Universiteit Utrecht

Master's Thesis in Theoretical Physics

Classicalization and Observational signatures of cosmological fluctuations

April 20, 2017

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Abstract

Since the discovery of cosmic microwave background (CMB) predictions and measurements of this background have been improved. In this project the interaction of a quantum system, which is weakly coupled to an environment, during inflation is discussed. This environment leads to the classicalization of the quantum behavior of the initial system. The quantum system during inflation can be described by the Schwinger-Keldysh formalism together with the 2PI-formalism. In the end the self-masses are calculated, which is the first step of solving the equations of motion for the statistical propagator. These solutions could give more information over the spectrum of the CMB.

Contents

Notation

Standard formula we will use during a lot of equations. We will use Einstein notation to shorten a lot of equations.

The expansion of the universe can be described by the scale factor $a(t)$, and we will also use conformal time η, and the scale factor $a(\eta)$.

The FLRW-metric, standard and in conformal time, is:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \tag{1a}$$

$$= a^2(\eta)\big(-d\eta^2 + \delta_{ij}dx^i dx^j\big) \tag{1b}$$

$$d\eta \equiv \frac{dt}{a(t)}. \tag{1c}$$

The Hubble, and conformal Hubble Parameter are given by:

$$\text{Hubble} \qquad H(t) = \frac{\dot{a}(t)}{a(t)} \tag{2a}$$

$$\text{conformal Hubble} \qquad \mathcal{H}(\eta) = \frac{a'(\eta)}{a(\eta)}. \tag{2b}$$

The accent denotes a derivative with respect to the conformal time η.

The equations of motion are often described using the d'Alembartian operator, which in the context of General Relativity can be written as:

$$\Box\phi(x) = \frac{1}{\sqrt{-g}}\partial_\alpha\big[\sqrt{-g}g^{\alpha\beta}\partial_\beta\phi(x)\big]. \tag{3}$$

The Christoffel symbols are defined as:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\big[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\big]. \tag{4}$$

Using the Christoffel the Riemann curvature tensor can be written as:

$$R^\lambda_{\sigma\mu\nu} = \Gamma^\lambda_{\nu\sigma,\mu} - \Gamma^\lambda_{\mu\sigma,\nu} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho\nu\sigma - \Gamma^\lambda_{\nu\rho}\Gamma^\rho\mu\sigma. \tag{5}$$

Using the Riemann curvature tensor, the Ricci tensor can be written as $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$, and the Ricci-scalar can be found by contraction of the Riemann tensor: $R = g^{\mu\nu}R_{\mu\nu}$.

1 Introduction

One of the first discoveries that is interpreted as evidence for the Big Bang is the Cosmic Microwave Background (CMB). This was first discovered as a homogeneous radiation which was the same from every direction (isotropic) it was measured. The CMB consists of photons resulting from the Big Bang, from a period which is called recombination. When the universe was created, it was a violent and very hot environment consisting of a cosmic plasma. At some moment this cosmic plasma contained electrons, protons and photons in equilibrium. Due to the high density of electrons and photons, the mean free path of the photons in this plasma was very short. Therefore these photons prevented protons and electrons to form a hydrogen atom by Thompson scattering. As the universe cooled, it reached a point in which the reaction: $e^- + p^+ \to H + \gamma$ was energetically favorable. At this point the free electron density dropped, and when this lead to a mean free path of the photons being as big as the horizon size (H^{-1}) the photons did not react anymore with the cosmic plasma. The photons that existed then, can nowadays be observed. These photons have traveled freely until they were observed, and therefore they have approximately the same value in every direction. The temperature of these photons, when created, was around $T = 3000$ K. Due to the expansion of the universe this temperature dropped to $T \approx 2.7$ K. This is the reason the CMB was not observed until 1965, when it was first measured by Pensias and Wilson [13].

As the experimental tools became better the CMB turned out not to be completely isotropic due to small fluctuations in the density (inhomogeneities). These fluctuations originated from a period of inflation, which is an era of nearly exponential expansion of the universe. The evolution of these inhomogeneities eventually led to large scale structures. In this thesis we will look at the origin of these quantum fluctuations, and therefore we are interested in quantum behavior during inflation. Quantum fluctuations created during inflation result in measurable quantities. Calculations on this quantum system during inflation are done using the following action:

$$S[\phi,\chi] = \int d^D x \sqrt{-g}\Big(\frac{1}{2}\zeta\mathcal{R} + \mathcal{L}_0[\phi] + \mathcal{L}_0[\chi] + \mathcal{L}_{int}[\phi,\chi]\Big) \tag{6a}$$

$$\mathcal{L}_0[\phi] = -\frac{1}{2}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} - \frac{1}{2}m_\phi^2(t)\phi^2(x) \tag{6b}$$

$$\mathcal{L}_0[\chi] = -\frac{1}{2}\partial_\mu\chi(x)\partial_\nu\chi(x)\eta^{\mu\nu} - \frac{1}{2}m_\chi^2(t)\chi^2(x) \tag{6c}$$

$$\mathcal{L}_{\rm int}[\phi,\chi] = -\frac{\lambda}{3!}\chi^3(x) - \frac{1}{2}h\chi^2(x)\phi(x). \tag{6d}$$

The ϕ field is the inflaton-field, weakly coupled to the environment χ. The role of the environment will be explained in section 3, and this action will be discussed in section 5. Assuming that the observer can only detect Gaussian correlators or two-point functions, there are three correlators that can be measured:

$$\langle\hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t)\rangle \tag{7a}$$

$$\langle\hat{\pi}(\vec{x},t)\hat{\pi}(\vec{y},t)\rangle \tag{7b}$$

$$\frac{1}{2}\langle\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{y},t)\rangle\,. \tag{7c}$$

The first case is intensively considered in cosmology, via the power spectrum of the CMB. In this thesis we will focus on the second and third case. The two additional expectation values will give rise two four extra measurable numbers. The outline of this thesis is analog to the work done by Koksma, Prokopec and Schmidt in [10]. In section 2 we briefly discuss some general principles of cosmology, where the main point is the essence of a period of inflation during Big Bang. After that in section 3 we discuss a problem which inflation causes. This is the classical behavior of large scale structure that origins from a quantum system. Section 4.2 is the section which sets the bases for the calculations of an interacting quantum field theory in an FLRW universe. First we introduce the Schwinger-Keldysh formalism, which gives rise to multiple propagators. After that the 2PI formalism is explained, which will eventually be used in section 5. In that section we introduce an interacting quantum field theory on an FLRW background, which is between a quantum system (ϕ) and an environment (χ). Eventually it results in terms of renormalized self-masses, which can be checked against previous found results in [10]. Eventually we will discuss the found results, and possibilities of future research.

2 Cosmology

Cosmology is a really old topic of interest, in which already the Babylonians showed interest thousands of years ago. This still is a very active and hot topic in physics. Copernicus tried to explain the universe as a heliocentric system, with the Sun as its center, instead of the earth as the center of the universe. Together with numerous other observations Hubble discovered the existence of other galaxies and the fact that they were moving away. But with the introduction of the General Relativity in 1915 by Einstein this topic did get a giant boost, in which it still is. The basis of the current cosmology is the General Relativity, however there are still some new theories. In November 2016 Verlinde published his paper: *Emergent Gravity and the Dark Universe* [19], in which he describes a different theory. However the most accepted theory is still General relativity. Einstein originally included the cosmological constant in the standard action:

$$S_{EH} = \frac{1}{16\pi G} \int d^4xx\sqrt{-g}(R - 2\Lambda). \tag{8}$$

He removed it later. But in fact this constant explains the accelerated expanding. Before the expanding universe was observed, Friedmann realized that Einstein's equations could describe an expanding universe. The solution naturally implied that the Universe had a beginning, and that the universe was created from a singularity, which is a point with infinite density. This theory is currently the most accepted theory, but from the beginning it had its problems. It can describe the expansion of the universe, which is divided in multiple era's. For simplicity, now only consider the radiation and matter dominated era's and the values of the scale factor which belong to these two era's:

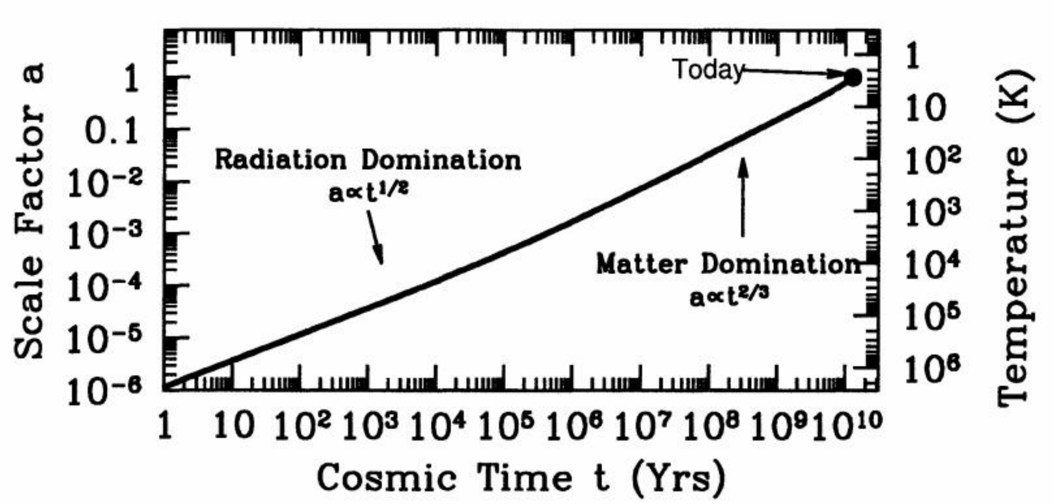

Figure 1: The evolution of the scale factor through the history of the universe. *source: [7]*

This model had four major problems, which are:

Problem 1: The horizon problem

To explain the horizon problem it is useful to introduce the comoving distance, which is defined as the distance that light could have traveled. The comoving distance depends on the scale factor:

$$\eta = \int_0^t \frac{dt'}{a(t')}. \tag{9}$$

This is the maximal distance information could have propagated. Therefore if two space-time points are separated by a distance greater than η, there is no way these points could have influenced each other. In other words these points are not causally connected. This value η is also called the comoving horizon. A very similar, but yet different, other scale is the Hubble radius, which is defined as:

$$R_H = \frac{1}{H}. \tag{10}$$

The Hubble radius is the boundary between particles that are moving slower and faster than the speed of light relative to the observer, at a given time ($c = 1$). The meaning of this distance is that two particles are not able to communicate anymore if separated by this distance. Comparing with the comoving distance, these particles could be causally connected in the past. Assuming, as before, there was only a radiation era, and after that a matter era. From the Big Bang, until the creation of the Cosmic Microwave Background (CMB), you can draw their future lightcone, and from the point of the creation, the CMB photons could move freely till we measured them. So drawing the lightcone and the Hubble sphere of the CMB photons at the time of measuring and creation of the CMB this in the following graph:
As seen in the figure above 2, most of the CMB photons have never been in causal contact with each other (roughly if they are within one degree). But the CMB is almost completely homogeneous, with only minor fluctuations. This suggests that all these points should have been in causal contact at one point in time.

Problem 2: The flatness problem

If we observe the universe at this moment, we observe an almost perfectly flat universe. Starting with the Friedmann equation, which governs the expansion in homogeneous and isotropic universe:

$$\begin{aligned} H^2 &= \frac{\rho_{tot}}{3M_P^2} - \frac{k}{a^2} & \qquad \rho_{tot} &= \rho_m + \rho_\gamma + M_P^2\Lambda + \rho_Q + \cdots \\ \Omega_{tot} &= \frac{\rho_{tot}}{\rho_{crit}} & \qquad \rho_{crit} &= 3M_P^2 H^2, \end{aligned} \tag{11}$$

where H is the Hubble parameter, ρ_{tot} and ρ_{crit} are the total and critical energy density, respectively. The total energy density depends on the energy density of matter(ρ_m),

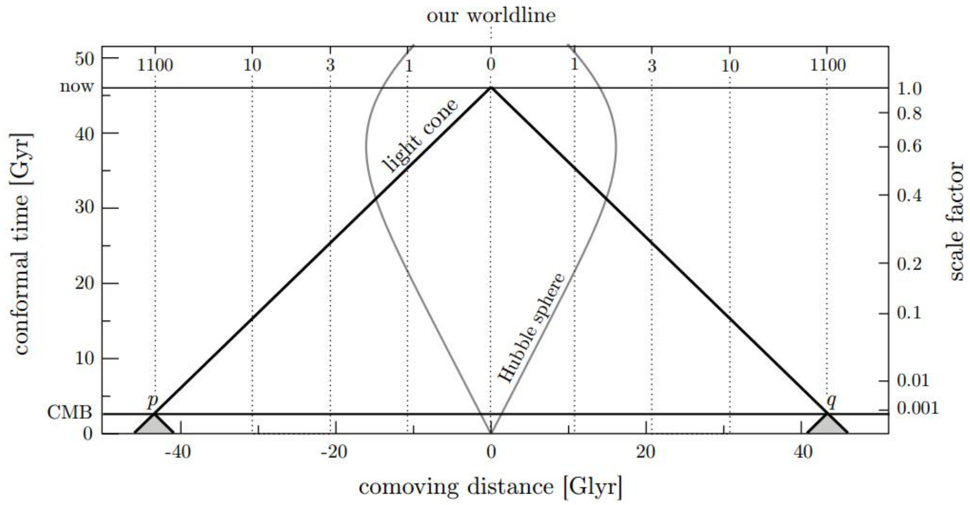

Figure 2: The horizon problem made visible in a sketch. *source: [1]*

radiation (γ), cosmological terms ($M_P^2\Lambda$) and quintessence (ρ_Q, which is related to dark energy). The dots include some terms that might still be missing. The density parameter Ω_{tot} depends on the total and critical density. The only remaining term is the reduced Planck Mass: $M_P = \frac{1}{\sqrt{8\pi G_N}}$. The fact that the universe is observed to be nearly flat means that the energy density $\Omega_0 \approx 1$, which is a critical value. In the radiation and matter dominated eras:

$$\Omega_{tot}(t) - 1 = \frac{k}{(H_{eq}a_{eq})^2}\left(\frac{a}{a_{eq}}\right)^2 \qquad \text{Radiation era,} \tag{12a}$$

$$1 - \Omega = \frac{k}{H_0^2}a \qquad \text{Matter era.} \tag{12b}$$

The values of a are as given in figure 1 [14]. The subscript 'eq' means the value of this variable at the matter-radiation equality, and H_0 is the Hubble constant. This means that if we now observe a nearly flat universe, the energy density Ω would even be closer to one in the beginning. This requires really specific initial conditions, which seems to be an unacceptable coincidence.

Problem 3. The homogeneity problem

Cosmologists often uses simplification assumptions, for example that the universe is homogeneous. At small scales this is obviously not true, but at larger and larger scales this assumption seems to be true. As showed in figure 3.

Keeping this in mind, and the observations of small temperature anisotropies in the CMB, we would expect that this inhomogeneity would also grow and be visible on large scales. This is because gravity is an attractive force, and therefore these inhomogeneities should grow. The fact that we do not observe this is a problem. This can be solved by

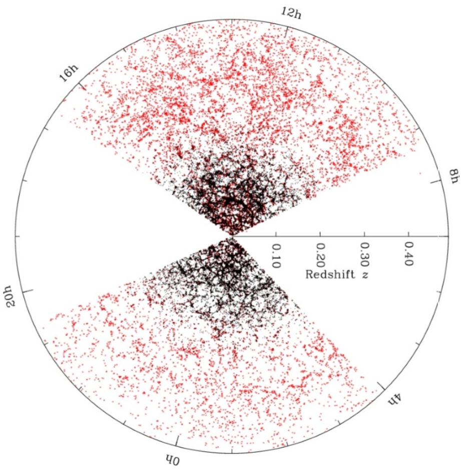

Figure 3: Galaxy distribution, which are non-uniform at small scales (forming bigger formations like galaxies), but on large scales and early times the universe becomes approximately uniform. *source: [1]*

placing us on a very special place in the universe. This is not appealing, because you will lose generality. Therefore it is assumed the universe is homogeneous at large scales.

Problem 4: The cosmic relics problem

In the early universe there were very different conditions, in which different type of relics could have been produced. There are two type of relics that could be created. The first are the topological defects such as magnetic monopoles, domain walls or cosmic strings. The second are particles, normal photons and neutrinos, but also more exotic particles as gravitons, axions and axinos, and lightest supersymmetric particles (LSP). Besides that these relics are very interesting, they are not observed (yet), and because some of these, if they exist, would have a very long lifetime, we should be able to observe them.

These are the four main problems with the Big Bang Theory, without inflation. Introducing a period in the early universe, which is called inflation, would immediately solve these problems.

2.1 Inflation

Starting at the creation of the universe, the Big Bang started in a singularity in space-time, from where it expanded. The specific details of the epochs just after the Big Bang are not known. One of the leading theories suggests that after the Big Bang there were different epochs, related to different temperatures. The global picture of the leading theory will be sketched. Just after the Big Bang, the behavior of the universe was dominated by quantum processes (Planck epoch), but as the universe cooled down to $T \sim 10^{17}$GeV, particles were created. When the universe cooled further to $T \sim 10^{16}$GeV the universe was filled with a dense plasma build up of all standard model particles, and possibly some yet unknown particles (for example dark matter particles). Then the period of inflation arrived, which was triggered by the separation of the strong force from the other elementary forces. This is the epoch in which the universe underwent an accelerated expansion, where the scale factor is written as: $a(t) = a_0 e^{H_i t}$. Before looking at the physics of this accelerated expansion, the problems which were stated before are discussed.

Problem 1: The horizon problem

Because in this epoch the expansion was exponentially, the comoving horizon also increased exponentially, where the Hubble radius stayed constant during inflation. Due to this big difference in evolution, particles could have been causally connected in the past, but when the comoving horizon increased exponentially the particles can nowadays be causally disconnected. They were not in the past. This is visualized in figure 4.

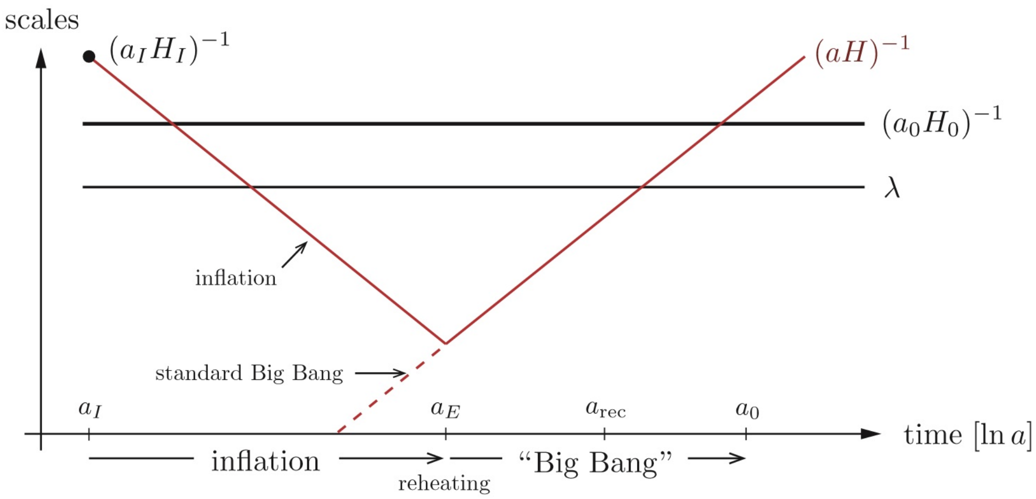

Figure 4: This figure shows the solution of the horizon problem through inflation. The comoving Hubble sphere shrinked during inflation, and after inflation it increased again as described before. The point where it started growing again is denoted as the reheating surface. All the photons of the CMB were causally connected in the past. The comoving separation of two particles with $(aH)^{-1}$ is denoted by λ. If $\lambda > (aH)^{-1}$ then the particles cannot communicate at this moment. *source: [1]*

Problem 2: The flatness problem

During inflation we can see that the energy density evolved different as in the radiation and matter era (12):

$$\Omega_{tot}(t) - 1 = \frac{k}{H_I^2}\left(\frac{1}{a}\right)^2 \qquad \text{(deSitter) inflation.} \qquad (13)$$

The consequence is that during inflation the factor $\Omega_{tot} - 1$ was reduced. Before inflation the initial conditions do not need to be fine tuned anymore, as long as inflation took long enough to reduce this term. The number of efolds that inflation had to last to solve the flatness problem can be calculated from:

$$N(t) = \log\left(\frac{a_e}{a(t)}\right). \tag{14}$$

Here a_e is the scale factor at the end of inflation, and every other time an underscore e is used, it is the value of that variable at the end of inflation. Now it is just a matter of plugging in the values at the end of inflation. At the end of inflation we enter the radiation dominated era, and that can be equated with the matter era, using both redshifts:

$$(z_e + 1)^2(\Omega_{tot} - 1)_e = (z_{eq} + 1)(\Omega_{tot} - 1)_0, \tag{15}$$

and using:

$$z_{eq} \approx 3230 \tag{16a}$$

$$(z_e + 1)^2 \approx \frac{H_I}{H_0}(z_{eq} + 1)^{\frac{1}{2}}. \tag{16b}$$

Then the number of e-folds that inflation should last is approximately 60, which solves the flatness problem.

Problem 3: The homogeneity problem

Gravity is an attractive force, therefore it this should lead to the growth of the inhomogeneities. Due to inflation this is not the case, the expanding background grows faster than the growth of inhomogeneities. The inhomogeneities before inflation will therefore be red-shifted away, and these perturbations are not observable today. The rapid growth of the universe did create inhomogeneities, for small scales these inhomogeneities lead to galaxy structures and other structures. For large enough scales, the expansion dominated the gravitation, resulting in a homogeneous and isotropic universe on large scales.

Problem 4: The cosmic relics problem

The solution for this problem is simply that the intense growth of the universe during inflation made the density of these relics very small, and therefore it is less likely to observe these topological defects or particles.

The physics of inflation is based on the period of accelerated expansion where $\ddot{a} > 0$, driven by a potential energy density $V(\phi)$ and a scalar field (inflaton)

$\phi(t, \vec{x})$. Because the inflaton is dynamical it has besides $V(\phi)$ also a kinetic energy density. The most common picture of inflation is the inflaton rolling down a potential hill, as shown in the figure on the right. When it reaches the minimum, the epoch of inflation ends, and the reheating begins. After that the universe is radiation dominated. This inflaton rolling down to the potentially minimum is bound to slow-roll conditions, such that inflation will last long enough.

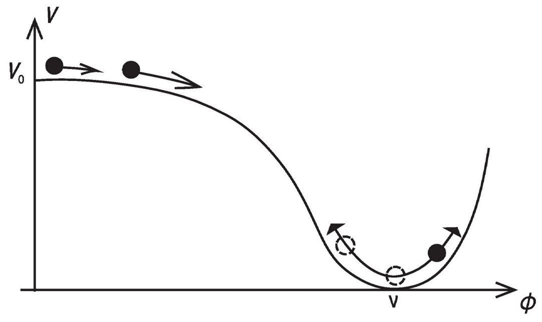

Figure 5: The inflaton rolling down a potential hill

These slow-roll conditions can be determined from the slow-roll parameters:

$$\epsilon = \frac{1}{2} M_P^2 \Big(\frac{V'}{V}\Big)^2, \qquad \eta = M_P^2 \Big(\frac{V''}{V}\Big). \tag{17}$$

The accent denotes a derivative with respect to ϕ, the slow-roll conditions are:

$$\epsilon << 1, \qquad \eta << 1. \tag{18}$$

In words this means that the potential energy must dominate the kinetic energy of the inflaton for inflation to take place long enough.

2.2 Quantum field theory in a FLRW universe

Quantum processes during the Big Bang can influence nowadays observations. Therefore applying quantum field theory on a curved background, in this case the FLRW background, is needed. Starting with a simple toy-model to get familiar with the theory, we can discuss more difficult problems later. Starting with a field ϕ, and a simple action:

$$S = \int d^D x (\mathcal{L}_M + \mathcal{L}_{NM} + \mathcal{L}_{EH}) \tag{19a}$$

$$\mathcal{L}_M = \frac{1}{2}\sqrt{-g}\Big(- \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \Big) \tag{19b}$$

$$\mathcal{L}_{NM} = -\frac{1}{2}\sqrt{-g}\zeta \mathcal{R} \phi^2 \tag{19c}$$

$$\mathcal{L}_{EH} = -\frac{1}{16\pi G}\sqrt{-g}\mathcal{R}. \tag{19d}$$

The first part is the standard expression of the action, which is an integral over the Lagrangian density. The second term $\mathcal{L}_{NM}$ is the contribution due to the non-minimally coupled scalar. The last term $\mathcal{L}_{EH}$ us the usual Einstein-Hilbert contribution due to the curved spacetime. In standard quantum field theory the background is a flat (Minkowski) background, and the last two terms will indeed be zero. In this case we are interested in a FLRW-background, which is given in (1).
The scale factor contains all information of the expansion of the universe, written as $a(t)$. In this case the potential is neglected in the $\mathcal{L}_M$, but it can also be included, but for now just consider the free case. The term in front of the Ricci Scalar (ζ) determines how the field is coupled to gravity. In appendix A you can see that if we demand conformal invariance we get a certain specific value for ζ: $\zeta = \zeta_c = \frac{D-2}{4(D-1)}$, where D is the number of dimensions. Therefore we can find three interesting cases, for $D = 4$:

1. $\zeta = 0$, Minimally coupled
2. $\zeta = \zeta_c = \frac{1}{6}$, Conformally coupled
3. $\zeta < 0$, Negatively coupled.

The action leads to the equations of motion, which are very similar to the Klein-Gordon equations:

$$\Box_x \phi - (m^2 + \zeta\mathcal{R})\phi = 0. \tag{20}$$

The d'Alembertian operator $\Box$ acts on field ϕ as:

$$\Box_x = \nabla^\mu \nabla_\mu = -\partial_t^2 + \frac{\nabla_i^2}{a(t)^2} - H(D-1)\partial_t. \tag{21}$$

D is the number of spacetime dimensions, and the last term is the non-obvious extra term, due to the curvature. The non-vanishing Christoffel symbols for this metric are:

$$\Gamma^0_{ij} = \dot{a} a \delta_{ij} \tag{22a}$$

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta_{ij}. \tag{22b}$$

With this metric and these Christoffel symbols, the Ricci scalar can easily be found:

$$\mathcal{R} = 2(D-1)\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\Big) = 2(D-1)\Big(\dot{H} + 2H^2\Big). \tag{23}$$

The FRLW metric can also be written in a conformal way as in equation 1, this changes the metric to a Minkowski like metric with an overall factor $a(\eta)$ in front of it. The d'Alembertian operator changes to:

$$\Box_\eta = \frac{1}{a^2}\Big(- \partial_\eta^2 + \nabla_i^2 - \frac{a'}{a}(D-2)\partial_\eta\Big). \tag{24}$$

The term $\frac{a'}{a}$ is often written as $\mathcal{H}$, known as the conformal Hubble parameter. The accent is a derivative with respect to the conformal time η. The Ricci scalar in terms of conformal time can be written as:

$$\mathcal{R} = \frac{2(D-1)}{a^2}\Big(\mathcal{H}' + 2\mathcal{H}^2\Big). \tag{25}$$

The action can now also be rewritten in conformal time using the definition of a new field and the following identity obtained by partial integration where the total derivatives will be zero:

$$\Phi = a^{\frac{1}{2}(D-2)}(\eta)\phi \tag{26a}$$

$$\int d\eta a'' a\phi^2 + a'^2\phi^2 + 2aa'\phi\phi' = \int d\eta a'' a\phi^2 - aa''\phi^2 - 2aa'\phi\phi' + 2aa'\phi\phi'$$

$$\rightarrow \int d\eta a'' a\phi^2 = -\int d\eta a'^2\phi^2 + 2aa'\phi\phi'. \tag{26b}$$

Therefore the action will, after some rewriting, be:

$$S = \int d^D x \frac{1}{2}\Big[\big(\partial_\eta\Phi\big)^2 - \big(\nabla\Phi\big)^2 - a^2\Big(m^2 + \Big[\zeta - \frac{D-2}{4(D-1)}\Big]\mathcal{R}\Big)\Phi\Big] + S_{EH}. \tag{27}$$

Now we can define an effective mass, which is:

$$m^2_{eff} = m^2 + \Big(\zeta - \frac{D-2}{4(D-1)}\Big)\mathcal{R}. \tag{28}$$

From this we can see that we have a critical value for ζ: $\zeta_c = \frac{1}{6}$ in $D = 4$, which is exactly the same as we found in appendix A (130).

3 Classicalization

There is an obvious difference between the way physics is described at quantum level, and the classical physics. Classical behavior of processes that started at quantum level result in a problem. One mechanism which explains the classical stochastic behavior arising from an underlying quantum theory is decoherence. Decoherence describes the process of classicalization of a quantum system. One of the reasons classicalization is an interesting phenomena is because of its history, the philosophical view on quantum mechanics, and the struggle to connect the quantum world with the world around us. The idea of decoherence was proposed by Zeh [20], and (amongst others) Zurek developed it further [21]. We first look at the quantum mechanics, and its problems to connect it with the everyday world.

3.1 Quantum mechanics versus Classical mechanics

Quantum mechanics is a well established theory that predicts the behavior of particles on the smallest scales, in which it is able to do predictions. The link with the classical familiar world is often hard to describe. Famous examples as Schrödinger's cat and many others make this unusual link between two different worlds visible. Just considering the linear Schrödinger equation $i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$, the evolution of $|\psi\rangle$ contains a lot of states, and due to superposition all these states can be linear combined to a possible quantum state. This should also be true for macroscopic states, resulting in a lot of similar problems as Schrödinger's cat. There are two main questions that the interpretation of quantum mechanics should be able to answer:

1. How do we explain the classical behavior and the quantum phenomena of macroscopic objects?
2. How can the classical and the quantum theory be connected?

There are many interpretations of quantum mechanics, two of them are described below.

Copenhagen Interpretation

Niels Bohr proposed this interpretation which imposes that particles are in a superposition until they are measured. Quantum mechanics would only occur up to some scale, which would be the border between the quantum and classical world. But macroscopic states cannot always be separated from the quantum world. For example non-classical squeezed states can describe oscillations of suitably prepared electromagnetic fields with macroscopic numbers of photons. There are more examples, but the fact that we can observe quantum phenomena means that both worlds are correlated. This means that the question rises whether there is a 'boundary'?

The many world interpretation

Hugh Everett developed the many world interpretation, in which the complete universe is described by quantum theory. Superpositions evolve via the Schrödinger equation, and whenever an interaction forces the superposition to fix the state, the universe is split into multiple branches. Each with a different weight. In this interpretation the classical behavior emerges from the complete system.

3.2 Decoherence as solution for classicalization

Decoherence is the phenomena which explains the classical stochastic behavior arising from an underlying quantum theory. The idea is that particles can be in superposition, but they still 'live' in an environment. Normally, considering quantum phenomena, there is an observer and a quantum system. The fact that there is a complete environment in which the observer and the quantum system live will have its consequences. The environment is weakly coupled to both the observer and the quantum system, and the interaction between the environment and the quantum system results in a classical stochastic system. Because the environment is weakly coupled, we can assume that this is below the threshold of the observer. Therefore an observer is unable to distinguish a pure state, and a state where the environment is traced out. Decoherence is the loss of coherence of a system in contact with an environment, which means that the coherence between quantum states disappears. This leads to a environment-induced superselection, which is called einselection. Einselection excludes states which have small weights. The einselected states do not change under interaction with the environment. This process will eventually lead to the classicalization of the states, because only the most common states survive. Many quantum properties, such as coherence and entanglement, of the system get lost or harder to observe, due to the decoherence. A schematic view of the environment only influencing the quantum system is given in the figure below:

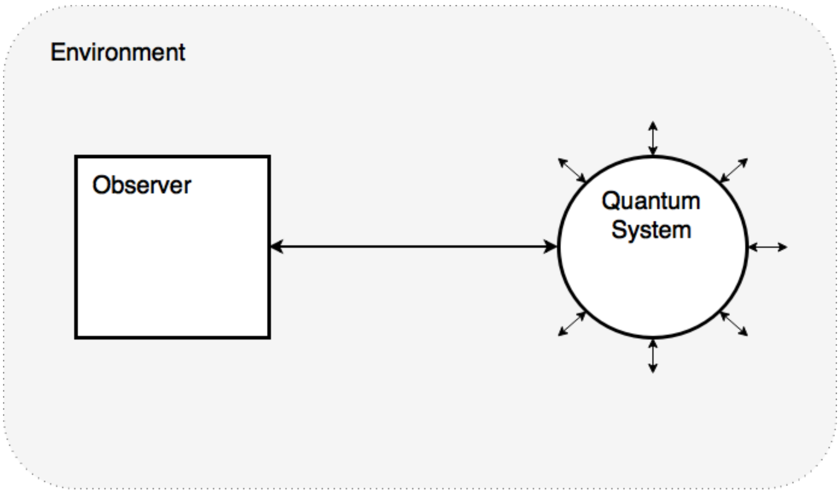

Figure 6: In standard quantum field theory the interaction of the quantum system with the environment is neglected, the principle of decoherence incorporates this. The main point of attention is that the observer does not interact with the environment.

In the case of observing the CMB, we are the observer, and we observe the quantum phenomena via the photons of the CMB. The quantum phenomena which we would observe via the CMB are the cosmological perturbations created during inflation. The cosmological perturbations, described by a $\delta\phi$ of an inflaton field which we will discuss later in the thesis. The inflaton field during inflation can be written as:

$$\hat{\phi} = \phi_0 + \delta\phi \tag{29}$$

The perturbations couple to the gravitational potential, and due to the accelerated expansion, these perturbations of the energy density are frozen out into super-Hubble size perturbations. Later, these fluctuations re-enter the Hubble horizon, as if they were non-causal perturbations. This idea is the same as the horizon problem which was discussed in section 2.1. In this case it has as a result that we can observe the primordial fluctuations of the inflaton via the power spectrum of the CMB.

3.3 Decoherence and entropy of cosmological perturbations

As mentioned in section 2.1, inhomogeneities lead to structure forming. These small inhomogeneities started as quantum fluctuations, after crossing the Hubble radius these fluctuations became classical and lead to structures in the universe. Through the mechanism of einselection these inhomogeneities became classical. Decoherence of quantum states results in the disentanglement of two states that were entangled before, and after the disentanglement these states can be described individually, without taking in account the full quantum state as before. The results of this disentanglement is an increase in entropy which provides us with a quantitative measure of decoherence and classicalization. The entropy originally is zero, since the state was a pure state. Calculations of the CMB are mainly focusing on tree level, resulting in a spectrum: $\langle\hat{\phi}^2\rangle$. in section 5 this will be discussed. For now, consider a bosonic density operator for a quantum mechanical system with only one bosonic degree of freedom as also done in [15]:

$$\hat{\rho}_B(t) = \frac{1}{Z}\exp\Big[-\frac{1}{2}\Big(\alpha\hat{\pi}^2 + \beta\{\hat{\phi},\hat{\pi}\} + \gamma\hat{\phi}^2\Big)\Big], \tag{30}$$

where $\hat{\phi}$ is the position operator and $\hat{\pi}$ is the momentum operator. The density operator depends on three time-dependent parameters α, β and γ, and the anti-commutator of the position and momentum operator $\{\hat{\phi},\hat{\pi}\}$. The factor of $\frac{1}{Z}$ is normalization factor. This energy density can result in extra information besides the spectrum $\langle\hat{\phi}^2\rangle$. From here it is a straightforward calculation, which is often done in statistical field theory. Due to diagonalizing the density operator, the bosonic creation and annihilation operators are defined as:

$$\hat{a}_B^\dagger = \sqrt{\frac{\sigma}{2\alpha}}\Big[\Big(1 - i\frac{\beta}{\sigma}\Big)\hat{\phi} - i\frac{\alpha}{\sigma}\hat{\pi}\Big] \tag{31a}$$

$$\hat{a}_B = \sqrt{\frac{\sigma}{2\alpha}}\Big[\Big(1 + i\frac{\beta}{\sigma}\Big)\hat{\phi} + i\frac{\alpha}{\sigma}\hat{\pi}\Big]. \tag{31b}$$

$\sigma \equiv \sqrt{\alpha\gamma - \beta^2}$; $\hat{a}_B^\dagger$ is the bosonic creation operator and $\hat{a}_B$ is the bosonic annihilation operator. The density operator can be rewritten using these operators:

$$\hat{\rho}_B = \frac{e^{-\frac{\sigma}{2}}}{Z} \exp\big[-\sigma \hat{a}_B^\dagger \hat{a}_B\big] = \frac{e^{-\sigma \hat{N}_B}}{Z'} \qquad\qquad Z' = Z e^{\frac{\sigma}{2}}. \tag{32}$$

$\hat{N}_B = \hat{a}_B^\dagger \hat{a}_B$ is the bosonic particle number, with the corresponding Fock-space basis: $|n_b\rangle$ which is defined in the usual way:

$$\hat{N}_B |n_B\rangle = n_B |n_B\rangle . \tag{33}$$

Taking the trace, and by demanding the trace of the density operator is one:

$$Z' = \mathrm{Tr}[Z'\hat{\rho}_B] == \sum_{n_B=0}^{\infty} \langle n_B| e^{-\sigma \hat{N}_B} |n_B\rangle = \sum_{n_B=0}^{\infty} e^{-\sigma n_B} = \frac{1}{1-e^{\sigma}}, \tag{34}$$

the average particle number can be written as:

$$\langle \hat{N}_B \rangle = \mathrm{Tr}\big[\hat{\rho}_B \hat{N}_B\big] = \frac{1}{e^{\sigma}-1} \equiv \bar{n}_B. \tag{35}$$

This agrees with the usual Bose-Einstein distribution if σ is set to: $\sigma = E/(k_b T)$. Using the density operator, the Gaussian correlators can be written as:

$$\langle \hat{\pi}^2 \rangle = -2\partial_\alpha \log(Z) = \Big(\bar{n}_B + \frac{1}{2}\Big)\frac{\gamma}{\sigma} \tag{36a}$$

$$\langle \hat{\phi}^2 \rangle = \Big(\bar{n}_B + \frac{1}{2}\Big)\frac{\alpha}{\sigma} \tag{36b}$$

$$\frac{1}{2}\langle\{\hat{\phi}, \hat{\pi}\}\rangle = \Big(\bar{n}_B + \frac{1}{2}\Big)\frac{-\beta}{\sigma}. \tag{36c}$$

Now the statistical propagator can be calculated.

$$F_\phi(t;t') = \frac{1}{2}\langle\{\hat{\phi}(t), \hat{\phi}(t')\}\rangle . \tag{37}$$

This propagator describes how states are populated, and therefore can be used to describe the phase space area occupied by a Gaussian state (in units of $\hbar$). This occupation area is Gaussian and can be written as [11]:

$$\begin{aligned}\frac{\Delta_\phi^2(t)}{2} &= 2\Big[\langle\hat{\phi}^2\rangle\langle\hat{\pi}^2\rangle - \langle\frac{1}{2}\{\hat{\phi},\hat{\pi}\}\rangle^2\Big] \\ &= 2\Big[F_\phi(t;t')\partial_t\partial_{t'}F_\phi(t;t') - \big(\partial_t F_\phi(t;t')\big)^2\Big]_{t=t'}. \end{aligned} \tag{38}$$

Plugging the correlators (36) in this expression leads to

$$\Delta_\phi(t) = 1 + 2\bar{n}_B(t) = \frac{1}{\tanh(\sigma/e)}. \tag{39}$$

When considering the free case, $\Delta_\phi(t) = 1$ as expected. Now the entropy can be expressed in terms of this occupation area:

$$S = -\mathrm{Tr}[\hat{\rho}\log(\hat{\rho})] = \frac{1+\Delta_\phi}{2}\log\left(\frac{1+\Delta_\phi}{2}\right) - \frac{1-\Delta_\phi}{2}\log\left(\frac{1-\Delta_\phi}{2}\right) \quad (40)$$

$$= (1+\bar{n}_B)\log(1+\bar{n}_B) - \bar{n}_B\log(\bar{n}_B). \quad (41)$$

The decoherence can only take place if the propagator is not free, and therefore the environment is needed to make the entropy non-zero. Thus the entropy differs if the process of classicalization takes place, and this results in a calculable variable. The pure state will become a squeezed state, but still with an occupation area of $\hbar$. As the system interacts with the environment the state will not be a pure state anymore, and the occupation area will increase. Therefore the entropy growth can be related to the occupation area as in equation 40. The difference in the phase space occupation area is shown in the following figure below:

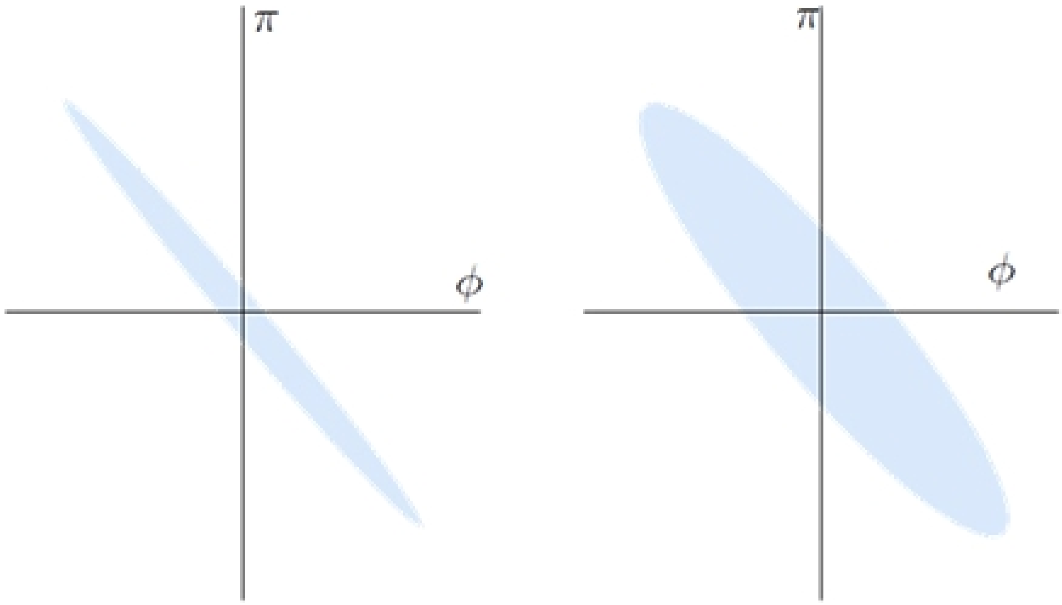

Figure 7: In the left figure the phase space area of a squeezed state is shown. Due to decoherence the phase pace area grows, which is shown in the right figure. This can be related to the entropy growth of the system (40).

4 Schwinger-Keldysh formalism

We are interested in quantum processes in the early universe. Therefore we need a formalism that describes out of vacuum quantum field theory on a curved background, especially the FLRW- background. Standard quantum field theory is a perturbation theory, where the system is in equilibrium. Therefore you start and end with a similar system. The transition amplitude can be calculated, which is the overlap between state $|\psi(t)\rangle$ and state $|\psi(t')\rangle$. In the case of an expanding universe the initial state and the final state are completely different, and this method is not applicable. The non-equilibrium quantum field theory was introduced by Schwinger [18] and Keldysh [8]. The Schwinger-Keldysh formalism is also known as the in-in formalism, or the closed time path (CTP) formalism.

Let's start with the expectation value of operators in the Heisenberg picture analog as done in [10]. An operator in the Heisenberg picture can be written as:

$$\hat{Q}(t) = \hat{U}^{\dagger}(t)\hat{Q}(t_0)\hat{U}(t) \tag{42}$$

$$\hat{U}(t) = e^{-i\hat{H}(t-t_0)/\hbar} \qquad , \hat{H} \text{ is time independent} \tag{43}$$

$$\hat{U}(t) = T\exp\Big(-i\int_{t_0}^{t} dt'\hat{H}(t')\Big) \qquad , \hat{H} \text{ is time dependent} \tag{44}$$

In this case $\hat{H}$ is the Hamiltonian, $U(t)$ is known as the time evolution operator and T is a time ordering operation. The expectation values can be written as:

$$\begin{aligned}\langle\hat{Q}(t)\rangle &= \mathrm{Tr}[\hat{\rho}(t_0)\hat{Q}(t)] \\ &= \mathrm{Tr}\bigg[\rho(\hat{t_0})\Big[\bar{T}\exp\Big(i\int_{t_0}^{t} dt'\hat{H}(t')\Big)\Big]\hat{Q}(t_0)\Big[T\exp\Big(-i\int_{t_0}^{t} dt'\hat{H}(t')\Big)\Big]\bigg].\end{aligned} \tag{45}$$

ρ is a given density operator, $t_0 < t$ and $\hat{H}$ is the Hamiltonian, The operation $\bar{T}$ is the opposite form T, an anti-time ordering operations. The problem with standard quantum field theory is the that this requires information of the 'out' state. In an expanding universe this out state is unknown. The generating functional, used to calculate expectation values, is normally defined as:

$$\mathcal{Z}[J^{\phi}] = \int \mathcal{D}\phi\, \exp\Big(iS[\phi] + i\int d^D x J^{\phi}(x)\phi(x)\Big). \tag{46}$$

In the Schwinger-Keldysh formalism this generating functional can be written as:

$$\begin{aligned}\mathcal{Z}[J^{\phi}_{+}, J^{\phi}_{-}, \rho(t_0)] &= \int \mathcal{D}\phi_0^{+}\mathcal{D}\phi_0^{-}\, \langle\phi_0^{+}|\,\hat{\rho}(t_0)\,|\phi_0^{-}\rangle \int_{\phi_0^{+}}^{\phi_0^{-}} \mathcal{D}\phi^{+}\mathcal{D}\phi^{-}\,\delta[\phi^{+}(t_f) - \phi^{-}(t_f)] \\ &\times \int \mathcal{D}\phi^{+}\mathcal{D}\phi^{-}\, \exp\bigg[i\int d^{D-1}x \int_{t_0}^{t_f} dt'\Big(\mathcal{L}(\phi^{+}, t') - \mathcal{L}(\phi^{-}, t') + J^{\phi}_{+}\phi^{+} - J^{\phi}_{-}\phi^{-}\Big)\bigg].\end{aligned} \tag{47}$$

The integration runs over a contour, consisting of a plus and a minus branch, where the time goes from t_0 to t_f. These boundaries can also be taken from minus infinity to plus infinity, if the end state is known. The contours are closely related to time and anti-time ordering operations when calculating the expectation value in equation 45. The first term in the generating functional (47) is an integral over the initial conditions at t_0, which are determined by the density operator $\hat{\rho}(t_0)$. The second integral forces the turning point of the contour at t_f to be the same: $\phi^+(t_f) = \phi^-(t_f)$. The last term is practically the same as the generating functional as we consider in equilibrium quantum field theory, but in this case there are two sources, one for each part of the contour displayed in the figure on the right: (J_+^ϕ and J_+^ϕ). With the definition of the generating functional (47) the expectation values of n-point functions can be found via:

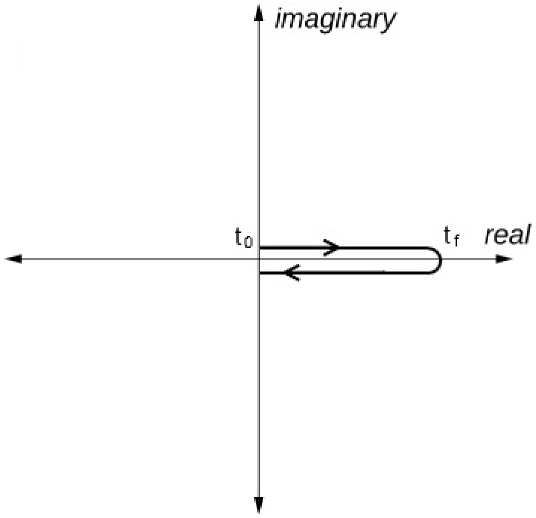

Figure 8: The Schwinger-Keldysh contour from the initial time t_0 to t_f.

$$\mathrm{Tr}\Big[\hat{\rho}(t_0)\bar{T}[\hat{\phi}(x_1)\dots\hat{\phi}(x_n)]T[\hat{\phi}(y_1)\dots\hat{\phi}(y_k)]\Big]$$
$$= \left[\frac{\delta^{n+k}\mathcal{Z}[J,\rho(t_0)]}{(-i\delta J_-^\phi(x_1))\dots(-i\delta J_-^\phi(x_n))i\delta J_+^\phi(y_1)\dots i\delta J_+^\phi(y_k)}\right]_{J=0}. \tag{48}$$

All fields have to be on the contour: $t_0 \leq x_j^0, y_j^0 \leq t_f$ for every j. For simplicity if a source J is written in the Schwinger-Keldysh formalism it means the sources with respect to all parts of the contour and all fields (in this case it is only one field ϕ).

4.1 Two-point functions

With this definition, the two-point functions can be written using the generating functional and the (anti)-time ordering operations. In the literature there are several different notations for these two-point functions (Δ^{ab} and G), the two-point functions are:

$$\begin{aligned}
\text{Feynman propagator} \qquad i\Delta_\phi^{++}(x,x') &= G_F \\
&= \langle T[\phi(x)\phi(x')]\rangle = \mathrm{Tr}\big[\hat\rho(t_0)\hat\phi^+(x')\hat\phi^+(x)\big] \\
&= \frac{\mathcal{Z}[J,\rho(t_0)]}{i\delta J_+^\phi(x) i\delta J_+^\phi(x')}\bigg|_{J=0}. \qquad (49a)
\end{aligned}$$

$$\begin{aligned}
\text{Dyson propagator} \qquad i\Delta_\phi^{--}(x,x') &= G_D \\
&= \langle \bar T[\phi(x)\phi(x')]\rangle = \mathrm{Tr}\big[\hat\rho(t_0)\hat\phi^-(x')\hat\phi^-(x)\big] \\
&= \frac{\mathcal{Z}[J,\rho(t_0)]}{i\delta J_-^\phi(x) i\delta J_-^\phi(x')}\bigg|_{J=0}. \qquad (49b)
\end{aligned}$$

$$\begin{aligned}
\text{Positive frequency} \qquad i\Delta_\phi^{-+}(x,x') &= G_+ \\
&= \langle \phi(x)\phi(x')\rangle = \mathrm{Tr}\big[\hat\rho(t_0)\hat\phi^-(x)\hat\phi^+(x')\big] \\
&= \frac{\mathcal{Z}[J,\rho(t_0)]}{\big(-i\delta J_-^\phi(x)\big) i\delta J_+^\phi(x')}\bigg|_{J=0}. \qquad (49c)
\end{aligned}$$

$$\begin{aligned}
\text{Negative frequency} \qquad i\Delta_\phi^{+-}(x,x') &= G_- \\
&= \langle \phi(x')\phi(x)\rangle = \mathrm{Tr}\big[\hat\rho(t_0)\hat\phi^-(x')\hat\phi^+(x)\big] \\
&= \frac{\mathcal{Z}[J,\rho(t_0)]}{i\delta J_+^\phi(x)\big(-i\delta J_-^\phi(x')\big)}\bigg|_{J=0}. \qquad (49d)
\end{aligned}$$

The positive and negative frequency two-point functions are also often called the Wightman functions. The free two-point functions obey a very similar equation as the Klein-Gordon equation(20):

$$\mathcal{D}_x i\Delta_{\phi,0}^{ab} \equiv \big(\Box_x - (m^2+\zeta\mathcal{R})\big) i\Delta_{\phi,0}^{ab} = ia\delta^{ab}\delta^D(x-x'). \qquad (50)$$

The four two-point functions are not completely independent since they can be written as:

$$i\Delta_\phi^{++}(x;x') = \theta(t-t')i\Delta_\phi^{-+}(x;x') + \theta(t'-t)i\Delta_\phi^{+-}(x;x') \qquad (51a)$$

$$i\Delta_\phi^{--}(x;x') = \theta(t'-t)i\Delta_\phi^{-+}(x;x') + \theta(t-t')i\Delta_\phi^{+-}(x;x') \qquad (51b)$$

$$i\Delta_\phi^{++}(x;x') + i\Delta_\phi^{--} = i\Delta_\phi^{-+}(x;x') + i\Delta_\phi^{+-}(x;x') \qquad (51c)$$

$$i\Delta_\phi^{-+}(x;x') = i\Delta_\phi^{+-}(x';x). \qquad (51d)$$

The zeroth component of x is as usual the time component: $x^0 = t$.

The Keldysh propagator

The four two-point functions in this formalism can be written in a 2×2 Keldysh two-point function matrix. This is also called the Keldysh propagator:

$$i\mathcal{G}_\phi = \begin{pmatrix} i\Delta_\phi^{++} & i\Delta_\phi^{+-} \\ i\Delta_\phi^{-+} & i\Delta_\phi^{--} \end{pmatrix}. \tag{52}$$

This propagator matrix obeys at tree level:

$$\mathcal{D}_x i\mathcal{G}_{\phi,0} = i\sigma^3 \delta^D(x-x'). \tag{53}$$

σ^3 is the third Pauli matrix $\sigma^3 = \mathrm{Diag}(1,-1)$. Making use of the four basic two-point functions (49) in this formalism, it is convenient to define other propagators.

The statistical two-point function

The statistical two-point function describes how states are populated in the Heisenberg picture. This two-point function is also called the Hadamard two-point functions. It can be written as an anti-commutator of two fields. Using the definition from equations (49), the statistical two-point functions can be written as:

$$F_\phi(x,x') = \frac{1}{2}\mathrm{Tr}\Big[\hat\rho(t_0)\{\hat\phi(x'),\hat\phi(x)\}\Big] = \frac{1}{2}\Big(i\Delta_\phi^{-+}(x,x') + i\Delta_\phi^{+-}(x,x')\Big). \tag{54}$$

Given some density matrix $\hat\rho(t_0)$.

The causal two-point function

The causal two-point function describes the number of accessible states, and is the commutator of the two fields. This two-point function is also called the Jordan two-point function:

$$i\Delta_\phi^c(x,x') = \mathrm{Tr}\Big[\hat\rho(t_0)\big[\hat\phi(x'),\hat\phi(x)\big]\Big] = i\Delta_\phi^{-+}(x,x') - i\Delta_\phi^{+-}(x,x'). \tag{55}$$

Rewriting these operators, all basic operators (49) can expressed only in the statistical and causal two-point functions:

$$i\Delta_\phi^{+-}(x,x') = F_\phi(x,x') - \frac{i}{2}\Delta_\phi^c(x,x') \tag{56a}$$

$$i\Delta_\phi^{-+}(x,x') = F_\phi(x,x') + \frac{i}{2}\Delta_\phi^c(x,x') \tag{56b}$$

$$i\Delta_\phi^{++}(x,x') = F_\phi(x,x') + \frac{1}{2}\mathrm{sgn}(t-t') i\Delta_\phi^c(x,x') \tag{56c}$$

$$i\Delta_\phi^{--}(x,x') = F_\phi(x,x') - \frac{1}{2}\mathrm{sgn}(t-t') i\Delta_\phi^c(x,x'). \tag{56d}$$

The propagator in presence of field condensates

In section 5 we will consider an action depending on two fields ϕ and χ. This formalism has some extra properties if the fields are coupled, which means that the + and − part of the contour can be of the different fields, resulting in cross terms in the two-point functions. The action that will be considered is the standard action of two fields with an interaction part:

$$\mathcal{L}_{\text{int}}[\phi, \chi] = -\frac{\lambda}{3!}\chi^3(x) - \frac{1}{2}h\chi^2(x)\phi(x).$$

With this interaction, the combined functional (non-free) propagators can be found by varying the action:

$$\tilde{\Delta}_{\phi}^{ab}(x; x') = \frac{\delta^2 S[\phi, \chi]}{\delta\phi(x)\delta\phi(x')}. \tag{57}$$

For the cross propagators the actions is varied once with respect to both fields, resulting in:

$$i\big(\tilde{\Delta}_{\phi;\phi}^{ab}\big)^{-1}(x, x') = i\big(\Delta_{\phi}^{ab}\big)^{-1} \tag{58a}$$

$$i\big(\tilde{\Delta}_{\chi;\chi}^{ab}\big)^{-1}(x, x') = \big(i\Delta_{\chi}^{ab}\big)^{-1} - (\lambda\chi^c(x) + h\phi^c(x))\delta(x - x')\delta^{ac} \tag{58b}$$

$$i\big(\tilde{\Delta}_{\phi;\chi}^{ab}\big)^{-1}(x, x') = -h\chi^a(x)\delta(x - x')\delta^{ab} \tag{58c}$$

$$i\big(\tilde{\Delta}_{\chi;\phi}^{ab}\big)^{-1}(x, x') = -h\chi^a(x)\delta(x - x')\delta^{ab}. \tag{58d}$$

This is formally correct, however in this thesis we will consider fields that are not coupled. The annoying cross terms will disappear,resulting in only two remaining full propagators, one for each field:

$$\begin{aligned} i\tilde{\Delta}_{\phi;\phi}^{ab}(x, x') &= i\Delta_{\phi}^{ab}(x, x') \\ i\tilde{\Delta}_{\chi;\chi}^{ab}(x, x') &= i\Delta_{\chi}^{ab}(x, x'). \end{aligned} \tag{59}$$

4.2 The two-particle irreducible (2PI) effective action

The 2PI formalism is developed by Cornwall, Jackiw and Tomboulis [6] in 1974, and therefore the 2PI effective action is sometimes called the CJT-effective action. This effective action differs from the usual one-particle irreducible (1PI) action. The 1PI effective action is a functional which depends on the one-point function (usually called ϕ), which contains all information of the quantum theory. In some situations the 1PI formalism is not the easiest way, and considering the 2PI formalism will make the problem easier. The 2PI effective action is a functional which not only depend on the one-point function ϕ, but also depends on the connected two point function Δ. This two-point function comes in by introducing a second bilinear source in the original definition of the effective action. From here it is possible to add even more sources to consider a NPI effective action. In the 2PI effective action the internal lines are represented by the full propagator, instead of the free propagator. Therefore an internal line of a simpler graph will be disregarded, because its correction is already taken in account. In other words, the only remaining graphs are the graphs where no non-trivial subgraphs can be isolated by cutting two internal lines, because these possible corrections are by definition already taken into account.

The first difference in constructing the 2PI effective action is the use of also two-point sources, instead of only one-point sources (J). The approach is very similar as in the 1PI formalism, starting with:

$$e^{iW[J,K]} = \int \mathcal{D}\phi^a \exp\Big[i \sum_{a,b=\pm} \Big(S[\phi] + aJ_a\phi^a + \frac{ab}{2}K_{ab}\phi^a\phi^b\Big)\Big]. \tag{60}$$

The fields can be found by varying W:

$$\frac{\delta W}{\delta J_a} = a\phi^a \qquad\qquad \frac{\delta W}{\delta K_{ab}} = \frac{ab}{2}\Big(\phi^a\phi^b + i\Delta_\phi^{ab}\Big). \tag{61}$$

Normally, in this formula, the last term is the full connected two point function. We will assume that fields are not coupled, resulting in the free propagator as extra term due to the variation with respect to the two-point source. The 2PI effective action is the double Legendre transform:

$$\Gamma[\phi, i\Delta] = W[J,K] - J_a\phi^a - \frac{1}{2}K_{ab}[\phi^a\phi^b + i\Delta_\phi^{ab}]. \tag{62}$$

The equations of motion for the fields and propagators can be constructed via the standard variational method, and setting afterwards the sources J and K to zero:

$$\frac{\delta\Gamma}{\delta\phi^a} = J_a - K_{ab}\phi^b \qquad\qquad \frac{\delta\Gamma}{\delta i\Delta_\phi^{ab}} = -\frac{1}{2}K_{ab}. \tag{63}$$

Now the effective action can be written as done in [2, 4, 10]:

$$\Gamma[\phi^a, i\Delta_\phi^{ab}] = S[\phi^a] + \frac{i}{2}\mathrm{Tr}\Big[\frac{\delta^2 S[\phi^a]}{\delta\phi^a\delta\phi^b}\Big]i\Delta_\phi^{ab} - \frac{i}{2}\mathrm{Tr}\Big[\log(i\Delta_\phi^{ab})\Big] + \Gamma^{\geq 2}[\phi, i\Delta^{ab}]. \tag{64}$$

$\Gamma^{\geq 2}$ contains the contributions from the 2PI effective action, which are the contribution to the effective action that contains two or more loops that are 2PI. To illustrate this formalism consider the following Lagrangian, with a Φ^4 interaction term:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2 - \lambda\Phi^4. \tag{65}$$

The interaction term is: $-\lambda\Phi^4$, and therefore the corresponding Feynman diagrams can be drawn, as in figure 9.

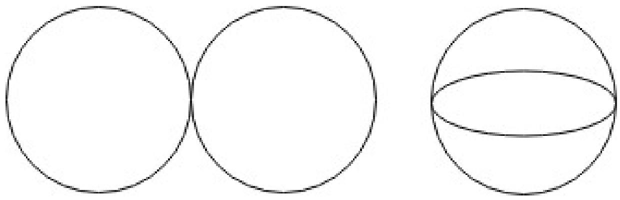

Figure 9: The two and three loop contribution to $\Gamma^{\geq 2}$

In the 1PI formalism there are more diagrams contributing to the effective action. For example the following diagram does contribute in the 1PI formalism, but doesn't contribute in the 2PI formalism:

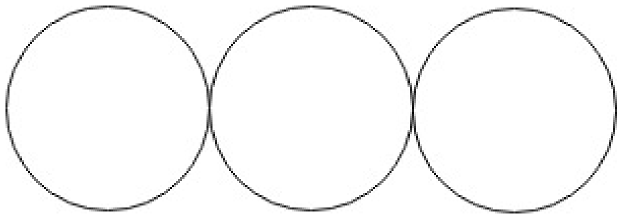

Figure 10: A tree loop contribution to the 1PI effective action, which does not contribute to the 2PI effective action.

The diagrams that do not contribute can be found by cutting any two of the internal lines of the diagram, it should remain connected. In figure 10 both internal lines can be cut, which leads to two disconnected diagrams. In the next section we will use the 2PI formalism, with a more relevant action for this thesis.

5 Interacting quantum field theory in an FLRW universe

In this section we examine the same action as done in the paper by Koksma, Schmidt and Prokopec in [10]. They looked at an interacting quantum field theory on a flat background, we are going to consider the same action on a FLRW background (1). This gives the opportunity to check the results by going from a FLRW metric to a Minkowski metric. We will discuss an inflaton field during inflation, as was briefly mentioned in section 3.3. From there one could see that there are three interesting expectation value

1. $\langle \hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t)\rangle$
2. $\langle \hat{\pi}(\vec{x},t)\hat{\pi}(\vec{y},t)\rangle$
3. $\frac{1}{2}\langle \{\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{y},t)\}\rangle$

The first one is often calculated, and results in the power spectrum of the CMB. This calculation is only at tree-level. The other two cases include loop terms, which might become larger in time and therefore they could be observable. The starting point is an interacting scalar field theory:

$$\begin{aligned} S[\phi,\chi] &= \int d^D x\sqrt{-g}\mathcal{L}[\phi,\chi] \\ &= \int d^D x\sqrt{-g}\Big(-\frac{1}{2}\zeta\mathcal{R} + \mathcal{L}_0[\phi] + \mathcal{L}_0[\chi] + \mathcal{L}_{\text{int}}[\phi,\chi]\Big) \qquad (66a) \\ \mathcal{L}_0[\phi] &= -\frac{1}{2}\partial_\mu\phi(x)\partial_\nu\phi(x)\eta^{\mu\nu} - \frac{1}{2}m_\phi^2(t)\phi^2(x) \qquad (66b) \\ \mathcal{L}_0[\chi] &= -\frac{1}{2}\partial_\mu\chi(x)\partial_\nu\chi(x)\eta^{\mu\nu} - \frac{1}{2}m_\chi^2(t)\chi^2(x) \qquad (66c) \\ \mathcal{L}_{\text{int}}[\phi,\chi] &= -\frac{\lambda}{3!}\chi^3(x) - \frac{1}{2}h\chi^2(x)\phi(x). \qquad (66d) \end{aligned}$$

In this action the following hierarchy is applied $\frac{h}{\lambda} \ll 1$. In this action the χ-field will act as an environment, as discussed in section 3. The ϕ-field will be an inflaton-field which interacts with the environment. Because we are interested in an expanding universe, and the interactions during inflation, this system is clearly not in equilibrium. Therefore the Schwinger-Keldysh formalism is used, together with the 2PI formalism (when discussed in section 4.2 the 2PI formalism was already generalized to the + and − contours). Instead of only considering one field, the action depends on two fields, resulting in more sources for the effective action. Equation (64) generalizes to:

$$\begin{aligned} \Gamma(\phi^a,\chi^a,i\Delta_\phi^{ab},i\Delta_\chi^{ab}) =& S[\phi,\chi] - \frac{i}{2}\text{Tr}\log\left[(i\Delta_\phi^{ab})\right] - \frac{i}{2}\text{Tr}\log\left[(i\Delta_\chi^{ab})\right] \\ &+ \frac{1}{2}\text{Tr}\Big[\frac{\delta^2 S[\phi^a,\chi^a]}{\delta\phi^a\delta\phi^b}\Big]i\Delta_\phi^{ab} + \frac{1}{2}\text{Tr}\Big[\frac{\delta^2 S[\phi^a,\chi^a]}{\delta\chi^a\delta\chi^b}\Big]i\Delta_\chi^{ab} \\ &+ \Gamma^{\geq 2}. \end{aligned} \qquad (67)$$

Because the perturbations are small, it is not necessary to compute all loop diagrams. The lowest order in which dissipation will occur is at two loops, therefore defining a truncation at two loops will be accurate enough. The effective action can be written as:

$$\Gamma[i\Delta_\phi^{ab}, i\Delta_\chi^{ab}] = \Gamma_0[i\Delta_\phi^{ab}, i\Delta_\chi^{ab}] + \Gamma_1[i\Delta_\phi^{ab}, i\Delta_\chi^{ab}] + \Gamma_2[i\Delta_\phi^{ab}, i\Delta_\chi^{ab}]. \tag{68}$$

The subscripts mean the number of loops. Because the first expression does not contain loops, the relevant expression is at tree level. The relevant equations for the ϕ field were already stated before in equations (52) and (53). For the χ field the equations are analogous. The diagrams that contribute at the one and two-loops are given in figure 11.

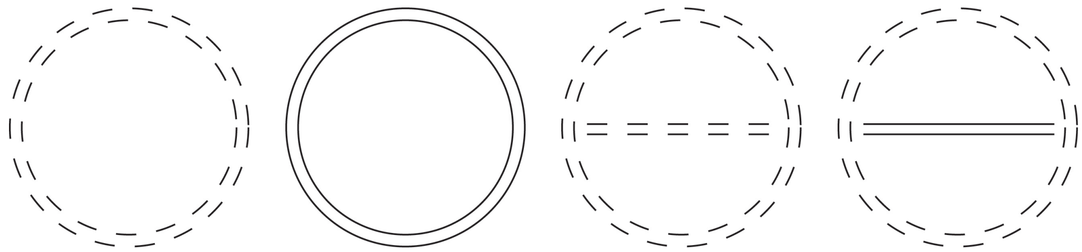

Figure 11: Contribution to the effective action of the interaction part of the Lagrangian given in (67), where the double dashed line is the (full) χ propagator, and the solid double line is the (full) ϕ propagator.

Because the expressions for the propagators are known, the diagrams can be written as equations. Therefore, for each loop contribution there is one definition:

$$\begin{aligned}\Gamma_0[i\Delta_\phi^{cd}, i\Delta_\chi^{ab}] = \int d^Dx d^Dx' \sqrt{-g}\Big[& \\ & \sum_{c,d=\pm} \frac{c}{2}\Big(\Box_x - (m_\phi^2 + \zeta\mathcal{R})\Big)\delta^D(x-x')\delta^{cd} i\Delta_\phi^{dc}(x';x) \\ & + \sum_{c,d=\pm} \frac{c}{2}\Big(\Box_x - (m_\chi^2 + \zeta\mathcal{R})\Big)\delta^D(x-x')\delta^{cd} i\Delta_\chi^{dc}(x';x)\Big]\end{aligned} \tag{69a}$$

$$\Gamma_1[i\Delta_\phi^{cd}, i\Delta_\chi^{cd}] = -\frac{i}{2}\mathrm{Tr}\ \mathrm{Log}[i\Delta_\phi^{cc}(x;x')] - \frac{i}{2}\mathrm{Tr}\ \mathrm{Log}[i\Delta_\chi^{cc}(x;x')] \tag{69b}$$

$$\begin{aligned}\Gamma_2[i\Delta_\phi^{cd}, i\Delta_\chi^{cd}] = \int d^Dx d^Dx' \sqrt{-g}\sqrt{-g'} \sum_{c,d=\pm} cd\Big[\frac{i\lambda^2}{12}\Big(i\Delta_\chi^{cd}(x';x)\Big)^3 & \\ + \frac{ih^2}{4}\Big(i\Delta_\chi^{cd}(x;x')\Big)^2 i\Delta_\phi^{cd}(x';x)\Big]. &\end{aligned} \tag{69c}$$

5.1 Equations of motion

From the effective action the equations of motion can be calculated. Starting with the standard expression, and simplifying this, the equations of motion can also be written in Fourier space, and expressed in conformal time. After the equations of motion are found for the standard propagators as defined in (49), they can be combined to construct equations of motion for the causal and statistical propagators. Starting with applying the standard variational principle:

$$\begin{aligned}\frac{\delta\Gamma[i\Delta_\phi^{ab},i\Delta_\chi^{ab}]}{\delta i\Delta_\phi^{ab}} =&\frac{a}{2}\sqrt{-g}\Big(\Box_x-(m_\phi^2+\zeta\mathcal{R})\Big)\delta^D(x-x')\delta^{ab}\\&-\sqrt{-g}\sqrt{-g'}\Big[\frac{i}{2}\big(i\Delta_\phi^{ab}(x;x')\big)^{-1}\\&\qquad+\frac{ih^2}{4}ab\big(i\Delta_\chi^{ab}(x;x')\big)^2\Big]=0\end{aligned} \tag{70a}$$

$$\begin{aligned}\frac{\delta\Gamma[i\Delta_\phi^{ab},i\Delta_\chi^{ab}]}{\delta i\Delta_\chi^{ab}} =&\frac{a}{2}\sqrt{-g}\Big(\Box_x-(m_\chi^2+\zeta\mathcal{R})\Big)\delta^D(x-x')\delta^{ab}\\&+\sqrt{-g}\sqrt{-g'}\Big[-\frac{i}{2}\big(i\Delta_\chi^{aa}(x;x')\big)^{-1}+\frac{i\lambda^2}{4}ab\big(i\Delta_\chi^{ab}(x';x)\big)^2\\&\qquad+\frac{ih^2}{2}ab\big(i\Delta_\chi^{ab}(x;x')\big)i\Delta_\phi^{ab}(x';x)\Big]=0.\end{aligned} \tag{70b}$$

These equations can be rewritten, and with the definition of self-masses these expressions are more familiar. The first equation can be rewritten by multiplying it by $(2ai\Delta_\phi^{bc}(x';x''))$, and integrate over x' and sum over b which lead to the equations of motion:

$$\begin{aligned}0=\int d^Dx'\sqrt{-g}\sum_{b=\pm}&\Big[\frac{a}{2}(\Box_x-(m_\phi^2+\zeta\mathcal{R}))\delta^D(x-x')\delta^{ab}(2ai\Delta_\phi^{bc}(x';x''))\\&-\sqrt{-g'}\frac{i}{2}\big(i\Delta_\phi^{ab}(x;x')\big)^{-1}(2ai\Delta_\phi^{bc}(x';x''))\\&+\sqrt{-g'}\frac{ih^2}{4}ab\big(i\Delta_\chi^{ab}(x;x')\big)^2(2ai\Delta_\phi^{bc}(x';x''))\Big],\\0=&a^2\sqrt{-g}(\Box_x-(m_\phi^2+\zeta\mathcal{R}))i\Delta_\phi^{ac}(x;x'')-ia\delta^{ac}\delta^D(x-x'')\\&+\int d^Dx'\sqrt{-g}\sqrt{-g'}\sum_{b=\pm}\frac{ih^2}{2}a^2b\big(i\Delta_\chi^{ab}(x;x')\big)^2\big(i\Delta_\phi^{bc}(x';x'')\big).\end{aligned}$$

The self-masses absorb the interacting parts by definition, which puts the equations of motion in a more traditional form:

$$\begin{aligned}\Big(\Box_x-(m_\phi^2+\zeta\mathcal{R})\Big)i\Delta_\phi^{ac}(x;x'')-&\int d^Dx'\sqrt{-g'}\sum_{b=\pm}biM_\phi^{ab}(x;x')i\Delta_\phi^{bc}(x';x'')\\&=\frac{ia\delta^{ac}\delta^D(x-x'')}{\sqrt{-g}}\end{aligned} \tag{71a}$$

$$\begin{aligned}\Big(\Box_x-(m_\phi^2+\zeta\mathcal{R})\Big)i\Delta_\chi^{ac}(x;x'')-&\int d^Dx'\sqrt{-g'}\sum_{b=\pm}biM_\chi^{ab}(x;x')i\Delta_\chi^{bc}(x';x'')\\&=\frac{ia\delta^{ac}\delta^D(x-x'')}{\sqrt{-g}}.\end{aligned} \tag{71b}$$

The self masses are:

$$iM_\phi^{ab}(x;x') = -\frac{ih^2}{2}\left(i\Delta_\chi^{ab}(x;x')\right)^2 = -2ab\frac{\delta\Gamma[i\Delta_\phi^{ac}, i\Delta_\chi^{ac}]}{\delta i\Delta_\phi^{ab}} \tag{72a}$$

$$\begin{aligned}iM_\chi^{ab}(x;x') =& -\frac{i\lambda^2}{2}\left(i\Delta_\chi^{ab}(x;x')\right)^2 - ih^2 i\Delta_\chi^{ab}(x;x')i\Delta_\phi^{ab}(x;x')\\ =& -2ab\frac{\delta\Gamma[i\Delta_\phi^{ac}, i\Delta_\chi^{ac}]}{\delta i\Delta_\phi^{ab}}.\end{aligned} \tag{72b}$$

As mentioned, the last two terms M_ϕ^{ab} and M_χ^{ab} are the self masses at one loop. The corresponding Feynman diagrams are perturbed diagrams given in figure 12.

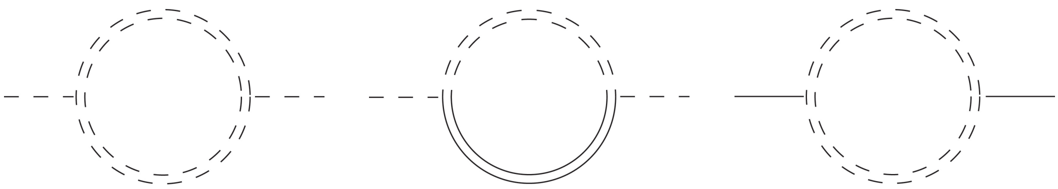

Figure 12: Contribution to the self masses up to one-loop, where the dashed double line is the full χ propagator, and the solid double line is the full ϕ propagator.

From the expressions in equation (72), the four equations of motion for the ϕ propagator, and the χ propagator can be found. Because both expressions are very similar, we only write down the equations of motion for the ϕ propagator:

$$\begin{aligned}&(\Box_x - (m_\phi^2 + \zeta\mathcal{R}))i\Delta_\phi^{++}(x;x') - \int d^Dy a^{D-1}\\ &\times\left[iM_\phi^{++}(x;y)i\Delta_\phi^{++}(y;x') - iM_\phi^{+-}(x;y)i\Delta_\phi^{-+}(y;x')\right] = \frac{i\delta^D(x-x')}{a^{D-1}}\end{aligned} \tag{73a}$$

$$\begin{aligned}&(\Box_x - (m_\phi^2 + \zeta\mathcal{R}))i\Delta_\phi^{--}(x;x') - \int d^Dy a^{D-1}\\ &\times\left[iM_\phi^{-+}(x;y)i\Delta_\phi^{+-}(y;x') - iM_\phi^{--}(x;y)i\Delta_\phi^{--}(y;x')\right] = \frac{i\delta^D(x-x')}{a^{D-1}}\end{aligned} \tag{73b}$$

$$\begin{aligned}&(\Box_x - (m_\phi^2 + \zeta\mathcal{R}))i\Delta_\phi^{+-}(x;x')\\ &-\int d^Dy a^{D-1}\left[iM_\phi^{++}(x;y)i\Delta_\phi^{+-}(y;x') - iM_\phi^{+-}(x;y)i\Delta_\phi^{--}(y;x')\right] = 0,\end{aligned} \tag{73c}$$

$$\begin{aligned}&(\Box_x - (m_\phi^2 + \zeta\mathcal{R}))i\Delta_\phi^{-+}(x;x')\\ &-\int d^Dy a^{D-1}\left[iM_\phi^{-+}(x;y)i\Delta_\phi^{++}(y;x') - iM_\phi^{--}(x;y)i\Delta_\phi^{-+}(y;x')\right] = 0.\end{aligned} \tag{73d}$$

These four equations are not independent of each other, and they are related via (51). Because the equations do only depend on $||x - x'||$, due to homogeneity, the Fourier transform over only the spatial part can be taken. This leads to the equations of motion in Wigner space , where the Wigner transform of a two point function is defined as the

Fourier transform with respect to the relative coordinate. There is a crucial difference with respect to the flat case, which is the FLRW-background. Its contribution is also in the d'Alembertian operator as it is defined in (3) and in FLRW-background the d'Alembertian operator can be written as in (21). The propagators become

$$i\Delta_\phi^{ab}(x;x') = \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} i\Delta_\phi^{ab}(\vec{k},t,t')e^{i\vec{k}(\vec{x}-\vec{x}')} \tag{74a}$$

$$i\Delta_\phi^{ab}(k,t;t') = \int d^{D-1}(\vec{x}-\vec{x}')i\Delta_\phi^{ab}(x;x')e^{-i\vec{k}(\vec{x}-\vec{x}')}. \tag{74b}$$

Plugging this in the equations of motion leads to:

$$\begin{aligned}\Big(\partial_t^2 + H(D-1)\partial_t + (m_\phi^2+\zeta\mathcal{R}) + \frac{k^2}{a^2(t)}\Big)i\Delta_\phi^{++}(k,t,t') \\ + \int dt_1 a^{D-1}(t_1)\Big[iM_\phi^{++}(k,t,t_1)i\Delta_\phi^{++}(k,t_1,t') \\ - iM_\phi^{+-}(k,t,t_1)i\Delta_\phi^{-+}(k,t_1,t')\Big] = \frac{-i\delta(t-t')}{a^{D-1}}\end{aligned} \tag{75a}$$

$$\begin{aligned}\Big(\partial_t^2 + H(D-1)\partial_t + (m_\phi^2+\zeta\mathcal{R}) + \frac{k^2}{a^2(t)}\Big)i\Delta_\phi^{--}(k,t,t') \\ + \int dt_1 a^{D-1}(t_1)\Big[iM_\phi^{-+}(k,t,t_1)i\Delta_\phi^{+-}(k,t_1,t') \\ - iM_\phi^{--}(k,t,t_1)i\Delta_\phi^{--}(k,t_1,t')\Big] = \frac{i\delta(t-t')}{a^{D-1}}\end{aligned} \tag{75b}$$

$$\begin{aligned}\Big(\partial_t^2 + H(D-1)\partial_t + (m_\phi^2+\zeta\mathcal{R}) + \frac{k^2}{a^2(t)}\Big)i\Delta_\phi^{+-}(k,t,t') \\ + \int dt_1 a^{D-1}(t_1)\Big[iM_\phi^{++}(k,t,t_1)i\Delta_\phi^{+-}(k,t_1,t') \\ - iM_\phi^{+-}(k,t,t_1)i\Delta_\phi^{--}(k,t_1,t')\Big] = 0\end{aligned} \tag{75c}$$

$$\begin{aligned}\Big(\partial_t^2 + H(D-1)\partial_t + (m_\phi^2+\zeta\mathcal{R}) + \frac{k^2}{a^2(t)}\Big)i\Delta_\phi^{-+}(k,t,t') \\ + \int dt_1 a^{D-1}(t_1)\Big[iM_\phi^{-+}(k,t,t_1)i\Delta_\phi^{++}(k,t_1,t') \\ - iM_\phi^{--}(k,t,t_1)i\Delta_\phi^{-+}(k,t_1,t')\Big] = 0.\end{aligned} \tag{75d}$$

For the χ fields analogous equations hold. This set of equations are solvable via a numerical approach, which only would give full general solutions. By making approximations one can extract specific information from this set of equations. The next step is considering the equations of motion in conformal time: $ad\eta = dt$, such that:

$$\begin{aligned}
\frac{-i\delta(\eta-\eta')}{a^D(\eta)} =& \frac{1}{a^2}\Big[\partial_\eta^2 + \mathcal{H}(D-2)\partial_\eta + \vec{k}^2 + a^2(m_\phi^2 + \zeta\mathcal{R})\Big] i\Delta_\phi^{++}(k,\eta,\eta') \\
&+ \int d\eta_1 a^D(\eta_1)\Big[iM_\phi^{++}(k,\eta,\eta_1) i\Delta_\phi^{++}(k,\eta_1,\eta') \\
&\qquad - iM_\phi^{+-}(k,\eta,\eta_1) i\Delta_\phi^{-+}(k,\eta_1,\eta')\Big] \qquad \text{(76a)} \\
+\frac{i\delta(\eta-\eta')}{a^D(\eta)} =& \frac{1}{a^2}\Big[\partial_\eta^2 + \mathcal{H}(D-2)\partial_\eta + \vec{k}^2 + a^2(m_\phi^2 + \zeta\mathcal{R})\Big] i\Delta_\phi^{--}(k,\eta,\eta') \\
&+ \int d\eta_1 a^D(\eta_1)\Big[iM_\phi^{-+}(k,\eta,\eta_1) i\Delta_\phi^{+-}(k,\eta_1,\eta') \\
&\qquad - iM_\phi^{--}(k,\eta,\eta_1) i\Delta_\phi^{--}(k,\eta_1,\eta')\Big] \qquad \text{(76b)} \\
0 =& \frac{1}{a^2}\Big[\partial_\eta^2 + \mathcal{H}(D-2)\partial_\eta + \vec{k}^2 + a^2(m_\phi^2 + \zeta\mathcal{R})\Big] i\Delta_\phi^{+-}(k,\eta,\eta') \\
&+ \int d\eta_1 a^D(\eta_1)\Big[iM_\phi^{++}(k,\eta,\eta_1) i\Delta_\phi^{+-}(k,\eta_1,\eta') \\
&\qquad - iM_\phi^{+-}(k,\eta,\eta_1) i\Delta_\phi^{--}(k,\eta_1,\eta')\Big] \qquad \text{(76c)} \\
0 =& \frac{1}{a^2}\Big[\partial_\eta^2 + \mathcal{H}(D-2)\partial_\eta + \vec{k}^2 + a^2(m_\phi^2 + \zeta\mathcal{R})\Big] i\Delta_\phi^{-+}(k,\eta,\eta') \\
&+ \int d\eta_1 a^D(\eta_1)\Big[iM_\phi^{-+}(k,\eta,\eta_1) i\Delta_\phi^{++}(k,\eta_1,\eta') \\
&\qquad - iM_\phi^{--}(k,\eta,\eta_1) i\Delta_\phi^{-+}(k,\eta_1,\eta')\Big]. \qquad \text{(76d)}
\end{aligned}$$

$\mathcal{H}$ is the conformal Hubble parameter as defined in equation (2). Using this set of equations, the equations of motion for the statistical (54) and causal (55) propagators can be found, rewriting equation (56) to:

$$F_\phi(x,x') = \frac{1}{2}(i\Delta_\phi^{-+}(x,x') + i\Delta_\phi^{+-}(x,x')) \qquad \text{(77a)}$$

$$i\Delta_\phi^c(x,x') = i\Delta_\phi^{-+}(x,x') - i\Delta_\phi^{+-}(x,x'). \qquad \text{(77b)}$$

Analog to the definition of the causal and statistical propagator, one can define the following masses:

$$M_\phi^{+-}(k,\eta,\eta') = M_\phi^F(k,\eta,\eta') - \frac{i}{2}M_\phi^c(k,\eta,\eta') \qquad \text{(78a)}$$

$$M_\phi^{-+}(k,\eta,\eta') = M_\phi^F(k,\eta,\eta') + \frac{i}{2}M_\phi^c(k,\eta,\eta') \qquad \text{(78b)}$$

$$M_\phi^{++}(k,\eta,\eta') = M_\phi^F(k,\eta,\eta') + \mathrm{sgn}(\eta-\eta')\frac{i}{2}M_\phi^c(k,\eta,\eta') \qquad \text{(78c)}$$

$$M_\phi^{--}(k,\eta,\eta') = M_\phi^F(k,\eta,\eta') - \mathrm{sgn}(\eta-\eta')\frac{i}{2}M_\phi^c(k,\eta,\eta') \qquad \text{(78d)}$$

Using the definitions in equation (51) we can write equation (76) as two equations of motion for the causal and statistical propagator:

$$\frac{1}{a^2}\Big[\partial_\eta^2+\mathcal{H}(D-2)\partial_\eta+\vec{k}^2+a^2(m_\phi^2+\zeta\mathcal{R})\Big]\Delta_\phi^c(k,\eta,\eta') - \int_{\eta'}^{\eta} d\eta_1 a^D(\eta_1)M_\phi^c(k,\eta,\eta_1)\Delta_\phi^c(k,\eta_1,\eta')=0 \tag{79a}$$

$$\frac{1}{a^2}\Big[\partial_\eta^2+\mathcal{H}(D-2)\partial_\eta+\vec{k}^2+a^2(m_\phi^2+\zeta\mathcal{R})\Big]F_\phi(k,\eta,\eta') - \int_{-\infty}^{\eta} d\eta_1 a^D(\eta_1)M_\phi^c(k,\eta,\eta_1)F_\phi(k,\eta_1,\eta') + \int_{-\infty}^{\eta'} d\eta_1 a^D(\eta_1)M_\phi^F(k,\eta,\eta_1)\Delta_\phi^c(k,\eta_1,\eta')=0 \tag{79b}$$

Solving the equations of motion for the causal and statistical propagator will give information of the accessibility and population of states. If one would only consider the propagator at tree level, this term leads to the usual power spectrum of the CMB, in the zero curvature gauge this is:

$$P_s(k,\eta)=\frac{H^2}{\dot{\phi}^2}\Big(\frac{k^3}{2\pi^2}F_\phi(k,\eta,\eta)\Big)\Big|_{k<Ha} \tag{80}$$

The propagator we are considering can diagrammatically be written as:

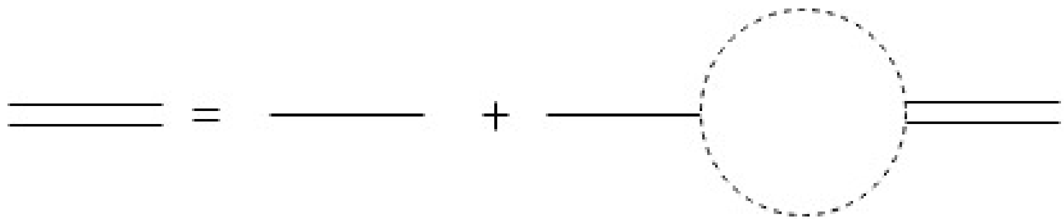

Figure 13: The full ϕ propagator (double solid line) is equal to the free propagator (solid line) and an extra term involving a χ-loop and the full propagator.

The additional information is contained in the right diagram of figure 13, and as obtained from the equations of motion (73), the general form is indeed a free part with additional integral over a propagator and mass term.

5.2 Renormalization of the self-masses

The equations of motions found in the previous section all depend on self-masses. Calculating the self-masses is the first step to solve these equations, the self-masses are defined in (72). Because the χ-field acts as an environment, the most interesting self-mass is the mass of the system (ϕ). Recall the definition of this mass:

$$iM_\phi^{ab}(x;x') = -\frac{ih^2}{2}\big(i\Delta_\chi^{ab}(x;x')\big)^2. \tag{81}$$

During inflation the assumption that our propagator is the Minkowski vacuum propagator cannot be justified. As mentioned in section 2.2, in a FLRW universe there are three interesting cases:

1. $m_\chi^2 + \zeta\mathcal{R} > 0$
2. $m_\chi^2 + \zeta\mathcal{R} = 0$
3. $m_\chi^2 + \zeta\mathcal{R} < 0$.

The first case is the most well-known case and already the solution for the propagator was found in 1968 by Chernikov and Tagirov [5]:

$$\begin{aligned} i\Delta_\chi(x;x') =& \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma\big[\frac{D-1}{2}+\nu_D\big]\Gamma\big[\frac{D-1}{2}-\nu_D\big]}{\Gamma\big[\frac{D}{2}\big]} \\ &\times\ {}_2F_1\Big[\frac{D-1}{2}+\nu_D,\frac{D-1}{2}-\nu_D,\frac{D}{2},1-\frac{y}{4}\Big], \end{aligned} \tag{82}$$

with

$$\nu_D^2 = \Big(\frac{D-1}{2}\Big)^2 - \frac{m^2}{H^2}. \tag{83}$$

This is fully infrared-finite, and therefore it is solvable.

In this thesis we will neglect the third case, and only consider the second case: $m_\chi^2+\zeta\mathcal{R}=0$. This means that the effective mass is considered zero, and we assume that the background during inflation is deSitter. In previous work, Onemli and Woodard [12] showed that this propagator can be written as:

$$\begin{aligned} i\Delta_{dS} =& \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg[-\sum_{n=0}^{\infty}\frac{1}{n-\frac{D}{2}+1}\frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+1)}\big(\frac{y}{4}\big)^{n-\frac{D}{2}+1} - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\big(\pi\frac{D}{2}\big) \\ &+\sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\big(\frac{y}{4}\big)^n + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\Bigg]. \end{aligned} \tag{84}$$

y can be written as:

$$y(x;x') = aa'H^2\Delta x_{ab}^2 \qquad ; \qquad y(x;x')\Big|_{\epsilon=0} = 4\sin^2\Big(\frac{1}{2}Hl(x;x')\Big). \tag{85}$$

In the Schwinger-Keldysh theorem, the distance between two spacetime points with a deSitter background are defined as:

$$\Delta x^2_{++}(x,x') = -(|\eta-\eta'|-i\epsilon)^2 + ||\vec{x}-\vec{x}'||^2 \tag{86a}$$
$$\Delta x^2_{+-}(x,x') = -(\eta-\eta'+i\epsilon)^2 + ||\vec{x}-\vec{x}'||^2 \tag{86b}$$
$$\Delta x^2_{-+}(x,x') = -(\eta-\eta'-i\epsilon)^2 + ||\vec{x}-\vec{x}'||^2 \tag{86c}$$
$$\Delta x^2_{--}(x,x') = -(|\eta-\eta'|+i\epsilon)^2 + ||\vec{x}-\vec{x}'||^2. \tag{86d}$$

Due to the Schwinger-Keldysh contour, the $+-$ and $-+$ do not need an absolute value sign. Choosing a different position for the poles (upper/lower half of complex plane), results in different contours for the integration, and a different propagator. Because the distance functions are in close relations with the propagators, it is convenient to consider only one case, and extend the solution. Therefore the full calculation will only be done once. The most obvious propagator to consider is the $++$ propagator, with a conformal time coordinate to make the metric Minkowski-like. Before just plugging this propagator into the self-mass equation (72), lets examine this propagator. Because eventually the number of dimensions will be set to $4+\tilde{\epsilon}$, this propagator consists of terms which disappear in this limit. Make sure the summations start both at $n=0$, after that set the first steps of dimensional regularization with the limit of $D \to 4+\tilde{\epsilon}$, where $\tilde{\epsilon}$ is infinitesimally small. This leads to:

$$\begin{aligned} i\Delta_{dS} = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg[&- \frac{1}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{-\frac{D}{2}+1} - \frac{1}{2-\frac{D}{2}}\Gamma\Big(1+\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-\frac{D}{2}} \\ &- \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big) + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa') \\ &+ \sum_{n=0}^{\infty}\frac{1}{n+1}\Big[\Big(-n-3-\frac{\tilde{\epsilon}}{2}+n+3+\tilde{\epsilon}\Big)\big(\frac{y}{4}\big)^{n+1}\Big]\Bigg]. \end{aligned} \tag{87}$$

The last line of this formula above goes to zero as $\tilde{\epsilon}$ goes to zero. Therefore we know that these terms will not contribute as $\tilde{\epsilon}$ goes to zero. The propagator can be written as:

$$\begin{aligned} i\Delta_{dS} = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Bigg[&- \frac{1}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{1-\frac{D}{2}} - \frac{1}{2-\frac{D}{2}}\Gamma\Big(1+\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-\frac{D}{2}} \\ &- \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big) + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\Bigg] + \mathcal{O}^{D\to 4+\tilde{\epsilon}}(\tilde{\epsilon}). \end{aligned} \tag{88}$$

For the self-mass the propagator squared is needed. The calculation of this propagator is done in appendix B with as result:

$$\begin{aligned}\left(i\Delta_{dS}\right)^2 =&\frac{H^{2D-4}}{(4\pi)^D}\Bigg[\frac{1}{(1-\frac{D}{2})^2}\Gamma^2\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-D}+4\log^2\Big(\frac{\sqrt{e}H^2\Delta x^2_{++}}{4}\Big)\\ &\qquad+\frac{2}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(1+\frac{D}{2})}{2-\frac{D}{2}}\big(\frac{y}{4}\big)^{3-D}\\ &\qquad+\frac{2}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{1-\frac{D}{2}}\\ &\qquad-\frac{2}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\big(\frac{y}{4}\big)^{1-\frac{D}{2}}\Bigg]\\ &+\mathcal{O}(D-4).\end{aligned}\tag{89}$$

This propagator squared is not renormalized yet, and due to terms as y^{2-D} the self-mass will have to be renormalized. This renormalization is very similar to the flat case as done in [10], and therefore this renormalization is also done in Appendix C. In the case of an FLRW-background, the identities that were used in appendix C are altered. The expressions are derived in [16] and also in appendix D. The identities which will be used are:

$$\big(\frac{y}{4}\big)^{-\alpha} = -\frac{1}{(\alpha-1)(\frac{D}{2}-\alpha)}\frac{\Box}{H^2}\big(\frac{y}{4}\big)^{1-\alpha}+\frac{D-\alpha}{\frac{D}{2}-\alpha}\big(\frac{y}{4}\big)^{1-\alpha}\ ,\text{ with }\alpha\neq\frac{D}{2}\tag{90a}$$

$$\frac{\Box}{H^2}\big(\frac{y}{4}\big)^{1-\frac{D}{2}} = \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D}i\delta^D(x-x')+\frac{D(D-2)}{4}\big(\frac{y}{4}\big)^{1-\frac{D}{2}}.\tag{90b}$$

Now all terms can be written in the limit $D\to 4+\tilde{\epsilon}$ which is done in Appendix E. Finally the propagator squared can be written as:

$$\begin{aligned}(i\Delta_{dS})^2 = \frac{H^{2D-4}}{(4\pi)^D}\Bigg[&\text{L.T.}(D)-\frac{\Box}{H^2}\Big[\big(\frac{4}{y}\big)\log\big(\frac{\mu^2 y}{H^2}\big)\Big]+\frac{4}{y}\Big(2\log\big(\frac{\mu^2 y}{H^2}\big)-1\Big)\\ &-8\log\big(\frac{y}{4}\big)+2\frac{\Box}{H^2}\log\big(\frac{y}{4}\big)-2\frac{\Box}{H^2}\log^2\big(\frac{y}{4}\big)+8\log(aa')\\ &+4\log(aa')\frac{\Box}{H^2}\log\big(\frac{y}{4}\big)+4\log^2\big(\frac{y}{4}\big)-8\log(aa')\log\big(\frac{y}{4}\big)\\ &+4\log^2(aa')+3\Bigg].\end{aligned}\tag{91}$$

The only divergence is now in the part which is called L.T., this is a local divergence, which can be canceled by adding a counter-term in the mass. As previously showed, the mass is of the form (28): $M^{ab}_\phi(x,x') = -\frac{ih^2}{2}\big(i\Delta^{ab}_\chi(x;x')\big)^2$. Therefore the counter-term which renormalizes the mass is:

$$iM^{++}_{\phi,c} = \frac{ih^2}{2}\frac{2(4\pi)^{D/2}}{(D-3)(D-4)\Gamma(\frac{D}{2}-1)(Ha)^D}\Big(\frac{2\mu}{H}\Big)^{D-4}i\delta^D(x-x').\tag{92}$$

The renormalized mass with $D \to 4 + \tilde{\epsilon}$ is:

$$
\begin{aligned}
iM^{++}_{\phi,Ren} = -\frac{ih^2}{2}\frac{H^4}{(4\pi)^4}\Big[& -\frac{\Box}{H^2}\Big[(\frac{4}{y})\log(\frac{\mu^2 y}{H^2})\Big] + \frac{4}{y}\Big(2\log(\frac{\mu^2 y}{H^2}) - 1\Big) \\
& - 8\log(\frac{y}{4}) + 2\frac{\Box}{H^2}\log(\frac{y}{4}) - 2\frac{\Box}{H^2}\log^2(\frac{y}{4}) + 8\log(aa') \\
& + 4\log(aa')\frac{\Box}{H^2}\log(\frac{y}{4}) + 4\log^2(\frac{y}{4}) - 8\log(aa')\log(\frac{y}{4}) \\
& + 4\log^2(aa') + 3\Big]
\end{aligned}
\tag{93}
$$

5.3 Fourier transform of the renormalized self-mass

The result found in position space is already solvable by a numerical integral, the only down side is that this numerical integral is very heavy to compute. Therefore even with a supercomputer the result will not be accurate enough, due to the fact that a small number of iterations can be computed. Therefore the Fourier transform of the renormalized mass will get rid of three spatial integrals. The fact that the mass is not renormalized in the Fourier space, comes from the fact that such a renormalization is done by cutoff. This cutoff is not diffeomorphism invariant, and therefore the result of the renormalization is not reliable. The Fourier transform of the self-mass is:

$$
iM^{++}_{\phi,Ren}(\vec{k},\eta,\eta') = \int d^{D-1}(\vec{x}-\vec{x}')iM^{++}_{\phi,Ren}(x,x')e^{-i\vec{k}(\vec{x}-\vec{x}')}. \tag{94}
$$

Before blindly trying to calculate the Fourier integral, one should observe the terms that appear in the renormalized mass (93). In fact we can use

$$
\frac{1}{(\Delta x)^2} = \frac{1}{4}\partial^2\log(\mu^2(\Delta x)^2), \tag{95a}
$$

$$
\frac{\log((\mu^2\Delta x)^2)}{(\Delta x)^2} = \frac{1}{8}\partial^2\Big[\log^2(\mu^2(\Delta x)^2) - 2\log(\mu^2(\Delta x)^2)\Big], \tag{95b}
$$

to write

$$
\begin{aligned}
(\frac{4}{y})\log\Big(\frac{\mu^2 y}{H^2}\Big) &= \frac{4}{aa'H^2\Delta x^2_{++}}\Big[\log(aa') + \log\Big(\mu^2\Delta x^2_{++}\Big)\Big] \\
&= \frac{4\log(aa')}{y} + \frac{1}{2aa'H^2}\partial^2\Big[\log^2\Big(\mu^2\Delta x^2_{++}\Big) - 2\log\Big(\mu^2\Delta x^2_{++}\Big)\Big].
\end{aligned}
\tag{96}
$$

The self-mass term becomes:

$$
\begin{aligned}
iM^{++}{}_{\phi,Ren} = -\frac{ih^2}{2}\frac{H^4}{(4\pi)^4}\Bigg[\\
&- \frac{\Box}{H^2}\Big[\frac{4\log(aa')}{y} + \frac{1}{2aa'H^2}\partial^2\Big(\log^2\big(\mu^2\Delta x^2_{++}\big) - 2\log\big(\mu^2\Delta x^2_{++}\big)\Big)\Big] \\
&+ 2\Big[\frac{4}{y}\big(\log(aa') - \frac{1}{2}\big) + \frac{1}{2aa'H^2}\partial^2\Big(\log^2\big(\mu^2\Delta x^2_{++}\big) - 2\log\big(\mu^2\Delta x^2_{++}\big)\Big)\Big] \\
&- 8\log\big(\frac{y}{4}\big) + 2\frac{\Box}{H^2}\log\big(\frac{y}{4}\big) - 2\frac{\Box}{H^2}\log^2\big(\frac{y}{4}\big) + 8\log(aa') + 4\log^2\big(\frac{y}{4}\big) \\
&+ 4\log(aa')\frac{\Box}{H^2}\log\big(\frac{y}{4}\big) - 8\log(aa')\log\big(\frac{y}{4}\big) + 4\log^2(aa') + 3\Bigg]
\end{aligned}
\tag{97}
$$

The Fourier integral is a huge integral over mainly three terms and (covariant) derivatives over these terms:

1. constant terms,
2. $\log\left(\frac{y}{4}\right)$ terms,
3. $\log^2\left(\frac{y}{4}\right)$ terms.

By applying a trick these integrals can be done. To check the correctness of this trick the integral over the constant term is done in appendix F and the integral over the $\log(\frac{y}{4})$ is done in appendix G. This different approach by applying a trick turns out to be more efficient. The calculation done in appendix G was already long, and the $\log^2$ would be worse (if it is even possible in that way). Therefore making use of a mathematical trick will make the integrals easier. Expanding the expressions:

$$
\left(y^2-1\right)^b = e^{b\log(y^2-1)} = 1 + b\log\left(y^2-1\right) + \frac{b^2}{2}\log^2\left(y^2-1\right) + \cdots \tag{98a}
$$

$$
\left(1-y^2\right)^b = e^{b\log(1-y^2)} = 1 + b\log\left(1-y^2\right) + \frac{b^2}{2}\log^2\left(1-y^2\right) + \cdots . \tag{98b}
$$

gives the opportunity to manually set b to zero, for which we indeed find the same integrals as we obtain.

$$
\int_0^\infty dyy\sin(k\Delta\eta y)\Big[\left(|1-y^2|\right)^b\Big]_{b=0} = \int_0^\infty dyy\sin(k\Delta\eta y) \tag{99a}
$$

$$
\int_0^\infty dyy\sin(k\Delta\eta y)\Big[\partial_b\Big(\left(|1-y^2|\right)^b\Big)\Big]_{b=0} = \int_0^\infty dyy\sin(k\Delta\eta y)\log\left(|1-y^2|\right) \tag{99b}
$$

$$
\int_0^\infty dyy\sin(k\Delta\eta y)\Big[\Big(\partial_b^2\left(|1-y^2|\right)^b\Big)\Big]_{b=0} = \int_0^\infty dyy\sin(k\Delta\eta y)\frac{\log^2\left(|1-y^2|\right)}{2}. \tag{99c}
$$

$$\vdots$$

The general integral can be written as two different integrals, using a regulator as before:

$$\underbrace{\int_0^1 dyy\sin(k\Delta\eta y)\Big[\big(1-y^2\big)^b\Big]}_{K(k\Delta\eta,b)}+\text{Im}\underbrace{\Big[\int_1^\infty dyye^{ik\Delta\eta y}\Big(\big(y^2-1\big)^b\Big)\Big]}_{L(k\Delta\eta,b)} \tag{100}$$

$$K(z,b)=\frac{\sqrt{\pi}}{4}\frac{\Gamma(1+b)}{\Gamma(\frac{5}{2}+b)}(k\Delta\eta)\ {}_0F_1\Big[\frac{5}{2}+b,-\frac{(k\Delta\eta)^2}{4}\Big] \tag{101}$$

$$L(z,b)=\frac{\Gamma(1+b)}{\sqrt{\pi}}\frac{K_{3/2+b}(-ik\Delta\eta)}{(\frac{ik\Delta\eta}{2})^{\frac{1}{2}+b}} \qquad \text{Im}(k\Delta\eta)>0 \text{ (Regulator)} \tag{102}$$

$$\begin{aligned}&=\frac{\Gamma(1+b)}{2\sqrt{\pi}}\Bigg[\Gamma\Big(\frac{3}{2}+b\Big)\frac{1}{(-ik\Delta\eta/2)^{2+2b}}\ {}_0F_1\Big[-\frac{1}{2}-b,\frac{-(k\Delta\eta)^2}{4}\Big]\\ &\qquad+\Gamma\Big(-\frac{3}{2}-b\Big)\Big(\frac{-ik\Delta\eta}{2}\Big)\ {}_0F_1\Big[\frac{5}{2}+b,\frac{-(k\Delta\eta)^2}{4}\Big]\Bigg].\end{aligned} \tag{103}$$

Writing $z=(k\Delta\eta)$, and using the following identities:

$$-i^{-2b}=e^{i\pi b}=\cos(\pi b)+i\sin(\pi b) \tag{104}$$

$$\Gamma(a)+\Gamma\Big(a+\frac{1}{2}\Big)=2^{1-2a}\sqrt{\pi}\Gamma(2a) \qquad \text{(duplication formula)} \tag{105}$$

$$\frac{1}{(\frac{-iz}{2})^{2+2b}}=-\frac{4}{z^2}\frac{\sin(\pi b)}{(\frac{z}{2})^{2b}} \qquad z\in\mathcal{R} \tag{106}$$

$$\Gamma\Big(-\frac{3}{2}-b\Big)=\frac{\pi}{\cos(\pi b)}\frac{1}{\Gamma(\frac{5}{2}+b)}. \tag{107}$$

Then:

$$\begin{aligned}\text{Im}\big[L(z,b)\big]=&-\Gamma(2+2b)\frac{\sin(\pi b)}{z^{2+2b}}\ {}_0F_1\Big[-\frac{1}{2}-b,\frac{-z^2}{4}\Big]\\ &-\frac{\Gamma(1+b)}{\Gamma(\frac{5}{2}+b)}\frac{\sqrt{\pi}z}{4\cos(\pi b)}\ {}_0F_1\Big[\frac{5}{2}+b,\frac{-z^2}{4}\Big].\end{aligned} \tag{108}$$

In total the integral becomes:

$$\begin{aligned}K(z,b)+\text{Im}\big[L(z,b)\big]&=-\Gamma(2+2b)\frac{\sin(\pi b)}{z^{2+2b}}\ {}_0F_1\Big[-\frac{1}{2}-b,\frac{-z^2}{4}\Big]\\ &\quad+\Bigg[\frac{\Gamma(1+b)}{\Gamma(\frac{5}{2}+b)}\frac{\sqrt{\pi}z}{4}\Big(1-\frac{1}{\cos(\pi b)}\Big)\Bigg]\ {}_0F_1\Big[\frac{5}{2}+b,\frac{-z^2}{4}\Big]\\ &=-\Gamma(2+2b)\frac{\sin(\pi b)}{z^{2+2b}}\ {}_0F_1\Big[-\frac{1}{2}-b,\frac{-z^2}{4}\Big]\\ &\quad-\Bigg[\frac{\Gamma(1+b)}{\Gamma(\frac{5}{2}+b)}\frac{\sqrt{\pi}z}{2}\Bigg(\frac{\sin^2(\frac{\pi b}{2})}{\cos(\pi b)}\Bigg)\Bigg]\ {}_0F_1\Big[\frac{5}{2}+b,\frac{-z^2}{4}\Big].\end{aligned} \tag{109}$$

Using this, the previous results can be checked. First we consider the constant term. Setting b to zero, it is immediately obtained that this term has no contribution as also stated in equation 153. The second term is the logarithmic term. This requires a derivative of (109) with respect to b, and after that set b to zero. The $\sin(b)$ term will kill almost all terms, only resulting in:

$$\int_0^\infty dyy\sin(k\Delta\eta y)\log\left(|1-y^2|\right) = -\Gamma(2)\frac{\pi}{(k\Delta\eta)^2}\underbrace{{}_0F_1\Big[-\frac{1}{2},\frac{-(k\Delta\eta)^2}{4}\Big]}_{\cos(k\Delta\eta)+(k\Delta\eta)\sin(k\Delta\eta)}.$$

This is indeed the same result as we obtained before in equation 164. The last term that still needs to be evaluated is the $\log^2$ term. Using the known integrals from equations 155 and 164:

$$L_0 = \frac{4\pi^2}{k}\int_0^\infty drr\sin(kr)\theta\Big(r^2-(\Delta\eta)^2\Big) = \frac{4\pi^2}{k^3}\Big(\sin\left(k\Delta\eta\right) - k\Delta\eta\cos(k\Delta\eta)\Big) \tag{110a}$$

$$\begin{aligned} L_1 &= \frac{4\pi}{k^3}(k\Delta\eta)^2\int_0^\infty dzz\sin(k\Delta\eta z)\log\left(|z^2-1|\right) \\ &= \frac{-4\pi^2}{k^3}\Big[k\Delta\eta\sin(k\Delta\eta)+\cos(k\Delta\eta)\Big]. \end{aligned} \tag{110b}$$

Following the derivation of the log term, the full Fourier integral can be split up in multiple parts, as done on the next page.

$$
\begin{aligned}
&\int d^{D-1}(\vec{x}-\vec{x}')\log^2\Big(\frac{aa'H^2\Delta x_{++}^2}{4}\Big)e^{-i\vec{k}(\vec{x}-\vec{x}')}\\
=&\frac{4\pi}{k}\int_0^\infty drr\sin(kr)\Bigg(\log\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big)+\log\Big(\big|\frac{r^2}{\Delta\eta^2}-1\big|\Big)+i\pi\theta(\Delta\eta^2-r^2)\Bigg)^2\\
=&\frac{4\pi}{k}\int_0^\infty drr\sin(kr)\Bigg(\underbrace{\log^2\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big)}_{\to 0}+\underbrace{2\log\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big)\log\Big(\big|\frac{r^2}{\Delta\eta^2}-1\big|\Big)}_{\text{known integral}}\Bigg)\\
&+\frac{4\pi}{k}\int_0^{\Delta\eta} drr\sin(kr)\Bigg(\underbrace{2i\pi\log\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big)}_{\text{known integral}}-\pi^2+\underbrace{2i\pi\log\Big(\big|\frac{r^2}{\Delta\eta^2}-1\big|\Big)}_{\text{Part of the log calculation}}\Bigg)\\
&+\frac{4\pi}{k}\int_0^\infty drr\sin(kr)\log^2\Big(\big|1-\frac{r^2}{\Delta\eta^2}\big|\Big)\\
=&\Bigg(\Big[2i\log\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big)-\pi\Big]+2i\Big[\mathrm{Ci}(2k\Delta\eta)-\gamma_E-\log(k\Delta\eta/2)\Big]\Bigg)L_0\\
&+\Bigg(2\log\Big[\frac{aa'H^2\Delta\eta^2}{4}\Big]+2i\Big[\mathrm{Si}(2k\Delta\eta)+\pi/2\Big]\Bigg)L_1\\
&+\frac{8i\pi^2}{k^3}\sin(k\Delta\eta)+\frac{4\pi}{k^3}(k\Delta\eta)^2\int_0^\infty dzz\sin(k\Delta\eta z)\log^2\left(|1-z^2|\right).
\end{aligned}
$$

The only remaining integral is the last integral. which can be evaluated using the trick of (98), and take the second derivative of the expansion with respect to b:

$$
\begin{aligned}
\partial_b^2\Big[K(z,b)+\mathrm{Im}[L(z,b)]\Big]_{b=0}=&\,2\Big[\frac{-2\phi(2)\pi}{z^2}+\frac{2\pi\log(z)}{z^2}\Big]\,{}_0F_1\Big[\frac{-1}{2},\frac{-z^2}{4}\Big]\\
&-2\frac{\pi}{z^2}\,{}_0F_1^{(1,0)}\Big[\frac{-1}{2}-b,\frac{-z^2}{4}\Big]\Bigg|_{b=0}-\frac{1}{3}z\pi^2\underbrace{{}_0F_1\Big[\frac{5}{2},\frac{-z^2}{4}\Big]}_{\frac{3}{z^3}\big(\sin(z)-z\cos(z)\big)}\,.
\end{aligned}
\tag{111}
$$

Plugging in the definition of the derivative of the hypergeometric function gives:

$$
{}_0F_1^{(1,0)}\Big[\frac{-1}{2},\frac{-z^2}{4}\Big]=-\psi(-\frac{1}{2}){}_0F_1\Big[\frac{-1}{2},\frac{-z^2}{4}\Big]+\sum_{k=0}^\infty\frac{\psi(k-\frac{1}{2})\Gamma(-\frac{1}{2})(-\frac{z^2}{4})^k}{k!\Gamma(k-\frac{1}{2})}.
$$

This can be rewritten using the recurrence relation for the digamma function:

$$
\psi(z+1)=\psi(z)+\frac{1}{z}. \tag{112}
$$

Using

$$\psi(\frac{1}{2}) = -2\log(2) - \gamma_E, \tag{113}$$

this results in:

$$\begin{aligned}
{}_0F_1{}^{(1,0)}\Big[\frac{-1}{2},\frac{-z^2}{4}\Big] &= -\psi\Big(-\frac{1}{2}\Big){}_0F_1\Big[\frac{-1}{2},\frac{-z^2}{4}\Big] \\
&+\Gamma\Big(-\frac{1}{2}\Big)\sum_{k=0}^{\infty}\frac{\psi(k+\frac{1}{2})\big(-\frac{z^2}{4}\big)^k}{k!\Gamma(k-\frac{1}{2})} - \Gamma\Big(-\frac{1}{2}\Big)\sum_{k=0}^{\infty}\frac{\big(-\frac{z^2}{4}\big)^k}{k!\Gamma(k+\frac{1}{2})} \\
&= -\Big(\psi(\frac{1}{2})+2\Big)\Big(\cos(z)+z\sin(z)\Big) - \Big(\text{Ci}(2z) - \log\big(\frac{z}{2}\big)\Big)\Big(\cos(z)+z\sin(z)\Big) \\
&+\Big(\text{Si}(2z)+\frac{\pi}{2}\Big)\Big(z\cos(z)-\sin(z)\Big) + 2\cos(z) \\
&=(2\log(2)+\gamma_E-2)\Big(\cos(z)+z\sin(z)\Big) \\
&-\Big[\Big(\cos(z)+z\sin(z)\Big)\Big(\text{Ci}(2z)+\log\big(\frac{z}{2}\big)\Big) \\
&+\Big(\sin(z)-z\cos(z)\Big)\Big(\text{Si}(2z)+\frac{\pi}{2}\Big)\Big] + 2\cos(z).
\end{aligned} \tag{114}$$

Adding up all terms results in:

$$\begin{aligned}
\frac{4\pi}{k^3}(k\Delta\eta)^2\int_0^\infty dyy\sin(k\Delta\eta y)\frac{\log^2\big(|1-y^2|\big)}{2} &= 2L_0\Big(\text{Si}(2k\Delta\eta)\Big) \\
&-2L_1\Big(\text{Ci}(2k\Delta\eta)+\gamma_E+\log\big(\frac{k\Delta\eta}{2}\big)\Big) - \frac{16\pi^2}{k^3}\cos(k\Delta\eta).
\end{aligned} \tag{115}$$

The complete expression for the this term is:

$$\begin{aligned}
L_2 &= \int d^{D-1}(\vec{x}-\vec{x}')\log^2\Big(\frac{aa'H^2\Delta x_{++}^2}{4}\Big) \\
&= 2L_0\Bigg\{\Big(\text{Si}(2k\Delta\eta)-\frac{\pi}{2}\Big) + i\Big[\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big)+\text{Ci}(2k\Delta\eta)-\gamma_E\Big]\Bigg\} \\
&+2L_1\Bigg\{\Big[-\text{Ci}(2k\Delta\eta)-\gamma_E+\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big)\Big]+i\Big[\text{Si}(2k\Delta\eta)+\pi/2\Big]\Bigg\} \\
&+\frac{8\pi^2}{k^3}\Big(i\sin(k\Delta\eta)-2\cos(k\Delta\eta)\Big).
\end{aligned} \tag{116}$$

The only remaining terms that are not Fourier transformed yet are the d'Alembertian terms. Using the known integrals, these integrals can also be found. There are three different terms containing a d'Alembertian operator. Starting with the most trivial term, which can directly be found by partial integrating twice:

$$\int_0^\infty d^{D-1}(x-x')\frac{\Box}{H^2}\log\big(\frac{y}{4}\big)e^{-i\vec{k}(\vec{x}-\vec{x}')} = \frac{-1}{aa'H^2}\Big(\partial_\eta^2+2\mathcal{H}\partial_\eta+k^2\Big)(L_1+iL_0). \tag{117}$$

As before $\mathcal{H}$ is the conformal Hubble parameter. This expression is already in the limit $D \to 4$. The second term is:

$$
\begin{aligned}
-\int_0^\infty d^{D-1}(x-x')\frac{\Box}{H^2}\log^2\big(\frac{y}{4}\big)e^{-i\vec{k}(\vec{x}-\vec{x}')} &= \frac{1}{H^2 aa'}\Big(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + k^2\Big) \\
\Big\{2L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta) - \frac{\pi}{2}\Big) + i\Big(\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big) + \mathrm{Ci}(2k\Delta\eta) - \gamma_E\Big)\Big] \\
+ 2L_1\Big[\Big(-\mathrm{Ci}(2k\Delta\eta) - \gamma_E + \log\big(\frac{aa'H^2\Delta\eta}{2k}\big)\Big) + i\Big(\mathrm{Si}(2k\Delta\eta) + \pi/2\Big)\Big] \\
+ \frac{8\pi^2}{k^3}\Big(i\sin(k\Delta\eta) - 2\cos(k\Delta\eta)\Big)\Big\}. \qquad (118)
\end{aligned}
$$

The last term is the most difficult term:

$$
-\frac{\Box}{H^2}\Big[\big(\frac{4}{y}\big)\log\Big(\frac{\mu^2 y}{H^2}\Big)\Big]. \qquad (119)
$$

Because these terms could be rewritten using (95) to (96), this results in integrals over the logarithm, logarithm squared and the remaining integral:

$$
\int_0^\infty dr \frac{4\log(aa')}{aa'H^2\Delta x_{++}^2} = \frac{\pi}{2}\frac{\log(aa')}{aa'H^2}\big(\cos(k\Delta\eta) + i\sin(k\Delta\eta)\big). \qquad (120)
$$

Using the definitions we introduced before in (110), this can be written as:

$$
\begin{aligned}
&\int d^{D-1}(\vec{x}-\vec{x}')\frac{-\Box}{H^2}\Bigg[\frac{4\log(aa')}{aa'H^2\Delta x_{++}^2} \\
&\qquad + \frac{1}{2aa'H^2}\partial^2\Big[\log^2\big(\mu^2\Delta x_{++}^2\big) + 2\log\big(\mu^2\Delta x_{++}^2\big)\Big]\Bigg]e^{ik(\vec{x}-\vec{x}')} \\
&= \frac{-1}{2(aa')^2H^4}\Big(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + k^2\Big)\Big(\partial_\eta^2 + k^2\Big) \\
&\quad \Big\{2L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta) - \frac{\pi}{2}\Big) + i\Big(\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big) + \mathrm{Ci}(2k\Delta\eta) - \gamma_E\Big)\Big] \\
&\quad + 2L_1\Big[-\mathrm{Ci}(2k\Delta\eta) - \gamma_E - 1 + \log\Big(\frac{\mu^2\Delta\eta}{2k}\Big) + i\Big(\mathrm{Si}(2k\Delta\eta) + \pi/2\Big)\Big] \\
&\quad + \frac{8\pi^2}{k^3}\Big(-2\cos(k\Delta\eta) + i\sin(k\Delta\eta)\Big)\Big\} \\
&+ \frac{1}{(aa')^2H^4}\frac{8\pi^2}{k}\Big(\partial_\eta^2 + 2\mathcal{H}\partial_\eta + k^2\Big)\log(aa')\big(\cos(k\Delta\eta) + i\sin(k\Delta\eta)\big). \qquad (121)
\end{aligned}
$$

The total Fourier transform (93) in $D=4$ then can be written using (110)(115)(117)(118)(121)

$$
\begin{aligned}
iM^{++}_{\phi,Ren} = & \frac{ih^2}{2}\frac{1}{(4\pi)^4}\frac{1}{(aa')^2}\Big(\partial_\eta^2+2\mathcal{H}\partial_\eta+k^2\Big)\Bigg\{\Big(\partial_\eta^2+k^2\Big)\Bigg[\\
& L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta)-\frac{\pi}{2}\Big)+i\Big(\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big)+\mathrm{Ci}(2k\Delta\eta)-\gamma_E-1\Big)\Big] \\
& +L_1\Big[-\mathrm{Ci}(2k\Delta\eta)-\gamma_E-1+\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big)+i\Big(\mathrm{Si}(2k\Delta\eta)+\pi/2\Big)\Big] \\
& +\frac{4\pi^2}{k^3}\Big(-2\cos(k\Delta\eta)+i\sin(k\Delta\eta)\Big)\Bigg] \\
& -\frac{8\pi^2}{k}\log(aa')\big(\cos(k\Delta\eta)+i\sin(k\Delta\eta)\big)\Bigg\} \\
& +\frac{iH^2h^2}{(4\pi)^4}\Bigg\{\Big(\partial_\eta^2+k^2\Big)\Bigg[\\
& L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta)-\frac{\pi}{2}\Big)+i\Big(\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big)+\mathrm{Ci}(2k\Delta\eta)-\gamma_E-1\Big)\Big] \\
& +L_1\Big[-\mathrm{Ci}(2k\Delta\eta)-\gamma_E-1+\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big)+i\Big(\mathrm{Si}(2k\Delta\eta)+\pi/2\Big)\Big] \\
& +\frac{4\pi^2}{k^3}\Big(-2\cos(k\Delta\eta)+i\sin(k\Delta\eta)\Big)\Bigg] \\
& -\frac{8\pi^2}{k}\big(\log(aa')-\frac{1}{2}\big)\big(\cos(k\Delta\eta)+i\sin(k\Delta\eta)\big)\Bigg\} \\
& \frac{-ih^2H^2}{(4\pi)^4}\Big[\frac{1}{(aa')^2}\Big(\partial_\eta^2+2\mathcal{H}\partial_\eta+k^2\Big)\Big]\Bigg\{ \\
& 2L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta)-\frac{\pi}{2}\Big)+i\Big(\big(-\frac{1}{2}-\log(aa')\big)+\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big)+\mathrm{Ci}(2k\Delta\eta)-\gamma_E\Big)\Big] \\
& -2L_1\Big[\big(\frac{1}{2}+\log(aa')\big)+\Big(\mathrm{Ci}(2k\Delta\eta)+\gamma_E-\log\big(\frac{aa'H^2\Delta\eta}{2k}\big)\Big)-i\Big(\mathrm{Si}(2k\Delta\eta)+\pi/2\Big)\Big] \\
& +\frac{8\pi^2}{k^3}\Big(i\sin(k\Delta\eta)-2\cos(k\Delta\eta)\Big)\Bigg\} \\
& -4ih^2\frac{H^4}{(4\pi)^4}\Bigg\{L_0\Bigg[\Big(\mathrm{Si}(2k\Delta\eta)-\frac{\pi}{2}\Big)+i\Big[\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big)+\mathrm{Ci}(2k\Delta\eta)-\gamma_E\Big]\Bigg] \\
& +L_1\Bigg[\Big[-1-\log(aa')-\mathrm{Ci}(2k\Delta\eta)-\gamma_E+\log\Big(\frac{aa'H^2\Delta\eta}{2k}\Big)\Big]+i\Big[\mathrm{Si}(2k\Delta\eta)+\pi/2\Big]\Bigg] \\
& +\frac{8\pi^2}{k^3}\Big(i\sin(k\Delta\eta)-2\cos(k\Delta\eta)\Big)\Bigg\}
\end{aligned}
$$

The other self masses

The other self-masses for the $+-$, $-+$ and $--$ cases can easily be found, using the definition of the different propagators. Because the different propagators result in different self-masses, they have to be calculated also, but using a clever trick this complete calculation is not needed four times. Starting with all four counter terms, remembering the definition of the different propagators from equation 86, using the similar identities as (90b), but for the $+-$, and the $-+$ case the constant term vanishes and for the $--$ case the constant term will get an extra minus sign. This leads to:

$$iM^{--}_{\phi,c} = -iM^{++}_{\phi,c} = -\frac{ih^2}{2}\frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D}i\delta^D(x-x') \tag{122}$$

$$iM^{-+}_{\phi,c} = iM^{+-}_{\phi,c} = 0. \tag{123}$$

From there the Fourier transform of the self-mass can be obtained. Dealing carefully with the Sine integral and the Cosine integral, using the following identities:

$$\sin(-z) = -\sin(z) \tag{124a}$$

$$\cos(-z) = \cos(z) \tag{124b}$$

$$\mathrm{Si}(-z) = -\mathrm{Si}(z) - \pi \tag{124c}$$

$$\mathrm{Ci}\big(-2k(-\Delta\eta + i\epsilon)\big) = -i\pi + \mathrm{Ci}(2k(-\Delta\eta)). \tag{124d}$$

Starting from the known result of the $++$ case, working towards the $--$ case, which includes the transformations:

$$\sin(k\Delta\eta) \to -\sin(k|\Delta\eta|) \tag{125a}$$

$$\mathrm{Si}(2k\Delta\eta) \to -\mathrm{Si}(2k|\Delta\eta|) - \pi \tag{125b}$$

$$\mathrm{Ci}(2k\Delta\eta) \to \mathrm{Ci}(2k|\Delta\eta|) - i\pi. \tag{125c}$$

Using these definitions, the integrals (L_i terms) become:

$$L_0^{--} = -L_0^{++} \tag{126a}$$

$$L_1^{--} = L_1^{++} \tag{126b}$$

$$L_2^{--} = \left(L_2^{++}\right)^{\dagger}. \tag{126c}$$

Using these definitions, the $--$ self-mass can be written as the complex conjugate of the $++$ case. This is what we should expect from the definition of the self-masses and the propagators from the beginning:

$$iM^{--}_{\phi,Ren} = \left(iM^{++}_{\phi,Ren}\right)^{\dagger}. \tag{127}$$

Because the propagators are not independent as stated in equations 51, and the self-mass is defined as in equation 81 the self-masses for the $+-$ and $-+$ cases can be written as:

$$iM^{+-}_{\phi,Ren} = \theta(\eta-\eta')iM^{--}_{\phi,Ren} + \theta(\eta'-\eta)iM^{++}_{\phi,Ren} \tag{128a}$$

$$iM^{-+}_{\phi,Ren} = \theta(\eta-\eta')iM^{++}_{\phi,Ren} + \theta(\eta'-\eta)iM^{--}_{\phi,Ren}. \tag{128b}$$

The masses can be rewritten in such a way that it is of the form as in [9]. The four renormalized masses can be combined to write the renormalized statistical and causal self masses (M^F_ϕ and M^c_ϕ respectively) as defined in equation 78. These self masses can then be used to solve the equations of motion for the statistical and causal propagator (79).

5.4 Reduce self-mass to Minkowski limit

A part of the answer can be checked by setting $a = 1$, and $H = 0$. This has as consequence that a Minkowski metric is again obtained. This results in:

$$
\begin{aligned}
iM^{++}{}_{\phi,Ren,flat}(\vec{k},\eta,\eta') = \frac{ih^2}{4}\frac{1}{(4\pi)^4}\Big(\partial_\eta^2 + k^2\Big)^2 \\
\Bigg\{2L_0\Big[\Big(\mathrm{Si}(2k\Delta\eta) - \frac{\pi}{2}\Big) + i\Big(\log\Big(\frac{\mu^2\Delta\eta}{2k}\Big) + \mathrm{Ci}(2k\Delta\eta) - \gamma_E - 1\Big)\Big] \\
+ 2L_1\Big[-\mathrm{Ci}(2k\Delta\eta) - \gamma_E - 1 + \log\Big(\frac{\mu^2\Delta\eta}{2k}\Big) + i\Big(\mathrm{Si}(2k\Delta\eta) + \pi/2\Big)\Big] \\
+ \frac{8\pi^2}{k^3}\Big(-2\cos(k\Delta\eta) + i\sin(k\Delta\eta)\Big)\Bigg\}.
\end{aligned}
$$

To be able to compare these results with the results found in [10], the expression needs to be rewritten, by taking $(\partial_\eta^2 + k^2)$. This gives an imaginary part which is:

$$
\begin{aligned}
\mathrm{Im}\Big[iM^{++}_{\phi,Ren,flat}(\vec{k},\eta,\eta')\Big] = \frac{h^2}{64k\pi^2}(\partial_\eta^2 + k^2) \\
\Big[\cos(\Delta\eta k)\Big(\gamma_E + \mathrm{Ci}(2k\Delta\eta) - \log\Big(\frac{\Delta\eta\mu^2}{2k}\Big)\Big) \\
+ \sin(\Delta\eta k)\Big(\mathrm{Si}(2k\Delta\eta) + \frac{\pi}{2}\Big)\Big],
\end{aligned}
$$

and the real part:

$$
\begin{aligned}
\mathrm{Re}\Big[iM^{++}_{\phi,Ren,flat}(\vec{k},\eta,\eta')\Big] = \frac{ih^2}{64k\pi^2}(\partial_\eta^2 + k^2)\Big[\\
\sin(\Delta\eta k)\Big(\gamma_E - \mathrm{Ci}(2k\Delta\eta) - \log\Big(\frac{\Delta\eta\mu^2}{2k}\Big)\Big) \\
+ \cos(\Delta\eta k)\Big(\mathrm{Si}(2k\Delta\eta) + \frac{\pi}{2}\Big)\Big].
\end{aligned}
$$

The result found in the flat in [10] by Koksma, Prokopec and Schmidt is:

$$
\begin{aligned}
iM^{++}_{\phi,Ren}(\vec{k},t,t') =& \frac{ih^2}{64k\pi^2}(\partial_\eta^2+k^2)\Big[\\
& \Big(\cos(k\Delta t) - i\sin(k|\Delta t|)\Big)\Big(\gamma_E + \log\Big(\frac{k}{2\mu^2|\Delta t|}\Big) - i\frac{\pi}{2}\Big) \\
& + \Big[\cos(k\Delta t) + i\sin(k|\Delta t|)\Big(\mathrm{Ci}(2k|\Delta t|) - i\mathrm{Si}(2k|\Delta t|)\Big)\Big] \\
\mathrm{Im}\big[iM^{++}_{\phi,Ren}(\vec{k},t,t')\big] =& \frac{h^2}{64k\pi^2}(\partial_\eta^2+k^2)\Big[\\
& \cos(k\Delta t)\Big(\gamma_E + \log\Big(\frac{k}{2\mu^2|\Delta t|}\Big) + \mathrm{Ci}(2k|\Delta t|)\Big) \\
& + \sin(k|\Delta t|)\Big(-\frac{\pi}{2} + \mathrm{Si}(2k|\Delta t|)\Big)\Big] \\
\mathrm{Re}\big[iM^{++}_{\phi,Ren}(\vec{k},t,t')\big] =& \frac{h^2}{64k\pi^2}(\partial_\eta^2+k^2)\Big[\\
& \sin(k\Delta t)\Big(\gamma_E + \log\Big(\frac{k}{2\mu^2|\Delta t|}\Big) - \mathrm{Ci}(2k|\Delta t|)\Big) \\
& + \cos(k|\Delta t|)\Big(\frac{\pi}{2} + \mathrm{Si}(2k|\Delta t|)\Big)\Big].
\end{aligned}
$$

This agrees, and verifies the methods used here which are different then the methods that were used in [10].

6 Discussion

The problem which led to this thesis is the quantum behavior of primordial fluctuations and the evolution of these fluctuations in time. The fact that during inflation the system was a quantum system, the system should still act as a quantum system. However, one should naturally expect that large scale structures should behave classically. In section 2 four problems of the Big Bang were described, and solved by introducing a period of inflation. So on one side inflation is needed to solve these problems of the Big Bang, on the other side is the problem of the classical behavior of the quantum inhomogeneity fluctuations in large scale structure theories due to inflation.

The classical behavior of these quantum inhomogeneities nowadays can be explained by the process of classicalization as described in section 3 . By introducing classicalization this quantum system underwent einselection, which has as result that only the most likely states survived. The einselection is triggered by the interaction of the quantum state with the weakly coupled environment, resulting in the classical behavior of large scale structures as we can observe nowadays.

In section 4 we described the Schwinger-Keldysh formalism together with the 2PI formalism. The Schwinger-Keldysh formalism is used to describe non-equilibrium quantum field theory. In this formalism the integration contour consists of two branches, with four corresponding two-point functions. The 2PI formalism is introduced, which is a functional depending on the one-point function ϕ, and the connected two point function Δ. This two-point function comes in by introducing a second bilinear source in the original definition of the effective action. In the 2PI effective action the internal lines are represented by the full propagator, instead of the free propagator.

The Schwinger-Keldysh formalism and the 2PI formalism were used in section 5 to describe two scalar fields interacting during inflation. That non-equilibrium quantum field theory is needed during inflation is due to the fact that in and out state are completely different. The specific action, which consists of two scalar fields interacting, (one which represented a system, and the other which represented the environment) can be described by the 2PI formalism. Because these scalar fields where considered living in an expanding universe, the expressions for the propagators and the self-masses quickly grew to large terms. In this section the Fourier-transform of the renormalized self-masses is found.

In section 5.4 we compared the result that we found of the self-mass with the self-mass of the case we would consider a Minkowski background as done in [10]. In this limit these terms are confirmed, which represent a nontrivial check of our result. The next step is to consider the Kadonoff-Baym equations (79), which will lead to a set of equations which are all set up to be integrated numerically, which could be done in future research.

The result of this numerical integral could provide information on the von Neumann entropy of the system, which therefore would be a sign of classicalization. Such a prediction, confirmed by measurements, could be of great value in cosmology. This could provide us with extra information over inflation and classicalization. As mentioned before some comparable calculations and numerical integrals are already done in the flat vacuum case [10], and the flat thermal case in [11]. This suggests that this indeed could lead to nonzero entropy. The effects of this particular action with a effective mass equals to zero is likely very small.

In this thesis we used a model of an inflaton (ϕ) weakly coupled to an environment (χ). For the effective mass of the environment field we could distinguish three cases:

1. $m_\chi^2 + \zeta\mathcal{R} > 0$
2. $m_\chi^2 + \zeta\mathcal{R} = 0$
3. $m_\chi^2 + \zeta\mathcal{R} < 0$.

Here we focused on the conformally coupled case: $m_\chi^2 + \zeta\mathcal{R} = 0$, and we did not focus on the other two cases. The first case $m_\chi^2 + \zeta\mathcal{R} > 0$ can also be discussed, and includes the propagator which is introduced in the paper by Chernikov and Tagirov [5]. The third case is the case that the effective mass is negative. These contributions are likely larger then the contributions of the case with an effective mass which is zero. For a full quantum description this calculation should be extended to this negative effective mass case, this can lead to better insight in the behavior of quantum fluctuations during inflation.

Appendices

A Conformal Invariance

As easily can be checked, the following action will not be conformal invariant:

$$S = \int d^D x \frac{1}{2}\sqrt{-g}(-\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2).$$

But if we add a non-minimally coupled mass term which is coupled to gravity through the Ricci scalar $\mathcal{R}$, we can find a specific form which is conformally invariant [3]:

$$S = \int d^D x \frac{1}{2}\sqrt{-g}(-\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - \zeta \mathcal{R} \phi^2).$$

Considering the transformations:

$$\begin{aligned} g_{\mu\nu} &\to g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}(x) \\ \phi(x) &\to \phi' = \Omega(x)^{\frac{2-D}{2}} \phi. \end{aligned} \tag{129}$$

Now we can look at how the action changes, and find a value for ζ for which this action is invariant. We can write this transformation, in a way we will only consider an infinitesimal shift:

$$\Omega(x) = 1 + \omega(x) \text{ , where } \omega \text{ is infinitesimal.}$$

Then (neglecting terms of ω^2or higher):

$$\begin{aligned}
\delta g_{\mu\nu}(x) &= 2\omega(x)g_{\mu\nu}(x) \\
\delta\phi(x) &= \frac{2-D}{2}\omega(x)\phi(x) \\
\delta\mathcal{R} &= (\delta g^{\mu\nu})\mathcal{R}_{\mu\nu} + g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} \\
\delta\mathcal{R}_{\mu\nu} &= \nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda} \\
g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} &= \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}) \\
\delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\nu}(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\lambda\mu} - \nabla_\gamma\delta g_{\mu\nu}) \\
g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} &= (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta g_{\mu\nu} \\
\delta\mathcal{R} &= (\delta g^{\mu\nu})\mathcal{R}_{\mu\nu} + (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta g_{\mu\nu} \\
&= -2\omega(x)\mathcal{R} + (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)2\omega(x)g_{\mu\nu} \\
&= -2\omega(x)\mathcal{R} + (2-2D)\Box\omega(x) \\
\delta(\mathcal{R}\phi^2) &= -D\omega(x)\mathcal{R}\phi^2 + (2-2D)\phi^2\Box\omega(x) \\
\delta\sqrt{-g} &= \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\delta g_{\mu\nu}) = D\omega(x) \\
\delta(\sqrt{-g}\phi^2) &= \sqrt{-g}(D\omega(x) + (2-D)\omega(x))\phi^2 = 2\sqrt{-g}\omega(x)\phi^2 \\
\delta(\sqrt{-g}\mathcal{R}\phi^2) &= \sqrt{-g}(2-2D)\phi^2\Box\omega(x) \\
\delta(g^{\mu\nu}\sqrt{-g}) &= \delta g^{\mu\nu}\sqrt{-g} + g^{\mu\nu}\delta\sqrt{-g} = \sqrt{-g}(D-2)g^{\mu\nu}\omega(x) \\
\delta(\sqrt{-g}\partial_\mu\phi\partial^\mu\phi) &= (D-2)\sqrt{-g}g^{\mu\nu}\Big[\omega(x)\partial_\mu\phi\partial_\nu\phi - \partial_\mu\phi\partial_\nu(\omega(x)\phi)\Big] \\
&= (2-D)\sqrt{-g}g^{\mu\nu}(\nabla_\mu\phi\nabla_\nu\omega(x)\phi).
\end{aligned}$$

Then the variation of the actions, allows us to partially integrate:

$$\begin{aligned}
\frac{\delta S}{\delta g^{\mu\nu}} &= \int d^Dx\nabla_\mu\Big[(D-2)\sqrt{-g}g^{\mu\nu}\Big(\frac{1}{2}\phi^2\nabla_\nu\omega(x)\phi\Big)\Big] + \\
&\int d^Dx\sqrt{-g}\Big[\frac{2-D}{2}\phi^2\Box\omega(x) - 2\omega(x)\phi^2m^2 + (2D-2)\zeta\phi^2\Box\omega(x)\Big] \\
0 &= \Big[\frac{2-D}{2} + (2D-2)\zeta\Big]\phi^2\Box\omega(x) - 2\omega(x)\phi^2m^2 \\
\zeta &= \frac{D-2}{4(D-1)} \qquad \text{if we neglect the mass term..} \qquad (130)
\end{aligned}$$

Therefore $\zeta_c = \frac{1}{6}$ is considered a critical value in $D = 4$, for which the scalar is conformally coupled.

B Dimensional regularization of the propagator squared

Starting from the propagator from equation 88:

$$i\Delta_{dS} = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Big[-\frac{1}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{1-\frac{D}{2}} - \frac{1}{2-\frac{D}{2}}\Gamma\Big(1+\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-\frac{D}{2}}$$
$$-\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big) + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\Big] + \mathcal{O}(D-4). \tag{131}$$

The propagator contains the term:

$$\frac{1}{\frac{D}{2}-2}\Gamma\Big(1+\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-\frac{D}{2}} = \frac{2}{D-4}\Gamma\Big(\frac{D}{2}+1\Big)e^{(2-\frac{D}{2})\log\left(\frac{y}{4}\right)}.$$

Then the Gamma functions and the exponent can be expand around $D = 4$ to get:

$$\frac{2}{D-4}\Big[\Big(\Gamma(3) + \frac{D-4}{2}\Gamma'(3) + \mathcal{O}((D-4)^2)\Big)\Big(1 - \frac{D-4}{2}\log\big(\frac{y}{4}\big) + \mathcal{O}((D-4)^2)\Big)\Big]$$
$$= \frac{2}{D-4}\Big[2 + (D-4)\Big(\psi(3) - \log\big(\frac{y}{4}\big)\Big)\Big] + \mathcal{O}(D-4),$$

where the ψ function is the digamma-function. The digamma-function is defined as:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

This expression is problematic if $D = 4$, but as will turn out by the definition of this propagator, the other terms will cancel the divergences. The term which should be rewritten is the cotangent, which is a 2π-periodic function. The cotangent term contains Gamma functions and trigonometric functions, which can be expanded around $D = 4$:

$$-\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big) = -\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\frac{\cos\left(\pi\frac{D}{2}\right)}{\sin\left(\pi\frac{D-4}{2}\right)}$$
$$= -\frac{\Gamma(3) + (D-4)\Gamma'(3) + \mathcal{O}(D-4)^2}{\Gamma(2) + \frac{(D-4)}{2}\Gamma'(2) + \mathcal{O}(D-4)^2}\pi\frac{1}{\pi\frac{D-4}{2} + \mathcal{O}(D-4)^3}$$
$$= \frac{-2}{D-4}\frac{\Gamma(3)}{\Gamma(2)}\Big(1 + (D-4)\psi(3) + \mathcal{O}(D-4)^2\Big)\Big(1 - \frac{(D-4)}{2}\psi(2) + \mathcal{O}(D-4)^2\Big)$$
$$= -\frac{4}{D-4}\Big(1 + (D-4)\psi(3) - \frac{(D-4)}{2}\psi(2)\Big) + \mathcal{O}(D-4).$$

These two results together with the logarithmic term in the propagator, can be written as one term that does not diverge as $D \to 4$:

$$
\begin{aligned}
&\frac{2}{D-4}\Big[2+(D-4)\Big(\psi(3)-\log\big(\frac{y}{4}\big)\Big)-2-(D-4)\big(2\psi(3)+\psi(2)\big)\Big]\\
&+2\log(aa')+\mathcal{O}(D-4)\\
=&\frac{2}{D-4}(D-4)\Big[\underbrace{\psi(2)-\psi(3)}_{=-\frac{1}{2}}-\log\big(\frac{y}{4}\big)\Big]+2\log(aa')+\mathcal{O}(D-4)\\
=&-1-2\log\big(\frac{y}{4}\big)+2\log(aa')+\mathcal{O}(D-4).
\end{aligned}
$$

Using the expression for y given in (85), the propagator can be written as:

$$
i\Delta_{dS}=\frac{H^2}{(4\pi)^2}\Big[\frac{-1}{1-\frac{D}{2}}\Gamma\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{1-\frac{D}{2}}-2\log\Big(\frac{\sqrt{e}H^2\Delta x_{++}^2}{4}\Big)\Big]+\mathcal{O}(D-4). \tag{132}
$$

The propagator squared is needed to calculate the self-mass. By squaring the propagator just found in (132), a lot of cross terms would be neglected. This because the limit $D\to 4$ made terms disappear, which will contribute when appearing in cross terms if the propagator is squared. The propagator found in equation 132 will be used, and will make the calculation easier. Before filling in every term, a schematic view of the propagator squared will make the terms which will vanish when $D\to 4$ visible:

$$
\begin{aligned}
i\Delta_{dS}=&\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Big[\alpha_{1-\frac{D}{2}}\big(\frac{y}{4}\big)^{1-\frac{D}{2}}+\Big(\alpha_{2-\frac{D}{2}}\big(\frac{y}{4}\big)^{2-\frac{D}{2}}+\alpha_{\cot}+\alpha_{\log}\Big)\\
&\qquad\sum_{n=0}^{\infty}\alpha_{n+3-\frac{D}{2}}\big(\frac{y}{4}\big)^{n+3-\frac{D}{2}}\Big].
\end{aligned}\tag{133}
$$

$$
\begin{aligned}
(i\Delta_{dS})^2=&\frac{H^{2D-4}}{(4\pi)^D}\Bigg[\alpha_{1-\frac{D}{2}}^2\big(\frac{y}{4}\big)^{2-D}+\Big(\alpha_{2-\frac{D}{2}}\big(\frac{y}{4}\big)^{2-\frac{D}{2}}+\alpha_{\cot}+\alpha_{\log}\Big)^2\\
&\Big(\sum_{n=0}^{\infty}\alpha_{n+3-\frac{D}{2}}\big(\frac{y}{4}\big)^{n+3-\frac{D}{2}}\Big)^2+2\alpha_{1-\frac{D}{2}}\alpha_{2-\frac{D}{2}}\big(\frac{y}{4}\big)^{3-D}\\
&+2(\alpha_{\cot}+\alpha_{\log})\alpha_{1-\frac{D}{2}}\big(\frac{y}{4}\big)^{1-\frac{D}{2}}\\
&+2*\alpha_{1-\frac{D}{2}}\sum_{n=0}^{\infty}\alpha_{n+3-\frac{D}{2}}\big(\frac{y}{4}\big)^{n+4-D}\\
&+2*\alpha_{2-\frac{D}{2}}\sum_{n=0}^{\infty}\alpha_{n+3-\frac{D}{2}}\big(\frac{y}{4}\big)^{n+5-D}\\
&+2*(\alpha_{\cot}+\alpha_{\log})\sum_{n=0}^{\infty}\alpha_{n+3-\frac{D}{2}}\big(\frac{y}{4}\big)^{n+3-\frac{D}{2}}\Bigg].
\end{aligned}\tag{134}
$$

Blindly filling in values would lead to a very big equation, but using the knowledge of the terms individually results in:

$$
\begin{aligned}
(i\Delta_{dS})^2 =&\frac{H^{2D-4}}{(4\pi)^D}\Bigg[\frac{1}{(1-\frac{D}{2})^2}\Gamma^2\Big(\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{2-D}+4\log^2\Big(\frac{\sqrt{e}H^2\Delta x_{++}^2}{4}\Big)\\
&\qquad+\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(1+\frac{D}{2})}{2-\frac{D}{2}}\big(\frac{y}{4}\big)^{3-D}\\
&\qquad+\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big)\big(\frac{y}{4}\big)^{1-\frac{D}{2}}\\
&\qquad-\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\big(\frac{y}{4}\big)^{1-\frac{D}{2}}\Bigg]\\
&+\mathcal{O}(D-4).
\end{aligned}
\tag{135}
$$

C Mass renormalization in $T = 0$, Minkowski universe

In this appendix the Minkowski case is considered. Using these calculations, the calculations in the FLRW universe will be easier to understand. This appendix follows the calculations which is done in section 3 of [10]. The self-masses can be found by looking at the propagators. Assuming that $m_\chi \to 0$ the propagator can be written as:

$$i\Delta_\chi^{++}(k^\mu) = \frac{-i}{k_\mu k^\mu + m_\chi^2 - i\epsilon} = \frac{-i}{k_\mu k^\mu - i\epsilon} \tag{136a}$$

$$i\Delta_\chi^{--}(k^\mu) = \frac{-i}{k_\mu k^\mu + m_\chi^2 + i\epsilon} = \frac{-i}{k_\mu k^\mu + i\epsilon} \tag{136b}$$

$$i\Delta_\chi^{+-}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(-k^0) = 2\pi\delta(k_\mu k^\mu)\theta(-k^0) \tag{136c}$$

$$i\Delta_\chi^{-+}(k^\mu) = 2\pi\delta(k_\mu k^\mu + m_\chi^2)\theta(k^0) = 2\pi\delta(k_\mu k^\mu)\theta(k^0). \tag{136d}$$

Due to divergences in the propagator, there are divergent terms in the self-mass (72). By first considering:

$$i\Delta_\chi^{++}(x;x') = \int \frac{d^D k}{(2\pi)^D} i\Delta_\chi^{++}(k^\mu) e^{ik(x-x')}. \tag{137}$$

This integral can be performed by two standard contour integrals, and the following identity :

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{i\vec{k}\vec{x}} f(k) = \frac{2}{(4\pi)^{(D-1/2)}} \times \int_0^\infty dk k^{D-2} \frac{J_{(D-3/2)}(kx)}{(\frac{1}{2}kx)^{(D-3/2)}} f(k). \tag{138}$$

$J(kx)$ is a Bessel function of the first kind, and $f(k)$ solely depends on $k = ||\vec{k}||$, resulting in

$$i\Delta_\chi^{++}(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{D/2}} \frac{1}{\Delta x^{D-2}(x,x')}. \tag{139}$$

$\Delta x^2(x,x')$ are frequently used distance function between two spacetime points in the Schwinger-Keldysh formalism. These are defined by:

$$\Delta x_{++}^2(x,x') = -(|t-t'| - i\epsilon)^2 + ||\vec{x} - \vec{x}'||^2 \tag{140a}$$

$$\Delta x_{+-}^2(x,x') = -(t-t' + i\epsilon)^2 + ||\vec{x} - \vec{x}'||^2 \tag{140b}$$

$$\Delta x_{-+}^2(x,x') = -(t-t' - i\epsilon)^2 + ||\vec{x} - \vec{x}'||^2 \tag{140c}$$

$$\Delta x_{--}^2(x,x') = -(|t-t'| + i\epsilon)^2 + ||\vec{x} - \vec{x}'||^2. \tag{140d}$$

We can then immediately write the self-mass using (72) :

$$\begin{aligned} iM_\phi^{++}(x,x') &= \frac{-ih^2}{2}\left(i\Delta_\chi^{++}(x,x')\right)^2 \\ &= \frac{-ih^2}{2} \frac{\Gamma(\frac{D}{2}-1)}{16\pi^D} \frac{1}{\Delta x_{++}^{2D-4}(x,x')}. \end{aligned} \tag{141}$$

This expression can be rewritten using two identities. These identities will be crucial for calculations, which we can use as a mathematical trick:

$$\frac{1}{\Delta x_{++}^{\beta}(x,x')} = \frac{1}{(\beta-2)}\frac{1}{(\beta-D)}\partial^2\frac{1}{\Delta x_{++}^{\beta-2}(x,x')} \qquad \text{for } \beta \neq \{2, D\} \tag{142a}$$

$$\partial^2\frac{1}{\Delta x_{\pm\pm}^{D-2}(x,x')} = \pm\frac{4\pi^{D/2}}{\Gamma(\frac{D-2}{2})}i\delta^D(x-x'). \tag{142b}$$

$$\partial^2\frac{1}{\Delta x_{\pm\mp}^{D-2}(x,x')} = 0. \tag{142c}$$

Using these identities, we can fill in $\beta = 2D-4$, and add and subtract the same term. The self-mass can be written as:

$$\begin{aligned} iM_{\phi}^{++}(x,x') = &-\frac{ih^2\Gamma^2(\frac{D}{2}-1)}{64\pi^D}\frac{1}{(D-3)(D-4)} \\ &\times\Big[\partial^2\Big(\frac{1}{\Delta x_{++}^{2D-6}(x,x')}-\frac{\mu^{D-4}}{\Delta x_{++}^{D-2}(x,x')}\Big) \\ &+\frac{4\pi^{D/2}\mu^{D-4}}{\Gamma(\frac{D-2}{2})}i\delta^D(x-x')\Big]. \end{aligned} \tag{143}$$

μ is introduced to use the identity written in (142b) in the same dimension as the other terms. If we know Taylor expand the term in the curly brackets around $D=4$ we can see what consequences this mathematical trick has:

$$\begin{aligned} iM_{\phi}^{++}(x,x') = &-\frac{ih^2\Gamma(\frac{D}{2}-1)}{16\pi^{D/2}}\frac{\mu^{D-4}}{(D-3)(D-4)}i\delta^D(x-x') \\ &-\frac{ih^2\Gamma^2(\frac{D}{2}-1)}{64\pi^D}\partial^2\Big[\frac{\log(\mu^2\Delta^2 x_{++}(x,x'))}{2\Delta_{++}^2(x,x')}+\mathcal{O}(D-4)\Big]. \end{aligned} \tag{144}$$

Now we can see that we only have one local $(D-4)^{-1}$ divergence, which we can solve by introducing counter terms. This counter mass-term can then be added by the original mass term, such that the divergences cancel.

$$iM_{\phi,ct}^{\pm\pm}(x,x') = \pm\frac{ih^2\Gamma(\frac{D}{2}-1)}{16\pi^{D/2}}\frac{\mu^{D-4}}{(D-3)(D-4)}i\delta^D(x-x'). \tag{145}$$

The $+-$ and the $-+$ term do not need to be renormalized, because they do not contain any divergences.

$$iM_{\phi,Ren}^{++}(x,x') = -\frac{ih^2\Gamma^2(\frac{D}{2}-1)}{128\pi^D}\partial^2\Big[\frac{\log\left(\mu^2\Delta x_{++}^2(x,x')\right)}{\Delta x_{++}^2(x,x')}\Big]. \tag{146}$$

This is the renormalized mass, for solving the dynamics we can look at the Fourier transform. But the point was to get familiar with the mathematical trick where we used equation 142a and 142b.

D Reducing the d'Alembertian divergence

In Appendix C a mathematical trick was used to deal with a divergent term. In 5.2 a generalization of this trick was used, these identities will be confirmed in this appendix:

$$(\frac{y}{4})^{-\alpha} = -\frac{1}{(\alpha-1)(\frac{D}{2}-\alpha)}\frac{\square}{H^2}(\frac{y}{4})^{1-\alpha} + \frac{D-\alpha}{\frac{D}{2}-\alpha}(\frac{y}{4})^{1-\alpha} \text{ ,with } \alpha \neq \frac{D}{2} \tag{147a}$$

$$\frac{\square}{H^2}(\frac{y}{4})^{1-\frac{D}{2}} = \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D} i\delta^D(x-x') + \frac{D(D-2)}{4}(\frac{y}{4})^{1-\frac{D}{2}} \tag{147b}$$

Starting with generalizing the first of the two (142a). As usual the partial derivatives will be changed to the d'Alembertian operator. Using the identity for non-singular functions $F(y)$:

$$\frac{\square}{H^2}F(y) = (4-y)yF''(y) + D(2-y)F'(y). \tag{148}$$

Substituting $F(y) = (\frac{y}{4})^{1-\alpha}$ and plug this in the identity above will lead to expressions that depend on $F'(y)$ and $F''(y)$:

$$\begin{aligned} F(y) &= (\frac{y}{4})^{1-\alpha} \\ D(2-y)F'(y) &= \frac{D(1-\alpha)}{2}(\frac{y}{4})^{-\alpha} - D(1-\alpha)(\frac{y}{4})^{1-\alpha} \\ (4-y)yF''(y) &= -(1-\alpha)\alpha(\frac{y}{4})^{-\alpha} + (1-\alpha)\alpha(\frac{y}{4})^{1-\alpha} \\ (\frac{y}{4})^{-\alpha} &= \frac{-1}{(\frac{D}{2}-\alpha)(\alpha-1)}\frac{\square}{H^2}(\frac{y}{4})^{1-\alpha} + \frac{(D-\alpha)}{(\frac{D}{2}-\alpha)}(\frac{y}{4})^{1-\alpha} \text{ ,with } \alpha \neq \frac{D}{2} \end{aligned}$$

This confirms the first identity, for the second identity $F(y)$ is set to: $\left(\frac{y}{4}\right)^{1-\frac{D}{2}}$, which makes it singular, Therefore the identity in equation (148) cannot be used. Starting with the identity of the flat case 142b:

$$\partial^2 \frac{1}{\Delta x_{++}^{D-2}(x,x')} = \frac{4\pi^{D/2}}{\Gamma(\frac{D-2}{2})} i\delta^D(x-x'). \tag{149}$$

Plugging in the definition of the d'Alembartian operator and the definition of y (85) this can be written as the second:

$$\frac{\square}{H^2}(\frac{y}{4})^{1-\frac{D}{2}} = \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D} i\delta^D(x-x') + \frac{D(D-2)}{4}(\frac{y}{4})^{1-\frac{D}{2}}.$$

This only is true for the case that Δx_{++} is used, the other cases are:

$$\begin{aligned} \frac{\square}{H^2}(\frac{y_{--}}{4})^{1-\frac{D}{2}} &= -\frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D} i\delta^D(x-x') + \frac{D(D-2)}{4}(\frac{y_{--}}{4})^{1-\frac{D}{2}} \\ \frac{\square}{H^2}(\frac{y_{\pm\mp}}{4})^{1-\frac{D}{2}} &= \frac{D(D-2)}{4}(\frac{y_{\pm\mp}}{4})^{1-\frac{D}{2}}. \end{aligned}$$

E Self-mass terms in the $D \to 4+\tilde{\epsilon}$ limit

This appendix is very similar to appendix C. Because the self-mass is defined as the propagator squared, the propagator squared terms are the interesting terms. Using the definitions of the identities derived in appendix D, the first term can be rewritten by applying the same trick, adding zero in the form of equation 90b:

$$\begin{aligned}
(\frac{y}{4})^{2-D} &= -\frac{1}{(D-3)(2-\frac{D}{2})}\frac{\Box}{H^2}(\frac{y}{4})^{3-D} + \frac{2}{2-\frac{D}{2}}(\frac{y}{4})^{3-D} \\
&= \frac{2}{(D-3)(D-4)}\Bigg\{\Big[\frac{\Box}{H^2} - \frac{(D-3)(D-4)}{2}\frac{4}{D-4}\Big](\frac{y}{4})^{3-D} \\
&\quad \underbrace{+(\frac{2\mu}{H})^{D-4}\Bigg(\Big[\frac{D(D-2)}{4} - \frac{\Box}{H^2}\Big](\frac{y}{4})^{1-\frac{D}{2}} + \frac{(4\pi)^{D/2} i\delta^D(x-x')}{\Gamma(\frac{D}{2}-1)(Ha)^D}\Bigg)}_{=0,\text{ The factor }(2\mu/H)^{D-4}\text{is introduced to make the counter-term not dependent on}H}\Bigg\} \\
&= \frac{2}{(D-3)(D-4)}\frac{\Box}{H^2}\Big[(\frac{y}{4})^{3-D} - (\frac{2\mu}{H})^{D-4}(\frac{y}{4})^{1-\frac{D}{2}}\Big] \\
&\quad - \frac{4}{D-4}(\frac{y}{4})^{3-D} + \frac{D(D-2)(\frac{2\mu}{H})^{D-4}}{2(D-3)(D-4)}(\frac{y}{4})^{1-\frac{D}{2}} \\
&\quad \underbrace{+\frac{2(4\pi)^{D/2}}{(D-3)(D-4)\Gamma(\frac{D}{2}-1)(Ha)^D}(\frac{2\mu}{H})^{D-4} i\delta^D(x-x')}_{\text{From now we will call this: L.T.}} \\
&\overset{(D\to 4+\tilde{\epsilon})}{=} \text{L.T.}(D) - \frac{\Box}{H^2}\Big[(\frac{4}{y})\log(\frac{\mu^2 y}{H^2})\Big] + \frac{4}{y}\Big(2\log(\frac{\mu^2 y}{H^2}) - 1\Big)
\end{aligned}$$

The L.T. term is a local divergent term, these terms can eventually be canceled when the mass is renormalized. All Gamma functions will also be expanded, therefore using:

$$\Gamma\Big(\frac{D}{2}+1\Big) \to \Gamma(3)\Big(1+\frac{\tilde{\epsilon}}{2}\phi(3)\Big) + \mathcal{O}(\tilde{\epsilon}^2) \tag{150}$$

$$\Gamma\Big(\frac{D}{2}\Big) \to \Gamma(2)\Big(1+\frac{\tilde{\epsilon}}{2}\phi(2)\Big) + \mathcal{O}(\tilde{\epsilon}^2). \tag{151}$$

The second term of the propagator squared (89) becomes:

$$\begin{aligned}
&\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(1+\frac{D}{2})}{2-\frac{D}{2}}(\frac{y}{4})^{3-D} \\
&= \frac{8\Gamma(\frac{D}{2})\Gamma(1+\frac{D}{2})}{(D-2)(D-4)}\Big[\frac{2}{(D-4)(D-6)}\frac{\Box}{H^2}(\frac{y}{4})^{4-D} - \frac{6}{D-6}(\frac{y}{4})^{4-D}\Big] \\
&\overset{(D\to 4+\tilde{\epsilon})}{=} \underbrace{2\frac{\Box}{H^2}1}_{=0} - 24\log(\frac{y}{4}) - 4\frac{\Box}{H^2}\log^2(\frac{y}{4}) + 12(\phi(2)+\phi(3)) \\
&\quad + 4(\phi(2)+\phi(3))\frac{\Box}{H^2}\log(\frac{y}{4}) + \frac{1}{\tilde{\epsilon}}\Big[24 + 8\frac{\Box}{H^2}\log(\frac{y}{4})\Big] + \mathcal{O}(\tilde{\epsilon}).
\end{aligned}$$

The third term in the propagator squared formula is the cotangent term. This term is already expanded in the derivation of the propagator (132). Using that result, and combine it with the expansion of the other term:

$$\begin{aligned}
&\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big)(\frac{y}{4})^{1-\frac{D}{2}}\\
=&\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\pi\frac{D}{2}\Big)\\
&\times\Big[\frac{-2}{(D-4)}\frac{\Box}{H^2}(\frac{y}{4})^{\frac{4-D}{2}}+(\frac{D}{2}+1)(\frac{y}{4})^{\frac{4-D}{2}}\Big]\\
\overset{(D\to4+\tilde{\epsilon})}{=}&\frac{8}{\tilde{\epsilon}}(-1+\frac{\tilde{\epsilon}}{2})\Big[1+\frac{\tilde{\epsilon}}{2}\phi(2)\Big]\Big[1+\tilde{\epsilon}\phi(3)-\frac{\tilde{\epsilon}}{2}\phi(2)\Big]\\
&\times\Big[\Big(-\frac{2}{\tilde{\epsilon}}\frac{\Box}{H^2}+(3+\frac{\tilde{\epsilon}}{2})\Big)\Big(1-\frac{\tilde{\epsilon}}{2}\log(\frac{y}{4})+\frac{\tilde{\epsilon}^2}{8}\log(\frac{y}{4})\Big)\Big]+\mathcal{O}(\tilde{\epsilon})\\
=&8+12\log(\frac{y}{4})+4\frac{\Box}{H^2}\log(\frac{y}{4})+2\frac{\Box}{H^2}\log^2(\frac{y}{4})\\
&-24\phi(3)-8\phi(3)\frac{\Box}{H^2}\log(\frac{y}{4})+\frac{1}{\tilde{\epsilon}}\Big[-24-8\frac{\Box}{H^2}\log(\frac{y}{4})\Big].
\end{aligned}$$

Combining the two previous results, we can directly obtain that the only remaining term is:

$$-12\log(\frac{y}{4})+2\frac{\Box}{H^2}\log(\frac{y}{4})-2\frac{\Box}{H^2}\log^2(\frac{y}{4})+2.$$

The last term in the propagator squared formula is the logarithmic term:

$$\begin{aligned}
&-\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')(\frac{y}{4})^{1-\frac{D}{2}}\\
=&-\frac{2}{(1-\frac{D}{2})}\Gamma\Big(\frac{D}{2}\Big)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\log(aa')\\
&\times\Big[\frac{-2}{(D-4)}\frac{\Box}{H^2}(\frac{y}{4})^{\frac{4-D}{2}}+(\frac{D}{2}+1)(\frac{y}{4})^{\frac{4-D}{2}}\Big]\\
\overset{(D\to4+\tilde{\epsilon})}{=}&12\log(aa')+4\log(aa')\frac{\Box}{H^2}\log(\frac{y}{4}).
\end{aligned}$$

In total we can write the propagator squared as:

$$\begin{aligned}
(i\Delta_{dS})^2=\frac{H^{2D-4}}{(4\pi)^D}\Bigg[&\text{L.T.}(D)-\frac{\Box}{H^2}\Big[(\frac{4}{y})\log(\frac{\mu^2 y}{H^2})\Big]+\frac{4}{y}\Big(2\log(\frac{\mu^2 y}{H^2})-1\Big)\\
&-8\log(\frac{y}{4})+2\frac{\Box}{H^2}\log(\frac{y}{4})-2\frac{\Box}{H^2}\log^2(\frac{y}{4})+8\log(aa')\\
&+4\log(aa')\frac{\Box}{H^2}\log(\frac{y}{4})+4\log^2(\frac{y}{4})-8\log(aa')\log(\frac{y}{4})\\
&+4\log^2(aa')+3\Bigg].
\end{aligned}\tag{152}$$

F Fourier transform of the constant terms of the renormalized self-mass

In section 5 we found a renormalized mass (93), in this appendix we will compute the spatial Fourier integral over the constant terms. Starting with the simple integrals, writing $(\vec{x}-\vec{x}')$ in terms of r, the constant terms are:

$$\begin{aligned}
&\int d^3(x-x')(8\log(aa')+4\log^2(aa')-4)e^{-i\vec{k}(\vec{x}-\vec{x}')}\\
=&2\pi\int_0^\infty drr^2\int_0^\pi d\theta\sin(\theta)e^{-ik\cos(\theta)r}(8\log(aa')+4\log^2(aa')-4)\\
=&-2\pi\int_0^\infty drr^2\int_1^{-1} d(\cos\theta)e^{-ik\cos(\theta)r}(8\log(aa')+4\log^2(aa')-4)\\
=&4\pi(8\log(aa')+4\log^2(aa')-4)\int_0^\infty drr^2\frac{\sin(kr)}{kr}\\
=&\frac{4\pi}{k}(8\log(aa')+4\log^2(aa')-4)\int_0^\infty drr\sin(kr).
\end{aligned}$$

Even in the constant term integrals there appears an divergent integral which can be evaluated using analytic extensions such that the ultraviolet divergences can be neglected.

$$\begin{aligned}
\int_0^\infty drr\sin[kr]&=\lim_{\delta\to0}\int_0^\infty dr\frac{re^{(ik-\delta)r}-re^{(ik-\delta)r}}{2i}\\
&=\frac{1}{2i}\lim_{\delta\to0}\Big[\frac{e^{(ik-\delta)r}(-1-\delta r+ikr)}{(\delta-ik)^2}+\frac{e^{(-ik-\delta)r}(1+\delta r+ikr)}{(\delta+ik)^2}\Big]_0^\infty.
\end{aligned}$$

The real part of the exponent is negative, and therefore will converge if $r\to\infty$ and the result is:

$$\begin{aligned}
&\lim_{\delta\to0}\frac{1}{2i}\Big[\frac{1}{(ik-\delta)^2}-\frac{1}{(ik+\delta)^2}\Big]\\
=&\lim_{\delta\to0}\Big[\frac{2k\delta}{(k^2+\delta^2)^2}\Big]=\lim_{\delta\to0}\partial_k\frac{-\delta}{(k^2+\delta^2)}=-\pi\partial_k\delta(k).
\end{aligned}\tag{153}$$

Terms up to $\delta(k)$ and its derivatives are not interesting in this model. Because fields proportional to $\delta(k)$ can be absorbed into the scalar field condensate, which is defined as the expectation value of the scalar field operator. It is possible to write this as a homogeneous and an inhomogeneous part: $\phi=\phi_0+\delta\phi$. The field condensate ϕ_0 has been set to zero. Therefore the constant terms do not contribute to $iM^{++}_{\phi,Ren}(\vec{x},\eta,\eta')$.

G Fourier transform of the $\log(\frac{y}{4})$ terms of the renormalized self-mass

As in appendix F, we look at the found renormalized mass (93), in this appendix we will compute the spatial Fourier integral over the logarithmic terms. For the log term the definition of y (given in equation (85)) is used. Because the ++ propagator is used, the ϵ comes with a minus i. The term can be split up into two parts as showed in this calculation. Therefore the logarithm will be written as:

$$\log(-\Delta\eta^2 + r^2 - i\tilde{\epsilon}) = \log\left(|-\Delta\eta^2 + r^2|\right) + i\pi\theta(\Delta\eta - r), \tag{154}$$

where θ is the Heaviside step function. The integral becomes:

$$\begin{aligned}
&\int d^{D-1}(\vec{x}-\vec{x}')\log(\frac{y}{4})e^{i\vec{k}(\vec{x}-\vec{x}')} \\
&= \frac{4\pi}{k}\int_0^\infty dr r\sin(kr)\log\Big(\frac{aa'H^2(-\Delta\eta^2+r^2)}{4}\Big) \\
&= 0 + \frac{4\pi}{k}\int_0^\infty dr r\sin(kr)\Big[\log\big(|\frac{r^2}{(\Delta\eta)^2} - 1|\big) + i\pi\theta(\Delta\eta^2 - r^2)\Big] \\
&= \frac{4\pi}{k}\Big[(\Delta\eta)^2\int_0^\infty dz z\sin(k\Delta\eta z)\log\big(|z^2-1|\big) + i\pi\int_0^{\Delta\eta} dr r\sin(kr)\Big].
\end{aligned}$$

Starting with the last integral, which is the easiest, and for simplicity it is called L_0:

$$iL_0 = \frac{4i\pi^2}{k}\int_0^\infty dr r\sin(kr)\theta\Big(r^2 - (\Delta\eta)^2\Big) = \frac{4i\pi^2}{k^3}\Big(\sin\left(k\Delta\eta\right) - k\Delta\eta\cos(k\Delta\eta)\Big). \tag{155}$$

The other integral is actually the interesting part, and it can be split into two regions:

$$\int_0^1 dz z\sin(k\Delta\eta z)\log\left(1-z^2\right) + \int_1^\infty dz z\sin(k\Delta\eta z)\log\left(z^2-1\right) \tag{156}$$

The first integral is done in [17]:

$$\begin{aligned}
z^2\zeta(z) \equiv& z^2\int_0^1 z'dz'\sin(zz')\log(1-z'^2) \\
=& 2\sin(z) - \Big[\cos(z) + z\sin(z)\Big]\Big[\text{Si}(2z) + \pi/2\Big] \\
&+ \Big[\sin(z) - z\cos(z)\Big]\Big[\text{Ci}(2z) - \gamma_E - \log(z/2)\Big],
\end{aligned} \tag{157}$$

with

$$\mathrm{Si}(z) = -\int_{z}^{\infty} \frac{\sin(t)}{t} dt \tag{158a}$$

$$\mathrm{Ci}(z) = -\int_{z}^{\infty} \frac{\cos(t)}{t} dt. \tag{158b}$$

The second integral can be written as:

$$\begin{aligned}
&(\Delta\eta)^2 \int_{1}^{\infty} dz z \sin(k\Delta\eta z) \log\left(z^2 - 1\right) \\
=&(\Delta\eta)^2 \int_{1}^{\infty} dz z \sin(k\Delta\eta z) \Big[\log\left(z-1\right) + \log\left(z+1\right) \Big] \\
=&(\Delta\eta)^2 \int_{0}^{\infty} dv (v+1) \sin\Big(k\Delta\eta(v+1)\Big) \log(v) \\
&+ (\Delta\eta)^2 \int_{2}^{\infty} dw (w-1) \sin\Big(k\Delta\eta(w-1)\Big) \log(w) \\
=&(\Delta\eta)^2 \int_{0}^{\infty} dv \frac{(v+1)}{2i} \Big[e^{ik\Delta\eta v} e^{ik\Delta\eta} - e^{-ik\Delta\eta v} e^{-ik\Delta\eta} \Big] \log(v) \\
&+ (\Delta\eta)^2 \int_{2}^{\infty} dw \frac{(w-1)}{2i} \Big[e^{ik\Delta\eta w} e^{-ik\Delta\eta} - e^{-ik\Delta\eta w} e^{ik\Delta\eta} \Big] \log(w).
\end{aligned}$$

This results in four different integrals:

$$\begin{aligned}
(\Delta\eta)^2 \lim_{\delta\to 0} \Bigg\{ & e^{(ik\Delta\eta - \delta)} \int_{0}^{\infty} dv \frac{(v+1)}{2i} e^{(ik\Delta\eta - \delta)v} \log(v) \\
& - e^{(-ik\Delta\eta - \delta)} \int_{0}^{\infty} dv \frac{(v+1)}{2i} e^{(-ik\Delta\eta - \delta)v} \log(v) \\
& + e^{(-ik\Delta\eta + \delta)} \int_{2}^{\infty} dw \frac{(w-1)}{2i} e^{(ik\Delta\eta - \delta)w} \log(w) \\
& - e^{(ik\Delta\eta + \delta)} \int_{2}^{\infty} dw \frac{(w-1)}{2i} e^{(-ik\Delta\eta - \delta)w} \log(w) \Bigg\}.
\end{aligned} \tag{159}$$

These four integrals can be done in two steps. The last two integrals can be done using partial integration, as done on the next page.

$$
\begin{aligned}
&(\Delta\eta)^2 e^{(-ik\Delta\eta+\delta)} \int_2^\infty dw \frac{(w-1)}{2i} e^{(ik\Delta\eta-\delta)w} \log(w) \\
&- (\Delta\eta)^2 e^{(ik\Delta\eta+\delta)} \int_2^\infty dw \frac{(w-1)}{2i} e^{(-ik\Delta\eta-\delta)w} \log(w) \\
=&(\Delta\eta)^2 \frac{e^{(-ik\Delta\eta+\delta)}}{2i} \Big[e^{2(ik\Delta\eta-\delta)} \frac{\delta - ik\Delta\eta + 1}{(ik\Delta\eta-\delta)^2} \log(2) + \frac{e^{2(ik\Delta\eta-\delta)}}{(ik\Delta\eta-\delta)^2} \\
&\qquad - \int_2^\infty dw e^{(ik\Delta\eta-\delta)w} \frac{\delta - ik\Delta\eta - 1}{w(ik\Delta\eta-\delta)^2} \Big] \\
&- (\Delta\eta)^2 \frac{e^{(ik\Delta\eta+\delta)}}{2i} \Big[e^{-2(ik\Delta\eta+\delta)} \frac{\delta + ik\Delta\eta + 1}{(ik\Delta\eta+\delta)^2} \log(2) + \frac{e^{-2(ik\Delta\eta+\delta)}}{(ik\Delta\eta+\delta)^2} \\
&\qquad - \int_2^\infty dw e^{(-ik\Delta\eta-\delta)w} \frac{\delta + ik\Delta\eta - 1}{w(ik\Delta\eta+\delta)^2} \Big] \\
=&\frac{(\Delta\eta)^2}{2i} \Bigg[e^{ik\Delta\eta+\delta} \frac{\delta - ik\Delta\eta + 1}{(ik\Delta\eta-\delta)^2} \log(2) + \frac{e^{ik\Delta\eta-\delta}}{(ik\Delta\eta-\delta)^2} \\
&\qquad - \frac{e^{(-ik\Delta\eta+\delta)}(\delta - ik\Delta\eta - 1)}{(ik\Delta\eta-\delta)^2} \int_{2(ik\Delta\eta-\delta)}^\infty dz \frac{e^z}{z} \Bigg] \\
&- \frac{(\Delta\eta)^2}{2i} \Bigg[e^{-ik\Delta\eta-\delta} \frac{\delta + ik\Delta\eta + 1}{(ik\Delta\eta+\delta)^2} \log(2) + \frac{e^{-ik\Delta\eta-\delta}}{(ik\Delta\eta+\delta)^2} \\
&\qquad - \frac{e^{(ik\Delta\eta+\delta)}(\delta + ik\Delta\eta + 1)}{(ik\Delta\eta+\delta)^2} \int_{2(-ik\Delta\eta-\delta)}^\infty dz \frac{e^z}{z} \Bigg] \\
=&\frac{(\Delta\eta)^2}{(\delta^2 + (k\Delta\eta)^2)^2} \Bigg\{ e^{-\delta} \sin(k\Delta\eta) \\
&\qquad \times \Big[\Big(\delta^3 + \delta^2 + \delta k^2 \Delta\eta^2 - k^2 \Delta\eta^2\Big) \log(2) + \Big(\delta^2 - (k\Delta\eta)^2\Big) \Big] \\
&\quad + e^{-\delta} \cos(k\Delta\eta) \Big[\Big(\delta^2 k\Delta\eta + 2\delta k\Delta\eta + k^3(\Delta\eta)^3\Big) \log(2) + 2k\Delta\eta\delta \Big] \\
&\quad + \frac{e^{(ik\Delta\eta+\delta)}}{2i} \Big(\delta^3 - i\delta^2 k\Delta\eta - \delta^2 + \delta(k\Delta\eta)^2 + 2i\delta k\Delta\eta - i(k\Delta\eta)^3 + (k\Delta\eta)^2\Big) \\
&\qquad \times \int_{2(-ik\Delta\eta-\delta)}^\infty \Big[dz \frac{e^z}{z} \Big] \\
&\quad - \frac{e^{(-ik\Delta\eta+\delta)}}{2i} \Big(\delta^3 + i\delta^2 k\Delta\eta - \delta^2 + \delta(k\Delta\eta)^2 - 2i\delta k\Delta\eta + i(k\Delta\eta)^3 + (k\Delta\eta)^2\Big) \\
&\qquad \times \int_{2(ik\Delta\eta-\delta)}^\infty \Big[dz \frac{e^z}{z} \Big] \Bigg\}.
\end{aligned}
$$

The last two integrals can be rewritten using equation 158 and:

$$\int\limits_{a+\delta}^{b} f(x) = F(b) - F(a+\delta) = F(b) - F(a) - \delta F'(a) - \frac{\delta^2}{2}F''(a) + \cdots \tag{160}$$

$$\int\limits_{p}^{\infty} du \frac{e^{iu}}{u} = \int\limits_{p}^{\infty} \frac{\cos(u) + i\sin(u)}{u} = -\mathrm{Ci}(p) - i\mathrm{Si}(p) \tag{161}$$

$$\int\limits_{2(\pm ik\Delta\eta - \delta)}^{\infty} dz \frac{e^z}{z} = \int\limits_{\pm 2ik\Delta\eta}^{\infty} dz \frac{e^z}{z} \mp 2i\delta \frac{e^{\pm 2ik\Delta\eta}}{2ik\Delta\eta} + (2\delta)^2 \Big(\pm \frac{e^{\pm 2ik\Delta\eta}}{2ik\Delta\eta} - \frac{e^{\pm 2ik\Delta\eta}}{(2ik\Delta\eta)^2} \Big) + \cdots$$

Then we get:

$$\begin{aligned}
=& \frac{(\Delta\eta)^2}{(\delta^2 + (k\Delta\eta)^2)^2} \Bigg\{ e^{-\delta} \sin(k\Delta\eta) \Big[\Big(\delta^3 + \delta^2 + \delta k^2 \Delta\eta^2 - k^2\Delta\eta^2 \Big) \log(2) \\
&+ (\delta^2 - (k\Delta\eta)^2) \Big] \\
&+ e^{-\delta} \cos(k\Delta\eta) \Big[\Big(\delta^2 k\Delta\eta + 2\delta k\Delta\eta + k^3(\Delta\eta)^3 \Big) \log(2) + 2k\Delta\eta\delta \Big] \\
&+ e^{\delta} \sin(k\Delta\eta) \Big[-\delta^3 + \delta^2 - \delta(k\Delta\eta)^2 - (k\Delta\eta)^2 \Big] \mathrm{Ci}(2k\Delta\eta) \\
&+ e^{\delta} \cos(k\Delta\eta) \Big[\delta^2 k\Delta\eta - 2\delta k\Delta\eta + (k\Delta\eta)^3 \Big] \mathrm{Ci}(2k\Delta\eta) \\
&+ e^{\delta} \sin(k\Delta\eta) \Big[\delta^2 k\Delta\eta - 2\delta k\Delta\eta + (k\Delta\eta)^3 \Big] \mathrm{Si}(2k\Delta\eta) \\
&+ e^{\delta} \cos(k\Delta\eta) \Big[\delta^3 - \delta^2 + \delta(k\Delta\eta)^2 + (k\Delta\eta)^2 \Big] \mathrm{Si}(2k\Delta\eta) \\
&+ \frac{e^{(ik\Delta\eta+\delta)}}{2i} \Big(\delta^3 - i\delta^2 k\Delta\eta - \delta^2 + \delta(k\Delta\eta)^2 + 2i\delta k\Delta\eta - i(k\Delta\eta)^3 + (k\Delta\eta)^2 \Big) \\
&\times \Big[-2i\delta \frac{e^{2ik\Delta\eta}}{2ik\Delta\eta} + (2\delta)^2 \Big(\frac{e^{2ik\Delta\eta}}{2ik\Delta\eta} - \frac{e^{2ik\Delta\eta}}{(2ik\Delta\eta)^2} \Big) + \cdots \Big] \\
&- \frac{e^{(-ik\Delta\eta+\delta)}}{2i} \Big(\delta^3 + i\delta^2 k\Delta\eta - \delta^2 + \delta(k\Delta\eta)^2 - 2i\delta k\Delta\eta + i(k\Delta\eta)^3 + (k\Delta\eta)^2 \Big) \\
&\times \Big[+2i\delta \frac{e^{-2ik\Delta\eta}}{2ik\Delta\eta} + (2\delta)^2 \Big(-\frac{e^{-2ik\Delta\eta}}{2ik\Delta\eta} - \frac{e^{-2ik\Delta\eta}}{(2ik\Delta\eta)^2} \Big) + \cdots \Big] \Bigg\}.
\end{aligned}$$

Again there appears an expression which can be written as a delta function, as in equation 153. Using partial integration:

$$\frac{z^2\delta}{(\delta^2 + z^2)^2} = -\frac{\pi}{2} z \partial_z \delta(z) = \frac{\pi}{2}\delta(z). \tag{162}$$

Therefore the only important terms are the terms which do not contain a δ (between the brackets):

$$
\begin{aligned}
=&\frac{(\Delta\eta)^2}{\left(\delta^2+(k\Delta\eta)^2\right)^2}\Bigg\{\\
&\sin(k\Delta\eta)\Big[-k^2\Delta\eta^2\log(2)-(k\Delta\eta)^2-(k\Delta\eta)^2\mathrm{Ci}(2k\Delta\eta)+(k\Delta\eta)^3\mathrm{Si}(2k\Delta\eta)\Big]\\
&+\cos(k\Delta\eta)\Big[k^3(\Delta\eta)^3\log(2)+(k\Delta\eta)^3\mathrm{Ci}(2k\Delta\eta)+(k\Delta\eta)^2\mathrm{Si}(2k\Delta\eta)\Big]\Bigg\}.
\end{aligned}
$$

The two remaining integrals of expression 159 are written on the next page.

$$(\Delta\eta)^2 e^{(ik\Delta\eta-\delta)} \int\limits_0^\infty dv \frac{(v+1)}{2i} e^{(ik\Delta\eta-\delta)v} \log(v)$$

$$- (\Delta\eta)^2 e^{(-ik\Delta\eta-\delta)} \int\limits_0^\infty dv \frac{(v+1)}{2i} e^{(-ik\Delta\eta-\delta)v} \log(v)$$

$$= \lim_{\epsilon\to 0} (\Delta\eta)^2 \Big\{ e^{(ik\Delta\eta-\delta)} \Big[\frac{(1-(1+\epsilon)(ik\Delta\eta-\delta))}{2i(ik\Delta\eta-\delta)^2} \log(\epsilon) e^{(ik\Delta\eta-\delta)\epsilon}$$

$$- \int_\epsilon^\infty dv \frac{\big(-1+(1+v)(ik\Delta\eta-\delta)\big)}{2iv(ik\Delta\eta-\delta)^2} e^{(ik\Delta\eta-\delta)v} \Big]$$

$$- e^{(-ik\Delta\eta-\delta)} \Big[\frac{(1-(1+\epsilon)(-ik\Delta\eta-\delta))}{2i(ik\Delta\eta+\delta)^2} \log(\epsilon) e^{(ik\Delta\eta-\delta)\epsilon}$$

$$- \int_\epsilon^\infty dv \frac{\big(-1+(1+v)(-ik\Delta\eta-\delta)\big)}{2iv(ik\Delta\eta-\delta)^2} e^{(ik\Delta\eta-\delta)v} \Big] \Big\}$$

$$= \lim_{\epsilon\to 0} (\Delta\eta)^2 \Bigg\{$$

$$\frac{e^{-\delta}\log(\epsilon)}{\big(\delta^2+k^2\Delta\eta^2\big)^2} \Big[\sin(k\Delta\eta) \Big(\delta^2+\delta^3+\delta^3\epsilon-k^2\Delta\eta^2+\delta k^2\Delta\eta^2+\delta k^2\Delta\eta^2\epsilon \Big)$$

$$+ \cos(k\Delta\eta) \Big(2\delta k\Delta\eta+\delta^2 k\Delta\eta+\delta^2 k\Delta\eta\epsilon+k^3(\Delta\eta)^3+k^3(\Delta\eta)^3\epsilon \Big) \Big]$$

$$+ e^{(ik\Delta\eta-\delta)} \Bigg[\frac{e^{(ik\Delta\eta-\delta)\epsilon}}{2i(ik\Delta\eta-\delta)^2} - \frac{-1+ik\Delta\eta-\delta}{2i(ik\Delta\eta-\delta)^2} \int_{\epsilon(ik\Delta\eta-\delta)}^\infty dv \frac{e^v}{v} \Bigg]$$

$$- e^{(-ik\Delta\eta-\delta)} \Big[\frac{e^{(-ik\Delta\eta-\delta)\epsilon}}{2i(ik\Delta\eta+\delta)^2} - \frac{-1-ik\Delta\eta-\delta}{2i(ik\Delta\eta+\delta)^2} \int_{\epsilon(-ik\Delta\eta-\delta)}^\infty dv \frac{e^v}{v} \Big] \Bigg\}$$

$$= \lim_{\epsilon\to 0} (\Delta\eta)^2 \Bigg\{$$

$$\frac{e^{-\delta}\log(\epsilon)}{\big(\delta^2+k^2\Delta\eta^2\big)^2} \Big[\sin(k\Delta\eta) \Big(\delta^2+\delta^3+\delta^3\epsilon-k^2\Delta\eta^2+\delta k^2\Delta\eta^2+\delta k^2\Delta\eta^2\epsilon \Big)$$

$$+ \cos(k\Delta\eta) \Big(2\delta k\Delta\eta+\delta^2 k\Delta\eta+\delta^2 k\Delta\eta\epsilon+k^3(\Delta\eta)^3+k^3(\Delta\eta)^3\epsilon \Big) \Big]$$

$$+ \frac{e^{-\delta}}{\big((k\Delta\eta)^2+\delta^2\big)^2} \Bigg[\sin(k\Delta\eta) \Big(\delta^2-k^2\Delta\eta^2 \Big) + \cos(k\Delta\eta)(2\delta k\Delta\eta)$$

$$+ \frac{\delta^3+\delta^2+2i\delta k\Delta\eta+i\delta^2 k\Delta\eta-k^2\Delta\eta^2+\delta k^2\Delta\eta^2+ik^3\Delta\eta^3}{2i} \int_{\epsilon(ik\Delta\eta-\delta)}^\infty dv \frac{e^v}{v}$$

$$- \frac{\delta^3+\delta^2-2i\delta k\Delta\eta-i\delta^2 k\Delta\eta-k^2\Delta\eta^2+\delta k^2\Delta\eta^2-ik^3\Delta\eta^3}{2i} \int_{\epsilon(-ik\Delta\eta-\delta)}^\infty dv \frac{e^v}{v} \Bigg] \Bigg\}.$$

$$
\begin{aligned}
=&\lim_{\epsilon\to 0}(\Delta\eta)^2\Bigg\{\\
&\frac{e^{-\delta}\log(\epsilon)}{\left(\delta^2+k^2\Delta\eta^2\right)^2}\Big[\sin(k\Delta\eta)\Big(\delta^2+\delta^3+\delta^3\epsilon-k^2\Delta\eta^2+\delta k^2\Delta\eta^2+\delta k^2\Delta\eta^2\epsilon\Big)\\
&\qquad+\cos(k\Delta\eta)\Big(2\delta k\Delta\eta+\delta^2 k\Delta\eta+\delta^2 k\Delta\eta\epsilon+k^3(\Delta\eta)^3+k^3(\Delta\eta)^3\epsilon\Big)\Big]\\
&+\frac{e^{-\delta}}{\left((k\Delta\eta)^2+\delta^2\right)^2}\Bigg[\\
&\quad\sin(k\Delta\eta)\Big[\delta^2-k^2\Delta\eta^2+\Big(-\delta^3-\delta^2+k^2\Delta\eta^2-\delta k^2\Delta\eta^2\Big)\mathrm{Ci}\big(\epsilon k\Delta\eta\big)\\
&\qquad+\Big(2\delta k\Delta\eta+\delta^2 k\Delta\eta+k^3(\Delta\eta)^3\Big)\mathrm{Si}\big(\epsilon k\Delta\eta\big)\Big]\\
&\quad+\cos(k\Delta\eta)\Big[2\delta k\Delta\eta+\Big(-2\delta k\Delta\eta-\delta^2 k\Delta\eta-k^3(\Delta\eta)^3\Big)\mathrm{Ci}\big(\epsilon k\Delta\eta\big)\\
&\qquad+\Big(-\delta^3-\delta^2+k^2\Delta\eta^2-\delta k^2\Delta\eta^2\Big)\mathrm{Si}\big(\epsilon k\Delta\eta\big)\Big]+\mathcal{O}(\delta)\Bigg]\Bigg\}\\
=&\lim_{\epsilon\to 0}(\Delta\eta)^2\Bigg\{\frac{\log(\epsilon)}{\left(\delta^2+k^2\Delta\eta^2\right)^2}\Big[\sin(k\Delta\eta)\Big(-k^2\Delta\eta^2\Big)+\cos(k\Delta\eta)\Big(k^3(\Delta\eta)^3\Big)\Big]\\
&+\frac{1}{\left((k\Delta\eta)^2+\delta^2\right)^2}\Bigg[\\
&\sin(k\Delta\eta)\Big[-k^2\Delta\eta^2+\Big(k^2\Delta\eta^2\Big)\mathrm{Ci}\big(\epsilon k\Delta\eta\big)+\Big(k^3(\Delta\eta)^3\Big)\mathrm{Si}\big(\epsilon k\Delta\eta\big)\Big]\\
&+\cos(k\Delta\eta)\Big[\Big(-k^3(\Delta\eta)^3\Big)\mathrm{Ci}\big(\epsilon k\Delta\eta\big)+\Big(k^2\Delta\eta^2\Big)\mathrm{Si}\big(\epsilon k\Delta\eta\big)\Big]+\mathcal{O}(\delta)\Bigg]\Bigg\}.
\end{aligned}
$$

Taking the δ limit of the second integral in equation 156 results in:

$$
\begin{aligned}
&\int_1^\infty dz z\sin(k\Delta\eta z)\log\left(z^2-1\right)\\
=&\frac{\sin(k\Delta\eta)}{k^2}\Big[-\log(2)-2-\mathrm{Ci}(2k\Delta\eta)+(k\Delta\eta)\mathrm{Si}(2k\Delta\eta)\Big]\\
&+\frac{\cos(k\Delta\eta)}{k^2}\Big[k\Delta\eta\log(2)+k\Delta\eta\mathrm{Ci}(2k\Delta\eta)+\mathrm{Si}(2k\Delta\eta)\Big]\\
&+\frac{\sin(k\Delta\eta)}{k^2}\Big[\gamma_E+\log(z)-\frac{\pi}{2}k\Delta\eta\Big]\\
&+\frac{\cos(k\Delta\eta)}{k^2}\Big[-\gamma_E-k\Delta\eta\log(k\Delta\eta)-\frac{\pi}{2}\Big]\Big\}.
\end{aligned}
\tag{163}
$$

The integral is then found by adding equations (157) and (163),and for simplicity it is called L_1:

$$L_1 = \frac{4\pi}{k^3}(k\Delta\eta)^2 \int_0^\infty dz z \sin(k\Delta\eta z) \log\left(|z^2 - 1|\right)$$
$$= \frac{-4\pi^2}{k^3}\Big[k\Delta\eta \sin(k\Delta\eta) + \cos(k\Delta\eta)\Big]. \tag{164}$$

L_0 (155) and L_1 (164) together are the Fourier integral of the logarithmic terms.

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