# Promoting Students’ Problem-Solving Skills in Secondary Mathematics Education

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June 2017

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Number of ECTS: 30

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Possible journal: Mathematical Thinking and Learning

## Abstract

Mathematical thinking and especially problem solving is the core of doing mathematics, though it is not quite clear how teaching problem solving can be done in the classroom. The aim of this study is to learn how problem-solving skills of students in mathematics education can be promoted. We focused on explicitly providing a problem-solving model to students, problems which have to be experienced as real problems, and the role of the teacher in guiding the problem-solving process of students. A design-based research approach is adopted to develop a series of 36 lessons during nine weeks. The designed lessons were taught to five grade 8 classes by three teachers, including the researcher herself. A pre- and post-test was conducted with 121 students. Also mini-interviews with groups of four students about the awareness of students' own problem-solving process were carried out each week. Results show that the problem-solving skills of students significantly improved and the awareness of students' own problem-solving process and skills increased. Therefore, it can be concluded that a well-implemented multidimensional approach, focused on explicitly providing a problem-solving model, activities that are real problems, and the guidance of the teacher, promote problem-solving skills of students in secondary mathematics education.

There has been a fundamental shift in mathematics education from an emphasis on knowledge and procedural skills to a focus on the active process of extending and applying known concepts in new contexts and problem solving (National Research Council, 1990; Schoenfeld, 2007). This shift is based on the idea that mathematics is a social activity, in which a community of mathematicians engage in studying patterns to determine the nature or principles of theoretical and practical systems (Schoenfeld, 1992). Van Streun (2001) concluded that in mathematics education doing mathematics is not about reproducing readymade mathematics, but about promoting mathematical thinking and understanding and developing problem solving abilities and research skills.

Nowadays, mathematical thinking plays an important role in the Netherlands. With the introduction of the new curricula for upper secondary mathematics (grades 10-12) in 2015, mathematical thinking has received more attention in Dutch mathematics education. The curriculum reform committee cTWO (2007) distinguished mathematical thinking in a number of distinct central activities, namely modelling and practicing algebra, organizing and structuring, analytical thinking and problem solving, manipulating formulas, abstracting, and logical reasoning and giving proofs. The purpose of the attention to mathematical thinking is to find a balance between procedural knowledge and algebraic skills on the one hand and conceptual knowledge, which concerns the understanding of the underlying mathematical concepts, problem-solving skills and applying this knowledge in practical situations or mathematical terms, on the other (cTWO, 2013; Drijvers, 2011; Van Streun, 2014).

Drijvers (2015) proposed the following definition of mathematical thinking: thinking about how you can use mathematical tools to solve a problem. A problem is really a problem when no readymade approach is known. Drijvers (2015) considered problem solving, modelling and abstracting as the main aspects of the central activities of mathematical thinking. Problem solving has played an important role in mathematics (Voskoglou, 2008; Lester, 1994; Wilson, Fernandez, & Hadaway, 1993). Halmos (1980) even claimed it is the heart of mathematics: "mathematician's main reason for existence is to solve problems, and (…) therefore, what mathematics *really* consists of is problems and solutions" (p. 519). Teaching problem solving to students is therefore essential and must not only occur in the examination years, but must start in the earlier years of education (Van Streun, 2014). The current study will focus on problem solving in lower secondary education.

Even though mathematical thinking and problem solving is described (cTWO, 2007; Drijvers, 2015; Van Streun, 2014), it is not yet known how problem-solving skills of students can be promoted in daily practice and how teachers can implement teaching problem solving in their lessons. This problem is very relevant nowadays. Mathematical thinking will be tested in the final examinations in 2017 (havo) and 2018 (vwo). Therefore, problem solving as part of mathematical thinking should be taught explicitly. According to Doorman et al. (2007) "problem solving in secondary mathematics education has only a marginal position" (p. 411) and work needs to be done. In addition, society, focused on knowledge, is increasingly shaped by the rapid emergence of ICT. This development suggests that other skills and competencies are necessary to function in society, the so-called '21st century skills'. Examples of such skills are: collaborating, constructing knowledge, using ICT for learning, problem solving, being creative and working methodically (Van den Oetelaar, 2012). Problem solving is one of these necessary skills. Also within mathematics, ICT ensures that mathematical calculating can be outsourced to software and therefore thinking about, interpreting and applying mathematical knowledge is more needed (Van Streun, 2001).

Problem solving is not new. Polya (1945) and Schoenfeld (1980, 1992, 2007) identified aspects of the problem-solving process and provided strategies in a model. Bor-de Vries and Drijvers (2015) gave starting points for and examples of learning activities and an appropriate classroom climate. Additionally, Mason (2000) and Stein, Engle, Smith and Hughes (2008) provided suggestions for the guidance of the teacher in learning problem solving to students. This knowledge about problem solving can be used to support the current study. However, it can be difficult for mathematics teachers to apply this knowledge in practice (Voskoglou, 2008). Problems, lists of strategies and suggestions for classroom activities are known, but a coherent program is not given (Lester, 1994). Therefore, the aim of the current study is gaining insight into how mathematical thinking, with the focus on problem solving, of students in mathematics education can be promoted.

## Theoretical Framework

Just as Carlson and Bloom (2005) "we regard problem solving as including situations in which an individual is responding to a problem that he or she does not know how to solve 'comfortably' with routine or familiar procedures" (p. 47). So problem solving is mainly about situations in which the problem solver does not know right away how to tackle the problem. Nevertheless, researchers emphasized the importance of well-connected basic knowledge and skills for successful problem solving (Lester, 1994; Voskoglou, 2008). Van Streun (2014) stated that the less basic knowledge and skills are present the more the working memory is burdened. This decreases the attention students can have for tackling a problem. Drijvers (2015) and the new Dutch examinations programs (Van Streun, 2014) also made a distinction between knowledge and skills that students should have ready and knowledge and skills that students can apply in non-routine situations. Therefore, it is important that students possess all kinds of basic tools that can be used in solving a problem.

To learn problem-solving skills to students adequately not only the process of problem solving has to be examined, but also the kind of problems that stimulate problem solving and the role of the teacher in teaching problem-solving skills.

### A Model for Problem Solving

Polya (1945) was one of the first researchers that was involved in studying the problem-solving process and creating a model for teaching problem solving to students. He provided questions and instructions for his model that consists of four phases (see Figure 1). First, students must understand the problem by looking for the principal parts of the problem, such as the unknown, the data or the condition. Drawing a figure or introducing suitable notation can help with understanding the problem. Second, students have to know in outline which computations they have to perform in order to solve the problem. Third, students have to fit the details in the general outline and check each step intuitively or formally. Fourth, students have to reconsider and re-examine the result and look if they can use the procedure or result for other problems.

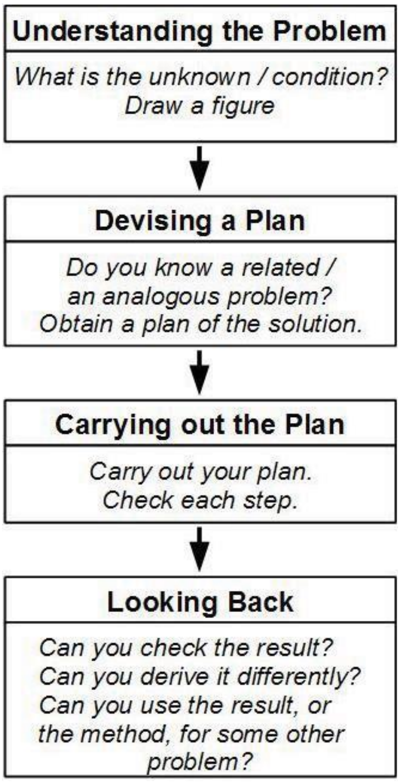



*Figure 1*. Polya's model for problem solving (Rott, 2012).

A few years later Schoenfeld (1980) added the phase Exploration to the model of Polya (see Figure 2). Schoenfeld stated that "Exploration is the heuristic "heart" of the strategy; it is in the exploratory phase that most of the problem-solving heuristics come into play" (p. 802). The phases Analysis, Implementation and Verification are very similar to the phases of the model of Polya. The phase Design differs a little bit from Polya's phase Devising a Plan. It is a controlled part of the model and "it entails keeping a global perspective on the problem and proceeding hierarchically" (Schoenfeld 1980, p. 802). The phases Analysis, Design and Exploration have a cyclic nature, because the problem solver can after exploring either return to design a plan or re-enter the Analysis phase.

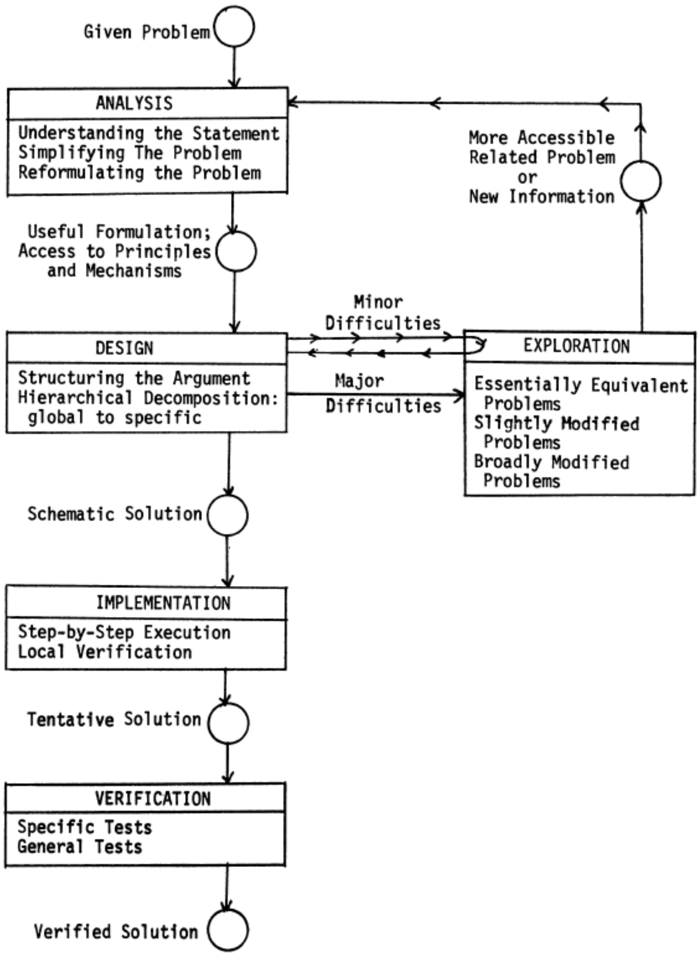



*Figure 2.* Schoenfeld's model for problem solving (Schoenfeld, 1980).

Wilson et al. (1993) pointed out that the phases of Polya are often presented as linear steps. Wilson and colleagues stated that "a framework is needed that emphasizes the dynamic and cyclic nature of genuine problem solving" (p. 61). They presented a dynamic and cyclic interpretation of Polya's stages (see Figure 3). The managerial processes of self-monitoring, self-regulating and self-assessment got a prominent place in this model. The managerial decisions between stages and the non-linear character are represented by arrows.

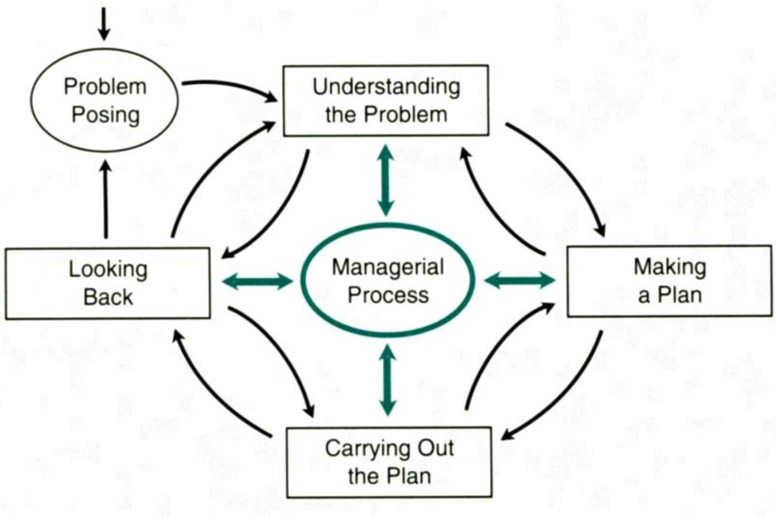



*Figure 3.* The model of Wilson and colleagues for problem solving (Fernandez, Hadaway, & Wilson, 1994).

The above mentioned models are mainly aimed at university students. Rott (2012) compared different models (among others the above) with each other. Rott investigated whether these models are suited to describe problem-solving processes of fifth graders (10 – 12 years old) in Germany. This resulted in a model with the following properties (see Figure 4):

- There should be a distinction between structured and unstructured behaviour (*Planning* and *Exploration*) as in Schoenfeld's model.
- It should be possible to intertwine *Planning* and *Implementation*.
- The framework should be able to display both linear and cyclic processes – with the majority of those processes being linear.
- Managerial activities and self-regulatory decisions should be included as a major part as in Wilson's model.

(Rott, 2012, p. 105, 106)

The managerial decisions are displayed in arrows and the steps of Planning and Implementation are intertwined.

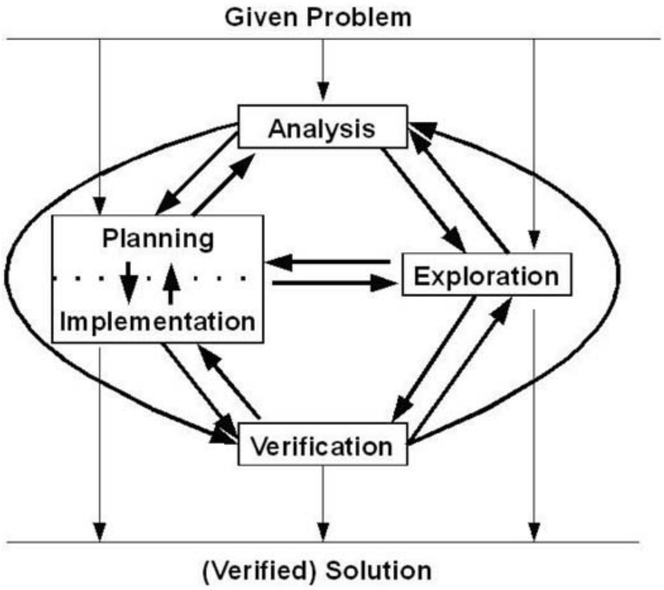



*Figure 4.* Rott's model for problem solving (Rott, 2012).

The target group of the study of Rott (2012) fits best with the target group of the current study and Rott's model is both cyclic and include managerial decisions. Therefore, this model has been used as the basis for the model of the current study. The language of Rott's model was translated in Dutch and the formulation of the model was adapted for a better understanding by the students. Several heuristics used by Polya (1945) and Schoenfeld (1980), but also heuristics from our own experience were added. The model used in the current study is presented in Figure 5.

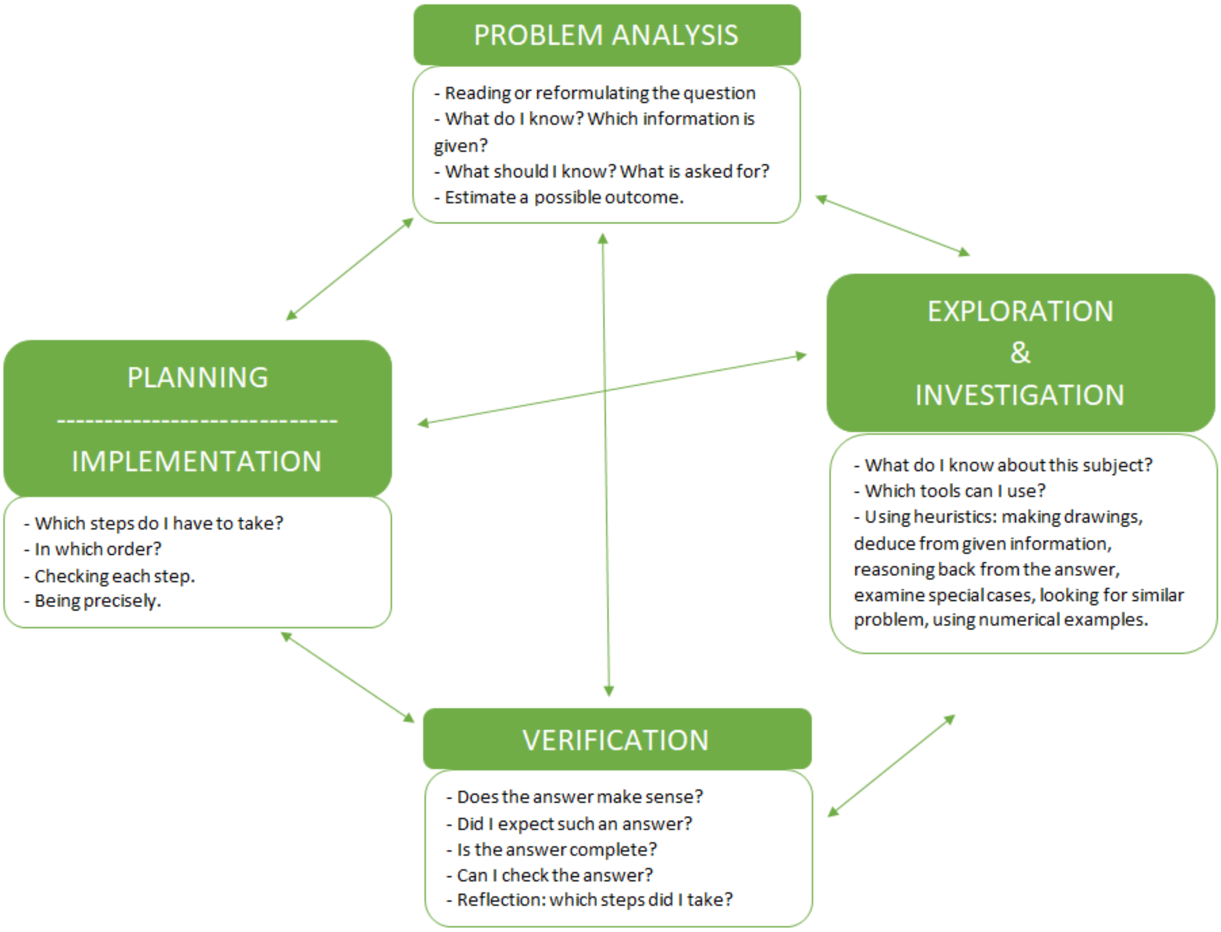



*Figure 5.* The adapted and used model of the current study.

It is important that students become aware of their own thinking process, use of possible heuristics and meta-level processes. Explicit instruction is therefore needed (Schoenfeld, 1992; Silver, 1987). Mason (2000) added: "if students are helped to recognize their own directed, prompted, and spontaneous use of heuristics, they are more likely to know to use them again in the future" (p. 102). Therefore, the first design criterion is: *the model should be explicitly provided during the lessons as a guideline for problem solving and not as a step-by-step model.*

## Learning Activities

Problems are the core of problem solving. To what extent a task is experienced as a problem depends on the characteristics of the problem solver and is related to the knowledge and skills of the person working on it (Drijvers, 2015; Voskoglou, 2008). Schoenfeld (1983) made the following distinction between a problem and an exercise:

> A problem is only a *Problem* (as mathematicians use the term) if you don't know how to go about solving it. A problem that holds no "surprises" in store, and that can be solved comfortably by routine or familiar procedures (no matter how difficult!) is an *exercise*. (p. 41)

Bor-de Vries and Drijvers (2015) investigated, by working together with several teachers, what a teacher can do to encourage mathematical thinking of students. This resulted in practical tips for and characteristics of suitable learning activities and teacher guidance. Bor-de Vries and Drijvers pointed out that in selecting or designing activities it is important to connect to prior knowledge and experience of students and to differentiate if necessary. They named the following characteristics of activities that are problems: the activity has a surprising element, the approach to solve the problem is unknown and asks for creativity, the activity is not too much structured, multiple steps are necessary to obtain a solution and every student has to be able to solve the problem to some extent. Concrete and practical tips for designing such activities are: adapting tasks from school textbooks by leaving out sub questions, forwarding more challenging tasks, looking critical at the context of an activity or task, variating in different activities, and designing a task with knowledge and skills from previous chapters. These suggestions led to the second design criterion: *learning activities in the lessons should be non-routine and experienced as problems by the students*.

**The Role of the Teacher**

Teaching problem solving is not only about providing a model and real problems to students, but also about the guidance of the teacher. Lester (1994) stated that he “do not believe that any problem-centered mathematics curriculum has a chance of success unless the teacher's role in the curriculum is clearly and unambiguously spelled out” (p. 672). Silver (1987) and Fernandez et al. (1994) underline the importance of teachers as role model in enhancing problem-solving skills of students. Teachers need to exemplify and discuss their actions and thoughts as they solve a problem and "focus not only on what is being done but also on why the choice was made" (Silver, 1987, p. 56). Mason (2000) elaborated on the importance of asking questions in stimulating students' thinking and fading of these questions:

> A sequence of directed or focused questions which over time gradually become more general and more indirect as prompts until they disappear altogether, can have the effect of transferring initiative from teacher to student, of becoming part of each student's inner monitor. (p. 100)

Beside modelling teacher's own problem solving process and asking questions, orchestrating whole-class discussions can advance mathematical learning in cognitively demanding tasks (Stein, Engle, Smith, & Hughes, 2008). A model for discussion facilitation consists of the following five practices:

> (1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students' responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class make mathematical connections between different students' responses and between students' responses and the key ideas. (Stein et al., 2008, p. 321)

Bor-de Vries and Drijvers (2015) underline these suggestions and add for example the importance of creating a safe learning environment and giving enough time to think. These suggestions led to the third design criterion: *the teacher should make the problem solving process explicit, ask questions and fade the use of questions, and orchestrate whole-class discussions in a safe learning environment.*

## Research question

The current study focuses on the use of a problem-solving model, learning activities and the role of the teacher in order to teach problem-solving skills to students. The main research question of this study is:

> How can an intervention, including a problem-solving model, learning activities and teacher guidance promote problem-solving skills in mathematics of students in lower secondary education?

To investigate the research question, there are two sub questions:

1. How does the awareness of students of their own problem-solving process progress during the intervention?
2. What is the effect of the designed intervention on students' problem-solving skills?

With the awareness of students' own process is meant that students consciously know which steps they take in their process of solving a problem.

# Methods

## Design-based Research

For this study a design-based research approach was adopted. Lessons about three mathematical topics were designed. Each topic involved three phases: an exploration phase, a design phase and a test phase. In the exploration phase literature was used to study the problem-solving process and how students can learn to solve mathematical problems. This resulted in design criteria for the problem-solving model, learning activities and the guidance of the teacher. Then learning activities and accompanying materials were designed for the particular topic and lesson plans were made for the teachers. In the test phase mini-interviews and a pre- and post-test were conducted.

## Context and Participants

The current study was carried out at a Dutch high school, called de Werkplaats (the workplace), in Bilthoven. The organization of the education in this school is quite different from most schools in the Netherlands. In this school emphasis is placed on learning of the students through collaboration and working independently. For mathematics this means the students attend one lesson a week in which they get instruction from the teacher about the subject. We call these lessons instruction lessons. The students also attend three lessons a week in which they carry out tasks individually or in fixed groups under the guidance of the teacher. We call these lessons working lessons. This arrangement was taken into account during the design and implementation of the intervention.

Participants were 138 students of two 8$^{th}$-grade mathematics classes of havo/vwo (pre-higher vocational and pre-university schooling, respectively) and three 8$^{th}$-grade mathematics classes of vwo. The students were 12 to 14 years old. These mathematics classes were taught by three teachers, including the researcher herself. In design-based research it is very common "that researchers are involved in the teaching or work closely with teachers or trainers to optimize the learning environment" (Bakker & Van Eerde, 2015, p. 5). The teachers each had approximately five years of experience in secondary education and provided the commitment and investment required for a proper implementation of the intervention.

## Design Procedure

Pythagoras' theorem, calculating areas of triangles, parallelograms and circles, and factorizing and solving quadratic equations were selected as topics in the intervention. These topics corresponded to chapters in the regular mathematics textbook of the students and represented a variety in geometry and algebra. In this way students were asked to solve different kinds of problems. When choosing the topics the already existing knowledge and skills of the students and the practical feasibility in the curriculum were taken into account. The intervention lasted for nine weeks, three weeks per topic. In total thirty-six 40-minute lessons were taught. The intervention took place from January till March 2016.

**Intensive collaboration.** The three teachers (including the researcher) in this study collaborated intensively. They designed all the materials and lessons together and developed lesson plans. Multiple sessions took place during the intervention to evaluate and discuss the lessons. The teachers used observations of their own lessons to refine upcoming lessons. This approach was chosen, because when teachers have to collaborate intensively they feel responsible for the lessons, are very involved and can learn from each other. This contributed to the designed activities, to the implementation of the activities and eventually to the actual learning of the students. After all, the teacher has an important role in promoting problem-solving skills of students.

The three teachers also followed the course 'Mathematical Thinking Activities' on the University of Utrecht to professionalize their knowledge and skills of designing activities that promote mathematical thinking and their role as a teacher in guiding students. Apart from personal ways of teaching, the intervention is considered to be implemented in the same way.

**Lesson plans and justification.** During the design phase the three teachers formed ideas for the learning activities by reasoning back and forth between the three design principles of the model, learning activities and the guidance of the teacher, and the mathematical content of the different topics. They discussed learning materials and checked if students' prior knowledge matched with the learning materials. Gradually detailed lesson plans of each learning activity and task were made by the author of this study (see Appendix A). These lesson plans ensure the correct and same implementation of the different lessons by the teachers. Also a justification of each learning activity and task was made (see also Appendix A). This justification contains the rationale for designing a particular activity with a theoretical underpinning. In this way it was checked whether the design principles were met. Beside the justification various versions of the learning materials and activities were discussed with colleagues.

In the lesson plans the organization of the secondary school was taken into account. Students were used to working in fixed groups of about four students. During the three working lessons a week the students had to work on almost all tasks and activities in these groups. Working in small groups can realise mathematical level raising (Dekker & Elshout-Mohr, 1998) and therefore can be helpful when solving problems. Whole-class discussions about the problem-solving process was planned in the instruction lessons. Students were not allowed to take their work home, because it was important students faced the problems themselves or with their groupmates. In this way the important guidance of the teacher was optimally guaranteed.

**Providing the problem-solving model.** The model in Figure 5 was used to teach problem-solving skills to students. During the instruction lessons a student was asked to solve a problem out loud in front of the class. In a whole-class discussion it was discussed which steps can be taken to solve the problem. These steps were related to the model. In this way the model was explicitly discussed, but not just presented by the teacher as a step-by-step plan which the students had to follow. Students themselves constructed possible steps and if necessary the teacher added possible steps from the model.

In the beginning of the intervention most attention went to the Problem-Analysis phase. The students weren't used to have a structured, conscious approach to solve a problem and most of the time a problem analysis is the first step. After the first two weeks more focus was put on the Planning and Exploration phase. During some instruction lessons the students were asked only to think of a plan. They were not allowed to carry out the plan or calculate any solution in order to "emphasize the role of planning (…) and to develop a qualitative rather than a quantitative approach to the problem" (Silver, 1987, p. 55). During the last weeks of the intervention more attention went to the verification of the solution to a problem.

**Learning activities and tasks.** Various tasks from the regular mathematics textbook were adapted and used as an inspiration for designing of activities in this intervention. Also new tasks and activities based on suggestions in literature were designed to challenge the students to think about mathematics and tackle different problems. This led to different kinds of activities (see Appendix B). A lot of attention went to whether the different tasks were non-routine and real problems for students. Prior knowledge and experience of students with certain tasks was taken into consideration. Furthermore, tasks were designed in which various heuristics of the model were needed, so the students could really put different heuristics into practice.

Based on the literature, having basic knowledge and skills is a necessary condition for tackling problems. So for every topic both standard tasks, like solving an equation, and non-routine, real problems were designed. Students first had to practice with standard tasks before they could apply the basic knowledge and skills in new situations and contexts.

**The guidance of the teacher.** Beforehand, expected students' difficulties with and questions about the tasks and activities during the working lessons were devised. Possible questions and hints of the teachers as a reaction on these difficulties and questions were also constructed and got a place in the lesson plans. The support and guidance of the teacher was adapted to the particular student. The teachers started with general questions and hints and became more concrete when the student didn't understand it. The fading over time of asking questions and providing hints of the teachers was explicitly named and planned in the lesson plans with in mind that adapting the guidance to the student is most important. Also, the teachers explicitly referred to the model by asking questions that contained steps of the model or questions about which parts of the model students could use. Asking the question 'which question am I about to ask you?' is an example of fading the guidance.

During the instruction lessons the teachers provided whole-group discussions according to a model of five practices for orchestrating productive mathematical discussions (Stein et al., 2008; see Appendix A). Special care was paid to a safe learning environment by not judging students' answers and having room for mistakes, encouraging students to think aloud, giving time to think and showing teacher's own enthusiasm and way of thinking.

**Example of a designed task.** Now an example of a designed task and the corresponding lesson plan and justification will be discussed (see Figure 6). This task was taken from the regular mathematics textbook and was adapted by removing sub questions. Students experienced this task as a problem, because they had to take various steps themselves and the coordinate system was an unknown context for using Pythagoras' theorem. Students could make use of different parts of the model like looking for what is given and what is asked, making a sketch, looking for what they already know of the coordinate system, if the Pythagoras' theorem is useful, and looking for a similar problem. The teacher can point students to these parts of the model and also ask how the students know for sure that their answer is correct, if they answered the question, and if they can compute the answer more efficiently.

## 11 Coordinate system

The points $K(-32, 45)$, $L(17, -63)$, $M(89, 31)$ and $N(23, 40)$ form a quadrilateral.
Find out which side of these quadrilateral is the longest and which side is the shortest.

| | | |
|---|---|---|
| **Domein 1**<br>Week 4<br>10 min:<br><br>§5<br>Opgave<br>11<br>Assenstels<br>el | - Leerlingen kunnen tegen de volgende zaken aanlopen:<br>• De getallen zijn groot, dus ze kunnen minder gemakkelijk 'zien' hoe de punten in het assenstelsel geplaatst zijn.<br>• Ze zien uit een schets niet welke zijde waarschijnlijk het langst en kortst is.<br>• Ze zien niet in met welke driehoek je met Pythagoras een zijde kunt uitrekenen.<br>• Ze gaan de lengte van alle vier de zijdes uitrekenen, maar geven geen antwoord op de vraag.<br>- Tips en vragen van de docent:<br>• Schets?<br>• Welk gereedschap? Rechthoekige driehoek?<br>• Gegeven en doel?<br>• Hoe weet je 100% zeker dat dit de langste en kortste zijdes zijn?<br>• Heb je antwoord gegeven op de vraag?<br>• Kun je deze opgave nog efficiënter aanpakken? Stimuleer ze om 'lui' te zijn en snellere methodes te bedenken. | Doel:<br>De leerling kan de stelling toepassen in een assenstelsel en zijdes vergelijken. De leerling kan iets zeggen over de zekerheid van het antwoord en kan naar efficiëntere aanpakken zoeken.<br><br>PO model:<br>*Probleemverkenning*<br>• Vraag lezen/herformuleren • Wat weet ik? • Wat moet ik weten?<br>*Uitproberen & Onderzoeken*<br>• Schets maken (De leerlingen moeten zich oriënteren op waar de punten in het assenstelsel zich bevinden en welke zijdes langer of korter zijn.)<br>• Welk gereedschap kan ik gebruiken? Waar kan ik rechthoekige driehoeken maken?<br>*Plannetje maken & Stappen uitvoeren*<br>• Welke stappen moet ik zetten? • Tussendoor stappen checken.<br>*Controleren*<br>• Heb ik antwoord gegeven op de vraag?<br>• Weet ik zeker dat dit het juiste antwoord is?<br><br>Kenmerken opgave(n) (Bor-de Vries & Drijvers, 2015):<br>*Probleem oplossen*<br>- "Heeft de opgave iets 'fris', zit er een verrassingselement in" (p. 7)?<br>Ze moeten de stelling toepassen in een assenstelsel en niet alleen een zijde berekenen, maar zijdes vergelijken (welke is kortst en langst).<br>- "Meerdere (denk)stappen zijn nodig om tot een oplossing te komen" (p. 7).<br>Ze moeten bedenken hoe ze een zijde uitkunnen rekenen, bedenken hoe ze erachter kunnen komen welke kortste en langste is en dan verschillende zijdes uitrekenen of vergelijken.<br>- "Er zijn meerdere oplossingsstrategieën mogelijk" (p. 7).<br>Alle zijdes uitrekenen, twee langste zijdes en twee kortste zijdes uitrekenen en vergelijken, twee langste zijdes uitrekenen en twee kortste zijdes met elkaar vergelijken en beredeneren welke korter is. |

*Figure 6.* An example of a task translated to English and the corresponding lesson plan (in the second column) and justification (in the third column) in Dutch.

**Data**

The data consist of two kinds of mini-interviews and a pre- and post-test. An overview of the design of this study is summarized in Figure 7. The mini-interviews account for the first sub question. The amount of consciously named steps of the model by students and the extent to which students named these steps by themselves is measured. With the pre- and post-test the effect is measured, which accounts for the second sub question.

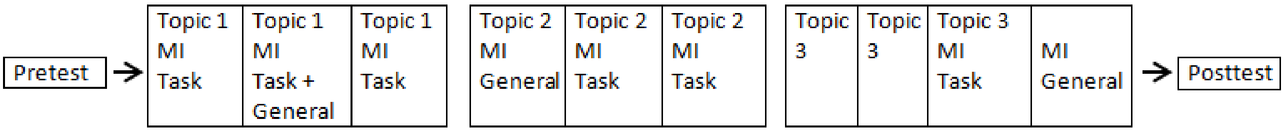



*Figure 7.* An overview of the design of the current study with a pre- and post-test and mini-interviews (MI) about a particular task and mini-interviews in general.

**Mini-interviews.** Almost every week during the working lessons mini-interviews were conducted. Multiple groups, consisting of about four students, in which they worked on the activities were interviewed. These mini-interviews were audiotaped and a trial was conducted before the intervention. The aim of the mini-interviews was to gain insight in the awareness of students about their own problem solving process and the progress over time.

Two kinds of mini-interviews were held. In a number of mini-interviews students were asked to explain aloud how they solved a non-routine task, which steps, related to the model, they took, and why these steps helped them to solve the problem. These mini-interviews are called 'task mini-interviews'. The questions were asked as open as possible to minimalize the effect of the interviewer. Sometimes follow-up questions were asked to clarify student's thoughts and get a more complete picture. No attention was paid whether the problem was solved in a mathematical right way. In the other mini-interviews students were asked which steps can be taken to solve a problem in general, not specified to a certain task. These mini-interviews are called 'general mini-interviews'. Follow-up questions about if they knew more steps to take and what they did first or last were asked.

**Pre- and post-test.** A week before the intervention began, a 40-minute pre-test with the students (n = 121) was conducted. The pre-test started with a general question about what the students do when they get a mathematical task they cannot solve right away, how they deal with this. The aim of the first question was to investigate which problem-solving skills the students are already aware of using. This first question was followed up by six non-routine tasks about different mathematical topics. These questions were adopted from the NRO-project 'Wiskundige denkactiviteit in wiskunde op havo en vwo' which was conducted in 2014-2015. This project was targeting grade 9, so some questions were removed or adapted depending on the already existing knowledge of students in grade 8. The aim of these questions was to document students' starting level concerning problem-solving skills. The students had enough mathematical knowledge and skills to answer the questions. However the tasks were non-routine, so the students had to use some problem-solving skills in order to tackle the tasks.

After the intervention a 40-minute post-test (n = 121) was conducted. The first question was the same as the first question in the pre-test. In this way the progress of awareness of used problem-solving skills could be investigated. This question was followed by five non-routine tasks also adopted from the post-test used in the NRO-project. These questions were not exactly the same questions as in the pre-test, but were about the same mathematical topics. The students had to use problem-solving skills as well to tackle the tasks.

A psychometric analysis was performed on the pre- and post-test (see Table 1). According to Van Berkel and Bax (2014) a few p-values are considered as low, but the most values are considered as good. The values of $r_{it}$ and $r_{it}$ are considered as very good, which means that the questions have a high discrimination. The reliability of the pre-test and post-test was α = .58 for both tests. This value is considered as low, maybe due to the few questions in the pre- and posttest.

Table 1

*Psychometric analysis of the pre- and post-test*

| | Question | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 |
| | | | Pre-test | | | |
| p-value | .19 | .17 | .49 | .43 | .14 | .26 |
| $r_{it}$ | .51 | .49 | .70 | .66 | .60 | .40 |
| $r_{ir}$ | .217 | .30 | .45 | .36 | .39 | .19 |
| | | | Post-test | | | |
| p-value | .79 | .11 | .36 | .32 | .38 | |
| $r_{it}$ | .47 | .52 | .69 | .67 | .68 | |
| $r_{ir}$ | .175 | .33 | .43 | .35 | .40 | |

**Data Analysis**

Data of the pre- and post-test was analysed using elementary statistical techniques in SPSS software. According to the Shapiro-Wilk and Kolmogorov-Smirnov tests the variable Improvement, which contains the standardized Posttest score (S_Posttest) minus the standardized Pretest score (S_Pretest), is normally distributed. The data set is of considerable size, so parametric techniques such as *t*-tests were applied (Field, 2009).

Data of the mini-interviews was analysed using ATLAS software. The mini-interviews were divided in task mini-interviews and general mini-interviews. Mentioned steps by students were linked to steps of the problem-solving model. Each step of the model got a label. The category Other consisted of steps or strategies that did not fit the model, like asking the teacher or a fellow student a question, consulting the theory of the textbook, skipping the task or thinking about the task in unspecific terms. These steps showed little or no active approach in solving the problem by the students and were therefore not seen as positive.

Thinking out loud was not always easy for students, so sometimes the interviewer had to ask further about what students meant or did. The level of student's initiative in mentioning these steps was rated with Minus, Zero or Plus:

- Minus: interviewer asked about a specific step, for example: 'What did you do with the solution to the problem?' or 'What did you do first?'
- Zero: interviewer asked further about other possible steps, for example: 'What other steps did you take?'
- Plus: students described by themselves a made step, for example: 'First, I read the question.'

The used labelling system is presented in Figure 8. The interrater reliability Kappa (κ = 0.772) is investigated. This means there was a substantial agreement between the two raters.

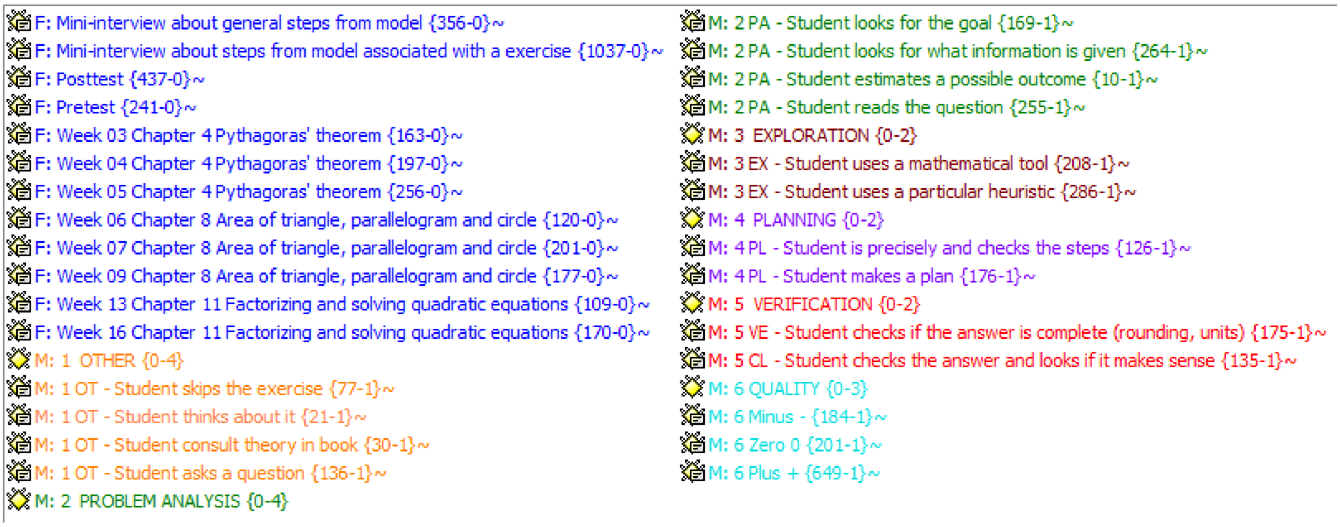



*Figure 8.* Used labelling system for analysing the mini-interviews.

## Results

The results consist of three parts: results of task mini-interviews, of general mini-interviews combined with the general question of the pre- and post-test, and the results of non-routine questions of the pre- and post-test. First we will examine the mini-interviews.

**Task Mini-Interviews**

As a global view on the data of the task mini-interviews, Figure 9 shows the average frequency per interview of the named steps of the main phases of the model, namely Problem Analysis, Exploration, Planning and Verification. Recall that the phase Problem Analysis consisted of reading the question, what is given and what is asked for. Exploration consisted of tools and heuristics. Planning consisted of a plan with different steps to take and the order of the steps and the phase Verification consisted of checking the answer and the completeness of the answer. Week 3 to 5 was about Pythagoras' theorem, week 7 and 9 was about calculating areas, and week 13 was about factorizing and solving quadratic equations. There is a gap between week 9 and 13, because of vacation and cancelled classes. In addition, factorizing and solving quadratic equations was very difficult for the students, so it was less possible to conduct mini-interviews about problem solving. First they needed time to master the basic skills. Yet week 13 is included, because it shows the progress over time.

Figure 9 shows the total amount of steps named by the students per week per interview. Over time the total number of steps increased. In the first week the students already name quite a few steps from the model. It seems that certain steps from the model weren't new to the students. The number of steps mentioned by students from the phase Problem Analysis stayed approximately the same over time. The number of steps from the phase Exploration first increased, then decreased a little. The number of steps from the phase Planning increased a little. Finally the number of steps from the phase Verification clearly increased.

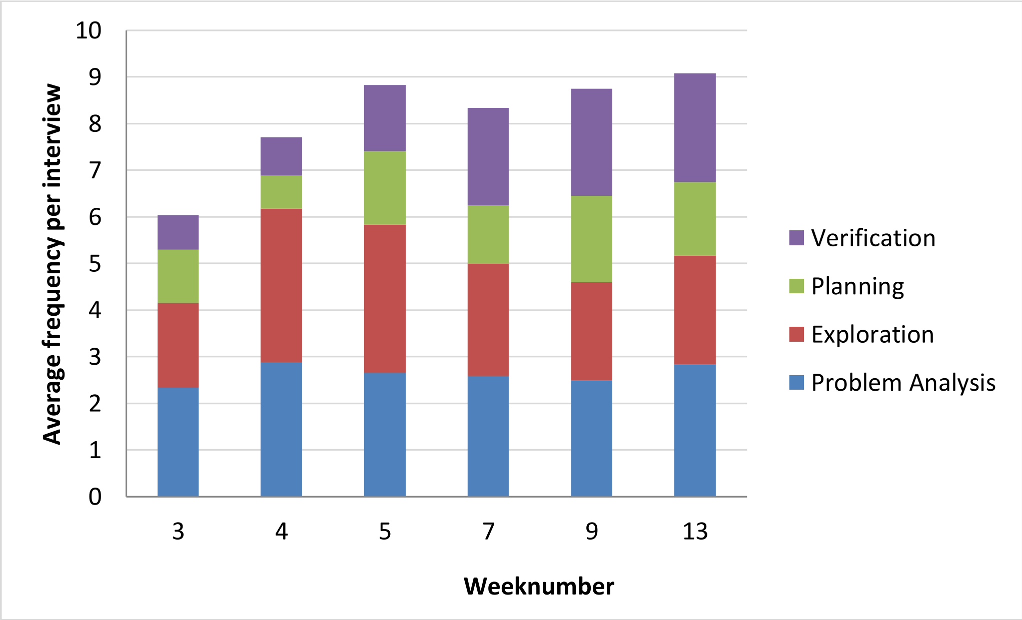



*Figure 9.* The average frequency per interview per week mentioned by the students of the phases of the model: Problem Analysis, Exploration, Planning and Verification.

To further investigate the data, Figure 10 shows the average frequency of steps per interview per week mentioned by the students for the phase Problem Analysis. Now the frequency of Problem Analysis was divided in Minus, Zero and Plus. Recall that with Plus, the interviewer had no influence on mentioning the used steps by the students, with Zero a little bit of influence and with Minus more influence. Over time the total average frequency remained approximately the same. The number of steps increased only slightly. The levels Minus, Zero and Plus did shift however. The average frequency per interview of Plus rose over time. This frequency almost doubled. The average frequency per interview of minus decreased.

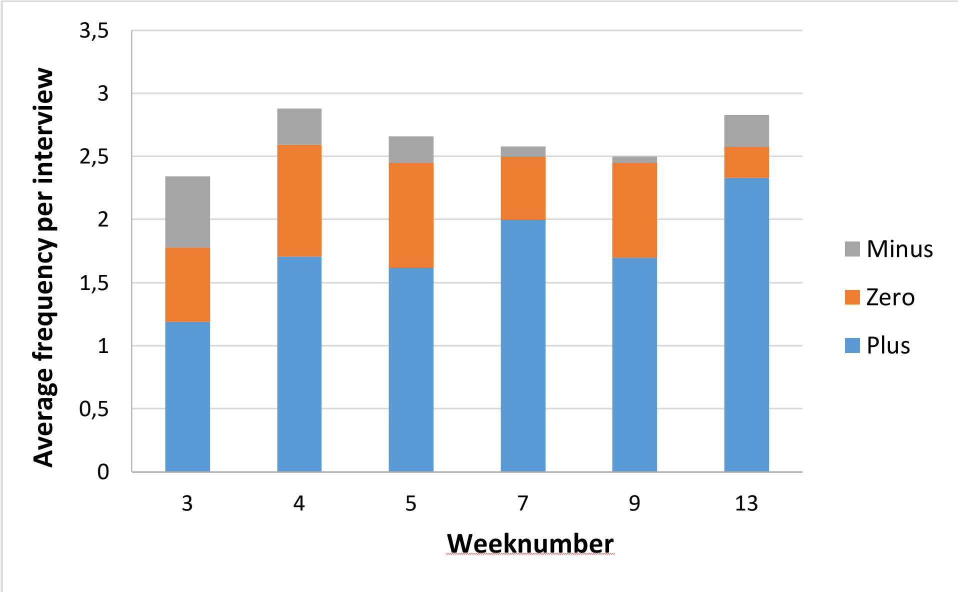



*Figure 10.* The average frequency per interview per week of the phase Problem Analysis divided in the levels Plus, Zero and Minus.

Figure 11 shows the average frequency of steps per interview per week named by the students for the phase Exploration. Over time the number of steps rose only slightly. In the first few weeks the number of steps per interview increased greatly, also due to the strong increase of Minus in week 4. However, in the next weeks the number of steps decreased. Over time the average frequency of Minus decreased a lot. The number of steps of Plus however, more than tripled.

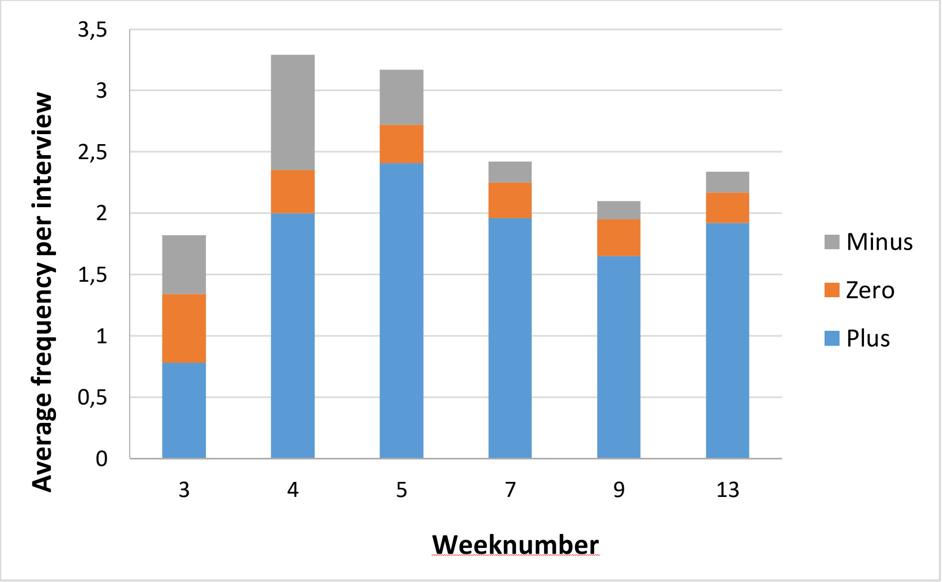



*Figure 11.* The average frequency per interview per week of the phase Exploration divided in the levels Plus, Zero and Minus.

Figure 12 shows the average frequency of steps per interview per week named by students for the phase Planning. Over time the total average frequency increased. This changed per week. In weeks 5, 9 and 13 the average frequency increased, in the other weeks it decreased. Furthermore, it can be seen that the average frequency of Minus decreased a bit over time. In the beginning the frequency of Zero was quite high, later the frequency decreased a little and in the last week the frequency even reduced to zero. The frequency of Plus rose strongly.

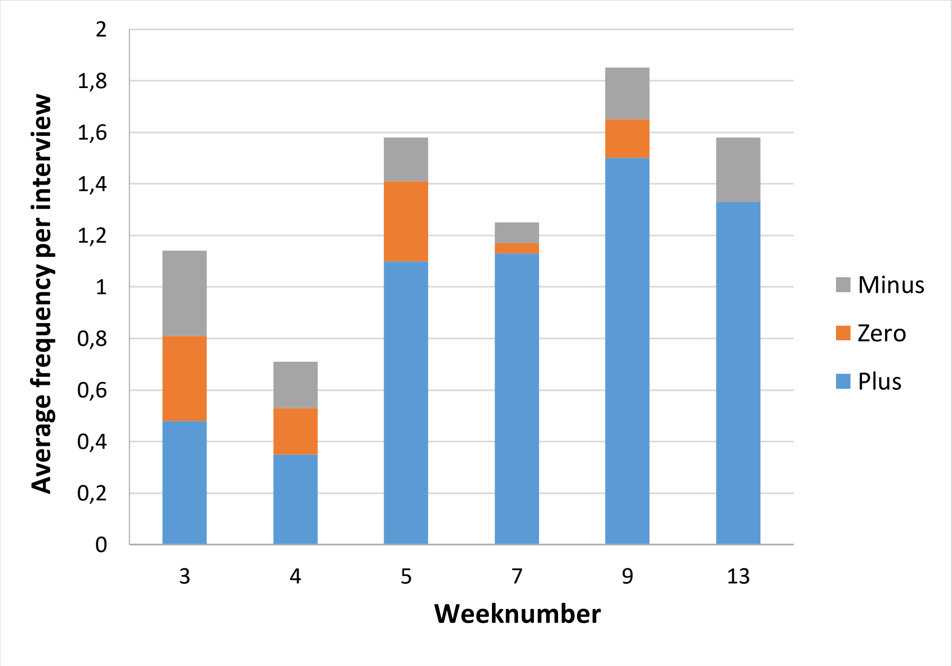



*Figure 12.* The average frequency per interview per week of the phase Planning divided in the levels Plus, Zero and Minus.

Figure 13 shows the average frequency of steps per interview per week mentioned by the students for the phase Verification. This figure is a bit different than the other figures. In the beginning the average frequency of Verification was still low, but over time this frequency constantly increased. In total, the average frequency tripled over time. Furthermore it can be seen that the average frequency of Minus stayed approximately the same and the frequency of Zero rose. The average frequency of Plus was at the end of the intervention about 11 times as high, but still not much more than one step per interview.

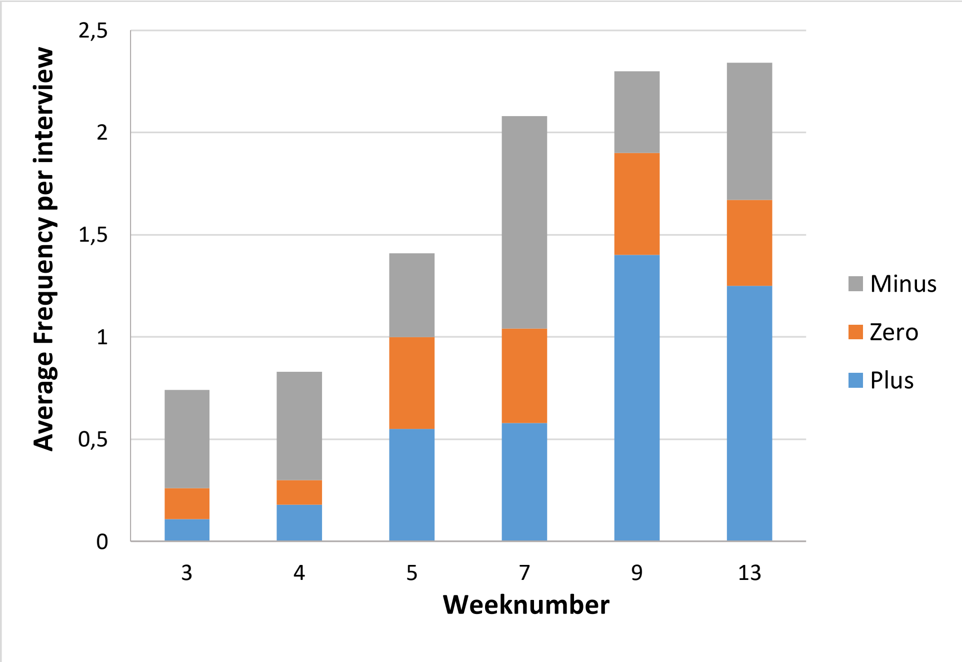



*Figure 13.* The average frequency per interview per week of the phase Verification divided in the levels Plus, Zero and Minus.

## General Mini-interviews and General Question in Pre- and Post-test

The figures above show the results of the task mini-interviews. Now we will look at the general mini-interviews and the question about which steps can be taken to solve a problem in general in the pre- and post-test. Figure 14 shows again the average frequency of steps named by students in the general mini-interviews per interview per week of the main parts of the model, but also the category Other. Recall that the category Other consisted of strategies that did not fit in the model, like asking the teacher or a fellow student a question or skipping the task. These strategies showed little or no active approach in solving the problems by the students.

Figure 14 shows that the total number of named steps increased. However, the number of steps in the category Other decreased significantly over time. This means that the average number of steps from the model increased strongly. Furthermore it can be seen that the average number of steps from the phase Problem Analysis stayed approximately the same.

This is in accordance with what we saw earlier. The number of steps from the phase Exploration also stayed approximately the same. The number of steps from the phase Planning stayed the same in weeks 4 and 6 and then increased. This is in accordance with Figure 12. Just as in the task mini-interviews the average number of steps from the phase Verification increased strongly over time.

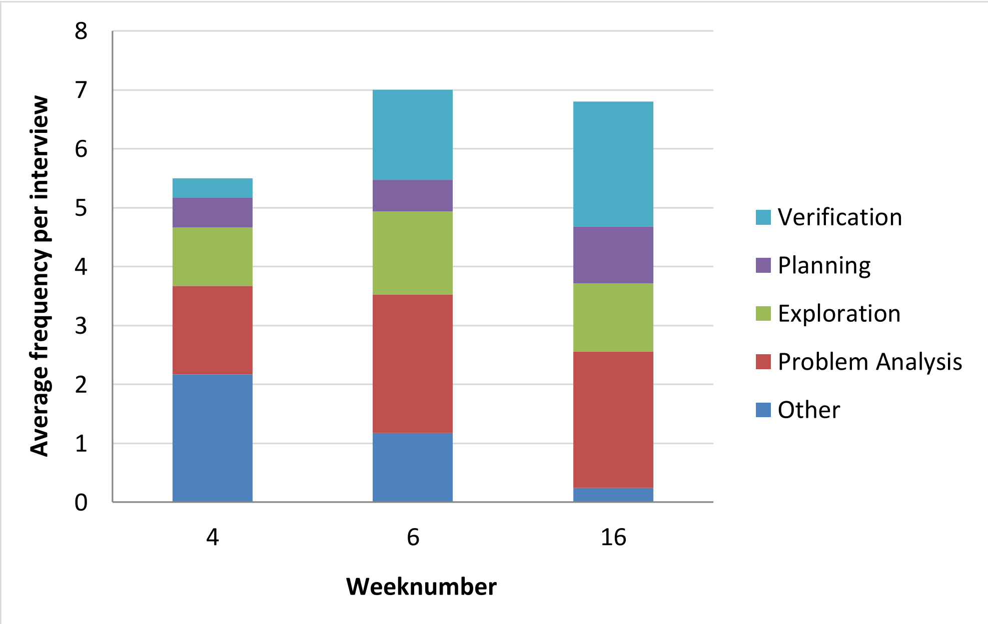



*Figure 14.* The average frequency per interview of the phases Problem Analysis, Exploration, Planning and Verification of the model and the category Other named in the general mini-interviews.

Figure 15 is about the general question from the pre- and post-test. This figure shows again the average frequency of the different parts of the model and the part Other. The average frequency is now displayed per student, because every student filled in the general question individually. In general, the total amount of steps of the different parts named by the students increased. Especially the frequency of the phase Problem Analysis from the model increased. Just like Figure 14 the frequency of the category Other decreased. Although this frequency decreased a lot, this frequency is still higher than the frequency of Verification and the same as the frequency of Planning and Exploration.

When Figure 14 and 15 are compared, it stands out that the total frequency of the mini-interviews about general steps is a lot higher than the total frequency of the general question in the pre- and post-test. This difference can possibly be explained by the difference between answering the general question individually or in groups. It is possible that students together knew more than alone and that they reminded each other of different steps. Beside that, students may gave less answers when they had to write it down than when they were asked to say it out loud. Also, in the interviews the students were encouraged to think of other steps.

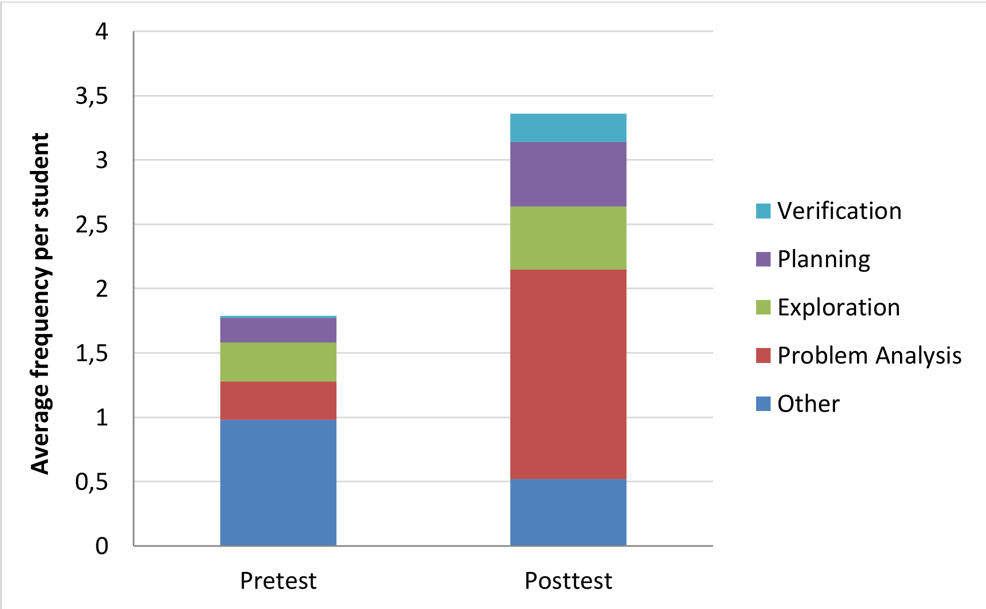



*Figure 15*. The average frequency per student of the phases Problem Analysis, Exploration, Planning and Verification of the model and the category Other named in the general question of the pre- and post-test.

### Pre- and Post-test

Now the results of the pre- and post-test will be discussed. Table 2 contains the descriptive statistics of the standardized pre-test score S_Pretest, the standardized post-test score S_Posttest and the variable Improvement, which is S_Posttest minus S_Pretest. This table also contains the significance of the *t*-test for paired samples, as well as the effect size

Cohen's $d$. The current study is based on a within-subject design, which means that the same group of participants is exposed to all the treatments in the experiment. Cohen's effect size $d_z$ tends to be an overestimation for this kind of design (Cookie Scientist, 2016, March 25; Dunlap, Cortina, Vaslow & Burke, 1996; Lakens, 2013), so the formula $d = t\sqrt{2(1-r)/N}$ (Dunlap et al., 1996; Dunst, Hamby & Trivette, 2004) was used. The means of the standardizes scores on the pre- and post-test confirmed that the students score higher in the post-test then in the pre-test. This difference was statistically significant. There was also a strong significant relationship between the standardized pre-test and the standardized post-test, $r$ = .694, $p$ (one-tailed) < .001.

Table 2

*Descriptives, t-test significance and Cohen's d for the pre- and post-test*

| Variable | N | Mean | SD | $t$(121) | Significance | Cohen's $d$ |
|---|---|---|---|---|---|---|
| S_Pretest | 121 | 2.2901 | 2.0809 | | | |
| S_Posttest | 121 | 3.9752 | 2.3504 | | | |
| Improvement | 121 | 1.0551 | 1.7520 | 6.624 | .000 | .47 |

## Conclusion

In the current study we set out to answer the following research question:

> How can an intervention, including a problem-solving model, learning activities and teacher guidance promote problem-solving skills in mathematics of students in lower secondary education?

The first sub question is how the awareness of students of their own problem-solving process progress through the intervention. The results show that over time the awareness of students' own problem-solving process is promoted and the level of naming these steps by themselves is improved. First, both the general mini-interviews as the general question in the pre- and post-test show that the average frequency of the four phases of the model, namely Problem Analysis, Exploration, Planning and Verification, together increased a lot. The average frequency of the four phases of the model named by the students in the task mini-interviews also increased. This means that during the intervention the students became more aware of different possible steps to take in solving a problem. Second, the average frequency of the category Other, obtained from the general mini-interviews and the general question in the pre- and post-test, decreased. In addition, the task mini-interviews show that for each phase of the model the frequency of Plus increased and the frequency of Minus decreased. So students name more steps by themselves without the influence of the interviewer.

The second sub question is what the effect is of the designed intervention on students' problem solving skills. The results show a significant improvement of the students on the post-test. According to Cohen (1988) an effect size of $d$ = 0.47 is of small to medium size. Several researchers, even Cohen himself, stated that these benchmarks should not be interpreted rigidly, but should be related to other effects in the literature and the context of the study (Hill, Bloom, Black, & Lipsey, 2008; Lakens, 2013; Vacha-Haase & Thompson, 2004). Hill et all. (2008) computed average annual gain in effect size from normed math tests in the United States for each grade. They gave an average effect size of 0.32 for grade 7 - 8. Lipsey and Wilson (1993) gave an overview of the mean treatment effect size found in meta-analysis studies of certain treatment areas. For example, Curbelo's mean effect size of 0.54 (as cited in Lipsey & Wilson, 1993) was found for instruction in problem-solving in science and mathematics vs. conventional instruction for K-12 students. Marcucci's mean effect size of 0.13 (as cited in Lipsey & Wilson, 1993) was found for systematic methods of teaching mathematics problem-solving to elementary and secondary students. Comparing the effect size of the current study to the effect sizes given in literature and considering the relatively short length of the intervention (nine weeks), it may be suggested that an improvement on the problem-solving skills of students has occurred and can be of practical significance. This being said, the mean of the variable S_Posttest is quite low, so work still needs to be done.

With regard to the research question we may conclude that problem-solving skills of students can be promoted by a multidimensional approach with the focus on three elements. First, the teachers explicitly provided a cyclic model for problem solving by giving students guidelines to solve a problem and not a ready-made roadmap. Second, we designed learning activities that really posed a problem, whereby students could not directly think of an answer or solution strategy. Third, attention was paid to the guidance of the teacher, where asking questions, giving hints adapted to the student, and orchestrating whole-class discussions play an important role. The intervention was accurately implemented due to extensive lesson plans, a theoretical justification for the designed activities and multiple meetings between the teachers and the researcher. Although classrooms are influenced by many aspects, we strongly believe that the progress of students in problem solving is caused by the intervention.

# Discussion

## Reflections on the Results

At this point, a number of additional reflections on the problem-solving model, learning activities and guidance of the teacher will be discussed.

**Problem-solving model.** The average frequency of steps of the phase Problem Analysis and thereby the awareness of the students of this phase increased in general only a little. A possible explanation can be that from the start and even before the intervention there has been a lot of attention for the steps from Problem Analysis. Furthermore these steps are intuitive, essential for every kind of task and students will keep on using these steps over time.

It is noteworthy that the average frequency of the phase Planning obtained from the task mini-interviews increased more in weeks 5, 9 and 13 compared to the other weeks. Specifically in these weeks the students were explicitly asked during the instruction lesson to first make a plan and not calculate anything. The total increase of the phase Verification also rhymes with the amount of attention that was paid to this phase during the intervention. This may suggests that explicitly providing attention to a particular phase of the model can promote the awareness of students and use of steps of this phase in their problem-solving process.

The fact that the awareness of the phase Verification is low in the beginning is acknowledged by the teachers. The teachers strongly experienced that when a student has solved a task, they were content with having found an answer and proceeded to the next task. Hardly any student checked their answer in the beginning. The students verified their solutions more over time, but the amount of influence of the interviewer still stayed approximately the same over time. So the verification of a solution remains a point of attention.

**Learning activities.** The current study show that it is important to not only provide non-routine tasks that form a real problem, but also routine tasks about basic skills. Especially in the chapter about factorization and quadratic equations it became clear that good mastery of basic skills as factorizing and solving equations was essential to being able to solve problems. Most students found this subject quite difficult and had to apply these basic skills too early. The students did not yet master these skills and therefore were unable to solve the problems given to them. This is consistent with the literature that stated that the toolbox of skills has to be well filled in order to be able to solve non-routine problems. An example of good practice can be found in the learning activities about the Pythagoras' theorem where students had to apply the theorem so many times that they ultimately knew it by heart.

**Guidance of the teacher.** The way of guiding by the teacher in this intervention was new to the students. They were not used to thinking on their own, searching for an appropriate approach to solve a problem and not getting a straight answer to their questions from the teachers. In the beginning this new way caused quite some resistance from the students.

Students reacted when they did not know right away how to tackle the activity as follows: "I don't understand it." or "I cannot do it, it is too difficult." and stopped trying. Frequently heard reactions were also: "Can you just explain the answer to me?" or "Just tell me how it works!" As the students got more guidelines handed by the model, they gained more confidence in their own ability to solve problems. They experienced that they certainly could take a number of steps in the problem-solving process and often even could solve the entire problem. Over time the teacher only had to confirm students' ideas and point to the phases of the model. As teacher asking the question "what did I do to help you?", helped the students see the major part they had done themselves. Some students even reacted that they were going to like mathematics in this way. It is important that every student experience success and confidence in his own ability. Adapted, faded hints and reflection questions from the teacher can boost confidence, overcome resistance and increase problem-solving skills.

**General.** After the intervention students and teacher were working like before the intervention with pre-structured exercises from the regular mathematics textbook. It was interesting to notice that both students and teachers made less use of what they learned during the intervention. The teachers went back to directly answering questions from students, asking little questions back and referring less to the phases of the model. The students made less use of the phases of the model, gave up more quickly and asked questions earlier to the teacher. This may mean that it is important to grind certain behaviour and to continue the intervention for a longer time to establish a more lasting effect. Lester (1994) standed out that "problem-solving ability develops slowly over a prolonged period of time" (p. 666). Besides, maybe it is harder to use certain behaviour and skills in an environment in which these behaviours and skills are not asked for. An educational approach that is consciously deployed on certain behaviours may invite and stimulate these behaviours more. So in order for students to really obtain problem-solving skills, it is important that problem solving is learned not just in the undergraduate or in a particular topic, but is interlaced throughout the curriculum in lower and upper secondary education.

**Considerations about the Current Study**

At this point, a number of considerations about this study will be discussed. The organisation of the education in the secondary school of the intervention is different from that in most schools in the Netherlands. This may have consequences for the generalizability of the results of this study. This school pays a lot of attention to cooperation in groups of students in all classes. The students are used to consult each other and ask each other questions while working on tasks. Furthermore the students get little instruction from the teacher and have to work autonomously on the assignments under the guidance of the teacher at school. On most schools this is less the case. Therefore, to be able to implement this intervention on other schools more attention to cooperation and working on assignments at school will be needed.

Almost every week the same kind of mini-interviews were conducted with students. This can have an impact on the results. However, this influence will only be positive on the awareness of students of the different components of the model. The questions of the mini-interviews corresponded to the questions the teachers constantly asked while guiding the students. Conducting the mini-interviews every week will therefore be an addition to this guiding. The level of asking concrete questions must be taken into account. As said earlier, students are not used to naming their thinking out loud. Therefore it was sometimes necessary as interviewer to ask more explicitly about different steps. One risk may be that students will give socially desirable answers, but have not actually taken the mentioned step. On the other hand, students may not have named certain steps in their process due to too open questions. Caution is paid to limit the influence of more concrete question to a minimal.

In design-based research it is very common “that researchers are involved in the teaching or work closely with teachers or trainers to optimize the learning environment" (Bakker & van Eerde, 2015, p. 5). Wong (1995) though states that being both a teacher and a researcher can be challenging. The purpose of a researcher or a teacher can contrast each other. The researcher of the current study had to change between the researching and teaching role in her own classes. This could have affected the data. However, the instruction lessons were film recorded for later observation and the mini-interviews were conducted when students themselves were at work for example. It was clear to students when they were expected to answer the questions of the mini-interview and when they could expect the teacher's help. So, the influence of the mini-interviews on the data has remained as limited as possible.

Finally, based on the results of the current study we recommend the implementation of a multidimensional approach, based on providing a problem-solving model, learning activities and the guidance of the teacher, because it can promote problem-solving skills of students in secondary education.

**Acknowledgement**

First, I would like to thank prof. dr. Paul Drijvers for all his time, encouragement and tremendously patience in supervising me. I would also like to thank my colleagues Mark Mulder and Valentijn de Marez Oyens for their support and involvement during all the phases of this study. I could not have done it without them. At last, the feedback from various colleagues and friends helped me enormously, so also thanks to them.

**References**

Bakker, A., & Van Eerde, D. (2015). An introduction to design-based research with an example from statistics education. In A. Bikner-Ahsbahs, C. Knipping & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 429-466). Netherlands: Springer.

Bor-de Vries, M., & Drijvers, P. (2015). *Handreiking denkactiverende wiskundelessen*. Utrecht: Freudenthal Instituut, Universiteit Utrecht

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.

Cohen, J. (1988). *Statistical power analysis for the behavioural sciences.* New York: Lawrence Earlbaum Associates.

Cookie Scientist. (2016, March 25). Five different "Cohen's d" statistics for within-subject designs [Web log post]. Retrieved from http://jakewestfall.org/blog/index.php/2016/03/25/five-different-cohens-d-statistics-for-within-subject-designs/

cTWO (2007). *Rijk aan betekenis. Visie op vernieuwd wiskundeonderwijs.* Utrecht: cTWO.

cTWO (2013). *Denken & doen, wiskunde op havo en vwo per 2015.* Utrecht: cTWO.

Dekker, R., & Elshout-Mohr, M. (1998). A process model for interaction and mathematical level raising. *Educational Studies in Mathematics, 35*(3), 303-314.

Doorman, M., Drijvers, P., Dekker, T., Van den Heuvel-Panhuizen, M., De Lange, J., & Wijers, M. (2007). Problem solving as a challenge for mathematics education in The Netherlands. *ZDM, 39*(5-6), 405-418.

Drijvers, P. (2011). Wat bedoelen ze toch met... denkactiviteiten? *Nieuwe Wiskrant, 31*(2), 38-41.

Drijvers, P. (2015). Kernaspecten van wiskundig denken. *Euclides, 90*(5), 4-8.

Dunlap, W. P., Cortina, J. M., Vaslow, J. B., & Burke, M. J. (1996). Meta-analysis of experiments with matched groups or repeated measures designs. *Psychological Methods, 1*(2), 170-177.

Dunst, C. J., Hamby, D. W., & Trivette, C. M. (2004). Guidelines for calculating effect sizes for practice-based research syntheses. *Centerscope*, *3*(1), 1-10.

Engle, R. A., & Conant, F. R. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*, 20(4), 399-483.

Fernandez, M. L.; Hadaway, N., & Wilson, J. W. (1994). Problem Solving: Managing It All. *The Mathematics Teacher*, *87*(3), 195 – 199.

Field, A. (2009). *Discovering statistics using SPSS* (3rd ed.). Los Angeles: Sage.

Halmos, P. R. (1980). The heart of mathematics. *The American Mathematical Monthly*, *87*(7), 519-524.

Hill, C. J., Bloom, H. S., Black, A. R., & Lipsey, M. W. (2008). Empirical benchmarks for interpreting effect sizes in research. *Child Development Perspectives*, *2*(3), 172-177.

Lakens, D. (2013). Calculating and reporting effect sizes to facilitate cumulative science: A practical primer for t-tests and ANOVAs. *Frontiers in Psychology*, *4*, 863.

Lester, F. K. (1994). Musings about mathematical problem-solving research: 1970-1994. *Journal for Research in Mathematics Education*, *25*(6), 660-675.

Lipsey, M. W., & Wilson, D. B. (1993). The efficacy of psychological, educational, and behavioral treatment: Confirmation from meta-analysis. *American Psychologist*, *48*(12), 1181.

National Research Council. (1990). *Reshaping school mathematics: A philosophy and framework for curriculum.* Washington, DC: National Academy Press.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, *31*(1), 97-111.

Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.

Rott, B., (2012). Models of the problem solving process – A discussion referring to the processes of fifth graders. In T. Bergqvist (Ed.), *Learning problem solving and learning through problem solving*: *proceedings from the 13th ProMath conference* (pp. 95-109). Umeå: UMERC.

Schoenfeld, A. H. (1980). Teaching problem-solving skills. *The American Mathematical Monthly, 87*(10), 794-805.

Schoenfeld, A. H. (1983). The wild, wild, wild, wild, wild world of problem solving (A review of sorts). *For the Learning of Mathematics, 3*(3), 40-47.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sensemaking in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York: MacMillan.

Schoenfeld, A. H. (2007). What is mathematical proficiency and how can it be assessed? In A. H. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 59-73). New York: Cambridge University Press.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem-solving instruction. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33-60). Hillsdale: Lawrence Erlbaum Associates.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313-340.

Vacha-Haase, T., & Thompson, B. (2004). How to estimate and interpret various effect sizes. *Journal of counseling psychology*, *51*(4), 473.

Van Berkel, H., & Bax, A. (2014). Toetsen: toetssteen of dobbelsteen. In H. van Berkel, A. van Bax, & D. Joosten-ten Brinke (Eds.), *Toetsen in het hoger onderwijs* (pp. 15-27). Houten: Bohn Stafleu van Loghum.

Van den Oetelaar, F., (2012). *21st century skills in het onderwijs.*

Van Streun, A. (2001). Hoe staat ons Nederlands wiskundeonderwijs ervoor? *Nieuw Archief voor Wiskunde*, *2*, 42-51.

Van Streun, A. (2014). *Onderwijzen en toetsen van wiskundige denkactiviteiten*. Enschede: SLO.

Voskoglou, M. G. (2008). Problem solving in mathematics education: Recent trends and development. *Quaderni di Ricerca in Didattica (Scienze Matematiche)*, *18,* 22-28.

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57 – 77). New York: Macmillian Publishing Company.

Wong, E. D. (1995). Challenges confronting the researcher/teacher: Conflicts of purpose and conduct. *Educational researcher*, *24*(3), 22-28.

## Appendix A: An example of lesson plans and verification

### Instruction lesson

| **Docent** | **Theorie/Verantwoording** |
|---|---|
| **Voorbereiding:**<br>- Laptops reserveren<br>- Tafels in groepjes van vier klaarzetten<br>- DWO aan klas koppelen<br>- Camera uit team 4 klaarleggen, weten hoe hij werkt en aan begin instructie aanzetten bij de computer, zodat zoveel mogelijk werkers in zicht zijn.<br>- Snelhechters met materiaal week 3 klaarleggen<br>- Filmpje op digibord klaarzetten.<br>- DWO op digibord openzetten<br><br>- Pagina op digibord met tabel om in te vullen<br>**Benodigdheden:**<br>Laat elk tweetal een laptop en pen meenemen naar de instructie.<br><br>**Min 0 – 5 Binnenkomst:**<br>Leerlingen komen binnen. Deel snelhechter bij de deur uit. Leg uit dat in periode 3 de wiskunde op een andere manier aangeboden wordt en dat daarom ze het materiaal in een snelhechter krijgen waar ze de antwoorden in op schrijven, geen boek nodig en dat ze de wiskunde hier op school gaan maken tijdens de wiskunde uren. Leg uit dat elke leerling een snelhechter krijgt waarin hij/zij steeds nieuw materiaal aan toe kan voegen. Elke week op het domein of in de instructie krijgen ze nieuw materiaal uitgedeeld die ze in hun snelhechter kunnen opbergen. Deze snelhechter laten de leerlingen op school, zodat ze de spullen altijd bij zich hebben en er ook echt onder de begeleiding van de docent aan werken.<br><br>Het werk is gepland voor 3 uren wiskunde, maar je hebt er 4 uur voor. Dit betekent dat je iets kunt uitlopen, maar dat je sowieso alles af kunt krijgen op school.<br><br>**Min 5 – 20 DWO:**<br>Leid kort in dat de leerlingen zelf gaan onderzoeken wat er opvalt en een vermoeden gaan opstellen. Duidelijk maken dat ze eerst de opdracht (blz. 2) goed moeten lezen.<br><br>- Mogelijke vermoedens leerlingen:<br>• Vierkant a + vierkant b = vierkant c<br>• Oppervlakte a + oppervlakte b = oppervlakte c<br>• $a^2 + b^2 = c^2$<br>• $a + b = c$ (fout)<br>- Loop de tweetallen langs. Beantwoord alleen vragen als je bij een tweetal bent.<br>- Tijdens het rondlopen het volgende doen:<br>• Technische vragen over Geogebra beantwoorden (kunnen ze elkaar ook mee helpen).<br>• Vragen stellen als: wat weet je al wel? Kun je naar een speciaal geval kijken? Wat valt op? Geldt dit altijd? Hoe kun je dat formuleren? Kun je dat wiskundiger formuleren?<br>• Als leerlingen een vermoeden hebben, stimuleren om dit zo wiskundig mogelijk te formuleren. | Doel:<br>De leerling onderzoekt zelf wat de stelling van Pythagoras is en hoe je daarmee een onbekende zijde kunt uitrekenen.<br><br>PO model:<br>*Onderzoeken en uitproberen*<br><br>Deze fase komt terug in het zelf vermoeden opstellen met DWO. De leerlingen gaan namelijk aan de slag met de volgende onderdelen:<br><br>• Wat weet ik van dit onderwerp?<br>• Speciaal geval (bijv. gehele getallen) zoeken.<br>• Vermoeden opstellen en controleren wanneer dit vermoeden geldt.<br><br>Kenmerken opgave(n) (Bor-de Vries & Drijvers, 2015):<br><br>Onderdeel DWO<br>*Probleem oplossen*<br>- "Een plan van aanpak is onbekend, de opgave vraagt om een nieuwe, inventieve methode, doet beroep op creativiteit" (p. 7). Het opstellen van een vermoeden is nieuw voor de leerlingen, dus een plan van aanpak is onbekend. Ze moeten zelf onderzoeken en ontdekken wat er gebeurt en daar creatief in zijn. Zo doorzien en onthouden ze de stelling gemakkelijker.<br>- "Leerlingen zijn in staat om een begin te maken, iets te proberen. De opgave is toegankelijk voor iedereen" (p. 7). Elke leerling kan namelijk het hoekpunt verschuiven en kijken wat er dan gebeurt.<br><br>*Abstraheren*<br>- "In de opgave wordt op basis van een aantal voorbeelden een algemene regel of patroon gegeneraliseerd" (p. 8). De leerlingen moeten namelijk zelf met verschillende rechthoekige driehoeken een algemene regel opstellen.<br><br>*Algemeen*<br>- "Structureer de opgave niet te veel voor. Geef leerlingen ruimte om te proberen en proefondervindelijk inzicht te ontwikkelen" (p. 10).<br>- "Laat leerlingen eerst een situatie verkennen met getallenvoorbeelden" (p. 10). |

| | |
|---|---|
| • Schrijf verschillende vermoedens en aanpakken kort op voor het klassikaal nabespreken.<br>**Min 20 – 30 Klassengesprek:**<br>- Geef beurten waarbij verschillende vermoedens aan bod komen. Kies ook leerlingen uit met creatieve of foute vermoedens. Maak een keuze in welke volgorde je de vermoedens behandelt. Bijvoorbeeld van minst wiskundig geformuleerd naar steeds meer.<br>- Schrijf de verschillende vermoedens op het bord. Vraag tijdens deze inventarisatie alleen naar hoe ze daar op zijn gekomen/het waarom. Geef voldoende denktijd!<br>- Vergelijk de vermoedens met elkaar door vragen te stellen als:<br>• Wat kan een nadeel/voordeel van deze formulering zijn?<br>• Hoe zeker ben je van je vermoeden? Waarom?<br>• Als leerlingen alleen de woorden 'vierkanten' of 'oppervlakte' gebruiken: wat kun je zeggen als je alleen een driehoek met zijdes gegeven heb? Welke stappen zet je dan?<br>• Hoe kun je dit vermoeden gebruiken?<br>• Kun je dit vermoeden altijd gebruiken of zitten er voorwaarden aan (positieve getallen, rechte hoek)? Op de DWO staat voor de docenten nog 'zomaar een driehoek'. Gebruik die om te laten zien dat de stelling niet werkt zonder een rechthoekige driehoek.<br>- Leid het klassengesprek naar de stelling van Pythagoras door een vermoeden aan de stelling van Pythagoras te linken. Bespreek neutraal wat Pythagoras als stelling heeft opgesteld zonder waardeoordeel over de gevonden vermoedens.<br><br>**Min 30 – 35 Filmpje (als eerste weglaten):**<br>- Laat filmpje zien: https://youtu.be/_e6w5GtkcGI<br><br>**Min 35 – 40 Tabel invullen (bij voorkeur niet weglaten):**<br>- Vul samen met de leerlingen de tabel op blz. 3 in. Benoem de zijde tegenover de rechte hoek als langste zijde.<br>- Benadruk het belang van netjes werken (geen foutjes) en het gebruik van de tabel (schuine/langste zijde).<br>- Laat elke leerling zelfstandig de stelling van Pythagoras invullen op blz. 4 bij opgave 1a. | Docent:<br><br>*Onderdeel Klassengesprek (Stein, Engle, Smith & Hughes, 2008)*<br>De vijf componenten:<br>1) Van tevoren mogelijke vermoedens van leerlingen bedenken.<br>2) Rondlopen door de klas en aantekeningen van vermoedens en aanpakken maken.<br>3) De leerlingen met verschillende vermoedens (goed of fout) een beurt geven.<br>4) Naar de volgorde van behandelen van de vermoedens kijken.<br>5) De verschillende vermoedens met elkaar vergelijken.<br><br>Algemeen:<br><br>- De ICT applet Geogebra is laagdrempelig en daarmee wordt snel inzichtelijk gemaakt wat er gebeurd. Ze kunnen zelf hoekpunt L verschuiven. Er is voor gekozen om de oppervlakte van de vierkanten erbij te zetten om de leerlingen een richting te geven. Zo kan de regelmatigheid sneller opvallen en hebben ze ook genoeg tijd om over de formulering na te denken.<br>- Het belang van de tabel invullen is: correcte antwoorden geven door nauwkeurigheid en het op de juiste plek zetten van de schuine zijde. |

## Working lessons

| **Docent** | **Theorie/Verantwoording** |
|---|---|
| **Domein algemeen**<br>- Loop de groepjes van vier à vijf leerlingen langs en stimuleer ze om netjes en met de tabel te werken. De leerlingen kunnen alleen vragen stellen aan de docent als hij/zij langskomt. Stimuleer het stellen van vragen binnen hun groepje.<br>- Blijf altijd vragen stellen en zeg nooit het antwoord voor. Ook in het geval dat het even niet lukt of de leerling er niets van snapt.<br>- Bedenk van tevoren waar de leerlingen tegen aan kunnen lopen of fout kunnen doen en bedenk daar geschikte vragen of hints bij.<br>- Schrijf opvallende aanpakken of uitspraken van leerlingen kort op, zodat je die kunt gebruiken in de instructies. | - Bor-de Vries en Drijvers (2015) wijzen op het belang van als docent niet direct beschikbaar te zijn voor het zelf ontdekken en uitproberen van de leerling. Het rondlopen van de docent en dan pas als leerling vragen mogen stellen sluit hierop aan.<br><br>- Volgens Bor-de Vries en Drijvers (2015) helpt het van tevoren bedenken van mogelijke fouten, obstakels en vragen bij het stellen van de juiste vragen tijdens het begeleiden. Ook helpt het stellen van een geschikte wedervraag, eerst algemeen en dan steeds specifieker als een leerling er nog niet uitkomt, om het denkproces van het kind op gang te laten komen (Mason, 2000). |
| **Domein 1:** week 3 10 min: opgave 1, 2abc, 3ab<br><br>- Leerlingen kunnen tegen de volgende zaken aanlopen:<br>• rechthoekszijde als schuine zijde uitrekenen;<br>• geen eenheid vermelden.<br>- Als de leerlingen er niet uit komen, stel dan alleen vragen of geef tips:<br><br>• Wat weet je? Wat moet je weten? Vul eens de tabel in. Hoe werkt de stelling en welke zijde moet waar staan?.<br>• Maak eens een schets.<br>• Eerst zo algemeen mogelijke vragen stellen en dan steeds specifieker. Wil je de leerlingen wijzen op verkeerd genomen zijdes of eenheid, doe dat dan.<br>- Controlevragen: kun je iets van deze orde van grootte qua zijde verwachten, is het antwoord volledig (denk aan wel/geen eenheid), kun je het antwoord controleren (invullen in de stelling)? Hoe kunnen deze vragen handig zijn?<br>- Bespreek de overeenkomsten en verschillen (wel/geen driehoek gegeven, wel/geen eenheid, zijdes enz.) van de deelvragen als je rondloopt. | <u>Doel:</u><br><br>Leerlingen kunnen verschillende zijdes uitrekenen met de stelling vanuit een gegeven driehoek of uit tekst.<br><br><u>PO model:</u><br><br>*Stappen uitvoeren*<br>Steeds moeten dezelfde stappen nauwkeurig uitgevoerd worden. Het 'plannetje' is (vaak) al duidelijk, het moet nog uitgevoerd worden.<br><br>*Controleren*<br>Tijdens het rondlopen en begeleiden, wijst de docent op de volgende onderdelen:<br><br>• Kan ik iets van deze orde van grootte verwachten?<br>• Is het antwoord volledig?<br>• Kan ik het antwoord controleren?<br><br><u>Algemeen:</u><br>- Om de stelling toe te kunnen passen in nieuwe abstracte of concrete situaties of om de stelling te kunnen bewijzen, is er een goed begrip van en soepele omgang met de stelling nodig om het werkgeheugen te ontlasten (Van Streun, 2012). Je gereedschapskist moet goed gevuld en van goede kwaliteit zijn om hem te kunnen gebruiken.<br>- Daarom moeten ze in domeinuur 1 week 3 veel oefenen. Dit oefenen gebeurt met 'saaie' opgaves die wel steeds over iets anders gaan: andere zijde, wel/niet plaatje gegeven, wel/niet eenheid gegeven, driehoeken gedraaid. In opgave 1 rekenen ze nog met gehele getallen, in opgave 2 ook met wortels. In opgave 3b is de rechte hoek een andere hoek. Door dit oefenen snappen ze de stelling en kunnen ze de stelling op verschillende manieren gebruiken of uitrekenen.<br>- Ze worden gevraagd om elke keer de tabel in te vullen om het belang (zie theorie/verantwoording bij opgave 1), namelijk nauwkeurigheid en de schuine zijde op de juiste plek, te bevorderen. |

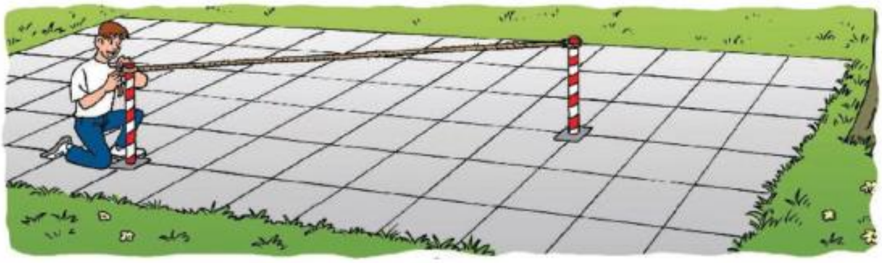
| | |
|---|---|
| **Domein 1:** week 3 15 min: opgave 2d, 3c, 4abc<br><br>- Leerlingen kunnen tegen de volgende zaken aanlopen:<br>• 2d: niet weten hoe ze de zijde moeten uitrekenen zonder rechte hoek of de rechte hoek over het hoofd zien en met zijde 2 en 3 en de stelling gaan rekenen.<br>• 3c: de hoek van 45˚ over het hoofd zien en met twee zijdes van 3 gaan rekenen, niet kunnen beginnen omdat ze niet weten hoe de driehoek eruit ziet en gelijkbenig over het hoofd zien.<br>• 4: de x en y omwisselen, bij c) de driehoek niet zien.<br><br>- Stel de volgende vragen of geef tips:<br>• Wat weet je? Wat wordt er gevraagd? Omschrijf de vraag eens in je eigen woorden.<br>• Maak eens een schets. Kun je de driehoek draaien, zodat het overzichtelijker wordt?<br>• Waar zit de rechte hoek? Kun je de stelling gebruiken of moet je iets anders doen?<br>• Welke eigenschappen heeft een gelijkbenige driehoek?<br>- Als ze bij vraag 2d, 3c of 4ab wel de stelling gebruiken, vraag dan eens of er nog een andere manier is om de opgave op te lossen en wat handiger is. | <u>Doel:</u><br><br>Leerlingen combineren voorkennis en de stelling om een probleem op te lossen. Ze nemen de tijd voor de probleemverkenning.<br><br><u>PO model:</u><br><br>*Probleemverkenning*<br>De volgende onderdelen komen naar voren:<br>• Vraag lezen/herformuleren • Wat weet ik? • Wat moet ik weten?<br><br>*Uitproberen & Onderzoeken*<br>• Schets/tekening maken<br><br><u>Kenmerken opgave(n) (Bor-de Vries & Drijvers, 2015):</u><br><br>*Probleem oplossen*<br>- "Is het een probleem, een niet-standaard opgave? Heeft de opgave iets 'fris', zit er een verrassingselement in" (p. 7)?<br><br>In opgave 2d hebben de leerlingen de stelling helemaal niet nodig en kunnen eerdere kennis over gelijkbenige driehoeken gebruiken.<br><br>Opgave 3c heeft een hoek van 45˚ en is een gelijkbenige driehoek.<br><br>In opgave 4 moeten ze nu de stelling toepassen in een assenstelsel i.p.v. bij een driehoek. Daarnaast hebben ze voor opgave 4a en b de stelling niet nodig.<br><br>- "Een plan van aanpak is onbekend, de opgave vraagt om een nieuwe, inventieve methode, doet beroep op creativiteit" (p. 7).<br>- "Er zijn meerdere oplossingsstrategieën mogelijk" (p. 7).<br>2d: gelijkbenig of de stelling en hoogte driehoek.<br>3c: stelling met KM en LM of stelling en hoogte driehoek. |

**Appendix B: Examples of designed learning activities**

# 4 Een probleem oplossen (instructie week 4)

## 10 Platbodem

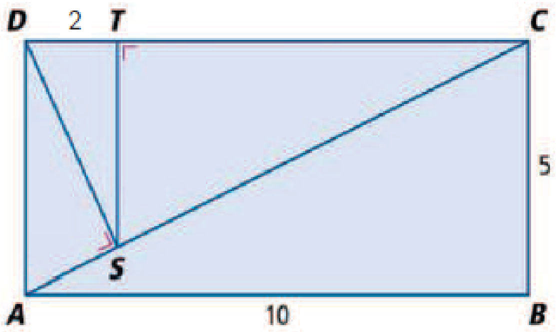
Van de platbodem hierboven is de lange mast 16 meter hoog. De andere mast is 9,5 meter hoog. Van de top van de lange mast wordt een vlaggenlijn strak gespannen naar de top van de andere mast. De masten staan 10 meter uit elkaar.
Bereken de lengte van de vlaggenlijn en leg uit op hoeveel decimalen het beste is om af te ronden.



## 12 Plein

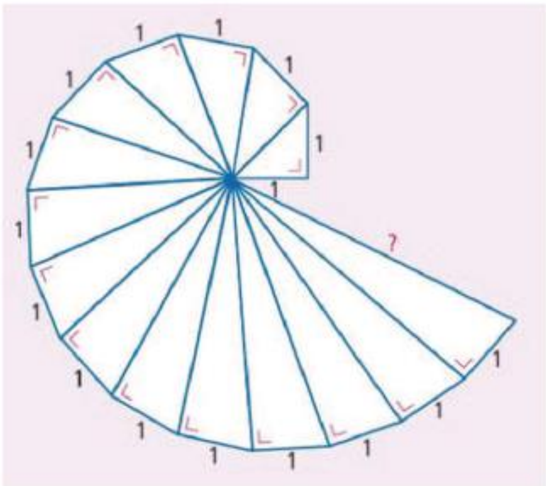
Op het plein hierboven wordt een touw gespannen tussen de paaltjes. De tegels op het plein zijn 50 cm bij 50 cm. Beide paaltjes zijn 70 cm hoog.

Nadat het touw is opgehangen zoals in de afbeelding wordt het strak gespannen van de bovenkant van het linker paaltje naar de onderkant van het rechter paaltje. Hoeveel cm moet dat touw minstens zijn?



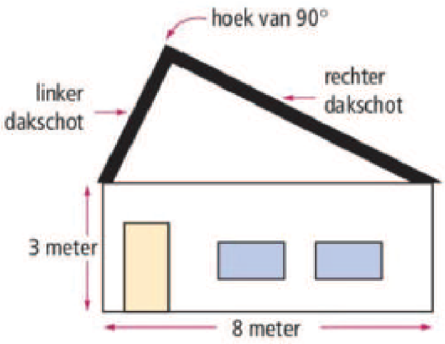

## 13 Gelijkvormig

Alle driehoeken die in bovenstaande rechthoek zitten, zijn gelijkvormig. Gebruik dit gegeven om de lengtes van de nog niet bekende lijnstukken te berekenen. Rond je antwoorden af op één decimaal.




## 14 Slakkenhuis

Hierboven is een soort slakkenhuis getekend. Het is opgebouwd uit 15 rechthoekige driehoeken waarvan telkens een rechthoekszijde 1 lang is.
Bereken de lengte van de zijde waar een vraagteken bij staat.




## 15 *Vakantiehuis (let op, deze opgave is extra voor als je klaar bent!)

Hierboven staat het ontwerp van een vakantiehuisje. Het linker dakschot en het rechter dakschot maken een hoek van 90° met elkaar. Verder is het rechter dakschot twee keer zo lang als het linker dakschot.
Bereken de lengte van het linker dakschot en van het rechter dakschot in cm nauwkeurig.

# 6 De stelling bewijzen (domeinuur 2 week 4)

We hebben de afgelopen week gezien wat de stelling van Pythagoras inhoudt en wat we bedoelen als we zeggen: $a^2 + b^2 = c^2$. Pythagoras was niet degene die de stelling heeft bedacht. Hij was wel degene die hem (voor zover wij weten) als eerste bewezen heeft. Inmiddels zijn er tientallen verschillende bewijzen van de stelling van Pythagoras bekend.
Dit uur krijg je met je werkgroepje drie verschillende bewijzen. De bewijzen zijn echter incompleet of door elkaar gehusseld. Per bewijs mogen jullie één hint aan je wiskundemedewerker vragen. Kunnen jullie de bewijzen weer in elkaar zetten? Zo ja, laat hem controleren door je wiskundemedewerker, dan krijgen jullie een nieuw bewijs en een Pythagoreïsch drietal.
Een wattus? Kijk maar naar het onderstaande theorieblokje.

**Theorie**

Een **Pythagoreïsch drietal** is een drietal gehele positieve getallen $a$, $b$ en $c$ waarvoor geldt: $a^2 + b^2 = c^2$. Het bekendste Pythagoreïsche drietal is: 3, 4 en 5.
Er geldt namelijk: $3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

Helaas! Ook met de Pythagoreïsche drietallen is iets misgegaan. Ook die zitten door elkaar of zijn incompleet. Gebruik de negen getallen die jullie gekregen hebben om twee goede Pythagoreïsche drietallen te vormen. Jullie houden dus drie getallen over.

Veel meetkundig plezier!