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Landau levels and holography

Bachelor Thesis

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Abstract

We explore the Landau levels in strongly coupled systems under an external magnetic field by means of the AdS/CFT duality. We start off by providing a classical computation of the landau levels where we follow the original computations of Lev Landau. Then we start looking at strongly coupled systems using the Field Theory/Gravity duality. Here we will follow the earlier papers of T.Albash and C.V.Johnson on the holographic approach to landau levels and fermi liquids where we look at the effects of an external magnetic field on the energy levels of 2+1 dimensional strongly coupled field theories holographically duel to charged AdS_4 black holes at zero temperature. We also compute the spectral functions and consider its solutions in the simplest case of $\omega = 0$.

Contents

1 Introduction

Classically one would expect the energy values of a charged particle in an external magnetic field to be continuously distributed. This has been proved to not be the case by the Soviet physicist Lev Landau [1]. Landau proved that charged particles under the influence of a magnetic field will exhibit quantized motion. Their motion will be quantized in the form of cyclotron orbits which results in the fact that the particles can only occupy orbits with discrete energy values. These energy values are called Landau levels.
This phenomena of quantized energy levels has many applications within Condensed matter physics [11, 10] and furthermore a direct and important result of this quantization is the Hall effect.
However there are also still open problems in physics concerning the landau quantization. Examples are the magnetic catalysis and inverse magnetic catalysis in quantum chromodynamics. So there are still a lot of exciting problems concerning the Landau levels of which we will study one in this thesis.

We can quite easily describe the energy dynamics for a single charged particle in a uniform magnetic field. A more challenging problem however is to try and describe the dynamics for a many body system of charged particles subject to a magnetic field. In the case when we consider charged fermions the system can be either weakly coupled or strongly coupled. In the case when the fermions are weakly interacting with each other we can describe the dynamics of the system using quantum field theory. An interesting thing to note here is that some at first complicated looking systems can actually be approximately described by considering the same system but then weakly coupled in free space which is the case for quasiparticles for example.
A question which then naturally arises is whether we can also describe the quantization for strongly interacting systems that cannot approximately be considered weak. The study of strongly interacting fermionic systems at finite density and temperature is a important but challenging task in condensed matter and high energy physics. The problem with studying strongly interacting systems is that analytic methods are limited or not available at all. Furthermore numerical simulations of fermions at finite density breaks down because of technical problems that go under the name of ”sign problems” see [14]
So we need a way to study these strongly interacting systems and in this thesis we resort to an unconventional method, namely the AdS/CFT duality. The AdS/CFT correspondence is a modern technique that promises us tools with which we can gain new insight in this problem. This correspondence promises us a duality between strongly interacting field theories and weak gravity. By mapping the physics from one side of the correspondence to the other we can try and solve the dynamics of our strongly coupled system in another theory where we have the tools to do so and then retrieve this information back.

The basic outline of this thesis is as follows: First we will compute the energy levels of a charged particle in a uniform magnetic field following the original computations of Landau. Where we show that indeed the energy levels of a charged particle moving in a magnetic field is quantized. Then in the next section we motivate the AdS/CFT correspondence and the way we can apply it to strongly coupled systems. After having done this we study the energy dynamics of a strongly coupled system under a magnetic field using a dual gravitational formulation of fermionic fields propagating in a 4 dimensional asymptotically AdS background. Here we will follow the earlier papers of Johnson and Albash [2 , 3] who already studied the dynamics of strongly interacting systems using holography.

2 Derivation of Landau levels

In this section we will derive the energy levels for a charged particle in a uniform magnetic field following Landau in 1930 (see his book [[1]] chapter XV "motion in a magnetic field"). We will then see that the particles can only have discrete energy values and hence their motion is also quantized.
We start off by defining the Hamiltonian. We know that in a classical system the Hamiltonian of a charged particle in a electromagnetic field is given by

$$H = \frac{1}{2m}(\mathbf{p} - \frac{q}{c}\mathbf{A})^2 + qV \tag{1}$$

In which q stands for the charge of the particle, $\mathbf{p}$ is its momentum, $\mathbf{A}$ is the vector potential and V is the scalar potential.

2.1 No spin case

First we will consider particles with no spin. Without the spin to affect the energy, the transition from the classical Hamiltonian to a quantum mechanical Hamiltonian is straightforwardly done by changing the momentum to a momentum operator in which $\mathbf{p} \to \hat{\mathbf{p}} = \frac{\hbar}{i}\nabla$. Hence our Hamiltonian becomes

$$H = \frac{1}{2m}(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A})^2 + qV \tag{2}$$

Now we are ready to start computing the energy levels. We will consider a free particle in a uniform magnetic field which is pointing in the z-direction. Since we consider a free particle the scalar potential will vanish. We take our vector potential like in 1 (chapter XV) which is

$$\mathbf{A} = \begin{pmatrix} -Hy \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

Using this vector potential we can write the Hamiltonian as

$$\hat{H} = \frac{1}{2m}(\hat{p}_x + \frac{qHy}{c})^2 + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} \tag{4}$$

Now we can write down the schrödinger equation corresponding to the Hamiltonian above which is

$$\frac{1}{2m}[(\hat{p}_x + \frac{qHy}{c})^2 + \hat{p}_y^2 + \hat{p}_z^2]\psi = E\psi \tag{5}$$

Since the Hamiltonian 4 is not explicitly dependent on x or y we immediately know that $\hat{p_x}$ and $\hat{p_z}$ commute with the Hamiltonian since these momenta take derivatives with respect to x and z. Since the x and z components of the momentum commute with the Hamiltonian they are conserved. Hence we can represent them by their eigenvalues p_x and p_z. In light of this we can take the following anzatz for the solution of 5 according to 1.

$$\psi = e^{\frac{i}{\hbar}(p_x x + p_z z)}\phi(y) \tag{6}$$

The next step is to substitute equation 6 into equation 5 and rewrite the equation into something that will come in handy later on

$$\frac{1}{2m}[(\hat{p}_x + \frac{qHy}{c})^2 + \hat{p}_y^2 + \hat{p}_z^2]\psi = E\psi \quad \rightarrow$$
$$\phi'' + \frac{2m}{\hbar^2}[E + \frac{\mu\sigma H}{s} - \frac{p_z^2}{2m} - \frac{1}{2m}(\frac{qHy}{c})^2 - \frac{p_x^2}{2m}]\phi(y) = 0 \tag{7}$$

Now we introduce $\omega = \frac{qH}{mc}$ and $y_0 = -\frac{cp_x}{qH}$ to finally get for the schrödinger equation:

$$\phi'' + \frac{2m}{\hbar^2}[(E - \frac{p_z^2}{2m}) - \frac{1}{2}m\omega^2(y - y_0)^2]\phi = 0 \tag{8}$$

Now we can actually immediately see what the allowed energies are since equation 8 looks just like the schrödinger equation for a harmonic oscillator

$$\phi'' + \frac{2m}{\hbar^2}[E - \frac{1}{2}m\omega^2 x^2]\phi = 0 \tag{9}$$

with allowed energies

$$E = (n + \frac{1}{2})\hbar\omega \tag{10}$$

Hence in our case of equation 8 we can say that the allowed energies are given by

$$E - \frac{p_z^2}{2m} = (n + \frac{1}{2})\hbar\omega \tag{11}$$

$$E = (n + \frac{1}{2})\hbar\omega + \frac{p_z^2}{2m} \tag{12}$$

For a particle moving in a plane perpendicular tot the direction of the magnetic field(which is in the z-direction) we have that $p_z = 0$ and hence the first term gives us the discrete energy levels. With this we have shown that the energy values for charged particles in a uniform magnetic field are quantized to discrete values.

2.2 Particles with spin

In the last case we looked at particles which have no spin. In order to implement the spin into the Hamiltonian we somehow need to account for the effect spin has on the energy. For this we will need to look at the magnetic moment of the particle which is given by

$$\hat{\mu} = \frac{\mu\hat{\mathbf{S}}}{s} \tag{13}$$

In which $\hat{\mathbf{S}}$ is the particle's spin, s stands for the magnitude of the spin and μ is a value characteristic to the particle. In order for us to implement the effect of the spin on the energy we need to add $-\hat{\mu}\cdot\mathbf{H}$ to the Hamiltonian (as in 1 chapter XV). Hence the Hamiltonian (1) will transform to

$$\hat{H} = \frac{1}{2m}(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A})^2 + qV - \hat{\mu}\cdot\mathbf{H} \tag{14}$$

In quantum mechanical systems with spin.
After plugging our vector potential and magnetic moment into 14 and expanding the square we get for our Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{p}_x + \frac{qHy}{c})^2 + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} - (\frac{\mu}{s})\hat{s_z}H \tag{15}$$

The next step will be to write the schrödinger equation for this Hamiltonian. For this we somehow want the Hamiltonian to be spin-independent so that we can write the schrödinger equation in its usual form coördinate form. In order to achieve this we note that

$\hat{s_z}\hat{H} - \hat{H}\hat{s_z} = [\hat{s_z}\frac{1}{2m}(\hat{p}_x + \frac{qHy}{c})^2 + \hat{s_z}\frac{\hat{p}_y^2}{2m} + \hat{s_z}\frac{\hat{p}_z^2}{2m} - \hat{s_z}(\frac{\mu}{s})\hat{s_z}H] - [\frac{1}{2m}(\hat{p}_x + \frac{qHy}{c})^2\hat{s_z} + \frac{\hat{p}_y^2}{2m}\hat{s_z} + \frac{\hat{p}_z^2}{2m}\hat{s_z} - (\frac{\mu}{s})\hat{s_z}H\hat{s_z}] = [\hat{s_z}\frac{1}{2m}(\hat{p}_x+\frac{qHy}{c})^2+\hat{s_z}\frac{\hat{p}_y^2}{2m}+\hat{s_z}\frac{\hat{p}_z^2}{2m}-(\frac{\mu}{s})H]-[\hat{s_z}\frac{1}{2m}(\hat{p}_x+\frac{qHy}{c})^2+\hat{s_z}\frac{\hat{p}_y^2}{2m}+\hat{s_z}\frac{\hat{p}_z^2}{2m}-(\frac{\mu}{s})H] = 0.$

Here we used the fact that the momenta commute with the spin and that $\hat{s}_z^2 = \hat{1}$ with $\hat{1}$ being the unit matrix. That the momenta commute with the spin can be quickly seen by

$$\hat{p}_x\hat{s}_z\mathbf{a} = \tfrac{\hbar}{i}\tfrac{d}{dx}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \tfrac{\hbar}{i}\tfrac{d}{dx}\begin{pmatrix}a_1\\ -a_2\end{pmatrix} = \tfrac{\hbar}{i}\begin{pmatrix}\frac{da_1}{dx}\\ -\frac{da_2}{dx}\end{pmatrix}$$

$$\hat{s}_z\hat{p}_x\mathbf{a} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\tfrac{\hbar}{i}\tfrac{d}{dx}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\tfrac{\hbar}{i}\begin{pmatrix}\frac{da_1}{dx}\\ \frac{da_2}{dx}\end{pmatrix} = \tfrac{\hbar}{i}\begin{pmatrix}\frac{da_1}{dx}\\ -\frac{da_2}{dx}\end{pmatrix}$$

Now since $\hat{s_z}$ commutes with the Hamiltonian this means that $\hat{s_z}$ is conserved and we can use its eigenvalue $\hat{s}_z = s_z$ to describe it. This means that the wave function's dependency on the spin is not important and we can write the schrödinger equation in its usual form.
So the schrödinger equation becomes

$$\frac{1}{2m}[(\hat{p}_x + \frac{qHy}{c})^2 + \hat{p}_y^2 + \hat{p}_z^2]\psi - \frac{\mu\sigma H}{s}\psi = E\psi \tag{16}$$

$$\phi'' + \frac{2m}{\hbar^2}[(E + \frac{\mu\sigma H}{s} - \frac{p_z^2}{2m}) - \frac{1}{2}m\omega^2(y - y_0)^2]\phi = 0 \tag{17}$$

Just like before we can see the resemblance between this equation and that of the harmonic oscillator 9 which has allowed energies 10. Hence we can say that the allowed energies for a particle with spin under a uniform magnetic field are

$$E + \frac{\mu\sigma H}{s} - \frac{p_z^2}{2m} = (n + \frac{1}{2})\hbar\omega \tag{18}$$

$$E = (n + \frac{1}{2})\hbar\omega - \frac{\mu\sigma H}{s} + \frac{p_z^2}{2m} \tag{19}$$

In which the first term gives us the discrete energy levels for a motion in a plane perpendicular to the direction of the field.

3 Ads/CFT duality and Correlations

As we mentioned before we are interested in studying the energy levels of strongly coupled systems subject to a magnetic field. Since analytic methods are limited and numerical simulations are also not an option we plan to resort to an unconventional method, namely the AdS/CFT duality. In this thesis we will not try to explain why the correspondence can be true since this will require detailed knowledge of string theory and d-branes which is beyond the scope of this thesis. Instead we assume the duality holds and try to explain how it connects the two theories and how we plan to use it to study strongly coupled systems. For this we looked at the following references [17, 12 ,18, 10].

In theoretical physics people often discover new and exciting things by realizing that certain concepts are in fact related to each other at a deep and fundamental level. Examples of such relations are dualities which relate two seemingly different quantum theories to each other by stating that the theories are in fact equivalent. The AdS/CFT correspondence is a duality albeit a different one from the one above. The Anti-de Sitter / Conformal field theory correspondence is a new type of duality which relates gravity theories in d dimensions on asymptotically Anti-de Sitter spacetimes to local field theories without gravity in d-1 dimensions. Here Anti-de Sitter spaces are maximally symmetric solutions of the Einstein equations with a negative cosmological constant. Examples of maximally symmetric Euclidean spaces are the sphere, flat space and the hyperboloid. The quantum field theory may be thought of as being defined on the conformal boundary of this Anti-de Sitter space. Since the duality relates gravity in d dimensions to field theories in d-1 dimensions it is also an important realization of the holographic principle. Which states that in gravitational theory , information stored in the volume are encoded in the boundary. An simple example is the way a 2 dimensional hologram encodes the information about a 3 dimensional object.

But What does it mean that two theories are dynamically equivalent? The correspondence states that the two theories are identical and therefore describe the same physics from two very different perspectives. If the AdS/CFT duality holds, it means that all the physics of one description can be mapped onto all the physics of the other. Hence we would like to be able to provide such a mapping. Here we will try and show what this mapping is. As we will see this map provides a one-to-one relation between operators in local field theories to fields in gravity.

3.1 Field-operator map

By considering the symmetries of the two theories and checking when they coincide we can find a precise one to one mapping between operators in local field theories to fields in gravity. One can then find that for a scalar field in AdS we have the following map:

$$m^2L^2 = \triangle(\triangle - d) \tag{20}$$

Where $\triangle$ is the conformal dimension. To make the mapping more explicit we can consider the boundary behavior of the supergravity fields [12, 17]. To do this we first consider the AdS side. Let us consider a scalar field in the AdS supergravity side for which we take the following simplified action:

$$S_{supergravity} = -\frac{C}{2}\int dzd^dx\sqrt{-g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2) \tag{21}$$

Where C is proportional to the rank of the gauge group N squared ($C \propto N^2$). And the mass is given by the AdS dictionary $m^2L^2 = \triangle(\triangle - d)$. Then the field equation given by the Klein-Gordon equation

$$(\Box - m^2)\phi = 0 \tag{22}$$

has two independent solutions which are characterized by their behavior as $z \to 0$

$$\phi(z) \sim z^{\triangle_+} \quad , \quad \phi(z) \sim z^{\triangle_-} \tag{23}$$

Where $\triangle_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2}$. Near the boundary $z \to 0$ we can expand the field as

$$\phi(x,z) \sim A(x)z^{\triangle_-} + B(x)z^{\triangle_+} \tag{24}$$

And we consider the supergravity fields ϕ to assume the boundary value:

$$A(x) = \lim_{z \to 0} \phi(x, z) z^{\triangle_+ - d} \tag{25}$$

Then we have the following map between generating functionals on both sides of the correspondence:

$$e^{-W[A(x)]} = e^{-\mathcal{S}_{supergravity}[\phi]|_{A(x)}} = \left\langle exp\left(\int d^d x A(x)\mathcal{O}(x)\right)\right\rangle_{CFT} \tag{26}$$

Where we assume that the operator $\mathcal{O}$ on the field theory side has dimension $\triangle$ and its source is given by $A(x)$. So in other words the field theory operators can be identified with a classical action on (d+1) dimensional Anti-de Sitter space, subject to the boundary condition that the (d + 1)-dimensional fields ϕ assume the boundary values $A(x)$.
Having this map between generating functionals we can perform holographic calculations of correlation functions of gauge invariant operators. Consider the family of operators $\mathcal{O}_i$ on the field theory side that correspond to the sources $A_i(x)$. We then obtain correlation functions from the generating functionals by taking derivatives with respect to the sources as:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle_{CFT} = -\frac{\partial^n W}{\partial A_1(x_1)\partial A_2(x_2)\ldots\partial A_n(x_n)}|_{A_i=0} \tag{27}$$

3.2 The correspondence for strongly coupled systems

Coming back to our original question: How do we actually use this correspondence to study strongly coupled systems subject to a magnetic field?
The Ads/CFT correspondence allows us to study strongly coupled field theories in which certain questions become computationally tractable and conceptually clearer. This is because in particular limits we can describe strongly coupled systems by considering its dual gravity theory which is then classical and weak. In the case of strongly coupled systems we can use the duality in the following way: In the case of a system with strongly interacting charged fermions we have no real way of computing the allowed energies of the system in quantum field theory. Instead what we do is consider this same system in its dual gravity. Since in the gravity dual we know how to solve the equations of motion. The way we go about doing this is by probing the bulk fermion onto the electrically and magnetically charged black hole. This way the bulk lives in the black hole without disturbing the curvature. In this black hole we have the Einstein-Maxwell action which contains the dynamics within the black hole. After we probed our bulk fermion onto the black hole we can solve the equations of motion within the gravity theory which is weak and classical. When we have found the relevant dynamics of the probe in the black hole we can retrieve this information by considering the bulk fermion near the ads-boundary where the field theory lives. Near the ads boundary we can expand the fields as powers of z as shown above where the information we need lives in the coefficients before these powers. Then the last step would be to extract the information from these coefficients. In order to do this we need to compute correlation functions which contain the relation between these coefficients. After we have determined these correlation functions we possess all the information regarding the dynamics of our system in field theory.

4 Review of existing literature: T.Albash , C.V. Johnson

In this section we will study the effects of an external magnetic field on the energy levels of 2+1 dimensional strongly coupled field theories holographic dual to charged AdS_4 black holes at zero temperature in weak gravity. Here we will we follow the work of Albash and Johnson in [2] and [3], who have already studied these field theories dual to gravity, and at some points provide a more detailed computation.

4.1 The black hole geometry and Einsteins equations

We consider a dyonic black hole which carries both electric as magnetic charges in a asymptotically AdS_4 spacetime. The metric is given by 2:

$$\begin{aligned} ds^2 &= \frac{L^2\alpha^2}{z^2}(-f(z)dt^2 + dx^2 + dy^2) + \frac{L^2}{z^2 f(z)}dz^2 \\ F &= 2H\alpha^2 dx \wedge dy + 2Q\alpha dz \wedge dt \\ f(z) &= 1 + (H^2 + Q^2)z^4 - (1 + H^2 + Q^2)z^3 = (1-z)(z^2 + z + 1 - (H^2 + Q^2)z^3) \end{aligned} \tag{28}$$

And the Einstein-Maxwell action for this background is:

$$S_{bulk} = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-G}\{R - 2\Lambda + \frac{3}{4}\frac{1}{\Lambda}F^2\} \tag{29}$$

In which Λ is the cosmological constant which we can rewrite using the length scale L which is related to Λ by $\Lambda = \frac{-3}{L^2}$. Furthermore R is the Ricci scalar and F is defined above in 28. The parameter α has dimensions of inverse length. The Einstein-maxwell action then becomes:

$$S_{bulk} = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-G}\{R + \frac{6}{L^2} - \frac{L^2}{4}F^2\} \tag{30}$$

Here $\kappa_4^2 = 8\pi G_N$ is the gravitational coupling. The mass and Hawking temperature of the black hole are given by [2]:

$$\begin{aligned} \epsilon &= \frac{\alpha^3 L^2}{\kappa_4^2}[1 + Q^2 + H^2] \\ T &= \frac{\alpha}{4\pi}[3 - (Q^2 + H^2)] \end{aligned} \tag{31}$$

We will now derive the Einsteins field equations from 30 and check whether these satisfy the fields given in 28.
We start off by writing the Lagrangian for the action 30 which is simply given by

$$\mathcal{L} = \frac{1}{2\kappa_4^2}\sqrt{-G}\{R + \frac{6}{L^2} - \frac{L^2}{4}F^2\} \tag{32}$$

The equations of motion are obtained by considering a variation with respect tot the metric $g^{\mu\nu}$:

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu g_{\mu\nu}} + \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} = 0 \tag{33}$$

The is the generalized form of the classical Euler-Lagrange equation for field theory. We immediately note that the first term is zero, since we have no explicit dependence on the derivative of the metric, and hence we only have to consider the second term. Knowing that R, G and F^2 have an dependency on the metric tensor $g^{\mu\nu}$ we get:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} &= \frac{1}{2\kappa_4^2}\{\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}}R + \sqrt{-G}\frac{\partial R}{\partial g^{\mu\nu}} + \frac{6}{L^2}\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}} - \frac{L^2}{4}\sqrt{-G}\frac{\partial F^2}{\partial g^{\mu\nu}} - \frac{L^2}{4}F^2\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}}\} = \\ &\frac{\sqrt{-G}}{2\kappa_4^2}\{\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}}\frac{R}{\sqrt{-G}} + \frac{\partial R}{\partial g^{\mu\nu}} + \frac{6}{L^2\sqrt{-G}}\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}} - \frac{L^2}{4}\frac{\partial F^2}{\partial g^{\mu\nu}} - \frac{F^2L^2}{4\sqrt{-G}}\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}}\} = 0\end{aligned} \tag{34}$$

Now we rewrite 34 by using the definitions $\frac{\partial\sqrt{-G}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{-G}g_{\mu\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$ in which $R_{\mu\nu}$ is the Ricci tensor. We then get for 34:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} &= \frac{\sqrt{-G}}{2\kappa_4^2}\{-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu} - \frac{3}{L^2}g_{\mu\nu} + \frac{F^2L^2}{8}g_{\mu\nu} + \frac{L^2}{4}\frac{\partial F^2}{\partial g^{\mu\nu}}\} = \\ &\frac{\sqrt{-G}}{2\kappa_4^2}\{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}[-\frac{6}{L^2} + \frac{F^2L^2}{4}] - \frac{L^2}{4}\frac{\partial F^2}{\partial g^{\mu\nu}}\} = 0\end{aligned} \tag{35}$$

Hence the Einstein field equations of motion reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -g_{\mu\nu}\frac{F^2L^2}{8} + \frac{L^2}{4}\frac{\partial F^2}{\partial g^{\mu\nu}} + \frac{3}{L^2}g_{\mu\nu} \tag{36}$$

Now we wish to check whether these equations satisfy the fields in 28. Computing the Einstein tensor (the left part of equation 36) is a lot of work to perform by hand hence we will compute the Einstein tensor using a mathematical code called the Riemannian Geometry & Tensor Calculus code (RGTC) (see 5). This code will compute the Einstein tensor for us after we plug in our metric. So the only thing that we need to do is to compute the right hand side of equation 36 by hand and check if this is equal to the terms we find for the Einstein tensor.
Plugging our metric into RGTC yields us the Einstein tensor:

$$\begin{pmatrix} -\frac{\alpha^2 f(3f-zf')}{z^2} & 0 & 0 & 0 \\ 0 & \frac{\alpha^2(6f-4zf'+z^2f'')}{2z^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha^2(6f-4zf'+z^2f'')}{2z^2} & 0 \\ 0 & 0 & 0 & \frac{3f-zf'}{z^2 f} \end{pmatrix} \tag{37}$$

We have to see whether the right hand side of 36 satisfies this. Here we will check the upper left (t^2) argument. The t^2 argument of the Einstein tensor is

$$-\frac{\alpha^2 f(3f-zf')}{z^2} = -\frac{\alpha^2 f((3+3(H^2+Q^2)z^4-3(1+H^2+Q^2)z^3)-(4(H^2+Q^2)z^4-3(1+H^2+Q^2)z^3)}{z^2} =$$

$$-\frac{\alpha^2 f(3-(H^2+Q^2)z^4)}{z^2} = \alpha^2 z^2 f^2[H^2+Q^2] - \frac{3\alpha^2 f}{z^2} \tag{38}$$

Using the definitions $F^2 = F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$ and $\frac{\partial F^2}{\partial g^{\mu\nu}} = 2F_{\mu\beta}F_{\nu}^{\beta}$ and considering their t^2 argument we can find:

$$\frac{3}{L^2}g_{tt} = -\frac{3}{L^2}\frac{L^2\alpha^2 f}{z^2} = -\frac{3\alpha^2 f}{z^2}$$

$$\frac{L^2}{4}\frac{\partial F^2}{\partial g^{\mu\nu}} = \frac{L^2}{2}F_{t\beta}F_t^{\beta} = \frac{L^2}{2}F_{tz}F_t^{z} = \frac{L^2}{2}F_{tz}F_{tz}g^{zz} = \frac{L^2}{2}\frac{z^2 f}{L^2}4Q^2\alpha^2 = 2z^2 fQ^2\alpha^2 \tag{39}$$

$$-g_{tt}\frac{L^2}{8}F^2 = \frac{L^4\alpha^2 f}{8z^2}F^2 = \frac{L^4\alpha^2 f}{8z^2}\frac{8z^4}{L^4}[H^2-Q^2] = \alpha^2 z^2 f[H^2-Q^2]$$

Hence the t^2 argument of the right hand side of equation 36 becomes: $\alpha^2 z^2 f^2[H^2+Q^2] - \frac{3\alpha^2 f}{z^2}$. We thus indeed see that the Einstein field equations of the action 30 satisfy the fields in 28.

4.2 Probing the fermion

In order to make things a bit more simpler we work in dimensionless units from now on by changing coördinates to:

$$t \to \frac{t}{\alpha} \quad , \quad x \to \frac{x}{\alpha} \quad , \quad y \to \frac{y}{\alpha} \quad , \quad A_t \to \alpha A_t \quad , \quad A_x \to \alpha^2 A_x \tag{40}$$

In this background our Dirac action becomes

$$S_D = \int d^4x\sqrt{-G}\; i\;(\overline{\mathbf{\Psi}}\;\Gamma^M \mathbf{D}_M \mathbf{\Psi} - m\overline{\mathbf{\Psi}}\mathbf{\Psi}) \tag{41}$$

In which $\mathbf{D}_M$ and ω_{abM} stand for the covariant derivative and the spin connection respectively. Given by

$$\mathbf{D}_M = \partial_M + \frac{1}{4}\omega_{abM}\Gamma^{ab} - iqA_M \tag{42}$$

$$\omega_{abM} = e_a^N \partial_M e_{bN} - e_{aN}e_b^O \Gamma_{OM}^N \tag{43}$$

Furthermore we choose our gauge in the following form:

$$\begin{aligned} A_t &= 2Q\alpha(z-1) \\ A_x &= -2H\alpha^2 y \end{aligned} \tag{44}$$

Here Γ^N_{OM} is the affine connection, capital letters like M stand for world indices and small letters are the tangent-space indices. We transform between these indices by acting with the vielbeins e^M_a. We can express these vielbeins in terms of the metric by

$$g_{MN} = \eta_{ab} e^a_M e^b_N \tag{45}$$

In which η_{ab} stands for the minkowski metric with signature (-+++). plugging in the terms for our metric as in 28 and considering only the diagonal terms of the vielbeins, since one can show that the vielbeins are in fact diagonal, we get

$$e^t_T = \frac{L}{z}\sqrt{f} \quad , \quad e^x_X = e^y_Y = \frac{L}{z} \quad , \quad e^z_Z = \frac{L}{z}\frac{1}{\sqrt{f}} \tag{46}$$

Furthermore we have the following relations between Gamma's:

$$\Gamma^{ab} = \frac{1}{2}[\Gamma^a, \Gamma^b] \quad , \quad \Gamma^M = e^M_a \Gamma^a \tag{47}$$

And we chose our Gamma matrices as:

$$\Gamma^t = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \Gamma^x = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \Gamma^y = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \Gamma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{48}$$

We consider the Lagrangian for the Dirac action 41 which is

$$\mathcal{L}_{Dirac} = \sqrt{-G}\; i\; (\overline{\mathbf{\Psi}}\; \Gamma^M \mathbf{D}_M \mathbf{\Psi} - m\overline{\mathbf{\Psi}}\mathbf{\Psi}) \tag{49}$$

And now we find the equations of motion by taking a variation with respect to the field $\overline{\mathbf{\Psi}}$:

$$\partial_\mu \frac{\delta \mathcal{L}_{Dirac}}{\delta \partial_\mu \overline{\mathbf{\Psi}}} + \frac{\delta \mathcal{L}_{Dirac}}{\delta \overline{\mathbf{\Psi}}} = 0 \tag{50}$$

We note that the first term is zero since there are no explicit field derivatives in the Lagrangian. The second term yields us the Dirac equation of motion:

$$\Gamma^M \mathbf{D}_M \mathbf{\Psi} - m\mathbf{\Psi} = 0 \tag{51}$$

In order to reduce the equation 42 we need to find the relevant spin connection terms in our background. For this we use the following relation between the spin connection and the vielbeins (see 6)

$$T^a = de^a + \omega^{ab} \wedge e^b \tag{52}$$

In which T^a stands for the torsion and we have the following relations

$$\begin{aligned} e^a &= e^a_M dx^M \\ \omega^{ab} &= \omega^{ab}_M dx^M \end{aligned} \tag{53}$$

The affine connection can be expressed in terms of the metric in the following way (see 7 chapter "Relation between metric and the affine connection")

$$\Gamma^N_{OM} = \frac{1}{2} g^{NN} \{ g_{ON,M} + g_{MN,O} - g_{OM,N} \} \tag{54}$$

In our background we see that the affine connection is symmetric in its lower indices. Hence the torsion term in 52 vanished (see 6) and we are left with the relation

$$de^a + \omega^{ab} \wedge e^b = 0 \tag{55}$$

We can compute the relevant terms of our spin connections using 53 and 55 as follows:

$$\begin{aligned} &de^x = \partial_N e^x_M dx^N \wedge dx^M = \partial_z e^x_X dz \wedge dx = -\omega^{xb} \wedge e^b = -\omega^{xz}_X dx \wedge e^z_Z dz \rightarrow \\ &(e^x_X)' dz \wedge dx + \omega^{xz}_X e^z_Z dx \wedge dz = ((e^x_X)' - \omega^{xz}_X e^z_Z) dz \wedge dx \rightarrow \omega^{xz}_X = e^Z_z (e^x_X)' \end{aligned} \tag{56}$$

Using this same method we find that our non-zero spin connections are

$$\omega^{tz}_T = e^Z_z (e^t_T)' \quad , \quad \omega^{xz}_X = e^Z_z (e^x_X)' \quad , \quad \omega^{yz}_Y = e^Z_z (e^y_Y)' \tag{57}$$

Now we can plug these expressions into the covariant derivative 42 and get for our Dirac equation 51

$$\begin{aligned} &\Gamma^M \mathbf{D}_M \mathbf{\Psi} - m\mathbf{\Psi} = 0 \quad \rightarrow \quad [\ \Gamma^Z \partial_z + \Gamma^X (\partial_x + \frac{1}{2} e^Z_z (e^x_X)' \Gamma_{xz} - iqA_x) \ + \\ &\Gamma^Y (\partial_y + \frac{1}{2} e^Z_z (e^y_Y)' \Gamma_{yz}) + \Gamma^T (\partial_t + \frac{1}{2} e^Z_z (e^t_T)' \Gamma_{tz} - iqA_t) \ - m \ ] \mathbf{\Psi} = 0 \end{aligned} \tag{58}$$

Using the expressions for the vielbeins, which we computed before 46 , we can compute the vielbein products in the equation above and obtain

$$e^Z_z (e^t_T)' = -\frac{1}{z} f + \frac{1}{2} f' \quad , \quad e^Z_z (e^x_X)' = e^Z_z (e^y_Y)' = -\frac{1}{z} \sqrt{f} \tag{59}$$

Furthermore we can use 47 to transform the Gamma matrices in the world representation to the tangent-space representation and we can write the Dirac equation as

$$\begin{aligned} &[\ \frac{\sqrt{f} z}{L} \Gamma^z \partial_z + \frac{z}{L} \Gamma^x (\partial_x - \frac{1}{2} \frac{\sqrt{f}}{z} \Gamma_{xz} - iqA_x) \ + \\ &\frac{z}{L} \Gamma^y (\partial_y - \frac{1}{2} \frac{\sqrt{f}}{z} \Gamma_{yz}) + \frac{z}{L\sqrt{f}} \Gamma^t (\partial_t + \frac{1}{2} (-\frac{f}{z} + \frac{f'}{2}) \Gamma_{tz} - iqA_t) \ - m \ ] \mathbf{\Psi} = 0 \end{aligned} \tag{60}$$

The gamma matrices are related to each other in the following way:

$$\begin{aligned} &\Gamma^x \Gamma_{xz} = \Gamma^x \Gamma^{xz} = \Gamma^z \\ &\Gamma^y \Gamma_{yz} = \Gamma^y \Gamma^{yz} = \Gamma^z \\ &\Gamma^t \Gamma_{tz} = -\Gamma^t \Gamma^{tz} = \Gamma^z \end{aligned} \tag{61}$$

Where we raised the indices by acting with the (-+++) signature Minkowski metric. Finally we introduce the following coördinate change for our field:

$$\mathbf{\Psi} = z^{3/2} f^{-1/4} e^{-i\omega t + k_x x} \begin{pmatrix} \phi_+(y,z) \\ \phi_-(y,z) \end{pmatrix} \tag{62}$$

Using this redefined field and the Gamma matrices 61 we reduce our Dirac equation 60 to

$$\sqrt{f} \partial_z \phi_+ - \frac{mL}{z} \phi_+ + \sigma \phi_- - iqA_x \sigma_1 \phi_- + i\sigma_1 k_x + \frac{1}{\sqrt{f}} \sigma_2 \omega \phi_- + \frac{1}{\sqrt{f}} q\sigma_2 A_t \phi_- = 0 \tag{63}$$

Which we can rewrite to get the following reduced form for our Dirac equations of motion:

$$\sqrt{f}(\partial_z - \frac{mL}{z\sqrt{f}})\phi_+ = i(i\sigma_2 u + i\sigma_3\partial_y - \sigma_1(2Hqy + k_x))\phi_- \tag{64}$$

Where u is given by

$$u = \frac{1}{\sqrt{f}}(\omega - 2qQ(1-z)) \tag{65}$$

We can perform the very same computation for the other field to obtain the following equations of motion:

$$\sqrt{f}(\partial_z \mp \frac{mL}{z\sqrt{f}})\phi_\pm = \pm i(i\sigma_2 u + i\sigma_3\partial_y - \sigma_1(2Hqy + k_x))\phi_\mp \tag{66}$$

4.3 Extracting the spectral function at zero Temperature

Since a lot of interesting physics like quasiparticle peaks, fermi level structure and critical behavior can be found by numerically studying the spectral functions, we stop to find an expression for these functions of interest in this section. Here we will consider the case of zero temperature. We note by looking at the temperature of the black hole 31 that for zero temperature we have a condition set on the field and charge. This condition is $Q^2 + H^2 = 3$. In order to find the spectral functions of interest we will follow the prescription of 8. For this prescription we need to consider the asymptotic behavior of our system near the ads-boundary $z = 0$. Near the ads-boundary $z \to 0$ the fields $\phi_\pm$ can be expanded as:

$$\begin{aligned} \phi_+ &= Az^m + Bz^{1-m} \\ \phi_- &= Cz^{m+1} + Dz^{-m} \end{aligned} \tag{67}$$

Plugging these expansions into the Dirac equations of motion 66 we get the following relation between the coefficients:

$$\begin{aligned} B &= \frac{i(i\sigma_2 u - \sigma_1(2Hqy + k_x) + i\sigma_3\partial_y)}{1-2m}D \\ A &= \frac{-i(i\sigma_2 u + \sigma_1(2Hqy + k_x) - i\sigma_3\partial_y)}{1+2m}C \end{aligned} \tag{68}$$

The prescription tells us that if there is a relationship between the coefficients which goes like $D = \mathcal{S}A$ that then the retarded greens function is given by [8]:

$$G_R = \mathcal{S}\ \sigma_2 \tag{69}$$

Now we will further decompose the fields as

$$\phi_\pm = \begin{pmatrix} \chi_\pm \\ \xi_\pm \end{pmatrix} \tag{70}$$

And obtain the following coupled equations of motion

$$\begin{aligned} \sqrt{f}(\partial_z \mp \frac{mL}{z})\chi_\pm &= \mp[-iu\xi_\mp + \partial_y\chi_\mp + i(2Hqy + k_x)\xi_\mp] \\ \sqrt{f}(\partial_z \mp \frac{mL}{z})\xi_\pm &= \mp[iu\chi_\mp - \partial_y\xi_\mp + i(2Hqy + k_x)\chi_\mp] \end{aligned} \tag{71}$$

Here we can introduce the fractions $i\frac{\chi_+(k)}{\xi_-(k)}, -i\frac{\xi_+(k)}{\chi_-(k)}$ as in the prescription 9 and use 69 with $D = \mathcal{S}A$ to obtain for the retarded greens function (for $m \geq 0$):

$$G_R = \lim_{\epsilon \to 0} \epsilon^{-2m} \begin{pmatrix} i\frac{\chi_+(k)}{\xi_-(k)} & 0 \\ 0 & -i\frac{\xi_+(k)}{\chi_-(k)} \end{pmatrix} \tag{72}$$

Here we note that the fields, which we defined in 70 in position space (y, z), we took in momentum space in the retarded greens function. In the prescription the fields where separated in a momentum and a position part which we cannot perform here. However we can always Fourier transform our fields and gain the momentum dependency which makes us believe that the prescription in 8 still holds.

4.4 Behavior near the horizon

In order to solve the Green's functions we need to find the boundary conditions near the horizon. Hence we now move away from the ads boundary and consider the equations and boundary conditions at the horizon. At the event horizon (z=1) we can expand the fields 70 as follows

$$\begin{aligned} \chi_\pm &= a_\pm(y,z) e^{\frac{i\omega}{6(1-z)}} (1-z)^{\frac{i(6qQ-4\omega)}{18}} \\ \xi_\pm &= b_\pm(y,z) e^{\frac{i\omega}{6(1-z)}} (1-z)^{\frac{i(6qQ-4\omega)}{18}} \end{aligned} \tag{73}$$

Plugging these field definitions into 71 yields us the following equations of motion:

$$\begin{aligned} \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right) a_+ &= \frac{mL\sqrt{f}}{z} a_+ + iub_- - \partial_y a_- - i(2Hqy + k_x) b_- \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right) a_- &= -\frac{mL\sqrt{f}}{z} a_- - iub_+ + \partial_y a_+ + i(2Hqy + k_x) b_+ \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right) b_+ &= \frac{mL\sqrt{f}}{z} b_+ - iua_- + \partial_y b_- - i(2Hqy + k_x) a_- \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right) b_- &= -\frac{mL\sqrt{f}}{z} b_- + iua_+ - \partial_y b_+ + i(2Hqy + k_x) a_+ \end{aligned} \tag{74}$$

Now we wish to find the boundary conditions present at the event horizon. For this we note that near the horizon the field f defined in 28 behaves as $6\ (1-z)^2$ where we used the condition obtained by considering zero temperature. Hence we have $\sqrt{f} = \sqrt{6}\ (1-z)$.
Plugging this into the equations of motion above we see that there are still terms that diverge at the horizon. These are the terms $\frac{i\omega}{\sqrt{6}(1-z)}$. In order to control its divergence at the horizon we need to have another term that can nullify its effect. We note that there is indeed such a term present in the equations of motion which is the first part of $u = \frac{1}{\sqrt{f}}(\omega - 2qQ(1-z))$. Now looking at the equations of motion shows us that we are restricted by the boundary conditions:

$$a_+(y,1) = b_-(y,1) \quad , \quad a_-(y,1) = -b_+(y,1) \tag{75}$$

Now we make the following redefinition's

$$\begin{aligned} A_+(y,z) &= b_-(y,z) - a_+(y,z) \quad , \quad A_-(y,z) = -i(a_-(y,z) + b_+(y,z)) \\ B_+(y,z) &= a_+(y,z) + b_-(y,z) \quad , \quad B_-(y,z) = i(b_+(y,z) - a_-(y,z)) \end{aligned} \tag{76}$$

In order to present our equations of motion in 74 in terms of the new fields we have to add and subtract the relevant equations and get the following redefined equations of motion:

$$
\begin{aligned}
\sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)A_+ &= -\frac{mL}{z}B_+ - iuA_+ + i\partial_y B_- + i(2Hqy+k_x)B_+ \\
\sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)A_- &= -\frac{mL}{z}B_- - iuA_- - i\partial_y B_+ - i(2Hqy+k_x)B_- \\
\sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)B_+ &= -\frac{mL}{z}A_+ + iuB_+ - i\partial_y A_- - i(2Hqy+k_x)A_+ \\
\sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)B_- &= -\frac{mL}{z}A_- + iuB_- + i\partial_y A_+ + i(2Hqy+k_x)A_-
\end{aligned}
\tag{77}
$$

In order to find the appropriate boundary conditions that restrict us near the horizon we have to expand the fields in 77 near the horizon like

$$
\begin{aligned}
A_\pm(y,z) &= A_\pm(y,1) - (1-z)\partial_z A_\pm(y,1) + (\mathcal{O}(1-z)^2) \\
B_\pm(y,z) &= B_\pm(y,1) - (1-z)\partial_z B_\pm(y,1) + (\mathcal{O}(1-z)^2)
\end{aligned}
\tag{78}
$$

And plug these into the equations of motion along with the definition for the field f which goes like $6(1-z)^2$ near the horizon. Doing this gives us the the following boundary conditions:

$$
\begin{aligned}
&A_+(y,1) = 0 \\
&\partial_z A_\pm(y,1) = \mp\frac{\sqrt{6}}{2\omega}(k_x B_\pm(y,1) + \partial_y B_\mp(y,1) + 2HqyB_\pm(y,1) \pm imB_\pm) \\
&\partial_z B_\pm(y,1) = \pm\frac{i}{\sqrt{6}}\left[(k_x \mp im)\partial_z A_\pm + \partial_z\partial_y A_\mp + 2Hqy\partial_z A_\pm\right] - \frac{i}{108}(48qQ - 23\omega)B_\pm
\end{aligned}
\tag{79}
$$

The terms in our retarded greens function 72 are ratio's of the fields $\chi_\pm$, $\xi_\pm$ which we would like to express in terms of our new fields $A_\pm$, $B_\pm$.
using the expansions in 73 we have that

$$
\begin{aligned}
G_+ &= i\frac{\chi_+(k)}{\xi_-(k)} = i\frac{a_+(k)}{b_-(k)} = i\frac{B_+(k) - A_+(k)}{B_+(k) + A_+(k)} \\
G_- &= -i\frac{\xi_+(k)}{\chi_-(k)} = -i\frac{b_+(k)}{a_-(k)} = i\frac{B_-(k) - A_-(k)}{B_-(k) + A_-(k)}
\end{aligned}
\tag{80}
$$

The idea now is that we can extract the interesting physics from the Green's function by determining $B_\pm$ and $A_\pm$ using the boundary conditions at the horizon. In this paper however we will only consider the simplest case of $\omega = 0$.

4.5 Separable solutions

From here on we only consider the case of zero mass $m = 0$. In this section we want to find solutions for the fields in equation 77 by separating the variables of the fields. This way we hope to find some interesting behavior that will allow us to speak about the energy levels of our system. We first consider a positive magnetic field $qH > 0$ and make the following coordinate change in the y-direction:

$$\eta = \sqrt{2Hq}\left(y + \frac{k_x}{2Hq}\right) \tag{81}$$

Using this coördinate change the y-dependent parts of the equations in 77 change to:

$$\begin{aligned} \partial_y B_\pm + (2Hqy + k_x)B_\mp &= \sqrt{2Hq}(\partial_\eta B_\pm + \eta B_\mp) \\ \partial_y A_\pm + (2Hqy + k_x)A_\mp &= \sqrt{2Hq}(\partial_\eta A_\pm + \eta A_\mp) \end{aligned} \tag{82}$$

plugging this into our equations of motion we get:

$$\begin{aligned} \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)A_+ &= -iuA_+ + i\sqrt{2Hq}(\partial_\eta B_\pm + \eta B_\mp) \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)A_- &= -iuA_- - i\sqrt{2Hq}(\partial_\eta B_\pm + \eta B_\mp) \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)B_+ &= +iuB_+ - i\sqrt{2Hq}(\partial_\eta A_\pm + \eta A_\mp) \\ \sqrt{f}\left(\partial_z + \frac{i\omega}{6(1-z)^2} + i\frac{-6qQ+4\omega}{18(1-z)}\right)B_- &= +iuB_- + i\sqrt{2Hq}(\partial_\eta A_\pm + \eta A_\mp) \end{aligned} \tag{83}$$

Now we can make two different ansätze for the solutions of these differential equations in the fields $A_\pm$ and $B_\pm$. The separable ansätze are:

$$\begin{aligned} &\text{Ansatz 1:}\ \ A_- = -A_+ = -Y_A(y)Z_A(z)\ \ ,\ \ B_- = B_+ = Y_B(y)Z_B(z) \\ &\text{Ansatz 2:}\ \ A_- = A_+ = Y_A(y)Z_A(z)\ \ ,\ \ B_- = -B_+ = -Y_B(y)Z_B(z) \end{aligned} \tag{84}$$

We start off by considering the first ansatz. following 16 and 15 we write

$$\begin{aligned} \partial_\eta Y_B + \eta Y_B &= \sqrt{2n}Y_A \\ \partial_\eta Y_A - \eta Y_A &= -\sqrt{2n}Y_B \end{aligned} \tag{85}$$

The solutions of these equations are:

$$Y_A = I_{n-1}(\eta)\ \ ,\ \ Y_B = I_n(\eta) \tag{86}$$

Where I_n is the Hermite function defined by

$$I_n(\eta) = N_n(\eta)H_n(\eta) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-\eta^2/2}H_n(\eta) = (-1)^n\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{\eta^2/2}\frac{d^n}{d\eta^n}e^{-\eta^2} \tag{87}$$

And we used the following identities for the Hermite function:

$$\begin{aligned} &N_{n-1} = \sqrt{2n}N_n \\ &H_{n+1}(\eta) = 2\eta H_n(\eta) - 2nH_{n-1}(\eta) \\ &\partial_\eta I_n = -\eta I_n + \sqrt{2n}I_{n-1} \\ &\partial_\eta I_{n-1} = \eta I_{n-1} - \sqrt{2n}I_n \end{aligned} \tag{88}$$

For completeness we define $I_{-1}(\eta) = 0$. Then our fields are given by:

$$\begin{pmatrix} A_+ \\ B_+ \\ A_- \\ B_- \end{pmatrix} = \begin{pmatrix} I_{n-1}Z_A \\ I_n Z_B \\ -I_{n-1}Z_A \\ I_n Z_B \end{pmatrix} \tag{89}$$

For the second ansatz we again follow the separable solutions of 16 and 15 and write:

$$\begin{aligned} \partial_\eta Y_B - \eta Y_B &= -\sqrt{2n} Y_A \\ \partial_\eta Y_A + \eta Y_A &= \sqrt{2n} Y_B \end{aligned} \tag{90}$$

In this case the solution is given by:

$$Y_A = I_n(\eta) \quad , \quad Y_B = I_{n-1}(\eta) \tag{91}$$

So now our fields in the second ansatz go like:

$$\begin{pmatrix} I_n Z_A \\ I_{n-1} Z_B \\ I_n Z_A \\ -I_{n-1} Z_B \end{pmatrix} \tag{92}$$

We can perform the same computations for a negative magnetic field $qH < 0$ but we need to change our coordinate transformation to

$$\eta = -\sqrt{-2Hq}\left(y + \frac{k_x}{2Hq}\right) \tag{93}$$

Considering the same ansatz as we took in 84 we get the following solutions for our fields with a negative magnetic field

$$\text{Ansatz 1:} \quad \begin{pmatrix} I_n Z_A \\ I_{n-1} Z_B \\ -I_n Z_A \\ I_{n-1} Z_B \end{pmatrix} \quad , \quad \text{Ansatz 2:} \quad \begin{pmatrix} I_{n-1} Z_A \\ I_n Z_B \\ I_{n-1} Z_A \\ -I_n Z_B \end{pmatrix} \tag{94}$$

So to summarize we considered both the positive and negative magnetic field case for both ansatz and obtained the following fields

$$\begin{aligned} qH > 0: &\quad \text{Ansatz 1:} \quad \begin{pmatrix} I_{n-1} Z_A \\ I_n Z_B \\ -I_{n-1} Z_A \\ I_n Z_B \end{pmatrix} \quad , \quad \text{Ansatz 2:} \quad \begin{pmatrix} I_n Z_A \\ I_{n-1} Z_B \\ I_n Z_A \\ -I_{n-1} Z_B \end{pmatrix} \\ qH < 0: &\quad \text{Ansatz 1:} \quad \begin{pmatrix} I_n Z_A \\ I_{n-1} Z_B \\ -I_n Z_A \\ I_{n-1} Z_B \end{pmatrix} \quad , \quad \text{Ansatz 2:} \quad \begin{pmatrix} I_{n-1} Z_A \\ I_n Z_B \\ I_{n-1} Z_A \\ -I_n Z_B \end{pmatrix} \end{aligned} \tag{95}$$

Now we finally have a means of talking about the energy levels of strongly coupled fermionic systems. Namely the n contained inside the Hermite polynomials quantize the motion of our particles. As a result the energy levels of our system under the influence of a magnetic field is quantized with each Landau energy level given by n.

There are some things worth noting here. First of all if we look at the equations 85 and 90 we note the similarity of these equations with the simple harmonic oscillator case for which we had the equation 9. Even though these equations clearly have their differences we cant help but noting their overall similar structure. The idea that the fermions we probed are behaving approximately harmonic is made even stronger by the solutions 86 and 91 which we found. The Hermite polynomials are the same solutions we find for the harmonic oscillator when solving its Schrodinger equation analytically by considering power series. Furthermore we see that flipping the magnetic field interchanges the η dependent parts. We also see that under a flip of the magnetic field the roles of the two ansatz interchange which gives us reason to believe that the two ansatz describe an aligned and anti-aligned system.

4.6 $\omega = 0$ case

Like we mentioned before a lot of interesting physics can be found by numerically studying the spectral functions. Hence we would like to perform some computations of the spectral functions in this section but due to time shortage we could unfortunately not explore the functions numerically.
We will compute the spectral functions 80 for the simplest case of $\omega = 0$, $H = 0$. Furthermore we recall that we applied the restriction $m = 0$. We can immediately note that in the case of $\omega = 0$ we no longer need the condition $A_\pm(y, 1) = 0$ since the diverging terms in ω no longer pose a problem.
For this case we need to redefine our fields in 73 as

$$\begin{aligned}\chi_\pm(z) &= a_\pm(z)\,(1-z)^{\frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6}}\\ \xi_\pm(z) &= b_\pm(z)\,(1-z)^{\frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6}}\end{aligned} \tag{96}$$

Note that because of the restriction $H = 0$ the y-dependency of the fields vanishes. Next we define the fields $A_\pm$, $B_\pm$ as before 76 , and gain the following equations of motion:

$$\begin{aligned}\sqrt{f}\left(\partial_z - \frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6(1-z)}\right)A_+ &= i\frac{2qQ(1-z)}{\sqrt{f}}A_+ + ik_xB_+\\ \sqrt{f}\left(\partial_z - \frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6(1-z)}\right)A_- &= i\frac{2qQ(1-z)}{\sqrt{f}}A_- - ik_xB_-\\ \sqrt{f}\left(\partial_z - \frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6(1-z)}\right)B_+ &= -i\frac{2qQ(1-z)}{\sqrt{f}}B_+ - ik_xA_+\\ \sqrt{f}\left(\partial_z - \frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6(1-z)}\right)B_- &= -i\frac{2qQ(1-z)}{\sqrt{f}}B_- + ik_xA_-\end{aligned} \tag{97}$$

Again by plugging the expanded form of the field f as $f = 6(1-z)^2$ into the equations of motion and expanding the equations of motion at the event horizon we get the following boundary conditions:

$$\begin{aligned}A_+(1) &= -\frac{1}{6\sqrt{k}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)B_+(1)\\ A_-(1) &= \frac{1}{6\sqrt{k}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)B_-(1)\end{aligned} \tag{98}$$

Now if we consider the case of $B_+(1) = B_-(1)$ we get $A_+(1) = -A_-(1)$. This then yields us the following green's functions at the event horizon:

$$\begin{aligned}G_+ = i\frac{\chi_+}{\xi_-} = i\frac{B_+ - A_+}{B_+ + A_+} = i\frac{1 + \frac{1}{\sqrt{6k}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)}{1 - \frac{1}{\sqrt{6k}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)} =\\ \frac{i + \frac{1}{\sqrt{6k}}\left(i2qQ - \sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)}{1 - \frac{1}{\sqrt{6k}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)} = \frac{i + \frac{1}{\sqrt{2}}\left(i2qQ - \sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)}{\sqrt{3k} - \frac{1}{\sqrt{2}}\left(2qQ + i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)}\end{aligned} \tag{99}$$

Only taking the real part of the greens function gives us:

$$G_{+} = -\frac{\sqrt{3k^2 - 2q^2Q^2}}{\sqrt{3}k - \sqrt{2}qQ} \tag{100}$$

In a similar way we find

$$G_{-} = \frac{\sqrt{3k^2 - 2q^2Q^2}}{\sqrt{3}k + \sqrt{2}qQ} \tag{101}$$

Another expansion for the fields we also could have taken is

$$\begin{aligned} \chi_{\pm}(z) &= a_{\pm}(z)\,(1-z)^{-\frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6}} \\ \xi_{\pm}(z) &= b_{\pm}(z)\,(1-z)^{-\frac{\sqrt{2}\sqrt{3k^2-2q^2Q^2}}{6}} \end{aligned} \tag{102}$$

Which then would have given us the following conditions at the event horizon:

$$\begin{aligned} A_{+}(1) &= -\frac{1}{6\sqrt{k}}\left(2qQ - i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)B_{+}(1) \\ A_{-}(1) &= \frac{1}{6\sqrt{k}}\left(2qQ - i\sqrt{2}\sqrt{3k^2-2q^2Q^2}\right)B_{-}(1) \end{aligned} \tag{103}$$

Again by taking $B_{+}(1) = B_{-}(1)$ we get $A_{+}(1) = -A_{-}(1)$ but this time with our Green's function given by:

$$G_{+} = \frac{\sqrt{3k^2 - 2q^2Q^2}}{\sqrt{3}k - \sqrt{2}qQ} \tag{104}$$

$$G_{-} = -\frac{\sqrt{3k^2 - 2q^2Q^2}}{\sqrt{3}k + \sqrt{2}qQ} \tag{105}$$

5 Conclusion

We have studied the energy levels of strongly coupled fermionic systems subject to a magnetic field by means of the AdS/CFT duality. Knowing that such strongly coupled systems are difficult to study both analytically and numerically with the usual tools we use to describe weakly coupled systems, we tried studying these systems by means of the AdS/CFT duality. For this we probed our bulk fermion into a dyonic black hole where we could solve the dynamics of the bulk using weak gravity in 4 dimensional AdS space. Here we studied the dynamics of the bulk propagating in 4d AdS space by considering separable solutions. We have found the discrete behavior of the probe fermion in the presence of a magnetic field where each n correspond to a different Landau level. Furthermore we noted that the solutions to the fields we found resemble those of the simple harmonic oscillator which is an interesting phenomena. Also we noted that in the different ansatze we could take a flip of the magnetic field interchanges the roles of the two ansatz which gives us reason to believe that the two ansatz describe an aligned and anti-algined system.

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