Universiteit Utrecht

Institute for Theoretical Physics

Master Thesis

Perturbations from Inflation from Model-independent Coset Construction

Author:
B. Oudemans BSc

Supervisor:
dr. E. Pajer

Thanks to:
dr. G.L. Goon
dr. L. Santoni

June 2017

Abstract

A model independent theory of curvature perturbations from inflation allows a fundamental understanding of the origins of structure formation and the Cosmic Microwave Background. In this thesis we focus on the necessities to achieve this systematic and model independent way to derive a theory of curvature perturbations. We derive such a theory by applying techniques from spontaneous symmetry breaking in field theory. We approach spontaneous symmetry breaking systematically with the coset construction. We present the necessary techniques and knowledge to apply the coset construction. Any theory that is time-dependent, inflation for example, spontaneously breaks time translations. With the coset construction we describe any theory with a spontaneous symmetry breaking of time translations, i.e. all cosmology described by a single degree of freedom. In particular the coset construction leads to a theory of curvature perturbations from the early universe. We propose the steps for deriving this in non-dynamical as well as dynamical gravity. As of yet the results do not comply with the heuristic theory. The bulk of this thesis provides background on the topics, providing an overview of the concepts that enable a researcher to derive a theory of spontaneously broken time translations in dynamical and non-dynamical gravity consistent with a heuristic theory.

Contents

Chapter 1

Introduction

Nature and symmetry are tied in a close relationship. The objects that we would associate with nature, like trees, animals or the night sky exhibit symmetry in some way. Trees and animals are almost exactly mirror-symmetric and spinning while looking up at the night sky does not change the impression it makes. A symmetry is doing something to an object such that the object looks the same as if we would have done nothing. Symmetries are of the utmost interest for a physicist. A symmetry can reveal striking properties in the laws of nature. The famous physicist Albert Einstein for example pondered on the symmetries of the Maxwell laws of electromagnetism and discovered special relativity. A symmetry is spontaneously broken when the symmetry of the equations of motion and of the Lagrangian is not a symmetry of the groundstate (or vacuum configuration). An object like a chair exhibits spontaneous symmetry breaking for example. The equations of motion of the atoms in the chair are rotationally invariant. The collective solution of the equations, i.e. the configuration that forms the chair, is not rotationally invariant. A side-note should be added for quantum mechanical systems. The subtlety is that the groundstate is a superposition of several groundstates, making the superposed groundstate unique and leaving the symmetry unbroken. The example of the chair does not hold for an isolated QM chair as in this case the quantum mechanical properties would conserve rotational symmetry. On the other hand, an infinite number of possible groundstates forming a continuous symmetry together ensures spontaneous symmetry breaking in quantum mechanics. The theories describing the quantum behaviour of degrees of freedom that arise from spontaneous symmetry breaking are effective for certain intervals of length or of energy. Many (existing) physical theories are effective at a certain scale. The chemist would for example not need to know about any substructures of atoms, like quarks or strong and weak nuclear forces, to describe the structure of a salt crystal. This scale effectiveness is usually the approach to modelling any physical structures, actions or predictions in science. This scale effectiveness is fundamental to the theories that we describe here. This thesis is on the subject of spontaneous symmetry breaking by inflation. The questions that we set out to be answered are: is there a general approach to describing curvature perturbations from spontaneous symmetry breaking of time translations? If this approach exists, can it be applied to a curved background? The thesis shows that a general approach is possible using the coset construction in spontaneous symmetry breaking. The research falls short on translating the construction to a curved background. There are several reasons [1] why spontaneous symmetry breaking (SSB) for a theory of curvature perturbations would be favourable:

- The theory is a systematic approach to the SSB of time translations, thereby taking into account all theories of inflation (that are slow rolling, i.e. vacuum expectation value of the field changes slowly).
- It is irrelevant what happens at microscopic scales. The scalar fluctuations that are responsible for the Cosmic Microwave Background (CMB) are the degrees of freedom that arise from this approach. Any fields like the inflaton are not necessary.
- The spontaneously broken symmetries are realised in a non-linear way. Non-linear symmetries provide non trivial relations among correlators through Ward identities. An example of a Ward identity is the disappearance of the longitudinal polarization of the photon, as it is non-physical.
- The Goldstone approach (Goldstones are the degrees of freedom in spontaneous symmetry breaking) is generally the most physical approach to describe degrees of freedom from a SSB.

Before describing what spontaneous symmetry breaking is and mentioning a couple of examples, we describe the inflating universe in part I. Inflation is a rapid expansion of spacetime in the early universe. This resulted in the stretching of quantum fluctuations that were present before and during inflation, which are responsible for the structure of the universe we see today. The rapid expansion during inflation takes place within a certain amount of time. This time dependence spontaneously breaks time translations. At the end of part I we describe a theory of the degrees of freedom in spontaneously broken time translations. This theory does not limit itself to inflation but generally describes all theories that demonstrate spontaneously broken time translations, i.e. all theories of cosmology. In Part II we cover spontaneous internal symmetry breaking in particle physics. An internal symmetry is a symmetry that does not transform the coordinates of the field. This is done for transformations that commute (Abelian) and for transformations that do not commute (non-Abelian). We derive the theory for the Goldstones of spontaneously broken symmetries that do not commute by using the coset construction. The coset construction is a systematic approach to a theory related to a symmetry breaking pattern. Instead of guessing the invariant objects from the remaining symmetries as we do at the end of the first part, we systematically compute them using the Goldstone degrees of freedom. The transformations are also made coordinate dependent (gauged) and the coset construction is used to describe spontaneous symmetry breaking of gauged transformations. The famous Higgs mechanism is an example of spontaneous internal gauged symmetry breaking. At high energies the electroweak symmetries are mixed. Only below the energy scale of 246 GeV the electroweak symmetry is spontaneously broken, giving the W-bosons and the Z-bosons (weak interactions) mass and leaving the photons massless. Theories that describe the quantum effects in spontaneously broken settings and that are effective in a certain energy regime are called effective field theories. In part II we describe the effective theory of spontaneous chiral symmetry breaking. The chiral symmetry is a symmetry that rotates the quark flavours u, d and s. These symmetries are spontaneously broken by the condensate and the eight Goldstone bosons are scalar meson fields. In part III we describe spacetime spontaneous symmetry breaking. The difference with internal SSB is that the symmetries are only transformations on the coordinates. It is possible to have degenerate degrees of freedom in spacetime spontaneous symmetry breaking. These are accounted for by application of the inverse Higgs constraint. We then compute the theory of scalar perturbations by spontaneous breaking of time translation, by applying the coset construction. The result of the third part is an approximation to the theory of spontaneously broken time translations in flat spacetime that we set out to derive. We approach the same theory but in a curved background by working with a membrane that spontaneously breaks the symmetry. We also do this by introducing covariant derivatives that conserve the diffeomorphism symmetries of gravity. A good extension would be to finish this approach in a curved background and compare the results with the findings we describe in part I. The prognosis is that not a lot of new physics would arise by this model independent approach as the degrees of freedom in a SSB of time translations have been extensively researched and measured.

Part I

Cosmology and Inflation

Chapter 2

A Review of Inflation

Throughout the thesis we use

$$c = \hbar = 1\,. \tag{2.1}$$

Before we cover the subject of inflation it is necessary to introduce the toolkit of a cosmologist. To grasp the spacetime of our universe the best bet is to look at the isometries underlying it [2]. The isometries are distance preserving transformations, and labeled homogeneity and isotropy. The former is invariance under translations in space and the latter is invariance under rotations of space. With these isometries in mind, Friedmann Lemaître Robertson and Walker derived a cosmological model with a time dependent scale factor $a(t)$, which is 1 today. This model is analogous to a raising bread in the oven, where the scale factor determines the relative size of the bread. A scale factor smaller than 1 would imply that the bread has shrunk. The metric of this theory is aptly named the FLRW-metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\phi d\phi^2)\right]\,, \tag{2.2}$$

where K denotes the curvature of spacetime. If K is zero, (2.2) would boil down to a flat spacetime metric with a scale factor. For a positive K it is positively curved. For a negative K, the factor in front of the measure dr^2 decreases when r increases and it is negatively curved. A coordinate transformation from the radius r to the coordinate χ can be applied to write the metric so that the measure for the radial component is $d\chi^2$ for all K. The transformation is defined by,

$$r^2 \equiv \Phi_k(\chi)\begin{cases}\sinh^2\chi & K=-1\,,\\ \chi^2 & K=0\,,\\ \sin^2\chi & K=1\,,\end{cases} \tag{2.3}$$

leading to,

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \Phi_k(\chi)(d\theta^2 + \sin^2\phi d\phi^2)\right]\,. \tag{2.4}$$

The scale factor is time dependent, therefore it is important to define a quantity,

$$H \equiv \frac{\dot{a}}{a}\,, \tag{2.5}$$

where a dot denotes the time derivative. H is the Hubble parameter, describing the expansion rate of the universe. For a collapsing universe the sign of H is negative and for an expanding universe it is positive. The inverse of the Hubble parameter is a measure for the cosmological time of the universe $H^{-1} \sim t$ and the cosmological size of the universe $H^{-1} \sim d$ if expansion had been linear. Now that we have the metric it is practical to define a conformal time τ, this is the time coordinate scaled with the scale factor,

$$\tau = \int \frac{dt}{a(t)}\,. \tag{2.6}$$

Conformal time enables us to write the metric as a spacetime with a conformal time dependent scale factor for the whole invariant measure,

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\chi^2\right]\,, \tag{2.7}$$

where the angles ϕ and θ have been fixed, made possible by the isotropy of the universe. Null geodesics $ds^2 = 0$ are now defined in a scale factor independent way. The null geodesic limits the maximal covered distance in a time $t - t_i$ by a particle p. This maximum is the horizon of a particle,

$$\chi_p = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a(t)}\,, \tag{2.8}$$

this horizon in the comoving frame is χ_p. Multiply by the scale factor $a(t)$ to recover the physical distance d_p at time t,

$$d_p(t) = a(t)\chi_p\,. \tag{2.9}$$

The particle horizon will turn out to play a pivoting role for the formulation of the theory of inflation. Before diving into inflation, the dynamics of spacetime are outlined in the following.

2.1 Gravitational Dynamics

The formula connecting the energy momentum of a physical system with the curvature of spacetime was derived by Albert Einstein in 1915. This will form the starting point of the section,

$$\frac{1}{M_{\rm Pl}^2} T_{\mu\nu} = \Lambda g_{\mu\nu} + G_{\mu\nu}\,. \tag{2.10}$$

The Einstein tensor is a relabeled combination of the metric, the Ricci scalar and the Ricci tensor. They depend on the metric and its derivatives, $G_{\mu\nu} \equiv$

$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. The left hand side of (2.10) is the content of the universe. This is divided into three epochs, matter domination, radiation domination and a cosmological constant. We describe these by a perfect fluid, with an energy density ρ and a pressure p. The distinguishing property between epochs is the parameter w, or the equation of state,

$$\frac{p}{\rho} \equiv w \,. \tag{2.11}$$

The equation of state parameter for the different epochs is,

$$w = \begin{cases} 0 & \text{in matter domination,} \\ \frac{1}{3} & \text{in radiation domination,} \\ -1 & \text{in cosmological constant domination.} \end{cases} \tag{2.12}$$

The energy-momentum tensor of a perfect fluid is,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \,. \tag{2.13}$$

With (2.10) we derive the Friedmann equations, named after their discoverer Friedmann,

$$H^2 + \frac{K}{a^2} = \frac{1}{3M_{\text{Pl}}^2}\rho \,, \tag{2.14}$$

$$\dot{H} + H^2 = -\frac{1}{6M_{\text{Pl}}^2}(\rho + 3p) \,. \tag{2.15}$$

Taking the time derivative of (2.14) and combining the equations, the continuity equation reads,

$$\frac{1}{6M_{\text{Pl}}^2}\frac{\dot{\rho}}{H} + \frac{K}{a^2} = -\frac{1}{2M_{\text{Pl}}^2}(\rho + p) \,, \tag{2.16}$$

rewriting it and setting $K = 0$,

$$\frac{d\ln\rho}{d\ln a} = -3(1 + w) \,, \tag{2.17}$$

where w is the equation of state parameter. Integrating it leads to,

$$\rho \propto a^{-3(1+w)} \,, \tag{2.18}$$

using (2.14) the scale factor reads,

$$a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & w \neq -1 \,, \\ e^{Ht} & w = -1 \,. \end{cases} \tag{2.19}$$

The composite energy density and pressure are,

$$\rho = \Sigma_i \rho_i \,, \qquad p = \Sigma_i p_i \,, \tag{2.20}$$

the index i covers the different substances. The critical density is the value of ρ in absence of spatial curvature and the cosmological constant set to zero,

$$\rho_c \equiv 3H^2 M_{\mathrm{Pl}}^2 \,. \tag{2.21}$$

The ratio between the energy density ρ and the critical energy density is

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \,. \tag{2.22}$$

Each i content has its own equation of state parameter w_i. This finalizes the introduction of some fundamental cosmological concepts. Looking at consequences of these fundamentals will lead to difficult questions concerning initial conditions of the universe. These are solveable by introducing the concept of inflation.

2.2 Fine tuned initial Conditions, a Feeling or a Fact?

Gravitational theory tells us that any inhomogeneities would attract and create larger ones in a big bang cosmology, so any initial inhomogeneity would grow exponentially and would dominate at this point. This is not what we observe in the universe. It seems that at the start, $\tau = 0$, the conditions of the universe were finely tuned, as there could have been causal contact between all observable parts in a big bang cosmology. The absence of causal contact between nearly homogeneous parts of the universe is also known as the horizon problem.

2.2.1 The Horizon Problem

The particle horizon between a time 0 and time t is,

$$\tau \equiv \int_0^t d\ln a \left(\frac{1}{aH}\right) \,, \tag{2.23}$$

$(aH)^{-1}$ is called the comoving Hubble radius. The comoving Hubble radius is the radius of a sphere whose surface is luminally recessing. Solving the integral with equation of state parameter w and using that $(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3w)}$ (from (2.14)) where H_0 is the value of the Hubble parameter now, the comoving horizon is proportional to,

$$\tau \propto a^{\frac{1}{2}(1+3w)} \,. \tag{2.24}$$

This means that the comoving horizon and the comoving Hubble radius grow with time in matter domination and in radiation domination,

$$\tau \propto \begin{cases} a & \text{Radiation Domination,} \\ a^{\frac{1}{2}} & \text{Matter Domination.} \end{cases} \tag{2.25}$$

Any comoving scale entering the horizon now is causally disconnected, yet approximately homogeneous by observation. This seems quite incongruous.

2.2.2 The flatness Problem

Consider the Friedmann equation (2.14), rewritten in terms of $\Omega = \frac{\rho}{\rho_c}$

$$1 - \Omega = \frac{-K}{(aH)^2}\,. \tag{2.26}$$

The growth of the comoving Hubble radius in matter and radiation domination implies that a small deviation from the value $\Omega = 1$ blows up the curvature K. Though observations show that the universe is approximately Euclidean flat space. Again, fine tuned initial conditions are one way of solving this issue. Although, these initial values are not necessary anymore with the introduction of inflation.

2.3 Inflation

The problems arising with the fine tuned initial conditions are linked to a comoving Hubble radius that is always increasing. An important distinction between comoving horizon and comoving Hubble radius has to be made. The comoving horizon describes the maximum distance that a massless particle has travelled. The comoving Hubble radius is the region on which the massless particle is receding by the speed of light. Any particle separated a distance larger than the comoving Hubble radius cannot communicate now. Postulating an era in which the comoving Hubble radius is shrinking as time passes puts every part of the universe inside the comoving Hubble radius at one point. Modes then left it as it was decreasing and later reentered as the radius increased in a matter or radiation dominated universe. The particle horizon is an integral of the comoving Hubble radius (2.23). A large comoving Hubble radius in the far past therefore implies a large comoving horizon now. The conditions for a decreasing comoving Hubble radius are synonymous with an accelerated expansion of the universe

$$\ddot{a} > 0 \quad \Leftrightarrow \quad \frac{d}{dt}(aH)^{-1} < 0\,. \tag{2.27}$$

We derive the right equivalence relation by taking the time derivative of the comoving Hubble radius,

$$\frac{d}{dt}(aH)^{-1} = -(aH)^{-2}a(H^2 + \dot{H})\,. \tag{2.28}$$

Inflation happens when $w < -\frac{1}{3}$. Now that we have the dynamics of an inflating universe, it enables us to revisit the flatness and the horizon problems from a new perspective.

2.3.1 Revisiting Horizon and flatness Problem

The comoving horizon is extended to large scales with a decreasing comoving Hubble radius as time passes. So the comoving horizon now contains all distances between particles that enter the horizon after inflation. The flatness problem revisited,

$$|1 - \Omega(a)| = \frac{1}{(aH)^2}\,, \tag{2.29}$$

shows that the energy density parameter now reaches 1 in a stable way as the comoving Hubble radius decreases. This way flatness of the universe is a consequence of inflation. Another way to look at this, is that any curvature that was there in the beginning drowned out when an exponential growth of the universe occurred.

2.4 Dynamics of Inflation

Inflation has to stop at a certain time. This time is expressed in the number of e-folds. An e-fold is the time in which the scale has grown by a factor e. Using the Friedmann Equation (2.15), we express the change in the Hubble parameter H,

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,, \qquad \frac{\ddot{a}}{a} = H^2(1-\epsilon)\,, \tag{2.30}$$

in an accelerated expansion $\ddot{a} > 0$ therefore $\epsilon < 1$. The measure of number of e-folds is N,

$$dN = d\ln a = H dt\,, \tag{2.31}$$

and we rewrite ϵ as

$$\epsilon = -\frac{d\ln H}{dN}\,. \tag{2.32}$$

This parameter is the tool to describe the conditions for the end of inflation. To model inflation we use a scalar field.

2.4.1 Scalar Field Inflation

A scalar field acts as an order parameter to make sense of the dynamics of inflation in most theories of inflation. The scalar field is called the inflaton. The simplest assumption is that the scalar field is minimally coupled to gravity. The action is a combination of the Einstein-Hilbert action and a coupled scalar field action S_ϕ,

$$S = S_{EH} + S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\,, \tag{2.33}$$

the potential $V(\phi)$ contains self-interactions. We model the inflaton as a perfect fluid. The energy momentum tensor is

$$T_{\mu\nu}^{(\phi)} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}\,. \tag{2.34}$$

Applying $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$ (with $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$) it is

$$T_{\mu\nu}^{(\phi)} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\sigma\phi\partial^\sigma\phi + V(\phi)\right). \tag{2.35}$$

Taking the field to be spatially homogeneous, i.e. independent of $\boldsymbol{x}$, $\phi(t,\boldsymbol{x}) = \phi(t)$, filling in the FLRW metric and applying (2.13) the pressure and energy density are,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{2.36}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.37}$$

To recap, accelerated expansion implies that the strong energy condition is violated. The equation of state parameter for the scalar field is denoted by w_ϕ and is

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,. \tag{2.38}$$

To have the scalar field drive accelerated expansion we need

$$w_\phi < -\frac{1}{3}\,, \tag{2.39}$$

so $V(\phi) \gg \frac{1}{2}\dot{\phi}^2$. Varying the action with respect to the field we get

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - \frac{dV}{d\phi} = 0\,, \tag{2.40}$$

and applying the same principles as before, the field equation is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,. \tag{2.41}$$

The Friedmann equation (2.14) with zero curvature is

$$H^2 = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{2.42}$$

It shows that the Hubble parameter H creates a significant friction to the dynamics of the scalar field. This friction makes sure that inflation takes some time to stop. The measure for how long this is, is expressed in terms of e-folds. The slow-roll conditions are used to define this measure.

2.4.2 Slow roll Inflation

We defined ϵ as,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}\,, \tag{2.43}$$

and call it the slow roll parameter. We can express it in terms of the equation of state parameter by using the Friedmann equations,

$$\epsilon = \frac{3}{2}(1 + w_\phi)\,, \tag{2.44}$$

and by using the field equations,

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}\,, \tag{2.45}$$

when $\epsilon < 1$ accelerated expansion occurs. This coincides with

$$\frac{1}{2}\dot{\phi}^2 \ll V(\phi)\,. \tag{2.46}$$

In the de Sitter limit exponential expansion $a = e^{Ht}$ occurs. The following are satisfied in this limit: $p_\phi \to -\rho_\phi$ and $\epsilon \to 0$. The expansion of the universe does not stop in de Sitter, as $\dot{\phi} = 0$. Inflation is quasi de Sitter as it fades out after a certain amount of time. The requirement for this is that the second time derivative of the scalar field is small relative to the two terms in the field equations,

$$|\ddot{\phi}| \ll |3H\dot{\phi}|, |\frac{dV}{d\phi}|\,. \tag{2.47}$$

We introduce another slow roll parameter η and express it in terms of ϵ

$$\eta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{1}{2\epsilon}\frac{d\epsilon}{dN}\,, \tag{2.48}$$

the condition is stated as $|\eta| < 1$. This means that the change of ϵ per e-fold N is just a fraction of the parameter itself. Most literature also parametrizes slow roll in terms of the potential and its derivatives with respect to ϕ, $V' \equiv \frac{dV}{d\phi}$,

$$\epsilon_V = \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \eta_V = \frac{V''}{V}\,, \tag{2.49}$$

where the requirements

$$\epsilon_V, |\eta_V| \ll 1\,, \tag{2.50}$$

hold for slow-roll inflation. Inflation ends when,

$$\epsilon(\phi_{end}) \equiv 1\,. \tag{2.51}$$

The number of e-folds before inflation ends is,

$$N = \int_a^{a_{end}} H dt = \int_\phi^{\phi_{end}} \frac{H}{\dot{\phi}} d\phi = \int_\phi^{\phi_{end}} \frac{d\phi}{\sqrt{2\epsilon}}\,, \tag{2.52}$$

it exceeds around 60 e-folds. This concludes the introduction to inflation. The topics covered were the horizon and flatness problem, the concept of inflation and single-field inflation. The next chapter covers perturbations of spacetime in general relativity and the role of inflation therein.

Chapter 3

Perturbations

The fundamentals of theoretical cosmology have been formulated in the preceding chapter. We can also derive an explanation for observed fluctuations in the large scale structures of the universe, the cosmic microwave background (CMB) and other perturbations from theoretical principles. We do this by applying perturbations to the background. The start of these inhomogeneities can be traced back to the period of inflation. The empty vacuum at microscopic scales is in actuality a fluctuating entity, due to the uncertainty principle. Spacetime at these scales is filled with excitations and annihilations. We derive the gravitational perturbations originating from inflation by applying these perturbations to the inflaton field, called $\delta\phi$. Any change in the inflaton field means a change in the energy momentum tensor. This in turn backreacts on spacetime itself. Therefore $\delta\phi$ has to be studied in connection to the perturbation of spacetime $\delta g_{\mu\nu}$ [3][4]. The field fluctuates with respect to a time dependent background,

$$\phi(\boldsymbol{x}, t) = \bar{\phi}(t) + \delta\phi(\boldsymbol{x}, t)\,, \tag{3.1}$$

as the unperturbed inflaton field is homogeneous $\bar{\phi}(\boldsymbol{x}, t) = \bar{\phi}(t)$. The bar denotes the background solution. The next sections will discuss perturbations and diffeomorphisms. Diffeomorphisms are spacetime dependent transformations on the coordinates and are gauge degrees of freedom in a gravitational theory.

3.1 Scalar, Vector and Tensor Decomposition

This section reviews the practice of describing the spacetime curvature of the universe as a small perturbation with respect to a time dependent background, the FLRW metric $\bar{g}_{\mu\nu}$ [5]. The labeling we use for the perturbation to the metric is $h_{\mu\nu}$,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,. \tag{3.2}$$

The indices are raised and lowered by using the background metric. Keeping in mind that $\delta(MM^{-1}) = 0$ for a matrix M ,

$$h^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}h_{\rho\sigma}\,. \tag{3.3}$$

Weinberg [5] has an extensive description of the perturbations on the most common tensors in General Relativity. They are not mentioned in this case, the conclusions do follow. The metric is decomposable into scalar, vector and tensor degrees of freedom due to spatial homogeneity and isotropy. The decomposition is denoted by latin capital letters,

$$h_{00} = -E\,, \tag{3.4}$$
$$h_{0i} = a\left[\partial_i F + G_i\right]\,, \tag{3.5}$$
$$h_{ij} = a^2\left[A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j + \partial_j C_i + D_{ij}\right]\,, \tag{3.6}$$

the degrees of freedom have a few constraints, $D_{ij} = D_{ji}$,

$$\partial_i C_i = \partial_i G_i = 0, \qquad \partial_i D_{ij} = 0, \qquad D_{ii} = 0\,. \tag{3.7}$$

The decomposition can be applied to the perturbation of the energy momentum tensor as well. The background values are: for the pressure $\bar{p}$, for the density $\bar{\rho}$ and for the four momentum $\bar{u}_\mu$. The elements of the decomposition are denoted by $\delta\rho$, δp and δu_i which is decomposed into the gradient of the velocity potential $\partial_i\delta u$ and a divergenceless vector δu_i^V, $\partial_i\delta u_i^V = 0$. Other terms include dissipative corrections: $\partial_i\partial_j\pi^S$, $\partial_i\pi_j^V + \partial_j\pi_i^V$ and π_{ij}^T. The equations that define these quantities are,

$$\delta T_{ij} = \bar{p}h_{ij} + a^2\left[\delta_{ij}\delta p + \partial_i\partial_j\pi^S + \partial_i\pi_j^V + \partial_j\pi_i^V + \pi_{ij}\right]\,, \tag{3.8}$$
$$\delta T_{i0} = \bar{p}h_{i0} - (\bar{\rho} + \bar{p})\left(\partial_i\delta u + \delta u_i^V\right)\,, \tag{3.9}$$
$$\delta T_{00} = -\bar{\rho}h_{00} + \delta\rho\,. \tag{3.10}$$

The constraints apply in the same manner,

$$\partial_i\pi_i^V = \partial_i\delta u_i^V = 0\,, \qquad \partial_i\pi_{ij}^T = 0\,, \qquad \pi_{ii}^T = 0\,. \tag{3.11}$$

With these equations we describe three classes of motion , a scalar mode, which can be viewed as a compression of spacetime by a change in potential, a vector mode, viewed as a vortex mode due to for example the dragging of the frame by a spinning object and tensor modes, viewed as radiation.

3.2 Gauge Transformations

The Einstein equations for the perturbed quantities show that only combinations of them present physical scalar and vector degrees of freedom, i.e. are invariant under diffeomorphisms. This will also be denoted as gauge invariant. To get a better understanding of the perturbed metric and the perturbed energy momentum tensor, we look at their transformations under the diffeomorphism,

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)\,, \tag{3.12}$$

where ϵ^μ is as small as the perturbations to the metric and energy momentum tensor. The metric transforms under a coordinate transformation $x \to x'$ as a two-tensor,

$$g'_{\mu\nu}(x') = g_{\lambda\kappa}(x)\frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\kappa}{\partial x'^\nu}\,. \tag{3.13}$$

Any change in the full metric $g_{\mu\nu}(x)$ is attributed to a change in the perturbation metric $h_{\mu\nu}(x)$. The field equations are invariant for a change in the metric perturbations under the diffeomorphisms ϵ,

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \Delta h_{\mu\nu}\,. \tag{3.14}$$

This change is defined as,

$$\Delta h_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x)\,. \tag{3.15}$$

Expanding the metric $g'_{\mu\nu}(x)$ to first order in ϵ and perturbations,

$$g'_{\mu\nu}(x) = g_{\kappa\lambda}(x'-\epsilon)\frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} \tag{3.16}$$

$$= g_{\mu\nu}(x) - \partial_\lambda \bar{g}_{\mu\nu}(x)\epsilon^\lambda - \bar{g}_{\kappa\nu}(x)\partial_\mu\epsilon^\kappa - \bar{g}_{\mu\lambda}\partial_\nu\epsilon^\lambda\,, \tag{3.17}$$

and plugging in the result,

$$\Delta h_{\mu\nu}(x) = -\partial_\lambda \bar{g}_{\mu\nu}(x)\epsilon^\lambda - \bar{g}_{\kappa\nu}(x)\partial_\mu\epsilon^\kappa - \bar{g}_{\mu\lambda}\partial_\nu\epsilon^\lambda\,. \tag{3.18}$$

Filling in the FLRW metric, then $\epsilon_i = a^2\epsilon^i$ and $\epsilon^0 = -\epsilon_0$ and the change in the perturbations in spatial and temporal components is,

$$\Delta h_{ij} = -\frac{\partial\epsilon_i}{\partial x^j} - \frac{\partial\epsilon_j}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0\,, \tag{3.19}$$

$$\Delta h_{i0} = -\dot{\epsilon}_i - \frac{\partial\epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i\,, \tag{3.20}$$

$$\Delta h_{00} = -2\dot{\epsilon}_0\,. \tag{3.21}$$

The field equations are invariant under the transformation of the energy momentum as well,

$$\delta T_{\mu\nu}(x) \to \delta T_{\mu\nu}(x) + \Delta\delta T_{\mu\nu}(x)\,. \tag{3.22}$$

In the same manner as the metric tensor, it is expressed linearly in ϵ and perturbations. A distinction has been made between δ signifying the perturbation on the background and Δ denoting the change of the perturbation under a gauge transformation. Analogously to the metric,

$$\Delta\delta T_{\mu\nu} = -\bar{T}_{\lambda\mu}(x)\partial_\nu\epsilon^\lambda(x) - \bar{T}_{\lambda\nu}(x)\partial_\mu\epsilon^\lambda(x) - \partial_\lambda\bar{T}_{\mu\nu}(x)\epsilon^\lambda(x)\,, \tag{3.23}$$

and similarly, filling in FRW,

$$\Delta\delta T_{ij} = -\bar{p}\left(\frac{\partial\epsilon_i}{\partial x^j} + \frac{\partial\epsilon_j}{\partial x^i}\right) + \frac{\partial}{\partial t}(a^2\bar{p})\delta_{ij}\epsilon_0\,, \tag{3.24}$$

$$\Delta\delta T_{i0} = -\bar{p}\dot{\epsilon}_i + \bar{\rho}\frac{\partial\epsilon_0}{\partial x^i} + 2\bar{p}\frac{\dot{a}}{a}\epsilon_i\,, \tag{3.25}$$

$$\Delta\delta T_{00} = 2\bar{\rho}\dot{\epsilon}_0 + \dot{\bar{\rho}}\epsilon_0\,. \tag{3.26}$$

The change Δ under a gauge transformation is expressible for each scalar, vector and tensor component introduced in the previous section individually. To manage that, a decomposition is necessary. ϵ is written in terms of a scalar degree ϵ^S and a divergenceless vector ϵ_i^V,

$$\epsilon_i = \frac{\partial \epsilon^S}{\partial x^i} + \epsilon_i^V \,, \qquad \partial_i \epsilon_i^V = 0 \,. \tag{3.27}$$

The change under a gauge transformation is now,

$$\Delta h_{00} = -2\dot{\epsilon}_0 \,, \tag{3.28}$$

$$\Delta h_{i0} = -\frac{\partial}{\partial t}\frac{\partial}{\partial x^i}\epsilon^S - \dot{\epsilon}_i^V - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\frac{\partial \epsilon^S}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i^V \,, \tag{3.29}$$

$$\Delta h_{ij} = -\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\epsilon^S - \frac{\partial \epsilon_i^V}{\partial x^j} - \frac{\partial \epsilon_j^V}{\partial x^i} - \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\epsilon^S - \frac{\partial \epsilon_j^V}{\partial x^i} + 2a\dot{a}\delta_{ij}\epsilon_0 \,. \tag{3.30}$$

Equating the degrees of freedom of $\Delta h_{\mu\nu}$ with the change to the scalar, vector and tensor degrees of freedom under a gauge transformation results in,

$$\Delta A = 2\frac{\dot{a}}{a}\epsilon_0 \,, \qquad \Delta B = -\frac{2}{a^2}\epsilon^S \,, \tag{3.31}$$

$$\Delta C_i = -\frac{1}{a^2}\epsilon_i^V \,, \qquad \Delta D_{ij} = 0 \,, \qquad \Delta E = 2\dot{\epsilon}_0 \,, \tag{3.32}$$

$$\Delta F = \frac{1}{a}\left(-\epsilon_0 - \dot{\epsilon}^S + \frac{2\dot{a}}{a}\epsilon^S\right) \,, \qquad \Delta G_i = \frac{1}{a}\left(-\dot{\epsilon}_i^V + \frac{2\dot{a}}{a}\epsilon_i^V\right) \,. \tag{3.33}$$

The same process is applied to the energy momentum tensor,

$$\Delta \delta p = \dot{\bar{p}}\epsilon_0 \,, \qquad \Delta \delta \rho = \dot{\bar{\rho}}\epsilon_0 \,, \qquad \Delta \delta u = -\epsilon_0 \,, \tag{3.34}$$

the change in a gauge invariant quantity is zero,

$$\Delta \pi^S = \Delta \pi_i^V = \Delta \pi_{ij}^T = \Delta \delta u_i^V = 0 \,. \tag{3.35}$$

There are two options now that we described the gauge transformations for each scalar, vector and tensor degree separately. The first option is to work exclusively with gauge invariant quantities and the second is to choose a gauge. The gauge for the tensor part is not fixable, the quantities D_{ij} and π_{ij}^T are already gauge invariant. The vector degrees of freedom π_i^V, δu_i^V, C_i and G_i can be combined to form gauge invariant quantities, π_i^V, δu_i^V and $\tilde{G}_i \equiv G_i - a\dot{C}_i$. Fixing a gauge for the vector part is choosing an ϵ for which either C_i or G_i vanishes. For the scalar degrees of freedom there are several options to fix the gauge.

3.3 Comoving Gauge

In this section we describe the relevant gauge fixing choice for the theory of inflation [6] [1]. Once we choose a gauge, a covariant Lagrangian density is obtained with the objects that are covariant in this gauge. The following gauge choice sets the fluctuations of the scalar field to zero,

$$\delta\phi(x) = 0\,, \tag{3.36}$$

this means that time diffeomorphisms are fixed,

$$\epsilon_0(x) = 0\,, \tag{3.37}$$

for all x spacetime coordinates. There is a function $\tilde{t}(x)$ that describes the spacetime 'slicing' for which the scalar perturbations are zero. Once the gauge is fixed this function coincides with the coordinate time t. This gauge is called the comoving gauge. In a way spacetime now moves to eliminate scalar perturbations. In some literature it is also called unitary gauge. This is a term used in spontaneous symmetry breaking of a gauge degree of freedom. Gauge symmetry breaking is similar to this gauge fixing, as in both cases certain degrees of freedom which are non-linearly realized by the symmetry are fixed to manifest the degree of freedom in other physical objects. In this case the perturbation of the inflaton field is gauge fixed and the degree of freedom is expressed by the metric. The spatial diffeomorphisms are not fixed,

$$x^i \to x^i + \epsilon^i(x)\,. \tag{3.38}$$

More terms can be added to a theory whose symmetry has decreased. Added terms only need to comply with the symmetries that are unbroken. The terms that comply with these symmetries are listed below.

- The Riemann tensor $R_{\mu\nu\rho\sigma}$ is invariant under all diffeomorphisms, so it is the usual diffeomorphism symmetry term. To combine it into a scalar it is contracted with covariant derivatives or with the completely antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$.

- Any function of $\tilde{t}$ becomes a function of the time t in the comoving gauge. Terms in the Lagrangian are therefore multiplied by generic functions $f(t)$.

- The gradient of $\tilde{t}$ is a delta function, $\partial_\mu\tilde{t} = \delta^0_\mu$. The contraction of any vector or tensor with this object, will result in a free upper index 0. An example of this is g^{00} , therefore powers of g^{00} are in the Lagrangian.

- The following unitary normal vector is perpendicular to the spatial hypersurface,

  $$n_\mu = \frac{\partial_\mu\tilde{t}}{\sqrt{-g^{\mu\nu}\partial_\mu\tilde{t}\partial_\nu\tilde{t}}}\,. \tag{3.39}$$

  We use it to write the spatial metric,

  $$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu\,, \tag{3.40}$$

this projects objects onto the spatial hypersurfaces by contraction. The extrinsic curvature is the covariant derivative of the normal vector projected onto the spatial hyperslices,

$$K_{\mu\nu} = h^{\sigma}_{\mu} \nabla_{\sigma} n_{\nu} \,, \tag{3.41}$$

the ν index is not contracted with the metric h, as it is already on the surface, $n^{\nu}\nabla_{\sigma} n_{\nu} = \frac{1}{2}\nabla(n^{\nu} n_{\nu}) = 0$. The extrinsic curvature is part of the Lagrangian. The Riemann tensor on the spatial hypersurfaces is equal to a combination of the extrinsic curvature tensor and the full Riemann term, projected on the surface. Therefore using one or the other in the Lagrangian is sufficient. In this case the extrinsic curvature is part of the Lagrangian.

In conclusion, the action consists of functions of the objects,

$$S = \int d^4x \sqrt{-g}\, F(R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \nabla_{\mu}, t) \,. \tag{3.42}$$

The linear terms in the action are functions of t,

$$S_{\text{lin}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R - \Lambda(t) - c(t) g^{00} \right] . \tag{3.43}$$

They are fixed by taking into account the equations of motion in an FLRW background. The equations are produced by evaluating the energy-momentum tensor,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\Lambda,c}}{\delta g^{\mu\nu}} = -c(t) g^{00} - \Lambda(t) + 2c(t) \delta^0_{\mu} \delta^0_{\nu} \,, \tag{3.44}$$

and filling it into the Einstein equations for the FLRW metric,

$$H^2 = \frac{1}{3M_{Pl}^2} [c(t) + \Lambda(t)] \,, \tag{3.45}$$

$$\dot{H} + H^2 = -\frac{1}{3M_{Pl}^2} [2c(t) - \lambda(t)] \,. \tag{3.46}$$

The values of the linear terms are,

$$c(t) = -M_{Pl}^2 \dot{H} \,, \tag{3.47}$$

$$\Lambda(t) = M_{Pl}^2 [3H^2 + \dot{H}] \,. \tag{3.48}$$

The higher order terms are powers of the objects subtracted by their background value, denoted by a bar. This ensures that any higher order term is invariant under spatial diffeomorphisms and is evaluated around the FLRW background. The background values of the objects are

$$\bar{g}^{00} = -1 \,, \tag{3.49}$$

$$\bar{K}_{\mu\nu} = a^2 H h_{\mu\nu} \,, \tag{3.50}$$

$$\bar{R}_{\mu\nu\rho\sigma} = 2(H + k) h_{\mu[\rho} h_{\sigma]\nu} + (\dot{H} + H^2) a^2 h_{\mu\sigma} \delta^0_{\nu} \delta^0_{\rho} \tag{3.51}$$

$$+ \text{ permutations in indices} \,, \tag{3.52}$$

where for a generic tensor $U_{[\mu\nu]\cdots} \equiv \frac{1}{2}(U_{\mu\nu\cdots} - U_{\nu\mu\cdots})$. The higher order terms are powers of: $\delta g^{00} = g^{00} + 1$, $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$, $\delta R = R_{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu\rho\sigma}$. Each power is multiplied by a function $M_i(t)$ which has a mass dimension compensating for the dimension of the operator. The action is,

$$\begin{aligned}
S = \int d^4x\sqrt{-g}[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2(3H^2 + \dot{H}) \\
&+ \frac{1}{2!}M_2(t)^4 (g^{00}+1)^2 + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 \\
&- \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu_\mu - \frac{\bar{M}_2(t)^2}{2}(\delta K^\mu_\mu)^2 \\
&- \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \cdots]\,.
\end{aligned} \tag{3.53}$$

This theory does not only describe the case for perturbations to a scalar field, but any theory of gravity with spontaneously broken time translations. Higher order terms, for instance the $\delta R_{\mu\nu\rho\sigma}$ term, are not mentioned above.

3.3.1 Restoring Time Diffeomorphisms

When a symmetry is spontaneously broken, it is non-linearly realized by a Goldstone field. The concept of symmetry breaking and Goldstone fields is more thoroughly explained in part II. The results of the Goldstone theorem applied to time translations symmetry breaking are stated here. The main difference here as opposed to the following chapters is that the Goldstone field is inserted by hand, also known as the Stückelberg trick. The method that we describe in the rest of this thesis is based on the coset construction and is used to derive the Goldstone field theory from the symmetry breaking pattern. The Stückelberg trick is done by replacing the time diffeomorphism parameter by a spacetime dependent field $\pi(x)$. Under a time diffeomorphism $\epsilon_0(x)$ it has the following transformation property, for

$$t \to t + \epsilon_0(x)\,, \tag{3.54}$$

then

$$\pi(x) \to \pi(x) - \epsilon_0(x)\,, \tag{3.55}$$

such that the combination

$$\pi(x) + t\,, \tag{3.56}$$

is time diffeomorphism invariant. The following example explains the application of the Stückelberg trick. We take the linear terms $A(t)$ and $B(t)g^{00}$. The original action is,

$$\int d^4x\sqrt{-g}\left[A(t) + B(t)g^{00}\right]\,. \tag{3.57}$$

The 00 element of the metric transforms under broken time diffeomorphisms as,

$$g^{00}(x) \to g'^{00}(x'(x)) = \frac{\partial x'^0(x)}{\partial x^\mu}\frac{\partial x'^0(x)}{\partial x^\nu} g^{\mu\nu}(x)\,. \tag{3.58}$$

Applying the Stückelberg trick $t \to t' \equiv t + \pi$ to the action,

$$\int d^4x\sqrt{-g}\left[A(t+\pi(x)) + B(t+\pi(x))\frac{\partial(t+\pi(x))}{\partial x^\mu}\frac{\partial(t+\pi(x))}{\partial x^\nu}g^{\mu\nu}(x)\right]. \tag{3.59}$$

The Stückelberg trick applied to the action that we derived previously in (3.53) results in restored time diffeomorphisms,

$$\begin{aligned} S = \int d^4x\sqrt{-g}[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2\dot{H}\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu} - M_{Pl}^2(3H^2+\dot{H}) \\ &+ \frac{1}{2!}M_2(t)^4(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1)^2 \\ &+ \frac{1}{3!}M_3(t)^4(\partial_\mu(t+\pi)\partial_\nu(t+\pi)g^{\mu\nu}+1)^3 + \cdots]\,. \end{aligned} \tag{3.60}$$

This method is a model independent description of a theory of spontaneous time diffeomorphism breaking [7]. The π field is related to perturbations of the scale factor and in turn to comoving curvature perturbations by multiplication with the Hubble parameter,

$$H\pi \propto \frac{\delta a}{a} \propto \zeta\,. \tag{3.61}$$

The comoving curvature perturbations ζ describe the origin of cosmological observables such as the CMB.

Part II

Spontaneous Symmetry breaking in QFT

Chapter 4

Goldstone Theorem and a complex Scalar Field

A spontaneous symmetry breaking of a physical system in a symmetric state under a certain action, a rotation for example, is characterized by three traits [8]:

(i) a parameter assuming a critical value breaks the symmetry, after that

(ii) the symmetric state is unstable and the system will reach an alternative ground state, which

(iii) is part of a continuous family of ground states.

A symmetry of a quantum field theory is characterized by a symmetry transformation on a field changes the Lagrangian by a total derivative [9]

$$\delta\mathcal{L} = \partial_\mu F^\mu \,, \tag{4.1}$$

where F_μ is some arbitrary function. For *any* transformation $\delta\phi$ the Lagrangian transforms as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)\,, \tag{4.2}$$

rewriting it

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\,, \tag{4.3}$$

where the equation of motion of the field is $\delta S = 0$. Then using (4.1) with (4.3),

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu\right) = 0\,, \tag{4.4}$$

we find for every symmetry transformation a current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - F^\mu(\phi)\,, \tag{4.5}$$

which is conserved

$$\partial_\mu j^\mu = 0\,. \tag{4.6}$$

The definition of the associated charge is

$$Q \equiv \int d^3x j_0(x)\,, \tag{4.7}$$

and this is conserved as well. This is proven by,

$$\frac{dQ}{dt} = \int d^3x \frac{dj_0(x)}{dt} = -\int d^3x \partial_i j^i(x) = -\int d\mathbf{A}\cdot\vec{j}(x) = 0\,, \tag{4.8}$$

as $\vec{j} \to 0$ when $|\vec{x}| \to \infty$, where $\mathbf{A}$ denotes a surface at infinity. In quantum mechanics the charge generates the transformation $\delta\phi$ with a commutator, in classical mechanics we use Poisson brackets. This is seen by,

$$[Q, \phi(y)] = \int d^3x \left[j^0(x), \phi(y)\right] = \int d^3x \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\delta\phi, \phi(y)\right]\,. \tag{4.9}$$

The conjugate momentum is defined as,

$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\,, \tag{4.10}$$

the commutation relation for the field $\phi(y)$ and its conjugate momentum is

$$[\phi(x), \pi(y)] = i\delta^{(3)}(x-y)\,. \tag{4.11}$$

This enables us to complete equation (4.9),

$$[Q, \phi(y)] = \int d^3x \,[\pi(x), \phi(y)]\,\delta\phi = -i\int d^3x \delta^{(3)}(x-y)\delta\phi = -i\delta\phi\,. \tag{4.12}$$

In the following we describe a group theory approach to symmetry transformations.

4.1 Group Theory

The groups that are covered are continuous symmetry groups, these are also called Lie groups. Groups are a set of elements with an action between them. A few prerequisites are needed to form a group, there has to be an identity operator, for every element an inverse, it has to be associative and any product of elements has to be in the group. The number of elements can be infinite, like in Lie groups. The group is Abelian when actions commute and non-Abelian when transformations of the group do not commute. The linearization of the action of a Lie group gives the Lie algebra. An exponential map connects the group with its related algebra in the case that the group is a matrix group. The transformation matrix $g \in G$ and the matrices T_a as elements of the Lie algebra are connected by,

$$g \equiv e^{i\alpha^a T_a} = 1 + i\alpha^a T_a + \cdots . \tag{4.13}$$

A finite dimensional and unitary matrix representation of the Lie group exists if and only if the group is a matrix Lie group. Only matrix Lie groups are handled in the cases studied. The elements of the Lie algebra are commonly called generators of the group. In the examples real fields are transformed into real fields, so the transformations g are real and unitary. The generators T_a are finite and hermitian, because the transformations are real and unitary. They are defined by hand to satisfy,

$$\mathrm{Tr}[T_a T_b] \equiv \delta_{ab} . \tag{4.14}$$

In this case up and down indices are indistinguishable. Closure of the group means for $g_1(\equiv e^{i\alpha_a T_a}) \in G, g_2(\equiv e^{i\alpha_b T_b}) \in G$ that $g_1 g_2 \in G$ and in terms of the generators is (by applying (4.13))

$$T_a T_b - T_b T_a = i c_{abd} T_d , \tag{4.15}$$

where c_{abd} is the form factor, a constant and antisymmetric under exchange of two indices. In spontaneous symmetry breaking we start with a full symmetry group G. The vacuum configuration then spontaneously breaks the symmetry to a subgroup H, denoting the group of preserved symmetries. A subgroup is a subset of elements of the full group that form a closed group by themselves, therefore it has its distinct generators. SSB then makes a division of the generators of G possible. Generators are divided into unbroken generators t_i, which are part of the Lie algebra of H and broken generators X_α, part of the Lie algebra of G/H. G/H is the quotient group or left coset of H in G and is defined by

$$G/H \equiv \{gH : g \in G\} . \tag{4.16}$$

The coset consists of the spontaneously broken symmetries. Note the indices that are used:

- $a, b, c...$ for generators that are part of the full Lie algebra,
- $i, j, k...$ for unbroken and
- $\alpha, \beta, ...$ for broken generators,

The generators of the full group are separated into the generators of the subgroup and of the coset,

$$T_a = \begin{cases} t_i & \text{unbroken generators} , \\ X_\alpha & \text{broken generators} . \end{cases} \tag{4.17}$$

H is a subgroup, so its algebra is closed under multiplication,

$$t_i t_j - t_j t_i = c_{ijk} t_k . \tag{4.18}$$

The commutator between broken and unbroken generators is proportional to the broken generators,

$$t_i X_\alpha - X_\alpha t_i = i c_{i\alpha\beta} X_\beta \,. \tag{4.19}$$

These properties are applied in the rest of this thesis.

4.2 Goldstone Theorem

Before formulating the Goldstone Theorem and its proof, we show that The charge Q is the generator of the symmetry. Let g act by definition on the field ϕ as,

$$g \cdot \phi \equiv \phi + \alpha\delta\phi + \cdots \,. \tag{4.20}$$

Where α is an infinitesimal parameter. The symmetry transformation is represented by a unitary operator, for a finite and Hermitian generator X

$$\Omega_g \equiv e^{i\alpha X} \,, \tag{4.21}$$

which acts on the field by conjugation,

$$(g \cdot \phi(x)) = \Omega_g^{-1}\phi(x)\Omega_g = e^{-i\alpha X}\phi(x)e^{i\alpha X} = \phi(x) - \alpha[X, \phi(x)] + \cdots \,. \tag{4.22}$$

The commutation relation,

$$[X, \phi(x)] = -i\delta\phi(x) \,, \tag{4.23}$$

holds for the generator $X = Q$. For non abelian symmetries this holds as well (for which the label a is added to denote multiple generators),

$$\Omega \equiv e^{i\alpha^a Q^a} \,. \tag{4.24}$$

This represents the symmetry transformation. A vacuum configuration of the field $|0\rangle$ and a transformation Ω are used to describe spontaneous symmetry breaking of a physical system. A symmetry that leaves the vacuum invariant acts like,

$$\Omega\,|0\rangle = |0\rangle \,. \tag{4.25}$$

The charge annihilates the vacuum,

$$Q^a\,|0\rangle = 0 \,. \tag{4.26}$$

This vacuum is a non-degenerate vacuum with respect to the symmetry. The symmetry is spontaneously broken, when there exists an infinite number of groundstates connected by the symmetry transformation are possible configurations. The transformation does not change the action (they are groundstates, i.e. produce the same minimum value of the action) but does change the vacuum state. Bringing the groundstate to a different groundstate means that,

$$\Omega\,|0\rangle = |0\rangle' \neq |0\rangle \,. \tag{4.27}$$

The charge does not annihilate the vacuum,

$$Q^a |0\rangle \neq 0 \,. \tag{4.28}$$

The action is invariant under the transformation

$$\delta S = 0 \,. \tag{4.29}$$

Goldstone bosons are associated with the theory when a symmetry is spontaneously broken, as stated by the Goldstone Theorem.

Goldstone Theorem Take a field $\phi_a(x)$, where a denotes the a'th vector element of the vector field. Let the field transform under non-Abelian symmetries. Its expectation value is non-vanishing around the vacuum,

$$\langle 0| \phi_a(x) |0\rangle \neq 0 \,, \tag{4.30}$$

and it is not a singlet, i.e. it is not invariant under the transformation $\Omega = e^{iQ^a \alpha^a}$,

$$[Q_a, \phi_b(x)] = -i\delta\phi(x) = -ic_{abc}\phi^c(x) \,. \tag{4.31}$$

If the charge Q^a of the transformation does not annihilate the vacuum,

$$Q^a |0\rangle \neq 0 \,, \tag{4.32}$$

then massless particles exist in the particle states of the theory. The number of massless particles is dependent on the conservation of symmetries $G \to H$ where G is the group of transformations under which the vacuum is invariant before spontaneous symmetry breaking and H is the group that still leaves the vacuum state invariant after spontaneous symmetry breaking. The number of massless particles is the dimension of the quotient group when Lorentz invariance is manifest,

$$\dim \{G/H\} = \dim \{G\} - \dim \{H\} \,. \tag{4.33}$$

Proof The vacuum expectation value is non-zero $\langle 0| \phi_a(x) |0\rangle \neq 0$, the state $\phi_b(x)$ is not a singlet $[Q_a, \phi_b(x)] = -ic_{abc}\phi^c(x)$ and the charge does not annihilate the vacuum,

$$\langle 0| [Q_a, \phi_b(x)] |0\rangle = \langle 0| Q_a\phi_b(x) - \phi_b(x)Q_a |0\rangle \neq 0 \,. \tag{4.34}$$

Writing the charge as an integral and inserting intermediate states $\sum_n |n\rangle \langle n|$,

$$\sum_n \int d^3y [\langle 0| j_a^0(y) |n\rangle \langle n| \phi_b(x) |0\rangle - \langle 0| \phi_b(x) |n\rangle \langle n| j_a^0(y) |0\rangle]|_{x^0=y^0} \neq 0 \,. \tag{4.35}$$

It is evaluated at $y^0 = x^0$ as the operators $\phi_a(x)$ and $j_a^0(y)$ act at the same time. Using translational invariance, $j_a^0(y) = e^{-ipy} j_a^0(0) e^{ipy}$, inserting this into (4.35) and integrating,

$$
\begin{aligned}
&\sum_n \int d^3y \left[\langle 0|\, j_a^0(0)\, |n\rangle \langle n|\, \phi_b(x)\, |0\rangle\, e^{ip_n y} - \langle 0|\, \phi_b(x)\, |n\rangle \langle n|\, j_a^0(y)\, |0\rangle\, e^{-ip_n y}\right]|_{x^0=y^0} \\
&= (2\pi)^3 \sum_n \delta^3(\mathbf{p}_n) \left[\langle 0|\, j_a^0(0)\, |n\rangle \langle n|\, \phi_b(x)\, |0\rangle\, e^{ip_{n0}y_0} - \langle 0|\, \phi_b(x)\, |n\rangle \langle n|\, j_a^0(y)\, |0\rangle\, e^{-ip_{n0}y_0}\right] \\
&= (2\pi)^3 \sum_n \delta^3(\mathbf{p}_n) \left[\langle 0|\, j_a^0(0)\, |n\rangle \langle n|\, \phi_b(x)\, |0\rangle\, e^{iM_n y_0} - \langle 0|\, \phi_b(x)\, |n\rangle \langle n|\, j_a^0(y)\, |0\rangle\, e^{-iM_n y_0}\right] \\
&\neq 0\,,
\end{aligned}
\tag{4.36}
$$

where M_n is the mass for the intermediate state n. To prove that $M_n = 0$ we have to show that (4.36) is independent of y_0. Taking the derivative of (4.34) with respect to y_0,

$$
\frac{\partial}{\partial y_0} \langle 0|\, [Q^a, \phi_b(x)]\, |0\rangle = \frac{\partial}{\partial y_0} \int d^3y\, \langle 0|\, [j_a^0(y), \phi_b(x)]\, |0\rangle\ . \tag{4.37}
$$

The conservation of the current, $\partial_\mu j_a^\mu = \partial_0 j_a^0(y) + \partial_i j_a^i(y) = 0$, integrated over space is

$$
\frac{\partial}{\partial y_0} \int d^3 j_a^0(0) = -\int d^3y \partial_i j_a^i(y)\,. \tag{4.38}
$$

Inserting this into (4.37) gives

$$
\begin{aligned}
&\frac{\partial}{\partial y_0} \int d^3y\, \langle 0|\, [j_a^0(y), \phi_b(x)]\, |0\rangle \\
&= -\int d^3y\, \langle 0|\, [\partial_i j_a^i(y), \phi_b(x)]\, |0\rangle \\
&= -\int d\boldsymbol{A} \cdot \langle 0|\, [\vec{j}_a(y), \phi_b(x)]\, |0\rangle\ .
\end{aligned}
\tag{4.39}
$$

The spatial part of the current $\vec{j}_a(y)$ is evaluated at the spatial boundary where the light cones of the fields $\phi_b(x)$ cannot overlap with it. This surface integral therefore vanishes by causality. In conclusion, the masses of the intermediate states are zero. The number of these states is equal to the number of charges that do not annihilate the vacuum after spontaneous symmetry breaking, i.e. is equal to $\dim\{G/H\}$.

Gaplessness and decoupling Before mentioning examples of spontaneous symmetry breaking we describe two special properties of Goldstone bosons in this paragraph. Firstly, in the low energy limit a Goldstone mode decouples from all interactions, because the state at zero momentum is indistinguishable from the vacuum. Secondly, the energy of a Goldstone mode vanishes at zero momentum, i.e. it is gapless. Gapless means that there is no energy to gap before the field is perturbed. The energy is

$$
E(p) = \sqrt{m^2 + p^2}\,, \tag{4.40}
$$

$m = 0$ for the Goldstone mode to ensure gaplessness. Gaplessness means that

$$\lim_{p \to 0} E(p) = 0\,, \tag{4.41}$$

with spatial momentum p, is ensured. The Goldstone theorem is applied in the next sections.

4.3 Abelian global internal Symmetries

We describe a toy UV complete model in this section as an example of spontaneous symmetry breaking. The internal $U(1)$ symmetry of the theory gets spontaneously broken. The field is then expanded around the nonzero vacuum expectation value to write the Goldstone boson part of the Lagrangian [10]. The UV theory is,

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - V(\phi, \phi^*)\,, \tag{4.42}$$

$$V = \frac{\lambda}{4}(\phi^*\phi - \frac{\mu^2}{\lambda})^2\,, \tag{4.43}$$

and is invariant under the $U(1)$ symmetry group. The transformation of the field is,

$$\phi \to e^{i\alpha}\phi\,. \tag{4.44}$$

The symmetry is called a global symmetry, because α is a constant. To calculate the current, apply Noether's theorem (4.5) with $\delta\phi = i\phi$. The current is

$$j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)\,. \tag{4.45}$$

The vacuum expectation values are the minima of the potential $V(\phi^*\phi)$, denoted by ϕ_0. The potential has

$$\begin{cases} \frac{\mu^2}{\lambda} \leq 0 & \text{one groundstate} & (|\phi_0| = 0)\,, \\ \frac{\mu^2}{\lambda} > 0 & \text{infinitely many groundstates} & (|\phi_0| = \frac{\mu}{\sqrt{\lambda}})\,. \end{cases} \tag{4.46}$$

In the last case, the groundstates can reach one another by a phase transformation

$$\phi_0 \to e^{i\alpha}\phi_0\,. \tag{4.47}$$

When this regime is reached the $U(1)$ symmetry is spontaneously broken, i.e. each vacuum expectation value is distinct from the other modulo $U(1)$. This example is also commonly described as a Mexican hat potential. The vacuum expectation value is defined as $v \equiv |\phi_0| = \frac{\mu}{\sqrt{\lambda}}$. The Goldstone mode becomes explicit in the Lagrangian as the field is written in terms of a radial component and a phase component,

$$\phi(x) = \chi(x) e^{i\theta(x)}\,. \tag{4.48}$$

Applying this to the Lagrangian (4.42),

$$\mathcal{L} = -\partial_\mu \chi \partial^\mu \chi - \chi^2 \partial_\mu \theta \partial^\mu \theta - V(\chi^2)\,. \tag{4.49}$$

At low energies the field is evaluated at $\phi_0(x) = ve^{i\theta(x)}$, this decouples the Goldstone field θ from other fields. Besides that, the Goldstone terms in the Lagrangian only depend on derivatives of the field, this implies gaplessness. Next we describe a Lagrangian that is produced by only looking at the spontaneous symmetry breaking pattern and that does not require knowledge of a full UV theory.

4.3.1 Phenomenological Lagrangian

In this subsection we derive a Lagrangian that is invariant under a non-linear symmetry transformation using only the transformation itself. This technique makes an underlying UV complete theory superfluous. It is also called constructing a phenomenological Lagrangian, as from the phenomenon, in this case an inhomogeneous symmetry transformation, the theory is constructed. Via this method one derives the quantum behaviour of the Goldstone field, i.e. the effective field theory. For example, the Goldstone field in the previous UV complete model transforms nonlinearly,

$$\theta(x) \to \theta(x) + \alpha\,, \tag{4.50}$$

with constant α. A Lagrangian that only depends on the derivatives of the field is covariant under this transformation. Written down to second order,

$$\mathcal{L}_{eff} = -\frac{1}{2}f_1^2\partial_\mu\theta\partial^\mu\theta - \frac{f_2}{2}\partial_\mu\partial_\nu\theta\partial^\mu\partial^\nu\theta + \frac{f_3}{4}\partial_\mu\theta\partial^\mu\theta\partial_\nu\theta\partial^\nu\theta + \cdots\,, \tag{4.51}$$

where f_1 is a dimension one quantity, as the angle θ is dimensionless. We apply Noether's Theorem (4.5) on the Lagrangian and derive the current. The non-linear transformation implies, $\delta\theta = -1$ and the current is,

$$j^\mu = f_1^2\partial^\mu\theta + f_2\partial_\nu(\partial^\mu\partial^\nu\theta) + f_3(\partial^\nu\theta\partial_\nu\theta)\partial^\mu\theta + \cdots\,. \tag{4.52}$$

To recap, this section started with a toy UV complete model and spontaneously broke the Abelian symmetry. The Goldstone mode is then written explicitly in the Lagrangian. After that, we describe how a phenomenological Lagrangian is obtained solely from the non-linear symmetry transformation. The next chapter is along the same lines, but for the case of non-Abelian spontaneous symmetry breaking.

Chapter 5

Non-Abelian spontaneous Symmetry breaking

The sections in this chapter cover a few examples and their phenomenological Lagrangian by making use of the group theory principles from the preceding chapter.

5.1 Spontaneous breaking of non-Abelian Symmetry

This section starts with a UV complete theory as in the Abelian case. A UV complete example model of the Lagrangian for N fields $\phi = \phi_i$ for $i = 1, 2, 3..., N$ is,

$$\mathcal{L} = -\partial_\mu \phi^T \partial^\mu \phi - V(\phi)\,. \tag{5.1}$$

The T denotes the transpose. The symmetry of the theory is the group of spacetime independent orthonormal rotations $O(N)$ between the fields ($O^T O = 1$ and $\partial_\mu O = 0$),

$$\phi(x) \to O\phi\,. \tag{5.2}$$

and the potential is invariant under the transformation O,

$$V(O\phi) = V(\phi)\,. \tag{5.3}$$

Spontaneous symmetry breaking occurs when the vacuum expectation value v is non-zero,

$$\langle\phi\rangle \equiv v \neq 0\,, \tag{5.4}$$

and conserves only the elements of the subgroup $H \leq O(N)$ where $h \in H$,

$$hv = v\,. \tag{5.5}$$

The Goldstone bosons are the field components of $\phi(x)$ that are aligned with the directions of the generators of the coset $O(N)/H$.

UV model non-Abelian spontaneous symmetry breaking The fields are $\phi(x) \equiv \{\phi_i\}$. Then writing the fields as,

$$\phi(x) = \Omega(\theta(x))\rho(x) \equiv e^{i\theta^\alpha(x)X^\alpha}\rho(x)\,. \tag{5.6}$$

The transformation $\Omega(\omega)$, for some constant ω can be seen as a symmetry transformation in the direction of the generators that are broken by the vacuum configuration. The component ρ is perpendicular to these directions. To recap, X_α are the broken generators, part of the algebra of the coset G/H and t_i are the unbroken generators part of the algebra of H. The transformation $g \in G$ is decomposed in these [11],

$$g \equiv e^{i\alpha^a T_a} = \exp\left[iv^i t_i\right]\exp\left[i\omega^\alpha X_\alpha\right]\,, \tag{5.7}$$

where ω^α and v^i are real parameters and the element of the subgroup $h \in H$ is

$$h \equiv e^{iv^i t_i}\,. \tag{5.8}$$

We require that the potential V has the property

$$V(\Omega(\theta(x))\rho(x)) = V(\rho(x))\,. \tag{5.9}$$

In the Abelian case this is synonymous to the potential being independent of the phase. The Goldstone fields are only expressed in the kinetic part of the Lagrangian. They are therefore gapless, i.e. there is no Goldstone field when $\partial_\mu\theta^\alpha(x) = 0$. The transformations $\theta^\alpha(x) \to \theta'^\alpha(x)$ and $\rho(x) \to \rho'(x)$ are defined by $\phi(x) \to \phi'(x) = g\phi(x)$,

$$g\Omega(\theta(x))\rho(x) = \Omega(\theta'(x))\rho'(x)\,, \tag{5.10}$$

let $\Omega(\theta') = \Omega'$. Then (not noting the spacetime dependence anymore),

$$(\Omega')^{-1}g\Omega\rho = \rho'\,, \tag{5.11}$$

the matrix $\gamma \equiv (\Omega')^{-1}g\Omega$ is the matrix that transforms ρ,

$$\gamma\rho = \rho'\,. \tag{5.12}$$

It is proveable that γ is in the subgroup H. Two cases of the symmetry transformation g are taken, $g = h$ to prove that $\gamma \in H$ and when $g \in G/H$. In each case we derive the transformation of the Goldstone field θ.

Case $g = h$ In the case that $g = h$ ($\omega = 0$ in (5.7)), because the commutator between broken and unbroken generators is proportional to the broken generators we have

$$h\Omega = \Omega' h\,, \tag{5.13}$$

$$e^{iv^i t_i}e^{i\theta(x)^\alpha X_\alpha} = e^{i\theta'(x)^\alpha X_\alpha}e^{iv^i t_i}\,. \tag{5.14}$$

This makes,

$$\Omega' = h\Omega h^{-1}\,, \tag{5.15}$$

(by writing this infinitesimally) we see that θ transforms linearly

$$\theta'^\alpha(x)X_\alpha = h\theta^\alpha(x)X_\alpha h^{-1}\,, \tag{5.16}$$

Case $g \in G/H$ To find out how the θ field transforms under a $g \in G/H$ we use infinitesimal notation. The transformation g is now (5.7) with $v^i = 0$,

$$g = 1 + \omega^\alpha X_\alpha + \cdots , \tag{5.17}$$

The linear change in the transformation of $\Omega(\theta)$ is expressed as Δ_α

$$\Omega(\theta') = \Omega(\theta)[1 + i\Delta_\alpha(\theta,\omega)X_\alpha + \cdots] . \tag{5.18}$$

Then using (5.11) to define the transformation of the Goldstone fields,

$$(\Omega')^{-1} g\Omega = \gamma , \tag{5.19}$$

$$g\Omega = \Omega'\gamma . \tag{5.20}$$

Where $\gamma \in H$. Using (5.18) we get

$$g\Omega = \Omega[1 + i\Delta_\alpha X^\alpha + \cdots]\gamma \tag{5.21}$$

taking it to first order (leaving out the matrix gamma for the transformation of θ),

$$e^{-i\theta\cdot X}(1 + i\omega \cdot X + \cdots)e^{i\theta\cdot X} = 1 + i\Delta^\alpha X_\alpha + \cdots . \tag{5.22}$$

Isolating the Δ_α

$$e^{-i\theta\cdot X}(\omega \cdot X)e^{i\theta\cdot X} = \Delta^\alpha X_\alpha , \tag{5.23}$$

$$\Delta_\beta = \mathrm{Tr}[X_\beta e^{-i\theta^\alpha X_\alpha}(\omega \cdot X)e^{i\theta^\gamma X_\gamma}] . \tag{5.24}$$

Where we've used that $\mathrm{Tr}[X_\alpha X_\beta] = \delta_{\alpha\beta}$ and the cyclicity property of the trace. We write the term within the trace infinitesimally to find the transformation of θ,

$$X_\beta e^{-i\theta^\alpha X_\alpha}(\omega\cdot X)e^{i\theta^\gamma X_\gamma} = X_\beta(1 - i\theta^\alpha X_\alpha + \cdots)(\omega\cdot X)(1 + i\theta^\gamma X_\gamma + \cdots) , \tag{5.25}$$

writing out the right hand side,

$$X_\beta X_\alpha \omega_\alpha + i\omega_\delta \theta^\gamma X_\beta(X_\delta X_\gamma - X_\gamma X_\delta) + \cdots , \tag{5.26}$$

using the commutation relation $X_\delta X_\gamma - X_\gamma X_\delta = ic_{i\delta\gamma}t_i + ic_{\alpha\delta\gamma}X_\alpha$ and that the $\mathrm{Tr}[X_\alpha t_i] = 0$ we find,

$$\Delta_\beta \approx \mathrm{Tr}[X_\beta X_\alpha \omega_\alpha + i\omega_\delta\theta^\gamma X_\beta(X_\delta X_\gamma - X_\gamma X_\delta) + \cdots] , \tag{5.27}$$

$$\Delta_\alpha \approx \omega_\alpha - c_{\alpha\beta\gamma}\omega_\beta\theta^\gamma + O(\theta^2) . \tag{5.28}$$

To show that,

$$\delta\theta^\alpha = (\theta')^\alpha - \theta^\alpha \propto \Delta^\alpha , \tag{5.29}$$

is engaging and not part of this thesis. Nevertheless, we can see from this that θ transforms as,

$$\delta\theta_\alpha = \omega_\alpha - c_{\alpha\beta\gamma}\omega_\beta\theta^\gamma + O(\theta^2) . \tag{5.30}$$

This implies that the Goldstone fields transforms inhomogeneously. Just as in the Abelian case. The field transformation of the Goldstone field is field dependent. This point is taken into account when constructing the Lagrangian.

5.2 Model independent Lagrangian

The field dependence of the transformation has to be taken into account in developing the kinetic part of the Goldstone field in the Lagrangian. Starting with the coset element and writing it down in the following combination,

$$\Omega^{-1}\partial_\mu\Omega\,. \tag{5.31}$$

This is called the Maurer-Cartan form. We find the transformation of $\Omega^{-1}\partial_\mu\Omega$ by using

$$\Omega' = g\Omega h^{-1}\,, \tag{5.32}$$

so that

$$\begin{aligned}\Omega^{-1}\partial_\mu\Omega \to &(\Omega')^{-1}\partial_\mu\Omega' = (g\Omega h^{-1})^{-1}\partial_\mu(g\Omega h^{-1})\\ &= h\Omega^{-1}\partial_\mu\Omega h^{-1} + h\partial_\mu h^{-1}\,.\end{aligned} \tag{5.33}$$

We apply $\partial_\mu(hh^{-1}) = (\partial_\mu h)h^{-1} + h\partial_\mu h^{-1} = 0\,,$

$$\Omega^{-1}\partial_\mu\Omega \to h\Omega^{-1}\partial_\mu\Omega h^{-1} - \partial_\mu h h^{-1}\,. \tag{5.34}$$

We seperate the Maurer-Cartan form, by defining it in terms of the generators t_i and X_α and fields A^i_μ and e^α_μ. This enables us to handle the transformation per generator,

$$\Omega^{-1}\partial_\mu\Omega \equiv -iA^i_\mu t_i + ie^\alpha_\mu X_\alpha\,. \tag{5.35}$$

The $\partial_\mu h^{-1}h$ only depends on t_i, as it is in the subgroup H. The fields $A^i_\mu(\theta)$ and $e^\alpha_\mu(\theta)$ transform as,

$$-iA^i_\mu(\theta)t_i \to -iA^i_\mu(\theta')t_i = h[-iA^i_\mu(\theta)t_i]h^{-1} - \partial_\mu h h^{-1}\,, \tag{5.36}$$

$$ie^\alpha_\mu(\theta)X_\alpha \to ie^\alpha_\mu(\theta')X_\alpha = h[ie^\alpha_\mu(\theta)X_\alpha]h^{-1}\,. \tag{5.37}$$

The field aligned with the broken generators transforms as a covariant quantity. Contractions of these fields with other covariant quantities are still covariant, the trace of these over the generators are in the Lagrangian. The fields aligned with the unbroken symmetries transform as gauge potentials. They are part of the covariant derivative. Both of these cases are shown below. For the rest of the derivations of the Lagrangian the reader is referred to the paper by C.P. Burgess. The main results are stated here. First we extract $\partial_\mu\theta$ from the fields,

$$e^\alpha_\mu = e^\alpha_\beta\partial_\mu\theta^\beta\,, \tag{5.38}$$

$$A^i_\mu(\theta) = A^i_\alpha(\theta)\partial_\mu\theta^\alpha\,. \tag{5.39}$$

The seperated fields are expanded and depend on θ in the following way,

$$A^i_\alpha(\theta) \approx \frac{1}{2}c_{i\alpha\beta}\theta^\beta + O(\theta^2)\,, \tag{5.40}$$

$$e^{\alpha}_{\beta}(\theta) \approx \delta_{\alpha\beta} - \frac{1}{2} c_{\alpha\beta\gamma}\theta^{\gamma} + O(\theta^2)\,. \tag{5.41}$$

Terms in the Lagrangian are G-invariant and Lorentz invariant. The covariant derivative is constructed with A^i_μ

$$(D_\mu e_\nu)^\alpha = \partial_\mu e^\alpha_\nu + c^\alpha_{i\beta} A^i_\mu e^\beta_\nu\,, \tag{5.42}$$

and transforms covariantly. The Lagrangian is made up of combinations of the terms,

$$\mathcal{L}(e_\mu, D_\mu, ...)\,, \tag{5.43}$$

so that the first term is

$$\mathcal{L}_{GB} = -\frac{1}{2} f_{\alpha\beta}\eta^{\mu\nu} e^\alpha_\mu e^\beta_\nu + (\text{higher-derivate terms})\,. \tag{5.44}$$

There are terms of dimension 4 and 5 in this first term

$$\begin{aligned} e^\alpha_\beta \partial_\mu \theta^\beta e^\gamma_\alpha \partial^\mu \theta^\gamma \approx\; & \partial_\mu\theta^\alpha\partial^\mu\theta_\alpha - c_{\alpha\beta\gamma}\theta^\gamma\partial_\mu\theta^\alpha\partial^\mu\theta_\beta \\ & + O(\text{higher-dimension})\,, \end{aligned} \tag{5.45}$$

where the dimension four term is the kinetic term. We now derived the Lagrangian for Goldstone bosons arising in non-Abelian spontaneous symmetry breaking by using the coset construction. Remark that there is a significant difference between the terms generated from the group invariant ingredients for Abelian and for non-Abelian symmetry breaking. This difference is in the dimensions of the terms. For the Abelian symmetry breaking, the terms depend on $\partial_\mu\theta$ and ∂_μ. The leading order term is a dimension four operator, the next to leading order is a dimension six operator. For the non-Abelian symmetry breaking (5.45) the next to leading order term is of the form

$$c_{\alpha\beta\gamma}\theta^\alpha(\partial_\mu\theta^\beta\partial^\mu\theta^\gamma)\,, \tag{5.46}$$

which is of dimension 5. Terms of dimension 6, 7 and so forth arise at higher order. Dimensional analysis [12] tells us why this difference is important. A Lagrangian made up from operators $\mathcal{O}_i$, where the i-th operator has dimension δ_i, has a scaling, for a process at scale E,

$$\int d^D x \mathcal{O}_i \sim E^{\delta_i - D}\,. \tag{5.47}$$

For $\delta_i > D$ then at low energies the term becomes less pronounced, this is called an irrelevant operator. If $\delta_i = D$, the term is called marginal, this means that it is independent of the energy scale and contributes the same in every regime. Lastly, if $\delta_i < D$ the term is dubbed relevant and its contribution grows as the energy scale drops. Applying this to the Lagrangians derived above, we find that in both cases the leading term is a marginal term. The difference is in the next to leading order term, here the non-Abelian case has a dimension five term significantly contributing at lower energies than for the Abelian case. Before proceeding with spacetime spontaneous symmetry breaking, we describe a famous example of spontaneous symmetry breaking.

5.2.1 Phenomenological Lagrangian for Abelian SSB by coset construction

Here we show how the coset construction is used in Abelian SSB. The spontaneous symmetry breaking pattern is $U(1) \to 1$. The coset element is

$$\Omega = e^{i\theta(x)} . \tag{5.48}$$

The Maurer Cartan one form is

$$\Omega^{-1}\partial_\mu\Omega \equiv iD_\mu\theta , \tag{5.49}$$

where the covariant derivative reduces to the partial derivative without any conserved symmetries. Calculating the left hand side

$$\Omega^{-1}\partial_\mu\Omega = i\partial_\mu\theta(x) . \tag{5.50}$$

The building blocks in the Lagrangian in this case are

$$\mathcal{L}(\partial_\mu\theta, \partial_\mu) . \tag{5.51}$$

We combine these in a Lorentz invariant way and add the dimension one quantity f to arrive at

$$\mathcal{L} = -\frac{1}{2}f_1^2\partial_\mu\theta\partial^\mu\theta - \frac{f_2}{4}\partial_\mu\partial_\nu\theta\partial^\mu\partial^\nu\theta - \frac{f_3}{4}\partial_\mu\theta\partial^\mu\theta\partial_\nu\theta\partial^\nu\theta + \cdots . \tag{5.52}$$

This is the same Lagrangian as in (4.51). It is the same symmetry breaking pattern, from which a single gapless excitation emerges.

5.3 Chiral Lagrangian

This is an instructive derivation of the Chiral Lagrangian. Three fields in QCD the u, d and s quarks are fairly light quarks. They approximately transform under the chiral $SU(3)_L \times SU(3)_R$. The symmetry would be exact if the fields were massless. The generators of $SU(3)_L \times SU(3)_R$ are,

$$\lambda_L^a = \frac{1}{2}(1+\gamma_5)\lambda^a , \qquad \lambda_R^a = \frac{1}{2}(1-\gamma_5)\lambda^a . \tag{5.53}$$

λ_a are the generators of $SU(3)$. The generators are normalized, $\mathrm{Tr}[\lambda_a\lambda_b] = \delta_{ab}$.

$$\lambda^a = \lambda_L^a + \lambda_R^a . \tag{5.54}$$

The generators of $SU(3) \times SU(3)$ can be written as

$$\lambda_a , \qquad \lambda_a\gamma_5 (= \lambda_L^a - \lambda_R^a) . \tag{5.55}$$

These generators are used in the SSB. The masses are negligible with respect to the energy scale that exceeds the binding energy between quarks. At this scale hadrons appear as these are QCD condensates. The hadrons are vacuum configurations of the quarks which spontaneously break the symmetry. The symmetry is spontaneously broken to a single $SU(3)$ by the formation of hadrons. The symmetry transformation for the three vector is

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \to \exp\left[i(v_a\lambda_a + \omega_a\lambda_a\gamma_5)\right] \begin{pmatrix} u \\ d \\ s \end{pmatrix} . \tag{5.56}$$

The broken and unbroken generators are

$$\begin{cases} \lambda_a & \text{unbroken } SU(3)\,, \\ \lambda_a\gamma_5 & \text{broken } SU(3)\,. \end{cases} \tag{5.57}$$

The distinction between broken and unbroken generators is made by the γ_5 matrix, so the indices $a, b, ...$ are used in both the broken and unbroken cases. The v_a and ω_a are real numbers, γ_5 is the product of the four anticommuting 4×4 Dirac- matrices. The generators of $SU(3)$ are stated explicitly in Weinberg's book [13]. Next, the $SU(3) \times SU(3)$ transformations are written as a product of the broken and unbroken transformations, with fields $\theta_a(x)$ and weight u_a respectively,

$$\exp\left[-i\gamma_5\theta_a\lambda_a\right]\exp\left[iu_a\lambda_a\right] . \tag{5.58}$$

The transformation of the Goldstone fields $\theta_a(x)$ is,

$$\begin{aligned} &\exp\left[i(v_a\lambda_a + \omega_a\lambda_a\gamma_5)\right]\exp\left[-i\gamma_5\theta_a\lambda_a\right] \\ &= \exp\left[-i\gamma_5\theta'_a\lambda_a\right]\exp\left[iu_a\lambda_a\right] \end{aligned} \tag{5.59}$$

In this case a different approach can be used to derive the transformation rule for the Goldstone fields. By splitting the transformation (5.59) in two independent parts. The $(1+\gamma_5)$ (denoted by L) and the $(1-\gamma_5)$ (R) part,

$$\varphi_a^L \equiv v_a + \omega_a\,, \qquad \varphi_a^R \equiv v_a - \omega_a\,, \tag{5.60}$$

(5.59) squared is

$$\begin{aligned} &\exp\left[i(\varphi_a^L\lambda_a(1+\gamma_5) + \varphi_a^R\lambda_a(1-\gamma_5))\right] \times \\ &\exp\left[(1+\gamma_5)\left(-i\theta_a\lambda_a\right) + (1-\gamma_5)\left(i\theta_a\lambda_a\right)\right] \\ =&\exp\left[(1+\gamma_5)\left(-i\theta'_a\lambda_a\right) + (1-\gamma_5)\left(i\theta'_a\lambda_a\right)\right] \times \\ &\exp\left[2iu_a\lambda_a\right] . \end{aligned} \tag{5.61}$$

Splitting this transformation into the L,

$$\begin{aligned} &\exp\left[i(\varphi_a^L\lambda_a)\right]\exp\left[-i\theta_a\lambda_a\right] \\ &= \exp\left[-i\theta'_a\lambda_a\right]\exp\left[iu_a\lambda_a\right] , \end{aligned} \tag{5.62}$$

and R parts,

$$\begin{aligned} &\exp\left[i(\varphi_a^R\lambda_a)\right]\exp\left[i\theta_a\lambda_a\right] \\ &= \exp\left[i\theta'_a\lambda_a\right]\exp\left[iu_a\lambda_a\right] . \end{aligned} \tag{5.63}$$

We derive the transformation of,

$$\Omega(\theta) \equiv \exp\left[2i\theta_a(x)\lambda_a\right] \,, \tag{5.64}$$

by multiplying (5.63) with the inverse of (5.62),

$$\Omega'(\theta) = \exp\left[i\varphi_a^R\lambda_a\right]\Omega(\theta)\exp\left[-i\varphi_a^L\lambda_a\right] \,. \tag{5.65}$$

Now that we have the transformation rule for the Goldstone fields we can write an invariant Lagrangian. The elements of $\Omega(\theta)$ are subject to constraints: $\Omega\Omega^\dagger = 1$ and $\det\Omega = 1$. The first term in the Lagrangian is (Maurer Cartan one form contracted with itself and in the trace),

$$\mathcal{L}_{2\text{derivatives}} = -\frac{1}{16}F^2\mathrm{Tr}\left\{\partial_\mu\Omega\partial^\mu\Omega^\dagger\right\} \,, \tag{5.66}$$

where the cyclicity of the trace ensures invariance under the transformation (5.65), F is an undetermined constant. To find the explicit behaviour of the meson fields we write $\theta_a(x)\lambda_a$ as a matrix (the matrices λ_a are generators of $SU(3)$, in a way this is writing fields along the generators),

$$\theta_a(x)\lambda_a = \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{bmatrix} \,, \tag{5.67}$$

and write (5.66) out,

$$\begin{aligned} \mathcal{L}_{kinetic} = &-\frac{1}{2}\partial_\mu\pi^0\partial^\mu\pi^0 - \partial_\mu\pi^+\partial^\mu\pi^- \\ &- \partial_\mu K^+\partial^\mu\bar{K}^- - \partial_\mu K^0\partial^\mu\bar{K}^0 - \frac{1}{2}\partial_\mu\eta^0\partial^\mu\eta^0 \,. \end{aligned} \tag{5.68}$$

This is the chiral Lagrangian describing the effective theory.

Chapter 6

Gauge Symmetries

In this chapter we will discuss the gauging of symmetries, i.e. making them local. This is done by adding spacetime dependence to the transformation parameter and introducing a new gauge field to the derivative such that it becomes a covariant derivative. Demanding that the covariant derivative on the field transforms linearly under the spacetime dependent transformation, we get the transformation rule for the gauge field. We will start with a section on Abelian gauge symmetries. The most famous example of a $U(1)$ gauge theory is the theory of quantum electrodynamics. Following that up is a section outlining non-Abelian gauge symmetries.

6.1 Abelian Gauge Symmetry

We introduce local Abelian gauge invariance to a theory as a prerequisite for the theory to be phase independent at each spacetime point [14]. In terms of a phase θ of a Dirac field ψ we write it as

$$\psi \to e^{i\theta}\psi \quad \text{(global)}\,, \qquad \psi \to e^{i\theta(x)}\psi \quad \text{(local)}\,. \tag{6.1}$$

To ensure that the theory is independent of its phase at each spacetime point we need to introduce a field which communicates the information regarding the phase differences, this is the gauge field. As an illustrating example we start with a free massive Dirac Lagrangian,

$$\mathcal{L} = -\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi\,. \tag{6.2}$$

This is invariant under global phase transformations. Making the transformation local we see that the derivative transforms differently,

$$\partial_\mu\psi(x) \to \partial_\mu(e^{iq\theta(x)}\psi(x)) = e^{iq\theta(x)}(\partial_\mu\psi(x) + iq\partial_\mu\theta(x)\psi(x))\,, \tag{6.3}$$

where q is introduced as a measure of the strength of the phase transformation. Now a modified derivative that transforms linearly is introduced,

$$D_\mu\psi(x) \to (D_\mu\psi(x))' = e^{iq\theta(x)}(D_\mu\psi(x))\,, \tag{6.4}$$

this D_μ is the covariant derivative. It must contain a field which compensates for the generation of the extra term in (6.3), we call this field A_μ. If we define,

$$D_\mu \psi(x) \equiv (\partial_\mu - iqA_\mu(x))\psi(x)\,, \tag{6.5}$$

we have the transformation,

$$\begin{aligned} D_\mu \psi(x) \rightarrow & (\partial_\mu - iqA'_\mu(x))(e^{iq\theta(x)}\psi(x)) \\ &= e^{iq\theta(x)}(\partial_\mu + iq\partial_\mu\theta - iqA'_\mu)\psi(x) \\ &= e^{iq\theta(x)}(\partial_\mu - iqA_\mu)\psi(x)\,. \end{aligned} \tag{6.6}$$

Evaluating this, we distill the transformation rule of the gauge field to compensate for the extra term. This rule is

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\theta(x)\,. \tag{6.7}$$

Now we that we have a new derivative which abides local gauge invariance we can write the Lagrangian as,

$$\mathcal{L} = -\bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,, \tag{6.8}$$

this Lagrangian is not interaction free anymore, as the fermion field now has an interaction term inside of the $\not{D}$. Repeated application of covariant derivatives yields new covariant quantities. An example of this is the commutator,

$$[D_\mu, D_\nu]\psi = -iq(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi\,, \tag{6.9}$$

and since ψ transforms covariantly and $[D_\mu, D_\nu]\psi$ is covariant, the commutator itself is covariant. The commutator is called the field strength tensor,

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{6.10}$$

which can be used to write a propagation term (non-interacting) for the gauge field in the Lagrangian. Combining it with the Lagrangian of the massive fermion,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \bar{\psi}\partial\psi - m\bar{\psi}\psi + iqA_\mu\bar{\psi}\gamma^\mu\psi\,, \tag{6.11}$$

bringing about a current for the gauge field, which is

$$J_\mu = iq\bar{\psi}\gamma_\mu\psi\,. \tag{6.12}$$

In the next section we will derive the theories for non-Abelian gauge symmetries is along the same lines.

6.2 Non-Abelian Gauge Symmetry

Consider a field which transforms under a Lie group G. For every group element we have a corresponding matrix $g \equiv U$. The transformation rotates the field as,

$$\psi(x) \rightarrow \psi'(x) = U\psi(x)\,. \tag{6.13}$$

We use T_a to denote the generators of the Lie algebra. To extend the group to local gauge transformations we make U spacetime dependent, $U(x)$. The derivative on the field transforms as,

$$\partial_\mu \psi(x) \to U(x)\partial_\mu \psi(x) + (\partial_\mu U(x))\psi(x)\,, \tag{6.14}$$

and in the same way as in the preceding section we introduce a gauge field which compensates for the extra term on the right,

$$D_\mu \psi(x) \equiv \partial_\mu \psi - A_\mu \psi\,, \qquad A_\mu \equiv A_\mu^a T_a\,, \tag{6.15}$$

by transforming under gauge transformations as,

$$A_\mu \to A'_\mu = U A_\mu U^{-1} + (\partial_\mu U)U^{-1}. \tag{6.16}$$

Taking the commutator of two covariant derivatives we get

$$[D_\mu, D_\nu]\psi = -(\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu])\psi \equiv -G_{\mu\nu}\psi\,, \tag{6.17}$$

where $G_{\mu\nu}$ is the field strength tensor for the non-Abelian gauge symmetry. In terms of the field A_μ^a ($G_{\mu\nu} \equiv G_{\mu\nu}^a T_a$) it is

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - c_{bc}^a A_\mu^b A_\nu^c\,, \tag{6.18}$$

where,

$$[T_a, T_b] = c_{ab}^c T_c\,. \tag{6.19}$$

The field strength tensor transforms according to,

$$G_{\mu\nu} \to G'_{\mu\nu} = U G_{\mu\nu} U^{-1}\,, \tag{6.20}$$

using this we construct the gauge Lagrangian. We replace derivatives with covariant derivatives to make the existing part of the Lagrangian gauge invariant. To incorporate a stand alone field strength term in the Lagrangian we use the trace operator for its cyclic property,

$$\mathcal{L}_{\text{gauge field}} = \frac{1}{4}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]\,, \tag{6.21}$$

and the Lagrangian plus a fermion field becomes,

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] - \bar{\psi}\not{D}\psi - m\bar{\psi}\psi\,, \tag{6.22}$$

with a current,

$$J_\mu^a = \bar{\psi}\gamma_\mu \psi T^a\,. \tag{6.23}$$

6.3 Gauge Symmetries and the Coset Construction

Applying the knowledge from the two preceding sections to the coset construction enables us to gauge any symmetry and work with it in its spontaneously broken state. If a group G with generators T_a is gauged, i.e. the measure of the parametrization is spacetime dependent, we replace the partial derivative by a covariant derivative in the Maurer-Cartan form,

$$\Omega^{-1}\partial_\mu\Omega \to \Omega^{-1}D_\mu\Omega \equiv \Omega^{-1}(\partial_\mu + iA_\mu^a T_a)\Omega\,. \tag{6.24}$$

This is invariant under the local transformations,

$$\Omega \to g(x)\Omega \qquad A_\mu \to g(x)A_\mu g^{-1}(x) + i(\partial_\mu g(x))g^{-1}(x)\,. \tag{6.25}$$

When broken symmetries are gauged we can fix the gauge by setting the Goldstone fields from the broken generators to zero, this is called the unitary gauge.

Part III

Spacetime spontaneous Symmetry breaking

Chapter 7

Formalisms

Before starting with the spontaneous breaking of spacetime symmetries, a few formalisms are explained in this chapter. Mainly the tetrad formalism and the Poincaré algebra are worked out.

7.1 Tetrad Formalism and Fermions in Gravity

We can replace all Lorentz tensors with objects which transform as tensors under general coordinate transformations, i.e. diffeomorphisms for most theories that we would like to gauge to incorporate gravity,[15] [16]. We then replace all derivatives by covariant derivatives and replace the η_{ab} minkowski metric by the general metric $g_{\mu\nu}$. This method however is only applicable to objects which behave like tensors under local Lorentz transformations. Spinors are objects which do not behave as tensors under these transformations and need a different approach. First we outline the tetrad formalism and after that we describe the spinor in gravity. Using the principle of equivalence we define at every point X a coordinate set that is local at this point, call it ξ^a_X. Note that it is only possible to cover the whole spacetime with a single coordinate system like this when it is a flat spacetime. The invariant quantity is given by,

$$ds^2 = \eta_{ab} dy^a dy^b \,, \tag{7.1}$$

and to make this quantity invariant under general coordinate transformations (GCT), we introduce fields $e^a_\mu(x)$. Then,

$$dy^a = e^a_\mu(x) dx^\mu \,, \tag{7.2}$$

where

$$e^a_\mu(x) \equiv \left(\frac{\partial \xi^a_X(x)}{\partial x^\mu} \right)_{x=X} \,, \tag{7.3}$$

with this, we can rewrite the invariant quantity as,

$$ds^2 = e^a_\mu(x) e^b_\nu(x) \eta_{ab} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu \,, \tag{7.4}$$

where dx^μ transforms as a vector under GCT. From this we derive the transformation rule for the $e_\mu^a(x)$,

$$e'^a_\mu(x) = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) e_\nu^a(x)\,. \tag{7.5}$$

This shows that the $e_\mu^a(x)$ describes a set of four vectors; this set is called the tetrad or vierbein. Using this formalism we derive an invariant measure, $dV = d^4y$,

$$d^4y = d^4x \left|\frac{\partial y}{\partial x}\right| = d^4x \det e_\mu^a\,, \tag{7.6}$$

and from the determinant of the metric we find the determinant of the vierbein,

$$g \equiv \det g_{\mu\nu} = \det(e_\mu^a e_\nu^b \eta_{ab}) = -(\det e_\mu^a)^2\,. \tag{7.7}$$

Matching it up, the invariant volume element becomes

$$d^4y = \sqrt{-g}d^4x\,. \tag{7.8}$$

This is the familiar integration measure for an action in a curved spacetime. By formulating the vierbein we now have a tool to promote any object with Lorentz indices (denoted by $a, b, c, ...$) to one with world indices (denoted by $\mu, \nu, \sigma, ...$) by contracting it with the vierbein. This enables us to generalize the spinors from flat spacetime to a general spacetime. Starting with a Dirac theory with Lorentz indices,

$$\mathcal{L} = \bar{\psi}(i\gamma^a\partial_a - m)\psi\,, \tag{7.9}$$

the Dirac field transforms as,

$$\psi \to S\psi \equiv \exp\left\{-\frac{iq}{2}\alpha^{ab}J_{ab}\right\}\psi\,. \tag{7.10}$$

The J_{ab} are the generators of the Lorentz group in spinorial representation,

$$J_{ab} = \frac{\sigma_{ab}}{2} = i\frac{[\gamma_a, \gamma_b]}{2}\,. \tag{7.11}$$

To add gravity the Lorentz transformations are gauged by making α^{ab} a function of spacetime and introducing a gauge field A_μ, such that the covariant derivative on the dirac field is,

$$D_\mu\psi \equiv (\partial_\mu - iqA_\mu)\psi \equiv (\partial_\mu - i\frac{q}{2}J_{ab}A_\mu^{ab})\psi\,. \tag{7.12}$$

To derive the transformation of the gauge field we demand that the covariant derivative on the field transforms linearly,

$$(D_\mu\psi)(x) \to S(x)(D_\mu\psi)(x)\,, \tag{7.13}$$

this implies,

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1}\,, \tag{7.14}$$

this gauge field A_μ is called the spin connection. We can now write the Dirac Lagrangian in a general spacetime by adding the vierbein and the spin connection to (7.9),

$$\mathcal{L} = \bar{\psi}(i\gamma^a e^\mu_a(x) D_\mu - m)\psi\,. \tag{7.15}$$

By making the known steps, adding a field strength tensor and the invariant volume element to the integration, we get the theory for a fermion field coupled to gravity.

7.2 Poincaré Group

The Poincaré algebra and the corresponding group express the symmetries of Minkowski spacetime. We derive the algebra in this chapter from first principles. From Einstein's principle of equivalence we know that for any two inertial reference frames denoted by coordinates x^μ and x'^μ, the invariant lengths have to be equal [17],

$$\eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}dx'^\mu dx'^\nu\,, \tag{7.16}$$

rewriting,

$$\eta_{\mu\nu}\left(\frac{\partial x'^\mu}{\partial x^\rho}\right)\left(\frac{\partial x'^\nu}{\partial x^\sigma}\right) = \eta_{\rho\sigma}\,. \tag{7.17}$$

The coordinate transformations $T(\Lambda, a)$ consist of two parts, the Lorentz transformations Λ^μ_ν and the constant translations a^μ,

$$T(\Lambda, a) : x \to x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu\,, \tag{7.18}$$

applying the equivalence principle (7.17),

$$\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma = \eta_{\rho\sigma}\,, \tag{7.19}$$

puts a constraint on the Lorentz transformation Λ. In the following is argued that the transformations $T(\Lambda, a)$ form a group. Firstly, the transformations $T(\Lambda, a)$ satisfy a composition rule. Starting with,

$$T(\bar{\Lambda}, \bar{a}) : \quad x'^\mu \to x''^\mu\,, \qquad x''^\mu = \bar{\Lambda}^\mu_\nu x'^\nu + \bar{a}^\mu\,, \tag{7.20}$$

and combining this with the transformation $x^\mu \to x'^\mu$,

$$x''^\mu = \bar{\Lambda}^\mu_\nu\Lambda^\nu_\rho x^\rho + \bar{\Lambda}^\mu_\nu a^\nu + \bar{a}^\mu\,, \tag{7.21}$$

is equivalent to applying the transformation,

$$T(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a}) = T(\bar{\Lambda}, \bar{a})T(\Lambda, a)\,. \tag{7.22}$$

To make the argument that these transformations constitute a group complete, we have to show that the Lorentz transformations have an inverse. The determinant of (7.19),

$$(\det\Lambda)^2 = 1\,, \tag{7.23}$$

shows that $(\Lambda^{-1})^\mu_\nu$ exists. The inverse is determined by the same equation,

$$(\Lambda^{-1})^\nu_\rho = \Lambda^\rho_\nu = \eta_{\nu\mu}\eta^{\rho\sigma}\Lambda^\mu_\sigma\,. \tag{7.24}$$

The transformations with $a = 0$ form a subgroup,

$$T(\bar{\Lambda}, 0)T(\Lambda, 0) = T(\bar{\Lambda}\Lambda, 0)\,, \tag{7.25}$$

called the homogeneous Lorentz group. The full group of transformations $T(\Lambda, a)$ is the inhomogeneous Lorentz group.

Poincaré algebra Transformations $T(\Lambda, a)$ induce linear transformations on vectors in physical Hilbert space $\Psi \in H$. Denoted by $U(\Lambda, a)$,

$$T(\Lambda, a) : \Psi \to U(\Lambda, a)\Psi\,, \tag{7.26}$$

satisfying the composition rule (7.22)

$$U(\bar{\Lambda}, \bar{a})U(\Lambda, a) = U(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})\,. \tag{7.27}$$

We derive its corresponding algebra to apply it to our coset construction of spontaneous spacetime symmetry breaking. To derive the algebra of a Lie group we look at infinitesimal transformations around the identity $\Lambda^\mu_\nu = \delta^\mu_\nu$ and $a^\mu = 0$,

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu\,, \qquad a^\mu = \epsilon^\mu\,, \tag{7.28}$$

the Lorentz constraint (7.19) puts a constraint on ω^μ_ν,

$$\eta_{\mu\nu}(\delta^\mu_\rho + \omega^\mu_\rho)(\delta^\nu_\sigma + \omega^\nu_\sigma) = \eta_{\rho\sigma}\,, \tag{7.29}$$

rewritten,

$$\eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + O(\omega^2) = \eta_{\rho\sigma}\,, \tag{7.30}$$

the constraint implies that $\omega_{\mu\nu}$ is anti-symmetric,

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{7.31}$$

An anti-symmetric 4 dimensional two tensor has $\frac{4\times 3}{2} = 6$ independent components, with the 4 components of ϵ^μ this makes a total of 10 independent components for an inhomogeneous Lorentz transformation. The linear transformation on a vector in physical Hilbert space is

$$U(1 + \omega, a) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\rho P^\rho + \cdots\,, \tag{7.32}$$

here $J^{\mu\nu}$ and P^μ are ω and ϵ independent. The sign for P^μ is a convention, as the theory does not distinguish between $\epsilon_\mu P^\mu$ and $-\epsilon_\mu P^\mu$. P^μ are the components of the energy-momentum operator and $J^{\mu\nu}$ are the components of Lorentz operators, these are spatial rotations and boosts. The transformation $U(1+\omega, a)$ is unitary, so $J^{\mu\nu}$ and P^μ are Hermitian and as $\omega_{\mu\nu}$ is anti-symmetric, $J^{\mu\nu}$ is as well. To derive the transformation properties we apply a transformation $U(\Lambda, a)$ and its inverse to the infinitesimal transformation,

$$U(\Lambda, a)U(1+\omega, a)U^{-1}(\Lambda, a)\,, \tag{7.33}$$

which to linear order is,

$$U(\Lambda, a)\left[\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\rho P^\rho\right]U^{-1}(\Lambda, a)\,. \tag{7.34}$$

The inverse of the transformation is obtained by using the transitive property of the transformation (7.27),

$$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1}a)\,, \tag{7.35}$$

so that,

$$U(\Lambda, a)U^{-1}(\Lambda, a) = U(\Lambda^{-1}\Lambda, \Lambda^{-1}a - \Lambda^{-1}a) = U(1, 0)\,. \tag{7.36}$$

We can use this inverse in the equation (7.33),

$$\begin{aligned} U(\Lambda, a)U(1+\omega, a)U^{-1}(\Lambda, a) &= U(\Lambda(1+\omega), \Lambda\epsilon + a)U(\Lambda^{-1}, -\Lambda^{-1}a) \\ &= U(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon + a - \Lambda(1+\omega)\Lambda^{-1}a) \\ &= U(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a) \end{aligned} \tag{7.37}$$

to analyze the transformation property of $J^{\mu\nu}$ and P^μ we compare this to linear order with the equation (7.34),

$$\begin{aligned} &U(\Lambda, a)\left[\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\rho P^\rho\right]U^{-1}(\Lambda, a) \\ &= \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu}J^{\mu\nu} - i(\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\rho P^\rho\,, \end{aligned} \tag{7.38}$$

To first order in ϵ and ω, the transformation rules for $J^{\mu\nu}$ and P^μ are

$$U(\Lambda, a)J^{\rho\sigma}U^{-1}(\Lambda, a) = \Lambda^\rho_\mu\Lambda^\sigma_\nu(J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu) \tag{7.39}$$

$$U(\Lambda, a)P^\rho U^{-1}(\Lambda, a) = \Lambda^\rho_\mu P^\mu \tag{7.40}$$

These rules clarify that $J^{\mu\nu}$ transforms as a tensor under Lorentz transformations (when $a = 0$) and P^μ as a vector. The operator P^μ is invariant under purely translations, but $J^{\mu\nu}$ is not. This can be viewed as a consequence of changing the origin with respect to the calculation of the angular momentum.

We derive the commutation relations between the operators in the last part of this chapter. The transformation of any operator X is,

$$U(1+\omega, a) X U^{-1}(1+\omega, a) = X + i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu, X\right] + \cdots . \tag{7.41}$$

Firstly, to first order in ω and ϵ (7.39) is,

$$\begin{aligned} &\Lambda^\rho_\mu \Lambda^\sigma_\nu (J^{\mu\nu} - a^\mu P^\nu + a^\nu P^\mu) \\ &= (\delta^\rho_\mu + \omega_{\mu\nu}\eta^{\nu\rho})(\delta^\sigma_\nu - \omega_{\mu\nu}\eta^{\mu\sigma})(J^{\mu\nu} - \epsilon^\mu P^\nu + \epsilon^\nu P^\mu) \\ &= \omega_{\mu\nu}\eta^{\nu\rho}J^{\mu\sigma} - \omega_{\mu\nu}\eta^{\mu\sigma}J^{\rho\nu} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho + O(\omega\epsilon) , \end{aligned} \tag{7.42}$$

and (7.40) is,

$$\Lambda^\rho_\mu P^\mu = (\delta^\rho_\mu + \omega_{\mu\nu}\eta^{\nu\rho})P^\mu = P^\rho + \eta^{\nu\rho}\omega_{\mu\nu}P^\mu . \tag{7.43}$$

Replacing X with $J^{\mu\nu}$ and with P^μ gives,

$$i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu, J^{\rho\sigma}\right] = \omega_{\mu\nu}\eta^{\nu\rho}J^{\mu\sigma} - \omega_{\mu\nu}\eta^{\mu\sigma}J^{\rho\nu} - \epsilon^\rho P^\sigma + \epsilon^\sigma P^\rho , \tag{7.44}$$

and,

$$i\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} - \epsilon_\mu P^\mu, P^\rho\right] = \omega_{\mu\nu}\eta^{\nu\rho}P^\mu , \tag{7.45}$$

splitting the equations in their ω and ϵ parts and taking into account the anti-symmetry of ω, the commutation rules are,

$$i\,[J^{\mu\nu}, J^{\rho\sigma}] = J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}\eta^{\mu\rho} - J^{\rho\nu}\eta^{\sigma\mu} + J^{\rho\mu}\eta^{\sigma\nu} , \tag{7.46}$$

$$i\,[P^\mu, J^{\rho\sigma}] = P^\sigma\eta^{\mu\rho} - P^\rho\eta^{\mu\sigma} , \tag{7.47}$$

$$[P^\mu, P^\rho] = 0 . \tag{7.48}$$

Which makes up the Lie algebra of the Poincaré group. The exponential maps for finite translations are represented by,

$$U(1, a) = e^{-ia_\mu P^\mu} , \tag{7.49}$$

and for Lorentz transformations by,

$$U(\Lambda_\xi, 0) = e^{\frac{i}{2}\xi_{\mu\nu}J^{\mu\nu}} , \tag{7.50}$$

combining them we get,

$$U(\Lambda_\xi, a) = e^{-ia_\mu P^\mu} e^{\frac{i}{2}\xi_{\mu\nu}J^{\mu\nu}} . \tag{7.51}$$

These are used in the coset construction of spacetime spontaneous symmetry breaking.

Chapter 8

Spacetime spontaneous Symmetry breaking and the Coset Construction

To recap, when a symmetry group is spontaneously broken the Goldstone fields are realized by the coset element and in effect by the Maurer-Cartan one form. The Maurer Cartan one form is created from the generators of the broken symmetries, i.e. non linearly realized symmetries. Translations are, regardless of the symmetry breaking pattern, non linearly realized symmetries. Therefore the generators of translations, P_μ are included into the Maurer Cartan one form [18].

8.1 Inverse Higgs Constraint

Breaking spacetime symmetries is different from breaking internal symmetries, because of the possibility of a degenerate number of degrees of freedom in the theory. In other words, the number of Goldstones is not equal to the dimension of the coset. To accurately predict the number of degrees of freedom we have to impose the Inverse Higgs constraint. The scheme for such a constraint is the following [6]. In general the commutator between a translation generator P, and a broken generator X_1, contains a broken generator X_2 and an unbroken generator t_1,

$$[P, X_1] \sim X_2 + t_1 \,. \tag{8.1}$$

A commutator containing a broken symmetry generator $X_2 \neq 0$, different from the broken symmetry generator in the commutator $X_2 \neq X_1$ implies that the inverse Higgs has to be applied. The inverse Higgs is a constraint that removes the redundant degrees of freedom. The component of the Maurer Cartan one form that is aligned with the broken generator X_2 is set to zero,

$$[\Omega^{-1}\partial_\mu \Omega]_{X_2} = 0 \,. \tag{8.2}$$

In most cases this corresponds to setting the covariant derivative of the Goldstone field of X_2 (denoted by π') in the direction of P to zero,

$$D_P\pi' = 0\,, \tag{8.3}$$

(this is the component of the Maurer Cartan form aligned with the generator X_2). Thereby expressing the Goldstone field of X_1 (denoted by π) in terms of derivatives on the Goldstone of X_2. A way of understanding the constraint and why it is necessary, is found in the low energy limit of the theory. The commutator (8.1) being non-zero implies that from the covariant derivative on the field π' a linear term of the Goldstone field π arises in the Lagrangian. This is synonymous to a mass term of the Goldstone π. This mass term creates a gap for low energy theories. By integrating out this gap we obtain an effective action. This is done by applying the equations of motion. The constraint is therefore a different approach to the same result by applying the equation of motion of π. The equation of motion would generally not be exactly the constraint (8.3), but some combination of these terms, which becomes a complicated term. This term describes the substition of the field π with derivatives of π'. We show the IH constraint for a toy UV example [19]. The theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2\,. \tag{8.4}$$

The theory is invariant under

$$\phi \to \phi' = \phi + \alpha + \beta_\mu x^\mu\,. \tag{8.5}$$

The α is removed by the derivative in the action and the β_μ in the action can be written as a total derivative,

$$\partial_\mu\phi\beta^\mu + \frac{1}{2}\beta_\mu^2 = \partial_\mu(\phi\beta^\mu + \frac{1}{2}\beta_\nu^2 x^\mu)\,. \tag{8.6}$$

The groundstate ϕ_0 spontaneously breaks the symmetry. This implies that two excitations exist, one for the transformation α and one for β. The excitations are the spacetime dependent realizations of these symmetries. In this case we have that any $\beta_\mu(x)x^\mu$ can be written as $\alpha(x) = \beta_\mu(x)x^\mu$. This means that the shifts are not independent and we have to apply inverse Higgs constraint.

8.2 SSB of Time Translations in non-dynamical Gravity

We derive a theory that describes scalar perturbations by SSB of time translations by defining the symmetry breaking pattern and the Maurer Cartan one form. We do this in non-dynamical gravity. The theory of spontaneously broken gauged time translation is the main goal. But a specific case where gravity is non-dynamical is done to see if the method is applied correctly. There are condensed matter systems that spontaneously break time translations without gravity. These theories with a single degree of freedom are described by the effective theory derived here. The symmetries are the elements of the Poincaré group. Boosts and time translations are related. When time translations are broken, Lorentz boosts are automatically broken. This is because boosts relate spacetime translations with each other. The broken generators are P_0 and K_i. Any non-linearly realized symmetry is in the Maurer Cartan one form.

The coset is $ISO(3,1)/SO(3)\times \mathbb{R}^3$ with an element:

$$\Omega \equiv e^{ix^iP_i}e^{i\pi(x)P_0}e^{i\eta^i(x)K_i}\,. \tag{8.7}$$

The Maurer Cartan form is defined as [20]

$$\Omega^{-1}\partial_\mu\Omega \equiv ie^0_\mu P_0 + ie^i_\mu P_i + ie^\nu_\mu D_\nu\eta^i K_i + e^\nu_\mu A^{ij}_\nu J_{ij}\,. \tag{8.8}$$

Evaluating it by using (8.7) turns out

$$\Omega^{-1}\partial_\mu\Omega = i(\delta^i_\mu\Lambda^\nu_i + \partial_\mu\pi\Lambda^\nu_0)P_\nu + e^{-i\eta^iK_i}\partial_\mu e^{i\eta^jK_j}\,. \tag{8.9}$$

Using equivalences described in formalisms of the Poincaré algebra we get

$$\Omega^{-1}\partial_\mu\Omega = i(\delta^i_\mu\Lambda^0_i+\partial_\mu\pi\Lambda^0_0)P_0+i(\delta^j_\mu\Lambda^i_j+\partial_\mu\pi\Lambda^i_0)P_i+iK_i(\Lambda^{-1}\partial_\mu\Lambda)^{0i}+\frac{i}{2}J_{ij}(\Lambda^{-1}\partial_\mu\Lambda)^{ij}\,. \tag{8.10}$$

Inverse Higgs Constraint The degree of freedom $\eta^i(x)$ is degenerate. To apply the Inverse Higgs constraint the degree of freedom is expressed in terms of the velocity vector that describes the boost, β^i defined as

$$\beta^i = \frac{\eta^i}{\eta}\tanh\eta\,, \tag{8.11}$$

with $\eta \equiv \sqrt{\vec{\eta}^2}$. We have

$$\Lambda^0_0 = \gamma, \qquad \Lambda^0_i = \gamma\beta_i, \qquad \Lambda^i_0 = \gamma\beta^i, \qquad \Lambda^i_j = \delta^i_j + (\gamma-1)\frac{\beta^i\beta_j}{\beta^2}\,, \tag{8.12}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$. The inverse Higgs is based on

$$[P_i, K_j] \subset \delta_{ij}P_0\,. \tag{8.13}$$

The field aligned with the generator P_0 is set to zero to remove the degenerate degree of freedom

$$e^0_i = 0\,. \tag{8.14}$$

Writing this out leads to

$$\begin{aligned} &\Lambda^0_i + \partial_i\pi\Lambda^0_0 = 0\\ &\gamma\beta_i + \partial_i\pi\gamma = 0\,,\\ &\beta_i = -\partial_i\pi\,. \end{aligned} \tag{8.15}$$

We interpret this statement as that any boost can be expressed by a spacetime dependent time translation. The building blocks of the Lagrangian are the invariant objects which arise from the Maurer-Cartan one-form. They are

$$\mathcal{L}(e_\mu^\nu, D_\mu)\,, \tag{8.16}$$

where the field in the covariant derivative is the field aligned with the unbroken symmetries J_{ij}

$$(D_\mu e_\nu)^\alpha = \partial_\mu e_\nu^\alpha + ic_{ij\beta}^\alpha A_\mu^{ij} e_\nu^\beta\,. \tag{8.17}$$

The form factor $c_{ij\beta}^\alpha$ is defined by

$$[P_\beta, J_{ij}] = ic_{ij\beta}^\alpha P_\alpha\,. \tag{8.18}$$

For the values of the form factors we refer to the Poincaré algebra. At lowest order the Lagrangian is

$$\mathcal{L}_{eff} = -\frac{1}{2} f_{\rho\sigma}\eta^{\mu\nu} e_\mu^\rho e_\nu^\sigma + \textit{higher order}, \tag{8.19}$$

Positivity of kinetic energy implies that $f_{\rho\sigma}$ has to be positive definite. Writing the Lagrangian out,

$$\mathcal{L}_{eff} = -f_{00}e_0^0 e_0^0 - f_{ij} e_0^i e_0^j + f_{kl}\eta^{ij} e_i^k e_j^l + \textit{higher order}. \tag{8.20}$$

In terms of the gapless excitations the elements are

$$e_0^0 = (\dot{\pi}+1)\gamma\,, \tag{8.21}$$

$$e_0^i = \Lambda_0^i + \partial_0\Lambda_0^i = \gamma\beta^i(1+\dot{\pi}) = -\gamma\partial^i\pi(1+\dot{\pi})\,, \tag{8.22}$$

$$e_i^j = \Lambda_i^j + \partial_i\pi\Lambda_0^j = \delta_j^i + (\gamma-1)\frac{\beta^j\beta_i}{\beta^2} + \partial_i\pi\gamma\beta^j = \delta_i^j + (\gamma-1)\frac{\partial^j\pi\partial_i\pi}{(\nabla\pi)^2} + \gamma\partial_i\pi\partial^j\pi\,. \tag{8.23}$$

This leads to the following theory in terms of the degree of freedom $\pi(x)$

$$\mathcal{L}_{eff} = -f_{00}(\dot{\pi}+1)^2\gamma^2 - f_{ij}\gamma^2\partial^i\pi\partial^j\pi(1+\dot{\pi})^2$$
$$+f_{ij}(\delta^{ij}+2(\gamma-1)\frac{\partial^i\pi\partial^j\pi}{(\nabla\pi)^2}+(2\gamma+(\gamma-1)^2+2\gamma(\gamma-1))\partial^i\pi\partial^j\pi+\gamma^2\partial^i\pi\partial^j\pi(\nabla\pi)^2)$$
$$+\textit{higher order}. \tag{8.24}$$

To find out if this is the correct Lagrangian, or that some combinations of parameters is necessary, we look at the transformations of the degrees of freedom $\pi(x)$ and $\eta^i(x)$. This is done by comparing it to a Heuristic Lagrangian of time translations SSB in flat spacetime.

8.2.1 Heuristic Lagrangian

The heuristic Lagrangian is a function of $t + \pi$ and its derivatives (similar to the Stückelberg trick). The heuristic theory of SSB of time translations in flat spacetime is made up of functions of $t + \pi$ and derivatives of this term. This is similar to the Stückelberg trick, but without gravity,

$$\mathcal{L} = \Lambda(t+\pi)+A_1(t+\pi)\partial_\mu(t+\pi)\partial^\mu(t+\pi)+A_2(t+\pi)((\partial(t+\pi))^2)^2+\textit{higher order}\,. \tag{8.25}$$

It contains powers of

$$(-1 - 2\dot{\pi} + \partial_\mu\pi\partial^\mu\pi)\,. \tag{8.26}$$

This Lagrangian is covariant under the transformation:

$$\pi(x) \to \pi'(x'(x)) = \pi'(t + c, x^i) = \pi(x) - c\,. \tag{8.27}$$

And Lorentz transformations

$$\pi(x) \to \Lambda^0_0(\alpha)\pi(x) - \Lambda^i_0(\alpha)x_i\,. \tag{8.28}$$

These are used to check the results of the coset constructed theory as they should transform in a similar fashion.

8.3 Transformation of Goldstone Fields

Transformation properties of the Goldstone fields follows from the action of an element $g \in G$ on the element of the coset G/H. In the case of spontaneously broken time translations and boosts the action of $g \in ISO(3,1)$ on the coset element is

$$g\Omega(\pi, \eta) = \Omega(\pi', \eta')h(\pi, \eta, g)\,, \tag{8.29}$$

where $h \in SO(3) \times \mathbb{R}^3$. We write the element h infinitesimally

$$h = 1 + if^i(\pi, \eta)P_i + i\rho^{ij}(\pi, \eta)J_{ij} + \cdots\,, \tag{8.30}$$

where f^i and ρ^{ij} denote the change in the parameters aligned respectively with the unbroken generators P_i and J_{ij}. The transformation is written infinitesimally

$$\Omega(\pi', \eta') = \Omega(\pi, \eta)(1 + i\Delta\pi P_0 + i\Delta\eta^i K_i + \cdots)\,, \tag{8.31}$$

where $\Delta\pi \approx \pi - \pi'$ and $\Delta\eta^i \approx \eta^i - \eta'^i$. We show the starting points of the transformation rules and work them out in seperate paragraphs. In the case that $g \in G/H$ is a time translation with parameter α

$$\begin{aligned}&(1 + i\alpha P_0 + \cdots)e^{i\pi(x)P_0}e^{ix^iP_i}e^{i\xi^i(x)K_i} = \\ &e^{i\pi(x)P_0}e^{ix^iP_i}e^{i\xi^i(x)K_i}(1 + i\Delta\pi P_0 + i\Delta\eta^i K_i + \cdots) \\ &\times(1 + if^i(\pi, \eta, \alpha)P_i + i\rho^{ij}(\pi, \eta, \alpha)J_{ij} + \cdots)\,.\end{aligned} \tag{8.32}$$

By arguing that the commutator of P_0 with any other generator of the Poincaré Algebra produces P_i and the generator itself we get

$$\Delta\pi P_0 + f^i P_i = e^{-i\xi^i(x)K_i} e^{-ix^i P_i} e^{-i\pi(x)P_0} (\alpha P_0) e^{i\pi(x)P_0} e^{ix^i P_i} e^{i\xi^i(x)K_i} . \quad (8.33)$$

In the case that $g \in G/H$ is a Lorentz boost

$$\begin{aligned}&(1 + i\alpha^j K_j + \cdots) e^{i\pi(x)P_0} e^{ix^i P_i} e^{i\xi^i(x)K_i} = \\ &e^{i\pi(x)P_0} e^{ix^i P_i} e^{i\xi^i(x)K_i} (1 + i\Delta\pi P_0 + i\Delta\eta^i K_i + \cdots) \\ &\times (1 + if^i(\pi,\eta,\alpha) P_i + i\rho^{ij}(\pi,\eta,\alpha) J_{ij} + \cdots) .\end{aligned} \quad (8.34)$$

Arguing that the commutator of K_i with other generators in the Poincaré group produces J_{ij}, P_i and the generator itself we get that

$$\Delta\eta^i K_i + if^i P_i + i\rho^{ij} J_{ij} = e^{-i\xi^i(x)K_i} e^{-ix^i P_i} e^{-i\pi(x)P_0} (\alpha^j K_j) e^{i\pi(x)P_0} e^{ix^i P_i} e^{i\xi^i(x)K_i} \quad (8.35)$$

We work these cases out in the following paragraphs and add the transformation under the unbroken spatial translations.

Transformations under Time Translations The transformation of the Goldstone fields under time translations is

$$\Delta\pi P_0 + f^i P_i = e^{-i\eta^i(x)K_i} e^{-ix^i P_i} e^{-i\pi(x)P_0} (\alpha P_0) e^{i\pi(x)P_0} e^{ix^i P_i} e^{i\eta^i(x)K_i} . \quad (8.36)$$

The generators of translations commute, so we get

$$\begin{aligned}&\Delta\pi P_0 + f^i P_i = e^{-i\eta^i(x)K_i} (\alpha P_0) e^{i\eta^i(x)K_i} \\ &= \alpha\Lambda^0_0 P_0 + \alpha\Lambda^i_0 P_i .\end{aligned} \quad (8.37)$$

Under a time translation $t \to t + \alpha$

$$\Delta\pi = \alpha\Lambda^0_0 , \qquad f^i = \alpha\Lambda^i_0 . \quad (8.38)$$

where $\Delta\pi \approx \pi - \pi'$. Guess from Heuristic Lagrangian:

$$\pi(t, x^i) \to \pi(t + \alpha, x^i) = \pi(t, x^i) - \alpha . \quad (8.39)$$

These are alike when $\gamma \approx 1$.

Transformations under Spatial Translations Commutator of spatial translations with boosts contains generators for time translations and spatial translations.

$$\begin{aligned}&\Delta\pi P_0 + f^i P_i = e^{-i\eta^i(x)K_i} (\alpha^i P_i) e^{i\eta^i(x)K_i} \\ &= \alpha^i\Lambda^0_i P_0 + \alpha^j\Lambda^i_j P_i .\end{aligned} \quad (8.40)$$

Under infinitesimal spatial translations $x^i \to x^i + \alpha^i$ the transformations are

$$\Delta\pi = \alpha^i \Lambda_i^0 \,, \qquad f^i = \alpha^j \Lambda_j^i \,. \tag{8.41}$$

In the heuristic Lagrangian the argument of the π field changes

$$\pi(x) \to \pi(t, x^i + \alpha^i) = \pi(x) + \alpha^i \partial_i \pi + \cdots \,. \tag{8.42}$$

These result correspond well with the transformation of the effective Lagrangian when the inverse Higgs is applied

$$\beta_i = -\partial_i \pi \tag{8.43}$$

and $\gamma \approx 1$.

Transformations under Boosts The commutator of the boost generator with the time translation generator is

$$[J_{0i}, P_0] \supset P_i, P_0 \,. \tag{8.44}$$

The boost generator commutes in the same manner with the spatial translation generators. The full expression for the transformation under boosts is

$$\begin{aligned} &\Delta\pi P_0 + \Delta\eta^i K_i + f^i P_i + \rho^{ij} J_{ij} = \\ &e^{-i\eta^i(x) K_i} e^{-ix^i P_i} e^{-i\pi(x) P_0} (\alpha^j K_j) \\ &\times e^{i\pi(x) P_0} e^{ix^i P_i} e^{i\eta^i(x) K_i} \,. \end{aligned} \tag{8.45}$$

The terms evaluated seperately are

$$e^{-i\pi(x) P_0} (\alpha^j K_j) e^{i\pi(x) P_0} = \alpha^j (K_j + \pi P_j) \,. \tag{8.46}$$

We use that spatial translations commute with themselves

$$e^{-ix^i P_i} (\alpha^j K_j) e^{ix^i P_i} = \alpha^j (K_j + x_j P_0) \,. \tag{8.47}$$

Taking all parts and using that $U(\Lambda)^{-1} J_{ab} U(\Lambda) = \Lambda_a^c \Lambda_b^d J_{cd}$

$$\begin{aligned} &\Delta\pi P_0 + \Delta\eta^i K_i + f^i P_i + \rho^{ij} J_{ij} = e^{-i\eta^i(x) J_{0i}} \alpha^j (J_{0j} + \pi P_j + x_j P_0) e^{i\eta^i(x) J_{0i}} \\ &= \alpha^j \Lambda_0^\mu \Lambda_j^\nu J_{\mu\nu} + \alpha^j \pi \Lambda_j^\mu P_\mu + \alpha^j x_j \Lambda_0^\mu P_\mu \,. \end{aligned} \tag{8.48}$$

The transformations under $x^\mu \to \Lambda(\alpha)^\mu_\nu x^\nu$ are

$$\Delta\pi = \alpha^j \pi \Lambda_j^0 + \alpha^j x_j \Lambda_0^0 \tag{8.49}$$

$$\Delta\eta^i = \alpha^j \Lambda_0^0 \Lambda_j^i - \alpha^j \Lambda_0^i \Lambda_j^0 \,, \tag{8.50}$$

$$f^i = \alpha^j \pi \Lambda_j^i + \alpha^j x_j \Lambda_0^i \tag{8.51}$$

and finally,

$$\rho^{ik} = \alpha^j \Lambda^i_0 \Lambda^k_j \tag{8.52}$$

Guess from Heuristic Lagrangian:

$$\pi(x) \to \Lambda^0_0(\alpha)\pi(\Lambda x) - \Lambda(\alpha)^i_0 x^i \,. \tag{8.53}$$

These do not seem to show a similar behaviour as in the other cases, even for an approximation.

8.3.1 Spatially homogeneous limit

By a spatially homogeneous approximation we may see if the theories coincide. The following approximations are made to the theories

$$\begin{aligned} \gamma &\approx 1\,, \\ \beta_i &\approx 0\,, \\ \partial_i \pi &\approx 0\,. \end{aligned} \tag{8.54}$$

In compliance with these approximations, the effective theory is

$$\mathcal{L}_{\text{eff. approx.}} = f_{ij}(\pi + t)\delta^{ij} - f_{00}(\pi + t)(1 + \dot{\pi})^2 + \text{higher order}\,. \tag{8.55}$$

The functions denoted by f are functions of $\pi + t$ as this is invariant under time translations. The same methodology applied to the Heuristic Lagrangian leads to

$$\mathcal{L}_{\text{Heur. approx}} = \Lambda(\pi + t) - A_1(t + \pi)(1 + \dot{\pi}) + A_2(t + \pi)(1 + \dot{\pi})^2 + \text{higher order}\,. \tag{8.56}$$

We can subtract a total derivative from this Lagrangian to cancel the term with A_1. This is

$$\begin{aligned} &\partial_t \left[\int^{t+\pi} A(x)dx \right] = \\ &\left(\frac{\partial}{\partial(t+\pi)} \int^{t+\pi} A(x)dx \right) \frac{\partial(t+\pi)}{\partial t}\,. \end{aligned} \tag{8.57}$$

The theory is

$$\mathcal{L}_{\text{Heur. approx}} = \Lambda(\pi + t) + A_2(t + \pi)(1 + \dot{\pi})^2 + \text{higher order}\,. \tag{8.58}$$

These results show that the theories have the same dynamics of the scalar degree in the spatially homogeneous limit. This concludes the dissertation on the theory of SSB of time translations in non-dynamical gravity. The present theory does not coincide perfectly with a heuristically derived theory. Further research is necessary to make a model independent theory that does have this property.

Chapter 9

Curvature perturbations in an FLRW Background

9.1 Spatial Hyperslices as Membranes in 3+1 Dimensions

The following approach is similar to SSB by membranes in [18]. It is mainly an exercise that does not yield significant results. The hyperslices in the paper [18] are $(d-1)$-dimensional where the $(d-1)$ dimensions include the time dimension. This is a correct approach to membranes in d-dimensions spontaneously breaking the $d-1$ spatial direction. But it does not correspond to a breaking of the time direction as the d'th dimension, as we would like it to do. This is because the derived action is defined only on the hypersurface. It is therefore an irrelevant approach to the effective theory that is our goal, as is defined over all spacetime. Nevertheless this part shows the approach up to the preliminaries of an effective theory. To use spatial hyperslices for deriving an effective theory in a curved background we distinguish four different coordinate indices, the indices are

- Greek indices $\mu, \nu, \rho, \sigma, \cdots$ for (curved) general spacetime,
- Latin indices $a, b, c, d, \cdots$ for (locally flat) Lorentz spacetime,
- Latin capital indices $A, B, C, D, \cdots$ for (curved) general spatial indices,
- Latin indices $i, j, k, l, \cdots$ for (locally flat) Lorentz spatial indices.

As before spatial rotations and translations are unbroken and time translations and boosts are broken

$$\text{Unbroken} = \begin{cases} P_i & \text{spatial translations}\,, \\ J_{ij} & \text{spatial rotations}\,. \end{cases} \tag{9.1}$$

and

$$\text{Broken} = \begin{cases} P_0 & \text{time translations}\,, \\ J_{0i} \equiv K_i & \text{boosts}\,. \end{cases} \tag{9.2}$$

The coset representative, which includes all translations as they are nonlinearly realized, is

$$\Omega = e^{i\pi(x)P_0} e^{iY^i(x)P_i} e^{i\eta^j(x)K_j} = e^{iY^a(x)P_a} e^{i\eta^i(x)K_i}\,. \tag{9.3}$$

The middle expression is used to make the degrees of freedom explicit, but we will use the right expression with $Y^a(x) = (\pi(x), Y^i(x))$ to calculate the Maurer-Cartan one-form. The Maurer-Cartan one-form is evaluated on the general spatial hyperslices $\Omega^{-1}\mathrm{D}_A\Omega$. This object can be seen as a projection of the Maurer Cartan in d-dimensions onto the $d-1$-dimensional spatial hypersurface,

$$\Omega^{-1}\mathrm{D}_A\Omega \equiv \partial_A Y^\mu \Omega^{-1}\mathrm{D}_\mu\Omega \tag{9.4}$$

We use the covariant derivative for the gauged $ISO(3,1)$ group in a curved background. The one form is

$$\partial_A Y^\mu \Omega^{-1}\mathrm{D}_\mu\Omega = \partial_A Y^\mu \Omega^{-1}\left(\partial_\mu + \tilde{e}^a_\mu P_a + \frac{i}{2}\omega^{ab}_\mu J_{ab}\right)\Omega\,. \tag{9.5}$$

The coset representative (9.3) is made up of two parts, one for the translations and one for the boosts. A good approach would be to work through the translations part first. This part is

$$e^{-iY^a(x)P_a}\left(\partial_\mu + \tilde{e}^a_\mu P_a + \frac{i}{2}\omega^{ab}_\mu J_{ab}\right) e^{iY^a(x)P_a}\,. \tag{9.6}$$

We rewrite it as

$$e^{-iY^a(x)P_a}\left((i\partial_\mu Y^a(x) + \tilde{e}^a_\mu)P_a + \frac{i}{2}\omega^{ab}_\mu J_{ab}\right) e^{iY^a(x)P_a}\,. \tag{9.7}$$

Using the commutation relations of the Poincaré group

$$\left(i\partial_\mu Y^a(x) + \tilde{e}^a_\mu\right) e^{-iY^a(x)P_a} P_a e^{iY^a(x)P_a} = (i\partial_\mu Y^a(x) + \tilde{e}^a_\mu)P_a\,, \tag{9.8}$$

we write the part with the generator of Lorentz transformations as

$$\frac{i}{2}\omega^{bc}_\mu(1 - iY^a(x)P_a + \cdots)J_{bc}(1 + iY^a(x)P_a + \cdots)\,. \tag{9.9}$$

Using the relations

$$[P_a, J_{bc}] = -i(P_b\eta_{ac} - P_c\eta_{ab})\,, \qquad \omega^{ab}_\mu = -\omega^{ba}_\mu\,, \tag{9.10}$$

we have firstly,

$$\frac{i}{2}\omega^{bc}_\mu(1-iY^a(x)P_a+\cdots)J_{bc}(1+iY^a(x)P_a+\cdots) = \frac{i}{2}\omega^{ab}_\mu J_{ab} + \frac{1}{2}\omega^{bc}_\mu Y^a(P_b\eta_{ac} - P_c\eta_{ab}) + \cdots\,, \tag{9.11}$$

and secondly keeping only the linear terms

$$ie_\mu^a P_a + \frac{i}{2}\omega_\mu^{ab} J_{ab}\,, \tag{9.12}$$

where we redefined the vierbein

$$e_\mu^a = \tilde{e}_\mu^a + \partial_\mu Y^a + \omega_\mu^{ab} Y_b\,. \tag{9.13}$$

Plugging this into the full Maurer-Cartan one-form

$$\partial_A Y^\mu \Omega^{-1} \mathrm{D}_\mu \Omega = \partial_A Y^\mu e^{-i\eta^i(x)K_i} \left(\partial_\mu + e_\mu^a P_a + \frac{i}{2}\omega_\mu^{ab} J_{ab}\right) e^{i\eta^i(x)K_i}\,. \tag{9.14}$$

Using as before that $U(\Lambda) \equiv e^{i\eta^i(x)K_i}$, $U(\Lambda)^{-1}P_a U(\Lambda) = \Lambda_a^b P_b$ and $U(\Lambda)^{-1}J_{ab}U(\Lambda) = \Lambda_a^c \Lambda_b^d J_{cd}$ we get

$$\partial_A Y^\mu \Omega^{-1} \mathrm{D}_\mu \Omega = i\partial_A Y^\mu e_\mu^b \Lambda_b^a P_a + \frac{i}{2}\partial_A Y^\mu (\Lambda^{-1})_c^a (\eta^{cd}\partial_\mu + \omega_\mu^{cd})\Lambda_d^b J_{ab}\,. \tag{9.15}$$

We express the result as a linear combination of all the generators, in terms of Goldstone fields and the coset vierbein this is

$$\partial_A Y^\mu \Omega^{-1} \mathrm{D}_\mu \Omega \equiv iE_A^i (P_i + \nabla_i \pi P_0 + \nabla_i \eta^j K_j) + \frac{1}{2} A_A^{ij} J_{ij}\,. \tag{9.16}$$

From this equation we can read off the fields, by equating the evaluation of the Maurer-Cartan to its definition

$$E_A^i = \partial_A Y^\mu e_\mu^a \Lambda_a^i = (\partial_A \pi e_0^a + e_A^a)\Lambda_a^i\,, \tag{9.17}$$

$$\nabla_i \pi = E_i^A \partial_A Y^\mu e_\mu^a \Lambda_a^0 = E_i^A (\partial_A \pi e_0^a + e_A^a)\Lambda_a^0\,, \tag{9.18}$$

$$\nabla_i \eta^j = E_i^A \partial_A Y^\mu [(\Lambda^{-1}\partial_\mu \Lambda)^{0j} + \omega_\mu^{cd}\Lambda_c^0 \Lambda_d^j]\,, \tag{9.19}$$

$$A_A^{ij} = \partial_A Y^\mu [(\Lambda^{-1}\partial_\mu \Lambda)^{ij} + \omega_\mu^{cd}\Lambda_c^i \Lambda_d^j]\,, \tag{9.20}$$

$$\tag{9.21}$$

The field aligned with the spatial rotations transforms as a connection. For this reason we incorporate this field into the covariant derivative that is used in the Lagrangian.

IH constraining The commutator of unbroken spatial translations and boosts contains broken time translations

$$[P_i, J_{0j}] = iP_0 \eta_{ij}\,. \tag{9.22}$$

We inverse Higgs constrain the covariant derivative along the generators of spatial translations on the field π

$$\nabla_i \pi = 0\,. \tag{9.23}$$

We write the constraint as

$$E_i^A (\partial_A \pi e_0^a + e_A^a)\Lambda_a^0 = 0\,. \tag{9.24}$$

We see that Λ_a^0 is orthogonal to the spatial hyperslices.

Effective Action The effective action is built up from several possible diffeomorphism invariant terms and their Lorentz invariant combinations. The building blocks are taken from the Maurer-Cartan form. The measure of integration is an invariant volume element defined by

$$d^4x\sqrt{-g} = d^4x\det\mathrm{E}\,. \tag{9.25}$$

The spatial part of the metric is defined as

$$h_{AB} \equiv e^a_\mu e^b_\nu \eta_{ab}\partial_A Y^\mu \partial_B Y^b = e^a_A e^b_B \eta_{ab} + e^a_0 e^b_0 \eta_{ab}\partial_A\pi\partial_B\pi\,. \tag{9.26}$$

This concludes the exercise of deriving an effective field theory by modelling the SSB of time translations as broken by spatial hyperslices. The derived action exists on the spatial hypersurface and does not yield a theory defined over all spacetime.

9.2 Preliminaries of a Coset Construction of SSB time translations in dynamical Gravity

To gauge the Poincaré transformations we introduce the gauge fields for spacetime translations $\tilde{e}^a_\mu$ and Lorentz transformations ω^{ab}_μ to the covariant derivative D_μ. The element of the gauged $ISO(3,1)/SO(3,1)$ coset is

$$\Omega = e^{iy^a(x)P_a}e^{i\eta^i K_i}\,. \tag{9.27}$$

We use a as the index of locally inertial coordinates $y^a(x)$ at some point within the patch that is described by the (curved) coordinates x and with $y^a = (\pi, y^i)$. We compute the translations in The Maurer-Cartan one form seperately

$$e^{-iy^aP_a}(\partial + \tilde{e}^a_\mu P_a + \frac{i}{2}\omega^{ab}_\mu J_{ab})e^{iy^aP_a} = ie^a_\mu P_a + \frac{i}{2}\omega^{ab}_\mu J_{ab}\,, \tag{9.28}$$

with

$$e^a_\mu = \tilde{e}^a_\mu + \partial_\mu y^a + \omega^{ab}_\mu y_b\,. \tag{9.29}$$

The full one-form is

$$\begin{aligned}\Omega^{-1}D_\mu\Omega &= ie^b_\mu\Lambda^a_b P_a + \frac{i}{2}J_{ab}[(\Lambda^{-1}\partial_\mu\Lambda)^{ab} + \omega^{cd}_\mu\Lambda^a_c\Lambda^b_d] \\ &\equiv iE^a_\mu(P_a + \nabla_a\eta^i K_i + \frac{1}{2}J_{ij}A^{ij}_a)\,.\end{aligned} \tag{9.30}$$

We read off the covariant parts

$$E^a_\mu = e^b_\mu\Lambda^a_b\,, \tag{9.31}$$

$$\nabla_a\eta^i = e^\mu_b\Lambda^b_a[(\Lambda^{-1}\partial_\mu\Lambda)^{0i} + \omega^{cd}_\mu\Lambda^0_c\Lambda^i_d]\,. \tag{9.32}$$

The inverse Higgs constraint is applied in a similar fashion as for the other spacetime SSB's.

Inverse Higgs constraint The commutator is

$$[P_i, K_j] \subset i\delta_{ij} P_0\,. \tag{9.33}$$

This implies that we set E_i^0 to zero to enforce the constraint and remove degenerate degrees of freedom. We make the π dependence explicit in the constraint,

$$\begin{aligned} E_i^0 &= 0\,, \\ e_i^b \Lambda_b^0 = (\tilde{e}_i^b + \partial_i y^b + \omega_i^{bc} y_c)\Lambda_b^0 &= 0\,, \\ (\tilde{e}_i^0 + \partial_\mu \pi + \omega_i^c y_c)\Lambda_0^0 + (\tilde{e}_i^j + \partial_j y^j + \omega_i^{jc})\Lambda_j^0 &= 0\,. \end{aligned} \tag{9.34}$$

These preliminaries may lead to an effective theory in dynamical gravity. But as the theory in non-dynamical gravity falls short we had to discontinue the research on this topic to first fix these issues before venturing into a more general case of the non-dynamical theory.

Chapter 10

Conclusion and Outlook

We recap the goal and method of the research in this chapter. Subsequently we describe the main conclusions and we formulate an outlook for further research.

10.1 Summary

Our main goal was to derive a model independent theory that describes curvature perturbations from a spontaneous symmetry breaking of time translations. We divided this goal into three parts: inflation, SSB in field theory and SSB of spacetime symmetries. In the first part we introduced the inflaton and worked out the requirements on the field and on the potential for extended inflation to occur. From this model we were able to observe that structure formation arises from the single degree of freedom in the early universe. These are the curvature perturbations described by the scalar degree of freedom in the metric. This enticed us to discover if there was a way to derive a theory of these perturbations without an underlying model. We approach such a model independent theory by looking at the spontaneous symmetry breaking pattern. We first derived the theory by manually introducing the degree of freedom. This top down approach is the Stückelberg trick. The coset construction introduces the degrees of freedom by looking at the invariant objects. We used the coset construction in field theory for Abelian and non-Abelian symmetries. As the main goal was the case of curvature perturbations in dynamical gravity we also looked at SSB of gauge symmetries. As dynamical gravity is the gauged version of spacetime symmetries. Finally we looked at the formalisms of spacetime SSB, and tried to derive an effective theory.

10.2 Conclusions

The main factor of this research was to build a complete understanding of spontaneous symmetry breaking and of the coset construction. We described these in the field theory part of the thesis. The coset construction of the Abelian

SSB is derived and coincides well with a model dependent version. The Chiral Lagrangian as an example was derived to test if the method was understood and that we were able to apply it. We applied the coset construction to the SSB of time translations in non dynamical gravity. This theory appeared to be incomplete. We tested this by looking at the transformations of the scalar degree of freedom, and comparing them to a phenomenological theory. The theories coincided for spatial translations after applying the inverse Higgs constraint. The boosts did not show this behaviour. This pointed to a misapprehension or a caveat in our approach. We specified the theory by taking the spatially independent limit to see if the theories coincide in that limit. They showed similar behaviour at this point. We had derived a more general case of SSB with dynamical gravity up to the inverse Higgs constraint but had to cease, as the approach does not in essence differ with respect to the non dynamical case. We can say with some certainty that this more general case would show similar issues and would not approach the EFT of inflation derived by the Stückelberg trick.

10.3 Outlook

The research in the field is ongoing. The coset construction is a relatively new technique, but has already been applied to a number of different cases. To understand and apply the coset construction in a systematic way without errors, so also for spacetime symmmetry breaking would be the first goal of research continuation. the second goal would be to use the coset construction for the SSB of time translation symmetry, in a consistent manner, thereby deriving a theory coinciding with the Heuristic theory. Applying the coset construction to dynamical gravity would subsequently be the main task. These results would then be compared to the theory that we derived using the Stückelberg trick. Furthermore, a better understanding of spacetime symmetry breaking and in particular the SSB by inflation would be possible. These results would thereby provide a fundamental conceptualization of curvature perturbations and in turn of the origins of structure formation in the universe.

Bibliography

[1] Clifford Cheung, Paolo Creminelli, A. Liam Fitzpatrick, Jared Kaplan, and Leonardo Senatore. The Effective Field Theory of Inflation. *JHEP*, 03:014, 2008.

[2] Daniel Baumann. Inflation. In *Physics of the large and the small, TASI 09, proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, Boulder, Colorado, USA, 1-26 June 2009*, pages 523–686, 2011.

[3] Antonio Riotto. Inflation and the theory of cosmological perturbations. *ICTP Lect. Notes Ser.*, 14:317–413, 2003.

[4] Tomislav Prokopec. Lecture notes on cosmology.

[5] Steven Weinberg. *Cosmology.* Oxford University Press, first edition, 2008.

[6] Yoshimasa Hidaka, Toshifumi Noumi, and Gary Shiu. Effective field theory for spacetime symmetry breaking. *Phys. Rev.*, D92(4):045020, 2015.

[7] Daniel Baumann. The effective theory of single-field inflation, 2012.

[8] Lewis H. Ryder. *Quantum Field Theory.* Cambridge University Press, second edition, 1996.

[9] Dr David Tong. Quantum field theory, 2006 and 2007.

[10] C. P. Burgess. Goldstone and pseudoGoldstone bosons in nuclear, particle and condensed matter physics. *Phys. Rept.*, 330:193–261, 2000.

[11] Sidney R. Coleman, J. Wess, and Bruno Zumino. Structure of phenomenological Lagrangians. 1. *Phys. Rev.*, 177:2239–2247, 1969.

[12] Joseph Polchinski. Effective field theory and the Fermi surface. In *Theoretical Advanced Study Institute (TASI 92): From Black Holes and Strings to Particles Boulder, Colorado, June 3-28, 1992*, pages 0235–276, 1992.

[13] Steven Weinberg. *The Quantum Theory of Fields*, volume II. Cambridge University Press, first edition, 1996.

[14] E. Laenen B. de Wit and J. Smith. Field theory in particle physics, January 2016.

[15] Steven Weinberg. *Gravitation and Cosmology: principles and applications of the general theory of relativity.* John Wiley & Sons, Inc., 1972.

[16] Mikhail M. Ivanov John F. Donoghue and Andrey Shkerin. Epfl lectures on general relativity as a quantum field theory, 2017.

[17] Steven Weinberg. *The Quantum Theory of Fields*, volume I. Cambridge University Press, first edition, 1996.

[18] Luca V. Delacrétaz, Solomon Endlich, Alexander Monin, Riccardo Penco, and Francesco Riva. (Re-)Inventing the Relativistic Wheel: Gravity, Cosets, and Spinning Objects. *JHEP*, 11:008, 2014.

[19] Tomáš Brauner and Haruki Watanabe. Spontaneous breaking of spacetime symmetries and the inverse Higgs effect. *Phys. Rev.*, D89(8):085004, 2014.

[20] Alberto Nicolis, Riccardo Penco, and Rachel A. Rosen. Relativistic Fluids, Superfluids, Solids and Supersolids from a Coset Construction. *Phys. Rev.*, D89(4):045002, 2014.