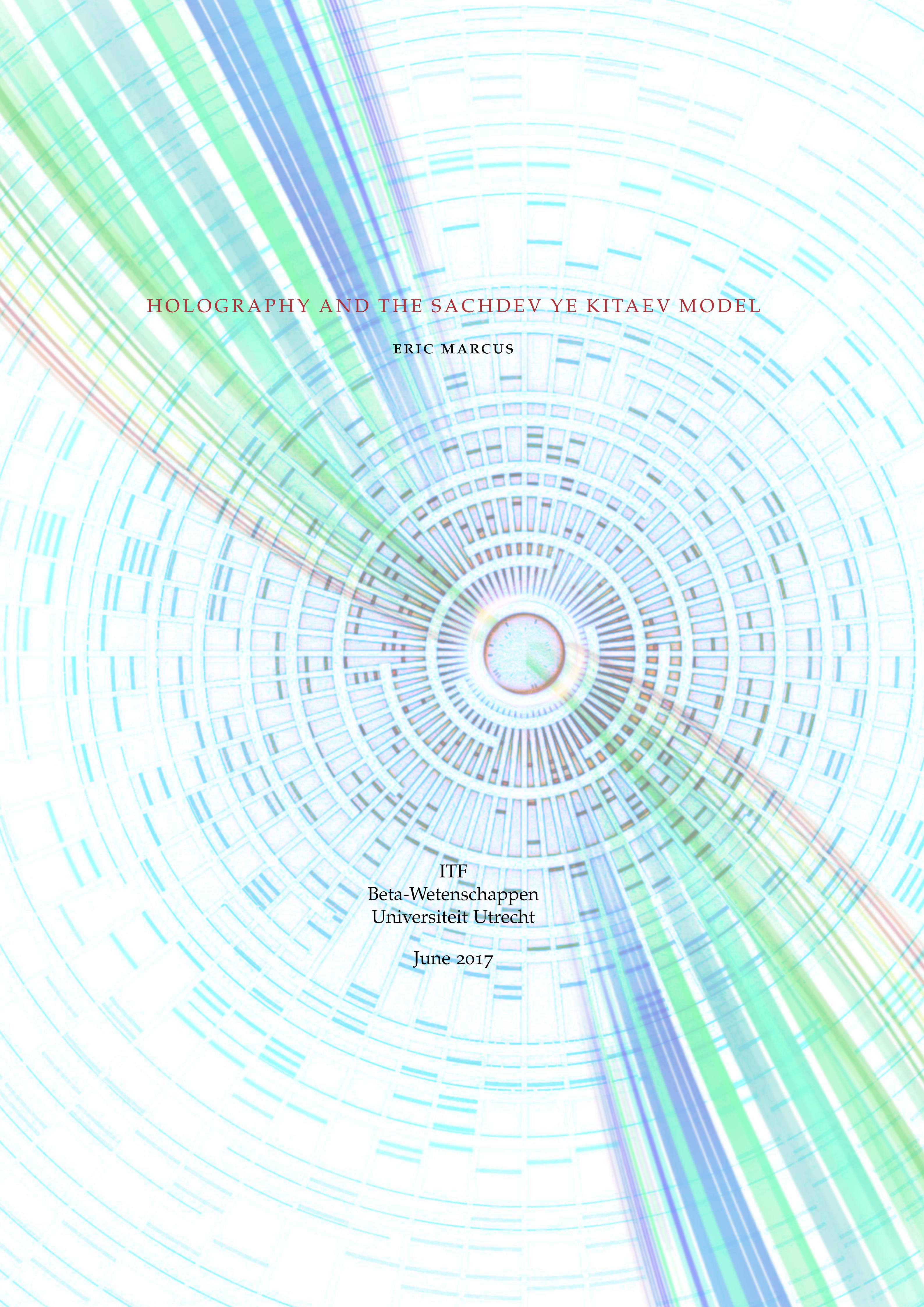

HOLOGRAPHY AND THE SACHDEV YE KITAEV MODEL

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The fact that we live at the bottom of a deep gravity well, on the surface of a gas covered planet going around a nuclear fireball 90 million miles away and think this to be normal is obviously some indication of how skewed our perspective tends to be.

— Douglas Adams, *The Salmon of Doubt: Hitchhiking the Galaxy One Last Time*

ABSTRACT

In this thesis we will review and discuss the Sachdev-Ye-Kitaev (SYK) model, its generalizations and related bulk models. We discuss the two and four point functions, the effective action and the Schwarzian. Also results regarding the (lack of a) current for the $O(N)$ symmetry and the choice of ensemble are presented. The generalizations that we (shortly) discuss are the supersymmetric extension and the tensor models.

The second part of the thesis consists of a discussion of a related bulk model, called the Almheiri-Polchinski model. We review its action, symmetries and the (black hole) solutions. In particular we generalize the solutions in the literature and show how these compare. The thesis is concluded by discussing the most important open questions in the holography of the SYK model.

CONTENTS

ACRONYMS AND ABBREVIATIONS

SY Sachdev-Ye

SYK Sachdev-Ye-Kitaev

AP Almheiri-Polchinski

EH Einstein-Hilbert

JT Jackiw-Teitelboim

NAdS Nearly Anti de Sitter

NCFT Nearly Conformal Field Theory

EFT Effective Field Theory

1 INTRODUCTION

1.1 ORIGIN AND MOTIVATION OF SYK MODEL

The Sachdev-Ye-Kitaev model was introduced by Kitaev in a series of talks [1]. It is a simplified version of an earlier model introduced by Sachdev and Ye [2] (hence the name). The SYK model is a quantum mechanical system consisting out of N Majorana fermions with random interactions between q fermions at the same time (usually $q = 4$).

There are several key features that make this such an intriguing model. Firstly, the 2,4 and 6 point functions are solvable at strong coupling: at large N one can sum over all the Feynman diagrams and obtain a closed form expression for the correlation functions. In principle this may also be true for higher order correlation functions, but these have (at the time of writing) not yet been computed. Secondly, the SYK model shows maximally chaotic behaviour. This chaos is quantified by the so called Lyapunov exponent [3]. For black holes in Einstein gravity this exponent has the maximal value $2\pi/\beta$ [4,5] with β the inverse temperature. In fact the SYK model also saturates this bound.
Lastly, the model has an emergent conformal symmetry. The second and last property seem to suggest that it has a holographic dual in some form of Einstein gravity.

Naively we could expect that the model has a full Virasoro symmetry group. However, the symmetry in the model is both spontaneously and explicitly broken. For this reason it is usually referred to as a $NCFT_1$ model, where the N stands for Nearly. The associated bulk models that we will discuss have the same symmetry breaking pattern and are referred to as $NAdS_2$ models.

The AdS/CFT correspondence [6,7], although widely studied and used, is still not completely microscopically understood. Due to the ability to completely solve several important correlation functions within SYK at strong coupling (and similar properties in the related bulk model [8]) we can hope to better understand this duality at a microscopic level.

1.2 THIS THESIS

In this thesis we will start by elaborately discussing the main features of the Sachdev-Ye-Kitaev model. The level of difficulty is aimed at a beginning graduate student with a solid basis in quantum field theory, general relativity and a short introduction to AdS/CFT (e.g. the first chapters of [9]).

Subjects that we will discuss include the most important aspects of the model such as the two and four point functions, the effective action and the Schwarzian action. Apart from this we also shortly discuss the $O(N)$ symmetry with its associated (lack of a) conserved current and how important the choice of the disorder average is. In particular we show that a large class of ensembles can reproduce the entire diagrammatic structure.

We also shortly discuss the supersymmetric version and tensor SYK models. In the second part of the thesis we discuss a dilaton gravitational model believed to be closely associated to the exact bulk dual (which is not currently known) of SYK: the Almheiri-Polchinski model. In this section we will see the similarities between the two models. We will also discuss in detail black hole solutions of this model. In particular we extend the work of [8] by considering the most general (black hole) solutions possible and show how these relate to those in [8] itself.

1.2.1 *Conventions*

Although we will mention mostly in the text when we use a particular convention for an easy overview we mention the most important ones here.

Throughout the thesis we will use the convention that $\hbar = c = k_b = 1$. In the second part of the thesis we will also adopt the units in which the AdS radius is equal to one. We will also almost always work in Poincare coordinates such that $z = 0$ corresponds to the boundary of the spacetime.

Lastly, we shall often identify a $SL(2, \mathbb{R})$ symmetry by the invariance under fractional transformations as:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \tag{1}$$

Where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. Strictly speaking this is of course a quotient of $SL(2, \mathbb{R})$ by $\mathbb{Z}_2$ due to the invariance of $(a, b, c, d) \mapsto$

$(-a, -b, -c, -d)$. But since we rarely (if ever) will need the difference between these two we shall call these fractional transformations $SL(2, \mathbb{R})$ transformations.

Part I

THE SYK MODEL

This part discusses the SYK and SYK-like models. We first give an introduction to the model as it was introduced by Kitaev. We discuss first the two point function, the four point function, the effective action and the $O(N)$ symmetry. We conclude this first chapter by discussing the reparametrizations and the associated Schwarzian action. The next three (shorter) chapters will discuss generalizations of the SYK model. Firstly we will consider choosing a different ensemble and note how this changes the model. Secondly we introduce the supersymmetric SYK model and lastly we discuss SYK tensor models.

2 THE SACHDEV-YE-KITAEV MODEL

In this chapter we will introduce the Sachdev-Ye-Kitaev (SYK) model. In particular we start by introducing the Hamiltonian, the disorder average and derive the equation of motion.
Afterwards we will discuss the two point function, for both free Majorana's and in the full interacting theory. In the limit of large N ('t Hooft limit) and the IR limit we can obtain an expression for this full two point function. After deriving these fundamental properties of the model we discuss yet another: the $O(N)$ symmetry that arises after performing the disorder average.
Once we have done this we will discuss an exact rewriting of our theory in terms of a non local action. This will allow us to discuss the theory of reparametrizations in the SYK model, describing the symmetry breaking process of the emergent conformal symmetry.

2.1 INTRODUCTION TO SACHDEV-YE-KITAEV MODEL

The SYK model [1] is a simplified version of the Sachdev-Ye model [2]. The model contains N Majorana fermions that randomly interact with $q \in 2\mathbb{Z}$ other Majorana fermions. In particular we will first discuss the case $q = 4$ where four fermions interact with each other. The Hamiltonian is then given by:

$$H = \frac{1}{4!} \sum_{ijkl} J_{ijkl} \chi_i \chi_j \chi_k \chi_l \,, \tag{2}$$

where χ denote the Majorana fermions, which obey the commutation relations: $\{\chi_i, \chi_j\} = \delta_{ij}$. Furthermore we have that J_{ijkl} is a completely anti-symmetric in all its indices (which follows from H being Hermitian and the anti commutation of the χ fields).
From H we also obtain the following Lagrangian:

$$L = \frac{1}{2} \chi_j \frac{d}{d\tau} \chi_j - H \,. \tag{3}$$

We see from here that the fermions χ have dimension 0 and the coupling in the Hamiltonian (for any q) has dimension 1. Hence, the coupling J_{ijkl} has the dimension of an energy scale. From this Lagrangian we can also derive the equation of motion for χ_i by simply using the Euler Lagrange equations (and using the anti-commutation relations):

$$\dot{\chi}_i = \frac{1}{3!} J_{iklm} \chi^k \chi^l \chi^m \,. \tag{4}$$

Lastly, the model has so-called quenched disorder where the couplings J_{ijkl} are randomly drawn from the distribution [1]:

$$P(J_{ijkl}) = \sqrt{\frac{N^3}{12\pi J^2}} \exp\left(\frac{-N^3 J_{ijkl}^2}{12 J^2}\right) \,. \tag{5}$$

Where J is thus the dimension 1 (energy) parameter that characterizes the distribution. To find the average $\langle J_{ijkl}^n \rangle$ ($n \in \mathbb{Z}^+$) we simply integrate over the probability distribution: (note that there is no sum in the expression below)

$$\langle J_{ijkl}^n \rangle = \int d(J_{ijkl})\, J_{ijkl}^n \, P(\, J_{ijkl}\,) \,. \tag{6}$$

This yields us the two results:

$$\langle J_{ijkl} \rangle = 0 \,, \tag{7}$$

$$\langle J_{ijkl}^2 \rangle = \tfrac{3!\, J^2}{N^3} \,. \tag{8}$$

We will also use $\overline{A}$ to denote the same averaging.

2.2 TWO POINT FUNCTIONS

In this section we will discuss the two-point function. First we consider the free two-point function and afterwards we will discuss the self energy in the large N limit. Using strong coupling we can obtain an expression for the full two point function.
We then consider the conformal symmetry of the equations. Using this symmetry we can also obtain a result for finite temperature.

2.2.1 *Free Majorana particles*

We will consider the two point functions in Euclidean time such that we can later on easily consider the case of finite temperature. In this formalism the two point function is defined as:

$$G_{ij}(\tau) \equiv \langle T\, \chi_i(\tau)\chi_j(0) \rangle = \langle\, \chi_i(\tau)\, \chi_j(0)\, \rangle\; \theta(\tau) - \langle\, \chi_i(0)\, \chi_j(\tau)\, \rangle\; \theta(-\tau) \,. \tag{9}$$

By introducing (anticommuting) sources and using (3) we obtain the generating functional for a free fermion as:

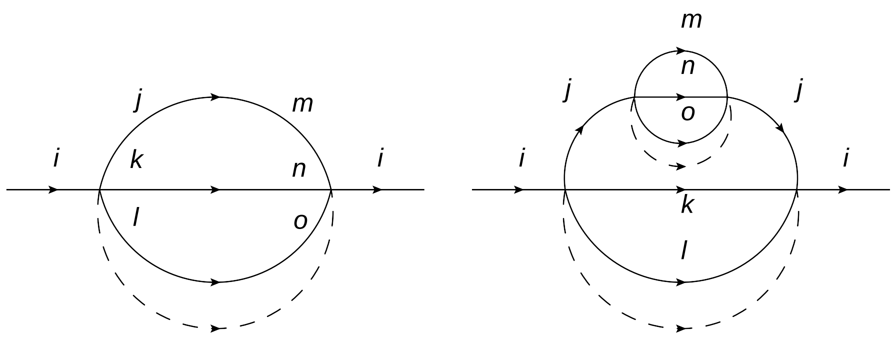

Figure 1: These diagrams will contribute at leading order in N. The dotted line indicates the disorder average which forces the indices to be equal. Note that the indices in the loops are summed over.

$$Z_0[J] = \int \mathcal{D}\chi_1\ \mathcal{D}\chi_2 \ldots \mathcal{D}\chi_N\ e^{-\int d\tau\left(\frac{1}{2}\chi_j \frac{d}{d\tau}\chi_j\right)+\int d\tau \chi_j J_j} . \tag{10}$$

We then compute the free two point function:

$$G_{0,ij}(\tau) = \frac{\delta}{\delta J_i(\tau)} \frac{\delta}{\delta J_j(0)} \ln\left(Z_0[J]\right)\Big|_{J=0} \tag{11}$$

$$= \frac{\delta}{\delta J_i(\tau)} \left[\left(\frac{1}{2}\int d\tau''\,\Delta(-\tau'')\,J_j(\tau'') - \frac{1}{2}\int d\tau'\,J_j(\tau')\,\Delta(\tau')\right)\right. \tag{12}$$

$$\left.\times \exp\left(\frac{1}{2}\int d\tau\,d\tau' J_k(\tau)\,\Delta(\tau-\tau')\,J_k(\tau')\right)\right]\Big|_{J=0} . \tag{13}$$

Now we need simply an expression for $\Delta(\tau)$,

$$\Delta(\tau) = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega\tau}}{\omega + i\epsilon} = -\theta(\tau) .$$

The most right-hand side can be proven using contour integration. Using this result we find that the two point function for a free Majorana fermion is:

$$G_{0,ij}(\tau) = \frac{1}{2}\,\text{sgn}(\tau)\,\delta_{ij} , \tag{14}$$

$$G_{0,ij}(\omega) = -\frac{1}{i\omega}\delta_{ij} , \tag{15}$$

where we also gave the Fourier transform.

2.2.2 *Disorder average and large N*

We now consider also the interactions and find an expression for the self energy (1PI). Due to the disorder average and the large N limit

many diagrams will be suppressed and we can find a simple expression for the self energy.
Consider for example the diagram in Fig 1 which will contribute at leading order. Note that the dotted line stands for the disorder average. The expression for the leftmost diagram is:

$$\frac{C}{(4!)^2} \sum_{\substack{jkl \\ mno}} \langle J_{ijkl} J_{imno} \rangle G_{0,jm}(\tau_1, \tau_2)\, G_{0,kn}(\tau_1, \tau_2)\, G_{0,lo}(\tau_1, \tau_2) =$$
$$= J^2 G_0(\tau_1, \tau_2)^3 \,, \tag{16}$$

where we made use of (8) and the combinatorial factor $C = \binom{4}{3}\binom{4}{3} 3!$.
Note however, that there exist also diagrams that don't contribute at this order. For example take the diagrams in Fig 2 which can be checked to contribute as $\frac{1}{N^d}$ with $d \neq 0$.

We can now generalize the expression for the self energy by realizing that the only kind of diagrams that contribute are those similar to (16). The diagrams need in general to have a disorder average over their "incoming" and "outgoing" lines and the lines must not cross any other lines in the diagram. We can thus construct the full two point function as shown in Fig 3. It is obvious that in this case the total expression for the self energy becomes [10,11]:

$$\Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3 \,, \tag{17}$$

where G now denotes the full two point function.
We can of course also express the two point function as a sum of all the 1PI diagrams as:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega) \,. \tag{18}$$

Together, (17) and (18) completely determine the full two point function. We can solve these equations in the strong coupling (or low energy) limit.

2.2.3 *Strong Coupling Limit*

In this limit we may ignore the first term that appears in (18) and hence we can obtain the following equation:

$$\int d\tau' G(\tau, \tau')\, \Sigma(\tau', \tau'') = -\delta(\tau - \tau'') \,. \tag{19}$$

Which, by using (17), becomes (notice the familiarity with the Schwinger-Dyson equations)

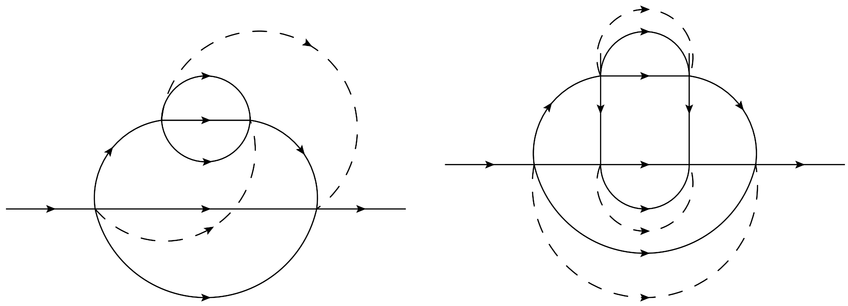

Figure 2: Here we show two diagrams that do not contribute at leading order in N the left diagram will contribute at N^{-5} and the right diagram as N^{-1}.

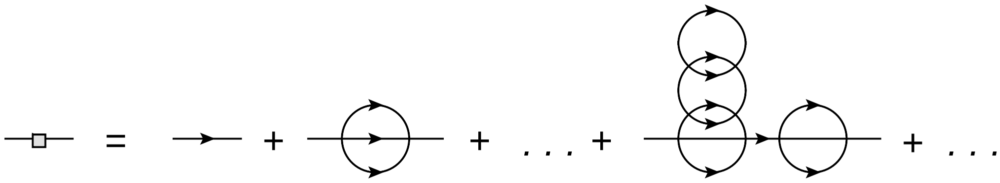

Figure 3: The full two point function is denoted by the line with the box. On the right side we omitted the dotted lines indicating the disorder averages. They are however all implemented in the same manner as shown in Fig 1.

$$J^2 \int d\tau' G(\tau,\tau')\, G(\tau',\tau'')^3 = -\delta(\tau-\tau'') \,. \tag{20}$$

We can now make an ansatz for G by using the conformal symmetry (which will be discussed below) and the form of G_0 [9,11]:

$$G_c(\tau) = A\frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}} \,, \tag{21}$$

where A and Δ are constants and the subscript c denotes the conformal limit. Indeed we can check that the expression is invariant under $SL(2,\mathbb{R})$ transformations. Plugging this into (20) gives (assuming translation invariance):

$$J^2 A^4 \int d\tau' \, \frac{\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{2\Delta}} \, \frac{\mathrm{sgn}(\tau'-\tau'')}{|\tau'-\tau''|^{6\Delta}} \,. \tag{22}$$

Now we can make use of the Fourier transform [11]:

$$\frac{\mathrm{sgn}(\tau)}{|\tau|^{2\Delta}} = \int \frac{d\omega}{(2\pi)}\, e^{-i\omega\tau}\, C(2\Delta)\, |\omega|^{2\Delta-1}\, \mathrm{sgn}(\omega) \,, \tag{23}$$

where $C(a)$ is given by:

$$C(a) = i\,2^{1-a}\sqrt{\pi}\,\frac{\Gamma(1-\frac{a}{2})}{\Gamma(\frac{1}{2}+\frac{a}{2})}\,.$$

Using this Fourier transform we get the following equation:

$$\frac{J^2 A^4}{(2\pi)}\,C(2\Delta)\,C(6\Delta)\int d\omega\, e^{-i\omega(\tau-\tau'')}\,|\omega|^{8\Delta-2}\,. \tag{24}$$

So we see that we should use in this case $\Delta = \frac{1}{4}$. In fact, when we have a q-point interaction (so in our case $q = 4$) we will find that $\Delta = \frac{1}{q}$, [11]. By considering the right-hand side of (20) we fix the constant A. We find thus the following result for the full two point function (in the strong coupling limit):

$$G(\tau) = \left(\frac{1}{4\pi J^2}\right)^{\frac{1}{4}}\frac{\mathrm{sgn}(\tau)}{\sqrt{|\tau|}}\,. \tag{25}$$

2.2.4 *Conformal Symmetry and Finite Temperature*

$\mathrm{Conf}(\mathbb{R}) \cong \mathrm{Diff}(\mathbb{R})$ follows from the fact that there is no notion of an angle in 1D. Every smooth transformation is conformal.

There is one interesting consequence of taking the IR limit for the equations defining the full two point function, emergent conformal symmetry. We can see that (20) has $\mathrm{Conf}(\mathbb{R}^1) \cong \mathrm{Diff}(\mathbb{R}^1)$ symmetry as follows:
Suppose that $G(\sigma,\sigma'')$ solves the equation:

$$J^2\int d\sigma' G(\sigma,\sigma')\,G(\sigma',\sigma'')^3 = -\delta(\sigma-\sigma'')\,.$$

We now let $\sigma = f(\tau)$ such that we obtain:

$$J^2\int\left|\frac{df}{d\tau'}\right| d\tau' G(f(\tau),f(\tau'))\,G(f(\tau'),f(\tau''))^3 = -\frac{1}{|f'(\tau'')|}\delta(\tau-\tau'')\,, \tag{26}$$

where we used that $\delta(f(x)-f(x_0)) = \frac{1}{|f'(x0)|}\delta(x-x_0)$. It now becomes clear that this is equal to (20) if (and only if):

$$G(\tau,\tau') = \left|f'(\tau)\,f'(\tau')\right|^{\Delta}\,G(f(\tau),f(\tau'))\,, \tag{27}$$

where in our current case $\Delta = \frac{1}{4}$. So we find that (20) is invariant under the reparametrization group $\mathrm{Diff}(\mathbb{R})$.

The symmetry is however spontaneously broken by the explicit solution for G in (25). It is obvious that this no longer has the full symmetry group but instead is only invariant under the subgroup

$SL(2,\mathbb{R})$.

This spontaneous breaking of the full Virasoro symmetry is part of a key ingredient of SYK models that will be discussed below in Section 2.6.

For now, let us realize that we can use this to compactify the domain to, for example, S^1. We pick $f(\tau) = e^{2\pi i t/\beta}$ [4] such that we map the line into the circle, and hence obtain a result for finite temperature:

$$G_\beta(\tau) = -\frac{\pi^{\frac{1}{4}}}{\sqrt{2\,\beta\,J}}\frac{1}{\sqrt{\sin(\frac{\pi\tau}{\beta})}}\,\mathrm{sgn}(\tau)\ . \tag{28}$$

2.2.5 $SL(2,\mathbb{R})$ *breaking (non-conformal) corrections*

In some cases further on we will need corrections to the conformal propagators computed above (valid in the IR limit). In order to find these we note that the true full two-point function interpolates between the $\mathrm{sgn}(\tau)$ in the UV and the conformal expression above in the IR.
An example of this (but of course not the correct result) is [10]:

$$G(\tau) = c\,\frac{\mathrm{sgn}(\tau)}{(|J\tau| + 1)^{2\Delta}}\ ,$$

with $c \in \mathbb{R}$ and Δ is as usual $\frac{1}{q}$. For the correct result one would have to solve (17) and (18) without taking the IR limit.
We can however use this formula above to compute the leading correction to the conformal propagator in the case $J|\tau| \gg 1$. To do this we simply Taylor expand:

$$G(\tau) = c\,\frac{\mathrm{sgn}(\tau)}{(J|\tau|)^{2\Delta}\,(1+\frac{1}{J|\tau|})^{2\Delta}} = c\,\frac{\mathrm{sgn}(\tau)}{(J|\tau|)^{2\Delta}}\left(1 - 2\Delta\frac{1}{J|\tau|} + \dots\right)\ . \tag{29}$$

So in the case $q = 4$ we see that the leading order correction is $-\frac{1}{2}\frac{1}{J|\tau|}$. It is important to see that this correction (from the UV) explicitly breaks the remaining $SL(2,\mathbb{R})$ of the original conformal propagator.

2.2.6 *The Case q=2*

Before we continue on with the $q = 4$ model we discuss quickly the case $q = 2$. In this case we have only two particle interactions and hence (18) and (17) reduce to:

$$\begin{aligned} \Sigma(\tau_1,\tau_2) &= J^2\, G(\tau_1,\tau_2) \\ \frac{1}{G(\omega)} &= -i\omega - \Sigma(\omega) \end{aligned} \tag{30}$$

In this case we see that it just reduces to a quadratic equation for $G(\omega)$ which we can solve by

$$\begin{aligned} G(\omega) &= \frac{i\omega}{2J^2}\left(-1+\sqrt{1+4\,\frac{J^2}{\omega^2}}\right) = , \\ &= \frac{-2}{i\omega + i\,\mathrm{sgn}(\omega)\,\sqrt{\omega^2+4J^2}}\,. \end{aligned} \tag{31}$$

Which agrees with [11]. The above result is valid only in the large N limit. As it turns out the $q = 2$ model can be solved for finite N by also summing also all diagrams where the disorder average dotted lines cross each other [12].

2.3 FOUR POINT FUNCTION

In this section we advance towards the four point function. The result (under certain conditions) will also have a relatively easy expression but the calculation is much more involved. This is the reason that we will not reproduce the entire calculation but instead give an overview of the derivation and results. Nevertheless, after reading this section most important steps and results will be understood. We will follow the approach as it can be found in [11]. For a complete computation involving all details see [10,11].
In the last subsection we will also very briefly mention the six point function, that was computed by Gross and Rosenhaus [13].

2.3.1 *Ladder diagrams*

Following the notation of [11] we consider the averaged four point correlation function as:

$$\frac{1}{N^2}\sum_{ij=1}^{N}\langle T\left(\chi^i(\tau_1)\,\chi^i(\tau_2)\,\chi^j(\tau_3)\,\chi^j(\tau_4)\right)\rangle = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\mathcal{F} + \dots\,. \tag{32}$$

Note that the indices are forced to come in pairs by the disorder average (as one can easily check). The first term on the right is the disconnected part that has aligned indices. The second term is the first in a power series in $\frac{1}{N}$. All the diagrams contributing at this order are ladder diagrams as shown Fig 4. Any other diagram will be suppressed by higher powers of $\frac{1}{N}$, which can be checked using the

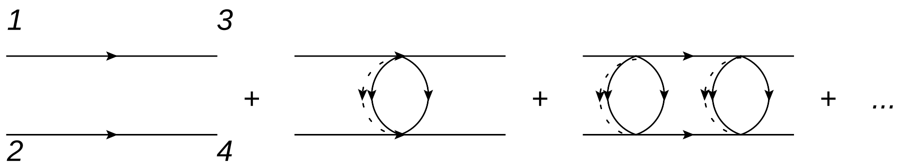

Figure 4: These diagrams represent the $\mathcal{F}$ term in the expansion for the four point function (32). The dotted lines indicate the disorder average. The numbers denote the time ($1 = \tau_1$ etc.). Of course, also the diagrams with $3 \leftrightarrow 4$ should be included with a minus sign.

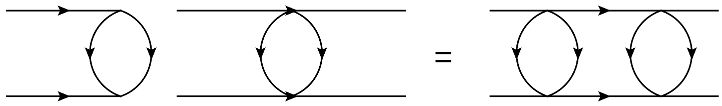

Figure 5: The kernel (leftmost diagram) acting on a diagram adds one 'rung' to the ladder. In this way we can generate all ladder diagrams in the expansion. Note we omitted the disorder average lines for clarity.

Feynman rules and the disorder average properties.

Since we have only ladder diagrams we can follow the standard technique for summing all of these by introducing a so called kernel [11], see Fig 5. We can find the explicit expression for the kernel for arbitrary q:

$$K(\tau_1,\tau_2,\tau_3,\tau_4) = -J^2\,(q-1)G(\tau_1,\tau_3)\,G(\tau_2,\tau_4)G(\tau_3,\tau_4)^{q-2}\,, \tag{33}$$

where G denotes two point function and $q-1$ arises as combinatorial factor. The kernel acts on a ladder diagram by adding a 'rung', see again Fig 5.

By using the kernel we see that any ladder diagram is obtained by consecutive multiplications of the kernel on the lowest order diagram. We call this diagram $\mathcal{F}_0$, the leftmost (disconnected) diagram in Fig 4 minus the same diagram with $\tau_3 \leftrightarrow \tau_4$. Hence we can find an explicit expression for $\mathcal{F}$ by the simple geometric series:

$$\mathcal{F} = \sum_{n=0}^{\infty} \mathcal{F}_n = \sum_{n=0}^{\infty} K^n \mathcal{F}_0 = \frac{\mathcal{F}_0}{1-K}\,. \tag{34}$$

Here we have inverted the matrix K. In order to understand how to do this properly we have to diagonalize K. But as it is defined right now, (33), it is not symmetric under $(\tau_1,\tau_2) \leftrightarrow (\tau_3,\tau_4)$. Hence we define the 'symmetric kernel' as follows:

$$\begin{aligned}\tilde{K}(\tau_1,\tau_2,\tau_3,\tau_4) = -J^2(q-1)\,|G(\tau_1,\tau_2)|^{\frac{q-2}{2}}\,G(\tau_1,\tau_3)\\ \times G(\tau_2,\tau_4)\,|G(\tau_3,\tau_4)|^{\frac{q-2}{2}}\,.\end{aligned} \tag{35}$$

This has the required symmetry and is enough to show that K has a complete set of eigenvectors. The actual inverting is the technical part of the computation and for details we refer to [10,11]. This concludes the discussion of the Feynman diagrams and we now move on to the result of the computation.

2.3.2 *Conformal four point function*

The final result for the computation of $\mathcal{F}$ as found in (32) in the IR limit is as follows [11]:

$$\mathcal{F} = \mathcal{F}_c + \mathcal{F}_{h=2} \,, \tag{36}$$

where $\mathcal{F}_c$ denotes a conformal contribution and $\mathcal{F}_{h=2}$ a non-conformal contribution.

Naively we would expect the following. Since we have emergent conformal symmetry in the IR the four point function should reduce to a sum over conformal blocks (see Section 2.3.2.1) multiplied by the $SL(2,\mathbb{R})$ invariant cross section:

$$\chi = \frac{\tau_{12}\,\tau_{34}}{\tau_{13}\,\tau_{24}} \,. \tag{37}$$

There is however one problem, SYK is merely a 'nearly' CFT [11]. In the IR we find that there exist several Goldstone modes associated to the spontaneous symmetry breaking (cf. Section 2.2.4). We will show later, Section 2.6.2.2 and Section 2.6.3, that these modes have eigenvalue 1 of the kernel K. Considering the expression (34) this can be seen to be a problem, since it will diverge.

It is this divergence that reminds us that we have a 'nearly' CFT in the IR and it is the origin of the non conformal term above. It arises because we will have to introduce non conformal corrections to the result that cancel this divergence (cf. Section 2.2.5). These non conformal corrections are expected in any 'nearly' CFT and are thus not unique to SYK [13]. In the rest of the section we will focus on the conformal part of (36). We therefore first review some basic aspects of conformal blocks and the relation to the two fermion OPE.

2.3.2.1 *Recap conformal blocks and two point OPE*

In this section we very shortly recap the main results for a four point function in a CFT that we shall need below. For a more thorough introduction and proofs of the results stated here see [14,15]. In this section we will assume that we have an arbitrary two dimensional

CFT.

Let us begin by stating the general form of the OPE of two (quasi) primary fields $\mathcal{O}_1$ and $\mathcal{O}_2$:

$$\mathcal{O}_i(z,\bar{z})\,\mathcal{O}_j(w,\overline{w}) = \sum_p \sum_{\{k,\bar{k}\}} C_{ij}^p \frac{\beta_{ij}^{p,\{k\}}\,\overline{\beta}_{ij}^{p,\{\bar{k}\}}\,\mathcal{O}_p^{\{k,\bar{k}\}}(\omega,\overline{\omega})}{(z-\omega)^{h_i+h_j-h_p-K}\,(\bar{z}-\overline{\omega})^{\bar{h}_i+\bar{h}_j-\bar{h}_p-\bar{K}}}\,, \tag{38}$$

where $\{k,\bar{k}\}$ labels all the descendant fields in the conformal family of $\mathcal{O}$. The β and $\overline{\beta}$ denote constants that can be completely fixed by the Virasoro symmetry. All the h_k denote the conformal weights of $\mathcal{O}_k$ and $K = \sum_i k_i$.

Lastly, the coefficients C_{ij}^p are the only unknown parameters, called the structure constants of the primary fields. They are related to the normalization of the three point function which is not fixed by conformal invariance.

Now consider an arbitrary four point function of four primary fields as:

$$G(z_1,\bar{z}_1,\ldots,z_4) = \langle \mathcal{O}_i(z_1,\bar{z}_1)\,\mathcal{O}_j(z_2,\bar{z}_2)\,\mathcal{O}_k(z_3,\bar{z}_3)\,\mathcal{O}_l(z_4,\bar{z}_4)\rangle\,. \tag{39}$$

Since we have (quasi) primary fields we always have the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry (holomorphic and antiholomorphic part). This means the four point function may only depend on crossing ratios:

$$\chi = \frac{z_{12}\,z_{34}}{z_{13}\,z_{24}}\,, \tag{40}$$

and a similar $\bar{\chi}$. This follows by using the $SL(2,\mathbb{R})$ symmetry to map $z_1 \mapsto 0$, $z_2 \mapsto x$, $z_3 \mapsto 1$ and $z_3 \mapsto \infty$ (and similar for the antiholomorphic part).

We may then use our above expression for the general form of the OPE, (38), and (for example) plug this in for the combinations $\mathcal{O}_i(z_1,\bar{z}_1)\,\mathcal{O}_j(z_2,\bar{z}_2)$ and $\mathcal{O}_k(z_3,\bar{z}_3)\,\mathcal{O}_l(z_4,\bar{z}_4)$. As a result the four point function can then be written as:

$$G(z_1,\bar{z}_1,\ldots,z_4) = \sum_p C_{ij}^p\, C_{lm}^p\, \mathcal{F}_{ij}^{lm}(p|\chi)\, \overline{\mathcal{F}}_{ij}^{lm}(p|\bar{\chi})\,, \tag{41}$$

where the C_{ij}^p are again the coefficients in the OPE, (38). The contributions of the descendants in the OPE yield the holomorphic and antiholomorphic factors $\mathcal{F}$ and $\overline{\mathcal{F}}$. These are called conformal blocks,

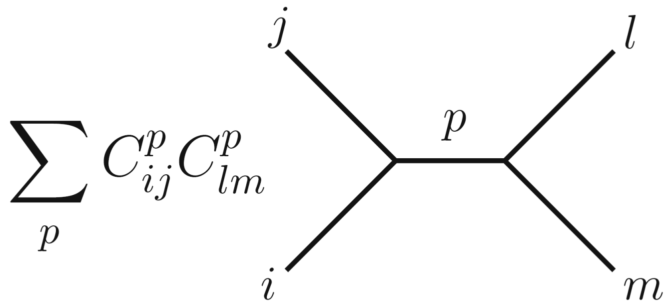

Figure 6: Four point functions in CFTs decompose into conformal blocks. The coefficients C^p_{ij} arise from the OPE and the figure represents the conformal block. The middle line, labeled p, denotes the intermediate descendant fields in the conformal family.

see also Fig 6. These are in general quite complicated factors but depend only on the cross ratio, the conformal weights of the involved operators and the central charge.

In the case of SYK we will have of course only one conformal block instead of a holomorphic and antiholomorphic one (since we have only one dimension). More explicitly, there is then only one cross ratio which depends on τ. So there will be one conformal block, dependent on the coordinate τ.

2.3.3 *Result conformal contribution*

The result for the conformal contribution to the four point function (in the IR) is as follows [11] (see also a convenient summary in [13]):

$$\mathcal{F}_c(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_{12})\, G(\tau_{34}) \sum_{n=1}^{\infty} c_n^2\, \chi^{h_n}\, {}_2F_1(h_n, h_n, 2\,h_n, \chi)\,, \tag{42}$$

where ${}_2F_1$ denotes a hypergeometric function, c_n are the OPE coefficients (we will see below) and χ denotes the cross ratio (37). So we see that the four point function is written as a sum over conformal blocks of operators with dimension h_n.

The h_n are the values of h for which the eigenvalue of the kernel operator, (33), is equal to one. These eigenvalues are denoted by k_c and hence the above boils down to $k_c(h_n) = 1$. The complicated expression for k_c is found to be [11]:

$$k_c(h) = -(q-1)\, \frac{\Gamma\left(\frac{3}{2} - \frac{1}{q}\right) \Gamma\left(1 - \frac{1}{q}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{q}\right) \Gamma\left(\frac{1}{q}\right)}\, \frac{\Gamma\left(\frac{1}{q} + \frac{h}{2}\right) \Gamma\left(\frac{1}{2} + \frac{1}{q} - \frac{h}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{1}{q} - \frac{h}{2}\right) \Gamma\left(1 - \frac{1}{q} + \frac{h}{2}\right)}\,. \tag{43}$$

Using that $z\Gamma(z) = \Gamma(z+1)$ one can easily derive that if $h = 2$ the $k_c(h = 2) = 1$ regardless of q. Hence it has always got this eigenvalue. The sum in (42) contains all the h_n with $h_n > 2$ since the $h = 2$ is taken care of in the non conformal part in (36).

Let us now state the OPE for the two Majorana fermions [11,13]:

$$\frac{1}{N}\sum_i \chi_i(\tau_1)\chi_i(\tau_2) = \frac{1}{\sqrt{N}}\sum_n c_n \frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta-h_n}}\mathcal{O}_n\left(\frac{\tau_1+\tau_2}{2}\right), \quad (44)$$

where c_n denote the OPE coefficients and $\mathcal{O}$ the primary descendants of the χ fields (see next section for an explicit expression). The OPE coefficients c_n can be computed to be:

$$c_n^2 = \frac{2q}{(q-1)(q-2)\tan\frac{\pi}{q}}\frac{(h_n-\frac{1}{2})}{\tan(\frac{\pi h_n}{2})}\frac{\Gamma(h_n)^2}{\Gamma(2h_n)}\frac{1}{k_c'(h_n)}, \quad (45)$$

where k_c' denotes the derivative of the quantity defined in (43).

Lastly, one may wonder if there is a general expression for the h_n. As it turns out we can solve (43) exactly in the large q limit such that we obtain:

$$h_n = 2n+1+2\epsilon_n\,, \qquad \epsilon_n = \frac{1}{q}\frac{2n^2+n+1}{2n^2+n-1}, \quad (46)$$

where $n \geqslant 1$ and $q \gg 1$. We have now a complete expression for the four point function as seen in (42) (plus the nonconformal contribution). We can however take short time limits in the expression and obtain a simpler form for the expression, as we will now show.

2.3.4 *Short time limits*

By considering the hypergeometric function ${}_2F_1(a,b,c,x)$ near $x = 0$ we find:

$${}_2F_1(a,b,c,x) \approx 1 + \frac{a\,b\,x}{c} + \mathcal{O}(x^2)\,. \quad (47)$$

Hence when we take $x \to 0$ we can replace the hypergeometric function by 1. This yields the idea to take the short-time limit $|\tau_{12}| \ll 1$ such that we obtain:

$$\mathcal{F}_c(\tau_1,\tau_2,\tau_3,\tau_4) = G(\tau_{12})G(\tau_{34})\sum_n c_n^2\left|\frac{\tau_{12}\tau_{34}}{\tau_{23}\tau_{24}}\right|^{h_n}. \quad (48)$$

We can then also take $|\tau_{34}| \ll 1$ we get:

$$\mathcal{F}_c(\tau_1, \tau_2, \tau_3, \tau_4) = G(\tau_{12})G(\tau_{34}) \sum_n c_n^2 \frac{|\tau_{12}|^{h_n} |\tau_{34}|^{h_n}}{|\tau_{24}|^{2h_n}} . \tag{49}$$

Indeed we see again that, as we expect from our recap section, the four point function is the sum of two point functions of descendants appearing in the $\chi_i \chi_i$ OPE.

These operators $\mathcal{O}_n$ have the following form [11,13]:

$$\mathcal{O}_n = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{k=0}^{2n+1} d_{nk}\, \partial_\tau^k \chi_i\, \partial_\tau^{2n+1-k} \chi_i . \tag{50}$$

Where the coefficients are chosen such that it is a primary field, see [13] for the explicit expression (we will not need it for the remainder of the section). It can also be seen that these bilinear operators are $O(N)$ invariant.

Lastly, we can find the explicit expression for $\langle \mathcal{O}_n(\tau)\, \mathcal{O}_m(\tau') \rangle$ by considering the fermion OPE ((44)) in the limit where $\frac{\tau_1+\tau_2}{2}$ is small (we follow the notation of [13]):

$$\frac{1}{N} \sum_i \chi_i(\tau_1)\, \chi_i(\tau_2) = \frac{1}{\sqrt{N}} \sum_n c_n\, \mathcal{C}_n(\tau_{12}, \partial_2)\, \mathcal{O}_n(\tau_2) , \tag{51}$$

where $\mathcal{C}_n$ is given by:

$$\mathcal{C}_n(\tau_{12}, \partial_2) = G(\tau_{12})\, |\tau_{12}|^{h_n} \left(1 + \frac{1}{2}\tau_{12}\partial_2 + \ldots\right) . \tag{52}$$

So by using the OPE in this limit ($|\tau_{12}| \ll 1$) we obtain:

$$\mathcal{F}_c = \sum_{n,m} c_n\, c_m\, \mathcal{C}_n(\tau_{12}, \partial_2)\, \mathcal{C}_m(\tau_{34}, \partial_4)\, \langle \mathcal{O}_n(\tau_2)\, \mathcal{O}_m(\tau_4) \rangle . \tag{53}$$

If we then compare with the previously derived expression (49) we conclude:

$$\langle \mathcal{O}_n(\tau_2)\, \mathcal{O}_m(\tau_4) \rangle = \frac{\delta_{n,m}}{|\tau_{24}|^{2h_n}} . \tag{54}$$

Now in conclusion: the conformal contribution to the four point function is found in (42) and the short time limit yields (49).

2.3.5 *Lyapunov behaviour*

As mentioned in the introduction, the SYK model saturates the chaos bound. Let us shortly mention how this arises in the four point functions. This chaos bound (see [3]) can be found using an out of time order correlation function. In particular one can consider two Hermitian operators V and W separated by a time distance t. The chaos can then be investigated by [3]:

$$F(t) = \mathrm{tr}\left[y\, W(t)\, y\, V(0)\, y\, W(t)\, y\, V(0)\right] \ , \tag{55}$$

where $y = \rho(\beta)^{1/4}$. So the thermal density matrix is split into the four factors of y. One may then furthermore show that for a large N CFT holographically described by Einstein gravity [3]:

$$F(t) = f_0 - \frac{f_1}{N^2} \exp\left(\frac{2\pi}{\beta} t\right) + \mathcal{O}(N^{-4}) \ . \tag{56}$$

Then the conjecture is that chaos in thermal quantum systems (with many degrees of freedom) can never develop faster than this above result (the Einstein gravity result). In chaotic systems the correlators are expected to grow exponentially [16]:

$$F_d - F(t) \propto \exp\left(\lambda_L\, t\right) \ , \tag{57}$$

where λ_L the Lyapunov exponent and F_d is the product of the disconnected correlators. This is introduced since due to translation invariance it will be time independent. Furthermore, at some time (between the 'scrambling' time and the 'dissipation' time, see [3]) we have that $F(t) \approx F_d$. The conjecture is then stated as:

$$\frac{d}{dt}\left(F_d - F(t)\right) \leqslant \frac{2\pi}{\beta}\left(F_d - F(t)\right) \ . \tag{58}$$

Or in terms of the Lyapunov exponent:

$$\lambda_L \leqslant \frac{2\pi}{\beta} \ . \tag{59}$$

As it turns out the SYK saturates this bound. Unfortunately this requires considering the $h = 2$ (non conformal) contribution to the four point function, see (36). In particular we would consider:

$$\mathrm{Tr}\left[y\, \chi_i(t)\, y\, \chi_j(0)\, y\, \chi_i(t)\, y\, \chi_j(0)\right] \ . \tag{60}$$

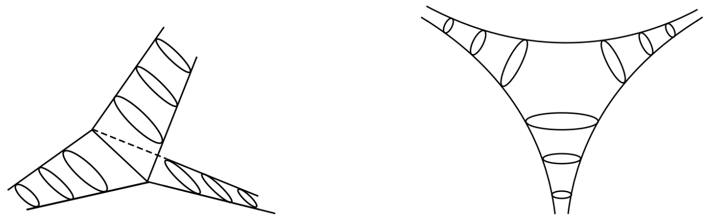

Figure 7: Here are the two contributing diagrams for the six point function. These are referred to as 'contact' and 'planar' diagrams respectively. Figure from [13].

Then, as derived in [11], the $h = 2$ contribution will yield a factor in the four point function as:

$$C\,\beta\,J\left(1 - \frac{\pi}{2}\cosh\frac{2\pi t}{\beta}\right)\,, \tag{61}$$

where C is a constant (see [11]). Then we see indeed the exponential behaviour with a Lyapunov exponent $\lambda_L = \frac{2\pi}{\beta}$, such that SYK saturates this bound.

2.3.6 *Six Point Function*

Let us now briefly mention the calculation of the six point function as done in [13]. The contributions to this correlation function consist out of two diagrams as shown in Fig 7.
The goal is to extract information of the interaction between three of the $O(N)$ invariant operators ((50)):

$$\langle \mathcal{O}_n(\tau_2)\,\mathcal{O}_n(\tau_4)\,\mathcal{O}_n(\tau_6)\rangle\,. \tag{62}$$

Where, as before, short time limits have been taken. One can then use the AdS/CFT dictionary to map these interactions to cubic interactions in the bulk. The final result they obtain for the six point function is in the form of a complicated triple sum over binomial coefficients [13].

2.4 EFFECTIVE ACTION

Now that we have discussed all the basic properties of the model we find a path integral representation over bilocal fields for the partition function. This can be used to find the free energy, the entropy and might be a good starting point for a holographic interpretation of the theory [11]. We note that the computation below also has a generalization including certain 'flavours' of χ's, see [12].

Now, in order to find the free energy (or action) we would use naively $-\beta F = \log Z$. Due to the disorder averaging, however, we have to consider $\overline{Z}$, and for the free energy $F = \overline{\log Z}$. The problem that arises now is that $\overline{\log Z} \neq \log(\overline{Z})$ in general.

The solution to this problem is the so called replica trick (see [17] for an introduction). This is most intuitively stated as:

$$\overline{\log Z} = \lim_{n \to 0} \frac{\overline{Z^n} - 1}{n} . \tag{63}$$

This is proven by simply using l' Hôpital's rule for the limit on the righthand side. The main idea of this equality is that we now have to calculate the disorder average over n copies of Z instead of the logarithm. This means the disorder average will boil down to doing Gaussian integrals.
The replica trick can also be written in a more useful way for us:

$$\overline{\log Z} = \lim_{n \to 0} \frac{1}{n} \log\left(\overline{Z^n}\right) , \tag{64}$$

which is proven by noting that in the righthand side n is small such that:

$$\lim_{n \to 0} \frac{1}{n} \log\left(\overline{Z^n}\right) \simeq \lim_{n \to 0} \frac{1}{n} \log\left(1 + n\,\overline{\log Z}\right) \simeq \overline{\log Z} , \tag{65}$$

where in the first step we used $A^n \simeq 1 + n \log A$ and that $\overline{b} = b$ for all $b \in \mathbb{R}$. In the second step we expanded the logarithm around $n = 0$. Finally we explicitly computed the limit such that only $\overline{\log Z}$ remains, since higher order terms would be proportional to n.

Following the above idea we will now compute the disorder average of M copies of the partition function.

$$\begin{aligned} \overline{Z^M} &= \int \mathcal{D}\chi_i^\alpha \, \mathcal{D}J_{ijkl} \exp\left\{-a \sum_{ijkl} J_{ijkl}^2\right\} \\ &\times \exp\left\{-\sum_{\alpha=1}^{M} \int d\tau \left(\frac{1}{2} \sum_i \chi_i^\alpha \frac{d}{d\tau} \chi_i^\alpha - \frac{1}{4!} \sum_{ijkl} J_{ijkl} \, \chi_i^\alpha \, \chi_j^\alpha \, \chi_k^\alpha \, \chi_l^\alpha\right)\right\} , \end{aligned} \tag{66}$$

Where the bar denotes the disorder average, α denotes the replica index and a is as in (5).

2.4.1 *Gaussian Integral over J*

We can now perform the integral over the coupling J_{ijkl} since it is a Gaussian:

$$\int \mathcal{D}J_{ijkl} \exp\left\{\sum_{\alpha}\sum_{ijkl}\left[\frac{1}{4!}\int d\tau\, J_{ijkl}\,\chi_i^\alpha\,\chi_j^\alpha\,\chi_k^\alpha\,\chi_l^\alpha - a\,J_{ijkl}^2\right]\right\} =$$
$$\exp\left\{\frac{1}{2}\,\frac{J^2N}{4}\sum_{\alpha\beta}\int d\tau_1\,d\tau_2\left(\sum_{i=1}^{N}\frac{1}{N}\,\chi_i^\alpha(\tau_1)\,\chi_i^\beta(\tau_2)\right)^4\right\}\,. \quad (67)$$

In general there would be a constant $c \in \mathbb{R}$ in front of the second line due to the normalization of a Gaussian integral. By the usual approach we have absorbed this into the measure.
In order to get the correct numerical factor in the exponential it is important to note that for any fixed $\{i, j, k, l\}$ (e.g. pick them out of $\{1, 2, 3, 4\}$) there exist 4! terms. This follows from the asymmetry in J_{ijkl} and the anticommutation of the χ fields. So in general one would find as factor:

$$\frac{a\,4!}{4\,(4!)^2\,a^2} = \frac{1}{2}\,\frac{J^2}{4\,N^3}\,.$$

In the case for general q interactions the factor changes by letting $4! \mapsto q!$, since then one has q indices in the coupling. Lastly, we have included the factor $\frac{1}{N}$ in the brackets such that we can make a convenient definition of a collective field later on.

Having completed the J_{ijkl} integral we have now the following expression for $\overline{Z^M}$:

$$\overline{Z^M} = \int \mathcal{D}\chi_i^\alpha\, \exp\left\{-\tfrac{1}{2}\left(\sum_{\alpha=1}^{M}\sum_{i=1}^{N}\int d\tau\,\chi_i^\alpha\,\tfrac{d}{d\tau}\chi_i^\alpha\right.\right.$$
$$\left.\left. -\,\tfrac{J^2N}{4}\sum_{\alpha\beta}\int d\tau_1\,d\tau_2\,\left(\sum_{i=1}^{N}\tfrac{1}{N}\chi_i^\alpha(\tau_1)\,\chi_i^\beta(\tau_2)\right)^4\right)\right\}\,. \quad (68)$$

2.4.2 *Bilocal Fields*

We see now that in (68) there exists a manifest $O(N)$ symmetry by $\chi_i\,\chi^i \to \chi\,O^T\,O\,\chi = \chi_i\,\chi^i$. This symmetry will be discussed in more detail in next section (Section 2.5). For now it leads us to introduce the bilocal field:

$$\widetilde{G}^{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{N}\sum_{i=1}^{N}\chi_i^\alpha(\tau_1)\,\chi_i^\beta(\tau_2)\,.$$

In order to introduce this into our partition function we need a delta function, which we can write as a path integral (think of the analogous $\delta(x) = \int dk e^{ikx}$):

$$\delta\left(\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)-\frac{1}{N}\sum_{i=1}^{N}\chi_i^{\alpha}(\tau_1)\,\chi_i^{\beta}(\tau_2)\right)\propto \tag{69}$$
$$\int d\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\,\exp\left\{-\frac{N}{2}\,\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\left(\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)-\right.\right.$$
$$\left.\left.\frac{1}{N}\sum_{i=1}^{N}\chi_i^{\alpha}(\tau_1)\,\chi_i^{\beta}(\tau_2)\right)\right\}\,,$$

where now $\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)$ acts as a Lagrange multiplier. So inserting this gives us the following expression:

$$\overline{Z^M}=\int \mathcal{D}\chi_i\,\mathcal{D}\widetilde{G}^{\alpha\beta}\,\mathcal{D}\widetilde{\Sigma}^{\alpha\beta}\,\exp\left\{-\sum_{\alpha\beta=1}^{M}\sum_{i=1}^{N}\frac{1}{2}\int d\tau_1\,d\tau_2\,\left[\chi_i^{\alpha}(\tau_1)\right.\right.$$
$$\times\left(\delta^{\alpha\beta}\delta_{\tau_{12}}\,\partial_\tau-\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\right)\left.\chi_i^{\beta}(\tau_2)\right] \tag{70}$$
$$-\frac{1}{2}\sum_{\alpha\beta}\int d\tau_1\,d\tau_2\,\left(N\,\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\,\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)-\right.$$
$$\left.\left.\frac{J^2N}{4}\left(\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)\right)^4\right)\right\}\,,$$

where we used the notation $\delta_{\tau_{12}}=\delta(\tau_1-\tau_2)$.

2.4.3 *Integrating out the fermions*

We see that (70) has now a quadratic dependence on the Majorana fields χ. This is a Gaussian integral such that we can use the standard result for Grassmann integrals except we now get a square root (think of the analogue between complex and real scalar fields):

$$\int \mathcal{D}\chi_i^{\alpha}\,\exp\left\{-\sum_{\alpha\beta=1}^{M}\sum_{i=1}^{N}\frac{1}{2}\int d\tau_1\,d\tau_2\,\left[\chi_i^{\alpha}(\tau_1)\left(\delta^{\alpha\beta}\delta_{\tau_{12}}\,\partial_\tau-\right.\right.\right.$$
$$\left.\left.\left.\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\right)\,\chi_i^{\beta}(\tau_2)\right]\right\}=$$
$$=\exp\left\{\frac{N}{2}\sum_{\alpha\beta}\log\det\left(\delta^{\alpha\beta}\partial_\tau-\widetilde{\Sigma}^{\alpha\beta}\right)\right\}\,.$$

This gives us then the following expression for the partition function:

$$\overline{Z^M}=\int \mathcal{D}\widetilde{G}^{\alpha\beta}\,\mathcal{D}\widetilde{\Sigma}^{\alpha\beta}\,\exp\left\{\frac{N}{2}\sum_{\alpha\beta}\log\det\left(\delta^{\alpha\beta}\partial_\tau-\widetilde{\Sigma}^{\alpha\beta}\right)\right\} \tag{71}$$
$$\times\exp\left\{\frac{1}{2}\sum_{\alpha\beta}\int d\tau_1\,d\tau_2\,\left(N\,\widetilde{\Sigma}^{\alpha\beta}(\tau_1,\tau_2)\,\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)-\right.\right.$$
$$\left.\left.\frac{J^2N}{4}\left(\widetilde{G}^{\alpha\beta}(\tau_1,\tau_2)\right)^4\right)\right\}\,.$$

The last step is to assume a replica symmetric saddle point $\tilde{G}^{\alpha\beta} = \delta^{\alpha\beta}\tilde{G}$. This is a valid assumption when we do not expect any spin-glass solutions [2,18]. By this procedure we get rid of all the α and β indices and it yields:

$$\overline{Z^M} = \int \mathcal{D}\tilde{G}\,\mathcal{D}\tilde{\Sigma}\,\exp\{-M\,S_{eff}\}\;, \tag{72}$$

where S_{eff} is now given by:

$$\begin{aligned} S_{eff} &= -\tfrac{N}{2}\log\det\left(\partial_\tau - \tilde{\Sigma}\right) \\ &+\tfrac{1}{2}\int d\tau_1\,d\tau_2\,\left(N\,\tilde{\Sigma}(\tau_1,\tau_2)\,\tilde{G}(\tau_1,\tau_2) - \tfrac{J^2 N}{4}\left(\tilde{G}(\tau_1,\tau_2)\right)^4\right)\;. \end{aligned} \tag{73}$$

Note that if we vary $\tilde{\Sigma}$ or $\tilde{G}$ we obtain (18) or (17) respectively. This shows us that we have found an exact rewriting of our theory in terms of path integrals over bilocal fields.

The computation for a general q-pt interaction is completely analogous to above and yields as a final result (see also [11,12]):

$$\begin{aligned} \overline{Z^M} = \int \mathcal{D}\tilde{G}\,\mathcal{D}\tilde{\Sigma}\,\exp&\left\{M\,\tfrac{N}{2}\,\log\det\left(\partial_\tau - \tilde{\Sigma}\right)\right\} \\ \times\exp&\left\{-\tfrac{M}{2}\int d\tau_1\,d\tau_2\,\left(N\,\tilde{\Sigma}(\tau_1,\tau_2)\,\tilde{G}(\tau_1,\tau_2) - \right.\right. \\ &\left.\left.\tfrac{J^2 N}{q}\left(\tilde{G}(\tau_1,\tau_2)\right)^q\right)\right\}\;. \end{aligned} \tag{74}$$

Lastly, we can now compute the leading order contribution to the free energy by evaluating at the saddle point and using our starting point (64) (which exactly divides away the M):

$$\begin{aligned} \tfrac{-\beta F}{N} = \tfrac{1}{2}\log\det\left(\partial_\tau - \Sigma\right) - \tfrac{1}{2}\int d\tau_1\,d\tau_2\,&\left[\Sigma(\tau_1,\tau_2)\,G(\tau_1,\tau_2) - \right. \\ &\left.\tfrac{J^2}{q}\,G(\tau_1,\tau_2)^q\right]\;. \end{aligned} \tag{75}$$

2.5 O(N) SYMMETRY

As a first application of the effective action we will discuss the $O(N)$ symmetry in the theory. Here we will consider when it arises and discuss the (lack of a) conserved current.

2.5.1 *Without interactions*

In this section we will shortly discuss the case of the free Majorana fermion to recap the procedure to find a conserved current associated to a continuous symmetry. In this case we will simply have an action as:

$$S = \frac{1}{2}\int d\tau\, \chi_i(\tau)\, \dot{\chi}^i(\tau)\ . \tag{76}$$

We now take an $O(N)$ transformation as $\chi^i \mapsto O^i{}_j\, \chi^j$ such that infinitesimally:

$$\delta\chi^i = \xi_a\, (T^a)^i{}_j\, \chi^j + \mathcal{O}(\xi^2)\,, \tag{77}$$

where now ξ_a is a (constant) parameter and T^a denote the generators of $O(N)$. We can then vary the action:

$$\delta S = \frac{1}{2}\int d\tau\ \left(\delta\chi_i(\tau)\,\dot{\chi}^i(\tau) + \chi_i(\tau)\,\delta\dot{\chi}^i(\tau)\right) = \tag{78}$$
$$= \frac{1}{2}\int d\tau\ \left(\xi_a\,(T^a)_i{}^j\,\chi_j(\tau)\,\dot{\chi}^i(\tau) + \chi_i(\tau)\,\frac{d}{dt}\left(\xi_a\,(T^a)^i{}_j\,\chi^j(\tau)\right)\right) = 0\ .$$

In the last step we use the antisymmetry of the generators. A convenient way to find the conserved current is now to assume that ξ is dependent on the time τ. This would yield:

$$\begin{aligned}\delta S = \tfrac{1}{2}\int d\tau\ &\Big(\xi_a(\tau)(T^a)_i{}^j\,\chi_j(\tau)\,\dot{\chi}^i(\tau) + \\ &\chi_i(\tau)\,\tfrac{d}{dt}\left(\xi_a(\tau)(T^a)^i{}_j\,\chi^j(\tau)\right)\Big) = \\ = \tfrac{1}{2}\int d\tau\ &\partial_\tau(\xi_a(\tau))\left(\chi_i(\tau)\,(T^a)^i{}_j\,\chi^j(\tau)\right) \stackrel{!}{=} 0\ .\end{aligned} \tag{79}$$

We may now partially integrate this last expression and use that the variation on the boundary points will be zero to obtain the conserved current:

$$J^a(\tau) = \frac{1}{2}\chi^i(\tau)\, T^a_{ij}\, \chi^j(\tau)\ . \tag{80}$$

2.5.2 *With Interactions*

In this section we will discuss the appearance of an $O(N)$ symmetry in SYK. We start with the path integral formulation of the partition function as stated in Section 2.4:

$$\begin{aligned}Z = \int \mathcal{D}\chi_i\, \mathcal{D}J_{ijkl}\, &\exp\left\{-a\sum_{ijkl} J^2_{ijkl}\right\} \\ \times \exp&\left\{-\int d\tau\left(\tfrac{1}{2}\sum_i \chi_i \tfrac{d}{d\tau}\chi_i - \tfrac{1}{4!}\sum_{ijkl} J_{ijkl}\,\chi_i\,\chi_j\,\chi_k\,\chi_l\right)\right\}\ .\end{aligned} \tag{81}$$

At this point one might naively think that we can reproduce the previous case by imposing certain transformation rules as:

$$\chi^i \mapsto O^i{}_j \, \chi^j \,, \tag{82}$$

$$J_{jklm} \mapsto O^{j'}{}_j \, O^{k'}{}_k \, O^{l'}{}_l \, O^{m'}{}_m \, J_{j'k'l'm'} \,. \tag{83}$$

However we have to realize that J_{ijkl} is not strictly a standard quantum variable [19] due to the constraint that it is static. We can for example not derive its equation of motion using the Euler Lagrange equations. One can also see that simply varying the action in the partition function above will yield terms with $\delta \, J_{ijkl}$ which will not drop out, preventing us from finding a conserved current.

This brings us to the idea that we perform the disorder average first, see also [12,13]. So let us, as in (68), perform the disorder average to obtain:

$$\begin{aligned} \overline{Z} = \int \mathcal{D}\chi_i \, \exp\Bigg\{ -\frac{1}{2} \Bigg(& \sum_{i=1}^{N} \int d\tau \, \chi_i(\tau) \, \frac{d}{d\tau} \, \chi_i(\tau) \\ & - \frac{J^2}{4\,N^3} \int d\tau \, d\tau' \, \left(\sum_{i=1}^{N} \chi_i(\tau) \, \chi_i(\tau') \right)^4 \Bigg) \Bigg\} \,, \end{aligned} \tag{84}$$

where we left out the explicit replica indices, since they won't play a role here any further. Note that we can generalize this very easily to the case of general q by changing the power $4 \mapsto q$ and similarly in the prefactor we change the 4 to q. We can now see the explicit $O(N)$ symmetry:

$$\begin{aligned} \delta S = \frac{1}{2} \int d\tau \, & \Bigg(\xi_a \, (T^a)_i{}^j \, \chi_j(\tau) \, \dot{\chi}^i(\tau) + \chi_i(\tau) \, \frac{d}{dt} \left(\xi_a \, (T^a)^i{}_j \, \chi^j(\tau) \right) \\ - \frac{J^2}{4\,N^3} \int d\tau' & \left(\xi_a (T^a)_j{}^k \, \chi_k(\tau) \, \chi^j(\tau') + \xi_a (T^a)^j{}_k \, \chi_j(\tau) \, \chi^k(\tau') \right)^4 \Bigg) = \\ & -\frac{1}{2} \, \frac{J^2}{4\,N^3} \int d\tau \, d\tau' \, \Big(\xi_a (T^a)_j{}^k \, \chi_k(\tau) \, \chi^j(\tau') + \\ & \qquad \xi_a (T^a)^k{}_j \, \chi_k(\tau) \, \chi^j(\tau') \Big)^4 = 0 \,. \end{aligned}$$

Note that the first line is simply the same as we did in the case without interactions; the terms simply cancel. In the third line we renamed some dummy indices and in the last step used again the antisymmetry of the generators.

We approach now the problem similarly as before and assume that the parameter ξ_a depends on the time τ. This yields us:

$$\delta S = \frac{1}{2}\int d\tau \;\Big(\partial_\tau(\xi_a(\tau))\left(\chi_i(\tau)\,(T^a)^i{}_j\,\chi^j(\tau)\right) \qquad (85)$$
$$-\frac{J^2}{4N^3}\int d\tau'\left(\xi_a(\tau)-\xi_a(\tau')\right)^4\;\left(\chi_k(\tau)\,\chi^j(\tau')\,(T^a)_j{}^k\right)^4\Big)\;.$$

We recognize the first term from the free case as before, the second term arises due to the interactions. It becomes however obvious that we can never find an expression of the form $\dot{\xi_a}(\tau)J^a(\tau)$ due to the bilocality of the action.

One might expand $\xi(\tau') = \xi(\tau) + \dot{\xi}(\tau)(\tau-\tau') + \ldots$, but still we won't find a proper expression for the current. So in fact we conclude: *although there is a continuous symmetry we can not find an associated conserved current*. Noether's theorem, however, is not violated in any way since it is formulated only for local actions.

2.6 THEORY OF REPARAMETRIZATIONS

Now we come to the main application of the effective action; we will discuss a hallmark feature of SYK and similar (nearly CFT) models: the symmetry breaking of the emergent conformal symmetry (see [10,11]).

We first explicitly show the emergent conformal symmetry in the above action, (75), in the IR limit. Then we will introduce fluctuations and obtain the effective action for the fluctuations.
In this action there appear zero modes: fluctuations of the conformal propagator that yield a zero action. We discuss the physical interpretation of these modes as Nambu-Goldstone modes.

Afterwards we will very shortly mention how the Goldstone modes cause the four point function to be infinite in the strict conformal limit. This means we will need to explicitly break the remaining $SL(2,\mathbb{R})$ symmetry (as in Section 2.2.5) to obtain a finite answer.

This explicit breaking means that we are in fact dealing with Pseudo-Goldstone bosons. We will introduce the non conformal corrections to the action for the Goldstones and obtain a Schwarzian action describing them. Lastly we will discuss the appearance of this Schwarzian in many different SYK-like theories and also mention the same pattern that arises in AdS_2 dilaton gravity which we will discuss extensively in the next part of this thesis.

2.6.1 $\mathrm{Diff}(\mathbb{R})$ *symmetry in the action*

The conformal symmetry is an emergent symmetry in SYK and hence it should appear in our action, (75), when we take the IR limit. In other words we disregard the ∂_τ in the first term:

$$\frac{S}{N} = -\frac{1}{2}\log\det(-\Sigma) + \frac{1}{2}\int d\tau_1\, d\tau_2 \left[\Sigma(\tau_1,\tau_2)\, G(\tau_1,\tau_2) - \frac{J^2}{q}\, G(\tau_1,\tau_2)^q\right] . \tag{86}$$

The first term is now trivially invariant under reparametrizations and hence we focus on the second. We take then the reparametrizations as:

$$\begin{aligned} \tau_1 &\mapsto f(\tau_1)\,, \\ \tau_2 &\mapsto f(\tau_2)\,. \end{aligned}$$

We then recall the transformation rules for G and Σ, see also (27):

$$\begin{aligned} G(\tau,\tau') &= |f'(\tau)\, f'(\tau')|^{\Delta}\; G(f(\tau), f(\tau'))\,, \\ \Sigma(\tau,\tau') &= |f'(\tau)\, f'(\tau')|^{\Delta(q-1)}\; \Sigma((f(\tau), f(\tau'))\,. \end{aligned}$$

Using all of this and plugging it in our above action, (86), we obtain for the second term:

$$\begin{aligned} \frac{1}{2}\int \left|\frac{df}{d\tau_1}\right|\left|\frac{df}{d\tau_2}\right| d\tau_1\, d\tau_2 \Bigg[&\left(\left|\frac{df}{d\tau_1}\right|\left|\frac{df}{d\tau_2}\right|\right)^{-\Delta(q-1)} \left(\left|\frac{df}{d\tau_1}\right|\left|\frac{df}{d\tau_2}\right|\right)^{-\Delta} \\ &\times \Sigma(\tau_1,\tau_2)\, G(\tau_1,\tau_2) - \frac{J^2}{q}\left(\left|\frac{df}{d\tau_1}\right|\left|\frac{df}{d\tau_2}\right|\right)^{-\Delta q} G(\tau_1,\tau_2)^q\Bigg] . \end{aligned} \tag{87}$$

Using now the fact that $\Delta = \frac{1}{q}$ we see indeed it reduces to the original term and hence we have proven that the action has the claimed $\mathrm{Diff}(\mathbb{R})$ symmetry.

2.6.2 *Fluctuations*

In this subsection and the following we will follow the approach of Maldacena and Stanford [11]. Our goal here is to derive an action for fluctuations on the bilocal operator $G(\tau,\tau')$ and discuss the physics that underlie it.

2.6.2.1 *Derivation of the action*

In this section we will denote the saddle point solutions by G and Σ, and the integration variables found in the action by $\tilde{G}$ and $\tilde{\Sigma}$.

We then consider fluctuations as follows [11]: $\tilde{G} = G + |G|^{\frac{2-q}{q}} g$ and $\tilde{\Sigma} = \Sigma + |G|^{\frac{q-2}{2}} \sigma$. The usefulness of the particular shape of these fluctuations will become obvious later. Notice also that the measure remains invariant $d\tilde{\Sigma} d\tilde{G} = d\sigma dg$. Plugging these into the action, (75), yields (note again that G and Σ are no longer integration variables):

$$\frac{S}{N} = -\frac{1}{2} \log\det\left(\partial_\tau - (\Sigma + |G|^{\frac{q-2}{2}} \sigma)\right) + \frac{1}{2} \int d\tau_1\, d\tau_2\, \Big[(\Sigma + |G|^{\frac{q-2}{2}} \sigma) \times (G + |G|^{\frac{2-q}{q}} g) - \frac{J^2}{q} (G + |G|^{\frac{2-q}{q}} g)^q\Big] \,. \tag{88}$$

We will now expand the action up to second order in g and σ. Note that since we are expanding around the saddle point, linear fluctuations will yield zero. Also we can ignore 'constant' terms, i.e. those that are independent of the path integral variables g and σ.

Let us start with the first term by using the standard result $\log\det A = \operatorname{Tr} \log A$ and expand the logarithm. The second order term becomes:

$$-\frac{1}{2} \operatorname{Tr}\left(-\frac{1}{2} \sigma |G|^{\frac{q-2}{2}} G\, G\, |G|^{\frac{q-2}{2}} \sigma\right) \,. \tag{89}$$

This reduces to the following integral expression:

$$-\frac{1}{4a} \int d\tau_1 d\tau_2 d\tau_3 d\tau_4\ \sigma(\tau_1, \tau_2)\, \tilde{K}(\tau_1, \tau_2, \tau_3, \tau_4)\, \sigma(\tau_3, \tau_4) \,, \tag{90}$$

where $a = J^2(q-1)$ and we defined the before, (35), the symmetric kernel used in the computation of the four point function:

$$\tilde{K}(\tau_1, \tau_2, \tau_3, \tau_4) = -a\, |G(\tau_1, \tau_2)|^{\frac{q-2}{2}}\, G(\tau_1, \tau_3)\, G(\tau_2, \tau_4)\, |G(\tau_3, \tau_4)|^{\frac{q-2}{2}} \,. \tag{91}$$

For the second term we get a product like $G^{q-2}|G|^{2-q}$ which reduces to $\operatorname{sgn}(G)^{q-2}$. However in our case $q - 2 \in 2\mathbb{Z}$ and hence it reduces to 1. So we expand up to second order and write out the other terms to obtain:

$$\frac{1}{2} \int d\tau_1 d\tau_2 \left(\sigma(\tau_1, \tau_2)\, g(\tau_1, \tau_2) - \frac{1}{2} J^2 (q-1)\, g^2(\tau_1, \tau_2)\right) \,. \tag{92}$$

So the total path integral becomes:

$$\int dg\, d\sigma\, \exp\left\{N \left(-\frac{1}{4a} (\sigma|\tilde{K}|\sigma) + \frac{1}{2} (\sigma|g) - \frac{1}{4} J^2 (q-1)\, (g|g)\right)\right\} \,, \tag{93}$$

where we introduced shorthand notations for the integrals above as:

$$(\sigma|\tilde{K}|\sigma) = \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \; \sigma(\tau_1,\tau_2)\,\tilde{K}(\tau_1,\tau_2,\tau_3,\tau_4)\,\sigma(\tau_3,\tau_4)\;,$$
$$(\sigma|g) = \int \; d\tau_1 d\tau_2\, \sigma(\tau_1,\tau_2)\, g(\tau_1,\tau_2)\;.$$

We can now complete the Gaussian integral over σ by the usual methods to obtain finally the action, similar to [11]:

$$\frac{S}{N} = \frac{J^2(q-1)}{4}\,(g|\tilde{K}^{-1} - 1|g)\;. \tag{94}$$

2.6.2.2 *Nambu-Goldstone Modes*

Now that we have obtained the action for the reparametrizations we can discuss some interesting properties of it. Although this above expression is valid for any energy, the interesting regime (as usual in SYK so far) is in the IR limit. In this limit we can use the expressions for the conformal propagator as in (21) to find also an explicit expression for the symmetric kernel.

In particular we see the possibility of the action yielding zero. This happens when g is an eigenfunction of the symmetric kernel with eigenvalue 1. Let us prove the existence of such eigenfunctions using the Schwinger-Dyson (SD) equations (17) and (19).

We recall that the SD-equations are invariant under conformal transformations as in (27). So suppose we take an infinitesimal transformation as $\tau \mapsto \tau + \epsilon(\tau)$. Then, if G_c is a solution to the SD-equations, also $G_c + \delta_\epsilon G_c$ is a solution. The explicit shape of $\delta_\epsilon G_c$ is (simply the infinitesimal form of (27))

$$\delta_\epsilon G_c(\tau,\tau') = \left(\Delta\epsilon'(\tau) + \Delta\epsilon'(\tau') + \epsilon(\tau)\partial_\tau + \epsilon(\tau')\partial_{\tau'}\right) G_c(\tau,\tau')\;. \tag{95}$$

We can thus plug this into the SD-equations and obtain the following result from the (19):

$$\delta_\epsilon G_c * \Sigma_c + G * \delta_\epsilon \Sigma_c = 0\;, \tag{96}$$

where $*$ denotes the involution: $(f*g)(\tau,\tau'') = \int d\tau' f(\tau,\tau')g(\tau',\tau'')$ and we left the explicit dependence on the two times absent. We can now involute from the right with $G_c = \Sigma_c^{-1}$ ((18) as $\omega \to 0$) to obtain:

$$\begin{aligned}\delta_\epsilon G_c + G_c * \left(J^2(q-1)\,G_c^{q-2}\,\delta_\epsilon G_c\right) * G_c = \\ (1-K_c)\delta_\epsilon G_c = 0\;,\end{aligned} \tag{97}$$

where we defined the kernel in the four point section as: (see (33))

$$K_c(\tau_1,\tau_2,\tau_3,\tau_4) = -J^2\,(q-1)G_c(\tau_1,\tau_3)\,G_c(\tau_2,\tau_4)G_c(\tau_3,\tau_4)^{q-2}\,, \tag{98}$$

where the subscript c denotes 'conformal', the G_c are solutions of the Schwinger Dyson equations that are invariant under conformal transformations. Note that we can rewrite this to contain the symmetric kernel ((35)) in which case it reduces to:

$$(1-\tilde{K})\,|G_c|^{\frac{q-2}{2}}\,\delta_\epsilon G_c = 0\,. \tag{99}$$

Hence we have now proven there exist eigenfunctions of $\tilde{K}$ with eigenvalue 1. In particular we have shown that these are exactly the reparametrizations of the conformal correlator. The particular shape of the eigenfunctions (so containing the $|G|$ term) also explains our choice of fluctuations in the beginning of this section.

The zero modes of the action have thus an interesting physical interpretation: the original emergent conformal symmetry of the action is spontaneously broken to $SL(2,\mathbb{R})$ by the solution G_c. The zero modes are the associated Nambu-Goldstone modes.
Note that the action, (94), is also zero when we choose $\epsilon \in SL(2,\mathbb{R})$ and evaluate it at $\delta_\epsilon G_c$. This is however unrelated to the Goldstone modes, and is due to the fact that in this case $\delta_\epsilon G_c = 0$. These fluctuations are simply not present in the path integral for G.

Now we will discuss the necessity for also explicitly breaking the conformal symmetry, which has a different origin than the spontaneous breaking.

2.6.3 *Divergent four point function*

In this section we will shortly mention the divergence in the four point function due to the above derived Goldstone modes, as hinted on before in Section 2.3.2.

Recall that we computed the four point function by using the kernel to sum all of the ladder diagrams:

$$\mathcal{F} = \sum_{n=0}^{\infty} \mathcal{F}_n = \sum_{n=0}^{\infty} K^n \mathcal{F}_0 = \frac{\mathcal{F}_0}{1-K}\,, \tag{100}$$

where $\mathcal{F}_0$ denotes the disconnected diagram in Fig 4 (minus the same diagram with $\tau_3 \leftrightarrow \tau_4$). It is now obvious that in the conformal limit the Goldstone modes, that have an eigenvalue 1 of K_c, will cause this expression to diverge.

Of course, the four point function should not diverge and hence it is a consequence of taking the IR limit. Hence we will need a correction to the kernel from the UV. The correction δK originates from the non conformal corrections to the propagator as in Section 2.2.5. This is the origin of the explicit symmetry breaking of the emergent conformal symmetry. It is hence also the origin of the term $\mathcal{F}_{h=2}$ in (36). The actual computation of this shift is involved and we wish not to explore it here, the details are found in [11]. We now want to obtain an action for the Goldstone modes after we incorporate the explicit breaking.

2.6.4 *Schwarzian action*

Now we want to find the explicit (nonzero) action for the reparametrizations $\tau \mapsto f(\tau)$ when including these corrections. This can be done exactly by using several computation heavy results from the four point function, see [11]. We give a short summary of this method in Appendix A.

There is however also a more intuitive (but equally valid) argument that we outline here. Let us first consider the case $T = 0$ and find the action that describes the dynamics of $f(\tau)$.
The action $S[f]$ must have the following properties:

A. If $f \in SL(2, \mathbb{R}) \iff f(\tau) = \frac{a\,\tau+b}{c\,\tau+d} \implies S[f] = 0$,

B. If $f \notin SL(2, \mathbb{R}) \implies S[f]$ is invariant under $f \mapsto \frac{a\,f+b}{c\,f+d}$.

Property A follows because we want the action to yield zero for such $f(\tau)$ since then $\delta G_c = 0$ (and hence the g in (94) is).
The second property, B, is a symmetry of the action with a similar origin. It follows because the zero temperature G_c is exactly invariant under $SL(2, \mathbb{R})$ transformations. Hence the transformed f is exactly the same as the original f from the viewpoint of our theory.

Now, from the effective field theory (EFT) (see e.g. [20]) point of view we can then argue as follows: we seek the action invariant under global $SL(2, \mathbb{R})$ transformations of lowest orders in derivatives.
We can rephrase this as follows, suppose F is a $SL(2, \mathbb{R})$ transformation of f:

$$F(\tau) = \frac{a\,f(\tau)+b}{c\,f(\tau)+d} \,. \tag{101}$$

Our goal is then to find the simplest combination of derivatives of F such that when we plug in the above equation, the result reduces to exactly the same combination of derivatives of f. This is best illustrated by doing the computation. The first two derivatives of F are:

$$F' = \frac{f'}{(c\,f+d)^2}\,, \tag{102}$$

$$F'' = \frac{f''}{(c\,f+d)^2} - \frac{2\,c\,(f')^2}{(c\,f+d)^3}\,, \tag{103}$$

where we omitted the dependence on τ. The first thing we notice is the term $1/(c\,f+d)^2$ which arises in both derivatives. A natural next step is thus to consider:

$$\frac{F''}{F'} = \frac{f''}{f'} - \frac{2\,c\,(f')^2}{(c\,f+d)}\,. \tag{104}$$

Note that the first term is indeed what we would want but the second term we still need to get rid of. By considering all expressions above it becomes clear we can not do this by any combination of F'' and F'. We are thus led to compute the third derivative:

$$\frac{F'''}{F'} = \frac{f'''}{f'} - \frac{6\,c\,f''}{(c\,f+d)} + \frac{6\,c^2\,(f')^2}{(c\,f+d)^2}\,, \tag{105}$$

where we followed the same logic as above and divided it by F'. We can now see a very big similarity between this term and $\frac{F''}{F'}$; consider:

$$\left(\frac{F''}{F'}\right)^2 = \left(\frac{f''}{f'}\right)^2 - \frac{4\,c\,f''}{(c\,f+d)} + \frac{4\,c^2\,(f')^2}{(c\,f+d)^2}\,. \tag{106}$$

Hence we see that we can make the following combination:

$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \equiv \{f,\tau\}\,. \tag{107}$$

Which indeed solves our stated problem. In the last step above we introduced a short hand notation for the particular transformation. In fact the above operation is called the 'Schwarzian derivative' on f with respect to τ. It is a well known operator with many applications. For example, it shows up in two dimensional CFT as we transform the energy momentum tensor under finite conformal transformations.

Anyhow, our EFT combined with the demanded symmetry yields us thus following 'Schwarzian action':

$$\frac{S}{N} = \frac{c}{J}\int d\tau\{f,\tau\}\,, \tag{108}$$

where $c \in \mathbb{R}$ and J is the dimension 1 parameter (as in (5)) to get correct dimensions. It is easy to check that this also satisfies the property A, as stated in the beginning of this section. Now in order to fix the constant $c \in \mathbb{R}$ we compare it with the action computed in [11] and write:

$$\frac{S}{N} = -\frac{\alpha_s}{\mathcal{J}} \int d\tau \{f, \tau\} \,, \tag{109}$$

where $\mathcal{J}^2 = J^2 \frac{q}{2^{q-1}}$ and α_s is a constant, but it does depend on the value of q. The constant α_s is numerically calculated and discussed in [11]. In particular, for large q it reduces to $\frac{1}{4\,q^2}$

Let us now discuss how the Schwarzian action looks like at finite temperature. This means we pick $f = \exp\left(\frac{2\pi i\,\tau}{\beta}\right)$ or $f = \tan\left(\frac{\pi\tau}{\beta}\right)$:

$$\frac{S_\beta}{N} = -\frac{\alpha_s}{\mathcal{J}} \int d\tau \left\{\exp\left(\frac{2\pi i\,\tau}{\beta}\right), \tau\right\} = -2\,\pi^2 \frac{\alpha_s}{\beta\,\mathcal{J}} \,, \tag{110}$$

where we added the subscript β to denote the finite temperature. The above result can also put into perspective as [11]:

$$-\beta\, F \supset 2\,\pi^2 \frac{\alpha_s\, N}{\beta\,\mathcal{J}} \,, \tag{111}$$

where the left hand side corresponds with the free energy. Note that the righthand side scales with temperature and thus says nothing about the zero temperature entropy; hence the $\supset$ sign.

We can now also go one step further and consider how reparametrizations on the circle itself behave. To do this we take not $f(\tau)$ as above but instead $f(g(\tau))$:

$$\frac{S_\beta}{N} = -\frac{\alpha_s}{\mathcal{J}} \int d\tau \left\{\exp\left(\frac{2\pi i\, g(\tau)}{\beta}\right), \tau\right\} \,. \tag{112}$$

To work this out we use the straightforwardly proven composition rule for the Schwarzian derivative:

$$\{f(g(\tau)), \tau\} = (g'(\tau))^2 \{f, g\} + \{g, \tau\} \,. \tag{113}$$

If we plug this into (112) and partially integrate the $\frac{g'''}{g'}$ term we obtain:

$$\frac{S_\beta}{N} = \frac{\alpha_s}{2\,\mathcal{J}} \int d\tau \left(\left(\frac{g''}{g'}\right)^2 - \left(\frac{2\,\pi}{\beta}\right)^2 (g')^2 \right) \,. \tag{114}$$

One can check that it indeed reduces to the zero temperature (109) as we take the limit $\beta \to \infty$ (and again partially integrate). Using this action we can also derive the action for small reparametrizations $\tau \mapsto \tau + \epsilon(\tau)$. To do this we take $g(\tau) = \tau + \epsilon(\tau)$. We then expand up to second order in ϵ and use the periodicity $\tau = \tau + \beta$:

$$\frac{S_\epsilon}{N} = \frac{\alpha_s}{2\mathcal{J}} \int_0^\beta d\tau \left((\epsilon''(\tau))^2 - \left(\frac{2\pi}{\beta}\right)^2 (\epsilon'(\tau))^2 \right) , \tag{115}$$

where the subscript indicates that it is valid for an infinitesimal transformation.

To conclude, in this section we have derived that the dynamics of the reparametrizations are described by a Schwarzian action. As it turns out this Schwarzian action is a hallmark feature of SYK-like models and is also found in the supersymmetric analog [21], in which case there is an explicit breaking of super reparametrization group sDiff. We will shortly discuss this in Chapter 4. Also in a recent 2D QFT generalization of SYK [22] it is found.
Moreover, in AdS_2 dilaton gravity models [23-26] (which will be thoroughly discussed in the next part) the same pattern of spontaneous and explicit breaking of symmetry is found. In this case the Schwarzian action can also be found in the boundary dynamics of the spacetime. [25,26].

3 CHOICE OF ENSEMBLE

3.1 THE TWO SIDES OF SYK

In this chapter we will discuss the choice of the particular ensemble for SYK and what the consequences are if we pick it differently.

3.1.1 *'Normal' SYK*

As mentioned before, see (5), the SYK model has a Gaussian disorder average. The coupling between the Majorana fermions, J_{ijkl}, is picked from the following ensemble:

$$P(J_{ijkl}) = \sqrt{\frac{N^3}{12\pi J^2}} \exp\left(\frac{-N^3 J_{ijkl}^2}{12J^2}\right) . \tag{116}$$

This specific ensemble is responsible for many of the properties that SYK has but we will show that several key features of the SYK model can be obtained from different ensembles. In particular, the entire melonic diagrammatic structure of SYK can be reproduced by a much wider class of ensembles.

3.1.2 *Binary ensemble*

To illustrate this claim we will create a different ensemble, the binary ensemble:

$$J_{ijkl} = \begin{cases} J_1, & \Pr(J_1) = p \\ J_2, & \Pr(J_2) = q \end{cases} , \tag{117}$$

where of course $q := 1 - p$ and $J_1, J_2 \in \mathbb{R}$. From this we can compute the expectation values of J_{ijkl} and J_{ijkl}^2:

$$\langle J_{ijkl} \rangle = p\, J_1 + q\, J_2 \, , \tag{118}$$
$$\langle J_{ijkl}^2 \rangle = p\, J_1^2 + q\, J_2^2 \, , \tag{119}$$
$$\langle J_{ijkl} \rangle^2 = p^2\, J_1^2 + 2\, p\, q\, J_1\, J_2 + q^2\, J_2^2 \, . \tag{120}$$

From this one may also check that we find the variance as:

$$\mathrm{Var}(J_{ijkl}) = p\, q\, (J_1 - J_2)^2 \, . \tag{121}$$

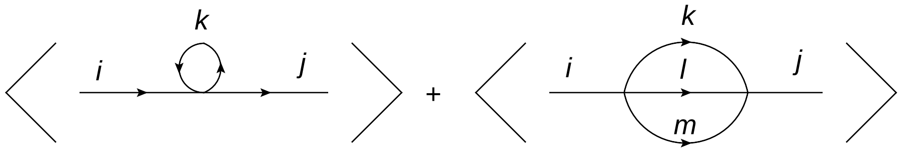

Figure 8: Here we show two diagrams that occur within SYK. The brackets around de diagrams denote the disorder average. To make SYK solvable we need the first diagram to yield zero and for the second that the incoming and outgoing lines have the same index. Due to the asymmetry of the coupling J_{ijkl} the former is already satisfied.

Now let us consider some basic two point diagrams within SYK, see Fig 8. There are now several demands that we must impose. As mentioned in the figure the first diagram vanishes due to the asymmetry in the coupling and hence this imposes no demands. The expression for the second diagram reads:

$$\frac{C}{(4!)^2}\sum_{klm}\langle J_{iklm}\, J_{jklm}\rangle\, G_0(\tau_1,\tau_2)^3 \overset{!}{=} J^2 G_0(\tau_1,\tau_2)^3\,, \tag{122}$$

where G_0, the free propagator, is as found in (14). We also chose the left vertex at τ_1 and the right one at τ_2. The combinatorial factor is $C = \binom{4}{3}\binom{4}{3}\,3!$. To reproduce the SYK structure (denoted with $\overset{!}{=}$ above, see (16)) we must thus demand:

$$\langle J_{iklm}\, J_{jklm}\rangle = \delta_{ij}\,\frac{3!\,J^2}{N^3}\,. \tag{123}$$

To reproduce the δ_{ij} we note first: suppose p and q are two random independent variables, then $E[p,q] = E[p]\,E[q]$ (E denotes the expectation value). So if $i \neq j$ we get:

$$\langle J_{iklm}\, J_{jklm}\rangle = \langle J_{iklm}\rangle\,\langle J_{jklm}\rangle\,,$$

and hence we obtain the demands:

$$\langle J_{ijkl}\rangle = p\,J_1 + q\,J_2 = 0 \implies \boxed{p\,J_1 = -q\,J_2}\,, \tag{124}$$

$$\langle J_{ijkl}^2\rangle = \boxed{p\,J_1^2 + q\,J_2^2 = \frac{3!\,J^2}{N^3}}\,. \tag{125}$$

We can solve this for J_1 and J_2 to obtain then the following ensemble:

$$J_{ijkl} = \begin{cases} \sqrt{\frac{1-p}{p}\frac{3!\,J^2}{N^3}}, & \Pr(J_1) = p \\ -\sqrt{\frac{p}{1-p}\frac{3!\,J^2}{N^3}}, & \Pr(J_2) = 1-p \end{cases}\,. \tag{126}$$

Now one may easily check: if we had started with this binary ensemble instead of the Gaussian one we would exactly reproduce the diagrammatic structure of SYK up to leading order in $\frac{1}{N}$. In particular we obtain exactly the same results for the four point function as derived in Section 2.3. This is due to the fact that the four point function only encounters products of $\langle J_{ijkl}^2 \rangle$ and never $\langle J_{ijkl}^n \rangle$ with $n > 2$. The same holds for higher n-point functions, see for example the calculation of the six-point function [13].

However, for the 'other side' of SYK: the bilocal effective action and the Schwarzian action, we run into some trouble with this choice of ensemble. We discuss this in the next section.

3.1.3 *Arbitrary ensemble*

Before we consider the effective action we can generalize the results of the previous section. By inspecting the expressions of the diagrams and the results in (124) and (125) we find that:

The diagrammatic structure of SYK is reproduced by any ensemble that has the following two properties:

- $\langle J_{ijkl} \rangle = 0$,
- $\langle J_{iklm} J_{jklm} \rangle = \delta_{ij} \frac{3! J^2}{N^3}$.

Now we can address the derivation of the effective action with this arbitrary ensemble $P(J_{ijkl})$ satisfying the two above conditions. We consider then the replica disorder averaged partition function (see (66) and the discussion above):

$$\begin{aligned} \overline{Z^M} &= \int \mathcal{D}\chi_i^\alpha \, \mathcal{D}J_{ijkl} \, P(J_{ijkl}) \qquad (127) \\ &\times \exp\left\{ -\sum_{\alpha=1}^{M} \int d\tau \left(\frac{1}{2} \sum_i \chi_i^\alpha \frac{d}{d\tau} \chi_i^\alpha - \frac{1}{4!} \sum_{ijkl} J_{ijkl} \chi_i^\alpha \chi_j^\alpha \chi_k^\alpha \chi_l^\alpha \right) \right\} . \end{aligned}$$

In the case of a Gaussian ensemble, such as in the ordinary SYK, $P(J_{ijkl})$ has the form $\exp(J_{ijkl}^2)$. This means that in that particular case we can simply perform the Gaussian integral over J_{ijkl} and solve the disorder average.

However if we pick an arbitrary ensemble like the binary ensemble this procedure is not always possible. In particular for the binary ensemble, $P(J_{ijkl})$ becomes $p\,\delta(J_{ijkl} - J_1) + q\,\delta(J_{ijkl} - J_2)$ where J_i can be seen in (126).

One can check that in this case we will not obtain the bilocal action as seen in (74). Hence we can also not directly derive the Schwarzian action as was done in [11].

To conclude; one might wonder how important is the disorder average for the model in the first place. As it turns out, the diagrammatic structure of SYK can also be found without using any notion of disorder average. This is what happens in the tensor models of SYK, which we will discuss very shortly in Chapter 5. Also for these tensor models it is, at the time of writing, not yet possible to find the effective action and the Schwarzian action.

4 SUPERSYMMETRIC SYK

In this section we wish to shortly introduce supersymmetry into SYK and discuss the consequences. It was first introduced by W. Fu et al. [21].
Note that just as the 'normal' SYK models we have again all to all interactions between the N Majorana fermions. Below we will consider the $\mathcal{N} = 1$ model. We describe in detail the Hamiltonian and obtain the Lagrangian in the superspace representation. Afterwards we will summarize some features analogous to SYK: the symmetries, the effective action and the Superschwarzian.

4.1 THE $\mathcal{N} = 1$ MODEL

4.1.1 *Hamiltonian*

Following [21] we introduce the following supercharge:

$$Q = \frac{i}{3!} \sum_{ijk=1}^{N} C_{ijk}\, \chi^i \chi^j \chi^k \,. \tag{128}$$

Following the standard procedures of supersymmetry for quantum mechanics (for an introduction see [27,28]) we note that

$$H = \frac{1}{2}\{Q, Q^\dagger\} \,.$$

Of course, in our case (Majorana fermions) we simply find that $Q^\dagger = Q$ and hence $H = Q^2$. Since H must be Hermitian it follows that also Q must be. Hence we find that C_{ijk} is a $N \times N \times N$ antisymmetric tensor with fixed real entries.
In some analogy with the standard SYK model (see (8)) we now take C_{ijk} to be a random Gaussian variables with:

$$\langle C_{ijk} \rangle = 0 \,, \tag{129}$$

$$\langle C_{ijk}^2 \rangle = \frac{2!\,J}{N^2} \,. \tag{130}$$

So let us now find the Hamiltonian of this model by computing first $\{Q, Q\}$:

$$\{Q, Q\} = -\frac{1}{(3!)^2} \sum_{ijklmn} C_{ijk}\, C_{lmn} \{\chi^i \chi^j \chi^k, \chi^l \chi^m \chi^n\} \,. \tag{131}$$

By using the anticommutation of the Majorana fermions we can rewrite this to:

$$\{Q,Q\} = -\frac{1}{(3!)^2}\sum_{aijkl}\left(3C_{akl}\,C_{ija} + 3C_{ail}\,C_{jka}\right. \\ \left. +3C_{aij}\,C_{kla}\right)\chi^i\chi^j\chi^k\chi^l\,. \tag{132}$$

Note that the first and third term are the same but we rewrite the third one as follows:

$$C_{aij}\,C_{kla} = -C_{aik}\,C_{lja}\chi^i\,(\delta^{kj} - \chi^j\chi^k)\,\chi^l = -C_{ija}\,C_{lja}\,\chi^i\chi^l \\ +C_{aik}\,C_{lja}\chi^i\chi^j\chi^k\chi^l\,,$$

so then we find for $H = \frac{1}{2}\{Q,Q\}$:

$$H \;= -\tfrac{1}{4!}\textstyle\sum_{aijkl}\left(C_{aij}\,C_{kla} + C_{ail}\,C_{jka} + C_{aik}\,C_{lja}\right)\chi^i\chi^j\chi^k\chi^l \\ +\tfrac{1}{4!}\textstyle\sum_{ijkl=1}^{N} C_{ijk}\,C_{ilk}\,\chi^j\chi^l\,. \tag{133}$$

For the second term we note that if $j \neq l$ we have a contraction of a symmetric tensor with an antisymmetric one. Hence in this context $\chi^j\chi^l$ will be equal to $\frac{1}{2}\delta^{jl}$. We can also rewrite the first term by noticing:

$$C_{a[ij}\,C_{kl]a} = 8\left(C_{aij}\,C_{kla} + C_{ail}\,C_{jka} + C_{aik}\,C_{lja}\right)\;,$$

where the brackets denote all possible antisymmetric permutations. So we define then the following quantities:

$$E_0 \equiv \tfrac{1}{8}\tfrac{1}{3!}\textstyle\sum_{ijk=1}^{N} C_{ijk}^2 = \tfrac{1}{8}\sum_{1\leqslant i<j<k\leqslant N} C_{ijk}^2\,, \tag{134}$$

$$J_{ijkl} \equiv -\tfrac{1}{8}\textstyle\sum_a C_{a[ij}\,C_{kl]a}\,. \tag{135}$$

We can then rewrite (133) to get our final expression for the Hamiltonian, obtaining a similar result as [21]:

$$H = E_0 + \frac{1}{4!}\sum_{ijkl=1}^{N} J_{ijkl}\,\chi^i\chi^j\chi^k\chi^l\,. \tag{136}$$

It is important to note that in this case J_{ijkl} are not the independent Gaussian variables, which constitutes an important difference between the supersymmetric model and the ordinary one.

4.1.2 *Superspace and Lagrangian*

Now we will obtain the Lagrangian for this supersymmetric model. In particular, we will start by deriving it in the superspace representation, for which we will denote the supercharge with $\mathcal{Q}$ [21]. The superspace representation arises by introducing an anticommuting coordinate for each supercharge in the model. So in our case we essentially map (for each t in the domain):

$$t \mapsto (t, \theta) \,,$$

where now θ is the anticommuting coordinate. In a model where $\mathcal{Q}^\dagger \neq \mathcal{Q}$ (so no Majorana fermions) one would also have the θ^* coordinate. Furthermore, we introduce the superfield

$$\Psi^i = \chi^i + \theta\, b^i \,, \tag{137}$$

with b a non-dynamical auxiliary field, that will linearize the supersymmetry transformations. This can be seen by considering the supercharge in this representation

$$\mathcal{Q} = \partial_\theta - \theta\, \partial_\tau \,. \tag{138}$$

Note that this supercharge satisfies the expected anticommutation relation $\frac{1}{2}\{\mathcal{Q}, \mathcal{Q}\} = -\partial_\tau = i\partial_t$, yielding the generator of time translations. Related to the supercharge by $t \mapsto -t$ (or taking right derivatives instead of left ones) is the covariant derivative:

$$D_\theta = \partial_\theta + \theta\, \partial_\tau \,. \tag{139}$$

Now we can determine how the superfield (and its components) change under supersymmetry transformations, which we will denote by δ_ϵ. For a general superfield Φ we have:

$$\delta_\epsilon \Phi = (\epsilon^* \mathcal{Q} + \epsilon \mathcal{Q}^\dagger)\, \Phi \,,$$

where ϵ and ϵ^* are (infinitesimal) anticommuting constant parameters. So in our case we have only the supercharge $\mathcal{Q}$ and the above reduces to:

$$\delta_\epsilon\, \Psi^i = \epsilon\, \mathcal{Q}\, \Psi^i = \epsilon b^i + \theta\, \epsilon\, \partial_\tau \chi^i \,. \tag{140}$$

Hence the χ and b fields transform as follows:

$$\delta_\epsilon \chi^i = \epsilon b^i \,,$$
$$\delta_\epsilon b^i = \epsilon\, \partial_\tau \chi^i \,. \tag{141}$$

The manifestly supersymmetric Lagrangian is then given by

$$\mathcal{L} = \int d\theta \left(-\frac{1}{2} \Psi^i D_\theta \Psi^i + \frac{i}{3!} C_{ijk} \Psi^i \Psi^j \Psi^k \right) , \tag{142}$$

where $D_\theta = \partial_\theta + \theta\, \partial_\tau$, the covariant derivative, is obtained by taking $t \mapsto -t$ in the supercharge. Instead of writing it in this manifestly symmetric way we can also first fill in the above expressions:

$$\mathcal{L} = \int d\theta \left[-\frac{1}{2} (\chi^i + \theta\, b^i) (\partial_\theta + \theta\, \partial_\tau) \left(\chi^i + \theta\, b^i\right) \right. \tag{143}$$
$$\left. + \frac{i}{3!} C_{ijk} \left(\chi^i + \theta\, b^i\right) \left(\chi^j + \theta\, b^j\right) \left(\chi^k + \theta\, b^k\right) \right] .$$

Then we can complete the Grassmann integral to obtain the same result as in [21]:

$$\mathcal{L} = \frac{1}{2} \chi^i \partial_\tau \chi^i - \frac{1}{2} b^i b^i + \frac{i}{2} C_{ijk} b^i \chi^j \chi^k \,. \tag{144}$$

We see that (as expected) the equation of motion for b^i is algebraic:

$$b_i = \frac{i}{2} C_{ijk} \chi^j \chi^k \,. \tag{145}$$

And since the Lagrangian is quadratic in this non-dynamical field b we can substitute the equation of motion back into the action (Lagrangian). This is because for example in the path integral formalism the b integral is then simply Gaussian and reproduces the classical result.

When we plug (145) into the Lagrangian, (144), we see indeed that the second and third term yield the four fermion interaction as seen in (136). Hence the dynamics described by this Lagrangian indeed reproduce those of the found Hamiltonian.
In order to get all constants exactly the same one would have to change the overall sign and add a term $\frac{1}{4!} C_{ijk}^2$, but neither changes the equations of motions and hence we ignore it here.

Let us now check the actual supersymmetry invariance of the Lagrangian in (144) using the supersymmetry transformations (141):

$$\delta_\epsilon \mathcal{L} = \epsilon\, \partial_\tau \left(-\frac{1}{2} \chi^i\, b^i + \frac{i}{3!}\, C_{ijk}\, \chi^i \chi^j \chi^k \right) + i\, C_{ijk}\, b^i\, b^j\, \chi^k \,. \tag{146}$$

The last term is zero by the total asymmetry of C_{ijk} and hence we see that the Lagrangian transforms as a total derivative. We will now continue to summarize some notable features of the supersymmetric model analogous to normal SYK. The rest of this chapter will however not contain any explicit calculations since the rest of the thesis focuses on 'normal' SYK and bulk models related to it.

4.2 EFFECTIVE ACTION, SYMMETRIES AND SUPERSCHWARZIAN

4.2.1 *Effective Action and Schwarzian*

In an analogous process as we followed in Section 2.4 one can integrate out the disorder and derive an effective action for the supersymmetric SYK model [21]:

$$Z = \int \mathcal{D}G_{\psi\psi} \mathcal{D}\Sigma_{\psi\psi} \mathcal{D}G_{bb} \mathcal{D}\Sigma_{bb}\, e^{-S_{eff}} \,, \tag{147}$$

$$\begin{aligned} \frac{S_{eff}}{N} &= -\frac{1}{2} \log \det \left[\partial_\tau - \Sigma_{\psi\psi} \right] + \frac{1}{2} \log \det \left[-1 - \Sigma_{bb} \right] + \\ &+ \frac{1}{2} \int d\tau_1\, d\tau_2\, \big[\Sigma_{\psi\psi}(\tau_1,\tau_2)\, G_{\psi\psi}(\tau_1,\tau_2) + \Sigma_{bb}(\tau_1,\tau_2)\, G_{bb}(\tau_1,\tau_2) - \\ &\qquad - J\, G_{bb}(\tau_1,\tau_2)\, G_{\psi\psi}(\tau_1,\tau_2)^2 \big] \,. \end{aligned} \tag{148}$$

Here, just as in (73), the Σ and G are introduced by Lagrange multipliers. Here we have of course two types of them due to the appearance of both a bosonic b and a fermionic ψ. Note also the similar kinetic terms, the lack of a ∂_τ term for b is due to it's lack of dynamics.

One may also write the effective action in superspace formalism. Recall that in the SYK case the Schwarzian action arose by considering fluctuations around the large N saddle point solutions. If one generalizes this process to the supersymmetric variant we obtain the so called Superschwarzian [21]:

$$\int d\tau\, d\theta\, S[\tau', \theta'; \tau, \theta] \,, \tag{149}$$

$$S[\tau', \theta'; \tau, \theta] = \frac{D^4\theta'}{D\theta'} - 2\, \frac{D^3\theta'\, D^2\theta'}{(D\theta')^2} \,, \tag{150}$$

where the D is the covariant derivative with respect to τ and θ as in (139).

4.2.2 *Symmetries*

The model has a symmetry breaking pattern analogous to the non supersymmetric SYK model. However, apart from the $\mathrm{Diff}(\mathbb{R})$ there is now also a supersymmetry. The transformations hence include the also the above introduced θ:

$$\tau \mapsto \tau'(\tau, \theta) \, , \tag{151}$$

$$\theta \mapsto \theta'(\tau, \theta) \, . \tag{152}$$

Together these make up the so called SDiff group. The bosonic part of these transformations is simply the group $\mathrm{Diff}(\mathbb{R})$:

$$\tau \mapsto \tau' = f(\tau) \, , \tag{153}$$

$$\theta \mapsto \theta' = \sqrt{\partial_\tau} \theta \, . \tag{154}$$

The first line is the symmetry as we had it in the original SYK model, the second line shows it's action on θ.

Similarly to SYK one might now expect the full symmetry to broken by the solutions, which it indeed does. As before there will be a manifest $SL(2, \mathbb{R})$ symmetry but in addition to this there will be two fermionic generators. So we need a graded algebra extension of $SL(2, \mathbb{R})$ with two fermionic generators: $OSp(1|2)$. So there is now a global superconformal group as residual symmetry.

4.3 CONCLUSIONS

The supersymmetric version of SYK has many features that generalize those of the original model. Of course there are also options to include $\mathcal{N} = 2$ supersymmetry, compute a Witten index (in the $\mathcal{N} = 2$ case) and compute the four point functions [21]. For elaboration on the results above and a thorough introduction see [21].

Apart from the first paper discussing the supersymmetry [21], many more have appeared. In [29] the notions above are extended to two dimensions and discussions of problems when doing so. The article [30] discusses a bi-local collective Superfield theory for both $\mathcal{N} = 1, 2$ supersymmetric SYK models. Lastly, in [31] the four point function in $\mathcal{N} = 2$ supersymmetric SYK is computed. In the next part, discussing bulk models, we will also shortly mention a supersymmetric model of the bulk. This would then be the analogue of the bulk model to this supersymmetric version of SYK.

5 TENSOR MODELS

In this section we will give a very short overview of the so called SYK tensor models. Tensor models were already studied before SYK, a useful reference is [32] for an overview. The application to SYK was noted by Witten [33] and further expanded by Gurau [34]. Afterwards these tensor models got a lot more attention and were expanded elaborately [35-41]. The main reason for considering these models is that we no longer have a disorder average.

Here we will only discuss the most rudimentary version and for more details we refer to the literature. First we will discuss the fields in the model and the action of the model. Afterwards we mention why these models may prove useful for finding a bulk dual.

5.1 INTRODUCTION

In the SYK tensor model we have $q \equiv D+1$ real fermionic (i.e. Majorana) fields $\psi^0, \dots, \psi^D$, where each of these has n^D real components [33]. Hence in total we have $N \equiv (D+1)\, n^D$ Majorana fields. Later on we will consider the large N, or equivalently the large n limit. The upper index $a \in \{0, \dots, D\}$ of the ψ^a is called the color of the field ψ^a. We will first discuss the symmetry group of the model.

5.1.1 *Group Structure*

We introduce for each of the $D+1$ real fields the Lie group $O(n)$. Then we demand that for every a the field ψ^a transforms in a real irreducible representation of $O(n)^D$.
In particular we demand it is in a vector representation, i.e. ψ^a transforms as the tensor product of vector representations of the individual $O(n)$ groups. Since the vector representation of $O(n)$ is n-dimensional it follows that indeed ψ^a has n^D real components.

Recall that every irrep of a direct product is isomorphic to the tensor product of irreps of the individual groups

Hence we conclude that every field ψ^a is a tensor with D indices, each of which can take n values.

We can write it more precisely as follows: let $a, b \in \{0, 1, \dots, D\}$ be distinct elements. For each such pair we introduce the group $G^{ab} = O(n)$. Note that a, b are unordered such that there is no difference between G^{ab} and G^{ba}. We then note that:

$$G_0 \equiv \prod_{a<b} G^{ab} \cong O(N)^{D(D+1)/2} \,, \tag{155}$$

where the power denotes direct products of the group. Then we demand that for every a: ψ^a transforms as an irreducible real representation G^{ab} for every $b \neq a$ and trivially under every G^{bc} with $b, c \neq a$. So indeed we have that the ψ fields transform as vectors under an $O(n)$ group as stated above.

Lastly we can find the faithfully acting total symmetry group. Using that the center of $O(n)$ is $\mathbb{Z}_2$ (which acts by a sign change on the vector representation) we find that the center of the above defined G_0 is

$$Z(G_0) = \mathbb{Z}_2^{D(D+1)/2} \,. \tag{156}$$

So then we can conclude that there is a certain subgroup of this center that acts trivially on all of the ψ fields. We have to subtract the number of times the $\mathbb{Z}_2$ acts non trivially, i.e. $\frac{D(D+1)}{2} - (D+1)$. This yields the faithfully acting symmetry group of the model:

$$G_0/\mathbb{Z}^{(D-2)(D+1)/2} \,. \tag{157}$$

5.1.2 *The Action*

Here we will follow the notation of [34]: we denote these indices of the fields ψ^c with a tuple $\mathbf{a}^c = \left(a^{cc'} \mid c' \in \mathcal{C} \setminus \{c\}\right)$. Here c denotes an index of ψ^c, $\mathcal{C} = \{0, \ldots, D\}$ and $a^{cc'}$ can take values 1 to n. The action proposed in [33] can then be written as:

$$S = \int d\tau \left(\left(\frac{1}{2} \sum_{c} \sum_{\mathbf{a}^c} \psi^c_{\mathbf{a}^c} \frac{d}{d\tau} \psi^c_{\mathbf{a}^c} \right) - \right. \tag{158}$$
$$\left. i^{q/2} j \sum_{\mathbf{a}^0, \ldots, \mathbf{a}^D} \psi^0_{\mathbf{a}^0} \cdots \psi^D_{\mathbf{a}^D} \prod_{c_1 < c_2} \delta_{a^{c_1 c_2}\, a^{c_2 c_1}} \right) \,,$$

where the factors of i make sure the Hamiltonian is hermitian and j is the (real) coupling (which we discuss more in the next section). The deltas at the end are introduced such that the total resulting term will be invariant under the above described symmetry group.
This can be explained as follows: for each G^{ab} exactly two of the fields transform as vectors under G^{ab} (namely ψ^a and ψ^b). For these two fields we contract the tensor indices corresponding to this particular G^{ab}. Continuing this for each pair yields a G-invariant term that can be written as above.

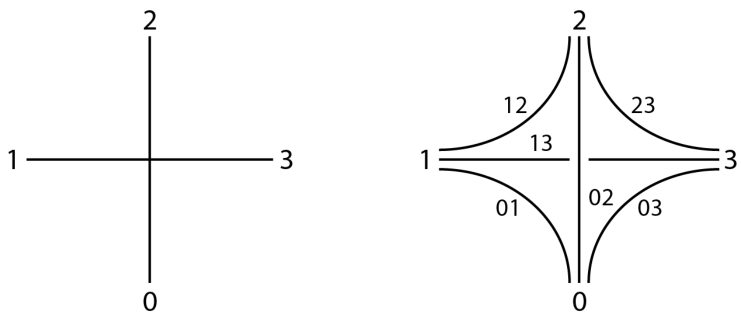

Figure 9: The left figure shows the four point vertex in the 'standard' way of drawing. The right figure includes the strands of each of the ψ fields. In our case we have $q = 4$ such that each ψ transforms non trivially under three copies of $O(n)$. The right hand figure shows this by including all the non trivial transformations, e.g. ψ^0 transforms as a vector under $G^{01} \times G^{02} \times G^{03}$.

5.2 DIAGRAMMATIC STRUCTURE

Let us now take $q = 4$ or equivalently $D = 3$ in order to compare with the most used version of SYK that also has $q = 4$ fields. We see then that there are now four Majorana fields $\psi^0, \dots, \psi^3$ in the model. Diagrammatically this means that we have one vertex; a four point function, see Fig 9.

The fields transform under three copies of $O(n)$, e.g. ψ^0 transforms in a vector representation of $G^{01} \times G^{02} \times G^{03}$. So the fields have thus three indices which can take n values. To explicitly show this diagrammatically for the four point vertex we follow [33] and draw the 'strands' of the fields, see again Fig 9.

When these strands make loops we will obtain a factor of n due to the n possible $O(n)$ indices each strand can take. Since each strand of type a, b has $a, b \in \{0, \dots, D\}$ it is obvious that we have $\frac{D(D+1)}{2}$ types of strand possible. In order to extract useful information about the large n (or N) limit we introduce the terminology [33]: let $\mathcal{F}_{ab}$ denote the number of strands of type a, b and $\mathcal{F} \equiv \sum_{a<b} \mathcal{F}_{ab}$.
Hence the sum over all loops in strands yields the following n factor:

$$n^{\mathcal{F}} = \prod_{a<b} n^{\mathcal{F}_{ab}} \,. \tag{159}$$

Let us consider the classical melonic diagram, Fig 10, with incoming (and hence outgoing) ψ^0. Using the above counting argument we see that we can make loops with strands 12, 13 and 23. Hence we find that $\mathcal{F} \equiv \sum_{a<b} \mathcal{F}_{ab} = 3$. Thus the diagram is proportional to $j^2 n^3$,

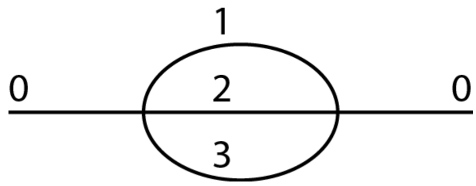

Figure 10: This is the 'classical' melonic diagram as it is also found in SYK. Note however that we have suppressed the strands in the figure for clarity. As one can check this diagram is proportional to $j^2 n^3 = J^2$. As in SYK, these melonic diagrams thus survive the large n limit and other (non melonic) diagrams will not.

with j the coupling of the vertex in the action.

In order to reproduce the SYK structure we thus conclude that $j = \frac{J}{n^{3/2}}$ with J some constant. This will mean that the melonic diagram, and hence all the generalizations (as seen in Fig 3) will survive the large n (or N) limit. Other diagrams will, just as in SYK, be proportional to some power of $\frac{1}{N}$ and drop out. Hence we are led to conclude that this *SYK-tensor model reproduces the diagrammatic structure of the original SYK model*. The same conclusions hold of course for the case of arbitrary $q \neq 4$. For a full discussion and derivation of the complete $\frac{1}{N}$ structure of this tensor model see [34].

5.3 CONCLUSIONS

Let us recall the discussion in Chapter 3 about the necessity of the disorder average. In that chapter we chose an arbitrary ensemble rather than a Gaussian one. Similarly as in this chapter we fixed the coupling constant such that we exactly reproduced the (leading order) diagrammatic structure of SYK.

We then raised the question; how important is the disorder average for obtaining the diagrammatic structure in SYK. As it turns out we do not need the disorder average at all to reproduce it, as is witnessed in this chapter.

This main reason for the interest in the tensor models is this lack of a disorder average. The reason is that the disorder average has no clear interpretation in the usual AdS / CFT correspondence. In particular the disorder averaged system is not really a quantum system (due to the lack of dynamics in the J_{ijkl} 'field'). This may lead to some difficulties in understanding black holes [33]. So the hope is that these tensor models will have a more clear interpretation in the holographic dual.

There are, however, similar issues as those we mentioned in Chapter 3. As of yet, there is no mention of any effective action describing this model in contrast to the bilocal effective action of SYK. Hence there is also no Schwarzian action. Both of these can be considered 'Hallmark' features of the SYK model and one might want these to be reproduced in any SYK-like model.

Part II

BULK MODELS

This part will consider the bulk models related to SYK models. In particular we will discuss the so called Almheiri-Polchinski model. The model introduces corrections on AdS_2 by dynamical dilaton gravity. We will discuss the equations of motion, solutions (in particular black holes) and the emergence of the Schwarzian action within this theory. Afterwards we will shortly mention other models relevant to SYK such as the explicit construction of the bulk [26]. Then we end the part with conclusions and a discussion of the most important results.

6 ADS$_2$ AND THE ALMHEIRI-POLCHINSKI MODEL

6.1 INTRODUCTION

In this section we will discuss a bulk model displaying many characteristic features of SYK. It is a 'nearly' AdS$_2$ space, with 'nearly' arising from the fact that we will include the leading correction on the AdS$_2$ geometry. This will be achieved by incorporating a dilaton field that explicitly breaks the symmetry.

When in the IR limit, the model will still correspond to pure AdS$_2$. We will see indeed that for large enough z (in Poincare coordinates) the dilaton will behave as a constant. However in the UV limit there will be a nontrivial dilaton profile regulating the gravitational backreaction.

6.1.1 *The action*

The model was first proposed by Jackiw [23] and Teitelboim [24] and hence is also known as the Jackiw-Teitelboim model. More recently Almheiri and Polchinski [8] also investigated this model.

Let us begin by first stating the Einstein-Hilbert action for pure AdS$_2$ [26]:

$$S_{EH} = \frac{1}{16\pi G}\left[\int_M d^2x\,\sqrt{-g}\,R + 2\int_{\partial M} dt\,\sqrt{-\gamma}\,K\right], \quad (160)$$

with R the Ricci scalar and K the extrinsic curvature. One of the main problems with pure AdS$_2$ (or in fact AdS$_2$ $\times$ X with X compact) is that the backreaction in the model is so strong that there can't exist any finite energy excitations [42,43]. Obviously, a model with features similar to SYK needs to allow finite energy states. In order to accomplish this we will introduce the leading order correction on AdS$_2$ geometry such that the UV geometry will regulate the backreaction and hence allow finite energy states to exist.

A model that correctly captures a large amount of situations in which AdS$_2$ arises from some higher dimensional geometry is the model studied by Almheiri and Polchinski. In addition to the (topological) action above, (160), we get the family of 1+1 dimensional models of dilaton gravity:

$$S = \frac{1}{16\pi G}\int d^2x\,\sqrt{-g}\left(\phi^2 R + \lambda(\nabla\phi)^2 - U_0(\phi)\right) + S_{\mathrm{matter}} \quad (161)$$

They are characterized by the value of λ and the potential $U_0(\phi)$. Note that ϕ^2 denotes the dilaton (we follow the notation of [8]) that yields the dynamics in 2D gravity. The matter action is described by the following matter fields Lagrangian [8]:

$$L_f = \frac{1}{32\pi G}\sqrt{-g}\,\Omega(\phi)\,(\nabla f)^2\,. \quad (162)$$

We will however focus on the case where $\Omega = 1$, as we will implement later on.

By action of a Weyl transformation $g_{ab} \mapsto g_{ab}\,\phi^{-\alpha/2}$ the action (161) changes by taking $\lambda \mapsto \lambda - \alpha$ and $U(\phi) \mapsto \phi^{-\alpha/2}\,U(\phi)$. Hence we can take without loss of generality $\lambda = 0$ and obtain:

$$\begin{aligned} S = \tfrac{1}{16\pi G}\int d^2x\,\sqrt{-g}\left(\phi^2 R - U_0(\phi) - \tfrac{\Omega(\phi)}{2}(\nabla f)^2\right) \\ + \tfrac{1}{8\pi G}\int dt\,\sqrt{-\gamma}\,\phi^2 K\,, \end{aligned} \quad (163)$$

where we also introduced the boundary term involving the extrinsic curvature K. This is introduced such that there are no boundary contributions to δS when varying with Dirichlet boundary conditions (the Gibbons Hawking York term). This action above will be the starting point of the investigation into the model. Note that ϕ^2 will still have a kinetic term originating from $\phi^2 R$.

We will ignore any higher order corrections to the pure AdS_2. Also, since (160) is topological, we will ignore this part of the total action. All dynamics and characteristic features are in (163). The model is then characterized by specifying the potential U_0.
As we will explain below, Section 6.3, the AP model will have the potential:

$$U_0(\phi) = 2 - 2\,\phi^2\,. \quad (164)$$

6.1.2 *Related models*

Before we continue investigating this particular model let us shortly mention several closely related models. These models also have the same action, (161), but different values of the parameters. Firstly there is the CGHS Model [44]. This corresponds to picking $\lambda = 4$ and $U(\phi) = -A\,\phi^2$ $(A > 0)$ in the action [8]. This is a renormalizable theory of quantum gravity coupled to a dilaton and matter in two dimensions. It allows for black hole formation, singularities and also admits Hawking radiation.

A second class of models are related to AdS_2 magnetic brane solutions [45-47]. For a discussion of how to incorporate magnetic fields in AdS/CFT in the first place see [48].

It arises as the bulk theory outside a stack of M2 branes subjected to a constant magnetic field. The geometry then interpolates between AdS_4 at high energies and $AdS_2 \times \mathbb{R}^2$ at low energies. These magnetic brane solutions correspond to taking $\lambda = 2$ and $U(\phi) = \frac{B^2}{\phi^2} - A\phi^2$ $(A > 0)$ [8]. The two dimensional model is obtained by dimensional reduction of the metric:

$$ds_4^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + \phi^2\left(dy_1^2 + dy_1^2\right) . \tag{165}$$

6.1.3 *Symmetries*

Let us now continue with the AP model and shortly discuss the symmetries of asymptotic, pure and the explicitly broken AdS_2. This is important as it will display a pattern exactly as we found it in the ordinary SYK model.
Let us for now consider only pure AdS_2, (160), without any explicit corrections.

Firstly we consider asymptotic AdS_2, in which case there is a complete reparametrization symmetry $t \mapsto t'(t)$ [49-51]. It can also be understood by considering a corresponding boundary theory [26]: since there is scale symmetry the stress tensor should be traceless. However, the stress energy tensor has only one component and hence it is zero everywhere. This indeed implies the full reparametrization symmetry.
More explicitly one can consider these asymptotic reparametrizations generated by $\zeta^t = \epsilon(t)$ and $\zeta^z = z\epsilon'(t)$. When plugged into the action the Gauss-Bonnet theorem implies that we always get the same action [26].

The full symmetry group is, however, spontaneously broken to a global $SL(2,\mathbb{R})$. This happens because it is only a asymptotic symmetry and the geometry of AdS_2 is only explicitly invariant under this subgroup. This can be seen by considering the (embedded) metric:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 , \tag{166}$$

which clearly has a $SO(1,2)$ symmetry. We then use that $SO(1,2) \cong SL(2,\mathbb{R})/\mathbb{Z}_2$. Following our usual convention we will not continue to write the quotient over $\mathbb{Z}_2$.

Lastly, let us consider the case where we include the corrections on the AdS$_2$ geometry. Obviously this explicitly breaks the symmetry of the system. We see thus a similar pattern as we saw in SYK: spontaneous breaking of full reparametrization ($\mathrm{Diff}(\mathbb{R})$) to $SL(2,\mathbb{R})$ and explicit breaking by non conformal corrections.
Just as in SYK the spontaneous breaking will lead to Goldstone modes. Once again these modes will correspond to the breaking of the coset $\mathrm{Diff}(\mathbb{R})/SL(2,\mathbb{R})$. In this case the coset structure followed because the reparametrizations of asymptotic AdS$_2$ are only different up to $SL(2,\mathbb{R})$ transformations.

6.1.4 *Conformal Gauge*

So we can now start to investigate the details of the model. In particular we want to derive the equations of motion. This will be done most easily by using the conformal gauge. In this section we shortly review it. In this gauge the metric reduces to:

$$ds^2 = -e^{2\omega(x^+,x^-)}\, dx^+ dx^- \,, \tag{167}$$

which yields also that $\sqrt{-g} = \frac{1}{2}\, e^{2\omega}$. The $SL(2,\mathbb{R})$ symmetry is also manifestly visible since the metric is invariant under:

$$x^\pm \mapsto \frac{a\, x^\pm + b}{c\, x^\pm + d} \qquad \mathit{with} \quad ad - bc = 1\,. \tag{168}$$

Let us also recap the nonzero components of the Christoffel Symbols:

$$\begin{aligned} \Gamma^+_{++} &= 2\,\partial_+\omega\,, \\ \Gamma^-_{--} &= 2\,\partial_-\omega\,. \end{aligned} \tag{169}$$

This leads to the following components of the Ricci tensor:

$$R_{+-} = R_{-+} = -2\,\partial_-\partial_+\omega\,,$$

and hence also the Ricci scalar as:

$$R = 8\, e^{-2\omega}\, \partial_-\partial_+\omega\,. \tag{170}$$

6.2 EQUATIONS OF MOTION

6.2.1 *Dynamical fields*

Let us now consider the equations of motion that follow from this model in the conformal gauge. The constraint equations will be obtained by varying the original action (163) with respect to the gauged components g^{++} and g^{--}.
To derive the equations of motion for the dynamical fields, however, we simply plug the conformal gauge into the action to obtain:

$$S = \frac{1}{16\pi G}\int d^2x \left(4\,\phi^2\,\partial_-\partial_+\omega - \frac{e^{2\omega}}{2}\,U_0(\phi) + \Omega\,\partial_+ f \partial_- f\right)\,. \quad (171)$$

We use the Euler Lagrange equations for f, ω and ϕ^2 respectively to obtain:

$$\frac{1}{\Omega}\,\partial_+(\Omega\,\partial_-)\,f + \frac{1}{\Omega}\,\partial_-(\Omega\,\partial_+)\,f = 0\,, \quad (172)$$

$$4\,\partial_+\partial_-\phi^2 - e^{2\omega}\,U_0(\phi) = 0\,, \quad (173)$$

$$4\partial_+\partial_-\omega - \frac{e^{2\omega}}{2}\,\partial_{\phi^2}U_0(\phi) - \partial_{\phi^2}\Omega(\phi)\,\partial_+ f\,\partial_- f = 0\,. \quad (174)$$

6.2.2 *Constraint equations*

Now for the constraint equations; let us ignore the prefactors and the matter part of the action for now and vary with respect to the metric:

$$\delta S = \int d^2x\,\sqrt{-g}\quad \left(-\tfrac{1}{2}\,g_{\mu\nu}\left(\phi^2\,R - U_0(\phi)\right)\,\delta g^{\mu\nu} + \phi^2\,R_{\mu\nu}\,\delta g^{\mu\nu} + \phi^2\,g_{\mu\nu}\,\delta R^{\mu\nu}\,\right)\,. \quad (175)$$

For the g^{++} and g^{--} components it is clear that only the last term will contribute and hence we need the standard result:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda\,\delta\Gamma^{\rho}_{\nu\mu} - \nabla_\nu\,\delta\Gamma^{\rho}_{\lambda\mu}\,. \quad (176)$$

Using this and the metric compatibility of the Levi-Civita connection yields:

$$\delta S = \int d^2x\,\sqrt{-g}\,\phi^2\,\nabla_\sigma\left(g^{\mu\nu}\,\delta\Gamma^{\sigma}_{\mu\nu} - g^{\mu\sigma}\,\delta\Gamma^{\lambda}_{\lambda\mu}\right)\,. \quad (177)$$

We can now partially integrate and set the boundary term to zero (it is not important for the equation of motion). Note also that the covariant derivative for any scalar field $\phi : M \to \mathbb{R}$ (M is of course some smooth manifold) reduces to the ordinary partial derivative. This yields:

$$\delta S = -\int d^2x \sqrt{-g}\, (\partial_\sigma \phi^2) \left(g^{\mu\nu}\, \delta\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\, \delta\Gamma^\lambda_{\lambda\mu} \right) . \tag{178}$$

Now we need to use the variation of the connection:

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2} \left(g_{\lambda\mu}\, \nabla_\nu (\delta g^{\lambda\sigma}) + g_{\lambda\nu}\, \nabla_\mu (\delta g^{\lambda\sigma}) - g_{\mu\alpha} g_{\nu\beta}\, \nabla^\sigma (\delta g^{\alpha\beta}) \right) \tag{179}$$

With this and the results for the conformal gauge (in particular the nonzero Christoffel symbols) in Section 6.1.4 we can compute the variation (178). Let us do the computation for δg^{++}, the computation for δg^{--} is completely analogous. Some useful relations are:

$$\nabla^+ \delta g^{++} = -2\, e^{-2\omega}\, \partial_-\, \delta g^{++} , \tag{180}$$

$$\nabla^- \delta g^{++} = -2\, e^{-2\omega}\, \partial_+\, \delta g^{++} - 8\, e^{-2\omega}\, (\partial_+ \omega)\, \delta g^{++} . \tag{181}$$

Plugging these two and (179) into (178) we obtain:

$$\delta S = -\frac{1}{2} \int d^2x\, \left[\partial_+ \left(e^{2\omega}\, \partial_+ \phi^2 \right) - 4 e^{2\omega}\, (\partial_+ \omega) \left(\partial_+ \phi^2 \right) \right] \delta g^{++} . \tag{182}$$

The two terms above can be combined into one as follows:

$$\frac{1}{2} \left[\partial_+ \left(e^{2\omega}\, \partial_+ \phi^2 \right) - 4 e^{2\omega}\, (\partial_+ \omega) \left(\partial_+ \phi^2 \right) \right] = \sqrt{-g}\, \left[e^{2\omega} \partial_+ \left(e^{-2\omega}\, \partial_+ \phi^2 \right) \right] . \tag{183}$$

We plug this in, reintroduce the correct prefactor ($\frac{1}{16\pi G}$) and add the matter action and obtain the final result for δg^{++}:

$$\delta S = -\frac{1}{2} \int d^2x \sqrt{-g} \left(\frac{1}{8\pi G}\, e^{2\omega}\, \partial_+ \left(e^{-2\omega}\, \partial_+ \phi^2 \right) + T_{++} \right) \delta g^{++} \tag{184}$$

where we used the definition of the energy momentum tensor as:

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$

Filling in the expression for the energy momentum tensor (which follows from the matter Lagrangian (162)), the constraint equation becomes:

$$-e^{2\omega}\, \partial_+ \left(e^{-2\omega}\, \partial_+ \phi^2 \right) = \frac{\Omega}{2}\, \partial_+ f\, \partial_+ f\, . \tag{185}$$

If we repeat all of the above steps for δg^{--} we obtain a very similar second constraint equation:

$$-e^{2\omega}\,\partial_-\left(e^{-2\omega}\,\partial_-\phi^2\right) = \frac{\Omega}{2}\,\partial_- f\,\partial_- f\,. \tag{186}$$

These two constraint equations together with (172), (173) and (174) give all the equations of motion in the model.

Let us now rewrite them for the case $\Omega = 1$ which corresponds to stating that the matter is independent of the dilaton. This will from now on always be the case. Then for an arbitrary energy momentum tensor the equations of motion become (note we don't have the equation of motion for f anymore):

$$4\partial_+\partial_-\omega - \frac{e^{2\omega}}{2}\,\partial_{\phi^2}U_0(\phi) = 0\,, \tag{187}$$
$$4\,\partial_+\partial_-\phi^2 - e^{2\omega}\,U_0(\phi) = 16\,\pi\,G\,T_{+-}\,, \tag{188}$$
$$-e^{2\omega}\,\partial_-\left(e^{-2\omega}\,\partial_-\phi^2\right) = 8\,\pi\,G\,T_{--}\,, \tag{189}$$
$$-e^{2\omega}\,\partial_+\left(e^{-2\omega}\,\partial_+\phi^2\right) = 8\,\pi\,G\,T_{++}\,. \tag{190}$$

Which is a similar result as obtained in [8]. Let us now find solutions for these equations of motion.

6.3 VACUUM SOLUTIONS

Before we can start imposing conditions and solving the equations we need to pick a certain potential $U_0(\phi)$. Considering the EOM for ω in (187) we see it would be quite fortunate to have $\partial_{\phi^2}U_0(\phi) \in \mathbb{R}$. This follows because then the ω decouples from ϕ^2 and describes a space of constant curvature. This will allow us to explicitly solve the model analytically. Of course this also forces the background metric to be AdS_2 regardless of the presence of matter.

These are the main reasons why the choice $U_0(\phi) = C - A\,\phi^2$ with $A, C \in \mathbb{R}^+$ is convenient [8]. In particular by suitably rescaling the fields and coordinates we can pick $U_0(\phi) = 2 - 2\phi^2$ that are convenient for calculations.

6.3.1 *Static Vacuum Solutions*

Let us start by considering vacuum solutions that are independent of the time t. This yields the equations:

$$2\,\partial_z^2\omega + e^{2\omega}\,\partial_{\phi^2}U_0(\phi) = 0\,, \tag{191}$$
$$\partial_z^2(\phi^2) + e^{2\omega}\,U_0(\phi) = 0\,, \tag{192}$$
$$\partial_z\left(e^{-2\omega}\,\partial_z\phi^2\right) = 0\,. \tag{193}$$

If we furthermore impose for the moment that $\phi(z) = \phi_c \in \mathbb{R}$ it follows from the above equations:

$$U_0(\phi_c) = 0\,, \tag{194}$$
$$2\,\partial_z^2\omega = -e^{2\omega}\,\partial_{\phi^2}U_0(\phi_C)\,. \tag{195}$$

Using $U_0(\phi) = 2-2\phi^2$ it follows that $\phi_c = 1$ and also $\partial_{\phi^2}U_0(\phi_c) = -2$. This yields the three static solutions for the spacetime:

$$e^{2\omega} = \frac{1}{z^2} \quad \text{or} \quad \frac{1}{\sin^2(z)} \quad \text{or} \quad \frac{1}{\sinh^2(z)} \tag{196}$$

Which represent AdS_2 (in Poincare coordinates), a Rindler subspace of the Poincare patch (or equivalently a black hole with event horizon at $z = \infty$) and AdS_2 in global coordinates. One way or another it shows us that when ϕ^2 approaches a constant we obtain a pure AdS_2 spacetime.

When we now assume that ϕ does depend on z we can integrate (193) to find:

$$\partial_z(\phi^2) = c_1\,e^{2\omega}\,. \tag{197}$$

We then follow [8] by introducing the pre-potential $U_0(\phi) = \partial_{\phi^2}W(\phi)$. Then we can rewrite (192):

$$c_1\,\partial_z^2(\phi^2) + \partial_z(\phi^2)\,\partial_{\phi^2}W(\phi) = \frac{d}{dz}\left(c_1\,\partial_z(\phi^2) + W(\phi)\right) = 0\,. \tag{198}$$

This can be integrated and rewritten to yield:

$$dz = c_1\frac{d(\phi^2)}{c_2 - W(\phi)}\,. \tag{199}$$

Using that $W(\phi) = 2\phi^2 - \phi^4$ for our chosen potential we can integrate the above equation for large ϕ. This yields us that

$$\phi^2 \propto \frac{1}{z}\,. \tag{200}$$

This shows us the $z \to 0$ behaviour of the dilaton in the static solution. We will see it behaves similarly in the non static case. We can also find by using (197) that:

$$e^{2\omega} \propto \frac{1}{z^2} . \tag{201}$$

To get a more intuitive picture of this behaviour we can lift the geometry back to four dimensions [8]. We recall from the introduction the action of the Weyl transform on the metric and also (165):

$$g_{ab} \mapsto g_{ab}\phi^{-\alpha/2} , \tag{202}$$
$$ds_4^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu + \phi^2(x)\,(dy_1^2 + dy_1^2) . \tag{203}$$

The first equation tells us (since we picked $\lambda = 0$) that we still have to include the z dependence from the $\phi^{\lambda/2}$:

$$e^{2\omega}_{\text{lift}} = e^{2\omega}\, \phi^{\lambda/2} \propto z^{-\frac{(8-\lambda)}{4}} . \tag{204}$$

This would yield:

$$ds_4^2 = \frac{1}{z^{\frac{8-\lambda}{4}}}\left(-dt^2 + dz^2\right) + \frac{1}{z}\left(dy_1^2 + dy_2^2\right) . \tag{205}$$

Now we define for convenience $L = \frac{8}{4+\lambda}$ and take the following collection of coordinate transformations:

$$z = \frac{1}{L}\,\tilde{z}^L , \tag{206}$$
$$t = L^{\frac{\lambda-8}{8}}\,\tilde{t} , \tag{207}$$
$$y_i = L^{-\frac{1}{2}}\,\tilde{y}_i , \tag{208}$$

where the last two simply scale out an irrelevant constant. When we apply these transformations we obtain:

$$ds_4^2 = \tilde{z}^{\frac{2\lambda}{4+\lambda}}\left(\frac{d\tilde{t}^2}{\tilde{z}^{\frac{16}{4+\lambda}}} + \frac{d\tilde{z}^2 + d\tilde{y}_1^2 + d\tilde{y}_1^2}{\tilde{z}^2}\right) . \tag{209}$$

We see that this is a space that is conformal to a Lifshitz spacetime with $z_{dyn} = \frac{8}{4+\lambda}$. Thus we can conclude that for large ϕ or equivalently small z the spacetime is that of conformal Lifshitz. In the next section we will see that for large z the spacetime (so also the dilaton behaviour) will reproduce pure AdS_2.

6.3.2 *Non-static Vacuum Solutions*

Let us now move on to the general solutions of the vacuum equations. The equations become as follows:

$$4\partial_+\partial_-\omega + e^{2\omega} = 0\,, \tag{210}$$
$$2\partial_+\partial_-\,\phi^2 + e^{2\omega}(\phi^2 - 1) = 0\,, \tag{211}$$
$$\partial_+\left(e^{-2\omega}\partial_+\phi^2\right) = 0\,, \tag{212}$$
$$\partial_-\left(e^{-2\omega}\partial_-\phi^2\right) = 0\,. \tag{213}$$

We can solve the equation for ω, (210) by:

$$e^{2\omega} = \frac{4}{(x^+ - x^-)^2}\,, \tag{214}$$

as one may easily check. In subsequent sections we will discuss the most general solution (which is a conformal transformation of it). For the solution of ϕ^2 we will have to work a bit harder.
We start by integrating the constraint equations (212) and (213):

$$\partial_+\phi^2 = \frac{4\,f(x^-)}{(x^+ - x^-)^2}\,, \tag{215}$$
$$\partial_-\phi^2 = \frac{4\,g(x^+)}{(x^+ - x^-)^2}\,, \tag{216}$$

where we used the above solution (214) and $f(x^-)$ and $g(x^+)$ are arbitrary functions. Now we derive both of these equations to x^- and x^+ respectively. We can then plug them into the equation of motion (211) to obtain:

$$\phi^2 = 1 - \frac{2\,\partial_- f(x^-)\,(x^+ - x^-) + 4\,f(x^-)}{(x^+ - x^-)^2}\,, \tag{217}$$
$$\phi^2 = 1 - \frac{2\,\partial_- g(x^+)\,(x^+ - x^-) - 4\,g(x^+)}{(x^+ - x^-)^2}\,. \tag{218}$$

Now we can find a restriction on the functions $f(x^-)$ and $g(x^+)$ by deriving again (215) with respect to x^- and (216) with respect to x^+. This time, however, we simply equate these two since they are equal. It results in:

$$(x^+ - x^-)\left(\partial_- f(x^-) - \partial_+ g(x^+)\right) = -2\left(f(x^-) + g(x^+)\right)\,. \tag{219}$$

Now we equate the x^- derivative of (217) with (216), the x^+ derivative of (218) with (215) and use the above (219) to get the constraint:

$$\partial_+^2 \, g(x^+) = -\,\partial_-^2 \, f(x^-)\,, \tag{220}$$

which shows us that:

$$g(x^+) = c_1 + c_2\, x^+ + c_3 (x^+)^2 \tag{221}$$
$$f(x^-) = c_4 + c_5\, x^- - c_3 (x^-)^2\,. \tag{222}$$

Lastly we equate (217) with (218) with the above functions to get the constraints $c_4 = -c_1$ and $c_5 = -c_2$. This yields us the final solution:

$$\phi^2 = 1 + \frac{a + b\,(x^+ + x^-) + c\, x^+ x^-}{x^+ - x^-}\,, \tag{223}$$

where we renamed the constants as $a = 4\,c_1$, $b = 2\,c_2$ and $c = 4\,c_3$ such that we obtain the same notation as in [8,25,26]. This solution shows us the expected behaviour. For very large z (the IR limit) and $b = c = 0$ (on which we will comment below) we see that it becomes constant and reproduces the pure AdS_2 solutions discussed in the previous section. So the AdS_2 behaviour holds in the neigbourhood of the Poincare horizons.

For small z (the UV limit), however, we have a non trivial dilaton profile proportional to $\frac{1}{z}$. This is where the Lifshitz boundary conditions hold (see (200) and below) and the spacetime will correspond to the conformal Lifshitz space. It is this dilaton profile that will allow us the finite energy excitations in the model. This is in contrast to pure AdS_2 where no finite energy solutions can exist [42].

Now we will first reproduce the solutions as found in [8]. To do this we will use the $SL(2,\mathbb{R})$ symmetry of the background metric to rewrite the solution for ϕ^2 in a more convenient form. However, we will later on in Section 6.7 consider the most general (black hole) solutions. These will generalize the results as found in [8] and have different behaviour.

So let us take the following fractional transformations:

$$x^\pm \mapsto \frac{A\, x^\pm + B}{C\, x^\pm + D}\,, \tag{224}$$

where the parameters obey $AD - BC = 1$. Plugging this into the above (223) we find that the parameters transform as:

$$\begin{aligned} a' &= a\,D^2 + c\,A\,B + c\,B^2\,, \\ b' &= a\,C\,D + c\,A\,B + b(A\,D + B\,C)\,, \\ c' &= a\,C^2 + 2\,b\,A\,C + c\,A^2\,. \end{aligned} \tag{225}$$

From which one may also find the explicit $SL(2, \mathbb{R})$ invariant:

$$b'^2 - a'\,c' = b^2 - a\,c\,. \tag{226}$$

Now, if we specifically pick the following parameters:

$$\begin{aligned} A = &\quad \mp\tfrac{1}{a}\sqrt{a^2 + b^2\,B^2 - a\,B^2\,c}\,, \\ C = &\quad \tfrac{1}{a}\left(\tfrac{b^2\,B}{a} - B\,c \pm \tfrac{1}{a}\sqrt{a^2 + b^2\,B^2 - a\,B^2\,c}\right)\,, \\ D = &\quad \tfrac{1}{a}\left(-b\,B \mp \sqrt{a^2 + b^2\,B^2 - a\,B^2\,c}\right)\,. \end{aligned} \tag{227}$$

Then we would obtain by defining $\mu = -\frac{a\,c - b^2}{a}$:

$$\phi^2 = 1 + \frac{a - \mu\,x^+\,x^-}{x^+ - x^-}\,. \tag{228}$$

Or if we had picked had picked the following parameters:

$$\begin{aligned} A = &\quad \mp\sqrt{a + b^2\,B^2 - a\,B^2\,c}\,, \\ C = &\quad \tfrac{b^2\,B}{a} - B\,c \pm \tfrac{1}{a}\sqrt{a + b^2\,B^2 - a\,B^2\,c}\,, \\ D = &\quad \tfrac{1}{a}\left(-b\,B \mp \sqrt{a + b^2\,B^2 - a\,B^2\,c}\right)\,. \end{aligned} \tag{229}$$

We now define $\mu = -(a\,c - b^2)$ and obtain the following expression:

$$\phi^2 = 1 + \frac{1 - \mu\,x^+\,x^-}{x^+ - x^-}\,. \tag{230}$$

In either case it is obvious that we need $ac - b^2 \neq 0$ to obtain a $\mu \neq 0$. We will see later on that the parameter μ is (modulo $8\pi G$) the mass of a black hole yielding the physical demand $\mu > 0$.

6.4 SINGULARITIES AND GENERAL SOLUTIONS

In this short section we discuss how singularities arise in the model and afterwards we discuss the most general solutions to the equations of motion. The above considered case can be seen as a specific choice of these solutions.

6.4.1 *Singularities*

In this section we will discuss singularities that arise in the model. If we consider the action, (163), we can find that the effective Newton constant is:

$$G_N^{eff} = \frac{G_N}{\phi^2} \,. \tag{231}$$

From this it becomes clear that when $\phi^2 \to 0$ there is a strong coupling singularity in the model [25]. When we are discussing singularities in the sections below this is the singularity we refer to.

6.4.2 *General solutions*

As shortly mentioned in the previous sections: the solutions of $e^{2\omega}$ and ϕ^2 were not the most general ones. In fact (as one may easily check) the following are also solutions:

$$e^{2\omega} = \frac{4\,\partial_+\omega^+(x^+)\,\partial_-\omega^-(x^-)}{(\omega^+(x^+) - \omega^-(x^-))^2} \,, \tag{232}$$

$$\phi^2 = 1 + \frac{a + b\,(\omega^+(x^+) + \omega^-(x^-)) + c\,\omega^+(x^+)\,\omega^-(x^-)}{\omega^+(x^+) - \omega^-(x^-)} \,. \tag{233}$$

The only demand on the functions $\omega^\pm(x^\pm)$ is that they are monotonic. This follows from the fact that $e^{2\omega}$ must be positive. We will see below (end of Section 6.6) that for physical reasons we require $a > 0$.
Note that also in particular the $SL(2,\mathbb{R})$ transformed ϕ^2 of previous section ((228)) has the following generalization:

$$\phi^2 = 1 + \frac{a - \mu\,\omega^+(x^+)\,\omega^-(x^-)}{\omega^+(x^+) - \omega^-(x^-)} \,. \tag{234}$$

For now we will focus on this (simpler) $SL(2,\mathbb{R})$ transformed solution of the equations of motion (as in [8]). We will see it allows already for several interesting spacetimes.

6.5 GLOBAL COORDINATES

Let us now put $\mu = 0$ in (234). Recall that this is the solution that behaves as conformal Lifshitz for small z and AdS_2 for large z. We can rewrite this to global coordinates by taking:

$$\omega^{\pm}(x^{\pm}) = \tan x^{\pm} \,. \tag{235}$$

This yields the following expressions for ϕ^2 and $e^{2\omega}$:

$$\phi^2 = 1 + a\,\frac{\cos\, x^+ \,\cos\, x^-}{\sin(x^+ - x^-)}\,, \tag{236}$$

$$e^{2\omega} = \frac{4}{\sin^2(x^+ - x^-)}\,. \tag{237}$$

These coordinates cover the whole space, so also outside of the Poincare patch. We may now investigate if there exist any singularities in this global vacuum. Let us first determine whether there are singularities within the Poincare patch. Then we would have:

$$\phi^2 = 1 + \frac{a}{x^+ - x^-} \overset{!}{=} 0\,. \tag{238}$$

At the end of next subsection we will see that $a > 0$ (such that the backreaction is regulated). Hence the above equation has no solutions such that there are no singularities within the Poincare patch.

Now, using our global coordinates, we consider the domain outside the Poincare patch and set (236) equal to zero. This yields:

$$a\,\cos x^+\,\cos x^- = -\sin(x^+ - x^-)\,, \tag{239}$$

which clearly has solutions and hence describes a non static singularity. However, we are not dealing with a black hole (see next section) and hence we have a naked singularity. In Fig 11 we sketch how this spacetime appears.

6.6 BLACK HOLES AND BACKREACTION

Here we will discuss black hole solutions within the model. In the first subsection we will study the solution as found at the end of Section 6.3.2:

$$\phi^2 = 1 + \frac{a - \mu\, x^+\, x^-}{x^+ - x^-}\,. \tag{240}$$

There we will study the behaviour of the singularity and the associated event horizon that arise with this solution.

Afterwards we will consider the most general (but $SL(2,\mathbb{R})$ transformed) solution (234). In particular we will construct a Penrose diagram for the spacetime.

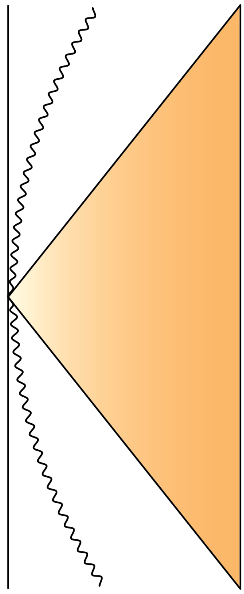

Figure 11: This is the Penrose diagram for the global coordinates as in (236). The coordinate t increases upwards and z increases to the left. The orange shaded part denotes the Poincare patch whilst the wiggly line indicates the naked singularity.

6.6.1 *Singularity*

Let us start with the solution (240) and consider the behaviour of the singularity, for which we set (240) equal to zero:

$$x^+ x^- = \frac{1}{\mu}\left(x^+ - x^-\right) + \frac{a}{\mu} \,. \tag{241}$$

We may also rewrite the equation in terms of t and z coordinates:

$$t^2 = z^2 + \frac{2}{\mu} z + \frac{a}{\mu} \,. \tag{242}$$

We note immediately that for large enough z the evolution will approach $t^2 = z^2$, the null line.

Now we consider the event horizons; these are defined to be the points of no return. More precisely: once an observer crosses this horizon he can never return and will inevitably fall into the singularity. From the above equation we see that for $z = 0$ we have that $t = \pm\sqrt{\frac{a}{\mu}}$.

If we take along these points on the $z = 0$ boundary the lines of $x^\pm$ having the constant values we obtain:

$$x^+ = \sqrt{\frac{a}{\mu}} \qquad \text{Future Horizon}\,, \tag{243}$$

$$x^- = -\sqrt{\frac{a}{\mu}} \qquad \text{Past Horizon}\,. \tag{244}$$

These are the horizons because whenever an observer crosses one of these lines, the singularity is inevitably in it's causal domain. This, in turn, follows due to the fact that the singularity eventually always behaves as $t^2 = z^2$ and starts on the boundary at $t = \pm\sqrt{\frac{a}{\mu}}$.

Now we wish to investigate the evolution of the singularity: is it timelike, null-like or spacelike. To solve this we first solve the equation for z:

$$z(t) = -\frac{1}{\mu} + \sqrt{-\frac{a}{\mu} + \frac{1}{\mu^2} + t^2}\,, \tag{245}$$

where we took the positive solution due to the fact that $z \geqslant 0$ (and $\mu > 0$). Now the requirement that $z \geqslant 0$ yields us that $t^2 \geqslant \frac{a}{\mu}$. We can then derive it with respect to the time such that we obtain:

$$\frac{dz}{dt}(t) = \frac{t}{\sqrt{t^2 - \frac{a}{\mu} + \frac{1}{\mu^2}}}\,. \tag{246}$$

This has different behaviour for the cases $\mu < \frac{1}{a}$ and $\mu > \frac{1}{a}$. Before we start on those consider $\mu = \frac{1}{a}$ which gives that the derivative is simply a sign function. This means that for $t \geqslant \sqrt{\frac{a}{\mu}}$ the singularity evolves as $t = z$ and for $t \leqslant -\sqrt{\frac{a}{\mu}}$ as $z = -t$.

Now for the case $\mu < \frac{1}{a}$ we see that the denominator will never be zero and the resulting function is smooth without singularities, see Fig 12. It becomes obvious that for all t we have $\frac{dz}{dt} < 1$ and hence in this case the evolution is completely timelike.

When $\mu > \frac{1}{a}$, however, we see that at $t^\pm = \sqrt{\frac{a}{\mu} - \frac{1}{\mu^2}}$ the derivative blows up. Fortunately this is not within the physical spacetime as $|t^\pm| < \sqrt{\frac{a}{\mu}}$, see also Fig 12. However when we compute the derivative at these endpoints:

$$\left.\frac{dz}{dt}\right|_{t=\pm\sqrt{\frac{a}{\mu}}} = \pm\sqrt{a\,\mu}\,. \tag{247}$$

So in this case we see that the speed of the singularity is always faster than light ($\mu > \frac{1}{a}$) and hence the evolution is completely spacelike. However, for both cases: at large enough t the derivatives will approach ± 1 and show similar behaviour. Also in the defining equation (242) it can be seen that this limit yields $t^2 = z^2$ and hence the evolution will approach that of a null line eventually.

Lastly, we may consider the acceleration of the singularity:

$$\frac{d^2 z}{dt^2}(t) = \frac{1 - a\,\mu}{(1 - a\,\mu + \mu^2 t^2)\,\sqrt{t^2 + \frac{1 - a\,\mu}{\mu^2}}}\,. \tag{248}$$

For which we find that at the boundary:

$$\left.\frac{d^2 z}{dt^2}\right|_{t=\pm\sqrt{\frac{a}{\mu}}} = \mu(1 - a\,\mu)\,. \tag{249}$$

So, depending on the mass of the black hole $\mu \lessgtr \frac{1}{a}$ it either accelerates or decelerates. It will do this asymptotically until it reaches $\left|\frac{dz}{dt}\right| = 1$.

6.6.2 *Penrose Diagram*

Now we want to create a Penrose diagram for the black hole spacetime. We therefore first consider the generalization of this ϕ^2 solution, which is (234), and take a function that compactifies the spacetime:

$$\omega^{\pm}(x^{\pm}) = \sqrt{\frac{a}{\mu}}\tanh\left(\sqrt{\frac{\mu}{a}}\,x^{\pm}\right)\,, \tag{250}$$

where the constants are chosen for later convenience (in the metric expressions). Note that now $-\sqrt{\frac{a}{\mu}} < \omega^{\pm} < \sqrt{\frac{a}{\mu}}$. The points endpoints of this region correspond with $x^+ - x^- = 2z \to \infty$. These coordinates yields us the explicit static form of the metric and dilaton:

$$\phi^2 = 1 + a\sqrt{\frac{\mu}{a}}\coth\left(\sqrt{\frac{\mu}{a}}\,(x^+ - x^-)\right)\,, \tag{251}$$

$$e^{2\omega} = \frac{4\,\mu}{a\,\sinh^2\left(\sqrt{\frac{\mu}{a}}\,(x^+ - x^-)\right)}\,. \tag{252}$$

Note that ϕ^2 does not go to zero in these coordinates, which means there are no singularities in this region. For this reason we call this region the exterior region. When we expand ϕ^2 near the boundary $x^+ - x^- = 0$ we obtain:

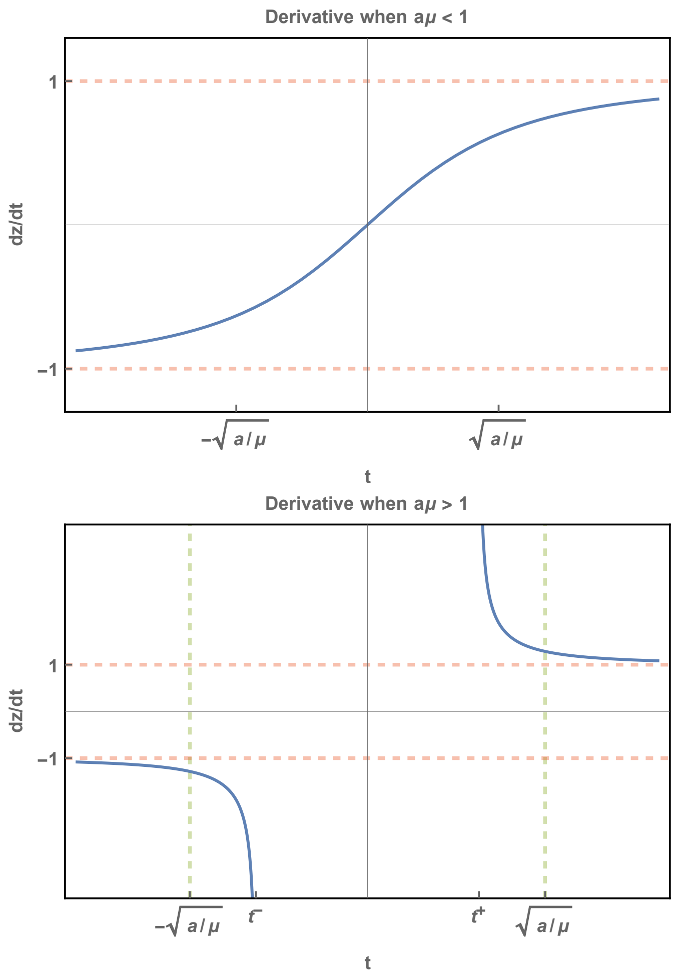

Figure 12: These two figures show the behaviour of the derivative $\frac{dz}{dt}$ for the two cases. The top figure shows the case $\mu < \frac{1}{a}$ and the bottom $\mu > \frac{1}{a}$. For both figures we keep in mind that the behaviour in the range $-\sqrt{\frac{a}{\mu}} < t < \sqrt{\frac{a}{\mu}}$ (in between the green dashed lines) is irrelevant since the singularity will not exist in this part of the spacetime. The red dashed lines shows the asymptotes of the graphs, so for large enough t the derivatives always approach ± 1. In the bottom figure $t^{\pm}$ denote the times for which the derivative blows up; note they lie within the green dashed lines. Lastly, for the top figure it is obvious that always $\frac{dz}{dt} < 1$ whilst for the bottom figure $\left|\frac{dz}{dt}\right|$ is largest (within the relevant region) at $t = \pm\sqrt{\frac{a}{\mu}}$.

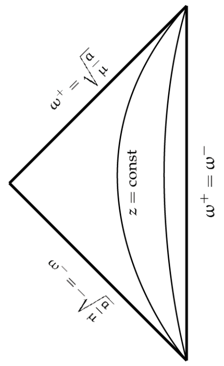

Figure 13: This is the Penrose diagram for the black hole spacetime in the exterior coordinates, (251). The curved lines denote lines of constant z. The vertical line $\omega^+ = \omega^-$ corresponds with $x^+ = x^-$ and hence $z = 0$. The diagonal lines denote the horizons $\omega^{\pm} = \pm\sqrt{\frac{a}{\mu}}$. These correspond with $z = \infty$ or $\rho = \sqrt{\frac{\mu}{a}}$ in the coordinates (257).

$$\phi^2 = 1 + \frac{a}{x^+ - x^-} + \frac{\mu\,(x^+ - x^-)}{3} + \dots \,. \tag{253}$$

This shows us that the dilaton still has the conformal Lifshitz boundary conditions in the exterior region.

To draw the diagram we consider lines of constant z. Using (250) we find:

$$2\,z = \sqrt{\frac{a}{\mu}}\left(\operatorname{arctanh}\left(\sqrt{\frac{\mu}{a}}\omega^+\right) - \operatorname{arctanh}\left(\sqrt{\frac{\mu}{a}}\omega^-\right)\right) \,. \tag{254}$$

Using this we can draw the Penrose diagram of the spacetime, see Fig 13.

Let us now make another transformation (following [8]) by:

$$\rho = \sqrt{\frac{\mu}{a}}\coth\left(\sqrt{\frac{\mu}{a}}\,(x^+ - x^-)\right) \,. \tag{255}$$

Now we rewrite $x^{\pm} = t \pm z$ and use:

$$d\rho^2 = \frac{4\,\mu^2}{a^2\,\sinh^4\left(\sqrt{\frac{\mu}{a}}\,2\,z\right)} \,. \tag{256}$$

To finally obtain the following expression for the metric and dilaton:

$$ds^2 = -\left(\rho^2 - \frac{\mu}{a}\right)\,d\tilde{t}^2 + \frac{d\rho^2}{\rho^2 - \frac{\mu}{a}} \tag{257}$$

$$\phi^2 = 1 + a\,\rho \,, \tag{258}$$

where $\tilde{t} = 2t$. This metric is the Schwarzschild metric for the spacetime except that in our case there is no singularity in the metric itself (it arises in the dilaton behaviour).

We can now find the horizons of the spacetime by using that $g_{tt} = 0$ at the horizon. This yields $\rho^2 = \frac{\mu}{a}$ such that $\rho = \sqrt{\frac{\mu}{a}}$ since $\rho \geqslant 0$ (due to $z \geqslant 0$). Plugging this into the solution (258) shows that it takes the constant value on the horizon:

$$\phi^2 = 1 + a\sqrt{\frac{\mu}{a}} \,. \tag{259}$$

In principle we have now obtained the Penrose diagram for the spacetime. However, we know that there is a singularity associated to the black hole solution and wish to find it in the spacetime diagram. Hence we will extend the $\omega^{\pm}$ coordinates using:

$$\phi^2 = 1 + \frac{a - \mu\omega^+\omega^-}{\omega^+ - \omega^-} \,. \tag{260}$$

So we have then an analogue of Poincare coordinates, but now for $\omega^{\pm}$. We can now completely repeat the previous subsection (Section 6.6.1) by replacing all the $x^{\pm}$ by $\omega^{\pm}$. The singularity arising in the above ϕ^2 solutions satisfies:

$$\omega^+\omega^- = \frac{1}{\mu}(\omega^+ - \omega^-) + \frac{a}{\mu} \,. \tag{261}$$

By introducing a rotation of the coordinate axes $\omega_z \equiv \frac{1}{2}(\omega^+ - \omega^-)$ and $\omega_t \equiv \frac{1}{2}(\omega^+ + \omega^-)$ one may also repeat the arguments involving z and t. In particular we find the event horizons related to the singularity:

$$\omega^+ = \sqrt{\frac{a}{\mu}} \qquad \text{Future Horizon}\,, \tag{262}$$

$$\omega^- = -\sqrt{\frac{a}{\mu}} \qquad \text{Past Horizon}\,, \tag{263}$$

which coincide with the horizons we found above in (257), justifying naming them horizons in the first place. Similarly we can investigate the evolution of the singularity:

$$\mu < \frac{1}{a} \qquad \text{Time-like}\,,$$

$$\mu = \frac{1}{a} \qquad \text{Null-like}\,,$$

$$\mu > \frac{1}{a} \qquad \text{Space-like}\,,$$

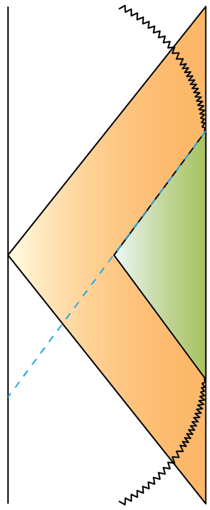

Figure 14: This is the black hole spacetime, consisting of (251) and the extended (260). We take $\mu < \frac{1}{a}$ such that the singularity is timelike. The orange shaded part denotes the 'Poincare' patch of the extended coordinates. The green shaded area is the exterior region of the black hole (Fig 13). In this region we still have the conformal Lifshitz boundary conditions and no singularity (wiggly line) appears. Lastly, the blue dashed line denotes the future horizon $\omega^+ = \sqrt{\frac{a}{\mu}}$.

and for all the cases (trivially for the Null-like case) the behaviour always approaches $\omega_z^2 = \omega_t^2$ for large enough ω_z (or equivalently ω_t).

In Fig 14 we sketch the spacetime with the blackhole for the case $\mu < \frac{1}{a}$. This means the evolution of the singularity will be time-like and approach $\omega_t^2 = \omega_z^2$ asymptotically. Graphically the difference between the time- and spacelike cases will be a concave line versus a convex one. Note that if we want a really 'correct' Penrose diagram we would have to compactify these new extend coordinates again (e.g. $\tilde{\omega} = \tanh(\omega)$).

6.6.3 *Backreaction*

Lastly let us discuss the backreaction. We can see from (242) the time at which the singularity reaches the boundary:

$$t = \pm\sqrt{\frac{a}{\mu}}\,. \tag{264}$$

From this it is immediately obvious that we want $a \geqslant 0$ to get a $t \in \mathbb{R}$. But in fact we need $a > 0$ since in that case the exterior region (see also Fig 14) with $-\sqrt{\frac{a}{\mu}} < t < \sqrt{\frac{a}{\mu}}$ will have no singularity. In this region we have the conformal Lifshitz spacetime and observers can 'live' without problems.

If we, however, had picked $a = 0$ we would have the pure AdS_2 case and we see that no part of the boundary survives as the singularity immediately reaches it. This yields the fact that AdS_2 does not admit any finite energy solutions [8,42] in contrast to the dynamical dilaton gravity.

6.7 MOST GENERAL BLACK HOLE SOLUTIONS

6.7.1 *Singularity*

In this section we investigate the most general solution, (233). Let us follow the same approach as we have taken so far, by first considering:

$$\phi^2 = 1 + \frac{a + b(x^+ + x^-) - d\,x^+ x^-}{x^+ - x^-}\,, \tag{265}$$

where as before $a, d > 0$, but now the $b \neq 0$ term is also taken into account. To find the singularity behaviour we set $\phi^2 = 0$ to get:

$$x^+ x^- = \tfrac{a}{d} + \tfrac{b}{d}(x^+ - x^-) + \tfrac{1}{d}(x^+ - x^-)\,, \tag{266}$$
$$t^2 - \tfrac{2\,b}{d}\,t = z^2 + \tfrac{2}{d}\,z + \tfrac{a}{d}\,, \tag{267}$$

where in the second line we rewrote it by $x^\pm = t \pm z$. We see that for large enough z (or t) the behaviour approaches the null lines $t^2 = z^2$, which was also the case for the $SL(2,\mathbb{R})$ transformed solutions. When we solve the equation for z we obtain:

$$z(t) = -\frac{1}{d} + \sqrt{\frac{1}{d^2} - \frac{a}{d} + t^2 - \frac{2\,b}{d}\,t}\,. \tag{268}$$

We can then again use that $z \geqslant 0$ to obtain the times at which the singularity appears on the boundary:

$$t^\pm_{z=0} = \frac{b}{d} \pm \sqrt{\frac{b^2}{d^2} + \frac{a}{d}}\,. \tag{269}$$

Recall that for the $SL(2,\mathbb{R})$ transformed solutions these would be $\pm\sqrt{\frac{a}{d}}$ (where d is then of course different). We can indeed check that the above reduces to this when $b \to 0$, which is effectively what the $SL(2,\mathbb{R})$ transformation does.

To figure out if the singularity behaves time, null or spacelike we compute again the derivative of z (compare with (246)):

$$\frac{dz}{dt}(t) = \frac{t - \frac{b}{d}}{\sqrt{t^2 + \frac{1}{d^2} - \frac{a}{d} - \frac{2\,b}{d}\,t}} \,. \tag{270}$$

At the points on the boundary we find:

$$\frac{dz}{dt}(t^{\pm}_{z=0}) = \pm\sqrt{b^2 + a\,d}\,. \tag{271}$$

It is interesting to note that $b^2 + a\,d$ is actually the $SL(2,\mathbb{R})$ invariant noted at the end of Section 6.3.2.

We see that we have the three cases where $b^2 + a\,d$ is either smaller than, equal or bigger than one. When equal to one, it will always evolve null-like. For the other two cases we have behaviour analogous to what is shown in Fig 12. The figures will be shifted from the origin by $\frac{b}{d}$ and the points $\pm\sqrt{a/\mu}$ will become the $t^{\pm}_{z=0}$ from (269). The top figure will correspond to the case $b^2 + a\,d < 1$ and $b^2 + a\,d > 1$ corresponds to the bottom figure.

6.7.2 *Penrose Diagram*

In this section we follow the procedure from Section 6.6.2 to obtain a Penrose diagram for the spacetime. To do so, we refer to (233) and use as before:

$$\omega^{\pm} = \sqrt{\frac{a}{d}}\tanh\left(\sqrt{\frac{d}{a}}x^{\pm}\right) , \tag{272}$$

which yields for the metric the same expression as before:

$$e^{2\omega} = \frac{4\,\mu}{a\,\sinh^2\left(\sqrt{\frac{\mu}{a}}\,(x^+ - x^-)\right)} \,. \tag{273}$$

Such that the discussion regarding the horizons remains unchanged, and they are still found at $\omega^{\pm} = \pm\sqrt{\frac{a}{d}}$. The solution for ϕ^2, however, changes:

$$\phi^2 = 1 + \sqrt{a\,d}\,\coth\left(\sqrt{\frac{d}{a}}\,(x^+ - x^-)\right) + b\,\frac{\sinh\left(\sqrt{\frac{d}{a}}(x^+ + x^-)\right)}{\sinh\left(\sqrt{\frac{d}{a}}(x^+ - x^-)\right)} \tag{274}$$

As before, one may check that when $b \to 0$ it reproduces the previous solution ((251)). This yields however a major qualitative difference: the dilaton is not static and goes to zero in this coordinate system. Both of these did not occur before. To find singularities we solve $\phi^2 = 0$ for z:

$$z_{\pm}(t) = \sqrt{\frac{a}{4\,d}} \log \left[\frac{-b \sinh\left(2\sqrt{\frac{d}{a}}\,t\right) \pm \sqrt{1 - a d + b^2 \sinh^2\left(2\sqrt{\frac{d}{a}}\,t\right)}}{1 + \sqrt{a\,d}} \right] \tag{275}$$

Recall that we have $z \geqslant 0$, we show first that the solution with the minus sign (z_-) above will never satisfy this. Let us define first some notation:

$$x(t) = \sinh\left(2\sqrt{\frac{d}{a}}\,t\right) , \tag{276}$$

$$I_{\pm}(t) = -b\,x(t) \pm \sqrt{1 - a d + b^2 x^2(t)} . \tag{277}$$

Then we can derive the following four demands:

$$\begin{aligned} a\,d &> 1 , \\ b^2 x^2(t) &\geqslant -(1 - a\,d) , \\ b\,x(t) &< 0 , \\ b\,x(t) + \sqrt{1 - a d + b^2 x^2(t)} + (1 + \sqrt{a\,d}) &\leqslant 0 . \end{aligned} \tag{278}$$

The first three arise from requiring positivity of $I_-(t)$ and the last one arises from demanding that the argument of the log in (275) is $\geqslant 1$. From these demands one can derive that:

$$-b\,x(t) < \sqrt{1 - a d + b^2 x^2(t)} + (1 + \sqrt{a\,d}) \leqslant -b\,x(t) , \tag{279}$$

which is obviously a contradiction, meaning there is no solution for $z_-(t) \geqslant 0$ in (275). So let us now consider $z_+(t)$ for which we obtain the following demands (we assume as always $a\,d > 0$):

$$\begin{aligned} b\,x(t) &< 0 , \\ b\,x(t) &\leqslant \sqrt{1 - a d + b^2 x^2(t)} - (1 + \sqrt{a\,d}) , \end{aligned} \tag{280}$$

arising from similar reasons as above. But now, in fact, they are satisfiable. To see this let us take this second demand and square the right hand side:

$$2\left(1+\sqrt{a\,d}\right)\left(1-\sqrt{1-ad+b^2x^2(t)}\right)+b^2\,x^2(t)\,, \tag{281}$$

from which we obtain then $b^2x^2(t) \geqslant a\,d$. Using that $|b\,x(t)| = -b\,x(t)$ we can solve this for t to obtain equality at:

$$t_{z=0} = -\frac{1}{2}\sqrt{\frac{a}{d}}\,\text{arcsinh}\left(\frac{\sqrt{a\,d}}{b}\right)\,. \tag{282}$$

So then we find that $\phi^2 = 0$ has solutions with $z \geqslant 0$ as follows:

$$z \geqslant 0 \text{ if } \begin{cases} t \leqslant t_{z=0} & b > 0 \\ t \geqslant t_{z=0} & b < 0 \end{cases}\,. \tag{283}$$

We can translate this back into the $\omega^{\pm}$ coordinates to obtain:

$$\omega^+(t_{z=0}) = \omega^-(t_{z=0}) = \sqrt{\frac{a}{d}}\,\tanh\left(-\frac{1}{2}\,\text{arcsinh}\left(\frac{\sqrt{a\,d}}{b}\right)\right)\,, \tag{284}$$

which corresponds to the (one) point on the boundary $\omega^+ = \omega^-$ where the singularity appears.

Lastly let us discuss the asymptotic behaviour of the singularity. To do this we consider the expression for $z_+(t)$ in (275) for very large t. We expand the sinh terms and use $b\,x(t) < 0$ to obtain:

$$z_+(t) \overset{t\gg 1}{\approx} \begin{cases} -t & b > 0 \\ t & b < 0 \end{cases}\,, \tag{285}$$

which in terms of the $\omega^{\pm}$ coordinates corresponds to $\omega^{\pm} = 0$ respectively.

6.7.3 *Extending Coordinates*

To obtain a figure of the entire spacetime we will again extend the $\omega^{\pm}$ coordinates by considering:

$$\phi^2 = 1 + \frac{a + b(\omega^+ + \omega^-) - d\,\omega^+\,\omega^-}{\omega^+ - \omega^-}\,. \tag{286}$$

We can then reproduce (as before) the first section Section 6.7.1 to find the singularity and its behaviour. By defining $2\omega_{t/z} = \omega^+ \pm \omega^-$ the singularity appears at:

$$\omega_z = -\frac{1}{d} + \sqrt{\frac{1}{d^2} - \frac{a}{d} + \omega_t^2 - \frac{2\,b}{d}\,\omega_t}\,, \tag{287}$$

which as usual approaches the null line $\omega_z^2 = \omega_t^2$ for large enough values of $\omega_{z/t}$. To find where the singularities appear at the boundary we have the equivalent of (269):

$$\omega^{\pm}_{t_{z=0}} = \frac{b}{d} \pm \sqrt{\frac{b^2}{d^2} + \frac{a}{d}}\,. \tag{288}$$

Now one may compare the seemingly unrelated (284) with these solutions and actually find:

$$\sqrt{\frac{a}{d}}\,\tanh\left(-\frac{1}{2}\,\text{arcsinh}\left(\frac{\sqrt{a\,d}}{b}\right)\right) = \begin{cases} \omega^{-}_{t_{z=0}} & b > 0 \\ \omega^{+}_{t_{z=0}} & b < 0 \end{cases}\,. \tag{289}$$

Such that the coordinates (as they should) give consistent results. To summarize, we have found that the singularity in this case exists outside of the horizons yielding a naked singularity in the exterior region. In Fig 15 we sketch this spacetime for the case $b^2 + a\,d < 1$ and $b > 0$.

6.8 SOLUTIONS WITH MATTER

We now consider adding matter to the model. As can be seen in (162) we consider the case where $T_{+-} = T_{-+} = 0$, which is true when a conformal field theory describes the matter action. For a consideration of the conformal anomaly see [8,25]. The equations of motion for ϕ^2 become (those for $e^{2\omega}$ remain unchanged):

$$2\,\partial_+\partial_-\phi^2 - e^{2\omega}\left(1-\phi^2\right) = 0\,, \tag{290}$$
$$-e^{2\omega}\,\partial_-\left(e^{-2\omega}\,\partial_-\phi^2\right) = 8\,\pi\,G\,T_{--}\,, \tag{291}$$
$$-e^{2\omega}\,\partial_+\left(e^{-2\omega}\,\partial_+\phi^2\right) = 8\,\pi\,G\,T_{++}\,. \tag{292}$$

The equations can be explicitly integrated to give the following expression [8,25]:

$$\phi^2(x^+,x^-) = 1 + \frac{a}{x^+ - x^-}\Big(1 + b(x^+ + x^-) + c\,x^+\,x^- - \frac{8\,\pi\,G}{a}\left(I_+(x^+,x^-) + I_-(x^+,x^-)\right)\Big)\,, \tag{293}$$

where a, b, c are constants and $I_\pm$ are as follows:

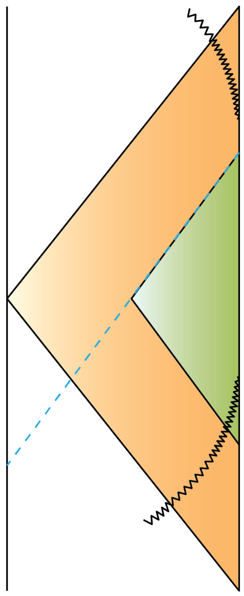

Figure 15: Here we show the spacetime (quite similar to Fig 14) of the most general ϕ^2 solution, see (233). In particular we have $b^2 + a\,d < 1$ such that the singularities evolve timelike and $b > 0$ which 'shifts' the singularities upward. For $b < 0$ they would be shifted in the other direction. As can be seen, the singularity can exist outside of the event horizons meaning we have a naked singularity in this solution.

$$I_+(x^+, x^-) = \int_{x^+}^{\infty} ds\,(s - x^+)\,(s - x^-)\,T_{++}(s)\;, \tag{294}$$

$$I_-(x^+, x^-) = \int_{-\infty}^{x^-} ds\,(s - x^+)\,(s - x^-)\,T_{--}(s)\;. \tag{295}$$

At this point one may wonder if we, as before, can replace the $x^\pm$ by monotonic functions $\omega^\pm(x^\pm)$. As it turns out this can indeed explicitly solve (290). However when we place this generalization in the constraint equations:

$$\begin{aligned} -e^{2\omega}\,\partial_\pm\left(e^{-2\omega}\,\partial_\pm\phi^2(\omega^+(x^+), \omega^-(x^-)\right) = \\ = 8\,\pi\,G\,T_{\pm\pm}\left(\omega^\pm(x^\pm)\right)\left(\partial_\pm\omega^\pm(x^\pm)\right)^2\;, \end{aligned} \tag{296}$$

from which we require $\partial_\pm\omega^\pm(x^\pm) = 1$, hence they are not in general solutions.

6.8.1 *Matter Pulse*

In order to illustrate the above solutions we throw a matter pulse in the geometry [8]. This is achieved by (for example) picking $T_{++} = 0$ and

$$T_{--} = E\,\delta(x^-)\,. \tag{297}$$

If we plug this into the solution (293) we find for ϕ^2 (see also Fig 16).

$$\phi^2 = 1 + \frac{e}{x^+ - x-}\left(1 + f\,(x^+ + x^-) + g\,x^+\,x^- - \frac{8\,\pi\,G\,E}{e}x^+\,x^-\right), \tag{298}$$

where e, f and g are constants. Before the matter pulse is thrown into the spacetime, however, there also exists a certain dilaton profile. We assume that there is no black hole before the matter pulse and hence use the solution found in the previous sections (Section 6.5):

$$\phi^2|_{E=0} = 1 + \frac{a}{x^+ - x^-}\,. \tag{299}$$

Note that the $E = 0$ follows from the fact that for no matter pulse the above ϕ^2 should be the solution in the spacetime. Comparing this with the solution (298) yields then $e = a$ and $f = g = 0$:

$$\phi^2 = 1 + \frac{a + 8\pi\,G\,E\,x^+\,x^-}{x^+ - x^-}\,. \tag{300}$$

This shows us that the parameter $-\mu$ in front of the x^+x^- term is always related to the mass of the black hole and hence $\mu > 0$. We will also show this later in calculation of the ADM mass.

6.9 BOUNDARY STRESS-TENSOR

Now that we have explored solutions for the model we wish to investigate the boundary dynamics. In particular we may compute the boundary stress tensor of the dual CFT [25]. The standard holographic prescription tells us to vary the bulk action with respect to the boundary metric (see [52]):

$$\langle T^{tt}\rangle = -\frac{2}{\sqrt{-\hat{\gamma}}}\frac{\delta S_{bulk}}{\delta\hat{\gamma}^{tt}} = \lim_{\epsilon\to 0}\frac{-2\,\epsilon}{\sqrt{-\gamma(\epsilon)}}\frac{\delta S_{bulk}(\epsilon)}{\delta\gamma^{tt}(\epsilon)}\,, \tag{301}$$

where the boundary metric $\hat{\gamma}$ is obtained by removing the prefactor in the 'original' boundary metric (the tt component of g):

$$\hat{\gamma}^{tt} = \lim_{\epsilon\to 0}\epsilon^2\,\gamma^{tt}(\epsilon)\,. \tag{302}$$

Note that we only have this tt component since the boundary is one dimensional. Hence there is also no momentum and we find that in fact that

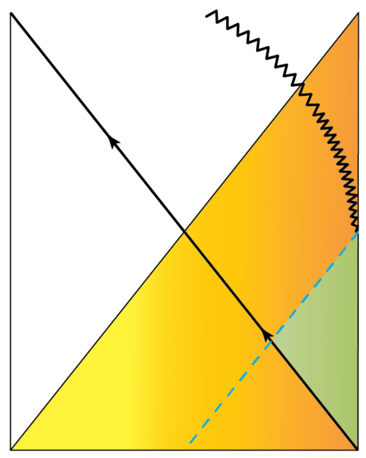

Figure 16: Here we show the spacetime due to an infalling matter pulse, (300). We only show the spacetime for $t \geqslant 0$, the wiggly line is once again the singularity in the black hole. Possible (naked) singularities due to the dilaton profile before the matter pulse are not drawn. The line with the arrow is the matter pulse travelling on $x^- = 0$. The dashed blue line is again the event horizon. The orange shaded area is the Poincare patch and the green area the exterior of the black hole.

$$T^{tt} = H\,. \tag{303}$$

Let us now compute the stress energy tensor using the 'standard' arguments (for more details see [8]). Our total renormalized action S_R will consist out of the normal action for the AP model and the counterterm action canceling the divergences:

$$S_R = \frac{1}{16\pi G}\left(\int d^2x\,\sqrt{-g}\,(\phi^2 R - U_0(\phi)) + 2\int dt\sqrt{-\gamma}\,\phi^2\,K\right) + S_{ct} \tag{304}$$

where S_{ct} denotes the counterterm action, sometimes we refer to the first two terms as S_G. To find this explicitly we consider the above action at the boundary. Hence we can use the solution for the 'exterior' of the black hole, see (251). Using the results from Section 6.1.4 and the fact that in this solution both ϕ^2 and $e^{2\omega}$ only depend on z the action can be written as:

$$S = \int dz\,dt\left[\frac{1}{8\pi G}\left(\partial_z\,\phi^2\,\partial_z\,\omega + (\phi^2 - 1)e^{2\omega}\right)\right] + S_{ct}\,. \tag{305}$$

We now expand the solutions (251) around the boundary $z = 0$ to obtain the asymptotic behaviour:

$$
\begin{aligned}
e^{2\omega} &\approx -\tfrac{4\mu}{3\,a} + \tfrac{1}{z^2} + \tfrac{16\,\mu^2 z^2}{15\,a^2} + \mathcal{O}(z^3)\,, \\
\phi^2 &\approx 1 + \tfrac{a}{2z} + \tfrac{2\mu z}{3} + \mathcal{O}(z^3)\,, \\
\omega &\approx -\log z - \tfrac{2\,\mu\, z^2}{3\,a} + \mathcal{O}(z^3)\,,
\end{aligned}
\tag{306}
$$

where we included the expansion of $omega$ for convenience. Using these we find:

$$
\frac{1}{8\pi G}\left(\partial_z\phi^2\,\partial_z\omega\right) \approx \frac{a}{16\,\pi\, G\, z^3} + \mathcal{O}(z)\,, \tag{307}
$$

$$
\frac{1}{8\pi G}\left((\phi^2-1)e^{2\omega}\right) \approx \frac{a}{16\,\pi\, G\, z^3} + \mathcal{O}(z)\,. \tag{308}
$$

Hence the divergent part of the action above is found as (we take $\epsilon \to 0$ as a regulator):

$$
\int dt\, \frac{a}{16\,\pi\, G\, \epsilon^2}\,. \tag{309}
$$

Hence we can take for the counterterm action:

$$
S_{ct} = \int dt\,\sqrt{-\gamma}\left(\frac{1}{8\,\pi\, G}\left(1-\phi^2\right)\right)\,, \tag{310}
$$

which is easily checked to cancel the divergence using the results above and that $\sqrt{\gamma} \approx \frac{1}{\epsilon} + \mathcal{O}(\epsilon)$ near the boundary. For the terms in the AP action S_G (everything except S_{ct}) the varying process is accomplished using the Hamilton Jacobi formalism to get [8]:

$$
\frac{-2\epsilon}{\sqrt{-\gamma(\epsilon)}}\frac{\delta S_G}{\delta\gamma^{tt}(\epsilon)} = \frac{2\epsilon}{\sqrt{-\gamma}}\,\frac{e^{2\omega}}{16\,\pi\, G}\,\partial_z\phi^2 = \frac{\epsilon\, e^{\omega}}{8\,\pi\, G}\partial_z\phi^2\,. \tag{311}
$$

For the counterterm action we get:

$$
\delta S_{ct} = -\frac{1}{2}\int dt \frac{1}{\sqrt{-\gamma}}\left(\frac{1}{8\,\pi\, G}\left(1-\phi^2\right)\right)\delta\gamma\,. \tag{312}
$$

Noting that $\gamma = \frac{1}{\gamma^{tt}}$ it follows that:

$$
\frac{\delta S_{ct}}{\delta\gamma^{tt}} = \frac{1}{2\sqrt{-\gamma}}\frac{1}{(\gamma^{tt})^2}\left[\frac{1}{8\,\pi\, G}\left(1-\phi^2\right)\right] = \frac{e^{3\omega}}{2}\left[\frac{1}{8\,\pi\, G}\left(1-\phi^2\right)\right] \tag{313}
$$

such that we obtain:

$$
\frac{-2\,\epsilon}{\sqrt{-\gamma(\epsilon)}}\frac{\delta S_{ct}}{\gamma^{tt}(\epsilon)} = -\epsilon\,\frac{e^{2\omega}}{8\,\pi\, G}\left(1-\phi^2\right)\,. \tag{314}
$$

Then we use (301) to find the final expression for the stress energy tensor:

$$\langle T^{tt} \rangle = \frac{\epsilon}{8\pi G} \left(e^{\omega} \partial_z \phi^2 - e^{2\omega} \left(1 - \phi^2 \right) \right) . \tag{315}$$

When we fill in the asymptotic expressions from (306) we find:

$$\langle T^{tt} \rangle = \frac{\epsilon}{8\pi G} \left(\frac{\mu}{\epsilon} - \frac{5\,\mu^2\,\epsilon}{3\,a} + \mathcal{O}(\epsilon^2) \right) = \frac{\mu}{8\pi G} . \tag{316}$$

6.9.1 *Schwarzian*

As mentioned in the part on SYK, the Schwarzian can be found in the boundary stress tensor of the AP model. However to see it arise there is one subtlety, the so called dynamical boundary time. Though we will not go into detail on this, it is worthwhile to see the Schwarzian emerge in this model.

In [8] it is argued that the time coordinate at the AdS_2 boundary becomes a dynamical variable when considering the model as a dual to a scalar field theory. Since we are interested in considering this model as a holographic dual we should use this notion. Following [25] we denote it by $\tau(t)$, where τ denotes the Poincare time coordinate. The coordinate t then describes the time evolution on the boundary of AdS_2 given by $x^+ = x^- \equiv t$.

The choice of coordinates means we investigate the boundary dynamics as some deformation relative to the vacuum solution ((230) with $\mu = 0$) [25]:

$$ds^2 = -\frac{4}{(x^+ - x^-)^2}\, dx^+\, dx^- \qquad , \qquad \phi^2 = 1 + \frac{a}{x^+ - x^-} . \tag{317}$$

Now we can define a 'new' (perturbed) boundary by considering functions $\omega^{\pm}(x^{\pm})$ of the original (above) coordinates. The precise form of these functions is irrelevant for the derivation of the Schwarzian. For an explicit example with scalar field holography, see [8]. The dynamical boundary time is defined as:

$$\omega^+(t) = \omega^-(t) \equiv \tau(t) . \tag{318}$$

One should think of $\tau(t)$ as the relation between the time t that a boundary observer experiences (at a large but fixed dilaton value) with the ordinary Poincare time coordinate. In order to study the dynamics of the boundary time, the usual procedure is to introduce an infinitesimal regulator ϵ such that the boundary is then located at $x^+ = t + \epsilon$ and $x^- = t - \epsilon$. So, the vacuum solution for ϕ^2 of above has the asymptotic value $\phi^2 = \frac{a}{2\epsilon}$ at the boundary. Now, using the above definition (318) we find:

$$\frac{1}{2}\left(\omega^+(t+\epsilon) + \omega^-(t-\epsilon)\right) = \tau(t)\ . \tag{319}$$

By equating the vacuum solution $\phi^2 = \frac{a}{2\epsilon}$ with the (still dynamical) ϕ^2 arising when plugging in this dynamical boundary time, one may derive an equation of motion for $\tau(t)$ [8,25]. This is, however, not relevant for our current purposes and noting that $x^+ - x^- = 2z$ we introduce a $z(t)$ as follows (see also [25]):

$$\frac{1}{2}\left(\omega^+(t+\epsilon) - \omega^-(t-\epsilon)\right) = \epsilon z(t)\ , \tag{320}$$

where the factor ϵ on the right hand side originates from the infinitesimal replacement of the boundary. The term $\epsilon\, z(t)$ denotes the distance between the dynamical boundary and the original. They are related by [25]:

$$z(t) = \frac{d\tau(t)}{dt}\ . \tag{321}$$

This is all we need, now we once again consider the expression for the energy momentum tensor (315). Similar to before we expand $e^{2\omega}$ and ϕ^2 (the vacuum solution with $\mu = 0$) near the boundary, however we now use that $z = \tau'$. Including all the relevant orders one finds [25]:

$$\begin{aligned} e^{2\omega} &= \frac{1}{\epsilon^2} + \frac{2}{3}\{\tau, t\} + \mathcal{O}(\epsilon^2)\ , \\ \phi^2 &= \frac{a}{2\epsilon} + 1 - \frac{1}{3}\{\tau, t\}\epsilon + \mathcal{O}(\epsilon^2)\ ; \end{aligned} \tag{322}$$

where, as usual, the brackets denote the Schwarzian derivative. We can plug these into (315) to get:

$$\langle T^{tt} \rangle = -\frac{a}{16\pi G}\{\tau, t\}\ . \tag{323}$$

Which indeed is the Schwarzian as claimed, showing another feature reminiscent of the SYK model.

6.10 BLACK HOLE THERMODYNAMICS

Now that we have obtained the stress energy tensor, (316), we can discuss the thermodynamics of the black hole solution. Using (303) we note that:

$$\langle T^{tt} \rangle = E \,. \tag{324}$$

Confirming that the ADM mass of black hole solutions is given by $8\,\pi\,G\,\mu$, as stated at the end of Section 6.3.2. Before we can continue with examining the entropy we need to calculate the temperature of the black hole. Note that for calculating the temperature we do not need the energy momentum tensor yet.

6.10.1 *Temperature*

The starting point of the calculation is the Schwarzschild metric in which the horizons are apparent ((257)). Following the standard procedure we then introduce the imaginary time $\tau = i\,t$:

$$ds^2 = 4\left(\rho^2 - \frac{\mu}{a}\right) d\tau^2 + \frac{d\rho^2}{\rho^2 - \frac{\mu}{a}} \,. \tag{325}$$

Then we assume we are close to the horizons $\rho \approx \frac{\mu}{a}$ such that we can expand:

$$g_{tt}(\rho) \quad \approx g'_{tt}\left(\tfrac{\mu}{a}\right)\left(\rho - \tfrac{\mu}{a}\right) \,, \tag{326}$$

$$g^{\rho\rho}(\rho) \quad \approx g'^{\rho\rho}\left(\tfrac{\mu}{a}\right)\left(\rho - \tfrac{\mu}{a}\right) \,, \tag{327}$$

which yields for the above metric, using $g_{tt} = 4\left(\rho^2 - \frac{\mu}{a}\right)$ and $g^{\rho\rho} = \rho^2 - \frac{\mu}{a}$:

$$ds^2 = g'_{tt}\left(\frac{\mu}{a}\right)\left(\rho - \sqrt{\frac{\mu}{a}}\right) d\tau^2 + \frac{d\rho^2}{g'^{\rho\rho}\left(\frac{\mu}{a}\right)\left(\rho - \frac{\mu}{a}\right)} \,. \tag{328}$$

We then introduce a new coordinate as:

$$R = \frac{2\sqrt{\rho - \sqrt{\frac{\mu}{a}}}}{\sqrt{g'^{\rho\rho}\left(\frac{\mu}{a}\right)}} \,, \tag{329}$$

from which we obtain then:

$$ds^2 = \frac{1}{4} R^2\, g'_{tt}\left(\frac{\mu}{a}\right)\, g'^{\rho\rho}\left(\frac{\mu}{a}\right) d\tau^2 + dR^2 \,. \tag{330}$$

Now we note that the metric seems to become singular at $R = 0$ which corresponds with the event horizon of the black hole. Since this is no special point (a coordinate singularity) we must demand that the metric is smooth at this point. This is achieved by interpreting (R, τ) as polar coordinates such that τ plays the role of an angular variable.

This means that the imaginary time coordinate is now periodic:

$$\tau \sim \tau + \frac{4\pi}{\sqrt{g'_{tt}\left(\frac{\mu}{a}\right)\, g'^{\rho\rho}\left(\frac{\mu}{a}\right)}} \,. \tag{331}$$

The temperature is then found by inverting this periodicity and plugging in the explicit g_{tt} and $g^{\rho\rho}$:

$$T = \frac{1}{\pi}\sqrt{\frac{\mu}{a}} \,. \tag{332}$$

6.10.2 *Entropy*

Now that we have obtained the temperature we can use the first law $dS = \frac{dE}{T}$ and use (316):

$$S = \frac{a\,\pi}{4\,G} T + C \,, \tag{333}$$

where C is an integration constant. One may compare this with the Bekenstein Hawking entropy [8], recalling that the area of S^0 is one and $\frac{1}{G_{eff}} = \frac{\Phi^2}{G}$:

$$S_{bh} = \left.\frac{\Phi^2}{4\,G}\right|_{z\to\infty} \,. \tag{334}$$

Now we use that Φ^2 takes a constant value on the horizon, see (259):

$$S_{bh} = \frac{a\,\pi}{4\,G} T + \frac{1}{4\,G} \,, \tag{335}$$

indicating that they agree when we pick $C = \frac{1}{4\,G}$. Lastly, as mentioned in [25], one may write the entropy once again in terms of E to get (we pick $C = 0$ here):

$$S = 2\pi\sqrt{\frac{a}{8\,\pi\,G} E} \,, \tag{336}$$

which is remnant of the Cardy formula for a 2D CFT on a circle with length L:

$$S_{\text{cardy}} = 2\pi \sqrt{\frac{c}{6} L E}\,, \tag{337}$$

where now c denotes the central charge of the CFT. We see that the two results agree when we take:

$$\frac{a}{8\pi G} = \frac{c}{6} L\,, \tag{338}$$

which would seem to indicate that a is perhaps some IR cut-off for the bulk [25].

7 OTHER MODELS AND CONCLUSIONS

7.1 SUPERSYMMETRY AND EXPLICIT CONSTRUCTION OF THE BULK

Now that we have extensively studied the AP model let us discuss some other models. First off let us discuss the supersymmetric extension of the AP model. The action then has the usual two spacetime coordinates denoted by x but also two anticommuting coordinates θ and becomes [53,54]:

$$S = -\frac{1}{16\pi G}\left[i\int_M d^2x\, d^2\theta\, E\, \Phi(R_{+-}-2) + 2\int_{\partial M} dt\, d\theta\, \Phi\, K\right] . \tag{339}$$

Here M denotes the supermanifold (with boundary) and ∂M its boundary. E is the superdeterminant of the vielbein in the superspace. R_{+-} is a superfield containing the usual curvature in the $\theta\bar{\theta}$ term when expanded. The Φ is the dilaton superfield and the last term in the action denotes the supersymmetric version of the Gibbons Hawking York term.
In [54] this boundary term is used to derive the Super-Schwarzian action by explicitly calculating the extrinsic curvature. This shows the analogy with the supersymmetric version of SYK as discussed in Chapter 4.

Another very interesting approach to find the bulk theory is discussed in [13] by Gross and Rosenhaus. We already mentioned this in Section 2.3.6. Their approach uses the $1/N$ finiteness of SYK to explicitly construct the bulk action. To start we recall the fermion OPE and the associated $O(N)$ invariant operators in it ((44) and (50)):

$$\frac{1}{N}\sum_i \chi_i(\tau_1)\chi_i(\tau_2) = \frac{1}{\sqrt{N}}\sum_n c_n \frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta-h_n}}\mathcal{O}_n\left(\frac{\tau_1+\tau_2}{2}\right),$$

$$\mathcal{O}_n(\tau) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{k=0}^{2n+1} d_{nk}\,\partial_\tau^k\chi_i\,\partial_\tau^{2n+1-k}\chi_i .$$

We can then use the standard AdS/CFT dictionary [9], that maps such $O(N)$ invariant operators to massive scalar fields ϕ_n in the bulk. This yields the action:

$$\int d^2x\sqrt{g}\sum_n\left(\frac{1}{2}(\partial\phi_n)^2 + \frac{1}{2}m_n^2\phi_n^2 + \frac{1}{\sqrt{N}}\lambda_{nmk}\phi_n\phi_m\phi_k\right) . \tag{340}$$

The mass is then related to the conformal dimension of the operator (see (54) and above) by [9,13] $m_n^2 = h_n(h_n - 1)$. The cubic coupling is in fact the main result of [13]. It is computed how exactly this coupling is related to the six point function in SYK which they also compute. Their method can in principle be repeated for computing also any higher order interactions (depending on the value of q).
It is hoped that finding the first few terms in the bulk Lagrangian will yield an understanding of the organizing principle. To understand this consider for example a Lagrangian for string theory containing an infinite number of fields, one for each mode of the string. Knowing that the amplitudes come from strings, the worldsheet is the organizing principle [13].

7.2 CONCLUSIONS AND DISCUSSION

In this thesis we have reviewed the SYK model (and generalizations) and its related bulk models. The main 'hallmark' features of SYK, the emergent conformal symmetry and the solvability at strong coupling, allowed one to calculate nearly anything in the model. It is these same properties that leads to a hope of better understanding the AdS/CFT conjecture in the first place.

We have in particular investigated the details of the ensemble and found that the entire diagrammatic structure of SYK can be reproduced by a large class of ensembles (Chapter 3). We also expanded the on the $O(N)$ symmetry and the lack of a current for it. In the investigation of the bulk, the AP model, we generalized the black hole solutions as obtained in [8].

SYK and its holographic dual remains a very active area of research. We have omitted many subjects, for example an approach to a 2d qft analog of SYK [22]. Also for the bulk there exist more models, such as a three dimensional view ($AdS_2 \times S^1/Z_2$) of the duality [55] in which the dilaton profile is seen as the third direction.

There remain many open questions regarding the SYK model and its dual. The natural question is of course the nature of the exact dual of SYK. Let us pose however some more detailed and straightforward questions. Firstly, the interpretation of the dilatonic black hole solutions in SYK should be explored. In particular the solutions we found in Section 6.7 may also carry an interesting interpretation.
Also regarding the bulk model, another problems is to integrate the above discussed construction of the bulk [13] into the AP model. One would have to take into account possible interactions between the

dilaton field and the scalars.

Lastly, for the SYK model itself it would be interesting to see if we can reproduce the effective action and the Schwarzian of SYK in the tensor models. Due to the lack of the disorder average, these tensor models are perhaps more suited for an exact bulk dual.

Part III

APPENDIX

APPENDIX I

A.1 A DIFFERENT METHOD TO OBTAINING THE SYK SCHWARZIAN

This method explains the main idea of the computation as done in [11], in contrast to the method in Section 2.6.4
To set the stage, start by considering a small reparametrization taking $\tau \mapsto \tau + \epsilon(\tau)$. We consider the action as:

$$\frac{S}{N} = \frac{J^2(q-1)}{4}\,(g|\tilde{K}^{-1} - 1|g)\,. \tag{341}$$

Afterwards we add in the explicit corrections on the kernel and the eigenfunctions. These are explicitly computed in chapter 3 of [11].

This yields in frequency space an action proportional to $n^2(n^2-1)$ (n is the analogue of k in the Fourier transform). After Fourier transforming this one obtains [11]:

$$\frac{S}{N} = \frac{\alpha_s}{2\mathcal{J}}\int_0^\beta d\tau \left((\epsilon''(\tau))^2 - \left(\frac{2\pi}{\beta}\right)^2 (\epsilon'(\tau))^2\right)\,, \tag{342}$$

where $\mathcal{J}^2 = J^2 \frac{q}{2^{q-1}}$ and α_s is a constant, but it does depend on the value of q [11]. Note this is the same action as we obtained in our procedure when considering small reparametrizations. The constant α_s is numerically calculated and discussed in [11]. In particular, for large q it reduces to $\frac{1}{4\,q^2}$ as also mentioned before.

The found action (342) thus turns the Goldstone modes into pseudo Goldstone modes by incorporating the explicit breaking of the symmetry.

Although this is a first step we wish to generalize to finite transformations $\tau \mapsto f(\tau)$. Let us consider the zero temperature case in which the domain for both τ and f is $\mathbb{R}$. We can now Taylor expand $f(\tau)$ around any point, we pick the origin, and rewrite:

$$f(\tau) = f(0) + f'(0)\left(\tau + \frac{1}{2}\frac{f''(0)}{f'(0)}\tau^2 + \ldots\right)\,. \tag{343}$$

When we consider $\tau \ll 1$, it can be considered a small reparametrization and we can compare it to the previous case. In particular let us compare the term in brackets to $\tau \mapsto \tau + \epsilon(\tau)$. Expanding ϵ up to second (around the origin) yields

$$\tau + \epsilon(0) + \epsilon'(0) + \frac{1}{2}\epsilon''(0)\,,$$

which, when compared to the terms within the bracket, yields us $\epsilon = \epsilon' = 0$ and $\epsilon'' = \frac{f''}{f'}$. We left out the point around which we expand since it was arbitrary to begin with.
The total transformation due to $f(\tau)$ thus consists out of the above choices for ϵ followed by a scaling f' and a translation f. Since these last two correspond to transformations due to $SL(2,\mathbb{R})$ we can ignore them (G_c is invariant). The generalization from the small reparametrization case thus becomes:

$$\epsilon'' \mapsto \frac{f''}{f'}\,. \tag{344}$$

Plugging this into (342) and partially integrating yields us (up to a total derivative term) [11]:

$$\frac{S}{N} = -\frac{\alpha_s}{\mathcal{J}} \int d\tau \{f, \tau\}\,, \tag{345}$$

where we introduced $\{f,\tau\} \equiv \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ as a shorthand notation for the Schwarzian derivative. This is thus the action that describes the pseudo Nambu-Goldstone modes.

There is a couple of things to note. Firstly, when we pick $f(\tau) = \frac{a\,\tau+b}{c\,\tau+d}$ (so a global $SL(2,\mathbb{R})$ transformation) we see that the action yields zero. This is exactly what we expect since for these transformations we know that G_c is invariant.
There is now, however, also the transformation $f(\tau) = \frac{a\,f+b}{c\,f+d}$. Under this (yet another $SL(2)$ transformation) the Schwarzian derivative, and thus action, remains invariant. This symmetry must of course be there, since the theory (in particular G_c) remains invariant under this. By a similar procedure as done in Section 2.6.4 we can obtain the infinitesimal action above by considering small reparametrizations on the circle.

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COLOPHON

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