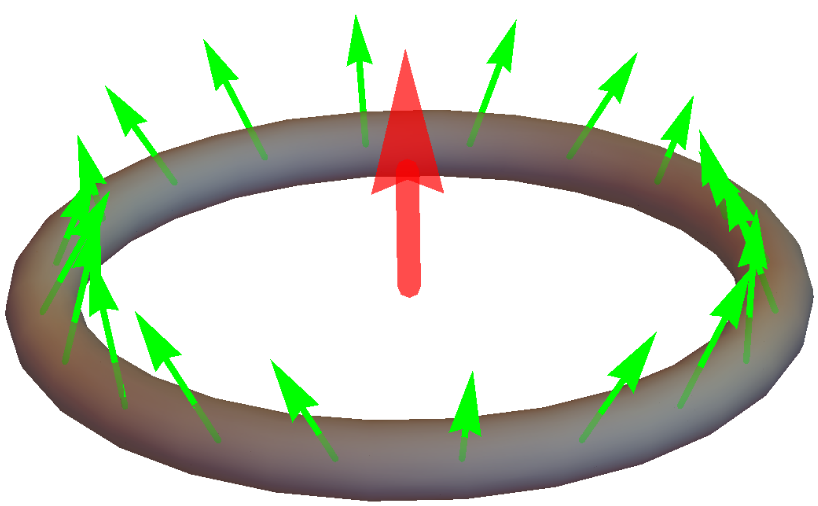
Universiteit Utrecht

Bachelor thesis

Superconductivity and Spin Superfluidity



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Abstract

When a magnetic field is applied to a ferromagnetic ring with easy-plane anisotropy, steady states with circulating spin currents occur. In a similar way, when a magnetic field is applied to a superconducting ring, circulating charge currents will appear. These currents are described by very similar hydrodynamic expressions. When a magnetic field is applied to a superconducting ferromagnet these hydrodynamic equations become coupled, which leads to different steady states.

Contents

1 Introduction

A relatively new and lively topic in physics concerns the way spin is transported in materials. In general, the study of these spins and often associated charge is called spintronics. So-called spin currents may have interesting properties, especially when they would be superfluid, which would mean that they would travel dissipationless through the material. These kinds of systems are interesting in general. Superconductivity, another phenomenon under the branch of superfluidity, has also drawn quite some attention the past century. On both subjects has been done extensive research, but what happens when a material has properties of two superfluid systems, is still not entirely clear. In this thesis we try to get a little closer to an answer to this uncertainty. We will start by working out the two systems separately. Guided by the article "Spin currents and spin superfluidity" by E. B. Sonin published in 2008 [1] we find states for spin currents in Section 2. Then, in Section 3, we derive steady states for charge currents guided by the lecture notes on superconductivity by D. Arovas and C. Wu updated last in 2015 [2]. We will find similar hydrodynamic equations of motion. In Section 4 we finally look at a ferromagnetic superconductor in which the equations of motion become coupled. This leads to differences in the steady states of both the spin and charge currents.

2 Magnetization Dynamics and Spin Superfluidity in Magnetic Insulators

In this section we will derive steady states for spin currents in magnetic insulators. A useful equation in this situation is the Landau-Lifshitz equation, which we will derive first. Then, we will look at what the magnetization in an easy-plane ferromagnet does, when a magnetic field is applied to it. In doing so we keep the exchange energy minimized. Then, finally, with the obtained information we will be able to find steady states.

2.1 Landau-Lifshitz Equation

In order to get an idea of how the magnetization dynamics of a certain material are described, we derive the Landau-Lifshitz equation. This equation gives a good and accessible description of the behavior of spin particles. In order to find this equation, we first look at a simple model of a single spin in a magnetic field. The Hamiltonian in such a case is written as follows.

$$\hat{H}[\hat{\boldsymbol{S}}] = -\Delta \boldsymbol{B} \cdot \hat{\boldsymbol{S}}. \tag{1}$$

In this equation $\boldsymbol{B}$ represents the magnetic field, $\hat{\boldsymbol{S}}$ is the spin operator and Δ is a parameter depending on the system. Let us now look at the expectation value of the spin operator, and in particular its derivative to the time t:

$$\frac{d}{dt}\langle \hat{\boldsymbol{S}} \rangle = \frac{d}{dt}\langle \psi(t)|\hat{\boldsymbol{S}}|\psi(t)\rangle, \tag{2}$$

with $\psi(t)$ being the wave function of the spin particle. If we now use the product rule for differentiation, we find

$$\frac{d}{dt}(\langle\psi(t)|)\hat{\boldsymbol{S}}|\psi(t)\rangle + \langle\psi(t)|\frac{d}{dt}\hat{\boldsymbol{S}}|\psi(t)\rangle. \tag{3}$$

Then, by applying the Schrödinger equation ($i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$) twice, we obtain:

$$\frac{1}{-i\hbar}\langle\psi(t)|\hat{H}\hat{\boldsymbol{S}}|\psi(t)\rangle + \langle\psi(t)|\hat{\boldsymbol{S}}\frac{1}{i\hbar}\hat{H}|\psi(t)\rangle = \frac{1}{i\hbar}\langle\psi(t)|[\hat{\boldsymbol{S}},\hat{H}]|\psi(t)\rangle, \tag{4}$$

with $[\hat{\boldsymbol{S}},\hat{H}]$ being the commutator between the Hamiltonian and the spin operator. Plugging in our first definition of the Hamiltonian, using the spin commutation relations and Einstein's summation notation we find the following equalities:

$$[\hat{S}_\alpha,\hat{H}] = [\hat{S}_\alpha, -\Delta\hat{S}_\beta B_\beta] = -\Delta B_\beta[\hat{S}_\alpha,\hat{S}_\beta] = -i\hbar\Delta\epsilon_{\alpha\beta\gamma}B_\beta\hat{S}_\gamma, \tag{5}$$

with $\epsilon_{\alpha\beta\gamma}$ being the Levi-Civita tensor. Using the definitions of this tensor we find

$$-i\hbar\Delta\frac{d}{dt}\langle\hat{\boldsymbol{S}}\rangle = \langle\psi(t)|\begin{pmatrix}B_z\hat{S}_y - B_y\hat{S}_z\\ B_x\hat{S}_z - B_z\hat{S}_x\\ B_y\hat{S}_x - B_x\hat{S}_y\end{pmatrix}|\psi(t)\rangle. \tag{6}$$

This looks exactly like a crossproduct between $\langle\hat{\boldsymbol{S}}\rangle$ and $\boldsymbol{B}$ with only a prefactor of a difference. So we get the following result.

$$\frac{d}{dt}\langle\hat{\boldsymbol{S}}\rangle = \Delta\langle\hat{\boldsymbol{S}}\rangle \times \boldsymbol{B}. \tag{7}$$

This equation implies that the spin vector could be revolving around the magnetic field vector in such a way that the angle between both vectors is kept constant. Furthermore, to rewrite this to a little more usable form, we see that the Δ and $\boldsymbol{B}$ can be replaced, as together they are the negative derivative of the Hamiltonian to the expectation value of the spin.

$$\frac{d}{dt}\langle\hat{\boldsymbol{S}}\rangle = \frac{d\hat{H}}{d\langle\hat{\boldsymbol{S}}\rangle} \times \langle\hat{\boldsymbol{S}}\rangle. \tag{8}$$

Then, we assume that the total magnetization $\boldsymbol{M}$ is simply related by a constant factor γ of the expectation value of the spin operator: $\boldsymbol{M} = \gamma\langle\hat{\boldsymbol{S}}\rangle$. And we find the Landau-Lifshitz equation, with the effective field $\boldsymbol{H}_{eff} = -\frac{\delta H}{\delta \boldsymbol{M}}$:

$$\frac{d\boldsymbol{M}}{dt} = \gamma\boldsymbol{H}_{eff} \times \boldsymbol{M}. \tag{9}$$

2.2 Magnetization in an Easy-plane Ferromagnet

Our next goal is to find how much the magnetization vector is tilted from the magnetic field vector in an easy-plane ferromagnet. An additional question is, whether there is a possibility of fully homogenuously magnetizing the ferromagnet in the B-direction.

And if so, at what conditions does this occur. To get an answer to these questions, we look at a ferromagnet with magnetization $\boldsymbol{M}$ in a homogenuous magnetic field B in the z-direction. We can then write the free energy in the following way.

$$F(\boldsymbol{M}) = \int d\boldsymbol{x} \left(-\frac{J}{2}\boldsymbol{M} \cdot \nabla^2 \boldsymbol{M} - \boldsymbol{B} \cdot \boldsymbol{M} + \frac{K}{2}\boldsymbol{M}^2 \right), \tag{10}$$

with J being the exchange constant and K is the parameter determining anisotrophy.

Then, we assume that the size of the magnetization vector is constant. It depends on the amount of spin particles per volume, not on the strength of the magnetic field. Using spherical coordinates we find

$$\boldsymbol{M} = M_S \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \tag{11}$$

with M_S being the size of the magnetization.

It is now possible to find the magnetization at which the exchange energy is at a minimum. In order to do this, we set the derivative of the free energy to zero,

$$\frac{\delta F}{\delta \boldsymbol{M}} = -J\nabla^2 \boldsymbol{M} - \boldsymbol{B} + K\boldsymbol{M} = 0. \tag{12}$$

If we take into account our first assumption $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, K will only be applied to M_z. Next to that, every change in the magnetization would contribute to a larger exchange energy, and therefore $\nabla^2 \boldsymbol{M}$ should equal zero. The equation is solved by

$$\begin{aligned} M_z &= \frac{B}{K}, \text{ for } B < M_S K \\ M_z &= M_S, \text{ for } B \geq M_S K, \end{aligned} \tag{13}$$

with the restriction that M_z does not exceed M_S, i.e. $|M_z|$ is only smaller than M_S for a $|B|$ smaller than $M_S K$. The Landau-Lifshitz equation (Equation 7) suggested that the magnetization would be revolving around B (in this case the z-axis), because the time-derivative of the spin vector is always perpendicular to the spin vector itself. So it is a good thing that our magnetization in the z-direction turns out to be constant. We proceed to fill in our new discovery in Equation 11.

$$\theta = \cos^{-1}\left(\frac{B}{M_S K}\right), \tag{14}$$

and thus

$$\sin\theta = \sqrt{1 - \left(\frac{B}{M_S K}\right)^2}. \tag{15}$$

If we now want a different way to calculate the level of "tiltiness" of the magnetization to magnetic field other than θ, we could calculate $\sqrt{M_x^2 + M_y^2}$, which is of course just the component of the magnetization in the xy-plane.

$$\sqrt{M_x^2 + M_y^2} = M_S \sqrt{\left(1 - \left(\frac{B}{M_S K}\right)^2\right) \left(\cos^2\phi + \sin^2\phi\right)} = M_S \sqrt{1 - \left(\frac{B}{M_S K}\right)^2}. \tag{16}$$

This would mean that if B were to exceed $M_S K$, then the entire ferromagnet would be homogenuously magnetized in the z-direction. This is also shown in Figure 1. If not, the situation described earlier, with the magnetization revolving the z-direction, would be applied. However, in this case, at a minimum of energy exchange, the magnetization would stand still at a certain angle from the z-axis.

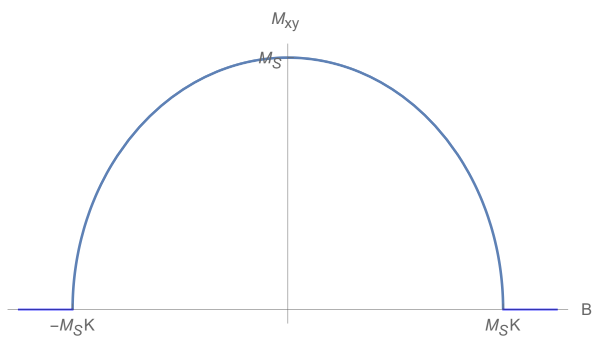

Figure 1: The component of the magnetization in the xy-plane (M_{xy}) plotted as a function of the magnetic field (B).

2.3 Spin Currents

We now have enough information to try to find spin currents in the material. In this case we are looking for stationary currents, that do not change in time (yet). We still take the same reasonable assumption that the size of the magnetization M_S is constant. Then, we introduce a new way of describing the magnitude in which the magnetization is tilted from the z-direction, that can be found in the next equation. At last, we take B to be not strong enough to fully magnetize the ferromagnet, so that there actually is an angle between these vectors.

$$\boldsymbol{M}(\boldsymbol{x}, t) = M_S \begin{pmatrix} \sqrt{\frac{2n}{s}} \cos\phi \\ \sqrt{\frac{2n}{s}} \sin\phi \\ \sqrt{1 - \frac{2n}{s}} \end{pmatrix}. \tag{17}$$

In this equation $n = n(\boldsymbol{x}, t)$ a measure of how much the magnetization is projected to the xy-plane per volume (it will often be considered constant) and s is the spin density equaling $\frac{S}{a^3}$, with a being the lattice size. Note that $\sqrt{\frac{2n}{s}}$ is chosen in such a way that n is not allowed to exceed half the spin density, as then M_z would become imaginary and M_x and M_y would possibly become larger than M_S. In Figure 2 the newly defined vector is shown.

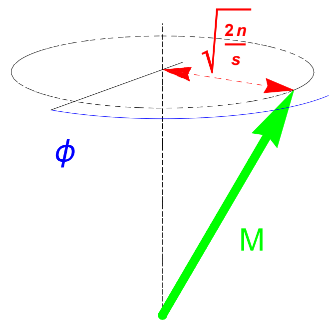

Figure 2: The Magnetization M for magnetic fields B smaller than $M_S K$.

We still take the free energy to be the same as in the previous section. Then, we are going to take the effective field $\boldsymbol{H}_{eff}$ from the Landau-Lifshitz equation to be the negative variational derivative of the free energy $-\frac{\delta F}{\delta \boldsymbol{M}}$. We have calculated this earlier on in the previous section. Now applying that our magnetic field is just in the z-direction, we find

$$\boldsymbol{H}_{eff} = -\frac{\delta F}{\delta \boldsymbol{M}} = \begin{pmatrix} J\nabla^2 M_x \\ J\nabla^2 M_y \\ J\nabla^2 M_z + B - KM_z \end{pmatrix}. \tag{18}$$

We can now plug this into the Landau-Lifshitz equation and we find three equations.

$$\frac{\partial \boldsymbol{M}}{\partial t} = \gamma \begin{pmatrix} (J\nabla^2 M_y)M_z - (J\nabla^2 M_z + B - KM_z)M_y \\ (J\nabla^2 M_z + B - KM_z)M_x - (J\nabla^2 M_x)M_z \\ (J\nabla^2 M_x)M_y - (J\nabla^2 M_y)M_x \end{pmatrix}. \tag{19}$$

We will now try to solve a combination of the x- and the y-equation. To keep everything a little accessible the following functions will be defined.

$$n_1(n) = \sqrt{\frac{2n}{s}}, \tag{20}$$

$$n_2(n) = \sqrt{1 - \frac{2n}{s}}. \tag{21}$$

We begin with working out some useful identities, which we will be using. They are also used in the Appendix Section 6.1.

$$\begin{aligned}\nabla^2 M_x = M_S \nabla^2 (n_1 \cos\phi) = \\ M_S(\cos\phi \nabla^2 n_1 - 2\sin\phi(\nabla n_1)\cdot(\nabla\phi) - n_1\cos\phi(\nabla\phi)^2 - n_1 \sin\phi \nabla^2\phi).\end{aligned} \tag{22}$$

And similarly for M_y:

$$\begin{aligned}\nabla^2 M_y = M_S \nabla^2 (n_1 \sin\phi) = \\ M_S(\sin\phi \nabla^2 n_1 + 2\cos\phi(\nabla n_1)\cdot(\nabla\phi) - n_1\sin\phi(\nabla\phi)^2 + n_1 \cos\phi \nabla^2\phi).\end{aligned} \tag{23}$$

These expressions will at this point speed up the process of writing out the x- and y-equations. First, the x-equation:

$$\frac{1}{\gamma}\frac{\partial M_x}{\partial t} = (J\nabla^2 M_y)M_z - (J\nabla^2 M_z + B - KM_z)M_y. \tag{24}$$

Using Equation 23 we find that the right-hand side of this equation equals

$$\begin{aligned}M_S^2 J n_2(\sin\phi\nabla^2 n_1 + 2\cos\phi(\nabla n_1)\cdot(\nabla\phi) - n_1\sin\phi(\nabla\phi)^2 + n_1\cos\phi\nabla^2\phi) \\ - M_S^2 J n_1 \sin\phi \nabla^2 n_2 - BM_S n_1 \sin\phi + KM_S^2 n_1 n_2 \sin\phi,\end{aligned} \tag{25}$$

while the left-hand side becomes

$$\frac{1}{\gamma}\frac{\partial}{\partial t}(M_S n_1 \cos\phi) = \frac{M_S}{\gamma}(\cos\phi\frac{\partial n_1}{\partial t} - n_1 \sin\phi\frac{\partial\phi}{\partial t}). \tag{26}$$

Then, we go through a similar process for the y-equation:

$$\frac{1}{\gamma}\frac{\partial M_y}{\partial t} = (J\nabla^2 M_z + B - KM_z)M_x - (J\nabla^2 M_x)M_z. \tag{27}$$

Now we can profit from using Equation 22 and we find the following for the right-hand side.

$$\begin{aligned}M_S^2 J n_1 \cos\phi \nabla^2 n_2 + BM_S n_1 \cos\phi - KM_S^2 n_1 n_2 \cos\phi \\ - M_S^2 J n_2(\cos\phi\nabla^2 n_1 - 2\sin\phi(\nabla n_1)\cdot(\nabla\phi) - n_1\cos\phi(\nabla\phi)^2 - n_1\sin\phi\nabla^2\phi),\end{aligned} \tag{28}$$

and the left-hand side:

$$\frac{1}{\gamma}\frac{\partial M_y}{\partial t} = \frac{M_S}{\gamma}(\sin\phi\frac{\partial n_1}{\partial t} + n_1\cos\phi\frac{\partial\phi}{\partial t}). \tag{29}$$

If we now use the simple identity $\sin^2\phi + \cos^2\phi = 1$, we add up these equations in such a way that only the time-derivative of ϕ remains. We combine these equations with the following factors.

$$\cos\phi \times (\mathrm{y-equation}) - \sin\phi \times (\mathrm{x-equation}). \tag{30}$$

So now, all we have left on the left-side of our total equation is:

$$\frac{M_S}{\gamma} n_1 \frac{\partial \phi}{\partial t}. \tag{31}$$

For the right-hand side, we are lucky that all terms become either shorter or cancel out. So what is left is less complicated than you might think at first.

$$M_S^2 J(n_1 \nabla^2 n_2 - n_2 \nabla^2 n_1 + n_1 n_2 (\nabla \phi)^2) + BM_S n_1 - KM_S^2 n_1 n_2. \tag{32}$$

Now finally solving this for the time-derivative of ϕ, we find:

$$\frac{\partial \phi}{\partial t} = \gamma M_S J(\nabla^2 n_2 - \frac{n_2}{n_1} \nabla^2 n_1 + n_2 (\nabla \phi)^2) + \gamma(B - KM_S n_2). \tag{33}$$

If we assume n is constant, the $\nabla^2 n_1$ and $\nabla^2 n_2$ fall out. The equation reduces to

$$\frac{\partial \phi}{\partial t} = \gamma M_S J \sqrt{1 - \frac{2n}{s}} (\nabla \phi)^2 + \gamma B - \gamma K M_S \sqrt{1 - \frac{2n}{s}}. \tag{34}$$

The result has a part depending on $\nabla \phi$, and a part that does not. Note that this constant part cancels out at a minimum in exchange energy ($M_z = \frac{B}{K}$).

In a similar way, with different sine and cosine factors we can find an expression for the time-derivative of n_1 and ultimately for n. Our new combination is the following one.

$$\cos \phi \times (\mathrm{x} - \mathrm{equation}) + \sin \phi \times (\mathrm{y} - \mathrm{equation}). \tag{35}$$

In the same way as before, we now find a left-hand side that does not contain the time-derivative of, in this case, ϕ anymore.

$$\frac{M_S}{\gamma} \frac{\partial n_1}{\partial t}. \tag{36}$$

The right-hand side again becomes a lot less complicated than before the equations were combined.

$$M_S^2 J(2n_2 (\nabla n_1) \cdot (\nabla \phi) + n_1 n_2 \nabla^2 \phi). \tag{37}$$

Now, we combine the two again, and find the following equation.

$$\frac{\partial n_1}{\partial t} = \gamma M_S J(2n_2 (\nabla n_1) \cdot (\nabla \phi) + n_1 n_2 \nabla^2 \phi). \tag{38}$$

Of course, we would rather have an equation just for n instead of one for n_1, which we defined solely for simplicity. So to obtain such an equation we have to derive the n_1's using the chain rule. Which gives us:

$$\frac{1}{\sqrt{2ns}} \frac{\partial n}{\partial t} = 2\gamma M_S J \sqrt{1 - \frac{2n}{s}} (\frac{1}{\sqrt{2ns}} (\nabla n) \cdot (\nabla \phi) + \frac{1}{2} \sqrt{\frac{2n}{s}} \nabla^2 \phi). \tag{39}$$

Which ultimately gives us the following expression for the time-derivative of n.

$$\frac{\partial n}{\partial t} = 2\gamma M_S J \sqrt{1 - \frac{2n}{s}} ((\nabla n) \cdot (\nabla \phi) + n \nabla^2 \phi). \tag{40}$$

This expression is valid for nonconstant n, since we did not assume that ∇n was zero in its derivation. Furthermore, unlike the equation of motion for ϕ, this equation does not have a constant part. This enables us to write it in the form of a divergence of a current $\boldsymbol{j}$.

$$\frac{\partial n}{\partial t} = -\nabla \cdot \boldsymbol{j} = -\nabla \cdot (-2\gamma M_S J \sqrt{1 - \frac{2n}{s}} n \nabla \phi). \tag{41}$$

In the Appendix (section 6.1) we discuss whether these two quantities are eachother's conjugate momentum.

Now consider a ferromagnetic ring (length L) in the xy-plane with a magnetic field going through it in the z-direction. We can now find solutions for the case in which the inclination of the magnetization is constant. This would mean that Equation 40 should equal zero. Because we now consider a ring, cylindrical coordinates will be used, with α the angle around the z-axis. Also, the equation becomes one-dimensional. We get

$$-2\gamma M_S J \sqrt{1 - \frac{2n}{s}} n \frac{2\pi}{L} \frac{\partial \phi}{\partial \alpha} = \text{constant}. \tag{42}$$

Still assuming that n is constant this means that the following must be true.

$$\phi = a\alpha + \phi_0, \tag{43}$$

with a and ϕ_0 being constant. Now applying our periodic boundary conditions to M_x we find:

$$\cos(a\alpha + \phi_0) = \cos(a(\alpha + 2\pi) + \phi_0), \tag{44}$$

which leads to

$$a\alpha = a(\alpha + 2\pi) + 2k\pi, \tag{45}$$

with k being an integer. This leaves the following expression for a.

$$a = k. \tag{46}$$

Now, inserting this result in our magnetization $\boldsymbol{M}$, we find a stationary spin current.

$$\boldsymbol{M}(\alpha) = M_S \begin{pmatrix} \sqrt{\frac{2n}{s}} \cos(k\alpha + \phi_0) \\ \sqrt{\frac{2n}{s}} \sin(k\alpha + \phi_0) \\ \sqrt{1 - \frac{2n}{s}} \end{pmatrix}. \tag{47}$$

Figure 3, 4 and 5 show examples of stationary solutions.

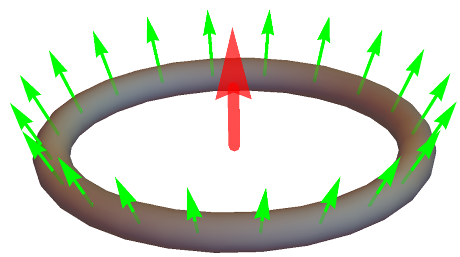

Figure 3: The ferromagnetic ring of length L, with the magnetic field (red arrow), and the magnetization (green arrows), for $k = 1$ and $\phi_0 = 0$.

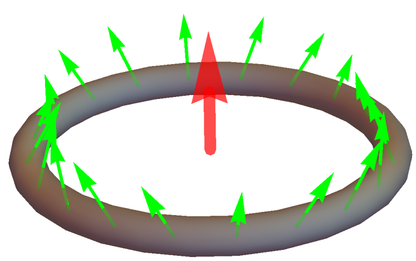

Figure 4: The ferromagnetic ring of length L, with the magnetic field (red arrow), and the magnetization (green arrows), for $k = 3$ and $\phi_0 = 0$.

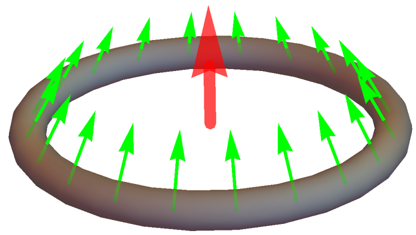

Figure 5: The ferromagnetic ring of length L, with the magnetic field (red arrow), and the magnetization (green arrows), for $k = 1$ and $\phi_0 = \pi$.

After having found these stationary currents, we are now also able to find some time-dependent currents, by filling in our newly found $\frac{\partial \phi}{\partial \alpha}$ in Equation 34 and keeping things one-dimensional. We find this ugly expression

$$\frac{\partial \phi}{\partial t} = \gamma \left(M_S \sqrt{1 - \frac{2n}{s}} \left(\frac{4k^2\pi^2}{L^2} J - K \right) + B \right). \tag{48}$$

As you can see, this does not depend on t nor ϕ, so the integration is simple. Having all the information we need, we can write down an equation for $\phi(x, t)$.

$$\phi(\alpha, t) = k\alpha + \gamma \left(M_S \sqrt{1 - \frac{2n}{s}} \left(\frac{4k^2\pi^2}{L^2} J - K \right) + B \right) t + \phi_0. \tag{49}$$

3 Charge Currents in Superconductors

For now, we leave the insulators behind us and we move on to conducting materials, to be more specific: superconductors. We attempt to find equations of quantities that play a role in superconductors which are similar to the expressions we already obtained for the magnetic insulators. To do so, we operate in a way that is comparable to the way we used in the previous section. Therefore, we first need a new expression for the free energy. We find

$$F(\Psi) = \int d\boldsymbol{x} \left(\Psi^* \left(-\frac{\hbar^2 \nabla^2}{2m} \right) \Psi + g|\Psi|^4 - \mu|\Psi|^2 \right), \tag{50}$$

with the total wave function Ψ, Ψ^* the complex conjugate of Ψ, g an interaction parameter, and μ the chemical potential. In this case the total wave function $\Psi = \sqrt{n}\psi$, with ψ being the wave function of a single electron, and n being the electron density. Finally m is twice the electron mass

We also want a magnetic field in our system, and therefore we need to introduce a vector potential $\boldsymbol{A}$, which obeys $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Then, in order to implement this in our free energy, we need to make the following adjustment.

$$-i\hbar\nabla \to -i\hbar\nabla + \frac{q}{c}\boldsymbol{A}. \tag{51}$$

Applying this, we get our new free energy,

$$\begin{aligned} F(\Psi) = \int d\boldsymbol{x} \Bigg(\Psi^* \frac{1}{2m} \Bigg(& -\hbar^2\nabla^2\Psi - \frac{i\hbar q}{c}\boldsymbol{A}\cdot\nabla\Psi \\ & - \frac{i\hbar q}{c}\nabla\cdot(\Psi\boldsymbol{A}) + \frac{q^2}{c^2}\boldsymbol{A}^2\Psi \Bigg) + g|\Psi|^4 - \mu|\Psi|^2 \Bigg). \end{aligned} \tag{52}$$

If we want to find equations of motion in the same form as we had earlier, we need to consider the Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi = \frac{\delta F}{\delta\Psi^*}. \tag{53}$$

In order to see why the $\frac{\delta F}{\delta \Psi^*}$ equals the right-hand side of the equation, we have to calculate it. Fortunately, this is not too hard, as the difficult part of the free energy is linearly related to Ψ^*. We get

$$\frac{\delta F}{\delta\Psi^*} = \frac{1}{2m}\left(-\hbar^2\nabla^2\Psi - \frac{i\hbar q}{c}\boldsymbol{A}\cdot\nabla\Psi - \frac{i\hbar q}{c}\nabla\cdot(\Psi\boldsymbol{A}) + \frac{q^2}{c^2}\boldsymbol{A}^2\Psi\right) + 2g|\Psi|^2\Psi - \mu\Psi. \tag{54}$$

We see that this does indeed look like some Hamiltonian acting on Ψ. To make sure we get two equations for quantities that are comparable to the quantities we used before, we rewrite Ψ using

$$\Psi = \sqrt{n}e^{i\theta}, \tag{55}$$

$$\Psi^* = \sqrt{n}e^{-i\theta}. \tag{56}$$

Before we work out each of the terms of Equation 53, we assume n is constant over space. This speeds up the process a lot, and we did this too in our earlier calculations. Then we work out each of the terms separately at first:

$$-\hbar^2\nabla^2\Psi = \sqrt{n}\left(-e^{i\theta}(\nabla\theta)^2 + ie^{i\theta}\nabla^2\theta = \left(i\nabla^2\theta - (\nabla\theta)^2\right)\Psi,\right. \tag{57}$$

$$-\frac{i\hbar q}{c}\boldsymbol{A}\cdot\nabla\Psi = \left(\frac{\hbar q}{c}\boldsymbol{A}\cdot\nabla\theta\right)\Psi, \tag{58}$$

$$-\frac{i\hbar q}{c}\nabla\cdot(\Psi\boldsymbol{A}) = \left(\frac{\hbar q}{c}\boldsymbol{A}\cdot\nabla\theta - \frac{i\hbar q}{c}\nabla\cdot\boldsymbol{A}\right)\Psi, \tag{59}$$

$$2g|\Psi|^2\Psi = 2gn\Psi, \tag{60}$$

and finally the left-hand side,

$$i\hbar\frac{\partial\Psi}{\partial t} = i\hbar\left(\frac{1}{2\sqrt{n}}\frac{\partial n}{\partial t}e^{i\theta} + i\frac{\partial\theta}{\partial t}\sqrt{n}e^{i\theta}\right) = \left(\frac{i\hbar}{2n}\frac{\partial n}{\partial t} - \hbar\frac{\partial\theta}{\partial t}\right)\Psi. \tag{61}$$

As we all can see these equations are all conveniently linear in Ψ. This allows us to devide both sides by Ψ and we obtain this result

$$\frac{i\hbar}{2n}\frac{\partial n}{\partial t} - \hbar\frac{\partial \theta}{\partial t} = \frac{\hbar^2}{2m}(\nabla\theta)^2 - \frac{i\hbar^2}{2m}\nabla^2\theta \\ + \frac{\hbar q}{mc}\boldsymbol{A}\cdot\nabla\theta - \frac{i\hbar q}{2mc}\nabla\cdot\boldsymbol{A} + \frac{q^2}{2mc^2}\boldsymbol{A}^2 + 2gn - \mu. \quad (62)$$

Since $\hbar$, n, θ, m, $\boldsymbol{A}$, q, c, μ, and g are all strictly real, this equation can be split up into two parts: a part for $\frac{\partial n}{\partial t}$ and a part for $\frac{\partial \theta}{\partial t}$. Sorting out the i's and deviding by some factors we find

$$\frac{\partial n}{\partial t} = -\frac{\hbar n}{m}\nabla\cdot(\nabla\theta + \frac{q}{\hbar c}\boldsymbol{A}), \quad (63)$$

and

$$\frac{\partial \theta}{\partial t} = -\frac{\hbar}{2m}(\nabla\theta + \frac{q}{\hbar c}\boldsymbol{A})^2 + \frac{\mu - 2gn}{\hbar}. \quad (64)$$

Just like before, our n equation depends on a divergence of a certain current,

$$\boldsymbol{j} = \frac{\hbar n}{m}\left(\nabla\theta + \frac{q}{\hbar c}\boldsymbol{A}\right), \quad (65)$$

and our θ equation (resembling the ϕ from the spin currents) depends on the length of a vector and some constant parts. It is also remarkable that the quantity $\nabla\theta + \frac{q}{\hbar c}\boldsymbol{A}$ can be found in both equations, taking the role of $\nabla\phi$ in Equation 40 and 34.

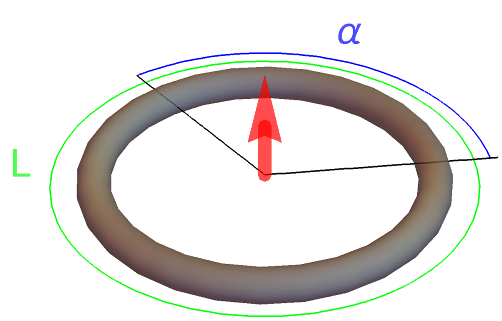

Figure 6: A superconducting ring (length L) with a magnetic field B through it and angle α around the z-axis.

Now we have found our equations of motion we return to our ring, but this time the ring is made out of a superconducting material instead of a magnetic insulator.

The length of our ring is again L and we still position it in the xy-plane having the z-axis at its center. We also apply a magnetic field of size B in the z-direction. Lastly, there is one additional difficulty that we need to take care of. In our non-conducting ring we did not need to take into account at what direction our current was going so we could get away with simple Cartesian coordinates. Due to our vector potential, we now do need to take into account the direction of the current and therefore it is easier to operate using cylindrical coordinates. This means the n and θ are now dependent on only one dimension, in this case α, the angle around the z-axis. Having said that, we can start finding our currents. We are interested in the case where n is constant again, like before.

$$\frac{\partial n}{\partial t} = -\frac{\hbar n}{m} \nabla \cdot (\nabla \theta + \frac{q}{\hbar c} \boldsymbol{A}) = 0. \tag{66}$$

In this equation our ∇ changes into $\frac{1}{r}\frac{\partial}{\partial \phi}$, which is the ϕ component of ∇ in a cylindrical coordinate system. We only need an expression for our vector potential in order to proceed. We find one in the Appendix (Section 6.2): Equation 133, which in one dimension is

$$A_\alpha = \frac{1}{2} B r, \tag{67}$$

with $r = \frac{L}{2\pi}$ the radius of our ring. We are now ready to solve the equation.

$$\frac{2\pi}{L}\frac{\partial \theta}{\partial \alpha} + \frac{q}{4\pi \hbar c} BL = \text{constant}, \tag{68}$$

which means that θ is linear in α and it should have the following form.

$$\theta(\alpha, t) = a\alpha + \theta(t) + \theta_0. \tag{69}$$

If we now apply our periodic boundary condition,

$$\sqrt{n} e^{i\theta(\alpha + 2\pi)} = \sqrt{n} e^{i\theta(\alpha)}, \tag{70}$$

we find a to be

$$a = \frac{\partial \theta}{\partial \alpha} = k, \tag{71}$$

with k being an integer and the amount of cycles in one ring. Now we know this we can also find the time-dependence of θ by plugging in our k into Equation 64. We find the result,

$$\frac{\partial \theta}{\partial t} = -\frac{\hbar}{2m}\left(\frac{2\pi k}{L} + \frac{q}{4\pi \hbar c} BL\right)^2 + \frac{\mu - 2gn}{\hbar}, \tag{72}$$

which can be written out, but it would only be getting uglier. Our full expression for θ is then

$$\theta(\alpha, t) = k\alpha + \left(-\frac{\hbar}{2m}\left(\frac{2\pi k}{L} + \frac{q}{4\pi \hbar c} BL\right)^2 + \frac{\mu - 2gn}{\hbar}\right) t + \theta_0. \tag{73}$$

Furthermore, we find that our periodic boundary conditions quantize the current that flows through the ring. This result is obtained by filling in our gradient and vector potential in Equation 65.

$$j = \frac{\hbar n}{m}\left(\frac{2\pi k}{L} + \frac{qBL}{4\pi\hbar c}\right). \tag{74}$$

Now we have found these solutions, we can also find the accompanying energy densities by filling in our discoveries into the equation of the free energy (Equation 52). First, let us handle the terms separately.

$$-\hbar^2\nabla^2\Psi = \frac{4\pi^2\hbar^2 k^2}{L^2}\Psi, \tag{75}$$

$$-\frac{i\hbar q}{c}\boldsymbol{A}\cdot\nabla\Psi = \frac{k\hbar qB}{2c}\Psi, \tag{76}$$

and

$$-\frac{i\hbar q}{c}\nabla\cdot(\Psi\boldsymbol{A}) = \frac{k\hbar qB}{2c}\Psi. \tag{77}$$

Filling these in leads to the following energy density ϵ.

$$\epsilon(k) = n\left(\frac{2\pi^2\hbar^2 k^2}{mL^2} + \frac{\hbar qBk}{2mc} + \frac{q^2B^2}{8mc^2} + gn - \mu\right). \tag{78}$$

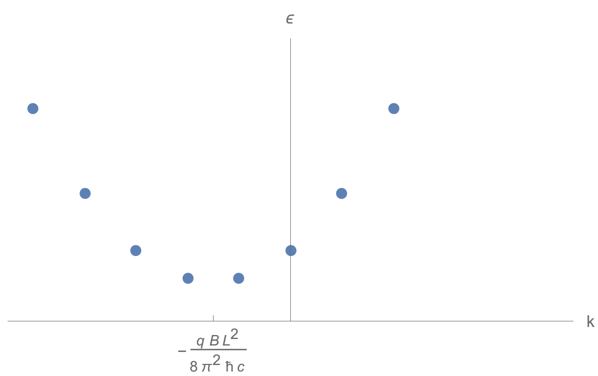

Figure 7: The energy density (ϵ) plotted as a function of k, the amount of cycles per ring.

In Figure 7 we see this energy density plotted. Interesting is that for $B \neq 0$ the most stable state is not found at $k = 0$. The minimum energy density is found at $k = -\frac{qBL^2}{8\pi^2\hbar c}$, and we discover that this belongs to no current at all, when we fill this result into our equation for j. In a case, such as the one shown in Figure 7, there is no value of k exactly in the equilibrium. This would mean that the system would tend to a superposition of two states; one with a small current clockwise and one with a small current anticlockwise.

4 Spin Superfluidity in Superconductors

Now we have discussed the two superfluid systems separately, we will now try to merge them together. First we compare the two systems and we realize how similar the two are. Then, we will couple the equations of motion, leading to modified states.

4.1 Comparison between Charge and Spin Currents

In the previous sections we have found equations of motion for several quantities. We can not deny that these equations possess some similarities. Our goal in this section is to develop a universal model that applies to both of the systems. The following equations will be generalised. Some indices are added to avoid confusion. Also, $M_S\sqrt{1-\frac{2n}{s}}$ is written back to M_z for simplicity. First the density equations:

$$\frac{\partial n_S}{\partial t} = -\nabla \cdot \left(-2\gamma M_z J n_S \nabla\phi_S\right), \tag{79}$$

$$\frac{\partial n_C}{\partial t} = -\nabla \cdot \left(\frac{\hbar}{m} n_C \left(\nabla\theta_C + \frac{q}{\hbar c}\boldsymbol{A}\right)\right). \tag{80}$$

Secondly, the phase/angle equations:

$$\frac{\partial \phi_S}{\partial t} = \gamma M_z J (\nabla\phi_S)^2 + \gamma B - \gamma K M_z, \tag{81}$$

$$\frac{\partial \theta_C}{\partial t} = -\frac{\hbar}{2m}\left(\nabla\theta_C + \frac{q}{\hbar c}\boldsymbol{A}\right)^2 + \frac{\mu - 2gn_C}{\hbar}. \tag{82}$$

To generate this model, it is useful to make a translation table, in which we define English words to be mathimatical expressions that have the same function in both equations for charge and spin currents.

	Spin	Charge
density	n_S	n_C
phase	ϕ_S	$-\theta_C$
gradient	$\nabla\phi_S$	$-\nabla\theta_C - \frac{q}{\hbar c}\boldsymbol{A}$
scale	γ	$\frac{1}{\hbar}$
stiffness	$2M_zJ$	$\frac{\hbar^2}{m}$
interaction	KM_z	$-2gn_C$
linear	B	$-\mu$

This table needs some explanation.

- The "density" is the easiest term. In both density equations we see it is found in the derivative and as a factor inside the divergence. It is also found in the phase equation of the charge currents, but since it is not in the one for spin currents, we ignore it for now.
- For the "phase" term, we take our angle ϕ to resemble the negative phase θ (something we also did in Appendix Section 6.1).

- The "gradient" is the only vector in our equations and it is represented in exactly the same way in both systems. Because our phase resembles a negative angle, we again accompany our charge gradient with a minus sign.
- We notice that all of our spin equation terms have a factor γ and nearly all of our charge equation terms have a factor $\frac{1}{\hbar}$. In the one term we do not find this immediately we force it out of our stiffness. We call this factor the "scale".
- If the factor in front of our gradient is large, then small changes in our phase over space result in relatively big changes in the the density over time. This factor we call the "stiffness".
- The "interaction" term comes from the part in our free energy that took care of the interaction of respectively the magnetization and the electron density. In it, we include the n_C we could not account for in the first item of this list.
- The "linear" term comes from the part of our free energy that depended linearly on respectively the magnetization and the electron density.

We added some minus signs to make sure that everything fits right. We get the following universal models.

$$\frac{\partial}{\partial t}\text{density} = -\nabla \cdot (-\text{scale} \times \text{stiffness} \times \text{density} \times \text{gradient})\,, \tag{83}$$

$$\frac{\partial}{\partial t}\text{phase} = \frac{1}{2} \times \text{scale} \times \text{stiffness} \times \text{gradient}^2 + \text{scale} \times \text{linear} - \text{scale} \times \text{interaction}. \tag{84}$$

4.2 Coupling the Equations

Until now, we have only discussed effects separately, but ultimately we are interested in the case in which we have a superconducting ferromagnet. In such material we would see both spin currents and charge currents and we are now interested in how they would affect one another. The equations of motion will have to be adjusted slightly to make these changes possible. To get a better idea of what information our current equations provide, we rewrite them after having derived the following velocities from Equation 41 and 65.

$$\boldsymbol{v}_S = -2\gamma M_z J \nabla \phi_S, \tag{85}$$

$$\boldsymbol{v}_C = \frac{\hbar}{m}\left(\nabla\theta_C + \frac{q}{\hbar c}\boldsymbol{A}\right). \tag{86}$$

We will now rewrite our equations of motion (Equation 79, 80, 81 and 82) into a version with a total derivative to the time ($\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$). The first part of this derivative looks at the local changes, while the second part looks at the changes that are dragged in by the current. First we identify the velocities in the equations.

$$\frac{\partial n_S}{\partial t} = -\nabla \cdot (n_S \boldsymbol{v}_S), \tag{87}$$

$$\frac{\partial n_C}{\partial t} = -\nabla \cdot (n_C \boldsymbol{v}_C), \tag{88}$$

$$\frac{\partial \phi_S}{\partial t} = -\frac{1}{2}\boldsymbol{v}_S \cdot \nabla \phi_S + \gamma(B - KM_z), \tag{89}$$

$$\frac{\partial \theta_C}{\partial t} = -\frac{1}{2}\boldsymbol{v}_C \cdot \left(\nabla \theta_C + \frac{q}{\hbar c}\boldsymbol{A}\right) + \frac{\mu - 2gn_C}{\hbar}. \tag{90}$$

Looking at it this way, the systems look even more analogous to eachother than discussed in the previous section. Reordering this, we find the following total derivatives.

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_S \cdot \nabla\right) n_S = -n_S \nabla \cdot \boldsymbol{v}_S, \tag{91}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_C \cdot \nabla\right) n_C = -n_C \nabla \cdot \boldsymbol{v}_C, \tag{92}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_S \cdot \nabla\right) \phi_S = \frac{1}{2}\boldsymbol{v}_S \cdot \nabla \phi_S + \gamma(B - KM_z), \tag{93}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_C \cdot \nabla\right) \theta_C = \frac{1}{2}\boldsymbol{v}_C \cdot \left(\nabla \theta_C - \frac{q}{\hbar c}\boldsymbol{A}\right) + \frac{\mu - 2gn_C}{\hbar}. \tag{94}$$

As we can see now, obviously, equations depend only on either $\boldsymbol{v}_S$ or $\boldsymbol{v}_C$. It is imaginable, though, that an electron current affects the way spin currents are constructed or propagated and the other way around. We introduce the dimensionless factors P and N. We use P as the part of $\boldsymbol{v}_C$ that contributes to $\boldsymbol{v}_S$ in the spin equations, and N is the part of $\boldsymbol{v}_S$ that contributes to $\boldsymbol{v}_C$ in the charge equations. In the spin equations, P is called the polarisation of the charge current. When a current is polarised (the spin ups and spin downs are not equally represented), the current contributes to a spin current. This effect is called the spin transfer torque [3]. We now make the following adjustments to the equations. For the spin equations:

$$\boldsymbol{v}_S \to \boldsymbol{v}_S + P\boldsymbol{v}_C, \tag{95}$$

and for the charge equations:

$$\boldsymbol{v}_C \to \boldsymbol{v}_C + N\boldsymbol{v}_S. \tag{96}$$

Moving back to the ring (length L) we earlier discussed for both cases, we find it to be easier to move to cylindrical coordinates with α the angle around the z-axis. Also we still have a homogenuous magnetic field. The problem becomes one-dimensional again.

$$\nabla \to \frac{2\pi}{L}\frac{\partial}{\partial \alpha}, \tag{97}$$

$$\boldsymbol{A} \to \frac{L}{4\pi}B. \tag{98}$$

These adjustments lead to the following one-dimensional equations.

$$\left(\frac{\partial}{\partial t} + \frac{2\pi}{L}(v_S + Pv_C)\frac{\partial}{\partial \alpha}\right) n_S = -\frac{2\pi}{L} n_S \frac{\partial}{\partial \alpha}(v_S + Pv_C), \tag{99}$$

$$\left(\frac{\partial}{\partial t}+\frac{2\pi}{L}(v_C+Nv_S)\frac{\partial}{\partial\alpha}\right)n_C=-\frac{2\pi}{L}n_C\frac{\partial}{\partial\alpha}(v_C+Nv_S), \tag{100}$$

$$\left(\frac{\partial}{\partial t}+\frac{2\pi}{L}(v_S+Pv_C)\frac{\partial}{\partial\alpha}\right)\phi_S=\frac{\pi}{L}(v_S+Pv_C)\frac{\partial\phi_S}{\partial\alpha}+\gamma(B-KM_z), \tag{101}$$

$$\left(\frac{\partial}{\partial t}+\frac{2\pi}{L}(v_C+Nv_S)\frac{\partial}{\partial\alpha}\right)\theta_C=(v_C+Nv_S)\left(\frac{\pi}{L}\frac{\partial\theta_C}{\partial\alpha}-\frac{qBL}{8\pi\hbar c}\right)+\frac{\mu-2gn_C}{\hbar}, \tag{102}$$

in which

$$v_S=-\frac{4\pi}{L}\gamma M_zJ\frac{\partial\phi_S}{\partial\alpha}, \tag{103}$$

$$v_C=\frac{\hbar}{m}\left(\frac{2\pi}{L}\frac{\partial\theta_C}{\partial\alpha}+\frac{qBL}{4\pi\hbar c}\right). \tag{104}$$

Our phase equations are still only valid in the assumption that n_S and n_C are constant over space. If they are constant over space they should also be constant over time if spin and charge are conserved. So we can take our density equations to equal zero. This leads to the following expressions.

$$v_S+Pv_C=c_1, \tag{105}$$

$$v_C+Nv_S=c_2, \tag{106}$$

with c_1 and c_2 constant velocities independent of α. Because these are two equations, v_S and v_C can be written out as a function of c_1, c_2, N and P, all of which are independent of α. This means v_S and v_C must both be independent of α. Similar to what we have seen earlier this means that ϕ_S and θ_C should be linear in α. Then, also similar to before, we apply the periodic boundary conditions and find the following expressions for ϕ_S and θ_C.

$$\phi_S(\alpha)=k_S\alpha+\phi_{S,0}, \tag{107}$$

$$\theta_C(\alpha)=k_C\alpha+\theta_{C,0}, \tag{108}$$

with k_S and k_C being integers. Now filling in the result in the phase equations we find

$$\frac{\partial\phi_S}{\partial t}=-\frac{\pi}{L}(v_S+Pv_C)k_S+\gamma(B-KM_z), \tag{109}$$

$$\frac{\partial\theta_C}{\partial t}=-\left(\frac{k_C\pi}{L}+\frac{qBL}{8\pi\hbar c}\right)(v_C+Nv_S)+\frac{\mu-2gn_C}{\hbar}, \tag{110}$$

with the velocities

$$v_S=-\frac{4\pi}{L}\gamma M_zJk_S, \tag{111}$$

$$v_C=\frac{\hbar}{m}\left(\frac{2k_C\pi}{L}+\frac{qBL}{4\pi\hbar c}\right). \tag{112}$$

Then, this leads to the functions for ϕ_S and θ_C.

$$\phi_S(\alpha,t)=k_S\alpha+\left(-\frac{k_S\pi}{L}(v_S+Pv_C)+\gamma(B-KM_z)\right)t+\phi_{S,0}, \tag{113}$$

$$\theta_C(\alpha,t) = k_C\alpha + \left(-\left(\frac{k_C\pi}{L} + \frac{qBL}{4\pi\hbar c}\right)(v_C + Nv_S) + \frac{\mu - 2gn_C}{\hbar}\right)t + \theta_{C,0}. \quad (114)$$

These results are exactly the same as earlier results (Equation 49 and 73), with the velocities adjusted like we did in this section.

Though this may seem like a trivial result, the important and remarkable thing is, that the states change due to this coupling. So the presence of two superfluid systems in a material does influence the way - in this case charge and spin - currents are decribed, assuming that $N, P \neq 0$.

5 Conclusion, Discussion and Outlook

In this thesis we have derived equations of motion for superfluid systems (spin superfluidity and superconductivity) as well as stationary states that obey these equations, keeping the densities constant. After we have done that separately, we have coupled the equations, guided by known effects like the spin transfer torque. The coupled equations lead to different states than the results of the separate cases. This means that our question whether the two systems would influence eachother can be answered affirmatively.

The conclusion that the states of a material in which both spin superfluidity and superconductivity occur *do* differ from the separate cases, does not necessarily imply that the coupled states derived in this thesis are *how* this influence is described in reality. Some effects are not taken into account and only the simplest states have been calculated, leaving a heap of solutions, that may have the preference of the real system, undiscussed. For example, the effects of dissipation have been left out and we have assumed the densities to be constant in both cases, which is not particularly a bad assumption and made the problems less difficult, but it may not be the actual way nature will govern this situation. From a different point of view, you could say that it may be hard to find a material that is actually fully described by the assumptions made in this thesis. It gives a good idea of what to expect, though.

These effects that have been left out could be added in further research in order to describe systems even better. Think about dissipation and solutions with nonconstant densities. Furthermore, in this thesis, we have only discussed constant magnetic fields. In further research one could possibly discuss changing magnetic fields, which would have an effect on both superfluid systems. Also, from an experimental point of view, experiments could be done on superconducting easy-plane ferromagnets to check whether the results from this thesis are in fact a good approximation of reality. Finally, one could research the effects different superfluid systems have on eachother and check whether they influence eachother in a similar way.

6 Appendix

6.1 Conjugate Momenta n and ϕ

In section 2.3 we found the following equations of motion for ϕ and n.

$$\frac{\partial \phi}{\partial t} = \gamma M_S J \sqrt{1 - \frac{2n}{s}} (\nabla \phi)^2 + \gamma B - \gamma K M_S \sqrt{1 - \frac{2n}{s}}, \tag{115}$$

$$\frac{\partial n}{\partial t} = 2\gamma M_S J \sqrt{1 - \frac{2n}{s}} n \nabla^2 \phi. \tag{116}$$

In these equations n was considered constant over space.

There is reason to think of the hypothesis, that one of these quantities is the conjugate momentum of the other one. To check that, we could consider working out the Hamilton equations and check whether they look like the expressions we have just found. The following equations should be looked at.

$$\frac{\partial n}{\partial t} \propto \frac{\delta F}{\delta \phi}, \tag{117}$$

$$\frac{\partial \phi}{\partial t} \propto \frac{\delta F}{\delta n}. \tag{118}$$

To do this, we first have to write out F in terms of n and ϕ. We will first use our earlier defined n_1 and n_2 again, before we write it all the way to n. By far the hardest part of Equation 10 is the first term. So let us look at that first.

$$-\frac{J}{2} \boldsymbol{M} \cdot \nabla^2 \boldsymbol{M} = -\frac{M_S^2 J}{2} \begin{pmatrix} n_1 \cos \phi \\ n_1 \sin \phi \\ n_2 \end{pmatrix} \cdot \nabla^2 \begin{pmatrix} n_1 \cos \phi \\ n_1 \sin \phi \\ n_2 \end{pmatrix}. \tag{119}$$

We see that our earlier found identities Equation 22 and 23 will come in handy again. Using those and again canceling several terms out we get the following expression for F.

$$F(\phi, n) = \int d\boldsymbol{x} \left(-\frac{M_S^2 J}{2} (n_1 \nabla^2 n_1 - n_1^2 (\nabla \phi)^2 + n_2 \nabla^2 n_2) - B M_S n_2 + \frac{K}{2} M_S^2 n_2^2 \right). \tag{120}$$

Let us first calculate the time-derivative of n, by calculating the variational derivative of F to ϕ. This should not be too hard as ϕ is hardly represented in the equation. It essentially comes down to this.

$$\delta F = \frac{M_S^2 J}{2} \frac{2n}{s} ((\nabla(\phi + \delta\phi))^2 - (\nabla \phi)^2). \tag{121}$$

This leads to the following derivative.

$$\frac{\delta F}{\delta \phi} = \frac{2 M_S^2 J}{s} n \nabla^2 \phi. \tag{122}$$

Which is in fact proportional to the earlier found $\frac{\partial n}{\partial t}$.

$$\frac{\partial n}{\partial t} = \frac{\gamma s}{M_S}\sqrt{1 - \frac{2n}{s}}\frac{\delta F}{\delta \phi}. \tag{123}$$

For the other derivative it is good to immediately assume that n is constant, because we did that to obtain the expression for the $\frac{\partial \phi}{\partial t}$ in the first place. The free energy then reduces to a less complicated form.

$$F = \int d\boldsymbol{x} \left(\frac{M_S^2 J}{2}\frac{2n}{s}(\nabla\phi)^2 \ - BM_S n_2 + \frac{K}{2}M_S^2 n_2^2 \right). \tag{124}$$

To derive this to n, we take the regular derivative to n_2 for the second term.

$$n_2(n + \delta n) = n_2(n) + \frac{\partial n_2}{\partial n}\delta n = n_2(n) - \frac{1}{s\sqrt{1 - \frac{2n}{s}}}\delta n. \tag{125}$$

This leaves us with the following

$$\frac{\delta F}{\delta n} = \frac{M_S^2 J}{s}(\nabla\phi)^2 + BM_S\frac{1}{s\sqrt{1 - \frac{2n}{s}}} - K\frac{M_S^2}{s}. \tag{126}$$

Now, we see that the first term is also proportional to $\frac{\partial \phi}{\partial t}$ with exactly the same factor as the previous proportionality.

$$\frac{\partial \phi}{\partial t} = \frac{\gamma s}{M_S}\sqrt{1 - \frac{2n}{s}}\frac{\delta F}{\delta n}. \tag{127}$$

If n and ϕ would be eachother's conjugate momenta, the proportionality factors should differ a minus sign. This is not the case. However, $\theta = -\phi$ is the conjugate momentum of n, as the equation of motion of n is linearly dependent on ϕ, where the equation of ϕ is quadratically dependent on ϕ. This means that a substitution of ϕ with $-\theta$ would give only one extra minus sign.

6.2 Vector Potential for a Homogenuous Magnetic Field

In this section of the Appendix we derive a vector potential for a homogenuous magnetic field in the z-direction like the one that is used in Section 3. We use cylindrical coordinates, as that is useful in that section. The vector potential is defined to be

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{128}$$

We also know we want $\boldsymbol{B}$ to equal $B\hat{\boldsymbol{z}}$. Using the cylindrical coordinates of a curl we find

$$\begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \\ \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \\ \frac{1}{r}\frac{\partial (rA_\phi)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \end{pmatrix}. \tag{129}$$

Due to symmetry $\boldsymbol{A}$ cannot depend on the angle ϕ nor z, as the field should be homogenuous. This means that only a part of the z-equation remains.

$$B = \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}. \tag{130}$$

Using the product rule we find the differential equation,

$$A_\phi + r\frac{\partial A_\phi}{\partial r} = Br, \tag{131}$$

with solution,

$$A_\phi = \frac{c}{r} + \frac{1}{2}Br, \tag{132}$$

with c an arbitrary constant. Since c can be chosen to be anything, we might as well set it to zero. We find the following vector potential.

$$\boldsymbol{A} = \begin{pmatrix} 0 \\ \frac{1}{2}Br \\ 0 \end{pmatrix}. \tag{133}$$

7 References

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