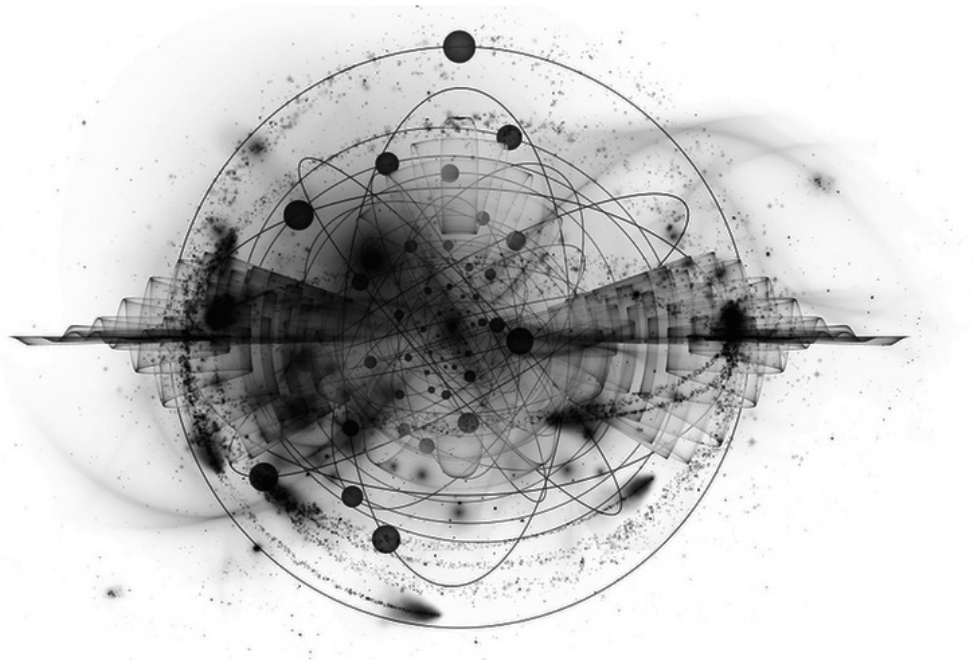
Utrecht University

Master Thesis

Theoretical Physics

Sublattice entanglement entropy for fermions



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ABSTRACT

In this thesis the entanglement entropy of fermions on sublattices is studied and contrasted with known results for bosons. By studying a $(1+1)$-dimensional periodic lattice and generalizing the notion and theory of circulant matrices to the broader class of phase circulant matrices, the results can be obtained analytically, thus providing a thorough understanding of entanglement entropy of fermion systems. To study the effect of long range coupling, a Lifshitz theory is adapted.

To start with, known results for the boson system have been reproduced to provide a background for the fermion results. The results for fermions show, firstly, a significant effect of the boundary conditions on the entanglement entropy. Secondly, a remarkable distinction between massive and massless fermions arises in the result that massless fermions generally have a maximal entanglement entropy. Most striking however, is the insight gained on the effect of long range coupling on the entanglement entropy, where the results for fermions strongly contrast the boson system: the entanglement entropy does not generally increase when long range coupling terms are added. We propose an explanation of this phenomenon by destructive interference. These results provide new and profound insights on entanglement of fermion systems.

ACKNOWLEDGMENTS

I would like to express my gratitude for all the assistance and intercourse I have had troughout the process of research and writing of this thesis. I am particularly grateful for the help and time from my supervisor Prof. Dr. Stefan Vandoren. Our weekly discussions were always very interesting and exciting and helped the steady pace of development of the thesis. I also wish to acknowledge the help I have had from discussions with my fellow students Maxim Faber, Yassir Awwad and Thijs van Gogh. Finally, I wish to thank my friends and family for their support and encouragement throughout my study.

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1

INTRODUCTION

Von Neumann wrote down the mathematical fundament of non relativistic quantum mechanics in 1932 [1]. But only later on, in 1935, it were Einstein, Podolsky and Rosen who uncovered a feature of the quantum theory that has no classical counterpart and is still being studied today: Quantum entanglement. This feature entails that the overall state of a composite quantum system with interactions between the subsystems is in general not a direct product of the states in the individual subsystems [2]. In that same year, Schrödinger realized "the best possible knowledge of a whole does not include the best possible knowledge of its parts" [3]. This phenomenon about the relation between knowledge of the system and knowledge of the subsystems was quantified around 1995 [4, 5] in terms of an entropy called the entropy of entanglement, or just entanglement entropy.

The current interest in quantum entanglement origins not only from theoretical and philosophical curiosity, but also from applications such as quantum cryptography [6, 7], quantum dense coding [8, 9], quantum teleportation [10] and the very exciting field of quantum computation [11]. On a more fundamental level, quantum entanglement may be responsible for macroscopic properties of solids [12] and can provide an order parameter for quantum phase transitions [13, 14, 15]. The above makes a clear case for the relevance of a good, analytic understanding of quantum entanglement.

Indeed, entanglement entropy is being studied intensively in quantum field theory and conformal field theory (a review of these fields of study can be found in [16] and [17] respectively). A well established result on the entropy of entanglement is the area law, which states that in a theory with local interactions, the entanglement entropy scales with the area between the considered subsystems [18]. This law was later seen to apply in conformal field theory as well [19], inviting the use of the AdS/CFT correspondence. Interestingly, the use of this correspondence relates the entanglement entropy to black hole physics, particularly, the area law results in Einsteins equation on gravity [20, 21]. Of special interest also is a class of conformal field theory which respect the so called Lifshitz scaling symmetry [22, 23], because at quantum critical points such physics occur [24, 25].

Often one has to rely on numerical methods to compute the measure of entanglement. This thesis however, attempts to contribute to the development

of an analytic understanding by specifying a particular composite system that allows us to compute the quantum entanglement analytically.

Before specifying the system, we first need to have a measure of quantum entanglement. This is provided by the von Neumann entropy [1, 5]. Suppose we have a composite system, composed of subsystems A and B, such that its corresponding Hilbert space can be written as $H = H_A \otimes H_B$. The composite system is in some state $|\psi\rangle$, which in general cannot be written as a direct product of states in subsystems A and B. Because $H = H_A \otimes H_B$,we can define the reduced density matrix of subsystem A from the total density matrix $\rho = |\psi\rangle\langle\psi|$, by tracing out the the eigenstates of H_B:

$$\rho_A = Tr_B(\rho). \tag{1}$$

Given the reduced density matrix ρ_A, the entanglement entropy is defined by the formula

$$S_A = -Tr(\rho_A \log \rho_A). \tag{2}$$

Thus, an analytic understanding of the entanglement entropy can be obtained by an analytic understanding of the reduced density matrix, or, more specifically, its eigenvalues. Fortunately, a useful method has been developed to obtain the eigenvalues of the reduced density matrix through the eigenvalues of the two point correlation matrices [26, 27, 28]. We adapt this method and apply it to specific systems where we can compute the eigenvalues of the correlation matrices analytically, by use of the mathematical formalism of circulant matrices and phase circulant matrices. This has already been done for bosons[29, 22, 23]. The main focus of this thesis is therefor to work out the same analytically solvable model of [29], but now for fermions and contrast with the established results of bosons. Of particular interest is the effect of adding long range interactions on the entanglement entropy. One would expect this to enhance the entanglement, as we have seen for bosons [22, 23]. But for fermions the situation is more delicate, as we shall see.

The system of study (taken from [29]) is a $(1+1)$-dimensional periodic lattice, containing N sites. The subsystem A, of which we wish to determine the entanglement with its complement, is defined by taking N_A evenly spaced lattice points. We define $p := N/N_A \in \mathbb{N}$ as the distance in number of sites between two particles of subsystem A and ϵ is the lattice spacing. In figure 1 two illustrations are given of such a system with $N = 12$ sites for $p = 2$ and $p = 3$. As a convention we will choose units $\hbar = c = 1$ throughout this thesis, except for some parts (there it will be mentioned explicitly) where this convention might obscure a proper dimensional analysis.

The entanglement entropy for fermionic sublattices has been studied in [30], where it is shown that spin and fermion representations generally lead to different reduced density matrices, and hence different entropies of entanglement. Some results of this study will be succesfully reproduced in this thesis. Sublattice entanglement is also studied in [14, 15] to obtain an order parameter for quantum phase transitions.

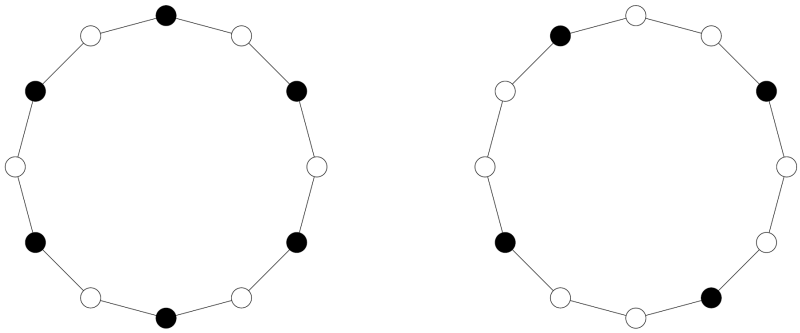

Figure 1.: Two illustrations of the system of study for $N = 12$ and $p = 2$ (left) or $p = 3$ (right). The black dots represent the subsystem A, and the white dots represent its complement. The figure is taken from [29].

This thesis is structured as follows. First, a new mathematical formalism of phase circulant matrices will be introduced. These phase circulant matrices are interesting on their own, but they will provide a crucial ingredient later on in the analytic study of the periodic fermion lattices. Then, the, already known [29, 22, 23] theory of periodic boson lattices is worked out and extended to provide a background for the study of periodic fermion lattices in the next section. Finally, once both theories for bosons and fermions are manifest, the results for both will be explored and compared graphically. These results turn out to be quite remarkable and lead to new insights, discussions and suggestions for future research, which will conclude this thesis.

2

CIRCULANT AND PHASE CIRCULANT MATRICES

A crucial ingredient of our study on the entanglement entropy is to be able to compute the eigenvalues of the two point correlation matrices analytically. In general, this not possible and numerical methods are used to determine the eigenvalues. In our specific system however, the correlation matrices are of a certain type which do allow us to determine the eigenvalues analytically. For the bosonic case, the correlation matrices turn out to be circulant matrices of which the theory is well known [31]. For the fermionic case the correlation matrices are a slightly different type of matrix, which we call phase circulant, of which the theory has been developed in this thesis. Apart from this physical application, the theory of circulant and phase circulant matrices is, on its own, interesting and will be described in this purely mathematical chapter.

2.1 CIRCULANT MATRIX

As the theory of circulant matrices is well known, here we only give its definition and results relevant for this thesis. The reader is referred to [31] for a more thorough treatment.

A *circulant matrix* is an $N \times N$ matrix characterized by the condition that its entries on one row are precisely a cyclic one position shift of of the entries of the row above it. Thus a circulant matrix is completely determined by its entries in the first row. A circulant matrix C can thus be written as

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ c_{N-2} & c_{N-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{pmatrix}. \tag{3}$$

One can also use the shorthand notation $C = circ(c_0, ..., c_{N-1})$.

The key result is that every circulant matrix $C = circ(c_0, ..., c_{N-1})$ has normalized eigenvectors

$$v_{(m)} \frac{1}{\sqrt{N}} (1, e^{2\pi i \frac{m}{N}}, e^{4\pi i \frac{m}{N}}, ..., e^{2(N-1)\pi i \frac{m}{N}})^T, \tag{4}$$

with corresponding eigenvalue

$$\lambda_m = \sum_{l=0}^{N-1} c_l e^{2\pi i \frac{ml}{N}}, \tag{5}$$

where m ranges from 0 to $N-1$. Most convenient about this result is that the eigenvectors $v_{(m)}$ do not depend on the entries c_l. As a consequence, all $N \times N$ circulant matrices can be diagonalized simultaneously so they commute and the sum or product of two circulant matrices is again a circulant matrix whose eigenvalues are the sum or product of the eigenvalues of the two separate matrices respectively.

2.2 PHASE CIRCULANT MATRIX

We will now generalize the notion of circulant matrices to phase circulant matrices, where the rows are defined from the first row together with a phase transformation: An $N \times N$ matrix P is called *phase circulant with phase* φ if it can be written as

$$P = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1}e^{i\varphi} & c_0 & c_1 & \cdots & c_{N-2} \\ c_{N-2}e^{i\varphi} & c_{N-1}e^{i\varphi} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & c_0 & c_1 \\ c_1 e^{i\varphi} & c_2 e^{i\varphi} & \cdots & c_{N-1}e^{i\varphi} & c_0 \end{pmatrix}, \tag{6}$$

with $\varphi \in [0, 2\pi)$. We shall denote such matrices P as $P = \mathrm{circ}_\varphi(c_0, c_1, ..., c_{N-1})$. So a phase circulant matrix with phase φ is circulant if and only if $\varphi = 0$. A matrix is called *anti-circulant* if it is circulant with phase π.

Let us compute the eigenvalues and eigenvectors of phase circulant matrices. Suppose $P = \mathrm{circ}_\varphi(c_0, c_1, ..., c_{N-1})$. The eigenvalue equation then reads

$$Pv = \lambda v, \tag{7}$$

where $v = (v_0, v_1, ..., v_{N-1})^T$ is the, yet to determine eigenvector, and λ the corresponding eigenvalue. This actually gives us N equations, for $i \in \{0, ..., N-1\}$:

$$\begin{aligned} \lambda v_i &= \sum_{l=i}^{N-1} v_l c_{l-i} + e^{i\varphi} \sum_{l=0}^{i-1} v_l c_{N-i+l} \\ &= \sum_{l=0}^{N-1-i} v_{l+i} c_l + e^{i\varphi} \sum_{l=N-i}^{N-1} v_{-N+l+i} c_l. \end{aligned} \tag{8}$$

In the summations in the second line we changed the dummy variables to $l \to l+i$ and $l \to -N+l+i$.

We intuitively guess the structure of v to be $v_i = \rho^i$ with ρ a variable. The eigenvalue equations will further determine ρ. Inserting this guess in our previous equation and dividing out ρ^i gives:

$$\lambda = \sum_{l=0}^{N-1-i} \rho^l c_l + \rho^{-N} e^{i\varphi} \sum_{l=N-i}^{N-1} \rho^l c_l. \tag{9}$$

So if we further demand

$$\rho^{-N} e^{i\varphi} = 1, \tag{10}$$

we get a concise eigenvalue equation. In fact, there are precisely N distinct solutions to this equation so that the obtained eigenvectors are distinct and span the entire solution space of P, which has at most N dimensions. This means that we have indeed found all solutions. The solutions are

$$\rho_m = e^{i\frac{\varphi+2\pi m}{N}}. \tag{11}$$

The subscript $m \in \{0, 1, ..., N-1\}$ indexes the N possible solutions for ρ of equation (10). This gives us our eigenvectors and eigenvalues:

$$v_{(m)} = \frac{1}{\sqrt{N}}(1, e^{i\frac{\varphi+2\pi m}{N}}, e^{2i\frac{\varphi+2\pi m}{N}}, ..., e^{(N-1)i\frac{\varphi+2\pi m}{N}})^T. \tag{12}$$

$$\lambda_m = \sum_{l=0}^{N-1} c_l e^{il\frac{\varphi+2\pi m}{N}}. \tag{13}$$

The prefactor of the eigenvector is to normalize $v_{(m)}^\dagger v_{(m)} = 1$. Notice that the eigenvectors again do not depend on the matrix coefficients c_i! So phase circulant matrices with the same phase can be diagonalized simultaneously. Hence, similar to circulant matrices, the sum or product of phase circulant matrices with the same phase is again a phase circulant matrix with the same phase and phase circulant matrices with the same phase commute.

2.3 PHYSICAL INTERPRETATION

As we shall see these two types of matrices and their desirable properties will provide us with convenient tools to compute the entanglement entropy from the correlation matrices. But on beforehand one can already derive from symmetry arguments that the correlation matrices are bound to be of the above described forms.

For suppose we study an infinite chain. The system has a discrete translational symmetry. Hence one would expect the two point correlators not to depend specifically on the site numbers of the two considered sites, but rather on the difference of these numbers. Hence the entries $c_{i,j}$ of the correlation matrix should be an explicit function of the difference $(i - j)$. Such matrices are called Toeplitz matrices (also reviewed in [31]) and the computation of their eigenvalues is an entire field of study.

Circulant matrices are a specific class of Toeplitz matrices, whose eigenvalues, as we have seen, can be computed very easily. The authors of [29] therefor sought a system with circulant correlation matrices. If one makes the system periodic, i.e. a ring, then indeed the correlation matrices should be circulant. To compute the reduced density matrix, we need to know the eigenvalues of the correlation matrix of the corresponding subsystem. Hence, we want the subsystem to respect a periodicity and discrete translational symmetry as well.

When considering (non-complex) bosons on a periodic lattice, one always has periodic boundary conditions. However, fermions on a periodic lattice need not respect periodic boundary conditions and can in general pick up any phase for example when there is a magnetic flux trough the ring as the Aharonov-Bohm effect demonstrates [32, 33]. This effect makes the correlation matrices of fermions phase circulant.

3

BOSONS

We will start with a brief explanation and derivation of the results for a system where the particles are bosons. These results were published in [29, 22, 23]. This chapter will provide a background for the following chapter, where fermions are studied, and will thus enable us to compare and contrast the results and their derivations.

The approach is the following: First, we will discretize the system and make it periodic. Then we can compute the correlation matrices using Fourier transformations. The correlation matrices can be used to determine the entanglement entropy because their eigenvalues are related to the eigenvalues of the reduced density matrix. Finally, a special class of systems with long range interactions, called Lifshitz theories, and their effect on the entanglement entropy is studied. This will give us the main results on the bosonic system which are demonstrated in chapter 5.

3.1 DISCRETIZATION

Our starting point is the Lagrangian density of a $(1+1)$-dimensional massive scalar field ϕ on a compact space with length L:

$$\mathcal{L} = -\frac{1}{2}\left((\partial_\mu \phi)^2 - m^2\phi^2\right). \tag{14}$$

From this expression the generalized momentum π follows

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi. \tag{15}$$

Hence the Hamiltionan becomes

$$H = \frac{1}{2}\int_0^L dx \left(\pi(x)^2 + (\partial_1 \phi(x))^2 + m^2\phi(x)^2\right), \tag{16}$$

with periodic boundary conditions $\phi(0) = \phi(L)$. Indeed, the scalar field has to be real, so this is the only possible boundary condition. This system has been studied in [16] for a subsystem consisting of a single interval.

We can now discretize the system, letting only the positions $x_j = j\epsilon$ be allowed, with $j \in \{0, 1, ..., N-1\}$ and ϵ the lattice spacing. Then we denote

$\phi_j = \phi(x_j)$ and $\pi_j = \pi(x_j)\epsilon$. The rescaling by ϵ makes the π_j dimensionless. The Riemann integral now becomes a finite sum:

$$\int_0^L dx \to \sum_{j=0}^{N-1} \epsilon. \tag{17}$$

Finally, the spatial derivative operator ∂_1 is discretized to the forward finite difference:

$$\partial_1 \phi(x_j) \equiv \lim_{\epsilon \to 0} \frac{\phi(x_j + \epsilon) - \phi(x_j)}{\epsilon} = \frac{\phi(x_{j+1}) - \phi(x_j)}{\epsilon}. \tag{18}$$

We choose the forward difference here, because we follow [29] where the same choice is made. A motivation for this choice is that it ensures that there will be nearest neighbor interactions in the theory. One could also choose to take the centered difference, which would result in a different theory. In appendix A the centered difference choice is worked out.

Putting the above together with the forward difference, the discretized Hamiltonian becomes:

$$H = \frac{1}{2\epsilon} \sum_{j=0}^{N-1} \left(\pi_j^2 + (\phi_{j+1} - \phi_j)^2 + m^2 \epsilon^2 \phi_j^2 \right) \tag{19}$$

3.2 THE CORRELATION MATRICES

As we will be interested in the entanglement entropy of the time independent ground state, we can fix $t = 0$. The Fourier transform of the scalar field ϕ is then given by

$$\phi(x_j) = \int \frac{dk}{2\pi} \tilde{\phi}_(k) e^{ikx_j}. \tag{20}$$

Demanding periodicity $\phi(x) = \phi(x + L)$ gives that the only momenta allowed are multiples of $2\pi/L$ and the above integral becomes a sum. We denote $\tilde{\phi}(2\pi k/L) = \tilde{\phi}_k$ for $k \in \mathbb{Z}$. As $x_j = j\epsilon$, we see the phases are integer multiples of $\frac{2\pi}{N}$ because $N = L/\epsilon$, so there are N different Fourier modes. By demanding the field is real we get $\tilde{\phi}_k^\dagger = \tilde{\phi}_{-k}$.

This all is similar for the momentum field, hence we obtain:

$$\begin{aligned} \phi_j &= \frac{1}{L} \sum_{k=0}^{N-1} \tilde{\phi}_k e^{ijk\frac{2\pi}{N}}, \\ \pi_j &= \frac{1}{L} \sum_{k=0}^{N-1} \tilde{\pi}_k e^{ijk\frac{2\pi}{N}}. \end{aligned} \tag{21}$$

The prefactor $1/L$ arises from the change of variables $k \to \frac{2\pi}{L}k$. By demanding the field is real we get $\tilde{\phi}_k^\dagger = \tilde{\phi}_{N-k}$, which we will denote as $\tilde{\phi}_{-k}$ for convenience. In this convention we have the commutation relations

$$[\tilde{\phi}_l, \tilde{\pi}_k] = i\epsilon L \delta_{-k,l}. \tag{22}$$

All other commutations vanish. Lets plug this in the Hamiltonian:

$$\begin{aligned} H =& \frac{1}{2\epsilon L^2} \sum_{j=0}^{N-1} \sum_{p,k=0}^{N-1} e^{ij(p+k)\frac{2\pi}{N}} \\ & \left(\tilde{\pi}_p \tilde{\pi}_k \left(m^2\epsilon^2 + (1 + e^{i(p+k)\frac{2\pi}{N}} - e^{ip\frac{2\pi}{N}} - e^{ik\frac{2\pi}{N}}) \right) \tilde{\phi}_p \tilde{\phi}_k \right) \\ =& \frac{N}{2\epsilon L^2} \sum_{k=0}^{N-1} \left(\tilde{\pi}_k^\dagger \tilde{\pi}_k + \left(m^2\epsilon^2 + 4\sin^2(\frac{\pi k}{N}) \right) \tilde{\phi}_k^\dagger \tilde{\phi}_k \right) \\ =& \frac{1}{2\epsilon^2 L} \sum_{k=0}^{N-1} \left(\tilde{\pi}_k^\dagger \tilde{\pi}_k + \epsilon^2 \omega_k^2 \tilde{\phi}_k^\dagger \tilde{\phi}_k \right). \end{aligned} \tag{23}$$

Here we used $\sum_{j=0}^{N-1} e^{ij(p+k)\frac{2\pi}{N}} = N\delta_{p,-k}$ and $2i\sin(x) = e^{ix} - e^{-ix}$, and we uncovered the dispersion relation

$$\omega_k^2 = m^2 + \frac{4}{\epsilon^2}\sin^2(\frac{\pi k}{N}). \tag{24}$$

Now we introduce the annihilator operators:

$$\tilde{a}_k = \frac{1}{\epsilon\sqrt{2L\omega_k}}(\epsilon\omega_k\tilde{\phi}_k + i\tilde{\pi}_k). \tag{25}$$

The creator operator is defined by complex conjugation. These satisfy the desired commutation relations $[\tilde{a}_l^\dagger, \tilde{a}_k] = \delta_{-k,l}$, using equation (22). We thus obtain the familiar expression:

$$H = \sum_{k=0}^{N-1} \omega_k \tilde{a}_k^\dagger \tilde{a}_k, \tag{26}$$

where we used $\omega_{N-k} = \omega_k$ and $\tilde{\phi}_k\tilde{\pi}_k^\dagger = \tilde{\phi}_{-k}^\dagger\tilde{\pi}_{-k}$ so $\sum_{k=0}^{N-1}(\tilde{\phi}_k^\dagger\tilde{\pi}_k - \tilde{\phi}_k\tilde{\pi}_k^\dagger) = 0$ by relabeling the second term.

We can now invert the creator and annihilator relations to obtain expressions for $\tilde{\phi}_k$ and $\tilde{\pi}_k$:

$$\begin{aligned} \tilde{\phi}_k =& \frac{(\tilde{a}_k + \tilde{a}_{-k}^\dagger)}{\sqrt{2}}\sqrt{\frac{L}{\omega_k}} \\ \tilde{\pi}_k =& -i\frac{(\tilde{a}_k - \tilde{a}_{-k}^\dagger)}{\sqrt{2}}\epsilon\sqrt{L\omega_k}. \end{aligned} \tag{27}$$

When deriving the correlation functions for ϕ-ϕ and π-π correlations, we will encounter expressions of the form,

$$\langle(\tilde{a}_p \pm \tilde{a}_{-p}^\dagger)(\tilde{a}_k \pm \tilde{a}_{-k}^\dagger)\rangle = \langle\tilde{a}_p\tilde{a}_k\rangle + \langle\tilde{a}_{-p}^\dagger\tilde{a}_{-k}^\dagger\rangle \pm (\langle\tilde{a}_p\tilde{a}_{-k}^\dagger\rangle + \langle\tilde{a}_{-p}^\dagger\tilde{a}_k\rangle). \tag{28}$$

The first two terms are zero since annihilators cannot act on the vacuum and created states are orthogonal to the vacuum. For the last two we use the commutation relations to obtain:

$$\begin{aligned} \langle\tilde{a}_p\tilde{a}_{-k}^\dagger\rangle &= \langle[\tilde{a}_p, \tilde{a}_{-k}^\dagger]\rangle + \langle\tilde{a}_{-k}^\dagger\tilde{a}_p\rangle \\ &= \delta_{p,-k}(1 + \langle N_{-k}\rangle) \\ \langle\tilde{a}_{-p}^\dagger\tilde{a}_k\rangle &= \delta_{-p,k}\langle N_k\rangle = \delta_{p,-k}\langle N_k\rangle. \end{aligned} \tag{29}$$

We introduced the number operator $N_k = \tilde{a}_k^\dagger \tilde{a}_k$ here. For ϕ-π correlations we will encounter

$$\langle (\tilde{a}_p \pm \tilde{a}_{-p}^\dagger)(\tilde{a}_k \mp \tilde{a}_{-k}^\dagger)\rangle = 0, \tag{30}$$

because the nonzero terms now cancel. So there will be no ϕ-π correlation.

We are interested in the correlation matrices of the subsystem A, so the indices of the field are multiples of p. For convenience we relabel $\phi_{ip} \to \phi_i$ and use $p/N = 1/N_A$. The entries of the two point correlation matrix for the ϕ field are

$$\begin{aligned}
\langle \phi_i \phi_j \rangle =& \frac{1}{L^2} \sum_{p,k=0}^{N-1} \langle \tilde{\phi}_p \tilde{\phi}_k \rangle e^{iip\frac{2\pi}{N_A}} e^{ijk\frac{2\pi}{N_A}} \\
=& \frac{1}{2L} \sum_{k=0}^{N-1} \frac{1}{\omega_k} (1 + \langle N_{-k}\rangle + \langle N_k \rangle) e^{-i(i-j)k\frac{2\pi}{N_A}} \\
=& \frac{1}{2N} \sum_{k=0}^{N-1} \frac{1}{\epsilon\omega_k} (\frac{1}{2} + \langle N_k \rangle)(e^{i(i-j)k\frac{2\pi}{N_A}} + e^{-i(i-j)k\frac{2\pi}{N_A}}) \\
=& \frac{1}{2N} \sum_{k=0}^{N-1} \frac{1}{\epsilon\omega_k} (1 + 2\langle N_k \rangle) \cos\left(\frac{2\pi(i-j)k}{N_A}\right).
\end{aligned} \tag{31}$$

Here we used a relabeling of $k \to -k$ and the equality $2\cos(x) = e^{ix} + e^{-ix}$. The calculation is similar for the π field and yields:

$$\langle \pi_i \pi_j \rangle = \frac{1}{2N} \sum_{k=0}^{N-1} \epsilon\omega_k (1 + 2\langle N_k \rangle) \cos\left(\frac{2\pi(i-j)k}{N_A}\right). \tag{32}$$

Notice that both correlation matrices only depend on the difference $(i-j)$. Furthermore, $\cos(\frac{2\pi k}{N_A}((i+1)-0)) = \cos(\frac{2\pi k}{N_A}(i-(N_A-1))$, so the correlation matrices are in fact linear combinations of circulant matrices, hence circulant themselves.

3.3 THE ENTANGLEMENT ENTROPY

One might expect that the correlation functions contain all the information about entanglement in the system. Here, we will make this intuition explicit by using the correlation functions to determine the reduced density matrix, from which the entanglement entropy can easily be computed (by use of equation (2)). This procedure is well known and can be found for example in [34, 28]. Here, some steps are highlighted that are relevant for this thesis. In this thesis we will go into more detail in the fermionic case, where a similar procedure applies.

Our theory is quadratic, so Wick's theorem holds and the corresponding Hilbert space can indeed be written as a direct product of the Hilbert spaces of the subsystems. We can write the reduced density matrix as [35]

$$\rho_A = \frac{e^- H_A}{Tr(e^- H_A)}. \tag{33}$$

Here, H_A is called the modular Hamiltonian of the subsystem A. In essence the above equation is the defining equation of H_A. Once we have obtained H_A we can relate its eigenvalues to the eigenvalues of the correlation matrices by use of

$$\langle \phi_i \phi_j \rangle = Tr(\rho \phi_i \phi_j), \tag{34}$$

for any bosonic field ϕ.

To find H_A we start out by distilling the relevant information of the correlation matrices. First, we construct a vector η of the sublattice positions ϕ_j and momenta π_j:

$$\eta = (\phi_0, ..., \phi_{N_A-1}, \pi_0, ..., \pi_{N_A-1}). \tag{35}$$

From this we define the $2N_A \times 2N_A$ correlation matrix C by $C_{\alpha,\beta} = \langle \eta_\alpha \eta_\beta \rangle$. By Wick's theorem, the reduced density matrix of our Gaussian system is completely determined by the two-point correlators, since for any operator $\mathcal{O}_A$ restricted to A we have

$$\langle \mathcal{O}_A \rangle = Tr(\rho_A \mathcal{O}_A). \tag{36}$$

This is the defining property of the reduced density matrix. Wick's theorem then allows us to write all expectation values of operators restricted to A in terms of the two-point correlators on A [35, 27].

We now wish to transform our basis η, preserving the commutation relations, to get a transformed correlation matrix whose eigenvalues can be related to the eigenvalues of the modular Hamiltonian H_A. Such a transformation (that preserves the commutation relations) is called symplectic. Fortunately, since the correlations functions are positive-definite, we can use Williamsons theorem [36, 37] to merge the information of the ϕ and π correlators. The theorem yields a symplectic transformation S, such that the matrix $\tilde{C} = S^T C S$ is diagonal with doubly degenerate eigenvalues. These eigenvalues are related to the eigenvalues of the correlation matrices [29, 28, 26]: If λ_l are the eigenvalues of $\tilde{C}$ and $\lambda_{\phi,l}$ and $\lambda_{\pi,l}$ the eigenvalues of the correlation matrices given in the previous subsection, then

$$\lambda_l = \sqrt{\lambda_{\phi,l} \lambda_{\pi,l}}. \tag{37}$$

The symplectic transformation gives us a new vector of positions and momenta $\tilde{\eta} = S\eta$, and thus new annihilation operators

$$\tilde{a}_i = \frac{(\tilde{\phi}_i + i\tilde{\pi}_i^\dagger)}{\sqrt{2}}. \tag{38}$$

Introducing the Williamson number operator $\tilde{n}_i = \tilde{a}_i^\dagger \tilde{a}_i$, the reduced density matrix factorizes as [38, 26]

$$\rho_A = \bigotimes_{l=0}^{N_A-1} (1 - e^{-\varepsilon_l}) e^{-\varepsilon_l \tilde{n}_l}. \tag{39}$$

Now all ingredients are in place to compute the entanglement entropy from the eigenvalues of the correlation matrices.

Since $\tilde{C}$ is diagonal, its eigenvalues are given by its entries on the diagonal. That is

$$\begin{aligned} \lambda_l &= \langle \tilde{\phi}_l \tilde{\phi}_l \rangle \\ &= \langle \frac{1}{2}(\tilde{a}_l \tilde{a}^\dagger + \tilde{a}_l^\dagger \tilde{a}_l) \rangle \\ &= \langle \tilde{a}_l^\dagger \tilde{a}_l \rangle + \frac{1}{2}, \end{aligned} \tag{40}$$

using the commutation relations. This shows that the symplectic eigenvalues are always greater than $\frac{1}{2}$. This bound from below will make the entanglement entropy well defined as we shall see next. The above derivation also allows us to relate λ_l and ε_l.

$$\begin{aligned} \langle \tilde{a}_l^\dagger \tilde{a}_l \rangle &= Tr(\rho_l \tilde{a}_l^\dagger \tilde{a}_l) \\ &= (1 - e^{-\varepsilon_l}) \sum_{n=0}^{\infty} e^{-\varepsilon_l n} n \\ &= \frac{1}{e^{-\varepsilon_l} - 1}, \end{aligned} \tag{41}$$

using the geometric series and the shorthand notation $\rho_l = (1 - e^{-\varepsilon_l}) e^{-\varepsilon_l \tilde{n}_l}$, which is the reduced density matrix of the l^{th} particle in this transformed basis. So

$$\varepsilon_l = \log\left(\frac{\lambda_l + \frac{1}{2}}{\lambda_l - \frac{1}{2}}\right). \tag{42}$$

Hence

$$\begin{aligned} S_A &= -Tr(\rho_A \log \rho_A) \\ &= -\sum_{l=0}^{N_A - 1} \sum_{n=0}^{\infty} (1 - e^{-\varepsilon_l}) e^{-\varepsilon_l n} \left(\log(1 - e^{-\varepsilon_l}) - \varepsilon_l n \right) \\ &= \sum_{l=0}^{N_A - 1} (\lambda_l + \frac{1}{2}) \log(\lambda_l + \frac{1}{2}) - (\lambda_l - \frac{1}{2}) \log(\lambda_l - \frac{1}{2}). \end{aligned} \tag{43}$$

This machinery to compute the entanglement entropy from the eigenvalues of the correlation matrices is applicable to any Gaussian system. In most systems these eigenvalues have to be computed numerically. But by choosing a system wherein the eigenvalues can be determined analytically, one can obtain analytical insight in the behavior of the entanglement entropy.

3.4 LONG RANGE INTERACTIONS

Intuitively one expects that adding interactions to any theory will increase the entanglement because it are precisely the interactions that cause entanglement in the first place. Indeed, two non interacting systems would have a total state that is the direct product of the states on the two systems, so the entanglement

entropy equals $-1\log(1) = 0$ as is immediate from equation (2). To investigate this intuition a special class of long range interaction theories, called Lifshitz theories, is studied. In Lifshitz theories, Lorentz symmetry is sacrificed to make place for the Lifshitz scaling symmetry: $x \to \lambda x$ and $t \to \lambda^z t$. The exponent $z \in \mathbb{Z}^+$ is called the dynamical exponent.

In particular, for the bosonic system, we will be concerned with a Hamiltonian of the form

$$H = \frac{1}{2}\int_0^L (\pi^2 + \alpha^2((\partial_x)^z\phi)^2 + m^2\phi^2)dx. \tag{44}$$

When $z = 1$ we obtain the familiar massive scalar field theory described previously for $\alpha = c$ the speed of light. Only when $m = 0$ this theory respects the Lifshitz scaling symmettry. Whereas previously we have set $\hbar = c = 1$ for simplicity, we will reintroduce SI units because the dimensional analysis now is less straightforward. The "speed of light" α for example, has units m^z/s.

The quantity S/k_B, with k_B the Boltzmann constant, is dimensionless, hence we expect it to be determined by dimensionless parameters. Apart from N, p and z the parameter including the mass and lattice spacing can be defined by $J_b = m\epsilon^z/\alpha$.

In this continuous system the above described expectation of the effect of adding long range interactions can be made explicit by studying the two point correlators. We have [23, 39]

$$\langle\phi(0)\phi(r)\rangle = \int \frac{dk}{2\pi}\frac{e^{ikr}}{2\omega_k}. \tag{45}$$

If $m = 0$ then $\omega_k = |k|^z$ so by use of the residue theorem we obtain for $z > 1$

$$\langle\phi(0)\phi(r)\rangle \sim r^{z-1}. \tag{46}$$

Hence, if z increases, the long range interactions increase as well. This corresponds with our intuition that adding interactions will increase the correlation. By the from of equation (43), we thus expect also the entanglement to increase as a function of z. Indeed, as our theory is still Gaussian, equation (43) still applies.

Indeed, the discretization procedure remains intact, but the result will now depend on z as the spatial derivative ∂_x operator, which becomes the forward finite difference, is now applied z times. Each application adds a neighbor to the interaction term obtained. The result is summarized in the discretized Hamiltonian

$$H = \frac{m\hbar}{2}\sum_{j=0}^{N-1}\left(\pi_j^2 + \frac{1}{J_b^2}\left(\sum_{n=0}^{z}\binom{z}{n}(-1)^n\phi_{j+z-r}\right)^2 + \phi_j^2\right), \tag{47}$$

which can be shown to be the correct expression by a proof by natural induction on z.

Using the Fourier transform we can again compute the correlation matrices. The previous method also applies here, since the system is still Gaussian, and the results of equations (31) and (32) thus hold, but the dispersion relation has changed. Using induction we obtain

$$\begin{aligned}\omega_k^2 &= m^2 + \frac{\alpha^2}{\epsilon^{2z}}\left(\sum_{r=0}^{z}\binom{z}{r}^2 + 2\sum_{s=1}^{z}\sum_{r=0}^{z-s}(-1)^s\binom{z}{r}\binom{z}{r+s}\cos(\frac{2\pi ks}{N})\right)\\ &= m^2 + \frac{\alpha^2}{\epsilon^{2z}}\left(\binom{2z}{z} + 2\sum_{s=1}^{z}(-1)^s\binom{2z}{z-s}\cos(\frac{2\pi ks}{N})\right),\end{aligned} \tag{48}$$

using Vandermonde's identity in the second line. Now, by using the symmetry of the binomial coefficients, we obtain

$$\sum_{s=1}^{z}(-1)^s\binom{2z}{z-s}\cos(\frac{2\pi ks}{N}) = \sum_{s=1}^{z}(-1)^s\binom{2z}{z+s}\cos(\frac{2\pi ks}{N}). \tag{49}$$

So

$$\binom{2z}{z} + 2\sum_{s=1}^{z}(-1)^s\binom{2z}{z-s}\cos(\frac{2\pi ks}{N}) = (-1)^z\sum_{s=0}^{2z}\binom{2z}{s}(-1)^s\cos(\frac{2\pi k(z-s)}{N}), \tag{50}$$

using $\cos(0) = 1$ and shifting the dummy variable $s \to s + z$. To further simplify this expression, one might already suspect the binomial theorem to be used. But first we need to make the cosine more workable. We use

$$\cos(\frac{2\pi k(z-s)}{N}) = \frac{1}{2}(e^{\frac{2\pi kz}{N}i}(e^{\frac{-2\pi k}{N}i})^s + e^{\frac{-2\pi kz}{N}i}(e^{\frac{2\pi k}{N}i})^s. \tag{51}$$

Then we can apply the binomial theorem

$$\begin{aligned}&\frac{(-1)^z}{2}\sum_{s=0}^{2z}\binom{2z}{s}\left(e^{\frac{2\pi kz}{N}i}(-e^{\frac{-2\pi k}{N}i})^s + e^{\frac{-2\pi kz}{N}i}(-e^{\frac{2\pi k}{N}i})^s\right) =\\ &\quad\frac{(-1)^z}{2}\left(e^{\frac{2\pi kz}{N}i}(1-e^{\frac{-2\pi k}{N}i})^{2z} + e^{\frac{-2\pi kz}{N}i}(1-e^{\frac{2\pi k}{N}i})^{2z}\right) =\\ &\qquad\frac{(-1)^z}{2}\left((e^{\frac{\pi k}{N}i} - e^{\frac{-\pi k}{N}i})^{2z} + (e^{\frac{-\pi k}{N}i} - e^{\frac{\pi k}{N}i})^{2z}\right) = (2\sin(\frac{\pi k}{N}))^{2z}.\end{aligned} \tag{52}$$

Thus the dispersion relation is

$$\omega_k^2 = m^2 + \frac{\alpha^2}{\epsilon^{2z}}(2\sin(\frac{\pi k}{N}))^{2z}, \tag{53}$$

a strongly simplified result. This modification to the dispersion relation is all that is needed to derive the results for this class of long range interaction theories.

4

FERMIONS

For the study of fermions we will follow a similar structure as we did for the bosons. We start out by setting up a familiar continuous theory and discretize it. Then we will determine the correlation matrices and successively show how the correlation matrices determine the entanglement entropy. Then we are ready to derive the first results. As it will turn out, these results are remarkable, so before stating the results, a sanity check is done to ensure the machinery works correctly. Finally, we will consider a fermionic Lifshitz theory. In the next section, the results of both the bosonic and fermionic systems will be compared and discussed.

4.1 DISCRETIZATION

Our convention for the metric is $\eta = diag(+1, -1)$. We must begin by introducing two gamma matrices γ^0 and γ^1 satisfying the 2 dimensional Clifford algebra:

$$\gamma^0\gamma^0 = 1, \qquad \gamma^1\gamma^1 = -1, \qquad \gamma^0\gamma^1 = -\gamma^1\gamma^0. \tag{54}$$

The following matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{55}$$

do the job.

In these conventions the Dirac Lagrangian density of a $(1+1)$-dimensional fermion field ψ is:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi, \tag{56}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$. The generalized momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger. \tag{57}$$

Hence the Hamiltionan becomes

$$\begin{aligned} H &= \int_0^L dx\bar{\psi}(x)(m - i\gamma^1\partial_1)\psi(x) \\ &= \int_0^L dx\left(m\bar{\psi}(x)\psi(x) - \frac{1}{2}i(\bar{\psi}(x)\gamma^1(\partial_1\psi(x)) - (\partial_1\bar{\psi}(x))\gamma^1\psi(x))\right), \end{aligned} \tag{58}$$

by integration by parts. To do integration by parts properly, we must ensure that the boundary terms indeed vanish. In fact, this will give us the boundary conditions: The chirality projection is defined from the gamma matrices:

$$P_\pm = \frac{\mathbb{1} \pm \gamma^5}{2}, \tag{59}$$

with $\gamma^5 = \gamma^0\gamma^1$ (indeed $(\gamma^5)^\dagger = \gamma^5$ and $(\gamma^5)^2 = \mathbb{1}$). That is:

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{60}$$

Thus we can conveniently write the fermion field in spinor representation:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \text{so } \bar{\psi} = \begin{pmatrix} \psi_-^\dagger & \psi_+^\dagger \end{pmatrix}. \tag{61}$$

The boundary term then is of the form:

$$\begin{aligned} \bar{\psi}\gamma^1\psi\Big|_0^L &= \psi_-^\dagger\psi_- - \psi_+^\dagger\psi_+\Big|_0^L \\ &= |\psi_-(L)|^2 - |\psi_-(0)|^2 - (|\psi_+(L)|^2 - |\psi_+(0)|^2), \end{aligned} \tag{62}$$

which vanishes when

- $\psi_+(0) = e^{-2\pi i\theta_+}\psi_+(L)$, and
- $\psi_-(0) = e^{-2\pi i\theta_-}\psi_-(L)$.

Where $\theta_\pm \in [0, 1)$. Notice that if $m \neq 0$ we must have $\theta_+ = \theta_- = \theta$ so that the Hamiltonian density respects the periodicity of our system. This point becomes clearer when one writes out the spinor components in the Hamiltonian density; The terms proportional to the mass are precisely the on-site coupling terms between the different spinor components. If these spinor components would have different boundary conditions, the phases will not cancel out and the Hamiltonian density no longer respects the periodicity. The fermion field indeed can pick up such a phase $2\pi\theta$ for example if there is a magnetic flux trough the ring by the Arahonov-Bohm effect [32, 33].

Now we discretize the Hamiltonian, letting the only allowed positions be multiples of ε. We will write $\psi_{\pm,j} = \psi_\pm(j\varepsilon)\sqrt{\varepsilon}$ (to make it dimensionless). For the spatial derivative, we take the centered finite difference

$$\partial_1\psi_{\pm,j} \to \frac{\psi_{\pm,j+1} - \psi_{\pm,j-1}}{2\epsilon}. \tag{63}$$

As we will see later on, the discretization of the spatial derivative will cause significant differences between fermions and bosons (where we took the forward finite difference). Whether we take the forward or centered difference, one can check that it has no effect on the Lagrangian and Hamiltonian for fermions

(see appendix A). The reason we choose the centered finite difference here, is that it will make the following calculations less cumbersome. Note that in the study of bosons, the choice of finite difference as discretization of the spatial derivative does significantly alter the theory; If one would choose the centered finite difference, there will be no nearest neighbor interactions but only next-nearest neighbor interactions. This is also discussed in appendix A. The discretized Hamiltonian now reads

$$\begin{aligned} H =& \frac{1}{\varepsilon}\sum_{j=0}^{N-1}\Big(m\varepsilon(\psi_{-,j}^\dagger\psi_{+,j}+\psi_{+,j}^\dagger\psi_{-,j}) \\ & -\frac{1}{2}i\left(\psi_{+,j}^\dagger(\psi_{+,j+1}-\psi_{+,j-1})-\psi_{-,j}^\dagger(\psi_{-,j+1}-\psi_{-,j-1})\right)\Big) \\ =& \frac{1}{\varepsilon}\sum_{j=0}^{N-1}\Big(m\varepsilon(\psi_{-,j}^\dagger\psi_{+,j}+\psi_{+,j}^\dagger\psi_{-,j}) \\ & -\frac{1}{2}i\left(\psi_{+,j}^\dagger\psi_{+,j+1}-\psi_{+,j+1}^\dagger\psi_{+,j}-(\psi_{-,j}^\dagger\psi_{-,j+1}-\psi_{-,j+1}^\dagger\psi_{-,j})\right)\Big). \end{aligned} \tag{64}$$

For the last equality we used a relabeling of $j \to j \pm 1$. The discretized boundary conditions are

- $\psi_{+,N} = e^{2\pi i\theta_+}\psi_{+,0}$, and
- $\psi_{-,N} = e^{2\pi i\theta_-}\psi_{-,0}$.

Massive Fermions

For now we assume $m > 0$. Hence $\theta_+ = \theta_- = \theta$. The massless Dirac field will be studied in a later subsection.

We obtain the field equations from the Euler-Lagrange formalism:

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = \left(i\gamma^\mu\partial_\mu - m\right)\psi = 0, \tag{65}$$

the Dirac equation. Note that if we would choose to work with the forward derivative, we would first have to partially integrate half of the spatial part of the Lagrangian to maintain hermiticity after discretizing.

General solutions to the Dirac equation are of the form $c_k(\omega)e^{i(kx+\omega t)}$. The discretized spatial derivative then becomes the centered finite difference:

$$\begin{aligned} \partial_1\psi_j =& \frac{\psi_{j+1}-\psi_{j-1}}{2\varepsilon} \\ =& \frac{1}{2\varepsilon}(e^{ik\varepsilon}-e^{-ik\varepsilon})\psi_j \\ =& \frac{i}{\varepsilon}\sin(k\varepsilon)\psi_j \equiv i\tilde{k}(k)\psi_j. \end{aligned} \tag{66}$$

Furthermore, periodicity of our lattice gives us that we must have $e^{ik(x+L)} = e^{i(kx+2\pi\theta)}$ so we relabel $k \to \frac{2\pi}{L}k$, so that $k \in \mathbb{Z} + \theta$. The period of $\tilde{k}$ then is $N/2$ if N is even, and N if N is odd. The Dirac equation thus gives the dispersion relation:

$$\omega_k^2 = m^2 + \tilde{k}(k)^2 = m^2 + \frac{1}{\varepsilon^2}\sin^2(\frac{2\pi k}{N}). \tag{67}$$

Notice the factor of 2 now absorbed in the sine, compared to the bosonic dispersion relation equation (24).

Similarly, we obtain the relation between the spinor components

$$\begin{pmatrix} m & \omega - \tilde{k} \\ \omega + \tilde{k} & m \end{pmatrix} \begin{pmatrix} c_{+,k}(\omega) \\ c_{-,k}(\omega) \end{pmatrix} = 0. \tag{68}$$

We set $\omega_k := +\sqrt{m^2 + \tilde{k}(k)^2}$, so the two allowed frequencies are $\omega = \omega_k$ and $\omega = -\omega_k$, for a given wavenumber k. Using $m = \sqrt{\omega - \tilde{k}}\sqrt{\omega + \tilde{k}}$ we find the relation

$$c_{+,k}(\omega)\sqrt{\omega + \tilde{k}} = -c_{-,k}(\omega)\sqrt{\omega - \tilde{k}}. \tag{69}$$

We can fill in $\omega = +\omega_k$ to find

$$c_k(+\omega_k) = a_k^\dagger \sqrt{\frac{\varepsilon L}{2\omega_k}} \begin{pmatrix} \sqrt{\omega_k - \tilde{k}} \\ -\sqrt{\omega_k + \tilde{k}} \end{pmatrix}, \tag{70}$$

and if we fill in $\omega = -\omega_k$, we find

$$c_k(-\omega_k) = b_k \sqrt{\frac{\varepsilon L}{2\omega_k}} \begin{pmatrix} \sqrt{\omega_k + \tilde{k}} \\ \sqrt{\omega_k - \tilde{k}} \end{pmatrix}. \tag{71}$$

We have chosen the prefactor $\sqrt{\frac{\varepsilon L}{2\omega_k}}$ such that the amplitudes a_k and b_k will become the annihilation operators for the two different modes after quantization. The reason to name the amplitude for the first mode $a_k^\dagger$ instead of a_k will become clear when we write down the Fourier transformed Hamiltonian. This relation between $c_k(+\omega_k)$ and $c_k(\omega_-)$, which is a direct consequence of the Dirac equation, will cause a significant difference with the bosons.

Now we can write down the Fourier transform of the discretized ψ field. The precise form will depend on the choice of boundary conditions for the two spinor components.

$$\begin{aligned} \psi_{+,j} &= \frac{1}{\sqrt{2N}} \sum_{k-\theta=0}^{N-1} \frac{1}{\sqrt{\omega_k}} e^{ijk\frac{2\pi}{N}} \left(a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k - \tilde{k}} + b_k e^{-i\omega_k t}\sqrt{\omega_k + \tilde{k}} \right), \\ \psi_{-,j} &= \frac{1}{\sqrt{2N}} \sum_{k-\theta=0}^{N-1} \frac{1}{\sqrt{\omega_k}} e^{ijk\frac{2\pi}{N}} \left(-a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k + \tilde{k}} + b_k e^{-i\omega_k t}\sqrt{\omega_k - \tilde{k}} \right). \end{aligned} \tag{72}$$

At this point we are ready to insert this in the Hamiltonian, but we can do a simple trick, simplifying the derivation significantly; We can now safely assume

that our fields will satisfy the Dirac equation equation (65). We can then use the following equality by rewriting the Dirac equation:

$$(m - i\gamma^1\partial_1)\psi = i\gamma^0\partial_0\psi. \tag{73}$$

So our Hamiltonian from equation (58) becomes:

$$H = \int_0^L dx\bar{\psi}i\gamma^0\partial_0\psi = \int_0^L dxi\psi^\dagger\partial_0\psi, \tag{74}$$

which we discretize to:

$$H = \sum_{j=0}^{N-1} i(\psi^\dagger_{+,j}\partial_0\psi_{+,j} + \psi^\dagger_{-,j}\partial_0\psi_{-,j}). \tag{75}$$

Now we insert the Fourier transforms of equation (72) to obtain:

$$\begin{aligned}
H =& \frac{1}{2N}\sum_{j=0}^{N-1}\sum_{k-\theta=0}^{N-1}\sum_{p-\theta=0}^{N-1}\frac{1}{\sqrt{\omega_p\omega_k}}e^{ij(k-p)\frac{2\pi}{N}}i \\
&\left(\left(a_p e^{-i\omega_p t}\sqrt{\omega_p - \tilde{k}(p)} + b_k^\dagger e^{i\omega_p t}\sqrt{\omega_p + \tilde{k}(p)}\right)\right. \\
&\quad \cdot (i\omega_k)\left(a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k - \tilde{k}(k)} - b_k e^{-i\omega_k t}\sqrt{\omega_k + \tilde{k}(k)}\right) \\
&+ \left(-a_p e^{-i\omega_p t}\sqrt{\omega_p + \tilde{k}(p)} + b_p^\dagger e^{i\omega_p t}\sqrt{\omega_p - \tilde{k}(p)}\right) \\
&\quad \left.\cdot(i\omega_k)\left(-a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k + \tilde{k}(k)} - b_k e^{-i\omega_k t}\sqrt{\omega_k - \tilde{k}(k)}\right)\right) \\
=& -\frac{1}{2}\sum_{k-\theta=0}^{N-1}\left(a_k a_k^\dagger(\omega_k - \tilde{k}) - b_k^\dagger b_k(\omega_k + \tilde{k}) - a_k b_k e^{-2i\omega_k t}m + b_k^\dagger a_k^\dagger e^{2i\omega_k t}m\right. \\
&\qquad \left. + a_k a_k^\dagger(\omega_k + \tilde{k}) - b_k^\dagger b_k(\omega_k - \tilde{k}) + a_k b_k e^{-2i\omega_k t}m - b_k^\dagger a_k^\dagger e^{2i\omega_k t}m\right) \\
=& \sum_{k-\theta=0}^{N-1}\omega_k(a_k^\dagger a_k + b_k^\dagger b_k - 1).
\end{aligned} \tag{76}$$

Where we used the equal time commutation relations

$$\begin{aligned}
&\{\psi_i, \psi_j^\dagger\} = \delta_{i,j}, \text{which is satisfied by} \\
&\{a_p, a_k^\dagger\} = \delta_{p,k} = \{b_p, b_k^\dagger\},
\end{aligned} \tag{77}$$

and all other commutations are equal to zero. This shows our operators a_k and b_k are indeed the proper annihilation operators, annihilating the left and right moving mode respectively, both with momentum $\frac{2\pi k}{N}$.

Let's introduce the two number operators $N_{a,k} = a_k^\dagger a_k$ and $N_{b,k} = b_k^\dagger b_k$, counting the number of modes with momentum k moving left and right respectively. Furthermore, we have:

$$\begin{aligned}
&\langle a_p a_k^\dagger\rangle = \langle\{a_p, a_k^\dagger\}\rangle - \langle a_k^\dagger a_p\rangle = \delta_{p,k}(1 - \langle N_{a,k}\rangle), \\
&\langle b_p^\dagger b_k\rangle = \delta_{p,k}\langle N_{b,k}\rangle,
\end{aligned} \tag{78}$$

and all other correlations vanish. Now we can compute the correlators for the spinor components. Again we choose the two sites of the correlators in our subsystem A, so they are multiples of p, and relabel $\psi_{ip} \to \psi_i$.

$$\begin{aligned}
\langle \psi^\dagger_{+,i}\psi_{+,j}\rangle &= \frac{1}{2N}\sum_{k-\theta=0}^{N-1}\frac{1}{\omega_k}e^{ik(j-i)\frac{2\pi}{N_A}}((1-\langle N_{a,k}\rangle)(\omega_k-\tilde{k})+\langle N_{b,k}\rangle(\omega_k+\tilde{k})),\\
\langle \psi^\dagger_{+,i}\psi_{-,j}\rangle &= \langle \psi^\dagger_{-,i}\psi_{+,j}\rangle = \frac{1}{2N}\sum_{k-\theta=0}^{N-1}\frac{m}{\omega_k}e^{ik(j-i)\frac{2\pi}{N_A}}(\langle N_{a,k}\rangle+\langle N_{b,k}\rangle-1),\\
\langle \psi^\dagger_{-,i}\psi_{-,j}\rangle &= \frac{1}{2N}\sum_{k-\theta=0}^{N-1}\frac{1}{\omega_k}e^{ik(j-i)\frac{2\pi}{N_A}}((1-\langle N_{a,k}\rangle)(\omega_k+\tilde{k})+\langle N_{b,k}\rangle(\omega_k-\tilde{k})).
\end{aligned} \tag{79}$$

Observe that for $k \in \mathbb{Z}+\theta$ and $n \in \{1,...,N_A\}$

$$e^{ik(N_A-n)\frac{2\pi}{N_A}} = e^{2\pi i\theta}e^{-ikn\frac{2\pi}{N_A}} \tag{80}$$

So the matrices in equation (79) are Toeplitz, and in general not circulant (only when $\theta = 0$ are they circulant), but they are phase circulant with phase $-2\pi\theta$. As the value of θ now is restricted to be either 0 of $1/2$, the correlation matrices are either circulant or anti-circulant.

Majorana Fermions

Suppose the considered fermions are Majorana fermions. We will first modify our Clifford algebra such that all entries are purely imaginary. This will ensure that the Majorana condition will demand that the spinor components are real. This can easily be achieved by setting

$$\tilde{\gamma}^0 = i\gamma^1 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \text{ and } \tilde{\gamma}^1 = i\gamma^0 = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{81}$$

The charge conjugation matrices are given by

$$C_+ = \pm\tilde{\gamma}^1, C_- = \pm\tilde{\gamma}^0, \text{ such that } C_\pm\tilde{\gamma}^\mu C_\pm^{-1} = \pm(\tilde{\gamma}^\mu)^T. \tag{82}$$

So the charge conjugated field is given by $\psi^c \equiv C_-\bar{\psi}^T = (\psi^\dagger)^T$. Solving the Dirac equation with this Clifford algebra gives the following Fourier transform of the fermion field:

$$\begin{aligned}
\psi_{+,j} &= \frac{1}{\sqrt{2N}}\sum_{k-\theta_+=0}^{N-1}\frac{i}{\sqrt{\omega_k}}e^{ijk\frac{2\pi}{N}}\left(a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k-\tilde{k}}+b_k e^{-i\omega_k t}\sqrt{\omega_k+\tilde{k}}\right),\\
\psi_{-,j} &= \frac{1}{\sqrt{2N}}\sum_{k-\theta_-=0}^{N-1}\frac{1}{\sqrt{\omega_k}}e^{ijk\frac{2\pi}{N}}\left(a_k^\dagger e^{i\omega_k t}\sqrt{\omega_k+\tilde{k}}-b_k e^{-i\omega_k t}\sqrt{\omega_k-\tilde{k}}\right).
\end{aligned} \tag{83}$$

The Majorana condition is $\psi^c = \psi$ which means $(\psi^\dagger)^T = \psi$, i.e. the fermion field has real spinor components. So θ_+ and θ_- have to be either 0 or 1/2. Furthermore, this condition will give a relation between the operators a and b. Let us check what the Majorana condition for the $+$ spinor component gives.

$$\begin{aligned}\psi^\dagger_{+,j} &= -\frac{1}{\sqrt{2N}} \sum_{k-\theta_+=0}^{N-1} \frac{i}{\sqrt{\omega_k}} e^{-ijk\frac{2\pi}{N}} \left(a_k e^{-i\omega_k t}\sqrt{\omega_k - \tilde{k}} + b_k^\dagger e^{i\omega_k t}\sqrt{\omega_k + \tilde{k}} \right) \\ &= \frac{1}{\sqrt{2N}} \sum_{k-\theta_+=0}^{N-1} \frac{i}{\sqrt{\omega_k}} e^{ijk\frac{2\pi}{N}} \left(a_{-k} e^{-i\omega_k t}\sqrt{\omega_k + \tilde{k}} + b_{-k}^\dagger e^{-i\omega_k t}\sqrt{\omega_k - \tilde{k}} \right),\end{aligned} \tag{84}$$

by change of variables $k \to -k$ and using $\omega_k = \omega_{-k}$, $\tilde{k}(-k) = -\tilde{k}(k)$. Now using the Fourier decomposition theory we obtain $b_k = a_{-k}$. Hence $N_{b,k} = N_{a,-k}$, so for the Majorana fermions, after discretizing, the correlators of equation (79) become:

$$\begin{aligned}\langle \psi^\dagger_{+,i}\psi_{+,j}\rangle &= \frac{1}{2N} \sum_{k-\theta_+=0}^{N-1} \frac{1}{\omega_k} e^{ik(j-i)\frac{2\pi}{N_A}} ((1-\langle N_{a,k}\rangle)(\omega_k - \tilde{k}) + \langle N_{a,-k}\rangle(\omega_k + \tilde{k})), \\ \langle \psi^\dagger_{+,i}\psi_{-,j}\rangle &= \langle \psi^\dagger_{-,i}\psi_{+,j}\rangle = \frac{1}{N} \sum_{k-\theta=0}^{N-1} \frac{m}{\omega_k} \cos(k(j-i)\frac{2\pi}{N_A})(\langle N_{a,k}\rangle - \frac{1}{2}), \\ \langle \psi^\dagger_{-,i}\psi_{-,j}\rangle &= \frac{1}{2N} \sum_{k-\theta_-=0}^{N-1} \frac{1}{\omega_k} e^{ik(j-i)\frac{2\pi}{N_A}} ((1-\langle N_{a,k}\rangle)(\omega_k + \tilde{k}) + \langle N_{a,-k}\rangle(\omega_k - \tilde{k})).\end{aligned} \tag{85}$$

Similarly, these matrices are also phase circulant with phase $-2\pi\theta_\pm$.

Massless Fermions

If we set $m = 0$ the spinor components decouple and the Dirac equation actually gives us two Weyl equations with opposite chirality. Filling in the plane wave ansatz $\psi_\pm = c_k(\omega_\pm)e^{i(kx-\omega_\pm t)}$, we obtain $(\omega_\pm \pm \tilde{k})\psi_\pm = 0$ thus the dispersion relation is

$$\omega_{\pm,k} = \mp\tilde{k}(k). \tag{86}$$

The discretization procedure and periodicity demand are similar to the massive case. Then the general solution for the components of ψ is

$$\psi_{+,j} = \frac{1}{\sqrt{N}} \sum_{k-\theta_+=0}^{N-1} e^{i(jk\frac{2\pi}{N} - \tilde{k}t)} b_k, \tag{87}$$

$$\psi_{-,j} = \frac{1}{\sqrt{N}} \sum_{k-\theta_-=0}^{N-1} e^{i(jk\frac{2\pi}{N} + \tilde{k}t)} a_k^\dagger. \tag{88}$$

Note that the two components may have different boundary conditions. a_k and b_k are the annihilation operators of the $-$ and $+$ spinor components

respectively. The use of the same characters a and b as is the massive case is somewhat misleading, because they are not the same; the a and b are the same operators for both the massive and massless case for those wavenumber indices k for which $\tilde{k}(k)$ is positive. Otherwise, the roles of a and b are interchanged between the two cases.

The massless Hamiltonian is

$$H = \sum_{k-\theta_+=0}^{N-1} \tilde{k}(k) b_k^\dagger b_k + \sum_{k-\theta_-=0}^{N-1} \tilde{k}(k)(a_k^\dagger a_k - 1). \tag{89}$$

Notice that the groundstate is not equal to the vacuum, because $\tilde{k}$ can be negative. This, however, is just a matter of when we call a state filled or empty. In fact, if we would redefine our annihilation operators in such a way that the groundstate is the vacuum (that is, $b_k^\dagger b_k \to (1 - b_k^\dagger b_k)$ whenever $\tilde{k}(k)$ is negative, and vice versa for a_k) then we would have the same creation and annihilation operators as in the massive case. So, physically the massless limit of the massive theory indeed is the correct limit. However, we have chosen to adopt this new notation for the massless case, to better understand the underlying physics.

Because the two Weyl spinors are decoupled, the correlations are easily computed to be

$$\begin{aligned}
\langle \psi_{+,i}^\dagger \psi_{+,j} \rangle &= \frac{1}{N} \sum_{k-\theta_+=0}^{N-1} e^{ik(j-i)\frac{2\pi}{N_A}} \langle N_{b,k} \rangle, \\
\langle \psi_{+,i}^\dagger \psi_{-,j} \rangle &= \langle \psi_{-,i}^\dagger \psi_{+,j} \rangle = 0, \\
\langle \psi_{-,i}^\dagger \psi_{-,j} \rangle &= \frac{1}{N} \sum_{k-\theta_-=0}^{N-1} e^{ik(j-i)\frac{2\pi}{N_A}} (1 - \langle N_{a,k} \rangle).
\end{aligned} \tag{90}$$

Which are, again, phase circulant with phase $-2\pi\theta_\pm$.

4.2 THE ENTANGLEMENT ENTROPY

The theory of phase circulant matrices developed in chapter 2 allows us to directly obtain the eigenvalues of the correlation matrices. Thus, we will now turn our attention to the computation of the entanglement entropy from these eigenvalues. Similar to the bosonic theory, we will use the eigenvalues to construct the reduced density matrix. Fortunately, this construction is less involved, as symplectic eigenvalues are not needed here [28].

Define the correlation matrix C of all two point correlation functions between the spinor components on the lattice sites of the subsystem:

$$C = \begin{pmatrix} C^{++} & C^{+-} \\ C^{-+} & C^{--} \end{pmatrix}, \tag{91}$$

with

$$C_{ij}^{\pm\pm} = \langle \psi_{\pm,i}^\dagger \psi_{\pm,j} \rangle \text{ , and } C_{ij}^{\pm\mp} = \langle \psi_{\pm,i}^\dagger \psi_{\mp,j} \rangle. \tag{92}$$

Recall that ψ_j is the fermion wave function on the $(jp)^{th}$ lattice site of the total system. One could also define $\eta = (\psi_{+,0}, ..., \psi_{+,N_A-1}, \psi_{-,0}, ..., \psi_{-,N_A-1})$ and subsequently define $C = \langle \eta^\dagger \eta \rangle$.

The two point correlators are relevant, because they define the reduced density matrix ρ_A of our subsystem A:

$$\langle \eta_i^\dagger \eta_j \rangle = Tr_A(\rho_A \eta_i^\dagger \eta_j), \tag{93}$$

where $\rho_A = Tr_{A^c}(\rho)$ with ρ the density matrix of the total system. Again, Wick's theorem gives us that all expectation values are determined from these two point correlation functions. We then can write

$$\rho_A = \mathcal{K} e^{-\mathcal{H}}, \tag{94}$$

with $\mathcal{K}$ a normalization constant ensuring $Tr(\rho) = Tr_A(\rho_A) = 1$ and

$$\mathcal{H} = \sum_{i,j} \chi_i^\dagger H_{i,j} \chi_j. \tag{95}$$

We have yet to determine the eigenvalues and eigenvectors of this matrix H. Suppose ϕ_m are the eigenvectors with eigenvalue ε_m. Then we can decompose

$$\chi_i = \sum_{m=0}^{2N_A-1} \phi_{m,i} c_m, \tag{96}$$

where $\phi_{m,i}$ is the i^{th} component of ϕ_m and c_m are new fermion operators satisfying the standard commutation relations. Then

$$H_{i,j} = \sum_{m=0}^{2N_A-1} \varepsilon_m \phi_{m,i} \phi_{m,j}^\dagger. \tag{97}$$

And

$$\mathcal{H} = \sum_{m=0}^{2N_A-1} \varepsilon_m c_m^\dagger c_m. \tag{98}$$

From demanding $Tr_A(\rho_A) = 1$, we can determine the normalization factor $\mathcal{K}$:

$$\begin{aligned} Tr_A(\rho_A) &= \mathcal{K} Tr(e^{-\sum_m \varepsilon_m c_m^\dagger c_m}) \\ &= \mathcal{K} \prod_m Tr_m(e^{-\varepsilon_m c_m^\dagger c_m}) \\ &= \mathcal{K} \prod_m (1 + e^{-\varepsilon_m}). \end{aligned} \tag{99}$$

So

$$\mathcal{K} = \prod_{m=0}^{2N_A-1} \frac{1}{1 + e^{-\varepsilon_m}}. \tag{100}$$

We used that since $\mathcal{H}$ is now in diagonal form, the operator $e^{-\sum_m \varepsilon_m c_m^\dagger c_m}$ is a tensor product of operators $e^{-\varepsilon_m c_m^\dagger c_m}$ and the trace of a tensor product is

the product over the traces. Furthermore we used that the operators c_m are fermionic, hence obey Pauli-Dirac statistics. So the only eigenvalues of $c_m^\dagger c_m$ are 0 and 1, which are non degenerate. We denoted Tr_m as the trace over the subspace of the entire Fock space of the m^{th} particle.

Next, we can compute the correlation matrix in terms of the eigenvalues of H using the reduced density matrix. Suppose $m \neq l$, then again using the product rule, we have

$$\begin{aligned} Tr_{m,l}(e^{-\varepsilon_m c_m^\dagger c_m} c_l^\dagger c_l) &= Tr_m(e^{-\varepsilon_m c_m^\dagger c_m}) Tr_l(c_l^\dagger c_l) \\ &= (1 + e^{-\varepsilon_m}). \end{aligned} \tag{101}$$

Furthermore,

$$Tr_l(e^{-\varepsilon_l c_l^\dagger c_l} c_l^\dagger c_l) = e^{-\varepsilon_l}. \tag{102}$$

So

$$\begin{aligned} Tr_A(\rho_A \eta_i^\dagger \eta_j) &= \mathcal{K} Tr_A(e^{-\sum_m \varepsilon_m c_m^\dagger c_m} \sum_l c_l^\dagger c_l) \phi_{l,i}^\dagger \phi_{l,j} \\ &= \mathcal{K} \sum_l Tr_A(e^{-\sum_m \varepsilon_m c_m^\dagger c_m} c_l^\dagger c_l) \phi_{l,i}^\dagger \phi_{l,j} \\ &= \mathcal{K} \sum_l e^{-\varepsilon_l} \prod_{m \neq l} (1 + e^{-\varepsilon_m}) \phi_{l,i}^\dagger \phi_{l,j} \\ &= \sum_l \frac{e^{-\varepsilon_l}}{1 + e^{-\varepsilon_l}} \phi_{l,i}^\dagger \phi_{l,j} \\ &= \sum_l \frac{1}{1 + e^{\varepsilon_l}} \phi_{l,i}^\dagger \phi_{l,j}. \end{aligned} \tag{103}$$

So $\phi_m^\dagger$ are the eigenstates of the correlation matrix C with eigenvalues $\lambda_m = \frac{1}{1+e^{\varepsilon_m}}$. Inverting this relation, we obtain

$$\varepsilon_m = \ln\left(\frac{1 - \lambda_m}{\lambda_m}\right). \tag{104}$$

So, at this point we must turn our attention to the eigenvalues λ_m of the correlation matrix C. We can consider two cases:

- $\mathbf{m = 0}$
  From equation (90) we immediately get $C^{\pm\mp} = 0$. Furthermore, $C^{\pm\pm}$ is phase circulant with phase $-2\pi\theta_\pm$. The plus and minus spinor components are now decoupled and the eigenvalues of $C^{\pm\pm}$ are given by:

$$\begin{aligned} \lambda_m^+ &= \sum_{l=0}^{N_A - 1} \frac{1}{N} \sum_{k - \theta_+ = 0}^{N-1} e^{-ikl\frac{2\pi}{N_A}} \langle N_{b,k} \rangle e^{il(m+\theta_+)\frac{2\pi}{N_A}} \\ &= \frac{1}{p} \sum_{n=0}^{p-1} \langle N_{b,\tilde{m}_+} \rangle \end{aligned} \tag{105}$$

Where we defined $\tilde{m}_+(n) = m + \theta_+ + nN_A$ and used $\sum_{l=0}^{N_A-1} e^{il(k-(m+\theta)\frac{2\pi}{N_A}} = N_a \sum_{n=0}^{p-1} \delta_{k,\tilde{m}_+}$ and $p = N/N_A$. Similarly,

$$\lambda_m^- = \frac{1}{p}\sum_{n=0}^{p-1}(1 - \langle N_{a,\tilde{m}_-}\rangle). \tag{106}$$

Notice that the explicit θ dependence has dropped out.

- **$\mathbf{m \neq 0}$**
  Now the off-diagonal blocks $C^{\pm\mp}$ do not vanish and the diagonalization becomes more involved. Fortunately, $m \neq 0$ couples the spinor components, and specifically makes $\theta_+ = \theta_- = \theta$ as noted before. This makes the matrices in equation (79) phase circulant with the same phase $-2\pi\theta$. The nice thing about phase circulant matrices, is that their eigenvectors (equation (12)) depend only on the phase and not on the matrix coefficients. Let P be the matrix whose columns are the eigenvectors in equation (12). This matrix diagonalizes all four blocks in C, hence the block diagonal matrix with two copies of P on the diagonal puts C in the following form:

$$C = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \Lambda_{++} & \Lambda_{+-} \\ \Lambda_{-+} & \Lambda_{--} \end{pmatrix} \begin{pmatrix} P^\dagger & 0 \\ 0 & P^\dagger \end{pmatrix}. \tag{107}$$

  The Λ matrices are the diagonal matrices with the eigenvalues of the corresponding block of C on its diagonal. We will now give their diagonal entries:

  - $\Lambda_{\pm\pm} = diag(\lambda_0^{\pm\pm}, ..., \lambda_{N_A-1}^{\pm\pm})$ where the $\lambda_m^{\pm\pm}$ are given by

$$\lambda_m^{\pm\pm} = \frac{1}{2p}\sum_{n=0}^{p-1}\frac{1}{\omega_{\tilde{m}}}\left((1 - \langle N_{a,\tilde{m}}\rangle)(\omega_{\tilde{m}} \mp \tilde{k}(\tilde{m})) + \langle N_{b,\tilde{m}}\rangle(\omega_{\tilde{m}} \pm \tilde{k}(\tilde{m}))\right). \tag{108}$$

    Where we defined $\tilde{m}(n) = m + \theta + nN_A$
  - $\Lambda_{+-} = \Lambda_{-+} = diag(\lambda_0^{+-}, ..., \lambda_{N_A-1}^{+-})$ with

$$\lambda_m^{+-} = \frac{1}{2p}\sum_{n=0}^{p-1}\frac{m}{\omega_{\tilde{m}}}(\langle N_{a,\tilde{m}}\rangle + \langle N_{b,\tilde{m}}\rangle - 1), \tag{109}$$

    by a similar calculation.

Now we are not there yet, since although the blocks of C are now simultaneously diagonalized, C itself is not yet in diagonal form. However, the problem of diagonalizing C has now been reduced to diagonalizing the following 2×2 matrices (see appendix B for a proof of this claim):

$$\begin{pmatrix} \lambda_m^{++} & \lambda_m^{+-} \\ \lambda_m^{+-} & \lambda_m^{--} \end{pmatrix}.$$

This can be done directly, as it amounts to finding the roots of a 2^{nd} degree polynomial. In general, the result is in general not that elegant however. Therefor, we will study some explicit systems later on to gain more insight. The general result is

$$\lambda_m^{\pm} = \frac{1}{2}\left(\lambda_m^{++} + \lambda_m^{--} \pm \sqrt{(\lambda_m^{++} + \lambda_m^{--})^2 - 4(\lambda_m^{++}\lambda_m^{--} - (\lambda_m^{+-})^2)}\right). \tag{110}$$

When we are considering the groundstate $\langle N_{a,k}\rangle = \langle N_{b,k}\rangle = 0$ for all k, observe that $\lambda_m^{++} + \lambda_m^{--} = 1$ as the $\tilde{k}(\tilde{m})$ terms cancel. Furthermore, when p is even we have $\tilde{k}(m+\theta+nN_A) = -\tilde{k}(m+\theta+(n+\frac{p}{2})N_A)$, whilst $\omega_{\tilde{m}(n)} = \omega_{\tilde{m}(n+\frac{p}{2})}$. So all terms in the sum over n cancel in equation (108), yielding $\lambda^{\pm\pm} = 1/2$. The θ dependence is only found in λ_m^{+-}. The result then becomes

$$\lambda_m^{\pm} = \frac{1}{2}\left(1 \pm \lambda_m^{+-}\right). \tag{111}$$

Entanglement Entropy

The entanglement entropy of the subsystem A is determined by the reduced density matrix:

$$S_A = -Tr(\rho_A \log \rho_A). \tag{112}$$

We can write ρ_A as a tensor product because we have diagonalized $\mathcal{H}$:

$$\rho_A = \bigotimes_{l=0}^{2N_A-1} \rho_l \text{ , with } \rho_l = \frac{e^{-\varepsilon_l c_l^\dagger c_l}}{1+e^{-\varepsilon_l}}. \tag{113}$$

Here, ρ_l is restricted to the particle with momentum l. This makes ρ_l a reduced density matrix of the l^{th} site. As $Tr_l(\rho_l) = 1$, also $Tr(\rho_A) = Tr_A(\rho_A) = \prod_l Tr_l(\rho_l) = 1$. This gives us

$$\begin{aligned}\rho_A \log(\rho_A) &= \bigotimes_{l=0}^{2N_A-1} \rho_l \log\Big(\bigotimes_{l=0}^{2N_A-1} \rho_l\Big) \\ &= \bigoplus_{l=0}^{2N_A-1} \Big(\bigotimes_{n=0}^{l-1} \rho_n\Big) \otimes \rho_l \log(\rho_l) \otimes \Big(\bigotimes_{n=l+1}^{2N_A-1} \rho_n\Big)\end{aligned} \tag{114}$$

So for the entanglement entropy we get

$$\begin{aligned}S_A &= -\sum_l \Big(\prod_{n=0}^{l-1} Tr_n(\rho_n)\Big) Tr_l(\rho_l \log \rho_l) \Big(\prod_{n=l+1}^{2N_A-1} Tr_n(\rho_n)\Big) \\ &= -\sum_l Tr_l(\rho_l \log \rho_l).\end{aligned} \tag{115}$$

Filling in the two eigenvalues of ρ_l, and using the relation in equation (104) we get:

$$\begin{aligned} S_A &= -\sum_m \frac{1}{1+e^{-\varepsilon_l}} \log \frac{1}{1+e^{-\varepsilon_l}} + \frac{e^{-\varepsilon_l}}{1+e^{-\varepsilon_l}} \log \frac{e^{-\varepsilon_l}}{1+e^{-\varepsilon_l}} \\ &= -\sum_l (1-\lambda_l^+)\log(1-\lambda_l^+) + \lambda_l^+ \log \lambda_l^+ \\ &\qquad + (1-\lambda_l^-)\log(1-\lambda_l^-) + \lambda_l^- \log \lambda_l^- \end{aligned} \tag{116}$$

with $\lambda_l^\pm$ from equations (105) and (106) ($m = 0$) or equation (110) ($m \neq 0$).

If we study the massless groundstate, the entanglement entropy takes on a particular form. For the case $m = 0$ we need to consider the Hamiltonian in equation (89) in order to determine the groundstate. We thus need to ask for what values of k, the value of $\tilde{k}$ is negative. That is, when $k > N/2$. For those k we have $\langle N_{a,k} \rangle = \langle N_{b,k} \rangle = 1$ in the ground state and the other expectation values are zero. So we can directly compute the eigenvalues from equations (105) and (106)

$$\lambda_m^+ = \begin{cases} \frac{\lfloor p/2 \rfloor}{p} & \text{if } m < N_A/2 \\ \frac{\lceil p/2 \rceil}{p} & \text{otherwise} \end{cases}, \tag{117}$$

$$\lambda_m^- = \begin{cases} \frac{\lceil p/2 \rceil}{p} & \text{if } m < N_A/2 \\ \frac{\lfloor p/2 \rfloor}{p} & \text{otherwise} \end{cases}. \tag{118}$$

Notice that $1 - \lambda_m^\pm = \lambda_m^\mp$. We can insert this in equation (116) to obtain

$$S_A = 2N_A \left(\frac{\lfloor p/2 \rfloor}{p} \log \left(\frac{p}{\lfloor p/2 \rfloor} \right) + \frac{\lceil p/2 \rceil}{p} \log \left(\frac{p}{\lceil p/2 \rceil} \right) \right). \tag{119}$$

So for even p, $S_A = 2N_A \log(2)$. Observe also that the result does not depend on the boundary conditions. This result was also found in [30] for the case when $p = 2$. The factor of 2 arises because we are studying two independent systems; in the massless limit the Dirac fermion splits into two Weyl fermions.

4.3 SANITY CHECK

Before looking at the results, we want to check is this general result is in line with a simple case where we can directly calculate the entanglement entropy. The direct approach will consist of manually writing down the reduced density matrix, by changing from the natural basis of our annihilation operators a_k and b_k to the basis of our fields ψ_j. This can then be checked with the above formula.

Consider the ground state of our system. Thus $\langle N_{a,k} \rangle = \langle N_{b,k} \rangle = 0$ for all k (that is, if $m \neq 0$. If $m = 0$ the ground state sets $\langle N_{a,k} \rangle = \langle N_{b,k} \rangle = 0$ for all $k < N/2$ and $\langle N_{a,k} \rangle = \langle N_{b,k} \rangle = 1$ for the other values of k). Take $N = p = 2$, so $N_A = 1$.Then $k := \tilde{k}(\theta) = -\tilde{k}(\theta + 1)$ and $\omega := \omega_\theta = \omega_{\theta+1}$.

Direct approach

Since the operators $\psi_{\pm,j}$ satisfy the fermionic anticommutation relations, they correspond to annihilation operators. We denote a ket state in the following convention:

$$|abcd\rangle = (\psi^\dagger_{-,1})^d(\psi^\dagger_{-,0})^c(\psi^\dagger_{+,1})^b(\psi^\dagger_{+,0})^a|0000\rangle, \tag{120}$$

with $a, b, c, d \in 0, 1$. We express the groundstate in terms of the quantum numbers associated with the ψ field.

$$|0\rangle := \frac{1}{2\omega}\left(m|0011\rangle + ke^{i\pi\theta}|0101\rangle + \omega|0110\rangle - \omega|1001\rangle - ke^{-i\pi\theta}|1010\rangle + m|1100\rangle\right),$$

We now want to compute the entanglement entropy of this state. The corresponding density matrix is $\rho = |0\rangle\langle 0|$. Now we trace out the subsystem of site 1. Once again care needs to be taken with permuting operators, since they anti commute. For example

$$\begin{aligned} Tr_1(|0110\rangle\langle 1100|) &= (id \times \langle 00|)\psi_{+,1}\psi^\dagger_{-,0}\psi^\dagger_{+,1}|0000\rangle\langle 0000|\psi_{+,0}\psi_{+,1}\psi^\dagger_{+,1}(id \times |00\rangle) \\ &= -|01\rangle\langle 10|, \end{aligned} \tag{121}$$

because we need to permute the two most left operators. After some more calculations like these, we obtain

$$\begin{aligned} \rho_0 = \frac{1}{4\omega^2}\big((m^2+\omega^2)(|01\rangle\langle 01| + |10\rangle\langle 10|) + k^2(|11\rangle\langle 11| + |00\rangle\langle 00|) \\ -2m\omega(|01\rangle\langle 10| + |10\rangle\langle 01|)\big), \end{aligned} \tag{122}$$

To determine the eigenvalues of ρ_0 we first compute how it acts on the obvious basis:

- $\rho_0|00\rangle = \frac{k^2}{4\omega^2}|00\rangle$,
- $\rho_0|01\rangle = \frac{1}{4\omega^2}((m^2+\omega^2)|01\rangle - 2m\omega|10\rangle)$,
- $\rho_0|10\rangle = \frac{1}{4\omega^2}((m^2+\omega^2)|10\rangle - 2m\omega|01\rangle)$,
- $\rho_0|11\rangle = \frac{k^2}{4\omega^2}|11\rangle$.

So the eigen-kets of ρ_0 are

- $|00\rangle$ with eigenvalue $\frac{k^2}{4\omega^2}$,
- $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ with eigenvalue $\frac{1}{4\omega^2}(\omega - m)^2$,
- $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ with eigenvalue $\frac{1}{4\omega^2}(\omega + m)^2$,
- $|11\rangle$ with eigenvalue $\frac{k^2}{4\omega^2}$.

Thus we can now compute the entanglement entropy

$$\begin{aligned} S_0 =& \frac{-1}{4\omega^2}\left((\omega-m)^2\log\left(\frac{(\omega-m)^2}{(2\omega)^2}\right)+(\omega+m)^2\log\left(\frac{(\omega+m)^2}{(2\omega)^2}\right)+2k^2\log\frac{k^2}{4\omega^2}\right) \\ =& -\left(\left(\frac{\omega-m}{\omega}-\frac{k^2}{2\omega^2}\right)\log\left(\frac{\omega-m}{2\omega}\right)+\left(\frac{\omega+m}{\omega}-\frac{k^2}{2\omega^2}\right)\log\left(\frac{\omega+m}{2\omega}\right)\right. \\ &\left.+\frac{k^2}{2\omega^2}\left(\log\left(\frac{\omega-m}{2\omega}\right)+\log\left(\frac{\omega+m}{2\omega}\right)\right)\right) \\ =& -\left(\frac{\omega-m}{\omega}\log\left(\frac{\omega-m}{2\omega}\right)+\frac{\omega+m}{\omega}\log\left(\frac{\omega+m}{2\omega}\right)\right). \end{aligned} \tag{123}$$

We used $m^2=\omega^2-k^2$, so $\frac{(\omega\pm m)^2}{2\omega^2}=\frac{\omega\pm m}{\omega}-\frac{k^2}{2\omega^2}$ and $k^2=(\omega+m)(\omega-m)$.

Check with general formula

Let us compare this with our general formula from equation (116). Filling in $\langle N_{a,\theta}\rangle=\langle N_{a,\theta+1}\rangle=\langle N_{b,\theta}\rangle=\langle N_{b,\theta+1}\rangle=0$ in equations (108) and (109), we obtain

- $\lambda^{++}=\frac{1}{2p}\sum_{n=0}^{p-1}\frac{1}{\omega_{\tilde{m}}}(\omega_{\tilde{m}}-\tilde{k}(\tilde{m}))=\frac{1}{2}$ because $k:=\tilde{k}(\theta)=-\tilde{k}(\theta+1)$ and $\omega:=\omega_\theta=\omega_{\theta+1}$.
- $\lambda^{--}=\frac{1}{2}$ similarly.
- $\lambda^{+-}=\frac{-1}{2p}\sum_{n=0}^{p-1}\frac{m}{\omega_{\tilde{m}}}=\frac{-m}{2\omega}$.

This is then inserted into equation (110) to obtain

$$\lambda^{\pm}=\frac{\omega\pm m}{2\omega}. \tag{124}$$

Notice that

$$1-\lambda^{\pm}=\frac{\omega\mp m}{2\omega}=\lambda^{\mp}. \tag{125}$$

So by use of equation (116) we find

$$S_0=-\left(\frac{\omega-m}{\omega}\log\left(\frac{\omega-m}{2\omega}\right)+\frac{\omega+m}{\omega}\log\left(\frac{\omega+m}{2\omega}\right)\right). \tag{126}$$

Which is in line with our direct computation.

Note that, as expected, the result for the massless limit in the direct approach also is correct compared to the formula for $m=0$, but the computation is a bit different: The eigenvalues in equations (105) and (106) immediately yield $S_0=2\log(2)$.

4.4 LARGE n LIMIT

The limit $N \to \infty$ and $N_A \to \infty$ keeping $p = 2$ fixed is studied in [29] for bosons. To compare the boson results with the fermion system, we take the same limit here. The ground state now is completely determined by $\langle N_{a,k} \rangle = \langle N_{b,k} \rangle = 0$. We have the following simplification from setting $p = 2$.

$$\begin{aligned} \tilde{k}(\tilde{m}(n)) &= \frac{1}{\varepsilon} \sin(\frac{2\pi(m + \theta + nN_A)}{N}) \\ &= \frac{(-1)^n}{\varepsilon} \sin(\frac{2\pi(m+\theta)}{N}). \end{aligned} \tag{127}$$

So $\omega_{\tilde{m}} = \omega_m$ does not depend on the index n anymore. Furthermore the summations in equations (108) and (109) now can be computed explicitly, giving for each $m \in \{0, ..., N_A - 1\}$

$$\lambda_m^{++} = \lambda_m^{--} = \frac{1}{2}, \tag{128}$$

$$\lambda_m^{+-} = \frac{-m}{2\omega_m}. \tag{129}$$

Thus our eigenvalues of the reduced correlation matrix are now of the same form as in the sanity check and the entanglement entropy becomes

$$S_A = -\sum_{m=0}^{N_A - 1} \frac{\omega_{m+\theta} - m}{\omega_{m+\theta}} \log\left(\frac{\omega_{m+\theta} - m}{2\omega_{m+\theta}}\right) + \frac{\omega_{m+\theta} + m}{\omega_{m+\theta}} \log\left(\frac{\omega_{m+\theta} + m}{2\omega_{m+\theta}}\right). \tag{130}$$

In the limit $N \to \infty$, whilst keeping $p = 2$ fixed, the sum becomes an integral

$$S_A = -2N_A \int_0^{\frac{1}{2}} \frac{\omega_x - m}{\omega_x} \log\left(\frac{\omega_x - m}{2\omega_x}\right) + \frac{\omega_x + m}{\omega_x} \log\left(\frac{\omega_x + m}{2\omega_x}\right), \tag{131}$$

with

$$\omega_x^2 = m^2 + \frac{1}{\varepsilon^2} \sin^2(2\pi x). \tag{132}$$

With the above formula, we can compute the entanglement entropy numerically. The results are shown in a later section.

4.5 LIFSHITZ DIRAC THEORY

A way to investigate long range interactions, is to extend the known Dirac Lagrangian to a Lifshitz scaled version. In this process, the scale invariance (which we had for $m = 0$) becomes somewhat skewed. At this point we will shortly abandon our $c = \hbar = 1$ simplification, because the units will eventually tell us which parameters determine the entanglement entropy. Our new Lagrangian will be

$$\mathcal{L}_{Lifshitz} = \bar{\psi}(\hbar\gamma^0 i\partial_0 + \hbar\alpha\gamma^1(i\partial_1)^z - \mu\alpha^2)\psi. \tag{133}$$

Here, α is the skewed "speed of light" with units m^z/s and μ is the skewed "mass" with units kg/m^{2z-2}. Thus we can define the parameter $J_f = \mu\alpha\varepsilon^z$, which has the same dimension as the entanglement entropy. The Lifshitz scale invariance reads

$$t \to \lambda^z t, \qquad x \to \lambda x. \tag{134}$$

The exponent z is called the *dynamical critical exponent*. The reason that i is also exponentiated in the second term in the Lagrangian, is to preserve Hermiticity (as we need to perform integration by parts z times to obtain the original Lagrangian after taking its Hermitean conjugate).

The intuition that adding long range interactions will increase the entanglement entropy is investigated less easily compared to the boson system. For the continuous system we have for example

$$\langle\overline{\psi(0)}\psi(r)\rangle = \int \frac{dk}{2\pi}\frac{\omega_k + k^z}{2\omega_k}e^{ikr}. \tag{135}$$

The additional term compared to equation (45) arises due to the summation over spinor components. If $m = 0$ then $\omega_k = k^z$ so the entanglement entropy does not scale with z as we had for bosons. In fact, we see that in the case of $m = 0$ the z dependence even drops out. This is also immediate from equations (105) and (106). This however does not compromise our intuition: In the bosonic case the entanglement entropy is unbounded in the sense that $S_A \to \infty$ as the eigenvalues of the correlation matrices $\lambda \to \infty$, which can be read of equation (43). The entanglement entropy for the fermionic system is given by equation (116), which is bounded and takes on its maximal value $S_A = 2N_A \log(2)$ if and only if all eigenvalues are equal to $\frac{1}{2}$. Thus we expect the eigenvalues of the correlation matrix to approach $\frac{1}{2}$ as z increases.

Translation

We can apply the same machinery of our previous work to this Lagrangian. The Lifshitz modification affects the theory at a fundamental level, overall derivations remain the same. We will derive its effect on the fundamental parameter (the dispersion relation) and thus obtain a translation to generalize our previous results.

The Euler-Lagrange formalism gives

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (\hbar\gamma^0 i\partial_0 + \hbar\alpha\gamma^1(i\partial_1)^z - \mu\alpha^2)\psi = 0. \tag{136}$$

We discretize and rescale (to make ψ dimensionless) and make the plane wave ansatz $\psi_j = c_k(\omega)e^{i(jk\frac{2\pi}{N}+\omega t)}$. Recall

$$\partial_1\psi_j = i\tilde{k}(k)\psi_j = \frac{i}{\varepsilon}\sin(\frac{2\pi k}{N}). \tag{137}$$

Hence

$$(i\partial_1)^z \psi_j = (-\tilde{k}(k))^z \psi_j. \tag{138}$$

This result is obtained more easily compared to the bosonic case. Using the equations of motion from equation (136) we obtain the dispersion relation

$$\hbar^2 \omega_k^2 = \alpha^4 \mu^2 + \hbar^2 \alpha^2 \tilde{k}(k)^{2z}. \tag{139}$$

A general solution will thus be a superposition of such plane waves. We obtain

$$\begin{aligned}
\psi_{+,j} &= \frac{1}{\sqrt{2N}} \sum_{k-\theta=0}^{N-1} \frac{1}{\sqrt{\omega_k}} e^{ijk\frac{2\pi}{N}} \left(a_k^\dagger e^{i\omega_k t} \sqrt{\omega_k + \alpha(-\tilde{k})^z} + b_k e^{-i\omega_k t} \sqrt{\omega_k - \alpha(-\tilde{k})^z} \right), \\
\psi_{-,j} &= \frac{1}{\sqrt{2N}} \sum_{k-\theta=0}^{N-1} \frac{1}{\sqrt{\omega_k}} e^{ijk\frac{2\pi}{N}} \left(-a_k^\dagger e^{i\omega_k t} \sqrt{\omega_k - \alpha(-\tilde{k})^z} + b_k e^{-i\omega_k t} \sqrt{\omega_k + \alpha(-\tilde{k})^z} \right).
\end{aligned} \tag{140}$$

At this point we can set $\hbar = \alpha = 1$ as a convention, but notice that this simplification actually depends on the value of z. The mass-dimension of μ then again is 1, but ε will have mass-dimension $-1/z$. Hence a natural dimensionless parameter is $\mu\varepsilon^z$.

The Hamiltonian remains of the same form as in equation (76), but now has different energy levels, since the dispersion relation has changed. It seems as though all previous results can be translated to results for this Lifshitz theory, just by replacing $\tilde{k} \to -(-\tilde{k})^z$.

This translation seems to conclude the calculations on our Lifshitz theory. But often it cannot be used in the final results in the previous sections. The translation must be applied earlier on in the machinery. As we will see, some peculiar situations will arise.

Massive fermions

We start with the massive case; a remarkable situation arises when $p = 2$ and z is even. In that case we have

$$\lambda_m^{\pm\pm} = \frac{\omega_{m+\theta} + \tilde{k}^z}{2\omega_{m+\theta}}, \tag{141}$$

$$\lambda_m^{+-} = \frac{-\mu}{2\omega_{m+\theta}}. \tag{142}$$

Thus using $\omega^2 = \mu^2 + \tilde{k}^{2z}$ we see that the eigenvalues of the correlation matrix become either 0 or 1. Hence when $p = 2$ and z is even $S_A = 0$. At first this would seem odd, but when considering what physical system we are studying in this case, the result makes complete sense. When we discretize the Lagrangian in equation (133) prior to making the plane wave ansatz we claim that the interaction between the sites given by the term proportional to

$$\psi_j^\dagger \partial_1^z \psi_j = \psi_j^\dagger \frac{1}{(2\varepsilon)^z} \sum_{n=0}^{z} (-1)^n \binom{z}{n} \psi_{j+z-2n}. \tag{143}$$

To prove the claim we will use natural induction on z. For $z = 1$ we immediately obtain the centered finite difference formula as in equation (66). Assume now the claim holds for $z - 1$, then

$$\begin{aligned}
\partial_1^z \psi_j &= \partial_1(\partial_1^{z-1}\psi_j) = \partial_1 \left(\frac{1}{(2\varepsilon)^{z-1}} \sum_{n=0}^{z-1} (-1)^n \binom{z-1}{n} \psi_{j+z-1-2n} \right) \\
&= \frac{1}{(2\varepsilon)^z} \sum_{n=0}^{z-1} (-1)^n \binom{z-1}{n} (\psi_{j+z-2n} - \psi_{j+z-2-2n}) \\
&= \frac{1}{(2\varepsilon)^z} \left(\sum_{n=0}^{z-1} (-1)^n \binom{z-1}{n} \psi_{j+z-2n} + \sum_{n=1}^{z} (-1)^n \binom{z-1}{n-1} \psi_{j+z-2n} \right) \\
&= \frac{1}{(2\varepsilon)^z} \left(\psi_{j+z} + (-1)^z \psi_{j-z} + \sum_{n=1}^{z-1} (-1)^n \left(\binom{z-1}{n} + \binom{z-1}{n-1} \right) \psi_{j+z-2n} \right) \\
&= \frac{1}{(2\varepsilon)^z} \left(\psi_{j+z} + (-1)^z \psi_{j-z} + \sum_{n=1}^{z-1} (-1)^n \binom{z}{n} \psi_{j+z-2n} \right) \\
&= \frac{1}{(2\varepsilon)^z} \sum_{n=0}^{z} (-1)^n \binom{z}{n} \psi_{j+z-2n}.
\end{aligned} \tag{144}$$

This concludes the proof. From equation (143) we see that site j has an interaction with sites $j + z - 2n$ for $n = 0, 1, ..., z$. That means that if z is even, there is no direct nearest neighbor interaction. If moreover N is even this means that all even sites do not interact with the odd sites. So if $p = 2$ we have a system with an even amount of sites and our subsystem consists of only the even sites. Hence there is no interaction between the subsystem and the rest of the system. As a result, the entanglement entropy vanishes.

A second way to legitimize this explanation is to check whether the nearest neighbor correlations vanish. Notice that for $p = 2$ we have

$$e^{ik\frac{2\pi}{N}} = e^{ik\frac{\pi}{N_A}} = -e^{i(k+N_A)\frac{\pi}{N_A}}, \tag{145}$$

whilst for even z, $(\tilde{k}(k))^z = (\tilde{k}(k + N_A))^z$. Hence, if we split the sum over k in equation (79) into two sums both ranging from 0 to $N_A - 1$ (with a change of summation variable), we see that the two thus obtained terms cancel. Hence the nearest neighbor correlations indeed vanish.

As a final check we can verify whether the ground state is indeed a product state in the product space of our subsystem and its complement. we do so for the simple case $p = N = 2$ and z is even. After some algebra we can rewrite our ground state $|0\rangle$ in the basis of the lattice sites 0 and 1. Using the same notation and conventions as in the previous sanity check we obtain the result

$$|0\rangle := \frac{1}{2\omega} \left((\omega - k^z)|0011\rangle + m|0110\rangle - m|1001\rangle + (\omega + k^z)|1100\rangle \right).$$

This is a product state, but this is not clear directly because one needs to keep track of permutations of the fermionic creation and annihilation operators as they anti commute. This results in the following factorization

$$|0011\rangle = |01\rangle_1 \times |01\rangle_0, \tag{146}$$
$$|0110\rangle = -|01\rangle_1 \times |10\rangle_0, \tag{147}$$
$$|1001\rangle = |10\rangle_1 \times |01\rangle_0, \tag{148}$$
$$|1100\rangle = |10\rangle_1 \times |10\rangle_0. \tag{149}$$

So the ground state factorizes as

$$|0\rangle := \frac{1}{2\omega}\left(\sqrt{\omega - k^z}|01\rangle_1 - \sqrt{\omega + k^z}|10\rangle_1\right) \times \left(\sqrt{\omega - k^z}|01\rangle_0 - \sqrt{\omega + k^z}|10\rangle_0\right).$$

Hence there is no entanglement.

Massless fermions

Note that the parameter z also has an important effect on the massless case: Considering this translation in the context of equation (119) one might be temped to say that z does not affect the entanglement entropy in the massless case. This is not true because there is actually a subtle effect of the parameter z on the entanglement entropy of massless fermions. Namely, when z is even the value of $-(-\tilde{k})^z$ is always negative, so in the ground state $\langle N_{a,k}\rangle = \langle N_{b,k}\rangle = 1$ for all k. Thus for even z we get that $S_A = 0$ and for odd z the result in equation (119) still applies.

5

RESULTS

At this point we have derived all the necessary formulas, which will now be analyzed graphically. The results for bosons are well known [29, 22, 23] and their main purpose here will be to compare with the results for fermions, which are new. We will start out with the results for the theory that contains only nearest neighbor interactions, before turning our attention to the Lifshitz theories that contain long range interactions.

5.1 NEAREST NEIGHBOR INTERACTION

For both the boson and fermion system, the dimensionless parameters that determine the entanglement entropy (which in our convention $\hbar = c = 1$ is also dimensionless) are N, p and $m\epsilon$, and for fermions we also have the boundary conditions determined by $\theta_{\pm}$. Due to the symmetry of our system, which is also reflected in equations (43) and (116), we expect the entropy to scale linearly with $N_A = N/p$ in the large N limit, keeping p fixed. For the dependence on $m\epsilon$, we expect that the entanglement entropy decreases as $m\epsilon$ increases; an increase of $m\epsilon$ means that either m increases or $1/\epsilon$ decreases. Whereas m couples on-site particles, $1/\epsilon$ couples nearest neighbors. On-site coupling is expected to decrease the entanglement entropy and off-site interactions is expected to increase the entanglement entropy.

Bosons

As will become clear, the above intuition indeed applies. In figure 2 the entanglement entropy is plotted as a function of $m\epsilon$ for different values of the subsystem size N_A. The figure confirms our expected behavior of the entanglement entropy as a function of $m\epsilon$. Observe also that for large values of $m\epsilon$ the entanglement entropy becomes extensive, i.e. a linear function of N_A (as S_A is divided by N_A on the vertical axis).

In figure 3 the entanglement entropy is plotted as a function of the fraction $p = N/N_A$ for a fixed value of N and various values of $m\epsilon$. Notice that the entanglement entropy thus is an increasing function of p. This is understood in the following way: As p increases and N remains fixed, our subsystem A

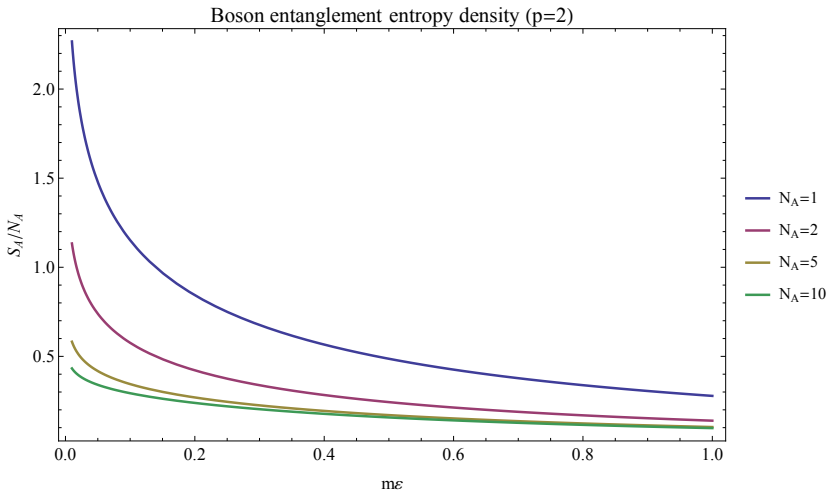

Figure 2.: The entanglement entropy density S_A/N_A for bosons for $p = 2$ fixed, plotted as a function of the mass m times the lattice spacing ϵ, for different subsystem sizes N_A. The entanglement entropy decreases as $m\epsilon$ increases. Furthermore, for large values of $m\epsilon$ the entanglement entropy becomes extensive as the graphs converge.

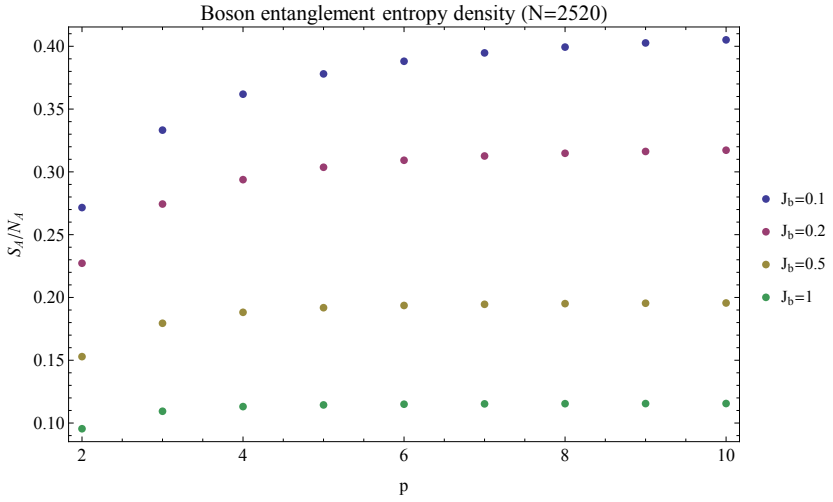

Figure 3.: The entanglement entropy density S_A/N_A for bosons for $N = 2520$ (the least common multiple of the numbers 2 up to 10), plotted as a function of p, for different values of $J_b = m\epsilon$. As p increases, so does the entanglement entropy.

becomes smaller, hence its complement becomes bigger and thus will contain more particles that can entangle with the particles in A. On the other hand, if $p = 2$ the two systems have equal size and thus the entanglement is minimal.

Another interesting situation arises when we take the large N limit $N \to \infty$ and $N_A \to \infty$ while keeping $p = 2$ fixed. In this limit the discrete steps of

k/N that are summed over in the computation of the entanglement entropy in equation (43) become a continuous parameter x and the sum becomes an integral, similar to what we have worked out in chapter 4 for the large N limit of the fermion system. The result in plotted in figure 4.

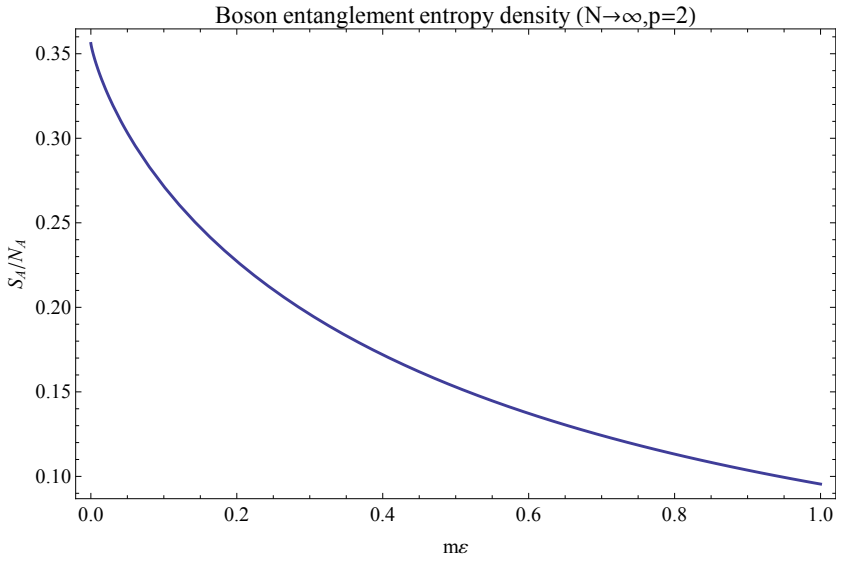

Figure 4.: The entanglement entropy for bosons density S_A/N_A for the large N limit plotted as a function of $m\epsilon$, while keeping $p = 2$ fixed.

Fermions

In the fermion system we have an additional parameter $\theta_\pm$ that effects the entanglement entropy. The graphs in figure 5 show this effect. Some remarkable situations arise.

First of all, when $p = 2$ and $\theta = 0$ there seems to be a discontinuity as a function of $m\epsilon$ at $m\epsilon = 0$. In this specific situation we have that $\omega_k = \sqrt{m^2 + \frac{1}{\epsilon^2}\sin^2(\pi k/N_A)} = 0$ when k is a multiple of N_A. Considering equation (76) we see that the groundstate becomes degenerate. Recall that for the massless fermions we chose a different definition of the annihilation operators a_k and b_k compared to the massive case. When $\theta \neq 0$, the massless limit yields the same groundstate for both conventions, but when $\theta = 0$, the two groundstates are different. From equations (105) and (106) we see that we are now allowed to choose a different massless groundstate, such that the eigenvalues are 0 or 1 instead of $1/2$, yielding the entanglement entropy of the massless limit.

Secondly, when N is even, the entanglement entropy is maximized for $\theta = 1/2$ (anti-periodic boundary conditions) and minimized for $\theta = 0$ (periodic boundary conditions). But when N is odd, $\theta = 1/2$ becomes a minimum

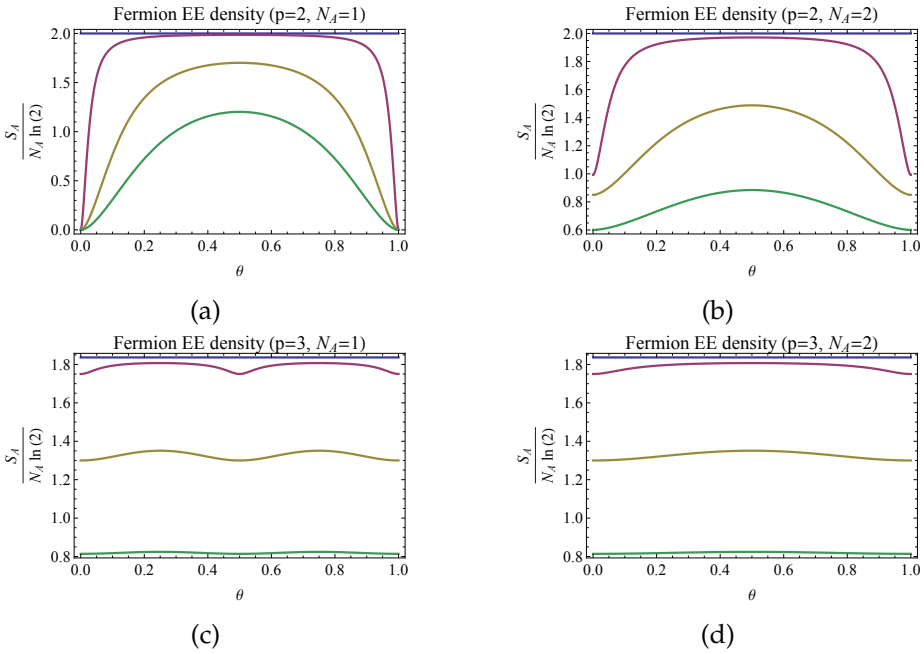

Figure 5.: Graphs of the entanglement entropy (EE) density for fermions S_A/N_A as a function of the boundary condition parameter θ for $p = 2$ ((a) and (b)) or $p = 3$ ((c) and (d)) and $N_A = 1$ ((a) and (c)) or $N_A = 2$ ((b) and (d)).The different curves show different values of J_f: $J_f = 0$ (blue), $J_f = 0,1$ (purple), $J_f = 0,5$ (orange) and $J_f = 1$ (green).

and $\theta = 1/4, 3/4$ are the new maxima. As N increases the same behavior is exhibited, but the amplitude of the effect decreases. Mathematically, this result is a direct consequence of the considerations resulting in equation (111): when p is even, the θ dependence comes from the eigenvalues λ_m^{+-} in equation (109), which shows the same behavior. When p is odd, only the $n = 0$ term contributes in the sum over $\tilde{k}$ in equation (108), which, for even N has period 1 as a function of θ and for odd N has period $1/2$. Thusfar is the reach of the mathematical interpretation.

For the physical interpretation, one can see from the Dirac Lagrangian that the boundary conditions indeed effect the entanglement. For simplicity, consider the system where $p = 2$ and $N_A = 1$. In the Dirac Lagrangian the interaction term on site 0 is proportional to $\psi_0^\dagger(\psi_1 - \psi_{-1})$. Now if we would have periodic boundary conditions ($\theta = 0$) then $\psi_1 = \psi_{-1}$ and we see that the interactions cancel, whereas for anti-periodic boundary conditions ($\theta = 1/2$), $\psi_1 = -\psi_{-1}$, the interactions are equal and add up, thus maximizing the entanglement.

A third remark we wish to make is that indeed the θ dependence drops out in the massless case. We have seen this as a consequence of the θ dependence being divided out in the correlation functions. This observation provides a further hint to the physical interpretation of the θ dependence of the entanglement

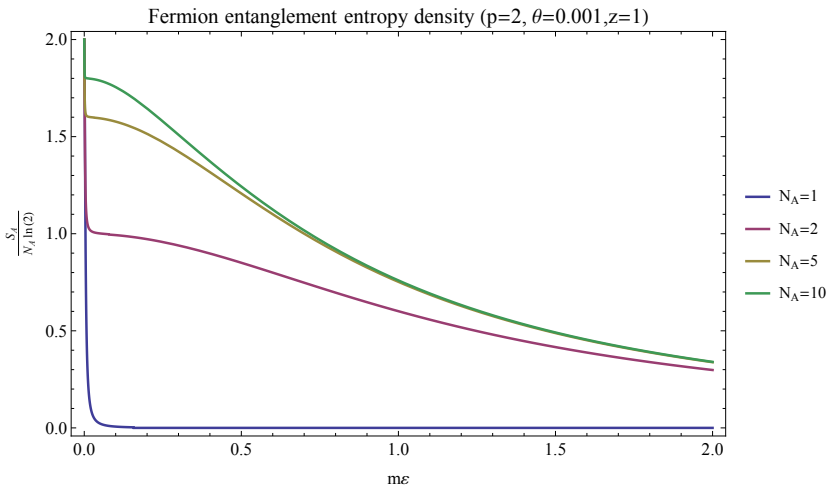

Figure 6.: The entanglement entropy density for fermions plotted as a function of $m\epsilon$, for different subsystem sizes N_A. The entanglement entropy decreases as $m\epsilon$ increases, similar to bosons. Also the same extensive behavior is shown.

entropy: θ moderates the effect of the coupling between the plus and minus spinor components. However, a proper understanding of the θ dependence still needs to be further developed.

Finally, the graphs in figure 5 already exhibit the predicted behavior as a function of $m\epsilon$ and p, which is made more explicit in figures 6 and 7. Notice that in figure 7 the increase as a function of p indeed is observed when one considers odd and even p separately. This alternating behavior as a function of the parity of the amount of sites between the sites of our subsystem has also been observed in [30] (we refer specifically to the fermion representation of figure 1) where a graph is given of the entanglement entropy for a subsystem consisting of two sites on an infinite chain of massless fermions as a function of the amount of sites between them.

In the large N limit the same behavior is shown for both bosons and fermions, as becomes clear from figure 8.

At this point we can already state a fundamental difference between the boson system and fermion system: The Hilbert space of fermions is finite dimensional due to the Pauli exclusion principle, whilst it is infinite dimensional for bosons. As a result, the entanglement entropy for fermions is bounded by $2N_A \ln(2)$, whereas for bosons it is unbounded and generally diverges for $m\epsilon \to 0$.

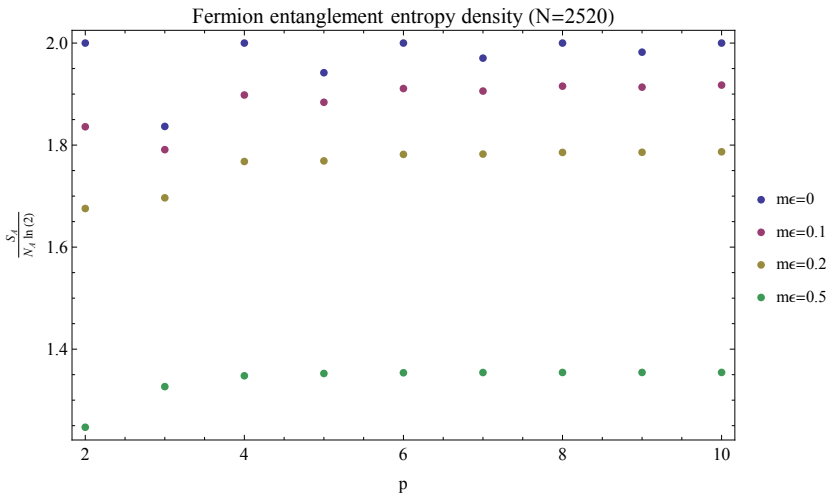

Figure 7.: The entanglement entropy for fermions density for $N = 2520$ (the least common multiple of the numbers 2 up to 10), plotted as a function of p, for different values of $J_f = m\epsilon$. For $J_f = 0$ we indeed read off the result of equation (119). Considering odd and even p separately, we see that the entanglement entropy is an increasing function of p.

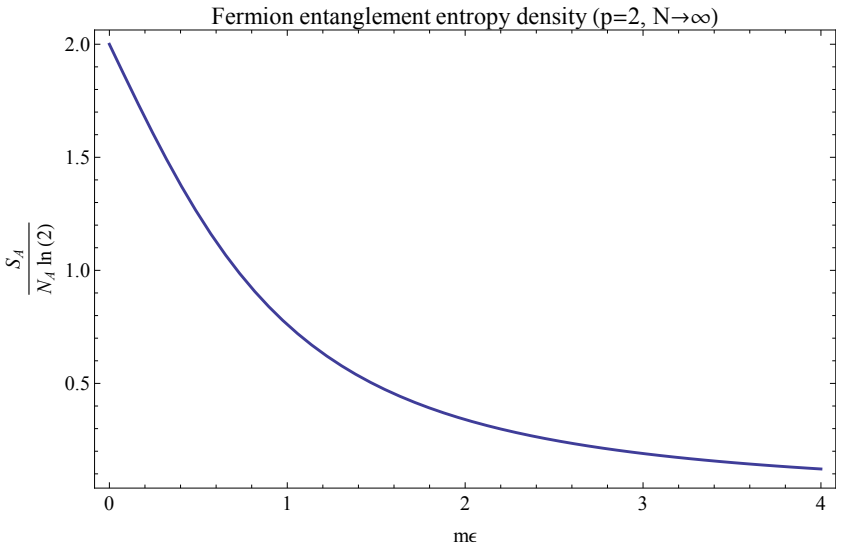

Figure 8.: The entanglement entropy for fermions density in the large N limit, plotted as a function of $m\epsilon$.

5.2 LONG RANGE INTERACTION

Very recently, the results for bosonic Lifshitz theories were published [22, 23]. Some of these results will be reproduced here, before we display and describe the results for fermionic Lifshitz theories. The main interest of this subsection is the dependence on the dynamical critical exponent z. In particular, the large N limit is interesting to compare for both systems, since only the parameters z and J determine the entanglement entropy.

Bosons

In figure 9 the entanglement entropy is plotted as a function of z, which clearly confirms our intuition that the entanglement entropy increases as z increases. Also in the large N limit, as is shown in figure figure 10, this behavior is exhibited. In fact, as $J_b \to 0$ the entanglement entropy seems to become a linear function of z.

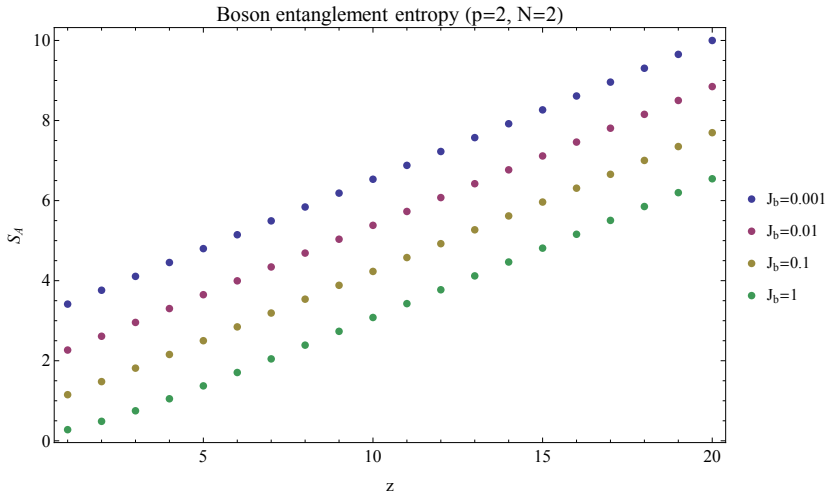

Figure 9.: The entanglement entropy for bosons as a function of the dynamical critical exponent z, for different values of $J_b = m\epsilon^z$.

Fermions

As we have discussed in section 4.5 for $p = 2$ and z is even, the entanglement entropy vanishes for fermions. In the graphs of the entanglement of fermions in the Lifshitz theory, we have therefor omitted the point corresponding to even values of z. In figure 11 the entanglement entropy for fermions in the large N limit is plotted as a function of z.

This figure shows some remarkable results. As we have already derived mathematically (see section 4.5) the entanglement entropy does not depend in

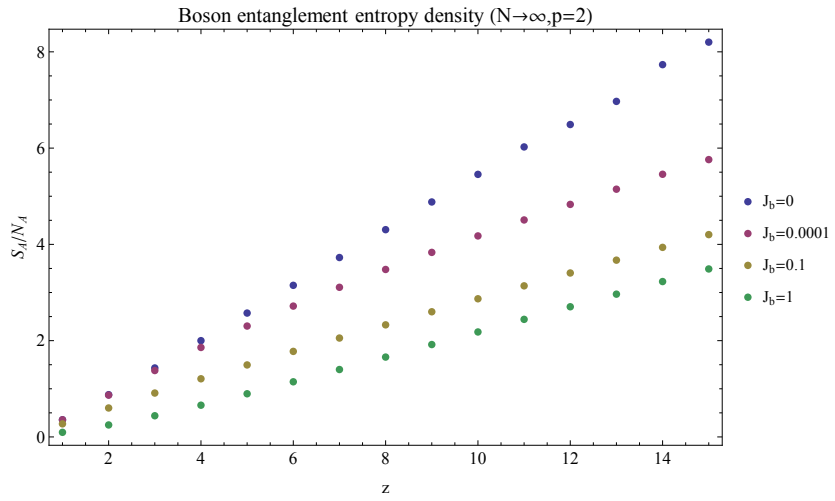

Figure 10.: The entanglement entropy for bosons as a function of the dynamical critical exponent z in the large N limit with $p = 2$ fixed, for different values of $J = m\epsilon^z$. The entanglement entropy seems to become a linear function of z as $J \to 0$.

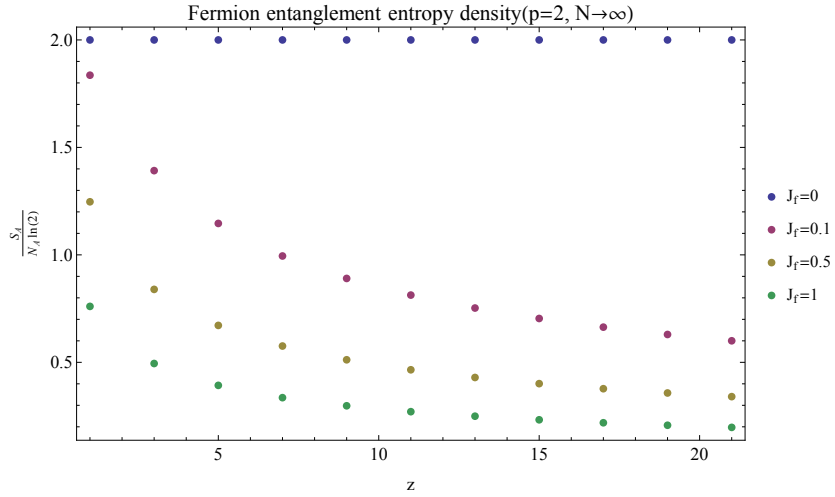

Figure 11.: The entanglement entropy density for fermions in the large N limit, keeping $p = 2$ fixed, plotted as a function of z, for different values of $J = \mu\epsilon^z$. Remarkably, the entanglement entropy density decreases as a function of z.

z in the massless limit: It depends only on the parity of z, being maximal when z is odd and zero when z is even. Most striking, however, is the observation that the entanglement entropy density decreases as z increases, contradicting our intuition. Mathematically, this is immediately clear from equations (131) and (132) where $\sin(2\pi x)$ now is generalized to $-(-\sin(2\pi x))^z$. As $z \to \infty$,

we have that $\sin^{2z}(2\pi x) \to 0$ almost everywhere (as the set of points where $\sin(2\pi x) = 1$ has measure zero), so $\omega\epsilon^z \to J_f$, so indeed $S_A \to 0$ when $J_f \neq 0$.

The physical understanding of this phenomenon is more involved. By studying the two point correlators in equation (79) in the same limit, we get that $\omega_k \to m$ so, as $\sum_{k-\theta=0}^{N-1} e^{ik(j-i)\frac{2\pi}{N_A}} = N\delta_{i,j}$, the correlators all converge to $\frac{1}{2}\delta_{i,j}$. Hence the correlation is lost. We see that this $z \to \infty$ limit is equivalent to taking the large J_f limit. This hints at what is going on physically when z increases: If we look at the interaction terms in the Lifshitz Lagrangian of equation (133), we have that the on site coupling parameter is proportional to

$$\mu\alpha^2, \tag{150}$$

whereas the interactions have a coupling parameter proportional to

$$\frac{\alpha}{(2\epsilon)^z}. \tag{151}$$

Now, we have defined $J_f = \alpha\mu\epsilon^z$ as a dimensionless parameter that defines the entanglement entropy for fermions, following the bosonic approach (where $J_b = \frac{\mu\epsilon^z}{\alpha}$). For the bosons this choice is indeed the slimmest definition of a dimensionless parameter one can make from μ, ϵ and α and conveniently it also assures that if we keep J_b fixed, the ratio between the coupling parameters of interactions and on-site coupling, which is $m^2/\frac{\alpha^2}{\epsilon^{2z}} = J_b^2$, remains fixed as well. Clearly, this is not the case for fermions when we fix J_f. In fact, the ratio diverges as 2^z. Hence, the result that the entanglement entropy density for fermions decreases as z increases when keeping J_f fixed, makes sense now.

If we wish to compare the situation for fermions properly with the boson system, we have to define a new dimensionless parameter $\tilde{J}_f$ from this ratio of coupling parameters:

$$\begin{aligned} \tilde{J}_f &= \mu\alpha^2 / \frac{\alpha}{(2\epsilon)^z} \\ &= 2^z\alpha\mu\epsilon^z \\ &= 2^z J_f. \end{aligned} \tag{152}$$

In figure 12 the entanglement entropy density in the large N limit is plotted as a function of z for different values of $\tilde{J}_f$. We see now that for massive fermions the entanglement entropy converges to some non-zero value.

We can compute the convergence limit analytically. Considering equation (131), we have to determine the limit of μ/ω_x as $z \to \infty$, keeping $\tilde{J}_f = 2^z\mu\epsilon^z$ fixed. For convenience, we first take the square of the inverse

$$\begin{aligned} \lim_{z\to\infty} \frac{\omega_x^2}{\mu^2} &= \lim_{z\to\infty} 1 + \frac{\sin^{2z}(2\pi x)}{2^{2(1-z)}} \\ &= 1 + \lim_{z\to\infty} \frac{1}{4}(2\sin(2\pi x))^{2z}. \end{aligned} \tag{153}$$

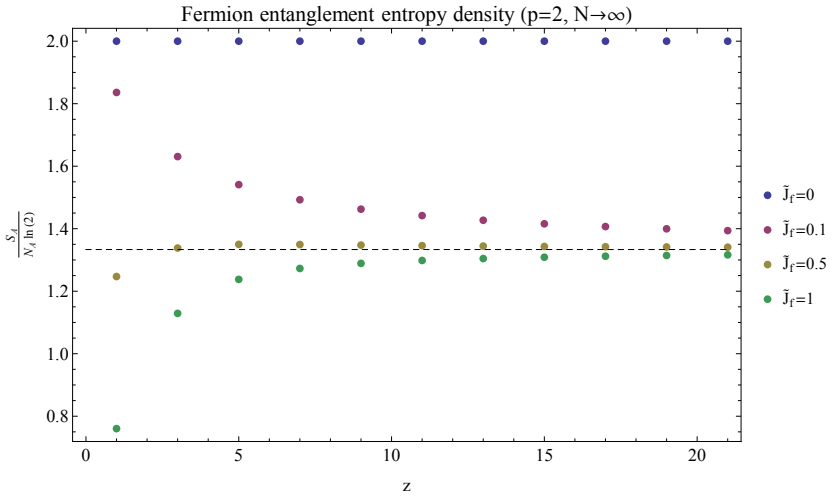

Figure 12.: The entanglement entropy density for fermions in the large N limit, keeping $p = 2$ fixed, plotted as a function of z, for different values of $\tilde{J}_f = 2^z \mu \epsilon^z$. For $\tilde{J}_f \neq 0$ the entanglement entropy density converges to $\frac{4}{3}\ln(2)$ (dashed line).

The limit of the latter term actually depends on x: when $2\sin(2\pi x) < 1$, the latter term vanshes and when $2\sin(2\pi x) > 1$, the latter term diverges. So inverting these results, and using $\arcsin(1/2) = \pi/6$, we obtain

$$\lim_{z\to\infty} \frac{\mu}{\omega_x} = \begin{cases} 0 & \text{, if } x > 1/12 \text{ or } x < 5/12, \\ \frac{2}{\sqrt{5}} & \text{, if} x = 1/12 \text{ or } x = 5/12, \\ 1 & \text{, if } x < 1/12, \text{ or } x > 5/12. \end{cases} \tag{154}$$

Inserting this in equation (131), we obtain

$$\begin{aligned} \lim_{z\to\infty} S_A &= -2 \lim_{z\to\infty} \int_0^{\frac{1}{2}} \frac{\omega_x - \mu}{\omega_x} \log\left(\frac{\omega_x - \mu}{2\omega_x}\right) + \frac{\omega_x + \mu}{\omega_x} \log\left(\frac{\omega_x + \mu}{2\omega_x}\right) \\ &= 4 \int_{1/12}^{5/12} \ln(2) \\ &= \frac{4}{3}\ln(2). \end{aligned} \tag{155}$$

This limit is shown in figure 12 by the dashed line.

However, this neat result still is not in line with the intuition that the entanglement entropy should increase as z increases. If a significant on-site coupling is present, increasing z will indeed increase the entropy, but even then, the entropy will never reach its maximum $2N_A \ln(2)$. The physical explanation is that the long range interactions that are added by the Lifshitz theory are actually interfering destructively with each other. To understand this phenomenon, consider $z = 3$. Then the Dirac Lagrangian contains a term proportional to

$$\psi_j^\dagger(\psi_{j+3} - 3\psi_{j+1} + 3\psi_{j-1} - \psi_{j-3}). \tag{156}$$

The nearest neighbor interaction is three times larger that the long range interaction, but of opposite sign! So the total interaction between site j and $j+3$ gets a direct contribution from the long range term, and an indirect contribution through a threefold nearest neighbor interaction of opposite sign. Hence, in general, the increase of z does not directly increase the correlations. Such destructive interference thus bounds the effectiveness of Lifshitz theories as a means to increase the entanglement entropy by adding long range interactions.

This results invites us to consider if a similar effect is present for bosons. Indeed, they do, but for bosons the interference is always constructive. For $z = 2$, the interaction term in the Hamiltonian is proportional to

$$(\phi_{j+2} - 2\phi_{j+1} + \phi_j)^2 = \phi_{j+2}^2 + 4\phi_{j+1}^2 + \phi_j^2 - 4\phi_{j+2}\phi_{j+1} - 4\phi_{j+1}\phi_j + 2\phi_{j+2}\phi_j. \tag{157}$$

We see that the nearest neighbor interactions always of opposite sign compared to the next nearest neighbor interaction. Hence, the indirect interaction through the twofold nearest neighbor interaction is of the same sign as the direct long range interaction.

To test this physical explanation, we will consider again a finite fermion system. Now boundary conditions will play a role. In the simple system $p = 2$, $N = 2$, this can be used to our advantage by choosing the boundary conditions to be anti periodic $\theta = 1/2$: Notice that the long range interaction between sites j and $j + 2n + 1$ and the direct nearest neighbor interaction differ a factor $(-1)^n$ in sign. In this system, such a long range interaction actually is a nearest neighbor interaction, which has gone round the system n times, picking up a factor $(-1)^n$ by anti periodicity. Hence, now all long range interactions, added by increasing z, constructively interfere, increasing the entanglement. In figure figure 13 the desired result is shown: the entanglement entropy now increases as z increases. The purpose of the figure is mainly to make a case for the explanation of the results in terms of interference.

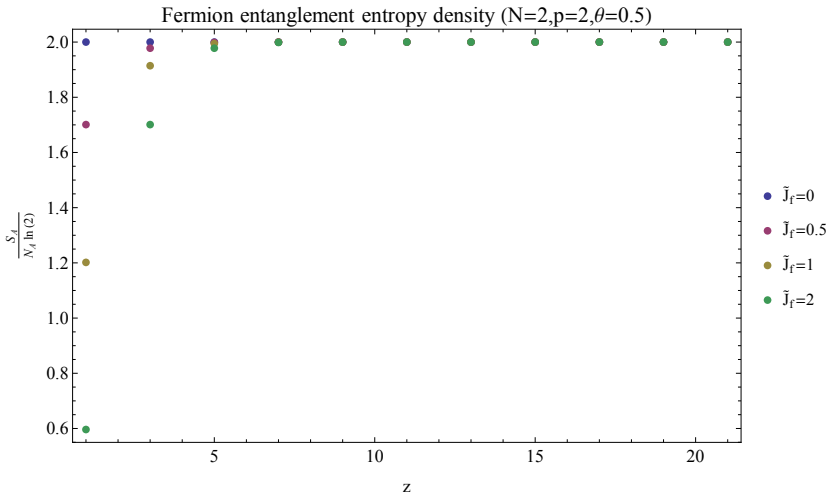

Figure 13.: The entanglement entropy density for fermions in the simplified system $N = p = 2$ with anti periodic boundary conditions $\theta = 1/2$, plotted as a function of z, for different values of $\tilde{J}_f$. The anti periodic boundary conditions insure that all interactions interfere constructively, thus an increase of z, which adds interactions, yields an increase of the entanglement entropy.

6

OUTLOOK

Before we conclude this thesis, we wish to highlight three interesting points of discussion that arise from the results. These considerations will also provide suggestions for future research.

We have seen multiple significant differences between the theory and results for for bosons and for fermions. A very subtle difference can be found in the dispersion relations (equations (24) and (67)). Though subtle, this difference has drastic consequences for the results as we have seen. Thus one may wonder what the origin is of this difference. For the fermions, we are effectively forced to consider a centered difference discretization of the spatial derivative (see appendix A for a detailed explanation of this claim). For bosons, we are free to choose either a centered of forward difference, the latter being the natural choice. One could however make the choice to take the centered difference for bosons to make its theory more comparable to that of fermions. This has also been done in appendix A. The result is satisfactory: the dispersion relation (equation (165)) now is the same for fermions and bosons and the same remarkable features arise as a consequence. In particular, similarly to fermions, a new dimensionless parameter $\tilde{J}_b$ was needed to provide a proper understanding of the z dependence.

The second point of discussion concerns the dependence of the entanglement entropy on the boundary conditions, characterized by θ. To some extent we have a good physical understanding of the effect of θ. But this understanding is not complete. The main issue is the relation between mass (on site coupling) and the boundary conditions: For massless fermions the entanglement entropy does not depend on θ. Why? We have seen that for $p = 2$ the entanglement entropy is completely determined by the behavior of λ_m^{+-} (see equation (111), which indeed vanishes when there is no mass. This could suggest that the effect of the boundary conditions is actually moderated through the coupling of the plus and minus spinor components. Another explanation may come from the boundedness of the entanglement entropy. In the massless case there are only off-site interactions in the Lagrangian and their strength may indeed be effected by θ, but since there is no on-site coupling, any effect of θ on the coupling strength may be canceled, by scaling the theory with an appropriate

scalar, yielding the same effective theory. The point is that there is no mass as a frame of reference to contrast the effect of θ against.

Thirdly, we have seen that adding longrange interactions by considering a Lifshitz scaled version of the fermion theory has led to the phenomenon of interference. This makes it harder to formulate a good intuition about the effect of adding long range interactions as becomes clear from the results. Therefor, we suggest to study other ways of adding long range interactions for future research.

In this thesis we have successfully produced a theoretical framework for the study of entanglement entropy in periodic fermion lattices. A crucial ingredient of this framework is the concept of phase circulant matrices, which extend the known theory of circulant matrices so that it can be applied to the correlation matrices of fermions. This has allowed us to determine the eigenvalues of the fermion two point correlation matrices analytically, thus providing analytic insight in the entanglement entropy. Equipped with these tools, we have studied both massive and massless systems and considered the continuum limit. Finally, we have added long range interactions by means of a Lifshitz scaling. Similar results for bosons were reproduced simultaneously, to be able to compare the fermion results.

As expected, for both fermions and bosons the entanglement entropy decreases as either the mass or the lattice spacing increase. Also, when we fix the total amount of lattice sites N, we see that the entanglement entropy density is an increasing function of the ratio $p = N/N_A$. This is also in line with our expectations, since an increase in p results in more particles in the complement of A that can entangle with the fewer particles in A. Furthermore, in the large N limit, the results of bosons and fermions are quite similar.

For fermions there is an additional degree of freedom that determines the entanglement entropy coming from the boundary conditions. This degree of freedom is characterized by the parameter θ and has a remarkable effect on the entanglement entropy, which is still not properly understood as we have mentioned above. Based on the results, we have attempted at some physical interpretations, which should be further investigated in future research.

In the simple system defined by $p = N = 2$, we can directly see an effect of θ as a moderator of the interference between the interactions. This phenomenon of interference is a key result of this thesis, whose implications become more clear when considering the theory of long range interactions. Where one would expect that adding interactions will always enhance the entanglement, we have seen that for fermions this is not the case. Our intuition is compromised by the interference between the interaction terms in our theory. This does not happen in the boson theory due to the square of the spatial derivative that occurs in the theory (even if we consider the centered difference as the discretization of the spatial derivative).

Another significant difference between the bosons and fermions is the fact that the Hilbert space of bosons is infinite dimensional, whereas the Hilbert

space of fermions is finite dimensional due to the Pauli exclusion principle. This results in the entanglement entropy being unbounded for bosons, whereas there is a clear bound for fermions. In this framework, the mass plays a delicate role. Mass is not just the on site coupling parameter, but it also provides the background for the off-site coupling, as the crucial parameter determining the entanglement entropy is the ration between the on-site and off-site coupling strength.

Although we have only considered the results for systems in their ground state, the machinery actually has been developed to produce the results for any specified state. An interesting case to study in future research would be the thermal state, i.e. the state that satisfies the Pauli-Dirac distribution. Apart from further investigating the points of discussion mentioned above, another suggestion for future research is to extend the developed machinery to higher dimensional lattices or, as has been done in [22, 23] for bosons, apply the machinery to a line segment subsystem. Furthermore, it might be interesting to investigate what these results translate to by the holographic principle.

The beauty of the developed formalism is that, although its results are remarkable, we can come to a thorough understanding because we have obtained them analytically, unearthing the mathematical structure that underlies entanglement phenomena.

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Appendices

A

FINITE DIFFERENCE APPROXIAMTION

The discretized dispersion relations in equations (24) and (67) differ in the subtle position of the factor 2 being inside or outside the sine. This is a consequence of the discretization of the spatial derivative. In general, one can choose to take the forward or centered difference as the discretization (the backward finite difference is equivalent to the forward choice). We will show first that for fermions, the two are equivalent and yield an effective centered difference due to the hermiticity of the Lagrangian.

For bosons however, the two choices yield different theories. The forward derivative is the natural choice and is worked out in chapter 3. Here we shall work out the effect of choosing the centered derivative. We shall see that for the centered derivative the dispersion relations become equal.

FERMIONS

Consider the Dirac Lagrangian density

$$\mathcal{L} = \overline{\psi} \left(i\gamma^{\mu} \partial_{\mu} - m \right) \psi. \tag{158}$$

The corresponding Lagrangian is hermitean, since we can integrate by parts, but this density is not. When we discretize, we have to consider the hermitian version of the Lagrangian density to preserve the symmetry that would otherwise be lost since we can no longer integrate by parts on a lattice. The proper Lagrangian to consider therefor is

$$L = \int dx \left(\bar{\psi} \left(i\gamma^{0} \partial_{0} - m \right) \psi + \frac{i}{2} \left(\overline{\psi} \gamma^{1} \partial_{1} \psi - (\partial_{1} \overline{\psi}) \gamma^{1} \psi \right) \right). \tag{159}$$

Now we discretize this expression by taking the forward finite difference as the spatial derivative. The only relevant term to consider is the last term.

$$\begin{aligned}\int dx \frac{i}{2}\left(\overline{\psi}\gamma^1\partial_1\psi - (\partial_1\overline{\psi})\gamma^1\psi\right) &\to \sum_j \frac{i}{2}\left(\overline{\psi}_j\gamma^1(\psi_{j+1}-\psi_j) - (\overline{\psi}_{j+1}-\overline{\psi}_j)\gamma^1\psi_j\right)\\ &= \sum_j \frac{i}{2}\left(\overline{\psi}_j\gamma^1\psi_{j+1} - \overline{\psi}_{j+1}\gamma^1\psi_j\right)\\ &= \sum_j \frac{i}{2}\left(\overline{\psi}_j\gamma^1(\psi_{j+1}-\psi_{j-1}\right).\end{aligned} \tag{160}$$

In the last equation we relabeled $j \to j-1$ in the second term.

We now work out the same calculation for the centered difference.

$$\begin{aligned}\int dx \frac{i}{2}\left(\overline{\psi}\gamma^1\partial_1\psi - (\partial_1\overline{\psi})\gamma^1\psi\right) &\to \sum_j \frac{i}{4}\left(\overline{\psi}_j\gamma^1(\psi_{j+1}-\psi_{j-1}) - (\overline{\psi}_{j+1}-\overline{\psi}_{j-1})\gamma^1\psi_j\right)\\ &= \sum_j \frac{i}{2}\left(\overline{\psi}_j\gamma^1(\psi_{j+1}-\psi_{j-1}\right).\end{aligned} \tag{161}$$

Where we relabeled the latter term on the first line $j \to j-1$ and $j \to j+1$ separately. Indeed the two Lagrangians are equal. Hence the Euler Lagrange equations obtained from the are equal. Thus the theories with either the forward or centered difference are equivalent.

BOSONS

This result for fermions does not hold for bosons. Suppose we take the centered difference when discretizing the spatial derivative term $(\partial_1\phi)^2$. We get

$$\begin{aligned}\int dx(\partial_1\phi)^2 &\to \sum_j \frac{1}{4\epsilon}(\phi_{j+1}-\phi_{j-1})^2\\ &= \sum_j \frac{1}{4\epsilon}(\phi_{j+1}^2 - 2\phi_{j+1}\phi_{j-1} + \phi_{j-1}^2).\end{aligned} \tag{162}$$

Hence there are no more nearest neighbor interactions making this theory differ significantly to the theory with the forward derivative. As a consequence, we see that if we choose $p = 2$ the entanglement entropy vanishes as we would expect, since our subsystem does not interact with the rest of the system.

However, the theory now does become more comparable to the fermion case. From the plane wave ansatz, we now have

$$\begin{aligned}\partial_1\phi_j =& \frac{\phi_{j+1}-\psi_{j-1}}{2\varepsilon}\\ =& \frac{1}{2\varepsilon}(e^{ik\varepsilon}-e^{-ik\varepsilon})\phi_j\\ =& \frac{i}{\varepsilon}\sin(k\varepsilon)\psi_j\end{aligned} \tag{163}$$

So, from the Klein-Gordon equation

$$(\partial^2 - m)\phi = 0, \tag{164}$$

and periodicity, we immediately obtain the new dispersion relation

$$\omega_k^2 = m^2 + \frac{1}{\epsilon^2}\sin(\frac{2\pi k}{N}), \tag{165}$$

which is the same as equation (67).

Working through this modification in the machinery for bosons, we obtain some interesting results. Firstly, by the loss of nearest neighbor interactions, we expect that when $p = 2$, there will be no entanglement entropy. Indeed a direct calculation shows that the eigenvalues of the correlation matrices vanish in this case, and thus the entanglement entropy vanishes as well.

Furthermore, considering long range interactions, we obtain a z dependence of the entanglement entropy density as plotted in figure 14. We now encounter the same phenomenon as for the fermions, as the entanglement entropy decreases when z increases. Similar to fermions, we need to define a new dimensionless constant $\tilde{J}_b$ as the ratio between the on-site coupling parameter of and the interaction coupling parameter. Hence we define $\tilde{J}_b = \frac{2^z m\epsilon^z}{\alpha}$ and consider the effect of z on the entanglement entropy keeping the value of $\tilde{J}_b$ fixed. The result is shown in figure 15. Indeed the expected behavior is obtained.

As for the phenomenon of interference, it still is constructive. Consider for example the case $z = 3$. The relevant interaction terms in the Hamiltonian are

$$-6\phi_{j+3}\phi_{j+1} - 18\phi_{j+1}\phi_{j-1} - 6\phi_{j-1}\phi_{j-3} + 6\phi_{j+3}\phi_{j-1} + 6\phi_{j+1}\phi_{j-3} - \phi_{j+3}\phi_{j-3}. \tag{166}$$

One can see immediately that the direct and indirect contributions are indeed of the same sign, hence the interference is always constructive.

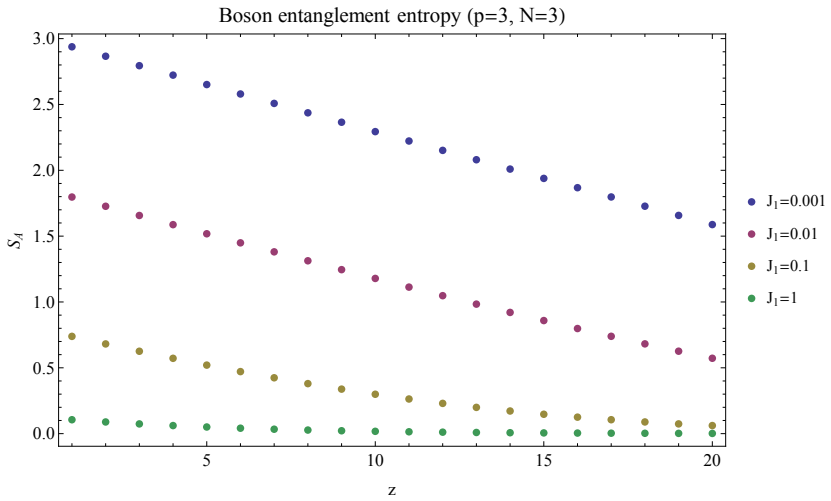

Figure 14.: The entanglement entropy density for bosons as a function of z for fixed values of $J_b = m\epsilon^z/\alpha$ in a theory that has been discretized with the centered derivative. Remarkably, as z increases, the entanglement entropy decreases due to the decrease of the interaction coupling parameter.

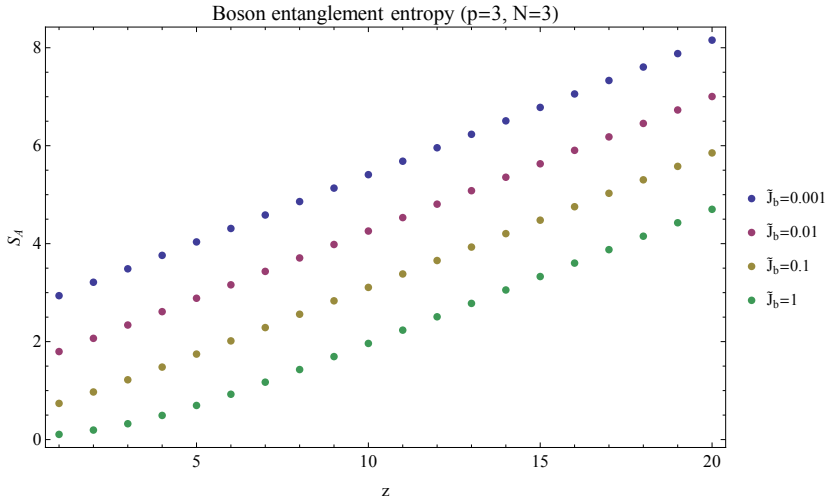

Figure 15.: The entanglement entropy density for bosons as a function of z for fixed values of $\tilde{J}_b = m\epsilon^z/\alpha$ in a theory that has been discretized with the centered derivative. As expected, increasing z now increases the entanglement.

B

DIAGONAL BLOCK MATRIX

In this appendix we show that a $2N \times 2N$ matrix, consisting of four diagonal $N \times N$ blocks is diagonalizable by transforming it to a $N \times N$ diagonal matrix, with the diagonal entries 2×2 blocks. So let M be a matrix of the following form:

$$M = \begin{pmatrix} m_1^{++} & 0 & \cdots & 0 & m_1^{+-} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_N^{++} & 0 & \cdots & 0 & m_N^{+-} \\ m_1^{-+} & 0 & \cdots & 0 & m_1^{--} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & m_N^{-+} & 0 & \cdots & 0 & m_N^{--} \end{pmatrix}. \tag{167}$$

We show by induction on N that there is an invertible matrix P such that

$$P^{-1}MP = \begin{pmatrix} m_1^{++} & m_1^{+-} & 0 & \cdots & 0 & 0 \\ m_1^{-+} & m_1^{--} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & m_N^{++} & m_N^{+-} \\ 0 & 0 & \cdots & 0 & m_N^{-+} & m_N^{--} \end{pmatrix}. \tag{168}$$

For $N = 1$, the result is immediate. So suppose it holds for $N - 1$. Let M be as above and let R be matrix obtained the $2N \times 2N$ identity matrix by interchanging the N^{th} and $(2N-1)^th$ column. Then

$$R^{-1}MR = \begin{pmatrix} m_1^{++} & 0 & \cdots & 0 & m_1^{+-} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & m_{N-1}^{++} & 0 & \cdots & 0 & m_{N-1}^{+-} & \vdots & \vdots \\ m_1^{-+} & 0 & \cdots & 0 & m_1^{--} & 0 & \cdots & 0 & \vdots & \vdots \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & m_{N-1}^{-+} & 0 & \cdots & 0 & m_{N-1}^{--} & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & m_N^{++} & m_N^{+-} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & m_N^{-+} & m_N^{--} \end{pmatrix}. \tag{169}$$

Now the 2×2 block down right is already in the desired form. For the $(2N-2) \times (2N-2)$ block in the upper left, we can use the induction hypothesis to obtain a matrix P_{N-1} that puts the upper block in the desired form. Extending P_{N-1} to Q by adding the 2×2 identity matrix to the right bottom, we thus obtain $P = RQ$ such that $Q^{-1}R^{-1}MRQ$ is indeed in the desired form. This concludes the proof.