} \frac{d^n}{dt^n} e^{-(z-t)^2} \Big|_{t=0} \\ &= \frac{d^n}{dt^n} e^{-t^2+2tz} \Big|_{t=0}. \end{aligned}$$

From definition (6.2) we can see that also

$$H_n(z) = \frac{d^n}{dt^n} \mathcal{H}(t) \Big|_{t=0},$$

which proves the desired equality. □

Using either definition (6.1) or the generating function, we may derive the following relations:

Lemma 6.2. *The Hermite polynomials satisfy the recurrence relations*

$$H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z) \tag{6.4}$$

$$\frac{d}{dz} H_n(z) = 2nH_{n-1}(z), \tag{6.5}$$

for all $1 < n \in \mathbb{N}$.

Proof. For the first relation, observe that

$$\begin{aligned} H_{n+1}(z) &= \frac{d^{n+1}}{dt^{n+1}} e^{-t^2+2tz}\Big|_{t=0} \\ &= \frac{d^n}{dt^n}(-2t+2z)e^{-t^2+2tz}\Big|_{t=0} \\ &= (-2n\frac{d^{n-1}}{dt^{n-1}} + 2z\frac{d^n}{dt^n})e^{-t^2+2tz}\Big|_{t=0} \\ &= -2nH_{n-1}(z) + 2zH_n(z). \end{aligned}$$

The second relation follows from:

$$\begin{aligned} \frac{d}{dz}H_n(z) &= \frac{d}{dz}\frac{d^n}{dt^n} e^{-t^2+2tz}\Big|_{t=0} \\ &= \frac{d^n}{dt^n} 2te^{-t^2+2tz}\Big|_{t=0} \\ &= t\frac{d^n}{dt^n} e^{-t^2+2tz}\Big|_{t=0} + 2n\frac{d^{n-1}}{dt^{n-1}} e^{-t^2+2tz}\Big|_{t=0} \\ &= 0 + 2nH_{n-1}(z). \end{aligned}$$

□

We can use these relations to derive the following results for the superconduction order parameter in the linear approximation

Corollary 6.1. *Let*[3] $\alpha \in \mathbb{R}$ *and define*

$$f_n(x) := H_n(x)e^{-\alpha x^2}. \tag{6.6}$$

Then we have

$$2nf_{n-1}(x) = 2\alpha x f_n(x) + \frac{d}{dx} f_n(x), \tag{6.7}$$

$$f_{n+1}(x) = 2(1-\alpha)x f_n(x) - \frac{d}{dx} f_n(x). \tag{6.8}$$

Proof. This is immediate from lemma 6.2. □

Next, we will derive an expression to write the product of Hermite polynomials as a sum:

Lemma 6.3. *Let* $a \in \mathbb{C}$*, then we have*

$$H_n(z-a)H_m(z+a) = \sum_{k=0}^{n+m} A_k(a,m,n)H_k(x), \tag{6.9}$$

[3] In the order parameter itself, we always have $\alpha = \frac{1}{2}$, but a later calculation will also require this result for a different value of α.

with

$$A_k = \sum_{2\alpha+\beta+\gamma=n+m-k} (-1)^{\beta}(2a)^{\beta+\gamma}2^{\alpha}\binom{n}{\alpha,\beta,n-\alpha-\beta}\binom{m}{\alpha,\gamma,n-\alpha-\gamma} \tag{6.10}$$

where

$$\binom{n}{k_1,\dots,k_r} = \frac{n!}{k_1!\dots k_r!}. \tag{6.11}$$

Proof. We have

$$\sum_{n,m}\frac{H_n(z-a)H_m(z+a)}{n!m!}t^n u^m = e^{-t^2-u^2+2t(z-a)+2u(z+a)} \tag{6.12}$$

$$= e^{-(t+u)^2+2(t+u)z+2tu+2a(u-t)}$$

$$= \sum_{\alpha}\frac{(2tu)^{\alpha}}{\alpha!}\sum_{r}\frac{H_r(z)}{r!}(t+u)^r\sum_{s}\frac{(2a(u-t))^s}{s!}$$

$$= \sum_{\alpha,r,s}\sum_{b=0}^{r}\sum_{\beta=0}^{s}(-1)^{\beta}2^{\alpha}\binom{r}{b}\binom{s}{\beta}\frac{H_r(z)}{r!}\frac{(2a)^s}{s!}t^{\alpha+b+\beta}u^{\alpha+r+s-\beta-d}$$

Now, let

$$\mu = r-b, \qquad \gamma = s-\beta,$$

then the above becomes

$$\sum_{\alpha,\mu,\gamma}\sum_{b=0}^{r}\sum_{\beta=0}^{s}(-1)^{\beta}2^{\alpha}(2a)^{\gamma+c}\frac{H_{\mu+b}(z)}{\mu!\gamma!b!\beta!}t^{\alpha+b+\beta}u^{\alpha+\mu+\gamma}.$$

Furthermore, we set

$$n = \alpha+b+\beta, \qquad m = \alpha+\mu+\gamma, \qquad k = m+n-2\alpha-\beta-\gamma,$$

so that the sum becomes

$$\sum_{n,m}t^n u^m \sum_{\substack{\alpha+b+\beta=n\\ \alpha+\mu+\gamma=m}}(-1)^{\beta}2^{\alpha}(2a)^{\gamma+\beta}\frac{H_{\mu+b}(z)}{\mu!\gamma!b!\beta!}$$

$$=\sum_{n,m}t^n u^m \sum_{\substack{\alpha+\beta\leq n\\ \alpha+\gamma\leq m}}(-1)^{\beta}2^{\alpha}(2a)^{\gamma+\beta}\frac{H_{m+n-2\alpha-\beta-\gamma}(z)}{(m-\alpha-\gamma)!\gamma!(n-\alpha-\beta)!\beta!}$$

$$=\sum_{n,m}t^n u^m \sum_{k}\sum_{\substack{\alpha+c\leq n\\ \alpha+\gamma\leq m\\ 2\alpha+\beta+\gamma=k-m-n}}(-1)^{\beta}2^{\alpha}(2a)^{\gamma+\beta}\frac{H_k(z)}{(m-\alpha-\gamma)!\gamma!(n-\alpha-\beta)!\beta!},$$

which proves the desired when we compare the terms in this sum with the ones in equation (6.12). $\square$

The author readily admits that this may not be the most handsome of expressions, but at least it works.

We can use the previous lemma to derive another expansion that is used to evaluate some integrals:

Proposition 6.1. *Let $a \in \mathbb{C}$, then we have*

$$H_n(x-a)H_{n+1}(x+a) - H_n(x+a)H_{n+1}(x-a) = 4a\sum_{k=0}^{2r} B_k(a,n)H_k(x), \tag{6.13}$$

where

$$B_k(a,n) := \sum_{\substack{0\le\alpha,\beta\le r\\ 2\alpha+2\beta+2\gamma=2n-k}} \frac{n!(n+1)!(-1)^\beta(2a)^{2\beta+2\gamma}2^\alpha}{\alpha!(n-\alpha-\beta-2\gamma)!(\beta+2\gamma+1)!(n-\alpha-\beta)!\beta!}, \tag{6.14}$$

which is zero when k is odd.

Proof. By lemma 6.3, we have

$$\begin{aligned}&H_n(x-a)H_{n+1}(x+a) - H_n(x+a)H_{n+1}(x-a)\\ =&\sum_{k=0}^{2n+1}(A_k(a,n+1,n) - A_k(-a,n+1,n))H_k(x)\end{aligned}$$

Upon inspection of the coefficients before the polynomials, we see that only the even terms survive:

$$\begin{aligned}&A_k(a,n+1,n) - A_k(-a,n+1,n)\\ =&\sum_{\substack{0\le\alpha,\beta\le r\\ 2\alpha+\beta+\gamma=2n+1-k}} \frac{n!(n+1)!(-1)^\beta 2^\alpha}{\alpha!(n-\alpha-\beta-2\gamma)!(\beta+2\gamma+1)!(n-\alpha-\beta)!\beta!}\left((2a)^{\beta+\gamma}-(-2a)^{\beta+\gamma}\right).\end{aligned}$$

When β and γ have the same parity, that term in the sum becomes zero. Therefore, we replace γ by $\beta+2\gamma+1$ to obtain

$$\begin{aligned}&A_k(a,n+1,n) - A_k(-a,n+1,n)\\ =&4a\sum_{\substack{0\le\alpha,\beta\le r\\ 2\alpha+2\beta+2\gamma=2n-k}} \frac{n!(n+1)!(-1)^\beta(2a)^{2\beta+2\gamma}2^\alpha}{\alpha!(n-\alpha-\beta-2\gamma)!(\beta+2\gamma+1)!(n-\alpha-\beta)!\beta!}\\ &= 4aB_k(a,n).\end{aligned}$$

Therefore

$$H_n(x-a)H_{n+1}(x+a) - H_n(x+a)H_{n+1}(x-a) =$$
$$= 4a\sum_{k=0}^{2r+1} B_k(a,n)H_k(x) = 4a\sum_{k=0}^{2r} B_k(a,n)H_k(x),$$

because $B_{2r+1} = 0$. □

With the use of lemma 6.2 and lemma 6.3, we can compute some antiderivatives and integrals that will turn out to be useful.

Proposition 6.2. *Let $a \in \mathbb{C}$, then we have*

$$\int^y xH_n(x+a)H_n(x-a)e^{-\frac{1}{2}(x-a)^2-\frac{1}{2}(x+a)^2}dx = -\sum_{k=0}^{2n} \frac{A_k(a,n,n)}{2}(H_k(y)+2kH_{k-2}(y))e^{-y^2-a^2}, \tag{6.15}$$

where we set

$$H_{-n}(x) = 0 \tag{6.16}$$

Proof. According to the lemmas 6.2 and 6.3, we have

$$xH_n(x-a)H_n(x+a) = \sum_{k=0}^{2n} A_k(a,n,n)xH_k(x) = \sum_{k=0}^{2n} \frac{A_k(a,n,n)}{2}\left(H_{k+1}(x) + 2kH_{k-1}(x)\right);$$

we also have

$$\begin{aligned}
\int^y H_k(x)e^{-\frac{1}{2}(x-a)^2-\frac{1}{2}(x+a)^2}dx &= \int^y H_k(x)e^{-x^2-a^2}dx \\
&= \int^y (-1)^k e^{x^2}\left(\frac{d^k}{dx^k}e^{-x^2}\right)e^{-x^2-a^2}dx \\
&= (-1)^k e^{-a^2}\frac{d^{k-1}}{dy^{k-1}}e^{-y^2} \\
&= -H_{k-1}(y)e^{-y^2-a^2}.
\end{aligned}$$

Combining the above, we find

$$\int^y xH_n(x-a)H_n(x+a)e^{-\frac{1}{2}(x-a)^2-\frac{1}{2}(x+a)^2}dx = -\sum_{k=0}^{2n} \frac{A_k(a,n,n)}{2}(H_k(y)+2kH_{k-2}(y))e^{-y^2-a^2}$$

□

We will need to compute two more integrals:

Proposition 6.3. *Let $a, b \in \mathbb{C}$ and $m, n \in \mathbb{N}_0$, then we have*

$$\int_{-\infty}^{\infty} H_m(x-a)H_n(x-b)e^{-(x-a)^2-(x-b)^2}dx$$
$$= \mathcal{I}(m,n,a,b)e^{-\frac{1}{2}(a-b)^2} := \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b)^2}\sum_{k=0}^{\left\lfloor\frac{m+n}{2}\right\rfloor}\sum_{l=\max(0,2k-n)}^{\min(2k,m)}\binom{2k}{l}\frac{(-2)^{-k}(-1)^m m!n!(a-b)^{m+n-2k}}{k!(m-l)!(n-2k+l)!} \tag{6.17}$$

and

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = \delta_{m,n}\sqrt{\pi}m!2^m. \tag{6.18}$$

Proof. For the first integral, we consider the sum

$$\sum_{t,u}\int_{-\infty}^{\infty}\frac{H_m(x-a)}{m!}\frac{H_n(x-b)}{n!}t^m u^n e^{-(x-a)^2-(x-b)^2} = \int_{-\infty}^{\infty} e^{-t^2-u^2+2t(x-a)+2u(x-b)-(x-a)^2-(x-b)^2}dx \tag{6.19}$$

Completing the square, this integral becomes a standard Gaussian integral that equals

$$\begin{aligned}\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b+t-u)^2} &= \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b)^2-\frac{1}{2}(t-u)^2+(a-b)(u-t)}\\ &= \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b)^2}\sum_{k=0}^{\infty}\sum_{l=0}^{2k}\binom{2k}{l}\frac{(-2)^{-k}t^l u^{2k-l}(-1)^l}{k!}\sum_{p,q}\frac{(a-b)^{p+q}(-t)^p u^q}{p!q!}\end{aligned}$$

We now set

$$m := l + p \qquad\qquad n := q + 2k - l,$$

then we should sum k from 0 to $\left\lfloor\frac{m+n}{2}\right\rfloor$, because $p, q \geq 0$. This means that our sum equals

$$\begin{aligned}&\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b)^2}\sum_{m,n}t^m u^n\sum_{\substack{l+p=m\\ q+2k-l=n\\ k\leq\left\lfloor\frac{m+n}{2}\right\rfloor\\ l\leq 2k}}\binom{2k}{l}\frac{(-2)^{-k}t^l u^{2k-l}(-1)^l}{k!}\frac{(a-b)^{p+q}(-t)^p u^q}{p!q!}\\ &\qquad = \sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}(a-b)^2}\sum_{k=0}^{\left\lfloor\frac{m+n}{2}\right\rfloor}\sum_{l=\max(0,2k-n)}^{\min(2k,m)}\binom{2k}{l}\frac{(-2)^{-k}(-1)^m(a-b)^{m+n-2k}}{k!(m-l)!(n-2k+l)!},\end{aligned}$$

where we replace p by $m-l$ and q by $n+l-2l$ to obtain the final equality. Comparing the terms with those of equation (6.19) we obtain equality (6.17).

For the second integral, observe that

$$\int_{-\infty}^{\infty} \sum_{m,n} \frac{H_m(x)H_n(x)}{m!n!} t^m u^n e^{-x^2} dx = e^{-t^2-u^2+2(t+u)x-x^2} dx \tag{6.20}$$
$$= \sqrt{\pi} e^{-t^2-u^2+(t+u)^2}$$
$$= \sum_m \frac{(2tu)^m}{m!},$$

and therefore

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2} dx = \delta_{m,n}\sqrt{\pi}m!2^m.$$

□

7 Abrikosov's method

In this chapter, we will study a second method to describe a lattice of vortex solutions in Ginzburg-Landau theory. This method is originally due to Abrikosov [3]. His treatment of the case of regular type II superconductors is reviewed and in some places supplemented in the first part of this chapter. In the second part, we will apply the method for the first time to the p-wave superconductor from chapter 2. This new case is much more involved than the original one, because the general formula for the order parameter becomes dependent on the strength of the Zeeman-term in the free energy. When we try to find the energetically optimal lattice, this leads to a different outcome than in the case that was studied by Abrikosov. Indeed, whereas the triangular lattice was always optimal in the original case, we now find a regime in which the rectangular lattice results in a lower free energy.

7.1 Regular type II superconductors

In this section, we will discuss some of the characteristics of superconductors of the second type, starting from the Ginzburg-Landau equations of motion (2.5) and (2.6). We will mostly follow the treatment in the original paper by Abrikosov [3] and the correction to that paper by Kleiner, Roth and Autler [22]. Some derivations that are especially long will be postponed until we apply Abrikosov's method to the p-wave superconductor described in second chapter.

As a starting point of our analysis, we will consider a two-dimensional superconductor or one that is completely isotropic in the z-direction, so that neither the order parameter ψ nor the magnetic potential A depends on z. Furthermore, we will assume that the magnetic field h is pointed along the z-axis.

We will first consider the regime where the magnetic field strength h is only slightly smaller than $h_{\text{crit},2}$. This means that we can use the linearised equations of motion (2.27) and (2.28) to derive an approximate expression for ψ. To somewhat manage the length of the expressions, we will use dimensionless units:

$$\begin{aligned} f &= \sqrt{\frac{g}{|\alpha|}}\psi \\ a &= \frac{A}{\sqrt{2}h_{\text{crit}}\lambda} \\ \tilde{h} &= \frac{h}{\sqrt{2}h_{\text{crit}}} \end{aligned} \tag{7.1}$$

Moreover, we will rescale our coordinates by a factor λ, replacing

$$(x, y) \mapsto \lambda(x, y) \tag{7.2}$$

In these units the free energy becomes

$$F = \frac{h_{\text{crit}}^2\lambda^2}{4\pi}\int d^2x \left\{-(\frac{i\partial_i}{\kappa}f^* - a_i f^*)(\frac{i\partial_i}{\kappa}f + a_i f) + \text{sign}(\alpha)|f|^2 + \frac{|f|^4}{2} + \tilde{h}^2\right\} \tag{7.3}$$

with corresponding equations of motion

$$\left(\frac{i\partial_i}{\kappa} + a_i\right)^2 f + \mathrm{sign}(\alpha)f + |f|^2 f = 0 \tag{7.4}$$

$$-\epsilon_{ijk}\epsilon_{klm}\partial_j\partial_k a_m = \frac{i}{2\kappa}\left(f^*\partial_i f - f\partial_i f^*\right) + a|f|^2. \tag{7.5}$$

Because we consider a system with temperature below T_c, we will assume in the remainder of this section that $\mathrm{sign}(\alpha) = -1$.

As we mentioned earlier, we will assume that $\tilde{h}$ is pointed along the z-axis. To realise this, we will choose a gauge with

$$a_1 = a_3 = 0, \qquad\qquad a_2 = h_0 x \tag{7.6}$$

for some $h_0 \in \mathbb{R}$. This choice of a_2 should be thought of as a first approximation, because the vortex-like solutions we will be looking for will have the magnetic field penetrating periodically through the superconductor. Substituting this a into equation (7.4) and neglecting the non-linear term, we find

$$\frac{-\partial_1^2}{\kappa^2} f + (\frac{i\partial_2}{\kappa} + h_0 x)^2 f = f. \tag{7.7}$$

We will first look for solutions that only depend on u and then build solutions depending on both u and v with it. If f depends only on u, then equation (7.7) becomes:

$$\frac{-\partial_1^2}{\kappa^2} f + h_0^2 x^2 f = f, \tag{7.8}$$

the Schrödinger equation for a harmonic oscillator with eigenvalue 1. This equation has (bounded) solutions whenever

$$h_0 = \frac{\kappa}{2n+1} \tag{7.9}$$

for some $n \in \mathbb{N}_0$. If this the case, then the solutions are given by

$$f_n(u) = C\exp(\frac{-\kappa^2 x^2}{4n+2}) H_n((2n+1)^{-1/2}\kappa x), \tag{7.10}$$

where H_n is the n^{th} Hermite polynomial, defined as

$$H_n(z) := (-1)^n e^{z^2} \frac{\partial^n}{\partial z^n} e^{-z^2}. \tag{7.11}$$

Now, suppose that f is a solution to equation (7.7) that does depend on both x and y. Then, we write[4]

$$f(x, y) = f_0(x)g(x, y) = C\exp(-\frac{\kappa^2 x^2}{2}). \tag{7.12}$$

[4]The reason we choose f_0 here is that it allows for the greatest value of h_0, which means that the approximation $|f|^2 f \approx 0$ is best in this case.

If we substitute this into equation (7.7) for $h_0 = \kappa$ and use the fact that f_0 solves equation equation (7.8), we obtain

$$\frac{-\partial_i^2}{\kappa^2} g + 2x(\partial_1 + i\partial_2)g = 0. \tag{7.13}$$

This equation is solved by any g that we can write as

$$g(u,v) = \tilde{g}(x+iy). \tag{7.14}$$

Because we are looking for doubly-periodic solutions, the function $\tilde{g}$ must (in particular) be periodic in v. This means that we can write it as

$$\tilde{g}(x+iy) = \sum_{n\in\mathbb{Z}} c_n \exp(ikny + knx), \tag{7.15}$$

so that f becomes

$$f(u,v) = \sum_{n\in\mathbb{Z}} C_n \exp(ikny - \frac{\kappa^2}{2}(x - \frac{kn}{\kappa^2})^2) = \sum_{n\in\mathbb{Z}} C_n \exp(ikny)\tilde{f}_n, \tag{7.16}$$

for some constants $C_n \in \mathbb{C}$. If we want this solution to be periodic in u as well, then we must make periodic choices for the constants C_n. In other words, there must be some $N \in \mathbb{N}$ such that

$$C_{n+N} = C_n \qquad\qquad \text{for all } n \in \mathbb{Z} \tag{7.17}$$

The choice of our N will be made by minimising the free energy, but before we can do that, we will need a better approximation for $\tilde{h}$: through equation (7.5), we can see that $\tilde{h}$ should depend on f and because the free energy contains terms of order $\mathcal{O}(|f|^4)$ that we will want to take into account when minimising, we should consider corrections to $\tilde{h}$ (and a) of order $\mathcal{O}(|f|^2)$.

Thus, we will write

$$\tilde{h} = \kappa\hat{e}_3 + \delta\tilde{h}, \qquad\qquad a = \kappa x\hat{e}_2 + \delta a, \tag{7.18}$$

while we still assume that $\tilde{h}$ is pointed along the z-axis and a along the y-axis. Substituting this into equation (7.5) and keeping terms up to order $\mathcal{O}(|f|^2)$, we find

$$-\epsilon_{ijk}\epsilon_{klm}\partial_j\partial_k\delta a_m = \frac{2i}{\kappa}\left(f^*\partial_i f - f\partial_i f^*\right) + \kappa x|f|^2\hat{e}_2 \tag{7.19}$$

Substituting our expression for f from equation (7.16), we find

$$-\partial_1\partial_2\delta a = \sum_{m,n\in\mathbb{Z}} C_m^* C_n \frac{ik(n-m)}{2\kappa}\exp(ik(n-m)y)\tilde{f}_m\tilde{f}_n \tag{7.20}$$

$$\partial_1^2\delta a = \sum_{m,n\in\mathbb{Z}} C_m^* C_n \left(\kappa x - \frac{k(n+m)}{2\kappa}\right)\exp(ik(n-m)y)\tilde{f}_m\tilde{f}_n \tag{7.21}$$

It is easy to verify (with a computer algebra package) that these equations are solved by

$$a(x,y) = H_0 x - \frac{1}{2\kappa}\int_{u_0}^{x} |f(x',y)|dx' \tag{7.22}$$

$$h(x,y) = \partial_1 a(x,y) = H_0 - \frac{1}{2\kappa}|f|^2, \tag{7.23}$$

where u_0 is an arbitrary constant and H_0 satisfies

$$|H_0 - \kappa| = \mathcal{O}(|f|^2). \tag{7.24}$$

The next step is to determine whether this a allows for the existence of solutions to the equation of motion for f. This time, however, we cannot neglect the $|f|^2 f$ term in the equation, because we are considering a up to $\mathcal{O}(|f|^2)$ and a appears multiplied by f. Instead, we will consider corrections to our original solution

$$f \mapsto f + \delta f = \sum_{n\in\mathbb{Z}} \exp(ikny)(C_n \tilde{f}_n + \delta\tilde{f}_n), \tag{7.25}$$

where

$$\delta f = \mathcal{O}(|f|^2). \tag{7.26}$$

Substituting this and (7.22) into equation (7.4), we find up to order $\mathcal{O}(|f|^2 f)$:

$$\begin{aligned}
0 &= \left(\frac{i\partial_i}{\kappa} + a_i\right)^2 (f+\delta f) - f - \delta f + (f+\delta f)|f+\delta f|^2 \\
&= \left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2} + (a_i - \kappa x\delta_{i2})\right)^2 (f+\delta f) - f - \delta f + f|f|^2 \\
&= \left(\left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2}\right)^2 + 2(a_i - \kappa x\delta_{i2})\left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2}\right)\right)(f+\delta f) + f\frac{i\partial_i}{\kappa}(a_i - \kappa x\delta_{i2}) \\
&\quad - f - \delta f + f|f|^2 \\
&= \left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2}\right)^2 \delta f + 2(a_i - \kappa x\delta_{i2})\left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2}\right) f + f\frac{i\partial_i}{\kappa} a_i - \delta f + f|f|^2,
\end{aligned} \tag{7.27}$$

where we used that f is a solution to the linearised equation of motion (7.7) and that $\partial_2 \kappa u \delta_{i2} = 0$ to obtain the last equality. This is a linear inhomogeneous differential equation for δf, where the homogeneous part is the familiar linearised equation of motion for f. When one works out the different parts of this equation, one finds:

$$\begin{aligned}
\left(\frac{i\partial_i}{\kappa} + \kappa x\delta_{i2}\right)^2 \delta f - \delta f &= 2x(\kappa - H_0)\kappa \sum_n \left(x - \frac{kn}{\kappa^2}\right) C_n e^{ikny}\tilde{f}_n + \\
&+ \sum_{m,n,p} C_m^* C_n C_p e^{ik(n-m+p)y}\left\{\left[x - \frac{k}{\kappa^2}\left(n + \frac{p-m}{2}\right)\right]\tilde{f}_n \int^x \tilde{f}_m \tilde{f}_p dx' - \tilde{f}_m\tilde{f}_n\tilde{f}_p\right\}
\end{aligned} \tag{7.28}$$

If we consider each n^{th} power of e^{iky} separately, we find[5] the equation

$$\left(\frac{kn}{\kappa}+\kappa x\right)^2\delta\tilde{f}_n-\frac{1}{\kappa^2}\partial_1^2\delta\tilde{f}_n-\delta\tilde{f}_n=2x(\kappa-H_0)\kappa\sum_n\left(x-\frac{kn}{\kappa^2}\right)C_n e^{ikny}\tilde{f}_n+ \tag{7.29}$$
$$+\sum_{m,p}C_m^*C_{n-p+m}C_p\left\{\left[x-\frac{k}{\kappa^2}\left(n-\frac{p-m}{2}\right)\right]\tilde{f}_{n-p+m}\int^x\tilde{f}_m\tilde{f}_p dx'-\tilde{f}_m\tilde{f}_{n-p+m}\tilde{f}_p\right\}$$

For a general inhomogeneous differential equation of the form

$$Af=B \tag{7.30}$$

for some self-adjoint operator A working on a Hilbert space, one can consider a solution f to the complete differential equation and a solution g to the homogeneous part of the equation. Then

$$0=\langle Ag,f\rangle=\langle g,Af\rangle=\langle g,B\rangle, \tag{7.31}$$

which means that a solution to the homogeneous part of the differential equation must be orthogonal to the inhomogeneous part. We can exploit this fact here, because we already know the solutions to the homogeneous part: these are simply the $\tilde{f}_n$. We will only explicitly perform the necessary integrations when we apply Abrikosov's method to the p-wave superconductor from the second chapter, because they are very long. These lead to the condition

$$\left(\frac{1}{2\kappa^2}-1\right)\sum_{k=1}^N\sum_{m,p}C^*_{m+p+k}C^*_kC_{k+m}C_{k+p}\exp\left(-\frac{k^2}{2\kappa^2}[m^2+p^2]\right)+\sqrt{2}\frac{\kappa-H_0}{\kappa}\sum_{k=1}^N|C_k|^2=0. \tag{7.32}$$

We will show that it is possible to write this as

$$\frac{\kappa-H_0}{\kappa}\overline{|f|^2}+\left(\frac{1}{2\kappa^2}-1\right)\overline{|f|^4}=0, \tag{7.33}$$

where the overline denotes the average over one unit cell. Recall that we required that $C_n=C_{n+N}$ for some $N\in\mathbb{N}$ in order to ensure periodicity of our solution in the x direction. Let $\bar{n}$ denote the equivalence class of n modulo N. Then we may split a sum over all integers into sums over the different equivalence classes

$$\sum_n=\sum_{n\in\bar{0}}+\ldots+\sum_{n\in\bar{N}-\bar{1}}. \tag{7.34}$$

[5] A more general case of this derivation will be treated in the next part, so we will postpone the details for now as to avoid being repetitive.

This means that

$$\overline{|f|^2} = \frac{\kappa^2}{2\pi N}\int_0^{\frac{2\pi}{k}}\int_0^{\frac{kN}{\kappa^2}}\sum_{n,m} C_m^* C_n e^{ik(n-m)y}\tilde{f}_m\tilde{f}_n dxdy \tag{7.35}$$

$$= \frac{\kappa^2}{kN}\int_0^{\frac{kN}{\kappa^2}}\sum_{n} C_n^* C_n \tilde{f}_m\tilde{f}_n dx$$

$$= \frac{\kappa^2}{kN}\int_0^{\frac{kN}{\kappa^2}}\left(\sum_{n\in\bar{0}} + \ldots + \sum_{n\in\bar{N}-\bar{1}}\right) C_n^* C_n \tilde{f}_n\tilde{f}_n dx$$

An individual term in this sum can be computed as follows

$$\sum_{n\in\bar{k}}\int_0^{\frac{kN}{\kappa^2}} C_n^* C_n \exp\left(-\kappa^2\left[x - \frac{kn}{\kappa^2}\right]\right) dx = |C_k|^2 \sum_{n\in\bar{k}} \int_{-\frac{kn}{\kappa^2}}^{\frac{k(N-n)}{\kappa^2}} \exp\left(-\kappa^2 u^2\right) du \tag{7.36}$$

$$= |C_k|^2 \int_{-\infty}^{\infty} \exp\left(-\kappa^2 u^2\right) du \tag{7.37}$$

$$= \frac{\sqrt{\pi}}{\kappa}|C_k|^2. \tag{7.38}$$

In this computation, we used that all C_n inside the summation are equal and substituted $u = x - \frac{kn}{\kappa^2}$ to obtain the first equality. Then, we 'glued' the separate integrals together before finally evaluating the canonical Gaussian integral. We thus find

$$\overline{|f|^2} = \frac{\kappa\sqrt{\pi}}{N}\sum_{k=1}^{N}|C_k|^2. \tag{7.39}$$

To compute $\overline{|\psi|^4}$, we can use the same trick:

$$\overline{|f|^4} = \frac{\kappa^2}{2\pi N}\int_0^{\frac{2\pi}{k}}\int_0^{\frac{kN}{\kappa^2}}\sum_{n,m,p,q} C_m^* C_n^* C_p C_q e^{ik(p+q-n-m)y}\tilde{f}_m\tilde{f}_n\tilde{f}_p\tilde{f}_q dxdy \tag{7.40}$$

$$= \frac{\kappa^2}{kN}\int_0^{\frac{kN}{\kappa^2}}\sum_{n,m,p} C_m^* C_n^* C_{n-p+m} C_p \tilde{f}_m\tilde{f}_n\tilde{f}_{n-p+m}\tilde{f}_p dx$$

$$= \frac{\kappa^2}{kN}\int_0^{\frac{kN}{\kappa^2}}\left(\sum_{n\in\bar{0}} + \ldots + \sum_{n\in\bar{N}-\bar{1}}\right)\sum_{m,p} C_m^* C_n^* C_{n-p+m} C_p \tilde{f}_m\tilde{f}_n\tilde{f}_{n-p+m}\tilde{f}_p dx, \tag{7.41}$$

where we can write the product of Gaussians as

$$\tilde{f}_m\tilde{f}_n\tilde{f}_{n-p+m}\tilde{f}_p = \exp\left(-2\kappa^2\left[x - \frac{k(n+m)}{2\kappa^2}\right]^2 - \frac{k^2}{2\kappa^2}[(m-p)^2 + (n-p)^2]\right) \tag{7.42}$$

If we substitute this into our expression for $\overline{|f|^4}$ and replace m by $m+p$ and then p by $p+n$ we find by the same method as before that

$$\int_0^{\frac{kN}{\kappa^2}} \sum_{n\in\bar{k}} \sum_{m,p} C^*_{m+n+p} C^*_n C_{n+m} C_{n+p} \exp\left(-2\kappa^2 \left[x - \frac{k(2n+m+p)}{2\kappa^2}\right]^2 - \frac{k^2}{2\kappa^2}[m^2+p^2]\right) dx \tag{7.43}$$

$$= \sum_{m,p} C^*_{k+m+p} C^*_k C_{k+m} C_{k+p} \sum_{n\in\bar{k}} \int_{-\frac{n}{\kappa^2}}^{\frac{k(N-n)}{\kappa^2}} \exp\left(-2\kappa^2 \left[u - \frac{k(m+p)}{2\kappa^2}\right]^2 - \frac{k^2}{2\kappa^2}[m^2+p^2]\right) du$$

$$= \sum_{m,p} C^*_{k+m+p} C^*_k C_{k+m} C_{k+p} \int -\infty^{\infty} \exp\left(-2\kappa^2 u^2 - \frac{k^2}{2\kappa^2}[m^2+p^2]\right) du$$

$$= \frac{1}{\kappa}\sqrt{\frac{\pi}{2}} \sum_{m,p} C^*_{k+m+p} C^*_k C_{k+m} C_{k+p} \exp\left(-\frac{k^2}{2\kappa^2}[m^2+p^2]\right)$$

and thus

$$\overline{|f|^4} = \frac{\kappa}{kN}\sqrt{\frac{\pi}{2}} \sum_{k=1}^{N} \sum_{m,p} C^*_{k+m+p} C^*_k C_{k+m} C_{k+p} \exp\left(-\frac{k^2}{2\kappa^2}[m^2+p^2]\right). \tag{7.44}$$

When we compare the equations (7.39) and (7.44) that we just derived with equation (7.32), we see that equation (7.33) does indeed follow.

If we now go back to the free energy (7.3) and use the equations of motion, we find that

$$F = \frac{h^2_{\text{crit}}\lambda^2}{4\pi} \int \left\{h^2 - \frac{|f|^4}{2}\right\} d^2x = \frac{h^2_{\text{crit}}\lambda^2}{4\pi} \int \left\{H_0^2 - \frac{H_0}{\kappa}|f|^2 + \frac{1}{2}\left(\frac{1}{2\kappa^2} - 1\right)\frac{|f|^4}{2}\right\} d^2x \tag{7.45}$$

Rather than minimise this Helmholz free energy, we will minimise the Gibbs free energy, that keeps the external field H_0 constant. According to [22] and [8], the two are related by

$$G = F - \frac{h^2_{\text{crit}}\lambda^2}{4\pi} \int 2H_0 h d^2x = \frac{h^2_{\text{crit}}\lambda^2}{4\pi} \int \left\{-H_0^2 + \left(\frac{1}{2\kappa^2} - 1\right)\frac{|f|^4}{2}\right\} d^2x. \tag{7.46}$$

One feature of interest that is worth pointing out is the fact that for $\kappa^2 = \frac{1}{2}$, which is the border between type I and type II superconductivity, the Gibbs free energy does not depend on f and therefore all possible lattices are degenerate. This is just like what we found when applying the method that led to the Liouville-like equation, where we had to require that $\kappa^2 = \frac{1}{2}$.

If we now take the average of G and use condition (7.33), we find that

$$\overline{G} = \frac{h^2_{\text{crit}}\lambda^2}{4\pi} \left\{-H_0^2 - \frac{(H_0-\kappa)^2}{2\kappa^2-1} \frac{\overline{|f|^2}^2}{\overline{|f|^4}}\right\}. \tag{7.47}$$

As we have seen in chapter 2, one always has $\kappa^2 > \frac{1}{2}$ in a regular type II superconductor, so $\overline{G}$ is minimal when $\frac{\overline{|f|^4}}{\overline{|f|^2}^2}$ is maximal.

The simplest case to study is that of the rectangular lattice, where we can take $N = 1$. This was carried out by Abrikosov and a minimum of $\frac{\overline{|f|^2}^2}{\overline{|f|^4}} = 1.18$ was found for $k = \kappa\sqrt{2\pi}$.

For the next simplest case, we set $N = 2$ and look choices of C_n that produce a triangular lattice. If n is even we set $C_n = C_0$ and if n is odd we set $C_n = C_1$. This case was carried out by Kleiner, Roth and Autler [22]. To ensure that the lattice is indeed triangular, we have to require that

$$|f(x + \frac{1}{2}L_x, y + \frac{1}{2}L_y)|^2 = |f(x, y)|^2 \tag{7.48}$$

where L_α is the length of the unit cell in de α direction. For $N = 2$, this means that $L_x = \frac{2k}{\kappa^2}$ and $L_y = \frac{2\pi}{k}$. If we work out equation (7.48), we find that

$$\begin{aligned} |f(x + \frac{1}{2}L_x, y + \frac{1}{2}L_y)|^2 = \sum_{m,n} \Big\{ & |C_0|^2 e^{ik(2n-2m)\left(y+\frac{\pi}{k}\right)+\frac{\kappa^2}{2}\left(x+\frac{k}{\kappa^2}+\frac{2km}{\kappa^2}\right)^2+\left(x+\frac{k}{\kappa^2}+\frac{2kn}{\kappa^2}\right)^2} + \\ & + C_0^* C_1 e^{ik(2n-2m-1)\left(y+\frac{\pi}{k}\right)+\frac{\kappa^2}{2}\left(x+\frac{k}{\kappa^2}+\frac{k(2m+1)}{\kappa^2}\right)^2+\left(x+\frac{k}{\kappa^2}+\frac{2kn}{\kappa^2}\right)^2} + \\ & + C_1^* C_0 e^{ik(2n+1-2m)\left(y+\frac{\pi}{k}\right)+\frac{\kappa^2}{2}\left(x+\frac{k}{\kappa^2}+\frac{2km}{\kappa^2}\right)^2+\left(x+\frac{k}{\kappa^2}+\frac{k(2n+1)}{\kappa^2}\right)^2} + \\ & + |C_1|^2 e^{ik(2n-2m)\left(y+\frac{\pi}{k}\right)+\frac{\kappa^2}{2}\left(x+\frac{k}{\kappa^2}+\frac{k(2m+1)}{\kappa^2}\right)^2+\left(x+\frac{k}{\kappa^2}+\frac{k(2n+1)}{\kappa^2}\right)^2} \Big\} \\ = \sum_{m,n} \Big\{ & |C_0|^2 e^{ik(2n-2m)y+\frac{\kappa^2}{2}\left(x+\frac{k(2m+1)}{\kappa^2}\right)^2+\left(x+\frac{k(2n+1)}{\kappa^2}\right)^2} - \\ & - C_0^* C_1 e^{ik(2n-2m-1)y+\frac{\kappa^2}{2}\left(x+\frac{2km}{\kappa^2}\right)^2+\left(x+\frac{k(2n+1)}{\kappa^2}\right)^2} - \\ & - C_1^* C_0 e^{ik(2n+1-2m)y+\frac{\kappa^2}{2}\left(x+\frac{k(2m+1)}{\kappa^2}\right)^2+\left(x+\frac{2kn}{\kappa^2}\right)^2} + \\ & + |C_1|^2 e^{ik(2n-2m)y+\frac{\kappa^2}{2}\left(x+\frac{2km}{\kappa^2}\right)^2+\left(x+\frac{k}{\kappa^2}+\frac{2kn}{\kappa^2}\right)^2} \Big\} \end{aligned} \tag{7.49}$$

If we require this to be equal to $|f(x, y)|^2$, we find that

$$C_0 = \pm i C_1 \tag{7.50}$$

Minimising the Gibbs free energy, we find $\frac{\overline{|f|^2}^2}{\overline{|f|^4}} = 1.16$ was found for $k = \kappa\sqrt{\sqrt{3}\pi}$. Therefore, the triangular lattice is energetically slightly favoured over the rectangular one.

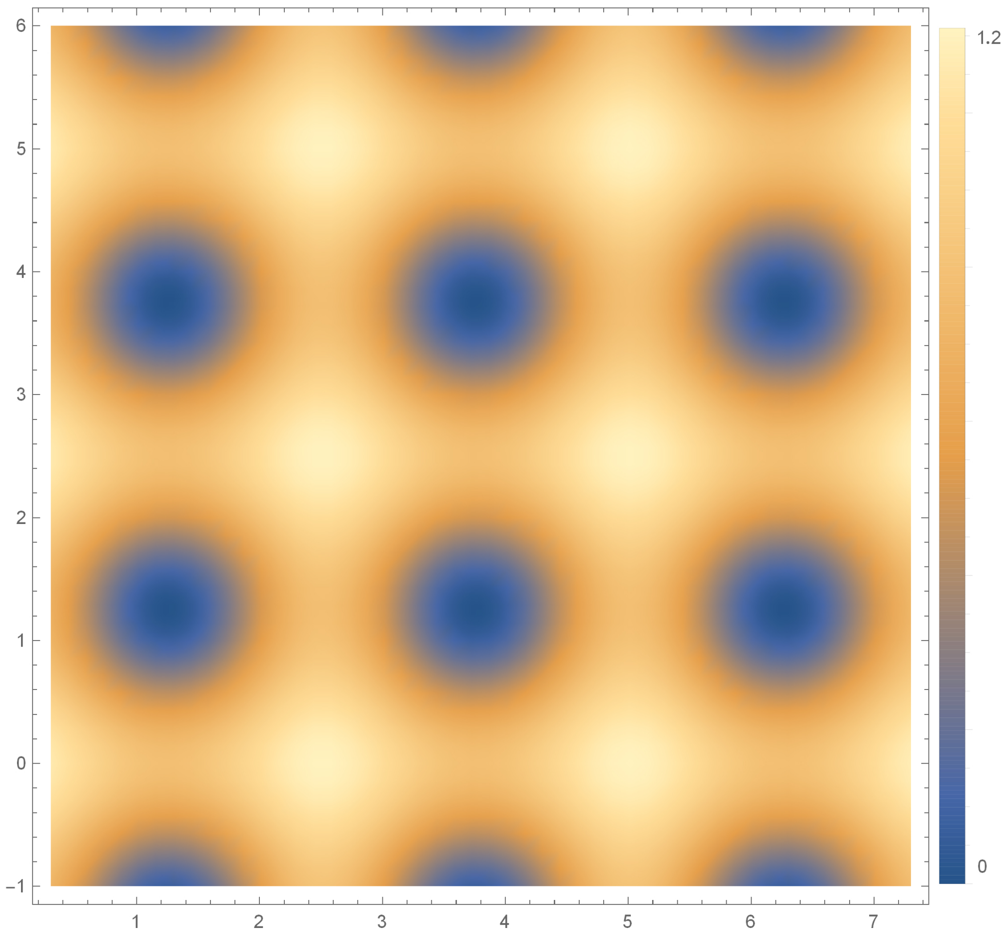

Figure 5: Rectangular lattice solution $|f|$ for $C_n = 1$, $\kappa = 1$ and $k = \kappa\sqrt{2\pi}$.

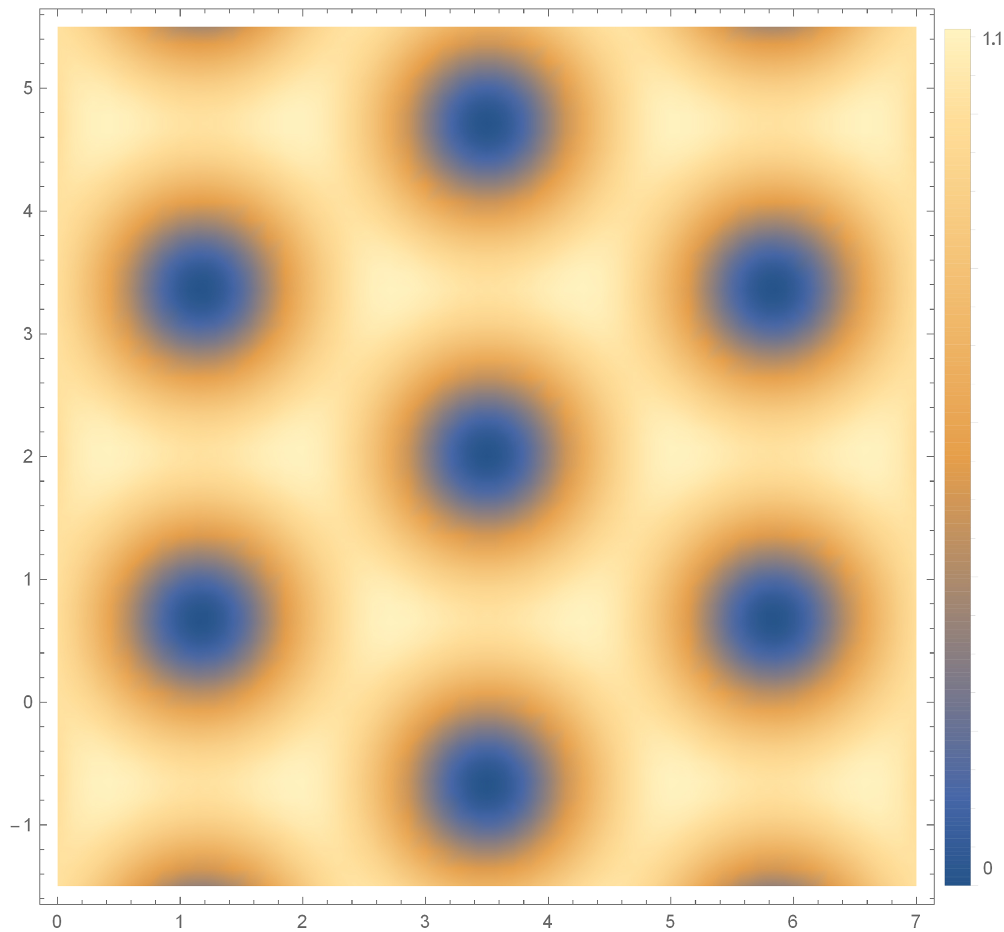

Figure 6: Triangular lattice solution $|f|$ for $|C_n| = 1$, $\kappa = 1$ and $k = \kappa\sqrt{\sqrt{3}\pi}$.

7.2 Application to a p-wave superconductor

We will now study Abrikosov's method applied to the p-wave superconductor described in chapter 2. Like before, we will introduce dimensionless units, to make the derivations more economical. Furthermore, we will assume linear magnetisation, so

$$M = \chi h \tag{7.51}$$

Then we set

$$\begin{aligned} \mu &= \frac{4\pi\chi + C_2}{4\alpha^2}\frac{\beta}{2\pi} \\ x &\mapsto 4ex\sqrt{\frac{\pi|\alpha|C_1}{\beta}} \\ y &\mapsto 4ey\sqrt{\frac{\pi|\alpha|C_1}{\beta}} \\ f &= 2\psi\sqrt{\frac{\beta}{|\alpha|}} \\ a &= \frac{2eA}{\sqrt{|\alpha|C_1}} \\ \tilde{h} &= h\sqrt{\frac{\beta}{4\pi\alpha^2}} \end{aligned} \tag{7.52}$$

If we furthermore define

$$\kappa = \frac{1}{4C_1}\sqrt{\frac{\beta}{\pi}}, \tag{7.53}$$

then the free energy (2.42) becomes

$$F = \frac{1}{32\pi e^2 C_1}\int d^2x \left\{ -(\frac{i\nabla f^*}{\kappa} - af^*)(\frac{i\nabla f}{\kappa} + af) - \mu\tilde{h}|f|^2 + \text{sign}(\alpha)|f|^2 + \frac{1}{2}|f|^4 + \tilde{h}^2 \right\}, \tag{7.54}$$

which leads to the following equations of motion

$$\left(\frac{i\nabla}{\kappa}f + a\right)^2 f = (1+\mu\tilde{h})f - f|f|^2 \tag{7.55}$$

$$-\nabla\times\nabla\times a = a|f|^2 + \frac{i}{2\kappa}(f^*\nabla f - f\nabla f^*) - \mu\begin{pmatrix}\partial_2 \\ -\partial_1 \\ 0\end{pmatrix}|f|^2. \tag{7.56}$$

Again, we be interested in the case of a strong magnetic field and we will start by solving the linearised equation of motion, ignoring the $f|f|^2$ term. We also again assume a constant magnetic field $\tilde{h} = b\hat{e}_2$ This time, it will be more convenient to immediately assume periodicity in the y-coordinate and write

$$f(x,y) = \sum_n e^{ikny} g_n\left(b\left(x - \frac{kn}{\kappa b}\right)\right). \tag{7.57}$$

Then the gradient term becomes

$$\left(-\frac{\Delta}{\kappa^2} + \frac{2ibx}{\kappa}\partial_2 + b^2x^2\right) g(x)h(y) = \sum_n e^{ikny}\left(\frac{-\partial_x^2}{\kappa^2} + \frac{k^2n^2}{\kappa^2} - \frac{2knbx}{\kappa} + b^2x^2\right) g_n(x - \frac{kn}{\kappa b}). \tag{7.58}$$

Now define

$$x_n := b(x - \frac{kn}{\kappa b}), \tag{7.59}$$

so that the gradient term (7.58) becomes

$$e^{ikny}\left(-\frac{b^2}{\kappa^2}\partial_1^2 + x_n^2\right) g_n(x_n). \tag{7.60}$$

We thus obtain the following linearised equation of motion

$$\left(-\frac{b^2}{\kappa^2}\partial_1^2 + x_n^2\right) g_n(x_n) = (1+\mu b) g_n(x_n). \tag{7.61}$$

This equation shares some similarities with the one that we obtained for the regular superconductor. However, the b-dependence of the right hand side will turn out to make a crucial difference when finding the solution that allows for the strongest magnetic field. Indeed, this equation has bounded solutions when

$$b = \frac{\kappa}{1+2r-\kappa\mu}, \qquad r \in \mathbb{Z}, \tag{7.62}$$

with the restriction that b should be positive if and only if r is positive. For simplicity, we will assume that $b > 0$. Then the solutions can be written as

$$g_n(x) = C_n H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) \exp\left(-\frac{\kappa b}{2}\left(x - \frac{kn}{\kappa b}\right)^2\right), \tag{7.63}$$

so the full solution becomes

$$f(x,y) = \sum_n C_n H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) \exp\left(ikny - \frac{\kappa b}{2}\left(x - \frac{kn}{\kappa b}\right)^2\right) = \sum_n C_n f_{r,n}(x) e^{ikny}. \tag{7.64}$$

Remark 7.1. *From now on, we will keep b fixed. This means that the same value for b will be used inside $f_{r,n}$ and $f_{s,n}$, even though r and s may be different. This will be convenient when using ladder operators in the parts to come.*

It is clear from equation (7.62) that the strongest magnetic field is obtained when

$$r = \left\lceil \frac{\kappa\mu - 1}{2} \right\rceil . \tag{7.65}$$

This means that more complicated solutions have to be taken into account than before. However, we will not despair and proceed to compute the correction δa the magnetic field up to $\mathcal{O}(|f|^2)$. Taking the first of the equations of motion for the vector potential (7.56), we find that

$$\begin{aligned}
-\partial_1\partial_2\delta a =& \frac{i}{2\kappa}\left(f^*\partial_1 f - f\partial_1 f^*\right) - \mu\partial_2|f|^2 \qquad (7.66)\\
=&\frac{i}{2\kappa}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}\Bigg\{ H_r\left(\sqrt{\kappa b}\left(x - \frac{km}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{km}{\kappa b}\right)^2} \partial_x H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{kn}{\kappa b}\right)^2} - \\
&- H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{kn}{\kappa b}\right)^2} \partial_x H_r\left(\sqrt{\kappa b}\left(x - \frac{km}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{km}{\kappa b}\right)^2}\Bigg\} - \mu\partial_2|f|^2\\
=&\frac{i}{2\kappa}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}\Bigg\{ H_r\left(\sqrt{\kappa b}\left(x - \frac{km}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{km}{\kappa b}\right)^2} \times\\
&\times\left[\kappa b\left(x - \frac{kn}{\kappa b}\right) H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{kn}{\kappa b}\right)^2} - \sqrt{\kappa b} H_{r+1}\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{kn}{\kappa b}\right)^2}\right]\\
&- H_r\left(\sqrt{\kappa b}\left(x - \frac{kn}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{kn}{\kappa b}\right)^2}\Bigg[\kappa b\left(x - \frac{km}{\kappa b}\right) H_r\left(\sqrt{\kappa b}\left(x - \frac{km}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{km}{\kappa b}\right)^2} - \\
&- \sqrt{\kappa b} H_{r+1}\left(\sqrt{\kappa b}\left(x - \frac{km}{\kappa b}\right)\right) e^{-\frac{\kappa b}{2}\left(x-\frac{km}{\kappa b}\right)^2}\Bigg]\Bigg\} - \mu\partial_2|f|^2\\
=&\frac{i}{2\kappa}\left(k(m-n) f_{r,m} f_{r,n} + \sqrt{\kappa b}\left(f_{r+1,m} f_{r,n} - f_{r+1,n} f_{r,m}\right)\right) - \mu\partial_2|f|^2.
\end{aligned}$$

To obtain the third equality, we used lemma 6.2 from chapter 6. Integrating this expression with respect to y, we find up to an integration constant that

$$\partial_1\delta a = \left(\frac{1}{2\kappa} + \mu\right)|f|^2 - \frac{1}{2}\sqrt{\frac{b}{\kappa}}\sum_{m,n} C_m^* C_n e^{ik(n-m)y} \frac{f_{r+1,m} f_{r,n} - f_{r,m} f_{r+1,n}}{k(n-m)}. \tag{7.67}$$

Next, we need to verify that this expression is consistent with the second equation of motion (7.56) for a. This one reads

$$\partial_1^2\delta a = bx|f|^2 + \frac{i}{2\kappa}(f^*\partial_2 f - f\partial_2 f^*) + \mu\partial_1|f|^2. \tag{7.68}$$

To do this, we first compute

$$\partial_1(f_{r+1,m}f_{r,n} - f_{r,m}f_{r+1,n}) = f_{r+1,m}\partial_1 f_{r,n} + f_{r,n}\partial_1 f_{r+1,m} - f_{r,m}\partial_1 f_{r+1,n} - f_{r+1,n}\partial_1 f_{r,m} \tag{7.69}$$

$$\begin{aligned}
=& f_{r+1,m}\left(\kappa b\left(x - \frac{kn}{\kappa b}\right) f_{r,n} - \sqrt{\kappa b} f_{r+1,n}\right) + f_{r,n}\left(-\kappa b\left(x - \frac{km}{\kappa b}\right) f_{r+1,m} + 2\sqrt{\kappa b}(r+1) f_{r,m}\right) + \\
&+ f_{r+1,n}\left(\kappa b\left(x - \frac{km}{\kappa b}\right) f_{r,m} - \sqrt{\kappa b} f_{r+1,m}\right) + f_{r,m}\left(-\kappa b\left(x - \frac{kn}{\kappa b}\right) f_{r+1,n} + 2\sqrt{\kappa b}(r+1) f_{r,n}\right) \\
=& - k(n-m)(f_{r,n}f_{r+1,m} + f_{r,m}f_{r+1,n}),
\end{aligned}$$

Where we used corollary 6.1 to obtain the second equality. Using this result, we find that the left hand side of equation (7.68) equals

$$\begin{aligned}
\partial_1^2 \delta a =& \frac{1}{2\kappa}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}(f_{r,m}\partial_1 f_{r,n} + f_{r,n}\partial_1 f_{r,m}) + \\
&+ \frac{1}{2}\sqrt{\frac{b}{\kappa}}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}(f_{r,n}f_{r+1,m} + f_{r,m}f_{r+1,n}) + \mu\partial_1|f|^2.
\end{aligned} \tag{7.70}$$

This leaves us with the task of computing the right hand side of equation (7.68). This yields

$$\begin{aligned}
& bx|f|^2 + \frac{i}{2\kappa}(f^*\partial_2 f - f\partial_2 f^*) + \mu\partial_1|f|^2 \\
=& \sum_{m,n} C_m^* C_n e^{ik(n-m)y}\left(bx - \frac{k(n+m)}{2\kappa}\right) f_{r,m}f_{r,n} + \mu\partial_1|f|^2 \\
=& \sum_{m,n} C_m^* C_n e^{ik(n-m)y}\left\{f_{r,n}\left(\frac{bx}{2} - \frac{km}{2\kappa}\right) f_{r,m} + f_{r,m}\left(\frac{bx}{2} - \frac{kn}{2\kappa}\right) f_{r,n}\right\} + \mu\partial_1|f|^2 \\
=& \sum_{m,n} C_m^* C_n e^{ik(n-m)y}\left\{f_{r,n}\left(\frac{1}{2\kappa}\partial_1 f_{r,m} + \frac{1}{2}\sqrt{\frac{b}{\kappa}} f_{r+1,m}\right) + f_{r,m}\left(\frac{1}{2\kappa}\partial_1 f_{r,n} + \frac{1}{2}\sqrt{\frac{b}{\kappa}} f_{r+1,n}\right)\right\} + \\
&+ \mu\partial_1|f|^2 \\
=& \frac{1}{2\kappa}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}(f_{r,m}\partial_1 f_{r,n} + f_{r,n}\partial_1 f_{r,m}) + \\
&+ \frac{1}{2}\sqrt{\frac{b}{\kappa}}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}(f_{r,n}f_{r+1,m} + f_{r,m}f_{r+1,n}) + \mu\partial_1|f|^2,
\end{aligned} \tag{7.71}$$

which we readily see to be equal to the left hand side. Therefore, we find that

$$\tilde{h}_3 = h_0 + \left(\frac{1}{2\kappa} + \mu\right)|f|^2 - \frac{1}{2}\sqrt{\frac{b}{\kappa}}\sum_{m,n} C_m^* C_n e^{ik(n-m)y} \frac{f_{r,n}f_{r+1,m} - f_{r,m}f_{r+1,n}}{k(n-m)} \tag{7.72}$$

$$a_2 = h_0 x + \left(\frac{1}{2\kappa} + \mu\right)\int^x d^2x|f|^2 - \frac{1}{2}\sqrt{\frac{b}{\kappa}}\sum_{m,n} C_m^* C_n e^{ik(n-m)y}\int^x d^2x \frac{f_{r,n}f_{r+1,m} - f_{r,m}f_{r+1,n}}{k(n-m)} \tag{7.73}$$

for some constant h_0. The next step is to determine an equation for the correction δf to the order parameter f that this magnetic field will cause. Using a derivation identical to the one that led to equation (7.27), we find up to $\mathcal{O}(f^3)$:

$$0 = \left(\frac{i\nabla}{\kappa} + bx\hat{e}_2\right)^2 \delta f + 2\left(a - bx\hat{e}_2\right)\left(\frac{i\nabla}{\kappa} + bx\hat{e}_2\right) f + f\frac{i\nabla}{\kappa}\left(a - bx\hat{e}_2\right) - \delta f + f|f|^2. \tag{7.74}$$

We will work out the inhomogeneous part of this differential equation. We have

$$\begin{aligned} &2\left(a - bx\hat{e}_2\right)\left(\frac{i\nabla}{\kappa} + bx\hat{e}_2\right) f = \qquad (7.75)\\ =&2x(h_0 - b)\left(bx - \frac{kn}{\kappa}\right)\sum_n C_n e^{ikny} f_{r,n} + 2\left(bx - \frac{kn}{\kappa}\right)\sum_{m,n,p} C_m^* C_n C_p e^{ik(n+p-m)y} f_{r,n} \times \\ &\times \int^x d\tilde{x}\left\{\left(\frac{1}{2\kappa} + \mu\right) f_{r,p}f_{r,p} - \frac{1}{2}\sqrt{\frac{b}{\kappa}}\frac{f_{r,p}f_{r+1,m} - f_{r,m}f_{r+1,p}}{k(p-m)}\right\}. \end{aligned}$$

and

$$\begin{aligned} f\frac{i\nabla}{\kappa}\left(a - bx\hat{e}_2\right) = &\sum_{m,n,p} C_m^* C_n C_p e^{ik(n+p-m)y} f_{r,n}\left\{\left(\frac{1}{2\kappa} + \mu\right)\frac{-k}{\kappa}(p-m)\int^x d\tilde{x} f_{r,m}f_{r,p}\right. \\ &\left. + \frac{k(p-m)}{2\kappa}\sqrt{\frac{b}{\kappa}}\int^x d\tilde{x}\frac{f_{r,p}f_{r+1,m} - f_{r,m}f_{r+1,p}}{k(p-m)}\right\}. \end{aligned} \tag{7.76}$$

We will add these terms as well as the $f|f|^2$-term and consider each power of e^{iky} separately. This means that we have to replace n by $n - p + m$ in the triple sums. Therefore, the inhomogeneous part equals

$$\begin{aligned} 2bx(h_0 - b)\left(x - \frac{kn}{\kappa b}\right) C_n f_{r,n} + \sum_{m,p} C_{n-p+m} C_m^* C_p &\left\{2\left(\frac{1}{2\kappa} + \mu\right) b\left(x - \frac{k}{\kappa b}\left(n - \frac{p-m}{2}\right)\right)\times\right. \\ &\times f_{r,n-p+m}\int^x d\tilde{x} f_{r,p}f_{r,m} \\ - b\sqrt{\frac{b}{\kappa}}\left(x - \frac{k}{\kappa b}\left(n - \frac{p-m}{2}\right)\right) f_{r,n-p+m}\int^x d\tilde{x}&\left.\frac{f_{r,p}f_{r+1,m} - f_{r,m}f_{r+1,p}}{k(p-m)} + f_{r,n-p+m}f_{r,m}r_{r,p}\right\} \end{aligned} \tag{7.77}$$

Again, the solution to the homogeneous part should be orthogonal to the inhomogeneous part. This means that we must multiply by $f_{r,n}$ and integrate. We will do so term by term, using the results from chapter 6 to evaluate the integrals in the end.

First, we evaluate the integral

$$\begin{aligned}
\int_{-\infty}^{\infty} x\left(x-\frac{kn}{\kappa b}\right) f_{r,n}^2 dx &= \int_{-\infty}^{\infty} \left(x-\frac{kn}{\kappa b}\right)^2 f_{r,n}^2 dx && (7.78)\\
&= \int_{-\infty}^{\infty} \left(\left(x-\frac{kn}{\kappa b}\right) H_r\left(\kappa b\left(x-\frac{kn}{\kappa b}\right)\right)\right)^2 \exp\left(-\kappa b\left(x-\frac{kn}{\kappa b}\right)^2\right) dx\\
&= (\kappa b)^{-3/2} \int_{-\infty}^{\infty} \left(xH_r(x)\right)^2 e^{-x^2} dx\\
&= \frac{(\kappa b)^{-3/2}}{4} \int_{-\infty}^{\infty} \left(H_{r+1}(x)+2rH_r(x)\right)^2 e^{-x^2} dx\\
&= (\kappa b)^{-3/2}\sqrt{\pi}2^{r-1}(2r+1)r! && (7.79)
\end{aligned}$$

To obtain the first equality, we used that $\frac{kn}{\kappa b}\left(x-\frac{kn}{\kappa b}\right) f_{r,n}^2$ is an odd function, so we could freely subtract it from the integrand. For the third equality, we made the substitution $\sqrt{\kappa b}\left(x-\frac{kn}{\kappa b}\right) \to x$. We used corollary 6.1 for the third equality and we used lemma 6.3 for the last equality.

The next integral that we need to compute is

$$\int_{-\infty}^{\infty}\left(x-\frac{k}{\kappa b}\left(n-\frac{p-m}{2}\right)\right)f_{r,n}f_{r,n-p+m}\int^{x}f_{r,p}f_{r,m}d\tilde{x}dx \tag{7.80}$$

$$= (\kappa b)^{-1}\int_{-\infty}^{\infty}yH_r\left(y-\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2}\right)H_r\left(y+\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2}\right)\times$$

$$\times\exp\left(-\frac{1}{2}\left(y-\frac{k}{\kappa b}\frac{p-m}{2}\right)^2-\frac{1}{2}\left(y+\frac{k}{\kappa b}\frac{p-m}{2}\right)^2\right)\int^{\frac{y}{\sqrt{\kappa b}}+\frac{k}{\kappa b}\left(n-\frac{p-m}{2}\right)}f_{r,p}f_{r,m}d\tilde{x}dy$$

$$= (\kappa b)^{-3/2}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{2r}\frac{1}{2}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_j(y)+\sum_{j=2}^{2r}jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_{j-2}(y)\right\}\times$$

$$\times H_r\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{m+p}{2}\right)\right)H_r\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{3p-m}{2}\right)\right)\times$$

$$\times\exp\left(-y^2-\left(y+\frac{k(n-p)}{\sqrt{\kappa b}}\right)^2-\frac{k^2(m-p)^2}{2\kappa b}\right)dy$$

$$= (\kappa b)^{-3/2}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{2r}\frac{1}{2}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_j(y)+\sum_{j=2}^{2r}jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_{j-2}(y)\right\}\times$$

$$\times\sum_{l=0}^{2r}A_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_l\left(y+\frac{k}{\sqrt{\kappa b}}(n-p)\right)\times$$

$$\times\exp\left(-y^2-\left(y+\frac{k(n-p)}{\sqrt{\kappa b}}\right)^2-\frac{k^2(m-p)^2}{2\kappa b}\right)dy$$

$$= (\kappa b)^{-3/2}\Bigg\{\sum_{j,l=0}^{2r}\frac{1}{2}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)A_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)\mathcal{I}\left(j,l,0,\frac{k(p-n)}{\sqrt{\kappa b}}\right)+$$

$$+\sum_{\substack{j=2\\l=0}}^{2r}jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)A_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)\mathcal{I}\left(j-2,l,0,\frac{k(p-n)}{\sqrt{\kappa b}}\right)\Bigg\}e^{-\frac{k^2(m-p)^2}{2\kappa b}-\frac{k^2(n-p)^2}{2\kappa b}}$$

$$= (\kappa b)^{-3/2}\mathcal{P}_1\left(r,\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},\frac{k}{\sqrt{\kappa b}}\frac{p-n}{2}\right)e^{-\frac{k^2(m-p)^2}{2\kappa b}-\frac{k^2(n-p)^2}{2\kappa b}}.$$

In the first step, we made the substitution $y=\sqrt{\kappa b}x-\frac{k}{\sqrt{\kappa b}}\left(n-\frac{p-m}{2}\right)$. Then, we used proposition 6.2 to obtain the second equality. The third equality follows from lemma 6.3 and the last equality from proposition 6.3.

The next integral is similar to the previous one:

$$\int_{-\infty}^{\infty}\left(x-\frac{k}{\kappa b}\left(n-\frac{p-m}{2}\right)\right)f_{r,n}f_{r,n-p+m}\int^{x}\frac{f_{r,p}f_{r+1,m}-f_{r+1,p}f_{r,m}}{k(p-m)}d\tilde{x}dx \tag{7.81}$$

$$= (\kappa b)^{-1}\int_{-\infty}^{\infty} yH_r\left(y-\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2}\right)H_r\left(y+\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2}\right)\times$$

$$\times\exp\left(-\frac{1}{2}\left(y-\frac{k}{\kappa b}\frac{p-m}{2}\right)^2-\frac{1}{2}\left(y+\frac{k}{\kappa b}\frac{p-m}{2}\right)^2\right)\times$$

$$\times\int^{\frac{y}{\sqrt{\kappa b}}+\frac{k}{\kappa b}\left(n-\frac{p-m}{2}\right)}\frac{f_{r,p}f_{r+1,m}-f_{r+1,p}f_{r,m}}{k(p-m)}d\tilde{x}dy$$

$$= (\kappa b)^{-3/2}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{2r}\frac{1}{2}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_j(y)+\sum_{j=0}^{2r}jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_{j-2}(y)\right\}\times$$

$$\times\left\{\frac{H_r\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{3p-m}{2}\right)\right)H_{r+1}\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{m+p}{2}\right)\right)}{k(p-m)}-\right.$$

$$\left.-\frac{H_{r+1}\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{3p-m}{2}\right)\right)H_r\left(y+\frac{k}{\sqrt{\kappa b}}\left(n-\frac{m+p}{2}\right)\right)}{k(p-m)}\right\}\times$$

$$\times\exp\left(-y^2-\left(y+\frac{k(n-p)}{\sqrt{\kappa b}}\right)^2-\frac{k^2(m-p)^2}{2\kappa b}\right)dy$$

$$= (\kappa b)^{-2}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{2r}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_j(y)+\sum_{j=2}^{2r}2jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)H_{j-2}(y)\right\}\times$$

$$\times\sum_{l=0}^{2r}B_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r\right)H_l\left(y+\frac{k}{\sqrt{\kappa b}}(n-p)\right)\times$$

$$\times\exp\left(-y^2-\left(y+\frac{k(n-p)}{\sqrt{\kappa b}}\right)^2-\frac{k^2(m-p)^2}{2\kappa b}\right)dy$$

$$= (\kappa b)^{-2}\left\{\sum_{j,l=0}^{2r}A_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)B_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r\right)\mathcal{I}\left(j,l,0,\frac{k(p-n)}{\sqrt{\kappa b}}\right)+\right.$$

$$\left.+\sum_{\substack{j=2\\l=0}}^{2r}2jA_j\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r,r\right)B_l\left(\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},r\right)\mathcal{I}\left(j-2,l,0,\frac{k(p-n)}{\sqrt{\kappa b}}\right)\right\}e^{-\frac{k^2(m-p)^2}{2\kappa b}-\frac{k^2(n-p)^2}{2\kappa b}}$$

$$= (\kappa b)^{-2}\mathcal{P}_2\left(r,\frac{k}{\sqrt{\kappa b}}\frac{p-m}{2},\frac{k}{\sqrt{\kappa b}}\frac{p-n}{2}\right)e^{-\frac{k^2(m-p)^2}{2\kappa b}-\frac{k^2(n-p)^2}{2\kappa b}}.$$

In the first step, we again made the substitution $y=\sqrt{\kappa b}x-\frac{k}{\sqrt{\kappa b}}\left(n-\frac{p-m}{2}\right)$. For the next step we applied proposition 6.2. Then we used lemma 6.3 and the last equality follows from proposition 6.3.

This leaves us with one last integral

$$\int_{-\infty}^{\infty} f_{r,m} f_{r,n} f_{r,p} f_{r,n-p+m} dx \tag{7.82}$$
$$= (\kappa b)^{-1/2} \int_{-\infty}^{\infty} H_r\left(y - \frac{km}{\sqrt{\kappa b}}\right) H_r\left(y - \frac{kn}{\sqrt{\kappa b}}\right) H_r\left(y - \frac{kp}{\sqrt{\kappa b}}\right) H_r\left(y - \frac{k(n-p+m)}{\sqrt{\kappa b}}\right) \times$$
$$\exp\left(-\frac{1}{2}\left(\left(y - \frac{km}{\sqrt{\kappa b}}\right)^2 + \left(y - \frac{kn}{\sqrt{\kappa b}}\right)^2 + \left(y - \frac{kp}{\sqrt{\kappa b}}\right)^2 + \left(y - \frac{k(n-p+m)}{\sqrt{\kappa b}}\right)^2\right)\right) dy$$
$$= (\kappa b)^{-1/2} \int_{-\infty}^{\infty} \sum_{j,l=0}^{2r} H_j\left(y - \frac{k(m+p)}{2\sqrt{\kappa b}}\right) A_j\left(\frac{k(p-m)}{2\sqrt{\kappa b}}, r, r\right) H_l\left(y - \frac{k(2n-p+m)}{2\sqrt{\kappa b}}\right) \times$$
$$A_l\left(\frac{k(p-m)}{2\sqrt{\kappa b}}, r, r\right) \exp\left(-\left(y - \frac{k(m+p)}{2\sqrt{\kappa b}}\right)^2 - \left(y - \frac{k(2n-p+m)}{2\sqrt{\kappa b}}\right)^2 - \frac{k^2(m-p)^2}{2\kappa b}\right) dy$$
$$= (\kappa b)^{-1/2} \sum_{j,l=0}^{2r} A_j\left(\frac{k(p-m)}{2\sqrt{\kappa b}}, r, r\right) A_l\left(\frac{k(p-m)}{2\sqrt{\kappa b}}, r, r\right) \mathcal{I}\left(j, l, \frac{k(p-n)}{\sqrt{\kappa b}}, 0\right) e^{-\frac{k^2(m-p)^2}{2\kappa b} - \frac{k^2(n-p)^2}{2\kappa b}}$$
$$= (\kappa b)^{-1/2} \mathcal{P}_3\left(r, \frac{k}{\sqrt{\kappa b}} \frac{p-m}{2}, \frac{k}{\sqrt{\kappa b}} \frac{p-n}{2}\right) e^{-\frac{k^2(m-p)^2}{2\kappa b} - \frac{k^2(n-p)^2}{2\kappa b}}.$$

First, we substituted $y = \sqrt{\kappa b}x$, then we applied lemma 6.3 to $H_r\left(y - \frac{km}{\kappa b}\right) H_r\left(y - \frac{kp}{\kappa b}\right)$ and to $H_r\left(y - \frac{kn}{\kappa b}\right) H_r\left(y - \frac{k(n-p+m)}{\kappa b}\right)$. This the put us in a position to use proposition 6.3 to evaluate the integral.

So, the condition on the inhomogeneous part (7.77) implies that

$$\sqrt{\pi} 2^r (2r+1) r! \kappa (h_0 - b) C_n + \sum_{m,p} C_{n-p+m} C_m^* C_p \Bigg\{ (1+2\kappa\mu) \mathcal{P}_1\left(r, \frac{k}{\sqrt{\kappa b}} \frac{p-m}{2}, \frac{k}{\sqrt{\kappa b}} \frac{p-n}{2}\right) -$$
$$- \mathcal{P}_2\left(r, \frac{k}{\sqrt{\kappa b}} \frac{p-m}{2}, \frac{k}{\sqrt{\kappa b}} \frac{p-n}{2}\right) + \kappa^2 \mathcal{P}_3\left(r, \frac{k}{\sqrt{\kappa b}} \frac{p-m}{2}, \frac{k}{\sqrt{\kappa b}} \frac{p-n}{2}\right) \Bigg\} e^{-\frac{k^2(m-p)^2}{2\kappa b} - \frac{k^2(n-p)^2}{2\kappa b}} = 0. \tag{7.83}$$

This expression can be simplified: we first replace $p-m$ by m and *then* replace $p-n$ by p. We thus find after multiplying by C_n^*

$$\sqrt{\pi} 2^r (2r+1) r! \kappa (h_0 - b) |C_n|^2 + \sum_{m,p} C_n^* C_{n-m} C_m^* C_{n+p} \Bigg\{ (1+2\kappa\mu) \mathcal{P}_1\left(r, \frac{km}{2\sqrt{\kappa b}}, \frac{kp}{2\sqrt{\kappa b}}\right) -$$
$$- \mathcal{P}_2\left(r, \frac{km}{2\sqrt{\kappa b}}, \frac{kp}{2\sqrt{\kappa b}}\right) + \kappa^2 \mathcal{P}_3\left(r, \frac{km}{2\sqrt{\kappa b}}, \frac{kp}{2\sqrt{\kappa b}}\right) \Bigg\} e^{-\frac{k^2 m^2}{2\kappa b} - \frac{k^2 p^2}{2\kappa b}} = 0. \tag{7.84}$$

We are now in a position to minimise the free energy. In this case, it is most convenient to minimise the Gibbs free energy, because the manipulations with the expected value of $\tilde{h}$ that are needed to minimise the Helmholz free energy are cumbersome. We will first minimise the Gibbs free energy for $C_n = C_{n+N}$ for some $N \in \mathbb{N}$. To make the notation a bit more compact, we set $K = \frac{k}{\sqrt{\kappa b}}$ and substitute it where it is economical. Let

$$B = \overline{\tilde{h}}, \tag{7.85}$$

then according to [8] and [22], the average Gibbs free energy is given by

$$\begin{aligned}\overline{G} = \overline{F} - 2h_0 B = -h_0^2 + & \\ \frac{\pi 2^{2r}(2r+1)^2(r!)^2\kappa^2(h_0-b)^2\left(|C_1|^2+...+|C_N|^2\right)^2 \times}{\left(\sum_{m,p} C_n^* C_{n-m} C_m^* C_{n+p}\left\{(1+2\kappa\mu)\mathcal{P}_1\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) - \mathcal{P}_2\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) + \kappa^2\mathcal{P}_3\left(r, \frac{Km}{2}, \frac{Kp}{2}\right)\right\} e^{-\frac{K^2m^2}{2}-\frac{K^2p^2}{2}}\right)^2} & \\ \times\left(\left(\frac{1}{2\kappa}+\mu\right)^2\overline{|f|^4} - \left(\frac{1}{2\kappa}+\mu\right)\sqrt{\frac{b}{\kappa}}\sum_{m,p}\frac{\overline{e^{ik(p-m)y}|f|^2(f_{r,p}f_{r+1,m}-f_{r+1,p}f_{r,m})}}{k(p-m)} + \right. & \\ \left. + \frac{b}{4\kappa}\sum_{m,n,p,q}\frac{\overline{e^{ik(p+q-m-n)y}(f_{r,p}f_{r+1,m}-f_{r+1,p}f_{r,m})(f_{r,q}f_{r+1,n}-f_{r+1,q}f_{r,n})}}{k^2(p-m)(q-n)} - \frac{1}{2}\overline{|f|^4}\right), &\end{aligned} \tag{7.86}$$

where we made use of the condition (7.84) after summing over $n = 1, ..., N$ to obtain the final expression.

We see that this leads to three more integrals that need to be carried out. This time however, we need to do the integrals over a single unit cell and the divide by the area of the cell in order to obtain the average. The first integral, however, is the average of $|f|^4$. Just like we will do for the other two integrals, we can use one summation to extend the integral from one unit cell to the complete x-axis. This means that[6]

$$\begin{aligned}\overline{|f|^4} &= \frac{\kappa b}{2\pi N}\sum_{n,m,p,q}\int_0^{\frac{2\pi}{k}}\int_0^{\frac{kN}{\kappa b}} C_m^* C_n^* C_p C_q e^{ik(p+q-m-n)y} f_{r,m}f_{r,n}f_{r,p}f_{r,q}\,dxdy \\ &= \frac{\kappa b}{k}\sum_{n=1}^{N}\sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m}\int_{-\infty}^{\infty} f_{r,m}f_{r,0}f_{r,p}f_{r,m-p}\,dx \\ &= \frac{\sqrt{\kappa b}}{k}\sum_{n=1}^{N}\sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m}\mathcal{P}_3\left(r, \frac{km}{2\sqrt{\kappa b}}, \frac{kp}{2\sqrt{\kappa b}}\right) e^{-\frac{k^2m^2}{2\kappa b}-\frac{k^2p^2}{2\kappa b}}.\end{aligned} \tag{7.87}$$

[6]This argument is by no means precise, but the author trusts it will become clear from the second and third integral, which require the same method.

The second integral is

$$\sum_{m,n,p,q} \int_0^{\frac{2\pi}{kN}} \int_0^{\frac{k}{\kappa b}} C_m^* C_n^* C_p C_{n-p+m} \frac{f_{r,n} f_{r,q} (f_{r,p} f_{r+1,m} - f_{r+1,p} f_{r,m})}{k(p-m)} e^{k(p+q-m-n)y} dx dy \tag{7.88}$$

$$= \frac{2\pi}{kN} \sum_{m,n,p} \int_0^{\frac{k}{\kappa b}} C_m^* C_n^* C_p C_{n-p+m} \frac{f_{r,n} f_{r,n-p+m} (f_{r,p} f_{r+1,m} - f_{r+1,p} f_{r,m})}{k(p-m)} dx$$

$$= \frac{2\pi}{kN\sqrt{\kappa b}} \sum_{m,n,p} C_m^* C_n^* C_p C_{n-p+m} \int_0^{KN} \frac{H_{r+1}\left(z - Km\right) H_r\left(z - Kp\right) - H_r\left(z-Km\right) H_{r+1}\left(z - Kp\right)}{k(p-m)} \times$$

$$\times H_r\left(z - Kn\right) H_r\left(z - K(n-p+m)\right) \times$$

$$\exp\left(-\frac{1}{2}\left((z-Km)^2 + (z-Kn)^2 + (z-Kp)^2 + (z-K(n-p+m))^2\right)\right) dz$$

$$= \frac{4\pi}{kN\kappa b} \sum_{m,n,p} C_m^* C_n^* C_p C_{n-p+m} \int_0^{KN} \sum_{j,l=0}^{2r} H_j\left(z - \frac{K(m+p)}{2}\right) B_j\left(\frac{K(p-m)}{2}, r\right) \times$$

$$\times H_l\left(z - \frac{K(2n-p+m)}{2}\right) A_l\left(\frac{K(p-m)}{2}, r\right) \times$$

$$\times \exp\left(-\left(z - \frac{K(m+p)}{2}\right)^2 - \left(z - \frac{K(2n-p+m)}{2}\right)^2 - \frac{K^2(m-p)^2}{2}\right) dy$$

$$= \frac{4\pi}{k\kappa b} \sum_{m,n,p} C_m^* C_n^* C_{p+n} C_{n+m} \int_0^{KN} \sum_{j,l=0}^{2r} H_j\left(z - Kn - \frac{K(m+2p)}{2}\right) B_j\left(\frac{Km}{2}, r\right) \times$$

$$H_l\left(z - Kn - \frac{Km}{2}\right) A_l\left(\frac{Km}{2}, r\right) \times$$

$$\times \exp\left(-\left(z - Kn - \frac{K(m+2p)}{2}\right)^2 - \left(z - Kn - \frac{km}{2}\right)^2 - \frac{K^2 m^2}{2}\right) dz$$

$$= \frac{4\pi}{kN\kappa b} \sum_{m,n,p} C_m^* C_n^* C_{p+n} C_{n+m} \int_{-Kn}^{K(N-n)} \sum_{j,l=0}^{2r} H_j\left(w - \frac{K(m+2p)}{2}\right) B_j\left(\frac{Km}{2}, r\right) H_l\left(w - \frac{Km}{2}\right) \times$$

$$A_l\left(\frac{Km}{2}, r\right) \exp\left(-\left(w - \frac{K(m+2p)}{2}\right)^2 - \left(w - \frac{Km}{2}\right)^2 - \frac{K^2 m^2}{2}\right) dz$$

$$= \frac{4\pi}{kN\kappa b} \sum_{n=1}^{N} \sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m} \sum_{j,l=0}^{2r} B_j\left(\frac{Km}{2}, r\right) A_l\left(\frac{Kb}{2}, r, r\right) \mathcal{I}\left(j, l, Kp, 0\right) e^{-\frac{K^2 m^2}{2} - \frac{K^2 p^2}{2}}$$

$$= \frac{4\pi}{kN\kappa b} \sum_{n=1}^{N} \sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m} \mathcal{P}_4\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) e^{-\frac{K^2 m^2}{2} - \frac{K^2 p^2}{2}}.$$

In the first step, we worked out the integral over y. Then, we substituted $z = \sqrt{\kappa b} x$ and replaced some instances where k appeared by K (to optimise the usage of space). After that, we used results 6.3 and 6.1 to reduce the integral to two Hermite polynomials. For the fourth equality, we first replaced m by $m+p$ and then p by $n+p$. Subsequently, we substituted $w = z - Kn$ so that we could 'glue' the integrals and use proposition 6.3 to obtain the sixth equality. Note that to do this, we also used that $C_n = C_{n+N}$ so that we could take C_n outside the sum over each equivalence class modulo N.

This leaves us with the third integral.

$$\sum_{m,n,p,q} \int_0^{\frac{2\pi}{kN}} \int_0^{\frac{k}{\kappa b}} C_m^* C_n^* C_p C_{n-p+m} \frac{(f_{r,p}f_{r+1,m} - f_{r+1,p}f_{r,m})(f_{r,q}f_{r+1,n} - f_{r+1,q}f_{r,n})}{k^2(p-m)(q-n)} e^{k(p+q-m-n)y} dxdy \tag{7.89}$$

$$= \frac{2\pi}{kN} \sum_{m,n,p} \int_0^{\frac{k}{\kappa b}} C_m^* C_n^* C_p C_{n-p+m} \frac{(f_{r,p}f_{r+1,m} - f_{r+1,p}f_{r,m})(f_{r,q}f_{r+1,n} - f_{r+1,q}f_{r,n})}{k^2(p-m)(q-n)} dx$$

$$= \frac{2\pi}{kN\sqrt{\kappa b}} \sum_{m,n,p} C_m^* C_n^* C_p C_{n-p+m} \int_0^{KN} \frac{H_{r+1}(z-Km) H_r(z-Kp) - H_r(z-Km) H_{r+1}(z-Kp)}{k(p-m)} \times$$

$$\times \frac{H_{r+1}(z-Kn) H_r(z-Kq) - H_r(z-Kn) H_{r+1}(z-Kq)}{k(q-n)} \times$$

$$\exp\left(-\frac{1}{2}\left((z-Km)^2 + (z-Kn)^2 + (z-Kp)^2 + (z-K(n-p+m))^2\right)\right) dz$$

$$= \frac{8\pi}{kN(\kappa b)^{3/2}} \sum_{m,n,p} C_m^* C_n^* C_p C_{n-p+m} \int_0^{KN} \sum_{j,l=0}^{2r} H_j\left(z - \frac{K(m+p)}{2}\right) B_j\left(\frac{K(p-m)}{2}, r\right) \times$$

$$\times H_l\left(z - \frac{K(2n-p+m)}{2}\right) B_l\left(\frac{K(p-m)}{2}, r\right) \times$$

$$\times \exp\left(-\left(z - \frac{K(m+p)}{2}\right)^2 - \left(z - \frac{K(2n-p+m)}{2}\right)^2 - \frac{K^2(m-p)^2}{2}\right) dy$$

$$= \frac{8\pi}{kN(\kappa b)^{3/2}} \sum_{m,n,p} C_m^* C_n^* C_{p+n} C_{n+m} \int_0^{KN} \sum_{j,l=0}^{2r} H_j\left(z - Kn - \frac{K(m+2p)}{2}\right) B_j\left(\frac{Km}{2}, r\right) \times$$

$$H_l\left(z - Kn - \frac{Km}{2}\right) B_l\left(\frac{Km}{2}, r\right) \times$$

$$\times \exp\left(-\left(z - Kn - \frac{K(m+2p)}{2}\right)^2 - \left(z - Kn - \frac{km}{2}\right)^2 - \frac{K^2m^2}{2}\right) dz$$

$$= \frac{8\pi}{kN(\kappa b)^{3/2}} \sum_{m,n,p} C_m^* C_n^* C_{p+n} C_{n+m} \int_{-Kn}^{K(N-n)} \sum_{j,l=0}^{2r} H_j \left(w - \frac{K(m+2p)}{2} \right) B_j \left(\frac{Km}{2}, r \right) H_l \left(w - \frac{Km}{2} \right) \times$$
$$B_l \left(\frac{Km}{2}, r \right) \exp\left(- \left(w - \frac{K(m+2p)}{2} \right)^2 - \left(w - \frac{Km}{2} \right)^2 - \frac{K^2m^2}{2} \right) dz$$
$$= \frac{8\pi}{kN(\kappa b)^{3/2}} \sum_{n=1}^{N} \sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m} \sum_{j,l=0}^{2r} B_j \left(\frac{Km}{2}, r \right) B_l \left(\frac{Kb}{2}, r, r \right) \mathcal{I}\left(j, l, Kp, 0 \right) e^{-\frac{K^2m^2}{2} - \frac{K^2p^2}{2}}$$
$$= \frac{8\pi}{kN(\kappa b)^{3/2}} \sum_{n=1}^{N} \sum_{m,p} C_m^* C_n^* C_{p+n} C_{n+m} \mathcal{P}_4 \left(r, \frac{Km}{2}, \frac{Kp}{2} \right) e^{-\frac{K^2m^2}{2} - \frac{K^2p^2}{2}}.$$

The steps that we took to perform this integral are identical to those for the previous one, with the exception that we only used corollary 6.1 and not lemma 6.3.

Combining the results of these three integrals, we find that the average Gibbs free energy (7.86) is given by

$$\overline{G} = -h_0^2 + \tag{7.90}$$
$$\frac{\pi 2^{2r}(2r+1)^2 (r!)^2 \kappa^2 (h_0 - b)^2 \left(|C_1|^2 + ... + |C_N|^2 \right)^2 \times}{\left(\sum\limits_{\substack{m,p \\ 1 \le n \le N}} C_n^* C_{n-m} C_m^* C_{n+p} \left\{ (1+2\kappa\mu)\mathcal{P}_1\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) - \mathcal{P}_2\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) + \kappa^2 \mathcal{P}_3\left(r, \frac{Km}{2}, \frac{Kp}{2}\right) \right\} e^{-\frac{K^2m^2}{2} - \frac{K^2p^2}{2}} \right)^2}$$
$$\times \sum_{\substack{n,q \\ 1 \le s \le N}} C_s^* C_{s-n} C_n^* C_{s+q} \left(\left[\left(\frac{1}{2\kappa} + \mu \right)^2 - \frac{1}{2} \right] \frac{1}{K} \mathcal{P}_3 \left(r, \frac{Kn}{2}, \frac{Kq}{2} \right) - \right.$$
$$\left. - \frac{2}{\kappa K} \left(\frac{1}{2\kappa} + \mu \right) \mathcal{P}_4 \left(r, \frac{Kn}{2}, \frac{Kq}{2} \right) + \frac{1}{\kappa^2 K} \mathcal{P}_5 \left(r, \frac{Kn}{2}, \frac{Kq}{2} \right) \right) e^{-\frac{K^2n^2}{2} - \frac{K^2q^2}{2}}.$$

In particular, we see that for large κ the average Gibbs free energy is proportional to

$$\frac{-2\left(|C_1|^2 + ... + |C_N|^2 \right)^2}{\sum\limits_{\substack{m,p \\ 1 \le n \le N}} C_n^* C_{n-m} C_m^* C_{n+p} K \mathcal{P}_3\left(r, \frac{Kn}{2}, \frac{Kq}{2}\right)}, \tag{7.91}$$

which is analogous to the case originally studied by Abrikosov.

When $r = 0$, equation (7.90) reduces to

$$\overline{G} = -h_0^2 + \frac{\sqrt{2\pi}\kappa (h_0 - b)^2 \left(|C_1|^2 + ... + |C_N|^2 \right)^2}{\sum_{1 \le s \le N}^{n,q} C_s^* C_{s-n} C_n^* C_{s+q} \left(3 - 2\frac{\kappa}{b} + 2\kappa^2 \right)^2 k e^{-\frac{k^2n^2}{2\kappa b} - \frac{k^2q^2}{2\kappa b}}} \left(\sqrt{\frac{b}{\kappa}} - 4\sqrt{\frac{\kappa}{b}} + 4\sqrt{\frac{\kappa^3}{b^3}} - 2\sqrt{\kappa^3 b} \right), \tag{7.92}$$

where we used that $\mu = \frac{1}{\kappa} - \frac{1}{b}$ when $r = 0$. If $\mu = 0$ and consequently $b = \kappa$, we immediately see that this expression is identical to the one obtains in the case that was originally studied by Abrikosov. One important thing to note here is the sign of the second term in the average. This is determined by the sign of the term behind the fraction. Assuming that $\kappa^2 > \frac{1}{2}$, we find that when

$$b < \frac{2\kappa}{1+\sqrt{2}\kappa} \tag{7.93}$$

the sign is positive, which means that we have to find a maximum of

$$\sum_{m,p} k e^{-\frac{k^2 n^2}{2\kappa b} - \frac{k^2 q^2}{2\kappa b}} \tag{7.94}$$

as a function of k to minimise the energy. Two maxima of this function exist: one corresponds to $k = 0$ and the other to $k = \infty$ and in both cases, the sum actually diverges. Moreover, none of these values for k actually corresponds to a real vortex lattice. For other values of $b > \frac{2\kappa}{1+\sqrt{2}\kappa}$, we have to find a minimum of (7.94), which is attained when

$$k = \sqrt{2\pi\kappa b} \qquad \text{for a rectangular lattice} \tag{7.95}$$

$$k = \sqrt{\sqrt{3}\pi\kappa b} \qquad \text{for a triangular lattice} \tag{7.96}$$

If $\kappa^2 < \frac{1}{2}$, the term behind the fraction in equation (7.92) becomes negative when

$$\frac{2\kappa}{1+\sqrt{2}\kappa} < b < \frac{2\kappa}{1-\sqrt{2}\kappa} \tag{7.97}$$

and non-negative otherwise. This means that a stable vortex lattice exist for b inside this interval when $k = \sqrt{2\pi\kappa b}$ in the case of a rectangular lattice and $k = \sqrt{\sqrt{3}\pi\kappa b}$ in the case of a triangular lattice, where the triangular lattice has a slightly lower free energy in all cases.

For any value of κ, the case where $b = \frac{2\kappa}{1\pm\sqrt{2}\kappa}$ corresponds to a degeneracy in the vortex lattice states, where all of the states share the same energy.

When $r = 1$, the situation is more complicated. Numerical investigations indicate that for certain (small) values of b, there are regimes of κ in which the triangular lattice is optimal, while the rectangular lattice is optimal in other regimes. For larger values of b, the rectangular lattice always seems to be optimal. We highlight two cases here. First, when $b = \kappa$ (so $\kappa\mu = 2$) the rectangular lattice always seems to yield the lowest free energy. There are two minima that yield roughly the same value for the free energy: one for $k \approx 1.43\sqrt{\kappa b}$ and one for $k \approx 4.38\sqrt{\kappa b}$. This makes sense, because one results in a lattice that is rotated $90°$ with respect to the other.

When $b = \frac{\kappa}{2}$ there is a phase in which the triangular lattice is optimal, as is depicted below. As soon as the rectangular lattice becomes energetically optimal, the value for k that yields the minimal free energy stabilises to $k \approx 1.42$ or $k \approx 4.40$. The optimal value for k in the triangular phase has a larger dependence on κ. It is depicted in the figure below.

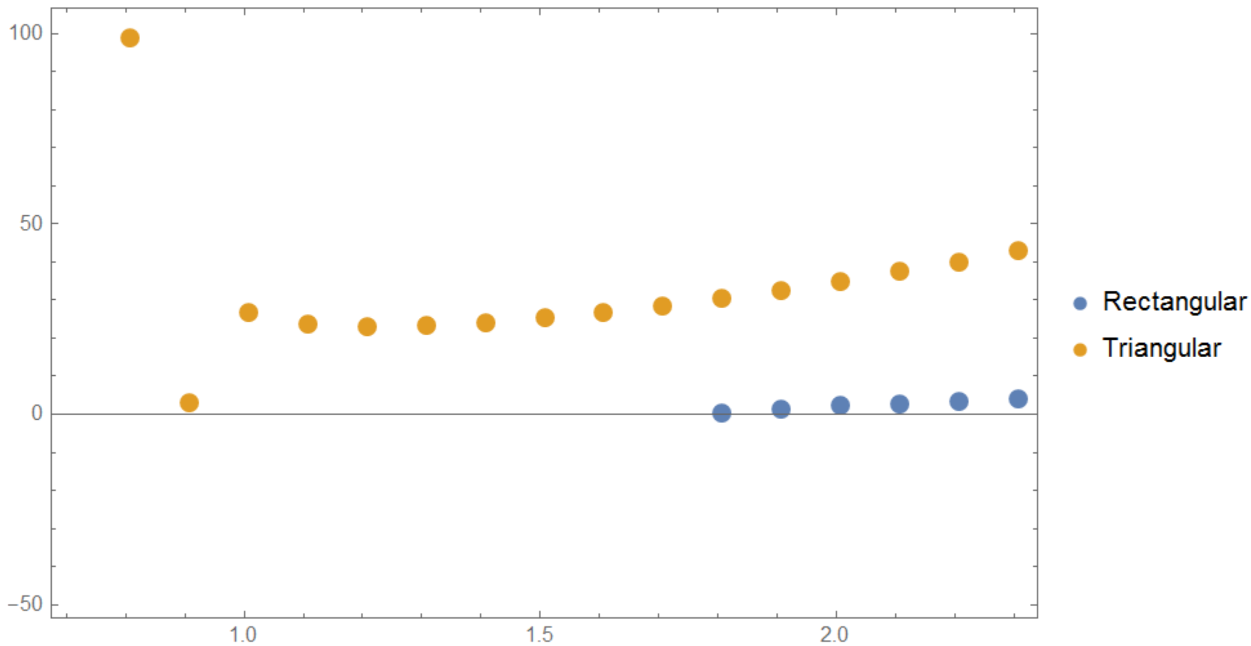

Figure 7: The minimal value of $\frac{-(h_0-b)^2}{\overline{G}+h_0^2}$ for $b = \frac{\kappa}{2}$ and $r = 1$ plotted as a function of κ. Only positive values are included. The sudden dip around $\kappa \approx 0.9$ cannot be explained on physical grounds.

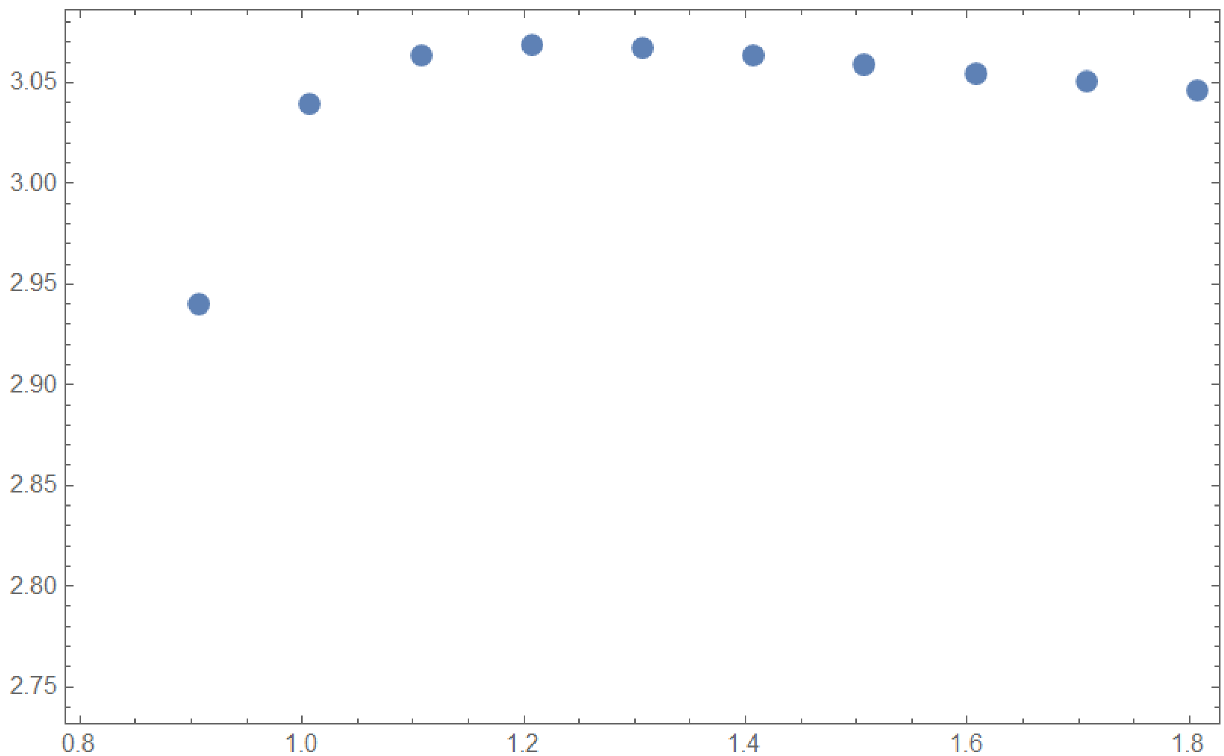

Figure 8: The optimal value for k in units $\sqrt{\kappa b}$ for the triangular lattice with $b = \frac{\kappa}{2}$ and $r = 1$.

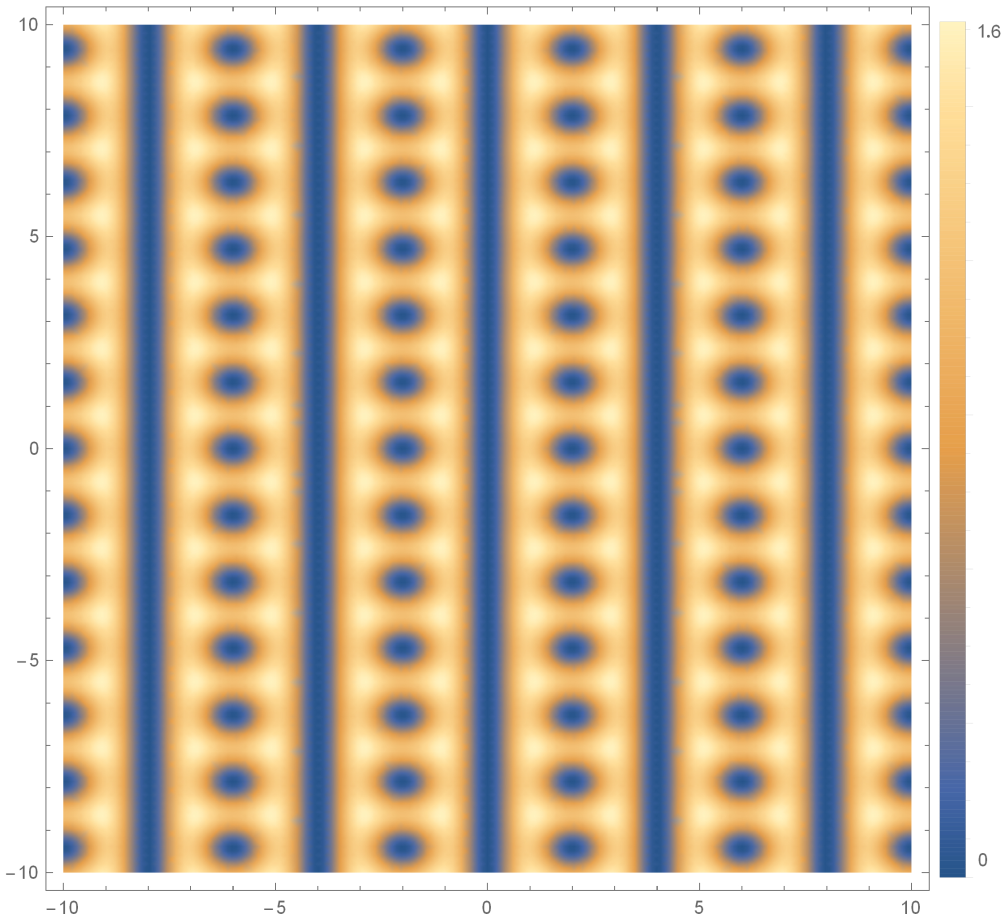

Figure 9: Rectangular lattice solution $|f|$ for $C_n = 1$, $r = 1$, $\kappa = 1$ and $k = 4.4\sqrt{\kappa b}$.

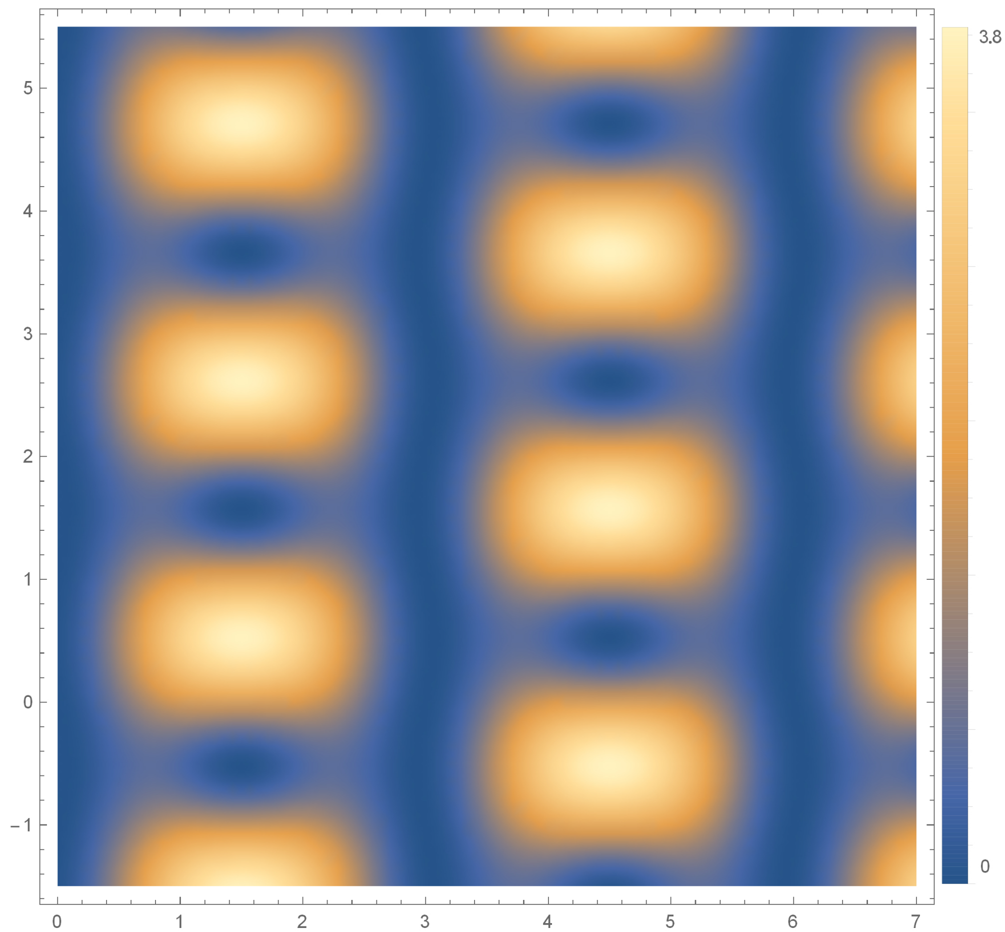

Figure 10: Triangular lattice solution $|f|$ for $|C_n| = 1$, $r = 1$, $\kappa = 1$ and $k = 3\kappa$.

8 Conclusion

In this thesis, we have studied two methods to find vortex lattices in a number of physical systems. The first method consists of solving the so-called self-dual equation and it leads to an infinite set of energetically degenerate solutions that are valid when certain constraints are placed on the physical parameters of the model. In particular, in the case of regular superconductors this constraint coincides with the border between type I and type II superconductors. One possible explanation for this phenomenon could be the fact that there is no energy cost associated with the formation of a domain wall inside the superconductor when $\kappa = \frac{1}{2}\sqrt{2}$.

To solve the Liouville-like equation that resulted from this first approach, we applied perturbation theory. Even though we could not prove that the solutions to the resulting series of differential equations actually converges, the results seem promising. They indicate that only a small correction to exact solution from the Liouville equation is necessary to obtain a solution to the perturbed equation.

In the last chapter, we studied Abrikosov's method of finding approximate vortex lattice solutions to the Ginzburg-Landau model for superconductors. For regular superconductors this was already carried out by Abrikosov himself, but we would like to point out one similarity with the other method: when $\kappa = \frac{1}{2}\sqrt{2}$ the resulting solutions are also degenerate and in this sense the self-dual method can be understood as a limiting case of Abrikosov's method. In our new application of this method to the p-wave superconductor of this thesis, we also found a degeneration in the energies of the solutions, but only in the simplest case where $r = 0$. The more complicated solutions that we studied did not have this degeneracy, but instead exhibited a phase transition from a triangular to a rectangular lattice for certain values of the μ parameter. This is unlike the situation in the case of traditional superconductors, where the triangular lattice is always optimal for $\kappa > \frac{1}{2}\sqrt{2}$.

One way to improve upon the second method that we studied in this thesis would be to find a way to minimise the free energy over all lattices simultaneously. This can in theory be done to write the approximate solution as a Fourier series in the x coordinate as well as the y coordinate, but may in practise result in expressions that are even more difficult to handle than the ones obtained here.

9 Acknowledgements

I would like to thank my supervisors Gunther en Rembert for their share in this thesis. When we started, I was very glad that I could combine research in condensed matter physics with my master in mathematics. I think that none of us had a very clear idea in what direction my thesis would be going to, especially because the article by Gunther *et alii*[20] had nothing to do with condensed matter physics. I appreciate the 'gamble' they were willing to take when they agreed to be my supervisors.

Next, I would like to thank Nikolas Akerblom for his suggestion to study the paper by Olesen in which the self-dual method was applied to superconductors. I would also like to thank Jason Frank for his help in finding numerical solution to the equations that arose when we did perturbation theory.

Finally, I would like to thank my friend Tom for keeping my spirits high and the pleasant walks we had in the botanic gardens during the time that we were both working in our thesis.

A Elliptic functions of the second kind

We will repeat the results from [21] here as a reference.

Definition A.1. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice. Then a meromorphic function f is said to be an elliptic function of the second kind with multipliers of unit modulus if there exist $\mu_1, \mu_2 \in \mathbb{C}$ with $|\mu_1| = |\mu_2| = 1$ if f satisfies*

$$f(z + \omega_j) = \mu_j f(z) \qquad j = 1, 2. \tag{A.1}$$

In [21], a complete classification of these functions if given:

Theorem A.1. *A meromorphic function f is an elliptic function of the second kind with multipliers of unit modulus μ_1, μ_2 if and only if*

There are constants

$$a_0, \dots, a_n \in \mathbb{C} \tag{A.2}$$

and parameters

$$z_1, \dots, z_n \in \{t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\} \tag{A.3}$$

such that f is of the form

$$f(z) = \left[a_0 + \sum_{k=1}^{n} a_k \frac{d^k\zeta}{dz^k}(z - z_0)\right] \frac{\sigma(z - z_0)^n}{\prod_{j=1}^{n} \sigma(z - z_j)} e^{\lambda z}, \tag{A.4}$$

with

$$z_0 = \frac{1}{2n\pi i}(\zeta(\omega_1/2)\log(\mu_2) - \zeta(\omega_2/2)\log(\mu_1)) + \frac{1}{n}\sum_{k=1}^{n} z_k \tag{A.5}$$

and

$$\lambda = \frac{1}{\pi i}(\zeta(\omega_1/2)\log(\mu_2) - \zeta(\omega_2/2)\log(\mu_1)), \tag{A.6}$$

where ζ and σ are Weierstrass' elliptic functions with periods ω_1 and ω_2.

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