Landau-Lifshitz-Gilbert Theory of Spin Current Transmission Through Ferromagnet's

Hendrik Snijder 3731502

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Supervised by R.A.Duine and Scott bender
Institute for Theoretical Physics
Utrecht University

Abstract

In this thesis we study the magnetic behavior of ferromagnet's (FM) and in particular the behavior of so-called magnons or spin waves. Furthermore we study anti ferromagnet's (AFM), with the ultimate goal of describing magnon spin transport through these systems. First we will observe the magnetic dynamics of the FM semi classical and quantum mechanically. Afterward we use collective coordinates to explore the behavior of the AFM close to the classical equilibrium position. From there we turn back to the FM to investigate small oscillations inside the bulk of an FM. After that we will expand our horizon and take a look at the boundaries of the FM. Combined with the behavior of the bulk, this leads to an expression for the spin current through the FM in terms of the spin accumulation entering and leaving the FM.

Contents

1 Introduction

In this thesis we study the magnetic behavior of ferromagnet's (FM) and anti ferromagnet's (AFM) away from equilibrium in particular, how a spin current through the FM may be engendered.
The dynamics of these systems at non-zero but not, too high temperatures are dominated by quantum mechanical effects, especially the electron configurations inside the magnets and how the electronic spins are oriented dominates. To describe these effects we will introduce *magnons* in the next subsection. After that we describe magnon dynamics. Therefore we will first observe the magnetic dynamics of the FM semi-classical and quantum mechanically suggested by [1] (Chapter 2)[1]. Afterward we use collective coordinates in Chapter 3[2] to explore the behavior of the AFM close to the classical equilibrium position, indicated by [2].When we have done that, we are able to investigate the current trough a ferromagnet. This will be our final goal. To do that we we turn back to the FM and using the equations of motion suggested by [3] to investigate small oscillations inside the bulk of an FM, which is sandwiched between an poor spin-sink and an strong spin-sink. From there we will take a look at the boundaries of the FM. Combined with the behavior of the bulk, this leads to an expression of the current trough the FM in terms of the spin accumulation. This will be done in Chapter 4. But first we make some general notes in the following section to get started.

1.1 Theoretical Background

In this section we will treat some general concepts and definitions in a qualitatively and intuitive manner. First we consider some magnetic characteristics and then we will say something about the spin currents through these magnets.

1.1.1 Magnetic Behavior Away From Equilibrium

The ideas in this subsection are indicated by [1]. Consider a magnetic material, then almost all the (microscopic-) characteristics of it depends on the local structure of the magnet, in particular the electronic configuration of the material is important. There are two very useful examples.

First of all we will define the **ferromagnet**. Inside the ferromagnet the electrons are arranged in such a way, that all the electrons have the same spin direction, which constitutes *equilibrium*. In equilibrium *i.e.* at zero temperature, the directions of the spins are static. If the system absorbs in some way more energy, *e.g.* by increasing the temperature of the magnet (not too much), here arise some oscillations around this static equilibrium, *i.g.* $0 < T < 1043K = Curietemperature$[5] for iron.
Now, let's consider a 2D-model of the ferromagnet and let one electron spin start precessing around the z-axis; then the neighboring spins will also start precessing, however they will pick up a little phase-shift. Hence an increasing of the internal energy of the ferromagnet creates standing waves inside the magnet. These spin-waves can described quantum mechanically. In that case the wavelike behavior is translated to the existence of an (quasi-)particle. This particle is called a **magnon**.
If the temperature becomes above the *Currie*-temperature the ferromagnetic ordering will be gone, *i.e.* the Ferromagnet has become a paramagnet. So general speaking we can say that if at

[1]The content of this chapter is composed with the help of Thomas Floss[6].
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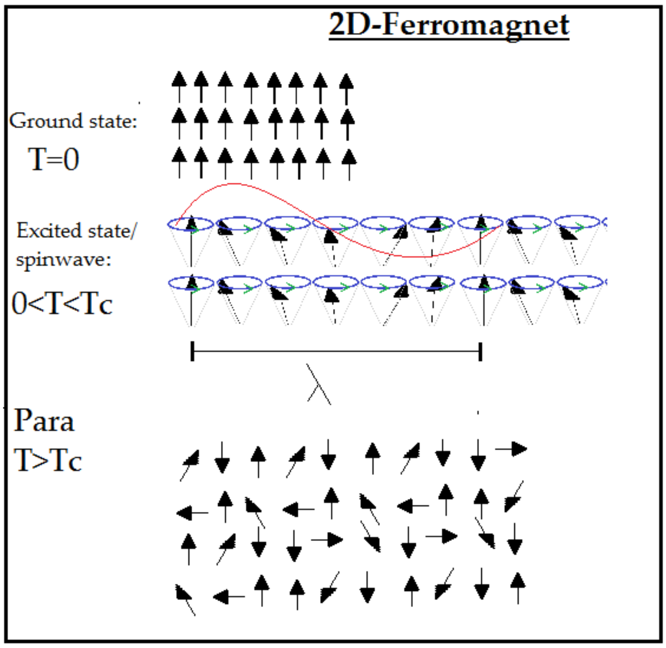

Figure 1: *An example of a 2D FM in different temperature regimes. The arrows denotes the electronic spin direction. In the ground state pointing along $\hat{z}$ ($T < T_C$). If the temperature is between zero and the Curie-temperature (T_C) the spins start precessing around the z-direction, these spin waves are denoted as magnons with wavelength lambda. If the temperature becomes above this critical temperature($T > T_C$), the FM becomes a paramagnet.*

$T = 0$ the nett magnetic moment is in the $\hat{z}$-direction. These three situations are given in fig. **??**.

On the other hand we can define an **fig:antiferromagnet**. This is an orderd magnet but instead of all spins pointing in the same direction, they alternate in opposite direction. There are two spin-lattices. The magnons in the AFM are slightly different from that one inside the ferromagnet, where they can described as a "vibration" of the spin lattice. In the case of two opposite lattices there are two spinwaves. Away from the zero temperature we can define the *Neël*-temperature, wich is the anti-ferromagnet analog of the Curietemperature. The charactaristics of this magnet are visualised in fig. 2.

In this thesis we are interested in the dynamics of magnons, hence we will work with Temperatures above zero but still far below the *currie/neël*-temperatures.

1.1.2 Currents of Magnons Trough Interfaces

Let us consider a magnetic insulator sandwiched between two non-magnetic metals see 3. There is a coupling between magnons in the insulator and the electronic spin currents in the non-magnetic metals[7]. The spin current depends on the spin pumping, the spin accumulation at the boundaries and the magnons transferring through the insulator[7]. If the left reservoir is a poor spin sink and the right a strong spin sink, then a current coming from the left, crosses the interface with the reservoir and the insulator, from where it causes spin waves inside the insulator, which transfer through the insulator. In Chapter 4 we will investigate the dynamics

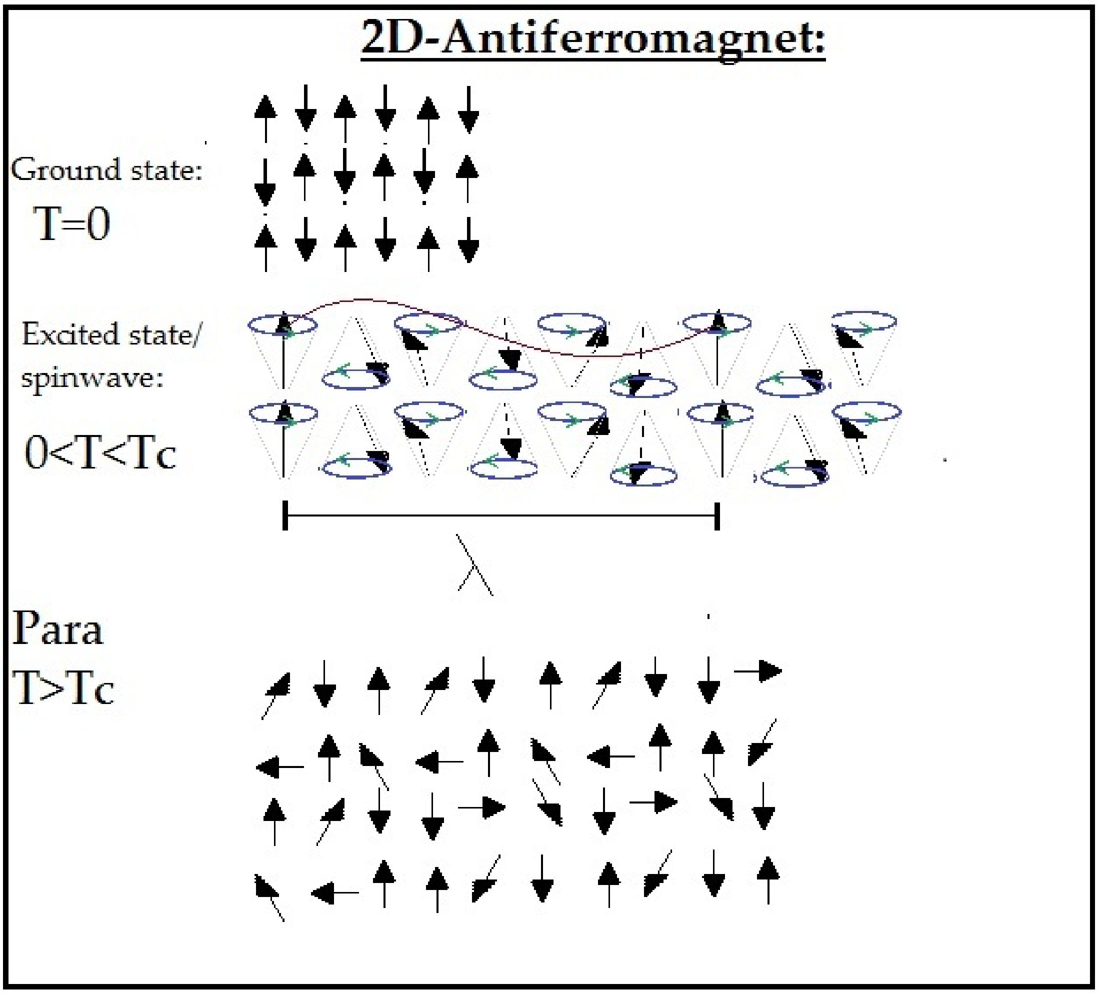

Figure 2: *An example of a 2D AFM in different temperature regimes. The arrows denotes the electronic spin direction. In the ground state they are alternating pointing up and down(*$T < T_C$*). If the temperature is between zero and the neël-temperature (*T_C*) we see the up and down lattices vibrating, which are spin waves with wavelength lambda. If the temperature becomes above this critical temperature(*$T > T_C$*), the AFM becomes a paramagnet.*

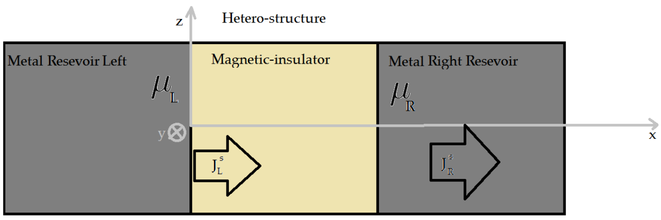

Figure 3: *Heterostructure. Here we see a magnetic insulator sandwiched between two metallic reservoirs. The left reservoir has a spin accumulation of* μ_L *and is an poor spin sink. The right reservoir has a spin accumulation of* μ_R *and is a strong spin sink. The spin current moves from the left to the right. This picture is self-produced, however it is based on ideas from [3],[4] and [7]*

of such a system, first the dynamics inside the bulk is treated. After that the current of such a structure will be investigated.

1.1.3 Some Extra Notes on the Rest of this Thesis

In this thesis we will further investigate the things mentioned in the previous sections in more detail. First we start with calculating the minimum energy for a magnon needed to exist in a ferromagnet (chapter 2). Later we take a look at the two different lattices inside the anti-ferromagnet and will derive expressions for spin dynamics in terms of the position and time(chapter 3). Next we will start with the dynamics inside the ferromagnet and try to find some general solution for the current trough the interface of an ferromagnetic heterostructure

2 Magnons in Ferromagnets

In this section we study the classical dynamics of of ferromagnet's. We will perform calculations suggested by [1]. Let us start with the Heisenberg-exchange Hamiltonian [1] i.e.

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j \tag{1}$$

where $\langle i, j \rangle$ denotes the sum over nearest neighbors. $J > 0$ Is the coupling constant for ferromagnet's.

2.1 The Semi-Classical Case

A semi-classical approximation means that we describe it both classical and quantum mechanically. If we start with classical the Hamiltonian (1) and treat it quantum mechanically, we may apply Ehrenfest's theorem, *i.e.*

$$\frac{d}{dt}\langle \hat{S}_\alpha \rangle = \frac{-i}{\hbar}\langle [\hat{S}_\alpha, \hat{H}] \rangle, \alpha \in \{x, y, x\}. \tag{2}$$

If we now plug in (1) into (2) and assume a one-dimensional chain of spins pointing along $\hat{z}$, we find

$$\frac{d}{dt}\langle \hat{S}_\alpha \rangle = J\frac{i}{\hbar}\big[\hat{S}_\alpha, \sum_i (\hat{S}_i \cdot \hat{S}_{i+1} + \hat{S}_i \cdot \hat{S}_{i-1})\big]. \tag{3}$$

Let's assume $\hat{S}_z = \hbar S$ and $\hat{S}_x, \hat{S}_y << \hat{S}_z$ [1]. If we now use $[\hat{S}_i^\alpha, \hat{S}_i^\beta] = i\hbar\epsilon_{\alpha\beta\gamma}\hat{S}_k^\gamma$[1], with $\epsilon_{\alpha\beta\gamma}$ the Levi-Civita-tensor, and plug everything in equation (3), where after we try an ansatz $\hat{S}_i^\alpha \propto e^{i\vec{k}\cdot\vec{R}_j - i\omega t}$[1] with $\vec{R}_j = j \cdot a\hat{x}$[1] and write everything in matrix form[1], we obtain

$$\begin{pmatrix} i\omega & 2SJ\hbar(1 - cos(k_x a)) \\ 2SJ\hbar(1 - cos(k_x a) & -i\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4}$$

This is true if the determinant of the matrix on the left side of equation (4) is zero. The determinant is zero if

$$\omega_k \equiv \omega(k) = 2SJ\hbar(1 - cos(ka)). \tag{5}$$

We call ω_k the *eigenvalue* or *dispersion* of the matrix in equation (4). According to that formulation we can obtain the corresponding eigenvectors:

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}. \tag{6}$$

The v_1 vector corresponds with the static case. So when the system is in state v_1 we write $A = v_{1,x}$ and $B = v_{1,y}$. This leads to $S_z = \hbar S$, $\hat{S}_i^x = 0$ and $\hat{S}_i^y = 0,$. The second vector, v_2, describes an eigenstate with magnons. If we set $\begin{pmatrix} v_{2,x} \\ v_{2,y} \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$ and let $a = 0$ we get the following expressions for the real parts of $\hat{S}_i^x$ and $\hat{S}_i^y$:

$$\hat{S}_i^x = \frac{1}{\sqrt{2}} sin(\omega t), \hat{S}_i^y = \frac{1}{\sqrt{2}} cos(\omega t). \tag{7}$$

From this we see that when the system is in eigenstate v_2, the spin on lattice side i, precess around the z-axis with period $\frac{2\pi}{\omega}$. If we now look at all the sites, we see that each site picks up an phase shift of ka. Hence we have described a spin wave. If we assume $k << 1$ we can approximate *omega* by $\omega \approx SJ\hbar(ka)^2$. ENERGY??

2.2 Holstein-Primakoff

In this section describe small amplitude excitations of eq. (1) quantum mechanically by performing the Holstein-Primakoff transformation. With that done we want to find again the energy for the magnons inside an ferromagnet.

We start again with eq. (1). As in [1] we define

$$\hat{S}_i^+ \equiv \hbar a_i \sqrt{2S - a_i^\dagger a_i}, \hat{S}_i^- \equiv \hbar a_i^\dagger \sqrt{2S - a_i^\dagger a_i}, \hat{S}_i^z \equiv \hbar S - \hbar 2S - a_i^\dagger a_i \tag{8}$$

such that $[a_i, a_j\dagger] = \delta_{ij}$. We may now define the counting operator $\hat{n} \equiv a_i^\dagger a_i$ such that $\langle \hat{n}_i \rangle \leq 2S$[1]. Then, by taking an Taylor expansion of (8) and plug it back in (1), we obtain [1]

$$\hat{H} = -J\hbar^2 \sum_i \sum_\delta S^2 - Sa_i^\dagger a_i - a_{i+\delta}^\dagger a_{i+\delta} S a_{i+\delta}^\dagger a_i + S a_i^\dagger a_{i+\delta}. \tag{9}$$

Defining $a_i^{(\dagger)} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{-\vec{k}\cdot\vec{r}_i} a_{\vec{k}}^{(\dagger)}$, we obtain by plugging $a_i^{(\dagger)}$ into (9) the following expression [1]:

$$\hat{H} = E_0 + \sum_{\vec{k}} \hbar\omega_{\vec{k}} a_{\vec{k}}^\dagger a_{\vec{k}}. \tag{10}$$

In this equation we have the ground state energy of a magnon in the chain $E_0 = -J\hbar^2 S^2 N \frac{z}{2}$ and $\omega_{\vec{k}} = SJ\hbar z(1 - cos(k_x a)$[1], where z is the number of pairs of nearest neighbors.

2.3 The Semiclasical Method Beside Holstein-Primakoff

As we have seen, we approached the dynamics inside the ferromagnet semi-classical and quantum mechanically. In the semi-classical case we used the classical Hamiltonian (1) to calculate the time evolution of the spin operator $\hat{\vec{S}}$ for small oscillations around the z-direction of the spin in an one-dimensional chain using the quantum mechanical Ehrenfest's theorem (3). From this time evolution we where able to calculate the frequency of the oscillations ω. From which we calculated the ground state of the vibration which had an energy $\hbar\omega > 0$. For the quantum mechanical treatment we made the classical Hamiltonian reasonable in a quantum mechanical sense. To do that we defined the spin raising-/lowering-operators which we plugged in (1). From there we again take a look at small oscillations from which we obtained that the lowest energy is given by $E_0 = -J\hbar^2 S^2 Nz/2$ and where the dispersion is given by $\omega_{\vec{k}} = SJ\hbar z(1 - cos(k_x a)$.

3 Magnons in Anti-Ferromagnets

Let's consider an AFM described by a lattice which consists of two sub lattices with magnetic moments $\vec{m}_1(\vec{r},t)$ and $\vec{m}_2(\vec{r},t)$. We define the total magnetization m and the unit Neel vector n as:

$$m(\vec{r},t) \equiv \vec{m}_1(\vec{r},t) + \vec{m}_2(\vec{r},t), \tag{11}$$

$$n(\vec{r},t) \equiv \frac{\vec{m}_1(\vec{r},t) - \vec{m}_2(\vec{r},t)}{\|\vec{m}_1(\vec{r},t) - \vec{m}_2(\vec{r},t)\|}. \tag{12}$$

Note that $\vec{n}$ is a unit vector wich gives the direction of the AFM ordering. Our goal is to find expressions for some small derivations of $\vec{m}$ and $\vec{n}$ around the z-axis. In that case our model is still corresponding with an AFM. But first we need some theory.

We will use the AFM-free energy suggested by [2], given by

$$U = \int [\frac{a}{2}\vec{m}^2 + \frac{A}{2}\sum_{i=x,y,z}(\partial_i\vec{n})^2 - \vec{H} * \vec{m}]d\vec{r}. \tag{13}$$

With a the homogeneous exchange constants and A inhomogeneous one. $\vec{H}$ represents an external magnetic field. We are interested in the behavior of the AFM without an external magnetic field, hence $\vec{H} = 0$. Now, from the equations (11) and (12) we can obtain the constraints (I) $|\vec{n}| = 1$ and (II) $\vec{m} \cdot \vec{n} = 0$. Also we define the effective fields[2] as

$$\vec{f}_n \equiv -\frac{\delta U}{\delta \vec{n}}, \vec{f}_m \equiv -\frac{\delta U}{\delta \vec{m}}. \tag{14}$$

With the constraints (I), (II) and $\vec{H} = \vec{0}$ we can calculate the variational derivatives of U in the $\vec{n}$- and $\vec{m}$-directions for small oscillations around the equilibrium of the AFM. With that we can construct the the linear effective field. Doing so, we get from

$$\vec{n} \longrightarrow \begin{pmatrix} \delta n_x \\ \delta n_y \\ nz \end{pmatrix}, \vec{m} \longrightarrow \begin{pmatrix} \delta m_x \\ \delta m_y \\ mz \end{pmatrix}, \tag{15}$$

and via the constraints (I) and (II) ,

$$\vec{n} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ 1 \end{pmatrix}, \vec{m} = \begin{pmatrix} \delta m_x \\ \delta m_y \\ 0 \end{pmatrix}. \tag{16}$$

This represents the AFM close to the equilibrium. Note that the time derivatives of the small oscillations are then given by

$$\partial_t\vec{n} = \begin{pmatrix} \partial_t\delta n_x \\ \partial_t\delta n_y \\ 0 \end{pmatrix}, \partial_t\vec{m} = \begin{pmatrix} \partial_t\delta m_x \\ \partial_t\delta m_y \\ 0 \end{pmatrix}. \tag{17}$$

If we now plug equation (16) in (14) and use $\vec{H} = 0$ we end up with

$$\vec{f}_n = A\vec{n} \times (\nabla^2\vec{n} \times \vec{n}) \simeq A \begin{pmatrix} -\delta\nabla^2 n_x \\ \delta\nabla^2 n_y \\ 0 \end{pmatrix} \tag{18}$$

and

$$\vec{f}_m = -a\vec{m} \simeq -a \begin{pmatrix} -\delta m_x \\ \delta m_y \\ 0 \end{pmatrix}, \tag{19}$$

where we neglect terms δ^2 and higher. Let us now introduce the equations of motion in this AFM suggestet by [2]:

$$\partial_t \vec{n} = \gamma \vec{f}_m \times \vec{n}; \partial_t \vec{m} = \gamma \vec{f}_n \times \vec{n}. \tag{20}$$

By equation (18) and (19) we can calculate equation (20). We obtain

$$\partial_t \vec{n} = -a\gamma \begin{pmatrix} \delta m_y \\ -\delta m_x \\ 0 \end{pmatrix} \tag{21}$$

and

$$\partial_t \vec{m} = \gamma A \begin{pmatrix} \nabla^2 \delta n_y \\ \nabla^2 \delta n_x \\ 0 \end{pmatrix}. \tag{22}$$

To understand the behavior of the AFM close to equilibrium we have to solve (21) and (22). Therefor we make the observation that both equations are wave equations. This suggests to try ansatz of the form $\vec{n} = (\alpha_x, \alpha_y)^t e^{-ikx-i\omega t}$ and $\vec{m} = (\beta_x, \beta_y)^t e^{-ikx-i\omega t}$. Plugging these in the equations of motion and writing it in a matrix-form equation leads us to

$$\begin{pmatrix} i\omega & 0 & 0 & -\gamma \\ 0 & i\omega & a\gamma & 0 \\ 0 & -A\gamma k^2 & i\omega & 0 \\ -A\gamma k^2 & 0 & 0 & i\omega \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \beta_x \\ \beta_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{23}$$

This equation is true if the determinant of the matrix on the left side is 0. With that we can calculate the dispersion relation of the magnons. Hence by setting the determinant= 0 we will find

$$\omega = \pm \gamma k \sqrt{aA}. \tag{24}$$

With these dispersions we can obtain the equations of $\vec{n}$ and $\vec{m}$, which are given by:

$$\vec{n} = \begin{pmatrix} \alpha_x \\ \alpha_y \\ 1 \end{pmatrix} e^{-ikx \pm i\gamma k\sqrt{aA}t} \tag{25}$$

$$\vec{m} = \begin{pmatrix} \beta_x \\ \beta_y \\ 0 \end{pmatrix} e^{-ikx \pm i\gamma k\sqrt{aA}t}. \tag{26}$$

Looking back on what we did in this section, we see that we have used the equations of motion (20) suggested by [2]. After making an approach of small oscillations of $\vec{n}$ and $\vec{m}$ around the equilibrium of the AFM we have calculated the dispersion relation and the oscillating vectors by plugging the approximated effective fields (18),(19) in to the free energy (13). As we see in the equations (25) and (26) the oscillations around the equilibrium of the AFM causes a wave-effect of the spindown- and spinup-lattice.

4 Magnon Spin Currents Through Ferromagnets

In this section we will use Green's functions to solve a linearized LLG (Laundau-Lifshitz-Gilbert)-equation applied on a ferromagnetic bulk, which describes the magnetic dynamics of such a system. After this semi-classical exercise we look at the boundaries of the ferromagnet. From there we are able to set some boundary conditions which, combined with the magnetic dynamics inside the ferromagnet, lead to an expression for the current through the ferromagnetic system. In this Chapter we consider a hetero-structure such as in figure 3, where the insulator is given by a ferromagnet.

4.1 Magnetic Motion in the Bulk

To get a sense of what is happening inside the ferromagnetic bulk, let us consider a ferromagnet sandwiched between two metals, a poor spin sink and an almost perfect one [ref 3,fig.1] and 3. In this section we neglect boundary conditions and focus on bulk dynamics. For this situation we can use the equation of motion(E.O.M.) suggested by ref.[3, eq 1], which is given by:

$$\frac{\partial}{\partial t}\vec{m} = -\gamma\vec{m} \times (\vec{H}_{eff} + \vec{h}_l) + \alpha\vec{m} \times \frac{\partial}{\partial t}\vec{m}. \tag{27}$$

This E.O.M. gives a recursive expression for the time-evolution of the unit magnetization vector $\vec{m} \equiv \frac{\vec{M}}{M_s}$, where $\vec{M}$ is the magnetization vector with length M_s [3]. Furthermore we have two constants, $-\gamma$ and α which represents the gyro magnetic ratio and Gilbert damping respectively[3]. The other two terms are given by:

$$\vec{H}_{eff} \equiv -\frac{\delta}{\delta\vec{M}}F = H_a\vec{z} + A_x\nabla^2\vec{m} + \vec{H}_r \tag{28}$$

and

$$\langle h_{l,i}(\vec{r},t)h_{l,j}(\vec{r'},t')\rangle = \frac{2\alpha}{\gamma M_s}k_BT(\vec{r})\delta_{ij}\delta(\vec{r}-\vec{r'})\delta(t-t'), \tag{29}$$

which eq.(28) represents the effective field and eq.(29) shows the correlator of the random Langevin field $\vec{h_l}$ [3]. In the expression for $\vec{H}_eff$, H_a and A_x denote constants due to the applied field and the exchange field respectively.
To make things simple and semi-classical, we need the following assumptions:
(i) The temperature is much lower than the Curie-temperature, so no spontaneous demagnetization can occur. On the other hand, we don't expect to low temperatures, otherwise quantum fluctuations become important [3].
(ii) We will take $T(\vec{r}) = constant$.
(iii) For the relativistic correction term we assume $\vec{H}_r = 0$, which is justified when $k_BT >> \hbar\gamma M_s$[3].
(iv) The average of $\vec{h_l}$ can be ignored when $\omega << \frac{k_BT}{\hbar}$ [3], which keeps the problem (quite) classical. Furthermore, we define the spin density $\vec{s} \equiv s\vec{n} \equiv -\frac{M_s}{\gamma}\vec{m}$, where $s = \frac{M_s}{\gamma}$ and $\vec{n} = -\vec{m}$, the magnitude of the spin density and its direction[3].

Now, if we rewrite the E.O.M. in terms of $\vec{s}$ and $\vec{n}$ we find:

$$0 = \frac{\partial}{\partial t}\vec{n} + \gamma\vec{n} \times (\vec{H}_{eff} + \vec{h}_l) + \alpha\vec{n} \times \frac{M_s}{\gamma}\frac{\partial}{\partial t}\vec{n}. \tag{30}$$

After doing that, we can multiply both sides by a factor $\frac{M_s}{\gamma} = s$ and use a slightly denser notion for the partial derivatives, we get:

$$0 = s\partial_t\vec{n} + \vec{n} \times (M_s\vec{H}_{eff} + M_s\vec{h}_l) + \alpha\vec{n} \times s\partial_t\vec{n}. \tag{31}$$

After, applying assumption (*iii*) on equation (28), defining $H \equiv M_s H_a$, $\vec{h} \equiv M_s \vec{h_l}$ and $A \equiv M_s A_x$, *ànd* rearranging some terms, we have:

$$0 = s(1 + \alpha\vec{n}\times)\partial_t\vec{n} + \vec{n} \times (H\hat{z} - A\nabla^2\vec{n} + \vec{h}). \tag{32}$$

This expression can be written as:

$$0 = s(1 + \alpha\vec{n}\times)\partial_t\vec{n} + \vec{n} \times (H\hat{z} + \vec{h}) - \vec{n} \times A\nabla^2\vec{n}. \tag{33}$$

Let us now define the spin current as:

$$\vec{J}_{s,i} \equiv -A\vec{n} \times \partial_i\vec{n}. \tag{34}$$

Now, if we take the derivative of $\vec{J}_{s,i}$ in the direction i and use $-A\partial_i\vec{n} \times \partial_i\vec{n} = 0$, we get:

$$\partial_i\vec{J_{s,i}} = -A\vec{n} \times \nabla^2\vec{n}. \tag{35}$$

In this equation we used Einsteins summation convention (E.S.C.). Since the cross product is compatible with scalar multiplication and A is a scalar, we can pull A over the $\times$-symbol and plug eq.(35) in eq.(33). Doing that, we end up with the following E.O.M.:

$$0 = s(1 + \alpha\vec{n}\times)\partial_t\vec{n} + \vec{n} \times (H\hat{z} + \vec{h}) + \partial_i\vec{J}_{s,i}, \tag{36}$$

which is a very useful expression in our discussion of spins and will be the starting point of our further calculations.

4.1.1 Linear Motion of the Bulk,

At this point we are ready to study the motion of $\vec{n}$ in the semi-classical approach. This means that we have to take the classical limit and differ a little from it to introduce the quantum effects. To do so, we assume that $\vec{n}$ is in equilibrium if $\vec{n} = -\hat{z}$, which is compatible with the notion of ferro magnetism in the classical manner. After setting $\vec{n}$ in the $-\hat{z}$-direction we can vary (due to the quantum mechanics) a little in the $x-$ and $y-$direction by a small change of δn_x and δn_y. This variation is random and it is determined by the stochastic term $\propto \vec{h}$ in eq.(36), given by eq.(29).

Let assume that

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \longrightarrow \tilde{\vec{n}} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ -\sqrt{1 - \delta n_x 2 - \delta n_y^2} \end{pmatrix}. \tag{37}$$

To get rid of the $\sqrt{}$-sign we perform a Taylor expansion of the second order in the $\hat{z}$-direction of $\tilde{\vec{n}}$. Hence,

$$\tilde{\vec{n}} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 + \frac{1}{2}\delta n_x 2 + \frac{1}{2}\delta n_y^2 \end{pmatrix} \; and \; \partial_t\tilde{\vec{n}} = \begin{pmatrix} \partial_t\delta n_x \\ \partial_t\delta n_y \\ \partial_t(-1 + \frac{1}{2}\delta n_x 2 + \frac{1}{2}\delta n_y^2) \end{pmatrix}. \tag{38}$$

If we assume that the variations are so small that we can neglect δ^2-terms, then the $\delta\vec{n}$'s are;

$$\tilde{\vec{n}} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \; and \; \partial_t\tilde{\vec{n}} = \begin{pmatrix} \partial_t\delta n_x \\ \partial_t\delta n_y \\ 0 \end{pmatrix}. \tag{39}$$

These equations can now plugged in easily in the equation of motion (36). So we acquire:

$$\vec{0} = s[1+\alpha \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times] \begin{pmatrix} \partial_t \delta n_x \\ \partial_t \delta n_y \\ 0 \end{pmatrix} + \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times [H\hat{z} + \vec{h}] - \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times A\nabla^2 \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix}. \tag{40}$$

If we use now the E.S.C., we can write this expression as

$$\vec{0} = s[1+\alpha \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times] \begin{pmatrix} \partial_t \delta n_x \\ \partial_t \delta n_y \\ 0 \end{pmatrix} + \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times [H\hat{z} + \vec{h}] - \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times A\partial_i^2 \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix}. \tag{41}$$

Since expression is quite long, we will calculate the terms one by one first. After doing that we can glue everything together. For the first term we end up with

$$s[1+\alpha \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times] \begin{pmatrix} \partial_t \delta n_x \\ \partial_t \delta n_y \\ 0 \end{pmatrix} \overset{\delta^2 \to 0}{\simeq} s \begin{pmatrix} \partial_t \delta n_x + \alpha \partial_t \delta n_y \\ \partial_t \delta n_y - \alpha \partial_t \delta n_x \\ 0 \end{pmatrix}. \tag{42}$$

If we now use $\vec{h} = (h_x h_y h_z)^t$, then second term appears to be

$$\begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times [H\hat{z} + \vec{h}] = \begin{pmatrix} \delta n_y (H + h_z) + h_y \\ -\delta n_x (H + h_z) - h_x \\ \delta n_x h_y - \delta n_y h_x \end{pmatrix}. \tag{43}$$

For the last expresion we derive

$$\partial_i \vec{J_{s,i}} = - \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \times A\partial_i^2 \begin{pmatrix} \delta n_x \\ \delta n_y \\ -1 \end{pmatrix} \overset{\delta^2 \to 0}{\simeq} -A \begin{pmatrix} \nabla^2 \delta n_y \\ -\nabla^2 \delta n_x \\ 0 \end{pmatrix}. \tag{44}$$

If we combine equation (41) (42) (43) and (44) we find

$$\vec{0} = s \begin{pmatrix} \partial_t \delta n_x + \alpha \partial_t \delta n_y \\ \partial_t \delta n_y - \alpha \partial_t \delta n_x \\ 0 \end{pmatrix} + \begin{pmatrix} \delta n_y H + h_y \\ -\delta n_x H - h_x \\ \delta n_x h_y - \delta n_y h_x \end{pmatrix} - A \begin{pmatrix} \nabla^2 \delta n_y \\ -\nabla^2 \delta n_x \\ 0 \end{pmatrix}, \tag{45}$$

where we used the the fact that h_z is small, *i.e.* $\delta n_x h_z = \delta n_y h_z = 0$. The resulting, Linearized equation contains all the information about the behavior of $\vec{n}$ in the semi-classical limit subject to our assumptions (i)-(iv). To get a real picture of the meaning of this behavior we have to solve (45) for δn_x and δn_y. But for we do this, let's switch to the complex plane by defining

$$\delta n_\pm \equiv \delta n_x \pm i \delta n_y, \tag{46}$$

which by plugging in (45) leads to

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s(\frac{\partial_t \delta n_+ + \partial_t \delta n_-}{2} + \alpha \frac{\partial_t \delta n_+ - \partial_t \delta n_-}{2i}) + \frac{\delta n_+ - \delta n_-}{2i} H + h_y - A\nabla^2 \frac{\delta n_+ - \delta n_-}{2i} \\ s(\frac{\partial_t \delta n_+ - \partial_t \delta n_-}{2i} - \alpha \frac{\partial_t \delta n_+ + \partial_t \delta n_-}{2}) - \frac{\delta n_+ + \delta n_-}{2} H - h_x + A\nabla^2 \frac{\delta n_+ + \delta n_-}{2} \\ \frac{\delta n_+ + \delta n_-}{2} h_y - \frac{\delta n_+ - \delta n_-}{2i} h_x \end{pmatrix}. \tag{47}$$

To lose this notation, we can write it as three separate equations and use $1/i = -i$ for the first line and multiply the second one by i. So the above equation is equivalent to:

$$0 = s(\frac{\partial_t \delta n_+ + \partial_t \delta n_-}{2} - \alpha \frac{i\partial_t \delta n_+ - i\partial_t \delta n_-}{2}) - \frac{i\delta n_+ - i\delta n_-}{2} H + h_y + A\nabla^2 \frac{i\delta n_+ - i\delta n_-}{2} \tag{48}$$

and

$$0 = s(\frac{\partial_t \delta n_+ - \partial_t \delta n_-}{2} - i\alpha \frac{\partial_t \delta n_+ + \partial_t \delta n_-}{2}) - i\frac{\delta n_+ + \delta n_-}{2} H - ih_x + iA\nabla^2 \frac{\delta n_+ + \delta n_-}{2} \tag{49}$$

and

$$\frac{\delta n_+ + \delta n_-}{2} h_y = \frac{i\delta n_- - i\delta n_+}{2} h_x. \tag{50}$$

Since $(\delta n_+)^* = \delta n_-$, we can solve everything for δn_+ and take the complex conjugate to find δn_-. This can be done by sum (48) and (49). So we get

$$0 = s(\partial_t \delta n_+ - i\alpha \partial_t \delta n_+) - i\delta n_+ H + h_y - ih_x + iA\nabla^2 \delta n_+. \tag{51}$$

Equivalently, we can write this in Schrödinger form:

$$(s - is\alpha)\frac{\partial}{\partial t}\psi = -iA\nabla^2 \psi + Hi\psi + ih_x - h_y, \tag{52}$$

by defining

$$\psi \equiv \delta n_+, \tag{53}$$

which depends on $\vec{x}$ and t. Note here that since $\vec{n} = \vec{n}(\vec{x}, t)$ we have $\delta n_\pm = \delta n_\pm(\vec{x}, t)$, hence ψ depends also on the location $\vec{x}$ and time t i.e $\psi = \psi(\vec{x}, t)$. Before we start solving this equation, we define $h_\pm \equiv h_x \pm ih_y$ and notice that $\frac{1}{i} = -i$ implies that $\frac{1-i\alpha}{i} = -(i + \alpha)$. If we multiply (52) by $\frac{1}{i}$, plug in $h_\pm$ and rearrange some terms we end up with

$$\hat{L}\psi(\vec{x}, t) \equiv \Big[- s(i + \alpha)\frac{\hat{\partial}}{\partial t} + A\hat{\nabla}^2 - (H + h_z)\Big]\psi(\vec{x}, t) = h_+(\vec{x}, t), \tag{54}$$

a linear non-homogeneous differential equation.

Before going any further, let us recap what we have done so far. We started with a general magnetic E.O.M. of the ferromagnetic bulk (27) suggested by[3]. We have treated the dynamics classically and we performed a linearization to obtain an equation to describe the magnetic dynamics of our system. After that, we switched to complex notation to find a useful Schrödinger-like expression, which is a solvable differential equation.

4.1.2 Finding Solutions Using Green's Functions

At this point we are ready to solve the E.O.M derived in the previous section. To do so, we will "build" the solution out of δ-functions i.e. by applying Green's functions to equation (54) whence we move to momentum space (Fourier space) to solve it. After that, we are able to calculate $\langle \psi^*(\vec{x}, t)\psi(\vec{x}, t)\rangle$, which is related to the average deviation of $\vec{n} = -\vec{m}$.

We introduce the Green's function $G(\vec{x} - \vec{x_0}, t - t_0)(= G(\vec{x}, t; \vec{x_0}, t_0))$, which is defined as

$$\psi(\vec{x}, t) \equiv \int_{\mathbb{R}^3} \int_{\mathbb{R}} G(\vec{x}, t; \vec{x_0}, t_0) h_+(\vec{x_0}, t_0) dt_0 d\vec{x_0}. \tag{55}$$

To find an explicit expression of G, we have to solve

$$\hat{L}G(\vec{x}, t; \vec{x_0}, t_0) = \delta(\vec{x} - \vec{x_0})\delta(t - t_0). \tag{56}$$

This is a simplified form of (54) depending on source terms i.e. δ-functions instead of h_+. With these source terms we can build ψ via (55). If we now write (56) in terms of the Fourier transforms of G and δ we obtain

$$\hat{L}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\tilde{G}(\vec{k},\omega)e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}=\int_{\mathbb{R}^3}\int_{\mathbb{R}}e^{+i\omega(t-t_0)}e^{+i\vec{k}(\vec{x}-\vec{x_0})}\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}, \tag{57}$$

where $\tilde{G}$ denotes the Fourier transform of $G(\vec{x},t)$. Note that the $\propto e^{+i\omega t}$ is not the conventional way of Fourier transform time in physics, nevertheless it is mathematically equivalent with what we wrote here since we have normalized the integrals. On the left hand side of (57) we can carry the operator $\hat{L}$ inside the integral. Hence we state

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}}\hat{L}\left[\tilde{G}(\vec{k},\omega)e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}\right]\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}=\int_{\mathbb{R}^3}\int_{\mathbb{R}}e^{+i\omega(t-t_0)}e^{+i\vec{k}(\vec{x}-\vec{x_0})}\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}. \tag{58}$$

Now we are able to drop the four integrals and the $\frac{1}{(2\pi)^4}$-factors on both sides. Doing that, and use $\hat{L}\in\{\hat{A};\hat{A}=\ \textit{linear operator s.t. } \hat{A}=c_1\partial_t+c_2\nabla^2+c_3 \textit{ and } c_1,c_2,c_3 \textit{ are constants}\}$ together with the fact that $\tilde{G}(\vec{k},\omega)$ is not a function of $\vec{x}$ and t, we can pull $\hat{L}$ over $\tilde{G}$ and write

$$\tilde{G}(\vec{k},\omega)\hat{L}\left[e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}\right]=e^{+i\omega(t-t_0)}e^{+i\vec{k}(\vec{x}-\vec{x_0})}. \tag{59}$$

This can now be written as

$$\tilde{G}(\vec{k},\omega)=\frac{e^{+i\omega(t-t_0)}e^{+i\vec{k}(\vec{x}-\vec{x_0})}}{\hat{L}\left[e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}\right]}=\frac{e^{+i\omega(t-t_0)}e^{+i\vec{k}(\vec{x}-\vec{x_0})}}{\left[i\omega(-s(i+\alpha))-A\vec{k}^2-H\right]e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}}. \tag{60}$$

If we now divide the right hand side by $-e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}$ and assume $\vec{x}_0=\vec{0}$ and $t_0=0$ we find the solution of G in terms of the Fourier transform, *i.e.*

$$\tilde{G}(\vec{k},\omega)=\frac{-1}{is\omega(i+\alpha)+A\vec{k}^2+H}, \tag{61}$$

so

$$G(\vec{x},t)=\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{-1}{is\omega(i+\alpha)+A\vec{k}^2+H}e^{+\vec{k}\cdot\vec{x}}e^{+i\omega t}\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}. \tag{62}$$

From here we can find an expression for the average $\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle$ by using eq. (55). To calculate this, we write $\tilde{G}(\vec{k},t)$ as

$$\tilde{G}(\vec{k},t)=\frac{1}{s}\frac{1}{\omega-\epsilon_k} \tag{63}$$

by defining $\epsilon_k\equiv\alpha i\omega+\frac{1}{s}(Ak^2+H)$ and $k\equiv||\vec{k}||_{\mathbb{R}^3}$. If we now use equation (55) we get the expression

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle=\left\langle\int_{\mathbb{R}^3}\int_{\mathbb{R}}G^*(\vec{x}-\vec{x'},t-t')h_+(\vec{x}',t')dt'd\vec{x}'\cdot\int_{\mathbb{R}^3}\int_{\mathbb{R}}G(\vec{x}-\vec{x''},t-t'')h_+(\vec{x}'',t'')dt''d\vec{x}''\right\rangle. \tag{64}$$

At this point we have to observe that $G*$ and G are not affected by randomness and that we have 8 integrals over 8 different variables. Furthermore we know that the integral over an average is the same as the average over an integral. Hence we can write

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle=\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}G^*(\vec{x}-\vec{x'},t-t')G(\vec{x}-\vec{x''},t-t'')\left\langle h_+(\vec{x}',t')h_+(\vec{x}'',t'')\right\rangle dt'd\vec{x}'dt''d\vec{x}''. \tag{65}$$

If we recall (29) and plug it in (65), we are left with the following equation for the average:

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} G^*(\vec{x}-\vec{x'},t-t')G(\vec{x}-\vec{x''},t-t'')\frac{2\alpha}{\gamma M_s}k_B T(\vec{r})\delta_{ij}\delta(\vec{x}'-\vec{x}'')\delta(t'-t'')dt'd\vec{x}'dt''d\vec{x}''. \tag{66}$$

The δ-functions that has appeared can now be killed by four out of the eight integrals i.e. if we apply $''f(\vec{a}) = \int_{\mathbb{R}} f(\vec{b})\delta(\vec{b}-\vec{a})d\vec{b}$ holds $\forall$ functions $f : \mathbb{R}^n \to \mathbb{R}$ on (66), we end up with

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}} G^*(\vec{x}-\vec{x''},t-t'')G(\vec{x}-\vec{x''},t-t'')2\alpha s k_B T dt''d\vec{x}''. \tag{67}$$

Hence,

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = 2\alpha s k_B T\int_{\mathbb{R}^3}\int_{\mathbb{R}} G^*(\vec{x}-\vec{x''},t-t'')G(\vec{x}-\vec{x''},t-t'')dt''d\vec{x}''. \tag{68}$$

Where we used assumption (ii) of section 4.1 and the fact that $s = \frac{M_s}{\gamma}$. If we now write the place-time dependent G's as a Fourier transform, we get the following expression:

$$\begin{aligned}\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = 2\alpha s k_B T\int_{\mathbb{R}^3}\int_{\mathbb{R}}\Big[\big[\int_{\mathbb{R}^3}\int_{\mathbb{R}}\tilde{G}(\vec{k},\omega)e^{+i\vec{k}\cdot(\vec{x}-\vec{x}'')}e^{+i\omega(t-t'')}\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}\big]^*\\ \big[\int_{\mathbb{R}^3}\int_{\mathbb{R}}\tilde{G}(\vec{k'},\omega')e^{+i\vec{k'}\cdot(\vec{x}-\vec{x}'')}e^{+i\omega'(t-t'')}\frac{d\omega'}{2\pi}\frac{dk'^3}{(2\pi)^3}\big]\Big]dt''d\vec{x}''.\end{aligned} \tag{69}$$

We can express this equation in terms of $\epsilon_k^{(*)}$ and put all the integrals together. This leads to

$$\begin{aligned}\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = 2\alpha s k_B T\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\\ \left[\frac{1}{s}\frac{1}{\omega-\epsilon_k^*}e^{-i\vec{k}\cdot(\vec{x}-\vec{x}'')}e^{-i\omega(t-t'')}\frac{1}{s}\frac{1}{\omega'-\epsilon_{k'}}e^{+i\vec{k'}\cdot(\vec{x}-\vec{x}'')}e^{+i\omega'(t-t'')}\right]\\ \frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}\frac{d\omega'}{2\pi}\frac{dk'^3}{(2\pi)^3}dt''d\vec{x}''.\end{aligned} \tag{70}$$

Taking the $\frac{1}{s^2}$-term outside the integral and collecting the powers of e gives

$$\begin{aligned}\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = \frac{2\alpha k_B T}{s}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\\ \left[\frac{1}{\omega-\epsilon_k^*}\frac{1}{\omega'-\epsilon_{k'}}e^{i(+\vec{k'}-\vec{k})\cdot(\vec{x}-\vec{x}'')}e^{i(+\omega'-\omega)(t-t'')}\right]\\ \frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}\frac{d\omega'}{2\pi}\frac{dk'^3}{(2\pi)^3}dt''d\vec{x}''.\end{aligned} \tag{71}$$

This equation can be manipulated by using $(2\pi)^n\delta(\vec{a}-\vec{b}) = \int_{\mathbb{R}^n} e^{i\vec{\kappa}\cdot(\vec{a}-\vec{b})}$ with $\vec{a},\vec{b}\in\mathbb{R}^n$ for the integrals over t and $\vec{x}$. Hence we have:

$$\begin{aligned}\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = \frac{2\alpha k_B T}{s}(2\pi)^4 \int_{\mathbb{R}^3}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}} \\ \left[\frac{1}{\omega-\epsilon_k^*}\frac{1}{\omega'-\epsilon_{k'}} e^{i(+\vec{k}'-\vec{k})\cdot\vec{x}} e^{i(+\omega'-\omega)t}\delta(\vec{k}-\vec{k}')\delta(\omega-\omega')\right] \\ \frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}\frac{d\omega'}{2\pi}\frac{dk'^3}{(2\pi)^3}. \\ = \frac{2\alpha k_B T}{s}(2\pi)^4\frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\left[\frac{1}{\omega-\epsilon_k^*}\frac{1}{\omega-\epsilon_{k'}}\right]\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3}.\end{aligned} \tag{72}$$

If we plug in the definition of $\epsilon_k^{(*)}$, we can calculate the integral over ω by using complex integration. The calculation of the integral is then given by:

$$\begin{aligned}\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle &= \frac{2\alpha k_B T}{s}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\left[\frac{1}{\omega-\epsilon_k^*}\frac{1}{\omega-\epsilon_k}\right]\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3} \\ &= \frac{2\alpha k_B T}{s}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\left[\frac{1}{[\omega-\frac{Ak^2+H}{s(1+i\alpha)}]}\frac{\frac{1}{(1+\alpha^2)}}{[\omega-\frac{Ak^2+H}{s(1-i\alpha)}]}\right]\frac{d\omega}{2\pi}\frac{dk^3}{(2\pi)^3} \\ &= \frac{2\alpha k_B T}{s}\int_{\mathbb{R}^3}\left[\frac{si}{2i\alpha(Ak^2+H)}\right]\frac{dk^3}{(2\pi)^3} \\ &= \int_{\mathbb{R}^3}\left[\frac{k_B T}{(Ak^2+H)}\right]\frac{dk^3}{(2\pi)^3}.\end{aligned} \tag{73}$$

Where we used Cauchy's complex integration formula and use the fact that $\omega \neq \frac{Ak^2+H}{s(1+i\alpha)}$. After a change to spherical coordinates and introducing a thermal cutoff Λ, we end up with:

$$\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle = \frac{k_B T}{2\pi^2}\int_0^\Lambda \frac{k^2}{(Ak^2+H)}dk. \tag{74}$$

The thermal cutoff is introduced since we have a finite system.

Let us summarize what we have done. We solved (54) by employing a technique referred to as Green's functions. After using Fourier transformations and δ-functions we obtained an equation for $\langle\psi^*(\vec{x},t)\psi(\vec{x},t)\rangle$, which characterizes equilibrium thermal fluctuations of the ferromagnetic spin density and is given by (74).

4.2 Boundary Conditions

In the previous section we solved a linearized E.O.M. for $\psi(\vec{x},t)\equiv\delta_n(\vec{x},t)$, which is derived in section 4.1.2 and given by eq. (54). No we start, again, with this expression and will focus on the deterministic part *i.e.* the part of this system that we can control. From there we again find an expression for the Fourier transform of ψ. After that we will take a look at the boundaries of the ferromagnet and from there we derive an equation to solve for the current trough the ferromagnet.

Recall eq.(54). Since we are interested in the deterministic properties we neglect h_+. Hence the operator $\hat{L}$ becomes

$$\hat{L} = \Big[-s(i+\alpha)\frac{\hat{\partial}}{\partial t} + A\hat{\nabla}^2 - H\Big]. \tag{75}$$

So the E.O.M considered is given by

$$\hat{L}\psi(\vec{x},t) = -s(i+\alpha)\frac{\hat{\partial}}{\partial t} + A\hat{\nabla}^2 - H\psi(\vec{x},t) = 0. \tag{76}$$

Before we can use this for the boundaries we have to solve eq. (76). Since this expression is easier than (54), we don't have to solve it by the use of Green's functions. Instead we will perform a Fourier transformation and express the solution in terms of the Fourier variables.

So, Fouriertransforming $\psi(\vec{x},t)$ $(=\psi(x,y,z,t))$ in the variables y,z and t gives

$$\psi(x,y,z,t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi(x,k_2,k_3,\omega)e^{ik_2y+ik_3z-i\omega t}d\omega dk_2 dk_3. \tag{77}$$

This expression can now be plugged in eq. (76). If we use the fact that the operator $\hat{L}$ can be carried into the integral and that it treats $\psi(x,k_2,k_3,\omega)$ as a constant, except for the ∂_x^2-part we end up with

$$\hat{L}\psi(\vec{x},t) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \big[i\omega s(i+\alpha)\widetilde{\psi}e^{\Gamma} + e^{\Gamma}A\partial_x^2\widetilde{\psi} - A(k_2^2+k_3^2)\widetilde{\psi}e^{\Gamma} - H\widetilde{\psi}e^{\Gamma}\big]d\omega dk_2 dk_3 = 0. \tag{78}$$

Where we defined $\widetilde{\psi} \equiv \psi(x,k_2,k_3,\omega)$ and $e^{\Gamma} \equiv e^{ik_2y+ik_3z-i\omega t}$ to make the expression simpler. Since we have a zero on the right side of eq.(78), we can drop the integrals, *i.e.*, we can state that the integrand is zero. Doing that and dropping powers of e leads to

$$i\omega s(i+\alpha)\widetilde{\psi} + A\partial_x^2\widetilde{\psi} - A(k_2^2+k_3^2)\widetilde{\psi} - H\widetilde{\psi} = 0. \tag{79}$$

This is an ordinary differential equation which can be solved by assuming $\psi(x,k_2,k_3,\omega) \propto ce^{i\lambda x}$ for some parameter λ and constant c. Note that $\lambda = \lambda(k_2,k_3,\omega)$. If we plug this assumption in eq.79 we find λ equals

$$\lambda_{1,2}(k_2,k_z,\omega) = \pm\sqrt{\frac{i\omega s}{A}(i+\alpha) - (K_2^2+k_3^2) - \frac{H}{A}}. \tag{80}$$

Since eq. (78) is a second order homogeneous differential equation, we can write the solution ψ in terms of λ *i.e.* we have the solution given by

$$\psi = ae^{i\lambda_1 x} + be^{i\lambda_2 x}. \tag{81}$$

More explicitly we can write

$$\psi(x,k_2,k_3,\omega) = ae^{+\sqrt{\frac{i\omega s}{A}(i+\alpha)-(K_2^2+k_3^2)-\frac{H}{A}}} + be^{-\sqrt{\frac{i\omega s}{A}(i+\alpha)-(K_2^2+k_3^2)-\frac{H}{A}}}. \tag{82}$$

With this expression for ψ we can calculate the current. Therefore, recall (34), which defines the spin current.To use this formula we need to write $\vec{J_{s,i}}$ in terms of ψ and linearize it by using eq. (37), *i.e.*,

$$\vec{J_{s,i}} \approx -A\begin{pmatrix}\delta n_x\\ \delta n_y\\ 1-\frac{\delta n_x^2+\delta n_y^2}{2}\end{pmatrix}\times\partial_i\begin{pmatrix}\delta n_x\\ \delta n_y\\ 1-\frac{\delta n_x^2+\delta n_y^2}{2}\end{pmatrix} = -A\big(\delta n_x\partial_i\delta n_y - \delta n_y\partial_i\delta n_x\big)\hat{z} = A\frac{i}{2}\big(\psi\partial_i\psi^* - \psi^*\partial_i\psi\big)\hat{z}, \tag{83}$$

where we thrown away δ^2 and higher.

We will leave this here for a moment and take a look at the boundaries. As [4] suggests we can distinguish the spin current entering and leaving the reservoir in terms of the spin accumulation. As before we consider an ferromagnetic insulator sandwiched between two metals. On the left side we have an metal reservoir L and on the right an reservoir R. The left one is a poor spin sink while the right reservoir is an excellent spin sink. We will neglect the imaginary parts of th spin mixing conductance.

$$\vec{J}^s_L = \Re g_L^{\uparrow\downarrow}\vec{n} \times \frac{(\tilde{\vec{\mu}}_s \times \vec{n})}{4\pi} \tag{84}$$

and

$$\vec{J}^s_R = \Re g_R^{\uparrow\downarrow}\frac{\hbar}{4\pi}\vec{\Omega}, \tag{85}$$

where $g_{L,R}$ is the spincurrent from the left/right and $\vec{\Omega} = \vec{n} \times \partial_t \vec{n}$[4] denotes the procession frequentie. In the theory of linearisation, *i.e.*, plugging eq. (38) in eq.(84) and eq.(85), we have

$$\vec{J}^s_L = \frac{1}{4\pi} g_L \vec{n} \times \left[(\mu\hat{z} - \hbar\vec{n} \times \partial_t\vec{n}) \times \vec{n}\right] \approx \frac{1}{4\pi} g_L \begin{pmatrix} \frac{\mu}{2}(\psi + \psi^*) + \frac{i\hbar}{2}(\partial_t\psi^* - \partial_t\psi) \\ \frac{i\mu}{2}(\psi - \psi^*) - \frac{\hbar}{2}(\partial_t\psi + \partial_t\psi^*) \\ 0 \end{pmatrix}, \tag{86}$$

and

$$\vec{J}^s_R = g_R \frac{\hbar\Omega}{4\pi}(\vec{n} \times \vec{\partial_t \vec{n}}) \approx g_R \frac{\hbar\Omega}{4\pi} \begin{pmatrix} i(\partial_t\psi - \partial_t\psi^*) \\ \partial_t\psi + \partial_t\psi^* \\ 0 \end{pmatrix}. \tag{87}$$

Here we defined $g_{L,R} \equiv \Re g_{L,R}$. Also we, again, used (53). Since these formulas for the spin currents are at zero temperature we have to modify them. To do that we add a term which allows some noise due to the ferromagnetic bulk. From there we can calculate the average current in terms of the above boundary conditions. In tradition of section 4.1 we denote the linearised quantummechanical randomness-term as

$$\vec{n} \times \vec{h}' \approx -\hat{z} \times \vec{h}' = \begin{pmatrix} h'_y \\ -h'_x \\ 0 \end{pmatrix}. \tag{88}$$

If we add this to our linearised expressions for the current entering resp. leaving the resevoir we end up with:

$$\vec{J}^s_L \approx \frac{1}{4\pi} g_L \begin{pmatrix} \frac{\mu}{2}(\psi + \psi^*) + \frac{i\hbar}{2}(\partial_t\psi^* - \partial_t\psi) + h'_y \\ \frac{i\mu}{2}(\psi - \psi^*) - \frac{\hbar}{2}(\partial_t\psi + \partial_t\psi^*) - h'_x \\ 0 \end{pmatrix} \tag{89}$$

and

$$\vec{J}^s_R \approx g_R \frac{\hbar\Omega}{4\pi} \begin{pmatrix} \frac{i}{2}(\partial_t\psi - \partial_t\psi^*) + h'_y \\ \frac{1}{2}(\partial_t\psi + \partial_t\psi^*) - h'_x \\ 0 \end{pmatrix}. \tag{90}$$

From here the average current can be calculated in terms of $\langle h' \rangle$ and ψ. These terms can be determined by using (29). To do that we have to solve

$$\langle \vec{J}_{s,i} \rangle = \langle \vec{J}^s_L \rangle \tag{91}$$

and

$$\langle \vec{J}_{s,i} \rangle = \langle \vec{J}^s_R \rangle. \tag{92}$$

After doing that we could plug in our result for ψ (eq. (82) and use some Fouriertransformations.

Let's start with eq (91). If we make (89) equal to (83) we find

$$\langle \vec{J}_{s,i} \rangle = \langle \vec{J}^s_L \rangle \tag{93}$$

$$\frac{i}{2}\begin{pmatrix} 0 \\ 0 \\ \langle \psi\partial_i\psi^* - \psi^*\partial_i\psi \rangle \end{pmatrix} = \frac{1}{4\pi} g_L \begin{pmatrix} \langle \frac{\mu}{2}(\psi+\psi^*) + \frac{i\hbar}{2}(\partial_t\psi^* - \partial_t\psi) + h'_y \rangle \\ \langle \frac{i\mu}{2}(\psi-\psi^*) - \frac{\hbar}{2}(\partial_t\psi + \partial_t\psi^*) - h'_x \rangle \\ 0 \end{pmatrix}. \tag{94}$$

No we can multiply the second row on both sides wit i and substract the second from the first. This leads to

$$\begin{cases} 0 = \frac{1}{4\pi} g_L \langle \mu\psi + i\hbar\partial_t\psi^* + h'_y + ih'_x \rangle \\ \langle \psi\partial_i\psi^* - \psi^*\partial_i\psi \rangle = 0 \end{cases}. \tag{95}$$

For the current leaving the resovoir, *i.e.* the current on the right, we can peform the same calculation. Doing that we end up with

$$\begin{cases} 0 = g_R \frac{\hbar\Omega}{4\pi} \langle \partial_t\psi - h'_x + ih'_y \rangle \\ \langle \psi\partial_i\psi^* - \psi^*\partial_i\psi \rangle = 0 \end{cases}. \tag{96}$$

Note that we multiplied the first row with i instead of the second. In both equations (95) and (96) we can sum the two rows that are left. This leads to the following two equations:

$$\begin{cases} Leftside : \langle \frac{1}{4\pi} g_L(\mu\psi + i\hbar\partial_t\psi^* + h'_y + ih'_x) - \psi\partial_i\psi^* + \psi^*\partial_i\psi \rangle = 0 \\ Rightside : \langle g_R \frac{\hbar\Omega}{4\pi}(\partial_t\psi - h'_x + ih'_y) - \psi\partial_i\psi^* + \psi^*\partial_i\psi \rangle = 0 \end{cases} \tag{97}$$

We will end our calculation here and make some final remarks about the general solution of this problem. First we investegated the magnetic motion inside the ferromagnet wich we can control given by (76) i.e. the motion inside the bulk of the ferromagnet in a deterministic manner. We solved this equation, with solution given by (82). From there we set some boundariesconditions to derive an expression (eq. (97) for ψ at the boundaries.

Now, we can formulate the general solution. Call ψ from the deterministic part, ψ_i and the ψ at the boundaries, *i.e.* ψ satifying eq. (97) we will call ψ_b. then the general solution for ψ is given by:

$$\psi(\vec{x}, t) = \psi_i(\vec{x}, t) + \psi_b(\vec{x}, t). \tag{98}$$

From reference [3] we now can calculate the spincurrent trough the ferromagnet well below the currietemperature. This is given by

$$j_s = A\Im\langle \psi\partial_x\psi \rangle = A\Im\langle (\psi_i(\vec{x}, t) + \psi_b(\vec{x}, t))\partial_x(\psi_i(\vec{x}, t) + \psi_b(\vec{x}, t)) \rangle. \tag{99}$$

Note that this current is given in the x-direction of the system.

5 Conclusions

As we have seen, we approached the dynamics inside the ferromagnet classically and quantum mechanically. In the classical case we used the classical Hamiltonian (1) to calculate the time evolution of the spin operator $\hat{\vec{S}}$ for small oscillations around the z-direction of the spin using the quantum mechanical Ehrenfest's theorem (3). From this time evolution we were able to calculate the frequency the oscillations ω. From this we calculated the ground state of the vibration which had an energy $\hbar\omega > 0$. For the quantum mechanical treatment we defined the spin raising-/lowering-operators which we plugged in (1). From there we again take a look at small oscillations from which we obtained that the lowest energy, which given by $E_0 = -J\hbar^2 S^2 Nz/2$ and where the dispersion is given by $\omega_{\vec{k}} = SJ\hbar z(1 - cos(k_x a)$. For further research it is possible to use the same methods but then take a look at the boundaries of the system.

Furthermore, we had seen that by using the equations of motion (20) suggested by [2]. After making an approach of small oscillations of $\vec{n}$ and $\vec{m}$ around the equilibrium of the AFM we have calculated the dispersion relation and the oscillating vectors by plugging the approximated effective fields (18),(19) in to the free energy (13). As we see in the equations (25) and (26) the oscillations around the equilibrium of the AFM causes a wave-effect of the spin down- and spin up-lattice. A suggestion for further research is to calculate the corresponding energy levels.

Finally we investigated a ferromagnetic insulator. First we looked at the bulk of the ferromagnet and derived an equation for $\langle\psi^*\psi\rangle$, which is a measure for the average deviation of the vector $\vec{n}$. We did this by using Landau-Lifshitz-Gilbert theory suggested by [3]. This result is given by (74). From there we looked at the same system with boundaries. From that we derived the general expression of the spin current through a ferromagnet given by (98) which can be solved.
An topic for further research will be to actually solve (96). That can be done by again using Green's functions and Fouriertransforming techniques. Furthermore the ferromagnet can be replaced by an AFM and one can try to derive the spincurrent in that case.

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