Utrecht University

Institute for Theoretical Physics

Master's Thesis

Gravitons on de Sitter modified by quantum fluctuations of a nonminimally coupled massive scalar

Author:
Roeland Koelewijn

Supervisor:
Dr. Tomislav Prokopec

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Utrecht University

Abstract

We investigate the one loop contributions to the graviton self-energy from a nonminimally coupled massive scalar perturbatively on a de Sitter background space-time. We assume a positive effective mass $m^2 + \xi R > 0$. We canonically quantize the scalar field and derive the de Sitter invariant Chernikov-Tagirov propagator. The graviton is defined as a small perturbation around a de Sitter background and the two one loop Feynman diagrams contributing to the graviton self-energy are computed through the effective action. We find that the diagram contributing to the nonlocal part of the graviton self-energy is proportional to the connected energy-momentum tensor correlator, where the expectation value is with respect to the de Sitter invariant Bunch-Davies vacuum. We employ dimensional regularization. The connected energy-momentum correlator is calculated and only partially renormalized due to the many divergences present. The second diagram is a local contribution to the graviton self-energy and is renormalized by the inverse gravitational constant and cosmological constant counterterms. The renormalized graviton self-energy can be used to quantum correct the linearized Einstein equation enabling us to investigate if nonminimally coupled massive scalars, produced during inflation, have an effect on dynamical gravitons and the force of gravity at one loop order. The connected energy-momentum correlator has importance on its own as it is the variance of the expectation value of the energy-momentum tensor and is needed to answer whether or not the expectation value of the energy-momentum tensor is a good description for backreaction.

Contents

Chapter 1

Introduction

The theory of primordial inflation, a period of rapid expansion in the early universe during which quantum fluctuations were magnified to macroscopic scales forming the seeds for density perturbations. These density perturbations later evolved into the temperature anisotropies in the cosmic microwave background (CMB) and the large-scale structure of the universe. Inflation is well supported by observational data from measurements to the cosmic microwave background by the WMAP and PLANCK collaborations [16][8][1].

However, matter and radiation explain only a small part of the total energy density in the universe. Indeed recent measurements show the density parameter for matter to be $\Omega_m = 0.308 \pm 0.012$, with baryonic matter $\Omega_B \approx 0.046$ and the remaining matter, dark matter, thought to be nonrelativistic particles of unknown origin (the radiation density today is negligible). This leaves roughly 69% of the energy contents of the universe unexplained. This unknown energy density is dubbed dark energy (DE) and is responsible for the accelerated expansion of the universe we see today. In the standard model of cosmology (ΛCDM) it is attributed to the cosmological constant and measurements constrain the equation of state of dark energy to $w_{DE} = -1.006 \pm 0.045$ compared to $w_\Lambda = -1$. One wonders if dark energy can be explained, like matter, from quantum fluctuations in the early universe.

In the work of Glavan and Prokopec the idea, that dark energy originates from the backreaction of quantum fluctuations originating in the primordial inflationay universe, is examined for three different scalar models [12][14][13]. They used semiclassical gravity, which quantizes the matter fields but leaves the background space-time, on which the matter fields live, classical. The one-loop corrected Einstein equation takes the form

$$G_{\mu\nu} = 8\pi G_N \left[T_{\mu\nu} + \langle \hat{T}_{\mu\nu} \rangle \right], \tag{1.1}$$

where $T_{\mu\nu}$ is the classical energy-momentum tensor of the scalar field and $\langle \hat{T}_{\mu\nu} \rangle$ is the expectation value of the energy-momentum tensor with respect to a given state of the quantum fluctuations. The Einstein equation (1.1) is treated perturbatively by separating it into its classical part and the first order corrections to the background space-time sourced by $\langle \hat{T}_{\mu\nu} \rangle$, which defines backreaction. The growth of these quantum fluctuations were studied on an FLRW background through the cosmological eras up to the beginning of the dark energy dominated era today. They found that the contribution from the backreaction of a nonminimally coupled very light massive scalar is a potentially good candidate for dark energy at late-times for the following parameter ranges: the scalar mass $m \lesssim H_0$, where H_0 is the Hubble parameter today; the nonminimal

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[1]Observed values of the density parameters, Hubble constant and equation of state parameter are the ones measured by the Planck collaboration 2015 results.

coupling parameter $|\xi| \sim 10^{-1} - 10^{-2}$, $\xi < 0$ (in this thesis we use $\xi > 0$ simplifying the calculations) and the number of e-foldings $N_I \sim 10^3 - 10^4$. They assumed that the expectation value of the energy-momentum tensor is a good description for backreaction, which needs to be confirmed. To check if $\langle T_{\mu\nu} \rangle$ is a good description for backreaction one would need to calculate the variance of the energy-momentum tensor, which is given by the connected energy-momentum tensor correlator (TT-correlator).

In the work by Park and Woodard [32] minimally coupled massless scalar contributions to the graviton self-energy on de Sitter were calculated. They used the result to calculate the effect of scalars, produced during inflation, on the propagation of dynamical gravitons and found no effect [31]. This result was reexamined by Leonard, Park, Prokopec and Woodard [24] and they found that inflationary scalars produced during inflation have no significant late-time effects on dynamical gravitons at one loop order. The graviton self-energy was also used by Park, Prokopec and Woodard to calculate quantum corrections to the gravitational potentials of a static massive point particle [30]. They found that the potentials were modified by a term $\propto \ln a$, the first such result for a fully renormalized analysis.

These works form the main motivation of this thesis to study gravitons on de Sitter modified by quantum fluctuations of a nonminimally coupled massive scalar and the TT-correlator. The central problems of this thesis are:

- Computation of the contributions to the graviton self-energy from nonminimally coupled massive scalars.

- Computation of the connected TT-correlator for the nonminimally coupled massive scalar.

The renormalized graviton self-energy can be used to quantum correct the linearized Einstein equation enabling us to investigate whether scalars, produced during inflation, have an effect on dynamical gravitons at one-loop order and how they effect the force of gravity as has already been done for the minimal massless case.

The TT-correlator is important to answer whether an expectation value of the energy-momentum tensor is a good description for backreaction, or if the distribution of energy densities and pressures vary a lot. If they do vary a lot; one must resort to stochastic gravity, where a stochastic source is added to the semiclassical Einstein equation to account for the fluctuations of the energy-momentum tensor [17]. It turns out that by calculating the graviton self-energy to one-loop order we obtain the TT-correlator automatically, making it the central object of this thesis. For the minimally coupled massive case the nonrenormalized TT-correlator was derived in [35].

In this thesis we work on de Sitter space and assume a de Sitter invariant vacuum, propagator, strictly positive effective mass $m^2 + \xi R > 0$ and stricly positive nonminimal coupling $\xi > 0$. We start with the introduction of some basic concepts from cosmology. We then introduce de Sitter space and discuss some of its geometric properties and show that it is a good approximation for inflation; moreover it is an exact model for a FLRW space-time entirely sourced by the cosmological constant. The nonminimally coupled massive scalar is introduced and canonically quantized. The equations of motion are exactly solved and the Chernikov-Tagirov propagator is derived. We define the graviton field as small fluctuations around a classical de Sitter background and derive the one loop corrections to the graviton self-energy. We find that the nonlocal contribution to the self-energy is proportional to the connected TT-correlator. We check our result for the nonrenormailzed TT-correlator by taking the minimal coupling limit $\xi \to 0$ and comparing it to [35] and taking the massless minimially coupled limit and comparing it to [32]. Our result agrees in both cases. We employ dimensional regularization, preserving general covariance, and extract differential operators, to localize the divergences onto delta function terms. This way

one can obtain a form of the self-energy suitable for use in the quantum corrected linear Einstein equation (6.2); however in the case of nonminimal coupling the TT-correlator contains many divergences making renormalization a cumbersome process, which we only partially complete. For the local contribution to the self-energy we find, unlike the massless minimally coupled scalar, a nonvanishing contribution [32]. We conclude this thesis by discussing our results and briefly discussing the result found in [30].

Chapter 2

The Homogeneous and Isotropic Universe

This chapter is a short introduction to some basics of general relativity and classical cosmology. We derive the Einstein field equations from the action functional. The matter content of the universe is modeled by a perfect fluid and the well known Friedmann equations are derived. We end this chapter by briefly discussing the history of our universe, which consists of roughly three distinct eras. The following conventions are used in this thesis unless explicitly mentioned otherwise. The temporal component of the metric comes with a minus sign and the spatial components carry a plus sign. For the geometric quantities we use the $(+++)$ convention [28] and we have taken $\hbar = c = 1$ unless mentioned otherwise. Most results are derived in D-dimensions; however, in this chapter we sometimes set $D = 4$ to make a connection to our four dimensional universe.

2.1 FLRW Space-Time

Observations show that, at sufficiently large scales ($\sim 100\,\mathrm{Mpc}$ and above), the universe is homogeneous and isotropic. This is known as the cosmological principle. Homogeneity tells us that the geometric properties of space are the same at all spatial locations. Isotropy tells us that we cannot single out any special direction in space. At smaller scales this is evidently not true, as there exist a significant degree of inhomogeneity in the form of galaxies, clusters, etc. The standard practice is to assume that these inhomogeneities can be ignored, and the matter distribution can be described by an average density when studying the large scale dynamics of the universe. The geometric properties of space are determined by the distribution of matter through the Einstein field equations. Furthermore, observational data suggests our universe is expanding and spatially flat (the spatial curvature of our universe is $|\Omega_K| < 0.005$). These observations are very well described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time, which is defined by the invariant line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j = a(\eta)^2\eta_{\mu\nu}dx^\mu dx^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ is the metric, $\eta_{\mu\nu}$ the Minkowski metric, a the scale factor, x^i spatial coordinates, t cosmological (physical) time and η conformal time. The cosmological and conformal time are related via $dt = ad\eta$, and the physical distance r_{phys} between two points at fixed coordinates x and x' is proportional to to the comoving distance χ through the relation $r_{phys} = a(t)\chi$.

Taking the derivative of this equation with respect to physical time t and evaluating it at $t = t_0$ yields $v(t_0) = H_0 r_{phys}(t_0)$ known as Hubble's law, where $H_0 = 67.8 \pm 0.9 \mathrm{km\, s^{-1}\, Mpc^{-1}}$ is the Hubble parameter today. Note that at each instant of time this metric is manifestly isotropic and homogeneous. In general we will work in D-dimensions to facilitate dimensional regularization and renormalization, which preserves general covariance, and is performed in chapter 5. The dynamics of the scale factor are governed by the Einstein field equations. The pure gravity part with cosmological constant is given by the Einstein-Hilbert action functional

$$S_{EH\Lambda}[g_{\mu\nu}] = \frac{1}{16\pi G_N} \int \mathrm{d}^D x \left(\sqrt{-g} R - 2\Lambda \right). \tag{2.2}$$

The vacuum Einstein field equations are then obtained via the principle of least action

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{2.3}$$

with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{2.4}$$

Where G_N is Newton's gravitational constant, $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. The geometric quantities are defined as follows:

- The Christoffel symbols,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}); \tag{2.5}$$

- The Riemann tensor,

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}; \tag{2.6}$$

- The Ricci tensor,

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}; \tag{2.7}$$

- The Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.8}$$

When matter fields are present the total action is given by

$$S = S_{EH\Lambda}[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}], \tag{2.9}$$

where $S_m[\phi, g_{\mu\nu}]$ is the unspecified matter content of the theory and ϕ are the matter fields. After Extremising (2.9) we obtain the equations of motion

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.10}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}, \tag{2.11}$$

is the energy-momentum tensor for the matter fields. Specifying the matter fields to be ideal classical fluids the energy-momentum tensor takes the form

$$T_{\mu\nu} = (\rho_m + p_m) u_\mu u_\nu + p g_{\mu\nu}, \tag{2.12}$$

where ρ_m and p_m are the energy density and pressure and u^μ is the four velocity normalized as $u^\mu u_\mu = -1$. The first Friedmann equation is obtained from the 00-component of (2.10) and reads

$$\frac{\mathcal{H}^2}{a^2} = \frac{8\pi G_N}{3} \times \frac{6}{(D-2)(D-1)}\rho. \tag{2.13}$$

The second Friedmann equation is obtained from the trace of (2.10) and the first Friedmann equation and reads

$$\frac{\mathcal{H}' - \mathcal{H}^2}{a^2} = -4\pi G_N \times \frac{2}{D-2}(\rho + p), \tag{2.14}$$

where $\mathcal{H} = \partial_\eta a/a = a'/a$ the conformal Hubble parameter, which is related to the Hubble parameter $H = \dot{a}/a = \mathcal{H}/a$ and ρ and p are the effective energy density and pressure defined as

$$\rho = \rho_m + \frac{\Lambda}{8\pi G_N} \quad , \quad p = p_m - \frac{\Lambda}{8\pi G_N}, \tag{2.15}$$

which reduce to ρ_m and p_m when there is no cosmological constant. Note that we can think of the cosmological constant term as either a geometric quantity on the left-hand side of (2.10) or as part of the energy-momentum tensor by moving it to the right-hand side. The Friedmann equations determine the evolution of the scale factor. Cosmological relevant fluids can be characterized by their barotropic equation of state relating their energy density and pressure, $p_m = w\rho_m$ where w is the equation of state parameter. For non-relativistic matter $w_M = 0$, for radiation $w_R = 1/3$ and for the cosmological constant $w_\Lambda = -1$. A fluid with energy density ρ_m and pressure p_m satisfies the covariant energy conservation law

$$\nabla^\mu T_{\mu\nu} = \rho_m' + (D-1)\mathcal{H}(\rho_m + p_m) = 0. \tag{2.16}$$

We can integrate equations (2.13) and (2.16) using the equation of state to obtain a relation between the scale-factor and physical time. For $D = 4$ space-time dimensions we find the following relations:

- For a radiation dominated universe $a \propto \sqrt{t}$,

- For a matter dominated universe $a \propto t^{2/3}$,

- For a universe dominated by the cosmological constant $a \propto \exp(Ht)$.

Next we introduce the slow-roll parameter defined as

$$\epsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2}, \tag{2.17}$$

which should not be confused with the pole prescription ε used in this thesis. For a spatially flat FLRW space-time dominated by an ideal fluid we can use (2.13) and (2.14) to express the slow-roll parameter in terms of the equation of state parameter

$$\epsilon = \frac{D-1}{2}(1+w), \tag{2.18}$$

which is related to the deceleration parameter q, which gives us information on how much the expansion of the universe is speeding up or slowing down, as

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = \epsilon - 1. \tag{2.19}$$

From this definition it is clear that the universe is decelerating for $\epsilon > 1$ and accelerating for $\epsilon < 1$. The history of our universe roughly consists of three distinct eras: the inflationary era ($0 < \epsilon_I \ll 1$), the radiation dominated era ($\epsilon_R = 2$) and the matter dominated era ($\epsilon_M = 3/2$), where we have taken $D = 4$. Note that $\epsilon_I \approx \epsilon_\Lambda = 0$, thus we see inflation can be well approximated by a flat FLRW space-time with $\epsilon = 0$, $H = \text{constant}$ and $a \propto \exp(Ht)$ [26]. In chapter 3 we will show that the de Sitter space-time describes exactly such a space-time.

Chapter 3

The de Sitter Space-Time

In this chapter we introduce the de Sitter space-time as a hyperboloid embedded in a Minkowski space-time of one spatial dimension higher, show that it corresponds to a FLRW space-time with exponential scale-factor and show that it is a good model for inflation. Once we have introduced all required properties of the de Sitter space-time, we introduce the action functional of the nonminimally coupled massive scalar and canonically quantize the fields using the Hamiltonian formalism. We follow the analysis of [13][12][14] closely where this procedure was done for a nonminimally coupled massive scalar field on a flat FLRW space-time. The energy-momentum tensor is derived, the equations of motion exactly solved and the Chernikov-Tagirov propagator calculated for a strictly positive effective mass to avoid any infrared divergences. The explicit value of the energy-momentum tensor is not relevant for this thesis; however, we will need the general form written in terms of the propagator, to derive the TT-correlator in appendix B. For those interested in the explicit value of the expectation value of the energy-momentum tensor we refer to [13][12][14], where it was calculated in terms of the mode functions for the inflationary, radiation dominated and matter dominated eras.

3.1   Maximally Symmetric Space-Time

A maximally symmetric space-time is a space-time that admits the maximum number of Killing vectors i.e. $D(D+1)/2$ [6]. All maximally symmetric space-times have constant curvature and the simplest example of such a space-time is the Minkowski space-time, which corresponds to a maximally symmetric space-time with zero curvature. The de Sitter space-time is the unique maximally symmetric space-time with positive curvature corresponding to a positive cosmological constant[1] [6]. Interest in this space-time follows from the fact that it is a good approximation to the universe during inflation ($\epsilon_{\mathrm{dS}} = 0$ compared to $0 < \epsilon_I \ll 1$). Moreover it is an exact model of a universe entirely sourced by the cosmological constant. Consider a $(D+1)$-dimensional Minkowski space-time. Then the D-dimensional de Sitter space-time can be represented by embedding the hyperboloid

$$\eta_{AB} X^A X^B = \frac{1}{H^2} \tag{3.1}$$

in $(D+1)$-dimensional Minkowski space-time Fig. 3.1. Here η_{AB} is the metric of the $(D+1)$-dimensional Minkowski space-time and the indices A and B run from 0 to D. H is a constant

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[1]The unique maximally symmetric space-time with negative curvature is the anti de Sitter space-time corresponding to a negative cosmological constant.
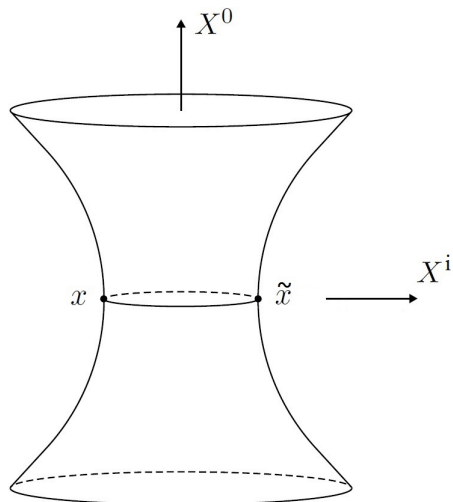
Figure 3.1: D-dimensional hyperboloid representing de Sitter space embedded in $(D + 1)$-dimensional Minkowski space. Every point $X^A(x)$ has an antipodal point $X^A(\tilde{x}) = -X^A(x)$. A straight line drawn between a point and its antipode passes through the origin, the analogue of a diameter of a sphere. Figure adapted from [35].

and can be related to the Hubble parameter, which we shall see shortly. Note that the de Sitter space-time is the set of points that lie at a constant distance from the origin, thus it is the Minkowskian analogue of the D-dimensional sphere. The embedding (3.1) makes manifest the isometries that leave the hyperboloid invariant i.e. the Lorentz transformations. We conclude that the isometry group of de Sitter is the Lorentz group also called the de Sitter group $O(1, D)$, which is generated by $D(D + 1)/2$ Killing vectors, the maximum number a manifold can have. We now introduce the so called flat slicing coordinates (t, x_i), $i = 1, \ldots, D - 1$ on de Sitter given by,

$$
\begin{aligned}
X_0 &= \frac{1}{H} \sinh(Ht) + \frac{H}{2} x_i x^i e^{Ht}, \\
X_i &= e^{Ht} x_i, \\
X_D &= \frac{1}{H} \cosh(Ht) - \frac{H}{2} x_i x^i e^{Ht}, \\
&-\infty < t < \infty, \ -\infty < x_i < \infty.
\end{aligned}
\tag{3.2}
$$

These coordinates do not cover the entire de Sitter manifold, indeed they cover only half as can be seen from the relation $X_0 + X_D > 0$. From (3.1) and (3.2) the metric induced by the embedding is found to be

$$
ds^2 = \eta_{AB} dX^A dX^B = -dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j \equiv g_{\mu\nu} dx^\mu dx^\nu.
\tag{3.3}
$$

This is exactly the spatially flat FLRW metric (2.1) with scale factor

$$
a(t) = e^{Ht}.
\tag{3.4}
$$

By taking the derivative of (3.4) one finds

$$
H = \frac{\dot{a}}{a},
\tag{3.5}
$$

which is indeed the Hubble parameter. Therefore, we can conclude that de Sitter space-time corresponds to a flat FLRW space-time with constant Hubble parameter, exponential scale factor and vanishing slow-roll parameter i.e. sourced entirely by the cosmological constant. Introducing conformal time $ad\eta = dt$ we can write the induced metric as

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}, \tag{3.6}$$

and the scale factor as

$$a(\eta) = -\frac{1}{H\eta}, \ \eta < 0, \tag{3.7}$$

which is the preferred form of the de Sitter metric in this thesis. For this range of η the metric covers the half of the de Sitter manifold corresponding to an expanding universe, which can be seen from the scale-factor in conformal time (3.7). The other half of the manifold is given by $\eta > 0$, which corresponds to a contracting universe, which we will not consider here.

For maximally symmetric spaces the geometric quantities take an especially simple form. The Riemann curvature tensor of a maximally symmetric space is given by

$$R_{\rho\sigma\mu\nu} = 2H^2 g_{\rho[\mu}g_{\nu]\sigma}, \tag{3.8}$$

where the square brackets denote normalized antisymmetrization

$$T_{[\mu\nu]} = \frac{1}{2}\big(T_{\mu\nu} - T_{\nu\mu}\big). \tag{3.9}$$

Contracting we find the Ricci tensor

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} = (D-1)H^2 g_{\mu\nu}. \tag{3.10}$$

Taking the trace we obtain the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = D(D-1)H^2. \tag{3.11}$$

The relation between the Hubble parameter and the cosmological constant follows from the Einstein equation. Indeed for

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{3.12}$$

to hold the cosmological constant must satisfy

$$H^2 = \frac{2\Lambda}{(D-1)(D-2)}. \tag{3.13}$$

The de Sitter invariant distance function[2] is defined as

$$1 - \frac{1}{2}Y(X;X') = H^2\eta_{AB}X^A X'^B, \tag{3.14}$$

which in flat coordinates in conformal form takes the form

$$\overline{y}(x;x') = \frac{-(\eta-\eta')^2 + ||\mathbf{x}-\mathbf{x}'||^2}{\eta\eta'} \equiv \frac{\Delta x^2(x;x')}{\eta\eta'}, \ 0 \le \overline{y} \le 4 \tag{3.15}$$

or using (3.7)

$$\overline{y}(x;x') = aa'H^2\Delta x^2(x;x'), \tag{3.16}$$

---

[2]In the literature one often encounters the definition $Z(x;x') = 1-\frac{1}{2}Y(x;x')$ for the invariant distance function.

with $a' = \partial_\eta a$. The function $\overline{y}$ is related to the geodesic distance ℓ between two points x and x' as

$$\overline{y}(x; x') = 4\sin^2\left[\frac{1}{2}H\ell(x; x')\right], \tag{3.17}$$

making the range of $\overline{y}$ evident. From (3.15) it is clear that for $\overline{y} < 0$, $\overline{y} = 0$ and $\overline{y} > 0$ the points x and x' are timelike, lightlike and spacelike separated respectively. We have denoted the classical invariant distance with a bar, $\overline{y}$, to discern it from the y used when doing quantum mechanical computations, which differs from $\overline{y}$ by a pole prescription.

We will now look at a special kind of isometry of de Sitter, the antipodal transformation. The antipodal transformation is a mapping that sends a point x to its antipodal point $\tilde{x}$, defined as $X^A(\tilde{x}) = -X^A(x)$ and maps a future directed curve into a past directed curve. Note that since $\eta < 0$ the antipodal point is not covered by our choice of coordinates (3.2). Clearly it is an isometry of the de Sitter hyperboloid (3.1). More descriptively, the antipodal point $\tilde{x}$ is opposite x such that a line drawn between these points passes through the center of the hyperboloid. From (3.14) and (3.16) we have the relation $\overline{y}(\tilde{x}; x') = 4 - \overline{y}(x; x')$. We note that if $\overline{y}(x; x') = 4$, x' lies on the lightcone of the antipodal point $\tilde{x}$ of x. For $\overline{y} > 4$ the antipodal point of x and x' are timelike separated.

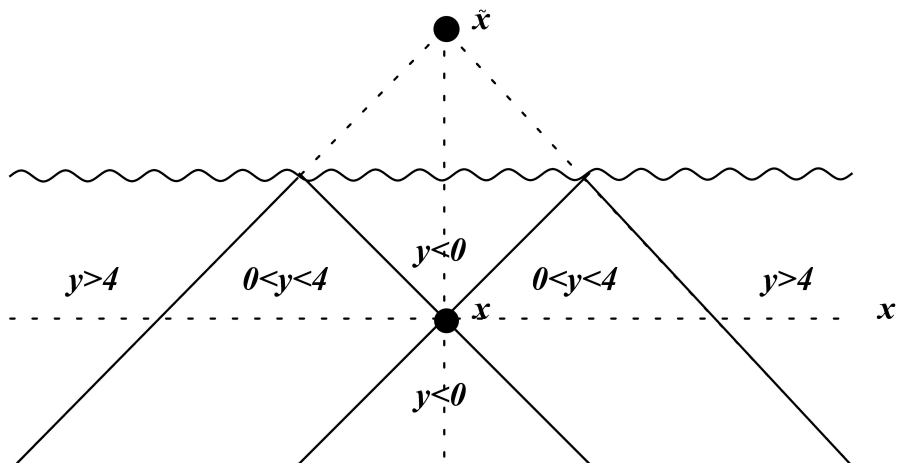In the remainder of this thesis our background geometry will be that of de Sitter with metric (3.6).

Figure 3.2: Causal structure of de Sitter in conformal coordinates (3.6),(3.2) The coordinates only cover the region $\eta < 0$. The wavy line is located at $\eta = 0$ and represents future infinity. Regions for various values of y are drawn. The lightcone of x is given by $y = 0$. For $y = 4$ the point x' lies on the lightcone of the antipodal point of x. Figure based on [19].

3.2 The Nonminimally Coupled Massive Scalar

In this section we introduce the action for the nonminimally coupled massive scalar field on a de Sitter space-time. We start with the quantization of the theory and then solve the equations of motion in terms of mode functions. Then we derive the quantized energy-momentum tensor for the nonminimally coupled scalar field and write it in terms of the expectation value of the fields through the method of pointsplitting. The results are kept in a sufficiently general form that

they are also valid for flat FLRW space-times. In section 3.3 we will exactly solve the equations of motion and derive the propagator on the de Sitter space-time.

The action for a nonminimally coupled massive scalar field in D-dimensions is given by

$$S_\phi = \int \mathrm{d}^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 \right], \tag{3.18}$$

where the dimensionless parameter ξ is the nonminimal coupling and m is the scalar field mass. The coupling is conformal if $\xi \equiv \xi_c = (D-2)/(4(D-1))$. For conformal coupling and $m = 0$ the scalar theory is left invariant under conformal transformations $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$. On a de Sitter background with metric (3.6) the Lagrangian density takes the form

$$\mathcal{L}_\phi = \frac{a^{D-2}}{2} \left[(\phi')^2 - (\nabla\phi)^2 - (ma)^2 \phi^2 - \xi a^2 R \phi^2 \right]. \tag{3.19}$$

To quantize this field we use the Hamiltonian formalism. First we must define a canonical conjugate momentum

$$\pi(x) = \frac{\partial \mathcal{L}_\phi}{\partial \phi'} = a^{D-2} \phi'(x), \tag{3.20}$$

then we define the Hamiltonian via the Legendre transform,

$$\begin{aligned} H[\phi, \pi; \eta) &= \int \mathrm{d}^{D-1}x \left[\pi(x)\phi'(x) - \mathcal{L}_\phi \right] |_{\phi' \to a^{2-D}\pi} \\ &= \frac{a^{D-2}}{2} \int \mathrm{d}^{D-1}x \left[\frac{\pi^2}{a^{2D-4}} + (\nabla\phi)^2 + (ma)^2 \phi^2 + \xi a^2 R \phi^2 \right]. \end{aligned} \tag{3.21}$$

Next we promote the fields ϕ and π to operators and impose canonical commutation relations,

$$[\hat{\phi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta^{D-1}(\mathbf{x} - \mathbf{x}') \tag{3.22}$$

$$[\hat{\phi}(\eta, \mathbf{x}), \hat{\phi}(\eta, \mathbf{x}')] = [\hat{\pi}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = 0. \tag{3.23}$$

Next we define the Hamiltonian operator as

$$\hat{H}(\eta) = H[\hat{\phi}, \hat{\pi}; \eta), \tag{3.24}$$

which determines the time evolution via the Heisenberg equations for the field operators,

$$\hat{\phi}'(x) = i[\hat{H}(\eta), \hat{\phi}(x)] = a^{2-D} \hat{\pi}(x) \tag{3.25}$$

$$\hat{\pi}'(x) = i[\hat{H}(\eta), \hat{\pi}(x)] = a^{D-2} \left[\nabla^2 \hat{\phi}(x) - (ma)^2 \hat{\phi}(x) - \xi a^2 R \hat{\phi}(x) \right]. \tag{3.26}$$

After combining these two equations we obtain the equations of motion for the field operator $\hat{\phi}$

$$\hat{\phi}'' + (D-2)\mathcal{H}\hat{\phi}' - \nabla^2\hat{\phi} + a^2(m^2 + \xi R)\hat{\phi} = 0. \tag{3.27}$$

Note that (3.27) is simply the Klein-Gorden equation for a nonminimally coupled scalar on a de Sitter background obtained by extremising (3.18)

$$(\Box - m^2 - \xi R)\hat{\phi} = 0, \tag{3.28}$$

where $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian and $\mathcal{H} = a'/a$ is the conformal Hubble parameter. Next we decompose the fields into Fourier modes,

$$\hat{\phi}(\eta, \mathbf{x}) = \frac{a^{\frac{2-D}{2}}}{(2\pi)^{\frac{D-1}{2}}} \int \mathrm{d}^{D-1}k \left[e^{i\mathbf{k}\cdot\mathbf{x}} U(k, \eta)\hat{b}(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x}} U^*(k, \eta)\hat{b}^\dagger(\mathbf{k}) \right], \tag{3.29}$$

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where the mode function $U(k, \eta)$ depends only on $k = ||\mathbf{k}||$, which is implied by the symmetries of our system. The equations of motion in terms of the mode function $U(k, \eta)$ take the form of a harmonic oscillator equation with time-dependent mass and is given by

$$U''(k, \eta) + \left[k^2 + \mathcal{M}^2(\eta) \right] U(k, \eta) = 0, \tag{3.30}$$

with

$$\mathcal{M}^2(\eta) = m^2 a^2 + \left(\xi - \frac{D-2}{4(D-1)} \right) R a^2. \tag{3.31}$$

The annihilation and creation operators are required to satisfy the commutation relations,

$$\left[\hat{b}(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}') \right] = \delta^{D-1}(\mathbf{k} - \mathbf{k}'), \tag{3.32}$$

$$\left[\hat{b}(\mathbf{k}), \hat{b}(\mathbf{k}') \right] = 0, \tag{3.33}$$

$$\left[\hat{b}^\dagger(\mathbf{k}), \hat{b}^\dagger(\mathbf{k}') \right] = 0. \tag{3.34}$$

These commutation relations together with the canonical commutation relations (3.22) and (3.23) fix the Wronskian normalization of the mode function

$$U(k, \eta) U'^*(k, \eta) - U^*(k, \eta) U'(k, \eta) = i. \tag{3.35}$$

The vacuum state $|\Omega\rangle$ is defined to be annihilated by the annihilation operators,

$$\hat{b}(\mathbf{k})|\Omega\rangle = 0, \quad \forall \mathbf{k}, \tag{3.36}$$

which implies there is no scalar field condensate

$$\langle \Omega | \hat{\phi}(x) | \Omega \rangle = 0, \tag{3.37}$$

nor can it be dynamically generated since we are working with effectively free fields [44] [41] [40]. The entire Fock space can now be constructed by acting on the vacuum state with the creation operators $b^\dagger(\mathbf{k})$ and once we have specified the mode functions and its derivative at some time it is completely determined. The vacuum state is required to not contain any infrared (IR) divergences and reduce to the Bunch-Davies vacuum [5][7] in the ultraviolet (UV), which reduces to the flat space positive frequency mode in the UV i.e.

$$U(k, \eta) \to U(k, \eta)_{BD} \overset{k \to \infty}{\sim} \frac{e^{-ik\eta}}{\sqrt{2k}}. \tag{3.38}$$

Now that the theory has been quantized we can derive the the energy-momentum tensor for the nonminimally coupled massive scalar in the usual way,

$$\hat{T}_{\mu\nu}(x) = \frac{-2}{\sqrt{-g(x)}} \frac{\delta S_\phi[\phi, g^{\mu\nu}]}{\delta g^{\mu\nu}(x)} \bigg|_{\phi \to \hat{\phi}}$$

$$= \partial_\mu \hat{\phi}(x) \partial_\nu \hat{\phi}(x) - \frac{1}{2} g_{\mu\nu}(x) g^{\alpha\beta}(x) \partial_\alpha \hat{\phi}(x) \partial_\beta \hat{\phi}(x)$$

$$- \frac{1}{2} g_{\mu\nu}(x) m^2 \hat{\phi}(x)^2 + \xi \left[G_{\mu\nu}(x) + g_{\mu\nu}(x) \Box - \nabla_\mu \nabla_\nu \right] \hat{\phi}(x)^2, \tag{3.39}$$

where in the derivation it is useful to recall

$$\delta R = \left[R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] \delta g^{\mu\nu}. \tag{3.40}$$
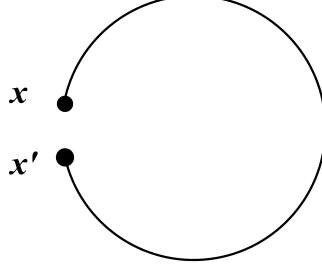
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Figure 3.3: Diagrammatic representation of pointsplitting $T_{\mu\nu}$. The coordinate x is split into two points x and x'; hence the name. After taking the coincidence limit the energy-momentum tensor $\langle T_{\mu\nu}(x)\rangle$ is returned. This method facilitates the computation of products of energy-momentum tensors, specifically the TT-correlator.

The expectation value of this operator with respect to the vacuum state $|\Omega\rangle$ can be written in terms of the the expectation value of the fields $\langle\hat{\phi}(x)\hat{\phi}(x')\rangle$. This method is called pointsplitting and a diagrammatic representation of this method is shown in Fig. (3.3).

$$
\begin{aligned}
\langle\hat{T}_{\mu\nu}(x)\rangle &= \left[\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{2}g_{\mu\nu}(x)g^{\alpha\beta}(x)\right]\langle\partial_{\alpha}\hat{\phi}(x)\partial_{\beta}\hat{\phi}(x)\rangle \\
&\quad + \left[\xi G_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)m^2\right]\langle\hat{\phi}(x)^2\rangle \\
&\quad\quad + \xi\left[g_{\mu\nu}(x)g^{\alpha\beta}(x) - \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\right]\langle\nabla_{\alpha}\nabla_{\beta}\hat{\phi}(x)^2\rangle \\
&= \Bigg(\left[(1-2\xi)\,\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{2}\left(1-4\xi\right)g_{\mu\nu}(x)g^{\alpha\beta}(x)\right]\partial_{\alpha}\partial'_{\beta} \\
&\quad + \xi G_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)m^2 \\
&\quad + \xi\left[g_{\mu\nu}(x)g^{\alpha\beta}(x) - \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)}\right]\left(\nabla_{\alpha}\partial_{\beta} + \nabla'_{\alpha}\partial'_{\beta}\right)\Bigg)\langle\hat{\phi}(x)\hat{\phi}(x')\rangle\Big|_{x'\to x} \\
&\equiv \tau_{\mu\nu}(x;x')\, i\Delta(x;x')|_{x'\to x}, \tag{3.41}
\end{aligned}
$$

This form will prove convenient in the computation of the TT-correlator in appendix B as it facilitates the computation of products of the energy-momentum tensor.

3.3 The Chernikov-Tagirov Propagator

The Chernikov-Tagirov propagator [7] is defined as

$$
i\Delta(x;x') = \langle\mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle, \tag{3.42}
$$

where $\mathcal{T}$ denotes time ordering

$$
\mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\} = \theta(\eta - \eta')\phi(x)\phi(x') + \theta(\eta' - \eta)\phi(x')\phi(x), \tag{3.43}
$$

with $\theta(x)$ the Heaviside step function defined as

$$\theta(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{2}, & \text{for } x = 0, \\ 1, & \text{for } x > 0, \end{cases} \tag{3.44}$$

and the expectation value is taken with respect to the vacuum $|\Omega\rangle$. If the vacuum is invariant under the full de Sitter group $O(1, D)$, which we have assumed, it follows that the propagator is invariant as well and can, up to a pole prescription, only depend on the two space-time points x and x' via the de Sitter invariant distance function $y(x; x')$ [2]. It is the analogue of the Feynman propagator of quantum field theory on Minkowski space-time. To find $i\Delta(x; x')$ we must first solve the equations of motion. The equations of motion for a nonminimally coupled massive scalar on de Sitter are

$$(\Box - m^2 - \xi R)\hat{\phi} = 0. \tag{3.45}$$

In section 3.2 we have shown that the equations of motion for the mode functions on de Sitter take the form of a harmonic oscillator. Up to this point those results are also valid for a general flat FLRW spacetime. We will now use the explicit value for R on de Sitter, which in contrast to a general flat FLRW space-time is a constant (3.10). Thus (3.30) becomes

$$[\partial_\eta^2 + k^2 + \mathcal{M}^2(\eta)]U(k, \eta) = 0, \tag{3.46}$$

where

$$\mathcal{M}^2(\eta) = a^2 H^2 \left(\frac{1}{4} - \nu^2 \right), \tag{3.47}$$

with

$$\begin{aligned}\nu^2 &= \left(\frac{D-1}{2} \right)^2 - \left(\frac{m^2}{H^2} + D(D-1)\xi \right) \\ &\equiv \left(\frac{D-1}{2} \right)^2 - M^2.\end{aligned} \tag{3.48}$$

This equation can be solved exactly in terms of Bessel functions. Defining $w = k/(aH)$ and $u = \sqrt{w}U$ (3.46) reads

$$\left(w^2 \frac{d^2}{dw^2} + w \frac{d^2}{dw^2} + w^2 - \nu^2 \right) u(w) = 0. \tag{3.49}$$

We recognize this equation as the Bessel differential equation which has solutions $J_\nu(w)$ and $Y_\nu(w)$, where $J_\nu(w)$ and $Y_\nu(w)$ are Bessel functions of the first and second kind respectively. Taking two linearly independent combinations of these solutions and transforming back to conformal coordinates we find

$$U(k, \eta) = \alpha_k \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta) + \beta_k \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(2)}(-k\eta), \tag{3.50}$$

where

$$\begin{aligned} H_\nu^{(1)}(w) &= J_\nu(w) + iY_\nu(w), \\ H_\nu^{(2)}(w) &= J_\nu(w) - iY_\nu(w), \end{aligned} \tag{3.51}$$

are the Hankel functions of the first and second kind and α_k and β_k are the Bogolyubov coefficients. The choice of $\sqrt{-\pi\eta/4}H_\nu^{(1)}$ and $\sqrt{-\pi\eta/4}H_\nu^{(2)}$ is such that they both satisfy the

Wronskian normalization condition separately. The full solution obeys the Wronskian normalization condition (3.35) for $|\alpha_k|^2 - |\beta_k|^2 = 1$. The mode function (3.50) must have the correct asymptotic behaviour (3.38). The asympotic expansion of the Hankel functions are given by (9.2.7) in [1]

$$\sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta) \to \frac{1}{\sqrt{2k}} e^{-ik\eta}, \tag{3.52}$$

$$\sqrt{\frac{-\pi\eta}{4}} H_\nu^{(2)}(-k\eta) \to \frac{1}{\sqrt{2k}} e^{ik\eta}. \tag{3.53}$$

We choose $\alpha_k = 1$ and $\beta_k = 0$ to obtain the correct asymptotic behaviour for the mode function corresponding to the Bunch-Davies vacuum. This is known as the adiabatic vacuum state of infinite order [33][34][27] and on de Sitter is also a Hadamard state [22]. Finally the mode function takes the form,

$$U(k, \eta) = \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta),$$

$$U^*(k, \eta) = \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)*}(-k\eta) = \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(2)}(-k\eta). \tag{3.54}$$

To derive the propagator we must calculate the expectation value $\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle$, which in terms of the mode function reads

$$\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle = (aa')^{\frac{2-D}{2}} = \int \frac{\mathrm{d}^{D-1}k}{(2\pi)^{D-1}} e^{i\mathbf{k}\cdot\Delta\mathbf{x}} \Big[\theta(\eta - \eta') U(k, \eta) U^*(k, \eta') \\ + \theta(\eta' - \eta) \times \text{h.c.} \Big], \tag{3.55}$$

where $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}'$ and $\Delta x \equiv ||\Delta\mathbf{x}||$. Switching to $(D-1)$-dimensional spherical coordinates and aligning the axes such that $\mathbf{k}\cdot\Delta\mathbf{x} = k\Delta x \cos\theta_{D-3}$ we obtain

$$\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle = \frac{(aa')^{\frac{2-D}{2}}}{(2\pi)^{D-1}} \int_0^\infty \mathrm{d}k\, k^{D-2} \int_0^\pi \mathrm{d}\theta_{D-3} \sin^{D-3}(\theta_{D-3}) e^{ik\Delta x \cos\theta_{D-3}} \int \mathrm{d}\Omega_{D-3} \\ \times \Big[\theta(\eta - \eta') U(k, \eta) U^*(k, \eta') + \theta(\eta' - \eta) \times \text{h.c.} \Big], \tag{3.56}$$

where the third integral is the surface integral of a $(D-3)$-dimensional sphere

$$\int \mathrm{d}\Omega_{D-3} = \frac{2\pi^{\frac{D-2}{2}}}{\Gamma\left(\frac{D-2}{2}\right)}. \tag{3.57}$$

The middle integral in (3.56) can be evaluated using the identity

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu + 1/2)\sqrt{\pi}} \int_0^\pi \mathrm{d}\theta\, (\sin\theta)^{2\nu} e^{iz\cos\theta}. \tag{3.58}$$

Taking $2\nu = D - 3$ and $z = k\Delta x$ and using (3.54, 3.7) we find

$$\langle \mathcal{T}\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle = \frac{\pi}{4}\frac{H^{D-2}}{2^{D-2}\pi^{\frac{D-1}{2}}}\int_0^\infty \mathrm{d}k\, k^{D-2}\frac{J_{\frac{D-3}{2}}(k\Delta x)}{\left(\frac{1}{2}k\Delta x\right)^{\frac{D-3}{2}}}\left[\theta(\eta - \eta')H_\nu^{(1)}(-k\eta)H_\nu^{(1)*}(-k\eta')\right.$$
$$\left.+\,\theta(\eta' - \eta)\times\text{h.c.}\right].$$
$$(3.59)$$

This integral can be evaluated by using (27) in [20], which we state here for completeness

$$\int_0^\infty \mathrm{d}x\, x^{\mu+1}J_\mu(cx)H_\nu^{(1)}(ax)H_\nu^{(1)*}(bx) = \frac{\Gamma\left(\mu + 1 + \nu\right)\Gamma\left(\mu + 1 - \nu\right)}{\pi^{\frac{3}{2}}\Gamma\left(\mu + \frac{3}{2}\right)}$$
$$\times \frac{\left(\frac{1}{2}c\right)^\mu}{(ab)^{\mu+1}}{}_2F_1\left(\mu + 1 + \nu, \mu + 1 - \nu; \mu + \frac{3}{2}; \frac{(a+b)^2 - c^2}{4ab}\right).$$
$$(3.60)$$

Using (3.60), (3.16) and noting that $y(x;x') = y(x';x)$ we find the Chernikov-Tagirov propagator

$$i\Delta(x;x') = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{D}{2}\right)}$$
$$\times {}_2F_1\left(\frac{D-1}{2} + \nu, \frac{D-1}{2} - \nu; \frac{D}{2}; 1 - \frac{y}{4}\right)$$
$$\equiv \frac{H^{D-2}}{(4\pi)^{D/2}}G(y),$$
$$(3.61)$$

where we have assumed a positive effective mass squared, $m^2 + \xi R > 0$, for which the propagator is de Sitter invariant; y is the de Sitter invariant distance modified by a pole prescription (3.64), which we will discuss shortly; and ${}_2F_1$ is the Gauss hypergeometric function. If the mass is not strictly positive or if $\epsilon = -\dot{H}/H^2 \neq 0$, the propagator develops de Sitter breaking contributions due to the particle creation in the deep infrared [20, 21] and thus becomes much more complicated. The divergent behaviour for $m^2 + \xi R = 0$ can be readily seen from (5.37) when taking $s \to 0$. We also note that for $D = 4$ and $m^2 + \xi R > 0$ the only divergence occurs when x and x' lie on the lightcone i.e. $y = 0$, which is expected as it coincides with the Minkowski result.

We will now discuss some subtleties in the derivation of the propagator. The propagator (3.61) satisfies the equation

$$(\Box - m^2 - \xi R)i\Delta(x;x') = i\frac{\delta^D(x - x')}{\sqrt{-g}};$$
$$(3.62)$$

however, in [18] it was shown that it also admits a solution of the form

$$i\Delta_2(x;x') = C\,{}_2F_1\left(\frac{D-1}{2} + \nu, \frac{D-1}{2} - \nu; \frac{D}{2}; \frac{y}{4}\right),$$
$$(3.63)$$

where C is some constant. For nonzero C the propagator has an additional singularity at $y = 4$ i.e. when x' lies on the lightcone of the antipodal point of x Fig. (3.2). Such a solution corresponds to the so called α-vacua, which are considered unphysical; therefore, we discard this solution [10][23][4][15][9].

We now define the pole prescription that sources the delta function in (3.62) correctly,

$$y(x;x') \equiv y_{++}(x;x') = \frac{-\left(|\eta - \eta'| - i\varepsilon\right)^2 + ||\mathbf{x} - \mathbf{x}'||^2}{\eta\eta'}.$$
$$(3.64)$$

Indeed the delta function is sourced entirely by the action of $\Box$ on terms proportional to $y^{1-D/2}$ in the expansion of the propagator as can be seen from (5.28). In the remainder of this thesis we will use the modified de Sitter length function as defined in (3.64). Had we not taken $\alpha = 1$ and $\beta = 0$, additional terms with different pole prescriptions would have entered the propagator as arguments of the hypergeometric function [18]. We list the other pole prescriptions for completeness

$$y(x;x')_{+-} = \frac{-(\eta - \eta' + i\varepsilon)^2 + ||\mathbf{x} - \mathbf{x}'||^2}{\eta\eta'} \tag{3.65}$$

$$y(x;x')_{-+} = \frac{-(\eta - \eta' - i\varepsilon)^2 + ||\mathbf{x} - \mathbf{x}'||^2}{\eta\eta'} \tag{3.66}$$

$$y(x;x')_{++} = \frac{-\left(|\eta - \eta'| + i\varepsilon\right)^2 + ||\mathbf{x} - \mathbf{x}'||^2}{\eta\eta'}. \tag{3.67}$$

Propagators with these pole prescriptions are called the Wightman propagators ($i\Delta_{+-}$, $i\Delta_{-+}$) and the anti-time ordered propagator ($i\Delta_{--}$). These terms would also bring additional singularities with them; thus, if one requires no additional singularities, de Sitter invariance and the vacuum state to be precisely Bunch-Davies, the only choice is $\alpha = 1$ and $\beta = 0$.

We conclude this chapter by calculating the energy-momentum tensor. It is convenient to define derivatives of the hypergeometric function $G(y)$,

$$\frac{d^n}{dy^n}G(y) \equiv \frac{d^n}{dz^n}G(y) = \left(-\frac{1}{4}\right)^n G_n(y), \tag{3.68}$$

where $z = 1 - y/4$ and

$$G_n(y) = \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} {}_2F_1\left(a+n, b+n; c+n; 1 - \frac{y}{4}\right), \qquad G_0(y) \equiv G(y). \tag{3.69}$$

Using (3.41), (3.68) and some identities from appendix A it is straightforward to show that

$$\langle T_{\mu\nu}(x)\rangle = k(D)\Gamma\left(1 - \frac{D}{2}\right)H^D g_{\mu\nu}, \tag{3.70}$$

where, $k(D)$ is a dimensionless function of D not divergent for $D = 4$. Note that the gamma function diverges for $D = 4$. We could immediately absorb this infinity into a cosmological constant counterterm, which will become clear in chapter 5, to obtain

$$\langle T_{\mu\nu}^{ren}(x)\rangle = \Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f g_{\mu\nu}, \tag{3.71}$$

Where $\Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f$ is some finite constant. The specific value of $\langle T_{\mu\nu}\rangle$ is not relevant for this work; however, it is important to note that it is nonzero and will result in a nonvanishing tadpole diagram as shown in chapter 4.

Chapter 4

Graviton Self-Energy One Loop Corrections

In this chapter we will derive the expressions for the one loop corrections to the graviton self-energy from a nonminimally coupled massive scalar. We start by expanding the full metric around a classical de Sitter background and derive the cubic and quartic vertices. We then derive the graviton self-energy through the effective action. We will find that the nonlocal part of the graviton self-energy is proportional to the connected TT-correlator, which we check against the known results in the literature in the minimally coupled case $\xi = 0$ [35] and in the minimally coupled massless case $m = 0$, $\xi = 0$ [32]. The local contribution is, unlike the minimally coupled massless case, nonvanishing.

4.1   Cubic and Quartic Vertices

In order to find the cubic and quartic corrections to the graviton self-energy we expand the full metric around the classical de Sitter background metric

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \kappa h_{\mu\nu}, \ \kappa^2 = 16\pi G_N, \tag{4.1}$$

where we define the graviton field $h_{\mu\nu}$ as a small perturbation around the classical de Sitter background $\overline{g}_{\mu\nu}$ (3.6) and κ is the loop counting parameter. The inverse metric is then given by

$$g^{\mu\nu} = \overline{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha{}^\nu + \mathcal{O}(h^3). \tag{4.2}$$

We want to expand the action (3.18), where $g_{\mu\nu}$ now is the full metric, up to second order in h and split the result into a zeroth, first and second order part. The zeroth order part of the action will be the action for the nonminimally coupled scalar with respect to the background metric, which contains no interactions. The first and second order part will contain the cubic and quartic interactions respectively. First we need to expand the determinant of the metric

$$\sqrt{-g} = \exp\left[\frac{1}{2}\operatorname{Tr}\ln(-g_{\mu\nu})\right] = \sqrt{-\overline{g}}\left(1 + \frac{\kappa}{2}h + \kappa^2\left[\frac{1}{8}h^2 - \frac{1}{4}h^{\alpha\beta}h_{\alpha\beta}\right]\right), \tag{4.3}$$

where indices are lowered and raised by the background metric $\overline{g}_{\mu\nu}$ and $h \equiv h^\mu{}_\mu = \overline{g}^{\mu\nu}h_{\mu\nu}$. For the expansion of R we refer to [28] and state the result

$$R^{(1)}_{\mu\nu} = \frac{\kappa}{2}\big(-\overline{\nabla}_\mu\overline{\nabla}_\nu h - \overline{\Box}h_{\mu\nu} + \overline{\nabla}^\alpha\overline{\nabla}_\nu h_{\alpha\mu} + \overline{\nabla}^\alpha\overline{\nabla}_\mu h_{\alpha\nu}\big) \tag{4.4}$$

$$R^{(2)}_{\mu\nu} = \frac{\kappa^2}{2}\bigg[\frac{1}{2}\overline{\nabla}_\mu h_{\alpha\beta}\overline{\nabla}_\nu h^{\alpha\beta} + h^{\alpha\beta}\big(\overline{\nabla}_\mu\overline{\nabla}_\nu h_{\alpha\beta} + \overline{\nabla}_\alpha\overline{\nabla}_\beta h_{\mu\nu} - \overline{\nabla}_\nu\overline{\nabla}_\beta h_{\alpha\mu} - \overline{\nabla}_\mu\overline{\nabla}_\beta h_{\alpha\nu}\big)$$
$$+ \overline{\nabla}^\beta h_\nu{}^\alpha\big(\overline{\nabla}_\beta h_{\alpha\mu} - \overline{\nabla}_\alpha h_{\beta\mu}\big)$$
$$- \big(\overline{\nabla}_\beta h^{\alpha\beta} - \frac{1}{2}\overline{\nabla}^\alpha h\big)\big(\overline{\nabla}_\nu h_{\alpha\mu} + \overline{\nabla}_\mu h_{\alpha\nu} - \overline{\nabla}_\alpha h_{\mu\nu}\big)\bigg], \tag{4.5}$$

where the covariant derivatives are taken with respect to the background metric. We are now in a position to expand all the quantities in (3.18). Using (4.2), (4.3), (4.4) and (4.5) we find the intermediate result

$$\sqrt{-g}R = \sqrt{-\overline{g}}\bigg\{\overline{R} - \kappa h^{\mu\nu}\overline{G}_{\mu\nu} - \big(\overline{g}_{\mu\nu}\overline{\Box} - \overline{\nabla}_\mu\overline{\nabla}_\nu\big)\kappa h^{\mu\nu} + \kappa^2 h^{\mu\nu}\bigg[\Big(\frac{1}{8}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma} - \frac{1}{4}\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma}\Big)\overline{R}$$
$$- \frac{1}{2}\overline{g}_{\rho\sigma}\overline{R}_{\mu\nu} + \overline{g}_{\nu\rho}\overline{R}_{\mu\sigma}\bigg]h^{\rho\sigma}$$
$$+ \kappa^2 h^{\mu\nu}\bigg[\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma}\overline{\Box} + \overline{g}_{\rho\sigma}\overline{\nabla}_\mu\overline{\nabla}_\nu - 2\overline{g}_{\mu\rho}\overline{\nabla}_\sigma\overline{\nabla}_\nu - \frac{1}{2}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}\overline{\Box} + \frac{1}{2}\overline{g}_{\mu\nu}\overline{\nabla}_\rho\overline{\nabla}_\sigma\bigg]h^{\rho\sigma}$$
$$+ \kappa^2\bigg[\frac{3}{4}\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma}\overline{\nabla}^\alpha h^{\mu\nu}\overline{\nabla}_\alpha h^{\rho\sigma} - \frac{1}{2}\overline{g}_{\mu\sigma}\overline{\nabla}_\rho h^{\mu\nu}\overline{\nabla}_\nu h^{\rho\sigma} - \overline{g}_{\mu\rho}\overline{\nabla}_\nu h^{\mu\nu}\overline{\nabla}_\sigma h^{\rho\sigma}$$
$$+ \overline{g}_{\rho\sigma}\overline{\nabla}_\nu h^{\mu\nu}\overline{\nabla}_\mu h^{\rho\sigma} - \frac{1}{4}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}\overline{\nabla}^\alpha h^{\mu\nu}\overline{\nabla}_\alpha h^{\rho\sigma}\bigg]\bigg\}, \tag{4.6}$$

where $\overline{R}$ and $\overline{R}_{\mu\nu}$ are the Ricci scalar and Ricci tensor of the background metric. Finally we find

$$S = \overline{S} + S^{(3)} + S^{(4)}, \tag{4.7}$$

with

$$\overline{S} = \int \mathrm{d}^D x\,\sqrt{-\overline{g}}\left[-\frac{1}{2}\overline{g}^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{1}{2}\xi\overline{R}\phi^2(x)\right], \tag{4.8}$$

which is precisely the action of the nonminimally coupled massive scalar on the de Sitter background. Secondly the first order part

$$S^{(3)} = \int \mathrm{d}^D x\,\sqrt{-\overline{g}}\frac{\kappa}{2}h^{\mu\nu}(x)\overline{T}_{\mu\nu}(x), \tag{4.9}$$

which contains cubic interactions between one graviton and two scalars (left diagram of Fig. 4.1), where $\overline{T}_{\mu\nu}(x)$ is precisely the energy-momentum tensor on the background metric (2.11)

$$\overline{T}_{\mu\nu}(x) = \frac{2}{\kappa\sqrt{-\overline{g}}(x)}\frac{\delta S}{\delta h^{\mu\nu}(x)}\bigg|_{h\to 0} = \partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}\overline{g}_{\mu\nu}(x)\overline{g}^{\alpha\beta}(x)\partial_\alpha\phi(x)\partial_\beta\phi(x)$$
$$- \frac{1}{2}\overline{g}_{\mu\nu}(x)m^2\phi(x)^2 + \xi\left[\overline{G}_{\mu\nu}(x) + \overline{g}_{\mu\nu}(x)\overline{\Box} - \overline{\nabla}_\mu\overline{\nabla}_\nu\right]\phi(x)^2. \tag{4.10}$$

Note the sign in the definition of $\overline{T}_{\mu\nu}$, which comes from the positive sign of the $h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ part of the expanded action. Lastly the second order part

$$S^{(4)} = \int \mathrm{d}^D x\,\sqrt{-\overline{g}}\int \mathrm{d}^D x'\,\frac{\kappa^2}{2}h^{\mu\nu}(x)V_{\mu\nu\rho\sigma}(x-x')h^{\rho\sigma}(x'), \tag{4.11}$$

Figure 4.1: The cubic vertex corresponding to graviton-scalar-scalar interactions to the left and the quartic vertex corresponding to graviton-graviton-scalar-scalar interactions to the right. Curly lines are gravitons and dashed lines are scalars.

which contains quartic interactions between two gravitons and two scalars (right diagram of Fig. 4.1), with

$$
V_{\mu\nu\rho\sigma}(x - x') = \frac{1}{\kappa^2 \sqrt{-\overline{g}}(x)} \frac{\delta^2 S}{\delta h^{\rho\sigma}(x') \delta h^{\mu\nu}(x)} \tag{4.12}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\Bigg\{ \left[\frac{1}{4}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma} - \frac{1}{2}\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu} \right] \left(\overline{g}^{\alpha\beta}\partial_\alpha\phi(x)\partial_\beta\phi(x) + m\phi^2(x) + \xi\overline{R}\phi^2(x) \right) \\
&\quad - \frac{1}{2}\left(\overline{g}_{\rho\sigma}\partial_{(\mu}\phi(x)\partial_{\nu)}\phi(x) + \overline{g}_{\mu\nu}\partial_{(\rho}\phi(x)\partial_{\sigma)}\phi(x) \right) \\
&\quad + \overline{g}_{\rho)(\nu}\partial_{\mu)}\phi(x)\partial_{(\sigma}\phi(x) + \overline{g}_{\mu)(\rho}\partial_{\sigma)}\phi(x)\partial_{(\nu}\phi(x) \\
&\quad + \xi\phi^2(x)\left[\overline{g}_{\rho)(\nu}\overline{R}_{\mu)(\sigma} + \overline{g}_{\mu)(\rho}\overline{R}_{\sigma)(\nu} - \frac{1}{2}\left(\overline{g}_{\rho\sigma}\overline{R}_{\mu\nu} + \overline{g}_{\mu\nu}\overline{R}_{\rho\sigma} \right) \right] \Bigg\} \delta^D(x - x') \\
&\quad - \frac{1}{2}\xi\Bigg\{ \left(\phi^2(x) + \phi^2(x') \right) \left[\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu}\overline{\Box} - 2\overline{g}_{\mu)(\rho}\overline{\nabla}_{\sigma)}\overline{\nabla}_{(\nu} - \frac{1}{2}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}\overline{\Box} \right] \\
&\quad + \left[\frac{1}{2}\phi^2(x) + \phi^2(x') \right] \overline{g}_{\mu\nu}\overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)} \\
&\qquad\qquad\qquad + \left[\frac{1}{2}\phi^2(x') + \phi^2(x) \right] \overline{g}_{\rho\sigma}\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)} \\
&\quad - \frac{3}{2}\phi^2(x)\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu}\overline{\Box} + 3\phi^2(x)\overline{g}_{\mu)(\rho}\overline{\nabla}_{\sigma)}\overline{\nabla}_{(\mu} \\
&\quad - \phi^2(x)\left[\overline{g}_{\mu\nu}\overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)} + \overline{g}_{\rho\sigma}\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)} \right] \\
&\quad + \frac{1}{2}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}\phi^2(x)\overline{\Box} \Bigg\} \delta^D(x - x') + \text{terms proportional to } \nabla\phi^2(x), \tag{4.13}
\end{aligned}
$$

where the terms proportional to $\nabla\phi^2(x)$ will vanish once we take the expectation value because $\langle \phi(x)^2 \rangle$ is a constant on de Sitter.
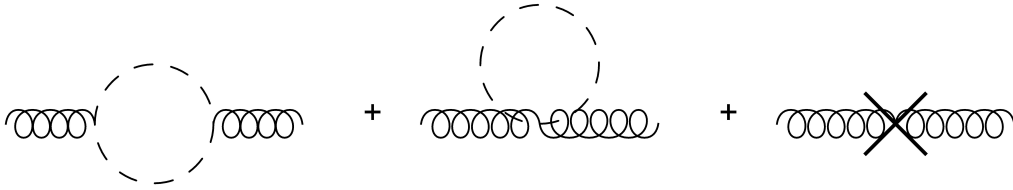
Figure 4.2: Diagrammatic representation of the graviton self-energy $-i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big](x;x')$ in the presence of a nonminimally coupled massive scalar to one loop order. The left diagram consists of two cubic vertices and is proportional to $\propto \langle \mathcal{T}\{\hat{\overline{T}}_{\mu\nu}(x)\hat{\overline{T}}_{\rho\sigma}(x')\}\rangle$. The middle diagram consists of one quartic vertex and gives a local contribution to the graviton self-energy. The right diagram is the counterterm vertex. Curly lines are gravitons and dashed lines are scalars.

4.2 One loop Corrections

The effective action can be obtained from

$$e^{i\Gamma[g]} = \int \mathcal{D}\phi\, e^{iS[\overline{g}+h,\phi]} = \int \mathcal{D}\phi\, e^{i(\overline{S}+S^{(3)}+S^{(4)})}$$
$$= \int \mathcal{D}\phi\, e^{i\overline{S}}\big[1 + iS^{(3)} + iS^{(4)} + \tfrac{1}{2}(iS^{(3)})^2 + \cdots\big] \tag{4.14}$$
$$= 1 + \langle iS^{(3)}\rangle + \langle iS^{(4)}\rangle + \tfrac{1}{2}\langle(iS^{(3)})^2\rangle + \cdots,$$

where we have expanded the interacting part of the action up to relevant order [19][45]. The metric is in principle a quantum field as well and one would expand the metric as a mean plus fluctuations i.e. $\hat{g} = \langle\hat{g}\rangle + \kappa h$. We are however expanding around a classical background $\overline{g} \neq \langle\hat{g}\rangle$, which as we will see shortly, has consequences. Moreover, one would similarly expand the scalar field $\phi = \langle\phi\rangle + \varphi$ and integrate out the fluctuations. However, in our case there is no scalar condensate as we discussed in chapter 3.2; therefore, the fluctuations are just ϕ. Next we expand the left-hand side of (4.14)

$$1 + i\Gamma[g] = 1 + i\Gamma[\overline{g}] + i\Gamma_h[\overline{g}] + i\Gamma_{hh}[\overline{g}] + i\Gamma_{\geq h^3}[\overline{g}], \tag{4.15}$$

where

$$i\Gamma_{hh} = -\frac{1}{2}\int \mathrm{d}^D x \int \mathrm{d}^D x'\, h^{\mu\nu}(x) i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big](x;x')h^{\rho\sigma}(x'), \tag{4.16}$$

with

$$-i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big](x;x') = -i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big]_{3pt}(x;x') + -i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big]_{4pt}(x;x') + \text{counterterms} \tag{4.17}$$

The graviton self-energy $_{\mu\nu}\Sigma_{\rho\sigma}(x;x')$, which is the sum of the diagrams in Fig. 4.2, can then be read off by comparing (4.14) with (4.15). Note that at this point we have not yet introduced counterterms; thus, the counterterm diagram is not included and the graviton self-energy is
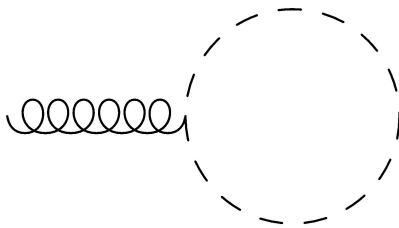
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Figure 4.3: The tadpole diagram corresponding to $\langle S^{(3)} \rangle$. It quantum corrects the classical background $\overline{g}_{\mu\nu}$ towards the true quantum average metric $\langle \hat{g}_{\mu\nu} \rangle$ to one loop order. Curly lines are gravitons and dashed lines are scalars.

therefore not yet renormalized. Writing down the expectation values in (4.14) explicitly we find

$$\frac{1}{2}\langle (iS^{(3)})^2 \rangle = -\frac{1}{2} \int \mathrm{d}^D x \, \sqrt{-\overline{g}(x)} \int \mathrm{d}^D x' \, \sqrt{-\overline{g}(x')} \frac{\kappa^2}{4} h^{\mu\nu}(x) \langle \mathcal{T}\{\hat{\overline{T}}_{\mu\nu}(x)\hat{\overline{T}}_{\rho\sigma}(x')\}\rangle h^{\rho\sigma}(x')$$
(4.18)

$$\langle iS^{(4)} \rangle = i \int \mathrm{d}^D x \sqrt{-\overline{g}(x)} \int \mathrm{d}^D x' \, \frac{\kappa^2}{2} h^{\mu\nu}(x) \langle V_{\mu\nu\rho\sigma}(x-x') \rangle h^{\rho\sigma}(x')$$
(4.19)

$$\langle iS^{(3)} \rangle = i \int \mathrm{d}^D x \sqrt{-\overline{g}(x)} \frac{\kappa}{2} h^{\mu\nu} \langle \overline{T}_{\mu\nu}(x) \rangle.$$
(4.20)

Note that the $S^{(3)}$ part contains a cubic vertex Fig. 4.1, thus $\langle (iS^{(3)})^2 \rangle$ contains two cubic vertices. The only diagram containing two cubic vertices at this order is the left diagram in Fig. 4.2 to which it corresponds. Similarly the $\langle S^{(4)} \rangle$ part gives rise to a quartic vertex Fig. 4.1 and relates to the middle figure of Fig. 4.2. Note that we also have a tadpole diagram Fig. 4.3 coming from the $S^{(3)}$ part. It is proportional to the energy-momentum tensor $\propto \langle \overline{T}_{\mu\nu} \rangle$, therefore nonzero since $\langle \overline{T}_{\mu\nu} \rangle \neq 0$ (3.70). The nonvanishing tadpole arises from the expansion around a classical background instead of the average of a quantum operator $\langle \hat{g} \rangle$ and it quantum corrects our classical background on de Sitter. Finally we can write down the nonrenormalized graviton self-energy

$$-i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big]_{3pt}(x;x') = -\frac{\kappa^2}{4}\sqrt{-\overline{g}(x)}\sqrt{-\overline{g}(x')}\langle \mathcal{T}\{\hat{\overline{T}}_{\mu\nu}(x)\hat{\overline{T}}_{\rho\sigma}(x')\}\rangle,$$
(4.21)

and

$$-i\big[_{\mu\nu}\Sigma_{\rho\sigma}\big]_{4pt}(x;x') = i\kappa^2 \sqrt{-\overline{g}(x)}\langle V_{\mu\nu\rho\sigma}(x-x') \rangle.$$
(4.22)

For the explicit expression of the self-energy we need the value of the TT-correlator and $\langle V_{\mu\nu\rho\sigma} \rangle$. The derivation of the TT-correlator $\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x')\}\rangle$ is a lengthy calculation and has been the main work of this thesis. Here we state the result and for the full calculation we refer to

appendix B where it is done in detail. The connected TT-correlator is given by

$$\langle \mathcal{T}\{\hat{\bar{T}}_{\mu\nu}(x)\hat{\bar{T}}_{\rho\sigma}(x')\}\rangle = \frac{H^{2D-4}}{(4\pi)^D}\Bigg\{[\partial_\mu y\partial_\nu y\partial'_\rho y\partial'_\sigma y]\Big[4\xi^2\frac{d^4G}{dy^4}G - 8\xi(1-2\xi)\frac{d^3G}{dy^3}\frac{dG}{dy}$$

$$+ 2(1-4\xi+6\xi^2)\Big(\frac{d^2G}{dy^2}\Big)^2\Big]$$

$$+ [\partial_{(\mu}y\partial_{\nu)}\partial'_{(\rho}y\partial'_{\sigma)}y]\Big[16\xi^2\frac{d^3G}{dy^3}G + 4(1-8\xi+12\xi^2)\frac{d^2G}{dy^2}\frac{dG}{dy}\Big]$$

$$+ [\partial_{(\mu)}\partial'_{(\rho}y\partial'_{\sigma)}\partial_{(\nu)}y]\Big[8\xi^2\frac{d^2G}{dy^2}G + 2(1-2\xi)^2\Big(\frac{dG}{dy}\Big)^2\Big]$$

$$+ H^2[\partial_\mu y\partial_\nu y\bar{g}'_{\rho\sigma} + \bar{g}_{\mu\nu}\partial'_\rho y\partial'_\sigma y]\Big[(2-y)[4\xi^2]\frac{d^3G}{dy^3}G$$

$$+ (2-y)\Big[D-2-8(D-1)\xi+4(4D-1)\xi^2\Big]\frac{d^2G}{dy^2}\frac{dG}{dy}$$

$$+ \Big[(1-M^2)+4(D-1+2M^2)\xi-4(3D-1+4M^2)\xi^2\Big]\Big(\frac{dG}{dy}\Big)^2 \qquad (4.23)$$

$$+ \Big[-M^2+8M^2\xi-4(D+1+4M^2)\xi^2\Big]\frac{d^2G}{dy^2}G\Big]$$

$$+ [H^4\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}]\Big[16\xi^2\frac{d^2G}{dy^2}G + \Big[2(D^2-D-4)-16(D^2-3)\xi+16(2D^2+2D-3)\xi^2$$

$$+ \frac{4y-y^2}{2}\Big(-(D-1)^2+2M^2+4((2D-1)(D-1)-4M^2)\xi$$

$$- 8(2D^2-2D+1-4M^2)\xi^2\Big)\Big]\Big(\frac{dG}{dy}\Big)^2$$

$$+ (2-y)\Big[-(D-1)M^2+8DM^2\xi-4(D-1+4(D+1)M^2)\xi^2\Big]\frac{dG}{dy}G$$

$$+ \Big[M^4-2M^2((D-1)+4M^2)\xi+2((D-1)^2+2(2D-3)M^2+8M^4)\xi^2\Big]G^2\Big]\Bigg\}.$$

Note that in the minimal coupling limit, when $\xi = 0$, the TT-correlator is much simpler as the terms that are most difficult to renormalize drop out. The nonrenormalized TT-correlator for a minimally coupled massive scalar on de Sitter was calculated by Perez-Nadal, Roura and Verdaguer [35]. Park and Woodard [32] used the result of Perez-Nadal, Roura and Verdaguer to check their calculation of the graviton self-energy for the minimally coupled massless scalar, which was derived by the use of Feynman rules, and found agreement. We will now compare our result (4.23) in the limit $\xi = 0$ with the result of Perez-Nadal, Roura and Verdaguer. The connected TT-correlator of Perez-Nadal, Roura and Verdaguer, denoted $F_{abc'd'}$, for the minimally coupled massive case is given by (28) in [35]:

$$F_{abc'd'} = P(\mu)n_an_bn_{c'}n_{d'} + Q(\mu)(n_an_b\bar{g}_{c'd'} + n_{c'}n_{d'}\bar{g}_{ab})$$
$$+ R(\mu)(4n_{(a}\bar{g}_{b)(c'}n_{d')} + S(\mu)2\bar{g}_{a(c'}\bar{g}_{d')b} + T(\mu)\bar{g}_{ab}\bar{g}_{c'd'},. \qquad (4.24)$$

where (un)primed indices always go with (un)primed coordinates. Note that their TT-correlator is expressed in five basis tensors that are different from the ones we use and the variable μ. They are defined as follows:

- $\mu(x; x')$ is the geodesic distance between x and x', which in our notation is $\mu(x; x') = H\ell(x; x')$;

- n_a and $n_{a'}$ are the unit tangent vectors to the geodesic at the points x and x', pointing outward, and defined as $n_a = \nabla_a \mu$ and $n_{a'} = \nabla_{a'} \mu$;

- $\overline{g}_{ab'}$ is the parallel propagator which parallel transports a vector from x to x' along the geodesic, which at coincidence becomes the metric;

- $\overline{g}_{ab}$ and $\overline{g}_{c'd'}$ are the metric tensors at the points x and x' respectively.

From (3.17) we find that the de Sitter invariant distance can be written in terms of $\mu(x; x')$ as

$$\cos(\mu) \equiv Z = 1 - \frac{y}{2}, \tag{4.25}$$

and from (4.25) we find the relation

$$\frac{dy}{d\mu} = \sqrt{4y - y^2}. \tag{4.26}$$

We can now convert their tensors into ours which was done in [32]. The results are

$$n_a = \frac{1}{H\sqrt{4y - y^2}} \partial_a y,$$

$$n_{b'} = \frac{1}{H\sqrt{4y - y^2}} \partial_{b'} y, \tag{4.27}$$

$$\overline{g}_{ab'} = -\frac{1}{2H^2}\left[\partial_a \partial_{b'} y + \frac{1}{4 - y}\partial_a y \partial_{b'} y\right],$$

where factors of H which are set to 1 in [35] have been restored. The five basis tensors can now be written in terms of our five basis tensors,

$$n_a n_b n_{c'} n_{d'} = \frac{1}{H^2(4y - y^2)^2} \partial_a y \partial_b y \partial_{c'} y \partial_{d'} y,$$

$$n_a n_b \overline{g}_{c'd'} + n_{c'} n_{d'} \overline{g}_{ab} = \frac{1}{H^2(4y - y^2)}\left[\overline{g}_{ab}\partial_{c'} y \partial_{d'} y + \partial_a y \partial_b y \overline{g}_{c'd'}\right],$$

$$4 n_{(a}\overline{g}_{b)(c'} n_{d')} = -\frac{2}{H^4(4y - y^2)}\partial_{(a} y \partial_{b)}\partial_{(c'} y \partial_{d')} y - \frac{2}{H^4(4y - y^2)(4 - y)}\partial_a y \partial_b y \partial_{c'} y \partial_{d'} y,$$

$$2\overline{g}_{a(c'}\overline{g}_{d')b} = \frac{1}{2H^4}\partial_{a)}\partial_{(c'} y \partial_{d')}\partial_{(b} y + \frac{1}{H^4(4 - y)}\partial_{(a} y \partial_{b)}\partial_{(c'} y \partial_{d')} y,$$

$$+ \frac{1}{2H^4(4 - y)^2}\partial_a y \partial_b y \partial_{c'} y \partial_{d'} y,$$

$$\overline{g}_{ab}\overline{g}_{c'd'} = \overline{g}_{ab}\overline{g}_{c'd'}. \tag{4.28}$$

Next we need the μ-dependent coefficients P, Q, R, S and T. They are [35]

$$P = 2\mathcal{G}_1^2,$$

$$Q = \mathcal{G}_1^2 + 2\mathcal{G}_1\mathcal{G}_2 - \frac{m^2}{H^2}\mathcal{G}'^2,$$

$$R = \mathcal{G}_1\mathcal{G}_2, \qquad (4.29)$$

$$S = \mathcal{G}_2^2,$$

$$T = \frac{1}{2}\mathcal{G}_1^2 - \mathcal{G}_1\mathcal{G}_2 + \frac{D-4}{2}\mathcal{G}_2^2 + \frac{m^2}{H^2}\mathcal{G}'^2 + \frac{1}{2}\frac{m^4}{H^4}\mathcal{G}^2,$$

where $\mathcal{G}$ is the minimally coupled massive scalar propagator

$$\mathcal{G}(\mu) = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D-1}{2}+\nu_D\right)\Gamma\left(\frac{D-1}{2}-\nu_D\right)}{\Gamma\left(\frac{D}{2}\right)}$$

$$\times\,{}_2F_1\left(\frac{D-1}{2}+\nu_D,\frac{D-1}{2}-\nu_D;\frac{D}{2};1-\frac{y}{4}\right), \qquad (4.30)$$

with

$$\nu_D^2 = \left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2}, \qquad (4.31)$$

and $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined as

$$\mathcal{G}(\mu) = \mathcal{G}''(\mu) - \mathcal{G}'(\mu)\csc\mu, \qquad (4.32)$$
$$\mathcal{G}_2(\mu) = -\mathcal{G}'\csc\mu, \qquad (4.33)$$

where primes on $\mathcal{G}$ are derivatives with respect to μ. By the chain rule and (4.26) we find

$$\frac{d\mathcal{G}}{d\mu} = \sqrt{4y-y^2}\frac{d\mathcal{G}}{dy}, \qquad (4.34)$$

$$\frac{d^2\mathcal{G}}{d\mu^2} = (4y-y^2)\frac{d^2\mathcal{G}}{dy^2} + (2-y)\frac{d\mathcal{G}}{dy}. \qquad (4.35)$$

We can now rewrite the expressions for P, Q, R, S and T (4.29) in terms of y and derivatives of $\mathcal{G}$ as follows

$$P = 2(4y - y^2)^2 \left(\frac{d^2\mathcal{G}}{dy^2}\right)^2 - 4y(4y - y^2)\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy} + 2y^2\left(\frac{d\mathcal{G}}{dy}\right)^2,$$

$$Q = -(4y - y^2)^2\left(\frac{d^2\mathcal{G}}{dy^2}\right)^2 - 2(2 - y)(4y - y^2)\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy} + (4y - y^2)\left(\frac{d\mathcal{G}}{dy}\right)^2$$
$$- \frac{m^2}{H^2}(4y - y^2)\left(\frac{d\mathcal{G}}{dy}\right)^2,$$

$$R = -2(4y - y^2)\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy} + 2y\left(\frac{d\mathcal{G}}{dy}\right)^2,$$

$$S = 4\left(\frac{d\mathcal{G}}{dy}\right)^2,$$

$$T = \frac{1}{2}\left[(4y - y^2)^2\left(\frac{d^2\mathcal{G}}{dy^2}\right)^2 + 2(2 - y)(4y - y^2)\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy} + [4(D - 4) - (4y - y^2)]\left(\frac{d\mathcal{G}}{dy}\right)^2\right]$$
$$+ \frac{m^2}{H^2}(4y - y^2)\left(\frac{d\mathcal{G}}{dy}\right)^2 + \frac{1}{2}\frac{m^4}{H^4}\mathcal{G}^2.$$

$$(4.36)$$

Using (4.28), (4.36) and the hypergeometric differential equation for $\mathcal{G}$

$$\left[(4y - y^2)\frac{d^2}{dy^2} + D(2 - y)\frac{d}{dy} - \frac{m^2}{H^2}\right]\mathcal{G}(y) = 0, \tag{4.37}$$

to reduce all second order derivative terms of the form $(4y - y^2)\frac{d^2\mathcal{G}}{dy^2}$, we can write the TT-correlator of [35] as

$$F_{\mu\nu\rho'\sigma'} = [\partial_\mu y \partial_\nu y \partial'_\rho y \partial'_\sigma y] 2\left(\frac{d^2\mathcal{G}}{dy^2}\right)^2 + [\partial_{(\mu}y\partial_{\nu)}\partial'_{(\rho}y\partial'_{\sigma)}y] 4\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy}$$
$$+ [\partial_{\mu)}\partial'_{(\rho}y\partial'_{\sigma)}\partial_{(\nu}y] 2\left(\frac{d\mathcal{G}}{dy}\right)^2$$
$$+ H^2\left[\partial_\mu y\partial_\nu y\overline{g}'_{\rho\sigma} + \overline{g}_{\mu\nu}\partial'_\rho y\partial'_\sigma y\right]\left[(2 - y)(D - 2)\frac{d^2\mathcal{G}}{dy^2}\frac{d\mathcal{G}}{dy} + \left(1 - \frac{m^2}{H^2}\right)\left(\frac{d\mathcal{G}}{dy}\right)^2 - \frac{m^2}{H^2}\frac{d^2\mathcal{G}}{dy^2}\mathcal{G}\right]$$
$$+ \left[H^4\overline{g}_{\mu\nu}\overline{g}'_{\rho\sigma}\right]\left[\left[2(D^2 - D - 4) - \frac{4y - y^2}{2}(D - 1)^2\right]\left(\frac{d\mathcal{G}}{dy}\right)^2 - (2 - y)(D - 1)\frac{m^2}{H^2}\frac{d\mathcal{G}}{dy}\mathcal{G} + \frac{m^4}{H^4}\mathcal{G}^2\right].$$

$$(4.38)$$

In this form we can compare it against our result (4.23) by noting that in the minimal coupling limit G (3.61) and $\mathcal{G}$ (4.30) are related by

$$\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}G(y) = \mathcal{G}(y). \tag{4.39}$$

Finally we can take the minimal coupling limit by making the replacements

$$\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} G(y) \to \mathcal{G}(y), \tag{4.40}$$

$$M^2 \to \frac{m^2}{H^2}, \tag{4.41}$$

$$\nu_D^2 \to \left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2}, \tag{4.42}$$

$$\xi \to 0, \tag{4.43}$$

in (4.23) to find that it reduces to the TT-correlator (4.38) derived in [35]. We will now take the massless limit of (4.38) and compare it to the result of Park and Woodard [32], who already checked their work against [35]; however, since we have used the hypergeometric equation to reduce higher derivative terms we cannot directly compare our result with theirs. The propagator for the minimally coupled massless scalar is given by

$$i\Delta_{\text{MMC}}(x; x') = A(y) + k \ln(aa'), \tag{4.44}$$

where $k \ln(aa')$ is a de Sitter breaking term as their exists no de Sitter invariant propagator for the massless minimally coupled scalar [3]. However, at one loop order the scalar contributions to the graviton self-energy only involve second derivatives of the propagator $\partial_\mu \partial_\nu' i\Delta_{\text{MMC}}(x; x')$ [32]; therefore, only derivatives of $A(y)$ will enter the graviton self-energy at one loop, which are de Sitter invariant. This fact will turn out to be crucial in obtaining the minimally coupled massless TT-correlator from (4.38). The function $A(y)$ has the expansion

$$A(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \left(\frac{y}{4}\right)^{1-\frac{D}{2}} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \left(\frac{y}{4}\right)^{2-\frac{D}{2}} - \pi \cot\left(\frac{\pi D}{2}\right) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \right. \tag{4.45}$$

$$\left. + \sum_{n=0}^{\infty} \left[\frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \left(\frac{y}{4}\right)^n - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-\frac{D}{2}+2} \right], \tag{4.46}$$

which obeys the equation

$$(4y - y^2) A''(y) + D(2-y) A'(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D)}{\Gamma(\frac{D}{2})}, \tag{4.47}$$

which looks almost like a hypergeometric differential equation. Note that in the massless limit the propagator $\mathcal{G}(y)$ (4.30) is ill defined, but its derivatives are not. Expanding $\mathcal{G}(y)$ by using the Gauss transformation formula (5.10) and the series expansion for the hypergeometric function (5.11) the formal series expansion for $\mathcal{G}(y)$ for $m = 0$ reads

$$\mathcal{G}(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1} \left(\frac{y}{4}\right)^{1-\frac{D}{2}} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2} \left(\frac{y}{4}\right)^{2-\frac{D}{2}} - \Gamma(0) \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \right. \tag{4.48}$$

$$\left. + \sum_{n=0}^{\infty} \left[\frac{1}{n} \frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})} \left(\frac{y}{4}\right)^n - \frac{1}{n-\frac{D}{2}+2} \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-\frac{D}{2}+2} \right]. \tag{4.49}$$

Note that while $\mathcal{G}(y)$ and $A(y)$ are not the same they differ only by a divergent but constant term in the massless limit; therefore, the derivatives of $\mathcal{G}(y)$ and $A(y)$ are identical and well defined for $m = 0$. As a result we can set $m = 0$ in all terms that contain only derivatives of $\mathcal{G}$. We

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must now deal with the terms proportional to $m^2/H^2\mathcal{G}$ in (4.23). In the small mass expansion $\mathcal{G}$ reduces to

$$\mathcal{G}(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{H^2}{m^2}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})},\tag{4.50}$$

it follows that

$$\lim_{m\to 0}\frac{m^2}{H^2}\mathcal{G}(y) = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})},\tag{4.51}$$

where we note it is the same as the constant on the right hand side of (4.47). The hypergeometric differential equation (4.37) then reduces to (4.47). Park and Woodard [32] and Perez-Nadal, Roura, Verdaguer [35] obtained the massless limit by discarding the term $M^4\mathcal{G}^2$ before taking the limit $m \to 0$, which is the only term that does not contain derivatives of $\mathcal{G}$ in the TT-correlator before applying the hypergeometric differential equation. This is justified because only derivatives of the propagator enter the self-energy at one loop order. In our case the application of the hypergeometric differential equation gives rise to additional factors that do not contain derivatives of G. Obtaining the correct massless limit would then require us to identify and discard the mass term that does not enter the TT-correlator from the application of the hypergeometric differential equation; and trace back the mass terms that do enter the TT-correlator through the hypergeometric differential equation, then take the limit $m \to 0$ so that we can apply (4.47). Not discarding the term $M^4\mathcal{G}^2$ and taking $m \to 0$ will not give the correct massless TT-correlator as it will add a constant contribution descending from the nonvanishing limit (4.51) nor will discarding all mass terms coming from the application of the hypergeometric function as the right hand side of (4.47) is nonzero! Because of these subtleties one might be better off using the TT-correlator in its form before higher derivatives were reduced through the hypergeometric differential equation, which we will do now. We can piece together the minimally coupled massless TT-correlator from appendix B by discarding the term $m^4/H^4\mathcal{G}$; setting $m = 0$ in all other terms since they come with derivatives of $\mathcal{G}$, which are well defined; and making the replacement $\mathcal{G}^{(n)}(y) \to A^{(n)}(y)$ to find

$$\langle\mathcal{T}\{\hat{\bar{T}}_{\mu\nu}(x)\hat{\bar{T}}_{\rho\sigma}(x')\}\rangle = [\partial_\mu y\partial_\nu y\partial'_\rho y\partial'_\sigma y]2\Big(\frac{d^2A}{dy^2}\Big)^2 + [\partial_{(\mu}y\partial_{\nu)}\partial'_{(\rho}y\partial'_{\sigma)}y]4\frac{d^2A}{dy^2}\frac{dA}{dy}$$

$$+ [\partial_{\mu)}\partial'_{(\rho}y\partial'_{\sigma)}\partial_{(\nu}y]2\Big(\frac{dA}{dy}\Big)^2$$

$$- H^2[\partial_\mu y\partial_\nu y\bar{g}'_{\rho\sigma} + \bar{g}_{\mu\nu}\partial'_\rho y\partial'_\sigma y]\Big[(4y - y^2)\Big(\frac{d^2A}{dy^2}\Big)^2 + 2(2 - y)\frac{dA}{dy}\frac{d^2A}{dy^2} - \Big(\frac{dA}{dy}\Big)^2\Big]\tag{4.52}$$

$$+ [H^4\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}]\Big[\frac{1}{2}(4y - y^2)^2\Big(\frac{d^2A}{dy^2}\Big)^2 + (2 - y)(4y - y^2)\frac{dA}{dy}\frac{d^2A}{dy^2}$$

$$+ [2(D - 4) - \frac{1}{2}(4y - y^2)]\Big(\frac{dA}{dy}\Big)^2\Big].$$

The result agrees with Park and Woodard [32].

For the expectation value of $V_{\mu\nu\rho\sigma}$ we need to calculate $\langle\phi(x)^2\rangle$ and $\langle\partial_\mu\phi(x)\partial_\nu\phi(x)\rangle$. Note that these are simply the propagator (3.61) and the second derivative of the propagator in the coincidence limit $(x' \to x)$ i.e.

$$\langle\phi(x)^2\rangle = i\Delta(x;x')|_{x'\to x}\tag{4.53}$$

$$\langle\partial_\mu\phi(x)\partial_\nu\phi(x)\rangle = \partial_\mu\partial'_\nu i\Delta(x;x')|_{x'\to x}.\tag{4.54}$$

Using the following identities

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c - a - b)}{\Gamma(c - a)\,\Gamma(c - b)}, \quad \mathrm{Re}(c - a - b) > 0, \tag{4.55}$$

$$\partial_\mu y(x; x') = aH(\delta^0{}_\mu y + 2a'H\Delta x_\mu), \tag{4.56}$$

$$\partial'_\nu y(x; x') = a'H(\delta^0{}_\nu y - 2aH\Delta x_\nu), \tag{4.57}$$

$$\partial_\mu \partial'_\nu y(x; x') = aa'H^2(\delta^0{}_\mu \delta^0{}_\nu y - 2a\delta^0{}_\mu H\Delta x_\nu + 2a'\delta^0{}_\nu H\Delta x_\mu - 2\eta_{\mu\nu}), \tag{4.58}$$

and (3.69) we find

$$\langle \phi^2(x) \rangle = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)\Gamma\left(\frac{1}{2} - \nu\right)} \Gamma\left(1 - \frac{D}{2}\right), \tag{4.59}$$

and

$$\langle \partial_\mu \phi(x) \partial_\nu \phi(x) \rangle = \frac{1}{2} \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)\Gamma\left(\frac{1}{2} - \nu\right)} \left[\frac{(D-1)^2}{4} - \nu^2\right] \Gamma\left(-\frac{D}{2}\right) H^2 \overline{g}_{\mu\nu}. \tag{4.60}$$

Using the results (4.59), (4.60) and (4.13) we find

$$\langle V_{\mu\nu\rho\sigma}(x - x') \rangle = -\frac{1}{2} \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)\Gamma\left(\frac{1}{2} - \nu\right)} \left\{ \right.$$

$$\left[\frac{1}{2} H^2 \left(\frac{(D-1)^2}{4} - \nu^2\right) \Gamma\left(-\frac{D}{2}\right)(D - 4) \right.$$

$$\left. + \Gamma\left(1 - \frac{D}{2}\right)\left(\frac{m^2}{H^2} + \xi(D-1)(D-4)\right)\right] H^2 \left[\frac{1}{4} \overline{g}_{\mu\nu} \overline{g}_{\rho\sigma} - \frac{1}{2} \overline{g}_{\mu(\rho} \overline{g}_{\sigma)\nu}\right]$$

$$+ \xi\Gamma\left(1 - \frac{D}{2}\right)\left[\frac{1}{2} \overline{g}_{\mu(\rho} \overline{g}_{\sigma)\nu} \overline{\Box} - \overline{g}_{\mu)(\rho} \overline{\nabla}_{\sigma)} \overline{\nabla}_{(\nu} - \frac{1}{2} \overline{g}_{\mu\nu} \overline{g}_{\rho\sigma} \overline{\Box}\right.$$

$$\left.\left. + \frac{1}{2}\left(\overline{g}_{\mu\nu} \overline{\nabla}_{(\rho} \overline{\nabla}_{\sigma)} + \overline{g}_{\rho\sigma} \overline{\nabla}_{(\mu} \overline{\nabla}_{\nu)}\right)\right]\right\} \delta^D(x - x'), \tag{4.61}$$

which is a local term. Now that we have the explicit expression for the nonrenormalized graviton self-energy we can begin to renormalize it, which we will do in the next chapter.

Chapter 5

Renormalization

The expressions we derived for the cubic and quartic corrections to the graviton self-energy contain ultraviolet divergences for $D = 4$. To deal with these divergences we employ dimensional regularization and renormalization, which allows us to maintain all the symmetries of the theory; thus, preserving general covariance [43]. The process is as follows. First we regularize the ultraviolet divergences by dimensional regularization, which automatically subtracts power-law divergences, then we renormalize the remaining logarithmic divergences $1/(D-4)$ by absorbing them into a counterterm action. It is important to note that one is generally interested in integrals of the self-energy that enter e.g. the quantum corrected linearized Einstein equation (6.2) which is given by

$$\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) - \int \mathrm{d}^4 x' \left[^{\mu\nu}\Sigma_{ret}^{\rho\sigma}\right](x;x')\kappa h_{\rho\sigma}(x') = \frac{\kappa^2}{2}T_{\text{lin}}^{\mu\nu}(x), \tag{5.1}$$

where the integration is over the coordinates x'^μ. Therefore the self-energy is renormalized when it is written in a form that is integrable in $D = 4$ space-time dimensions. Were it not for the divergent coefficients $1/(D-4)$, which are of course the ultraviolet divergences we want to extract, the contributions from the quartic vertices to the graviton self-energy (4.22)(4.61) would already be integrable. These divergent coefficients are easily renormalized as we will show in section 5.1. To renormalize the contribution of the cubic vertices (4.21) we will have to work a lot harder as there are many terms that are not integrable in $D = 4$ space-time dimensions (even terms with coefficients that are perfectly regular in $D = 4$). The procedure will be to extract differential operators and localize the divergences onto delta function terms, which can then be subtracted off using counterterms. The derivatives that act with respect to x^μ can be pulled out of the integral and the derivatives with respect to x'^μ can be partially integrated. In this thesis we will only renormalize the term proportional to G^2. For the renormalization of the G^2 part of the TT-correlator we follow a more general approach than [36] [11] and show that in the small mass expansion the renormalized results are the same. For the other products of the propagator the equivalence of both approaches is not clear.

5.1 Renormalizing the Quartic Contribution

The renormalization of the contributions to the graviton self-energy coming from the quartic vertices is fairly simple as (4.61) is already integrable with the exception of divergent constants of the type $\propto 1/(D-4)$ multiplying tensor structures. These divergences descend from the

gamma functions when $D = 4$. To make the divergences manifest we expand (4.61) around $D = 4$

$$\Gamma\left(1 - D/2\right) = \frac{2}{D-4} + \gamma_E - 1 + \mathcal{O}(D-4), \tag{5.2}$$

$$\Gamma\left(-D/2\right) = -\frac{1}{D-4} + \frac{3}{4} - \frac{1}{2}\gamma_E + \mathcal{O}(D-4), \tag{5.3}$$

where the logarithmic divergences are now evident and $\gamma_E = 0.577\ldots$ is the Euler constant. Since we have assumed $m^2 + \xi R > 0$ or equivalently $M^2 > 0$ (3.48) and $\epsilon = 0$ (2.17) (not to be confused with the pole prescription ε) there should only be divergences of the logarithmic type. However the overall factor of the quotient of gamma functions may diverge for general ν_D (in this chapter we have added the subscript D to the D-dependent index ν to differentiate between ν_D and $\nu_4 \equiv \nu$). To show that no divergences for our choice of ν exist we expand the overall factor around $D = 4$

$$\frac{\Gamma\left(\frac{D-1}{2} + \nu_D\right)\Gamma\left(\frac{D-1}{2} - \nu_D\right)}{\Gamma\left(\frac{1}{2} + \nu_D\right)\Gamma\left(\frac{1}{2} - \nu_D\right)} = 1 + \frac{D-4}{2}\left[\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right)\right], \tag{5.4}$$

where $\psi(z) = d/dz \ln\Gamma(x)$ is the digamma function. The digamma function is holomorphic on the entire complex plane minus 0 and the negative integers ($\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$). In four dimensions we have $\nu = \sqrt{9/4 - M^2}$ with $M^2 > 0$; thus, the digamma functions do not diverge and indeed the only divergences are of the logarithmic type. Going back to (4.61) we can immediately see that the $D - 4$ term in the second line cancels the divergence from $\Gamma(-D/2)$. In the third line one logarithmic factor is cancelled by the factor $D - 4$, however, the term proportional to m^2/H^2 remains logarithmically divergent. The last line also contains a logarithmic divergence. These divergences are eliminated by two counterterms, the cosmological constant counterterm and the inverse gravitational constant counterterm. The counterterm action takes the form

$$S_{ct} = \int \mathrm{d}^D x \sqrt{-g(x)}\left\{\Delta\left(\frac{1}{16\pi G_N}\right)R - 2\Delta\left(\frac{\Lambda}{16\pi G_N}\right)\right\}. \tag{5.5}$$

Taking the second variation of (5.5) we obtain

$$\frac{\delta^2 S_{ct}}{\delta h^{\rho\sigma}(x')\delta h^{\mu\nu}(x)} = \kappa^2 \sqrt{-\bar{g}}\left\{\Delta\left(\frac{1}{16\pi G_N}\right)\left\{(D-1)(D-4)H^2\left[\frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \frac{1}{2}\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}\right]\right.\right.$$

$$+ \left[\frac{1}{2}\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}\bar{\Box} - \bar{g}_{\mu)(\rho}\bar{\nabla}_{\sigma)}\bar{\nabla}_{(\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}\bar{\Box} + \frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{\nabla}_{(\rho}\bar{\nabla}_{\sigma)} + \bar{g}_{\rho\sigma}\bar{\nabla}_{(\mu}\bar{\nabla}_{\nu)}\right)\right]\right\}$$

$$- 2\Delta\left(\frac{\Lambda}{16\pi G_N}\right)\left[\frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \frac{1}{2}\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}\right]\right\}\delta^D(x - x'). \tag{5.6}$$

To cancel the divergences in (4.61) the counterterm coefficients must be

$$\Delta\left(\frac{1}{16\pi G_N}\right) = i\frac{1}{2}\xi\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)\Gamma\left(\frac{1}{2} - \nu\right)}\Gamma\left(1 - \frac{D}{2}\right) + \Delta\left(\frac{1}{16\pi G_N}\right)_f, \tag{5.7}$$

$$\Delta\left(\frac{\Lambda}{16\pi G_N}\right) = -i\frac{1}{4}m^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma\left(\frac{D-1}{2} + \nu\right)\Gamma\left(\frac{D-1}{2} - \nu\right)}{\Gamma\left(\frac{1}{2} + \nu\right)\Gamma\left(\frac{1}{2} - \nu\right)}\Gamma\left(1 - \frac{D}{2}\right) + \Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f, \tag{5.8}$$

where $\Delta\left(\frac{1}{16\pi G_N}\right)_f$ and $\Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f$ are finite parameters. We can now construct the renormalized quartic contribution to the graviton self-energy

$$
\begin{aligned}
-i\big[_{\mu\nu}\Sigma^{\mathrm{ren}}_{\rho\sigma}\big]_{4pt}(x;x') = ia^4\kappa^2\Bigg\{ &\left[\frac{H^4}{64\pi^2}\left(\frac{m^2}{H^2}\right)\left(\frac{m^2}{H^2}+12\xi-2\right)+2i\Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f\right] \\
&\times\left[\frac{1}{4}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}-\frac{1}{2}\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu}\right] \\
-i\Delta\left(\frac{1}{16\pi G_N}\right)_f &\left[\frac{1}{2}\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu}\overline{\Box}-\overline{g}_{(\mu|(\rho}\overline{\nabla}_{\sigma)}\overline{\nabla}_{|\nu)}-\frac{1}{2}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma}\overline{\Box}\right. \\
&\left.+\frac{1}{2}\left(\overline{g}_{\mu\nu}\overline{\nabla}_{(\rho}\overline{\nabla}_{\sigma)}+\overline{g}_{\rho\sigma}\overline{\nabla}_{(\mu}\overline{\nabla}_{\nu)}\right)\right]\Bigg\}\delta^4(x-x'), \quad (5.9)
\end{aligned}
$$

where we have taken $D=4$, which is integrable. Note that the finite remainder of the first term in (5.7) and (5.8) are multiplied by overall factors of ξ and m^2; thus, the finite counterterm parameters also contain overall factors of ξ and m^2. Therefore, in the minimally coupled massless case the contribution from the quartic vertices to the graviton self-energy vanishes, which agrees with [32].

5.2 Renormalizing the Cubic contribution

The TT-correlator (4.23) contains products of the scalar propagator (3.61) and up to its fourth order derivative, which compared to minimal coupling where the highest derivative is of second order, complicates matters considerably. To renormalize we must first write the propagator as the sum of two series around $y=0$; one with integer powers and one with non-integer powers. Using the following transformation formula

$$
\begin{aligned}
_2F_1&\left(\frac{D-1}{2}+\nu_D,\frac{D-1}{2}-\nu_D,\frac{D}{2},1-\frac{y}{4}\right) \quad (5.10)\\
&=\frac{\Gamma(\frac{D}{2})\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}-\nu_D)\Gamma(\frac{1}{2}+\nu_D)}\times{}_2F_1\left(\frac{D-1}{2}+\nu_D,\frac{D-1}{2}-\nu_D,\frac{D}{2},\frac{y}{4}\right)\\
&+\left(\frac{y}{4}\right)^{1-\frac{D}{2}}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}{\Gamma(\frac{D-1}{2}-\nu_D)\Gamma(\frac{D-1}{2}+\nu_D)}\times{}_2F_1\left(\frac{1}{2}+\nu_D,\frac{1}{2}-\nu_D,2-\frac{D}{2},\frac{y}{4}\right),
\end{aligned}
$$

and the series expansion of the hypergeometric function

$$
_2F_1(a,b;c;d)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}\left(\frac{y}{4}\right)^2, \quad (5.11)
$$

where $(z)_n=\Gamma(z+n)/\Gamma(z)$ is the Pochhammer symbol, the propagator can be written as

$$
\begin{aligned}
i\Delta(x;x')=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}&\left[\frac{\Gamma\left(\frac{D-3}{2}-\nu_D\right)\Gamma\left(\frac{D-3}{2}+\nu_D\right)}{\Gamma\left(\frac{1}{2}-\nu_D\right)\Gamma\left(\frac{1}{2}+\nu_D\right)}\left[\left(\frac{D-3}{2}\right)^2-\nu_D^2\right]\Gamma\left(1-\frac{D}{2}\right)\right.\\
&\times\sum_{n=0}^{\infty}\frac{\left(\frac{D-1}{2}+\nu_D\right)_n\left(\frac{D-1}{2}-\nu_D\right)_n}{\left(\frac{D}{2}\right)_n}\frac{\left(\frac{y}{4}\right)^n}{n!}\\
&\left.+\left(\frac{y}{4}\right)^{1-\frac{D}{2}}\Gamma\left(\frac{D}{2}-1\right)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+\nu_D\right)_n\left(\frac{1}{2}-\nu_D\right)_n}{\left(2-\frac{D}{2}\right)_n}\frac{\left(\frac{y}{4}\right)^n}{n!}\right], \quad (5.12)
\end{aligned}
$$

In the expansion of (5.12) we have to keep a certain amount of terms in D-dimensions, while the remaining ones can be resummed. The number of terms can be determined from the requirement that the self-energy needs to be integrable in $D = 4$ space-time dimensions i.e. the lowest power of the resulting $D = 4$ part of the TT-correlator must be $1/y$. For G^2 we have to keep only the term $\sim y^{1-D/2}$ in D-dimensions while the other terms can be set to $D = 4$ and resummed. Indeed for $G^2 \sim y^{2-D} + \mathcal{O}(1/y)$ only one term is not integrable in $D = 4$ space-time dimensions and needs to be renormalized. For GG' we need to keep the leading two terms in each propagator $\sim (y^{1-D/2} + [y^{2-D/2} + y^0])$ and $\sim (y^{-D/2} + y^{1-D/2})$, which leads to three terms that need to be renormalized $\sim (y^{1-D}, y^{2-D}, y^{-D/2})$. For GG'' we need to keep the leading three terms of each propagator in D-dimensions $\sim (y^{1-D/2} + [y^{2-D/2} + y^0] + [y^{3-D/2} + y^1])$ and $\sim (y^{-1-D/2} + y^{-D/2} + y^{1-D/2})$. The resulting product will have five terms that need to be renormalized $(y^{-D}, y^{1-D}, y^{2-D}, y^{-1-D/2}, y^{-D/2})$. G'^2 also requires us to keep three terms in D-dimensions $\sim (y^{-D/2} + y^{1-D/2} + [y^{2-D/2} + y^0])^2$ yielding four terms to renormalize $\sim (y^{-D}, y^{1-D}, y^{2-D}, y^{-D/2})$. GG''' and $G'G''$ require us to keep the leading four terms in D-dimensions giving rise to seven and six terms respectively that need to be renormalized. Finally we have GG'''', $G'G'''$ and $(G'')^2$ that require us to keep the leading five terms in D-dimensions leading to nine, eight and seven terms that need to be renormalized. The propagator products also come with factors of y and y^2, which will reduce the number of divergent terms. The message is clear, additional derivatives of the propagator complicate matters considerably and third and fourth order derivatives enter the TT-correlator entirely due to nonminimally coupling. In this thesis we will only renormalize G^2. The other factors are beyond the scope of this thesis and need to be renormalized in a later work.

Upon pulling out the $n = 0$ term in the second sum of (5.12) and reorganizing the expression we find

$$i\Delta(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \left\{ \Gamma\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-\frac{D}{2}} \right. \tag{5.13}$$

$$+ \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(\frac{D-3}{2}-\nu_D\right)\Gamma\left(\frac{D-3}{2}+\nu_D\right)}{\Gamma\left(\frac{1}{2}-\nu_D\right)\Gamma\left(\frac{1}{2}+\nu_D\right)} \left[\left(\frac{D-3}{2}\right)^2 - \nu_D^2\right] \Gamma\left(1-\frac{D}{2}\right) \right.$$

$$\times \frac{\left(\frac{D-1}{2}+\nu_D\right)_n \left(\frac{D-1}{2}-\nu_D\right)_n}{\left(\frac{D}{2}\right)_n} \frac{\left(\frac{y}{4}\right)^n}{n!}$$

$$\left. \left. + \Gamma\left(\frac{D}{2}-1\right) \frac{\left(\frac{1}{2}+\nu_D\right)_{n+1}\left(\frac{1}{2}-\nu_D\right)_{n+1}}{\left(2-\frac{D}{2}\right)_{n+1}} \frac{\left(\frac{y}{4}\right)^{n+2-\frac{D}{2}}}{(n+1)!} \right] \right\}.$$

We need this expression to zeroth order in $(D-4)$. It suffices to expand all parts of this expression to first order in $(D-4)$. Thus we have

$$\frac{\Gamma\left(\frac{D-3}{2}-\nu_D\right)\Gamma\left(\frac{D-3}{2}+\nu_D\right)}{\Gamma\left(\frac{1}{2}-\nu_D\right)\Gamma\left(\frac{1}{2}+\nu_D\right)} = 1 + \frac{D-4}{2}\left[\psi\left(\frac{1}{2}+\nu\right)+\psi\left(\frac{1}{2}-\nu\right)\right], \tag{5.14}$$

$$\left(\frac{D-3}{2}\right)^2 - \nu_D^2 = \left(\frac{1}{4}-\nu^2\right)\left[1+\frac{D-4}{2}\left(\frac{1-\delta\nu^2}{\frac{1}{4}-\nu^2}\right)\right], \tag{5.15}$$

$$\Gamma\left(1-\frac{D}{2}\right) = \frac{2}{D-4}\left[1+\frac{D-4}{2}\left(-1-\psi(1)\right)\right], \tag{5.16}$$

$$\Gamma\left(\frac{D}{2}-1\right) = 1 + \frac{D-4}{2}\psi(1), \tag{5.17}$$

where $\psi(1) = -\gamma_E$ and $\delta\nu^2 = 3 - 14\xi$. Expanding (5.13) around $D = 4$ we can show that all

logarithmic divergences cancel. Indeed to leading order in $\propto 1/(D-4)$ (5.13) can be written as

$$
\begin{aligned}
i\Delta(x;x') &= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\left\{\Gamma\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-\frac{D}{2}}\right. \\
&\quad +\frac{2}{D-4}\left(\frac{1}{4}-\nu^2\right)\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}+\nu\right)_n\left(\frac{3}{2}-\nu\right)_n}{(2)_n}\frac{\left(\frac{y}{4}\right)^n}{n!} \\
&\quad \left.-\frac{2}{D-4}\left(\frac{1}{4}-\nu^2\right)\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}+\nu\right)_n\left(\frac{3}{2}-\nu\right)_n}{(1)_n}\frac{\left(\frac{y}{4}\right)^n}{(n+1)!}\right\}+\mathcal{O}((D-4)^0) \\
&= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-\frac{D}{2}}+\mathcal{O}((D-4)^0)\,.
\end{aligned}
\tag{5.18}
$$

Next we expand to order $(D-4)^0$ and after some algebra we arrive at the following form for the propagator

$$
\begin{aligned}
i\Delta(x;x') &= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\left\{\Gamma\left(\frac{D}{2}-1\right)\left(\frac{y}{4}\right)^{1-\frac{D}{2}}\right. \\
&\quad +\left(\frac{1}{4}-\nu^2\right)\ln\left(\frac{y}{4}\right){}_2F_1\left(\frac{3}{2}+\nu,\frac{3}{2}-\nu;2;\frac{y}{4}\right) \\
&\quad +\left(\frac{1}{4}-\nu^2\right)\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}+\nu\right)_n\left(\frac{3}{2}-\nu\right)_n}{(2)_n}\frac{\left(\frac{y}{4}\right)^n}{n!} \\
&\quad \left.\times\left[\psi\left(\frac{3}{2}+\nu+n\right)+\psi\left(\frac{3}{2}-\nu+n\right)-\psi(1+n)-\psi(2+n)\right]\right\} \\
&\quad +\mathcal{O}(D-4)\,,
\end{aligned}
\tag{5.19}
$$

where we have kept only the $\sim y^{1-D/2}$ term in D-dimensions as discussed earlier. This expression is more general than the one used in [36] [11] in the sense that it is valid for arbitrary mass parameter $m^2+\xi R>0$ while the expansion in [36] [11] is a small mass expansion valid for $0<m^2+\xi R\ll H^2$. To renormalize G^2 we need the square of (5.19)

$$
\left(i\Delta(x;x')\right)^2 = \frac{H^{2D-4}}{(4\pi)^D}\left\{\Gamma\left(\frac{D}{2}-1\right)^2\left(\frac{y}{4}\right)^{2-D}+\frac{8}{y}f(y)+f(y)^2\right\},
\tag{5.20}
$$

where we have taken $D=4$ in the second and third term and $f(y)$ is defined as

$$
\begin{aligned}
f(y) = \left(\frac{1}{4}-\nu^2\right)&\left\{\ln\left(\frac{y}{4}\right){}_2F_1\left(\frac{3}{2}+\nu,\frac{3}{2}-\nu;2;\frac{y}{4}\right)+\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}+\nu\right)_n\left(\frac{3}{2}-\nu\right)_n}{(2)_n}\frac{\left(\frac{y}{4}\right)^n}{n!}\right. \\
&\left.\times\left[\psi\left(\frac{3}{2}+\nu+n\right)+\psi\left(\frac{3}{2}-\nu+n\right)-\psi(1+n)-\psi(2+n)\right]\right\}.
\end{aligned}
\tag{5.21}
$$

Note that for our choice of ν, $f(y)$ contains no divergences; moreover, the term proportional to y^{2-D} has finite coefficients for $D=4$, but is not integrable! To make it integrable one must extract a d'Alembertian operator. Before we can extract a d'Alembertian we must first know how the d'Alembertian acts on functions of y.

Writing out the d'Alembertian operator and using the relations of appendix A we find

$$\frac{\overline{\Box}}{H^2}F(y) = \frac{1}{H^2}\frac{1}{\sqrt{-\overline{g}}}\partial_\mu\left(\sqrt{-\overline{g}}\,\overline{g}^{\mu\nu}\partial_\nu y\right) = (4y - y^2)F''(y) + D(2-y)F'(y) - 4i\varepsilon\delta(\eta - \eta')F'(y)$$
$$- 2i\varepsilon Ha(\eta')\mathrm{sgn}(\eta - \eta')\big[2yF''(y) + DF'(y)\big].$$
(5.22)

When $F(y)$ is a nonsingular function i.e. when $F(y)$ contains no powers of $y^{1-D/2}$ we can take $\varepsilon = 0$ and (5.22) reduces to

$$\frac{\overline{\Box}}{H^2}F(y) = (4y - y^2)F''(y) + D(2-y)F'(y).$$
(5.23)

When

$$F(y) = y^{1-\frac{D}{2}} + \mathcal{O}(y^{2-\frac{D}{2}}),$$
(5.24)

the term $-4i\varepsilon\delta(\eta - \eta')F'(y)$ gives rise to a delta function in the limit $\varepsilon \to 0$ [29]. To show this we use (5.22) to find the action of the d'Alembertian on $F(y) = y^{1-\frac{D}{2}} + \mathcal{O}(y^{2-\frac{D}{2}})$ and obtain

$$\frac{\overline{\Box}}{H^2}F(y) = 4i\left(\frac{D}{2} - 1\right)\varepsilon\delta(\eta - \eta')\frac{1}{y^{\frac{D}{2}}} + \mathcal{O}(y^{1-\frac{D}{2}}) = \frac{4i(\frac{D}{2}-1)}{H^D\sqrt{-\overline{g}}}\frac{\varepsilon\delta(\eta - \eta')}{(\varepsilon^2 + ||\mathbf{x} - \mathbf{x}'||)^{\frac{D}{2}}} + \mathcal{O}(y^{1-\frac{D}{2}}).$$
(5.25)

Multiplying (5.25) with a test function $\varphi(x)$ that falls off sufficiently rapidly, integrating, taking the limit $\varepsilon \to 0$ and noting that terms $\mathcal{O}(y^{1-\frac{D}{2}})$ are integrable, we see that the only delta function comes from the term proportional to $\frac{\varepsilon\delta(\eta-\eta')}{(\varepsilon^2+||\mathbf{x}-\mathbf{x}'||)^{\frac{D}{2}}}$ as follows,

$$\lim_{\varepsilon\to0}\int \mathrm{d}^D x \frac{\varepsilon\delta(\eta - \eta')}{(\varepsilon^2 + ||\mathbf{x} - \mathbf{x}'||)^{\frac{D}{2}}}\varphi(x) = \lim_{\varepsilon\to0}\int \mathrm{d}^{D-1}x\frac{\varepsilon}{(\varepsilon^2 + ||\mathbf{x} - \mathbf{x}'||)^{\frac{D}{2}}}\varphi(\mathbf{x}, \eta')$$
$$= \lim_{\varepsilon\to0}\int \mathrm{d}^{D-1}z\frac{1}{(1 + |\mathbf{z}|^2)^{\frac{D}{2}}}\varphi(\varepsilon\mathbf{z} + \mathbf{x}', \eta') = \varphi(\mathbf{x}', \eta')\int \mathrm{d}^{D-1}r\frac{r^{D-2}}{(1 + r^2)^{\frac{D}{2}}}\int \mathrm{d}\Omega_{D-2} \quad (5.26)$$
$$= \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}\varphi(x'),$$

where we made the change of variables $\mathbf{z} = (\mathbf{x} - \mathbf{x}')/\varepsilon$ and switched to spherical coordinates (3.57). Note that (5.25) behaves exactly like a delta function times some constant when $\epsilon \to 0$; thus, we conclude

$$\lim_{\varepsilon\to0}\frac{4i(\frac{D}{2} - 1)}{H^D\sqrt{-\overline{g}}}\frac{\varepsilon\delta(\eta - \eta')}{(\varepsilon^2 + ||\mathbf{x} - \mathbf{x}'||)^{\frac{D}{2}}} = \frac{4\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2} - 1)}\frac{i\delta^D(x - x')}{H^D\sqrt{-\overline{g}}}.$$
(5.27)

Using (5.22), (5.27) and taking $\epsilon \to 0$ we find

$$\left[\frac{\overline{\Box}}{H^2} - \frac{D(D-2)}{4}\right]\left(\frac{y}{4}\right)^{1-\frac{D}{2}} - \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2} - 1)}\frac{i\delta^D(x - x')}{H^D\sqrt{-\overline{g}}} = 0.$$
(5.28)

Relation (5.28) plays a crucial role in localizing the divergences onto delta function terms. Before continuing with renormalization we compare our choice of $F(y)$ (5.24) with the propagator expansion (5.12). Note that they contain the same powers of y and that the delta function in

the propagator equation (3.62) is correctly sourced by the action of $\overline{\Box}$ on the term proportional to $y^{1-D/2}$ in the propagator.

We are now ready to reduce the degree of divergence of y^{2-D} by two by extracting a d'Alembertian [37][38][39]. Using relation (C.2), which derives from (5.23), and adding zero to it using (5.28) we can isolate the divergence,

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{\overline{\Box}}{H^2}\left[\left(\frac{y}{4}\right)^{3-D} - \left(\frac{y}{4}\right)^{1-\frac{D}{2}}\right] + \frac{D(D-2)}{2(D-3)(D-4)}\left(\frac{y}{4}\right)^{1-\frac{D}{2}}$$
$$- \frac{4}{D-4}\left(\frac{y}{4}\right)^{3-D} + \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{i\delta^D(x-x')}{(aH)^D}. \tag{5.29}$$

At first look every term in (5.29) has a divergence of the type $\propto 1/(D-4)$. However on closer inspection, expanding all factors of y around $D=4$, all but the divergent factor in front of the Dirac delta cancel. Indeed upon performing the expansion and setting $D=4$ for all terms that do not diverge we find

$$\Gamma\left(\frac{D}{2}-1\right)^2\left(\frac{y}{4}\right)^{2-D} = -4\frac{\overline{\Box}}{H^2}\left[\frac{\ln(y/4)}{y}\right] + \frac{8}{y}\left[\ln(y/4) - \frac{1}{2}\right]$$
$$+ \frac{2(4\pi)^{\frac{D}{2}}}{(D-3)(D-4)}\frac{i\delta^D(x-x')}{(aH)^D}\Gamma\left(\frac{D}{2}-1\right). \tag{5.30}$$

The first two terms of (5.30) are integrable and the divergence has been localized on a delta function. Note that we can move the d'Alembertian acting on the first term out of the integral over x' (5.1) making it truly local. To renormalize we subtract off the divergent term with local counterterms. For this particular divergence we introduce the cosmological constant counterterm action

$$S_{ct} = -2\int \mathrm{d}^D x \, \sqrt{-g}\Delta\left(\frac{\Lambda}{16\pi G_N}\right), \tag{5.31}$$

and taking its second variation we obtain

$$\frac{\delta^2 S_{ct}}{\delta h^{\rho\sigma}(x')\delta h^{\mu\nu}(x)} = -2\kappa^2\sqrt{-\overline{g}}\Delta\left(\frac{\Lambda}{16\pi G_N}\right)\left[\frac{1}{4}\overline{g}_{\mu\nu}\overline{g}_{\rho\sigma} - \frac{1}{2}\overline{g}_{\mu(\rho}\overline{g}_{\sigma)\nu}\right]\bigg\}\delta^D(x-x'), \tag{5.32}$$

which gives the correct tensor structure $\overline{g}_{\mu\nu}\overline{g}'_{\rho\sigma}$ together with an additional tensor structure, which we will discuss shortly. To cancel the divergence in the last line of (4.23) the counterterm coefficient must be

$$\Delta\left(\frac{\Lambda}{16\pi G_N}\right) = -iK_D\frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D/2-1)}{(D-3)(D-4)} + \Delta\left(\frac{\Lambda}{16\pi G_N}\right)_f, \tag{5.33}$$

where
$$K_D = M^4 - 2M^2\big((D-1)+4M^2\big)\xi + 2\big((D-1)^2 + 2(2D-3)M^2 + 8M^4\big)\xi^2 \tag{5.34}$$

and we define $K_4 \equiv K$. Recalling (4.21) the partially renormalized correction to the graviton

self-energy is given by

$$
\begin{aligned}
-i\big[{}_{\mu\nu}\Sigma^{\mathrm{ren}}_{\rho\sigma}\big]^{G^2}_{\mathrm{3pt}}(x;x') = &-\kappa^2\frac{(aa')^4}{4}\frac{H^8}{(4\pi)^4}\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}K\bigg(-4\frac{\overline{\Box}}{H^2}\Big[\frac{\ln(y/4)}{y}\Big]+\frac{8}{y}\Big[\ln(y/4)-\frac{1}{2}\Big] \\
&\qquad\qquad +\frac{8}{y}f(y)+f(y)^2\bigg) \\
&-\frac{\kappa^2}{2}a^4\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}\delta^4(x-x')\Delta\Big(\frac{\Lambda}{16\pi G_N}\Big)_f+\kappa^2 a^4\bar{g}_{\mu(\rho}\bar{g}'_{\sigma)\nu}\delta^4(x-x')\Delta\Big(\frac{\Lambda}{16\pi G_N}\Big)_f \\
&-i\kappa^2 a^D K_D\frac{H^D}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D/2-1)}{(D-3)(D-4)}\bar{g}_{\mu(\rho}\bar{g}'_{\sigma)\nu}\delta^D(x-x').
\end{aligned}
$$

(5.35)

Note that in the last line we traded a divergent term of the type $\propto\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}$ with one of the type $\propto\bar{g}_{\mu(\rho}\bar{g}'_{\sigma)\nu}$. This is perhaps no surprise since there is no guarantee that every divergent term can be cancelled independently. However the last term must cancel a divergent term elsewhere in (4.23) for the calculation to be correct. Moreover, the usual counterterms employed in gravity, which also contain this tensor structure, are R^2 and the Weyl tensor squared $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, which we will need for the full renormalization[1]. We mentioned before that the renormalization of the other propagator products is beyond the scope of this thesis; however, we can speculate. The tensor structure $\partial_{\mu)}\partial'_{(\rho}y\partial'_{\sigma)}\partial_{(\nu}y$ at coincidence is proportional to $\propto\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}$ and is a likely candidate. Our approach thus far was to expand the propagator around $D=4$ and then renormalize. In [36] [11] a small mass expansion preceeded the renormalization procedure. It is not clear if renormalizing then performing a small mass expansion or first performing the small mass expansion and then renormalizing yield the same result. We now show that for the G^2 term they are indeed equivalent. Because the first term in (5.19) has no dependency on M^2, the divergent term in (5.29) does not either. Thus we only need to show that the small mass expansion of (5.19) yields the same result as in [36] [11]. We define

$$
\nu=\frac{D-1}{2}-s=\frac{D-1}{2}-\frac{1}{D-1}\frac{m^2+\xi R}{H^2}+\mathcal{O}\Big(\frac{m^2+\xi R}{H^2}\Big)^2
$$

(5.36)

which corresponds to the deviation of ν_D from the minimally coupled, massless case. Expanding (5.19) in s we find

$$
\begin{aligned}
i\Delta(x;x')=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\bigg\{&\Gamma\Big(\frac{D}{2}-1\Big)\Big(\frac{y}{4}\Big)^{1-\frac{D}{2}}+(-2+3s)\ln\Big(\frac{y}{4}\Big)+\frac{2}{s}-4 \\
&+s\sum_{n=1}^{\infty}\Big[\frac{2-n(n+2)\ln(y/4)}{n^2}\Big(\frac{y}{4}\Big)^n\Big]\bigg\},
\end{aligned}
$$

(5.37)

where the infinite sum is finite and can be written as

$$
\sum_{n=1}^{\infty}\Big[\frac{2-n(n+2)\ln(y/4)}{n^2}\Big(\frac{y}{4}\Big)^n\Big]=2\mathrm{Li}_2\Big(\frac{y}{4}\Big)-\frac{y/4}{1-y/4}\ln\Big(\frac{y}{4}\Big)+2\ln\Big(\frac{y}{4}\Big)\ln\Big(1-\frac{y}{4}\Big),
$$

(5.38)

where $\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}z^k/k^n$ is the polylogarithm. This result agrees with [36]. For propagator products where more terms need to be kept in D-dimensions it is not clear if the two approaches yield the same result.

---

[1]When there is a scalar condensate one also needs the counterterm $R\phi^2$.

Chapter 6

Discussion and Outlook

In this thesis we have derived the one loop contributions to the graviton self-energy modified by a nonminimally coupled massive scalar on the de Sitter background and calculated the connected TT-correlator. It turned out that by performing the calculation of one, one can almost immediately obtain the other. Our case is de Sitter invariant and we assumed a strictly positive effective mass $m^2 + \xi R > 0$ and strictly positive nonminimal coupling $\xi > 0$. We found the two diagrams contributing to the self-energy through an expansion of the effective action, which relates the nonlocal part of the self-energy to the TT-correlator,

$$-i\big[{}_{\mu\nu}\Sigma_{\rho\sigma}\big]_{3pt}(x;x') = \frac{\kappa^2}{4}\sqrt{-g}\sqrt{-g'}\langle\mathcal{T}\{\hat{\bar{T}}_{\mu\nu}(x)\hat{\bar{T}}_{\rho\sigma}(x')\}\rangle. \tag{6.1}$$

We checked our result of the TT-correlator in the minimal coupling and massless minimal coupling limit against the results of Park and Woodard [32] and Perez-Nadal, Roura and Verdaguer [35] and found that they agreed. It is worth noting that first taking the massless limit $m \to 0$ does not give the same result as discarding the $m^4 G^2$ term then setting $m = 0$, with the latter resulting in the correct massless minimally coupled correlator. Discarding the $m^4 G^2$ term is justified because only derivatives of the propagator enter the self-energy at one loop order. The local contribution $-i\big[{}_{\mu\nu}\Sigma_{\rho\sigma}\big]_{4pt}(x;x')$ to the self-energy was renormalized through the inverse gravitational and cosmological constant counterterms. We found, unlike the minimally coupled massless case, the result to be nonvanishing. The nonlocal $-i\big[{}_{\mu\nu}\Sigma_{\rho\sigma}\big]_{3pt}(x;x')$ part proved harder to renormalize as it contains derivatives of the propagator of up to order four resulting in many divergent terms. In contrast, only derivatives of up to second order of the propagator enter in the TT-correlator of the minimally coupled massive scalar; thus the hardest terms to renormalize enter the TT-correlator through nonminimal coupling. The result is a partially renormalized result where we renormalized the part proportional to $\propto H^4 \bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}G^2$. We isolated the divergence by extracting d'Alembertians, localizing the divergence onto a delta function, and used the cosmological constant counterterm to remove this divergence since it gives rise to the correct tensor structure $\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}$; however, it also has an additional tensor structure $\propto \bar{g}_{\mu(\rho}\bar{g}'_{\sigma)\nu}$ which comes with a logarithmic divergence. This is perhaps no surprise since there is no guarantee that we can renormalize the TT-correlator term by term. Indeed this leftover term must cancel a divergence elsewhere for our analysis to be correct. Moreover, for the full renormalization the usual counterterms employed in gravity are R^2 and $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, which will be required to cancel terms that require the extraction of multiple d'Alembertians. In the case of a scalar condensate one also needs the counterterm $R\phi^2$.

The natural next step would be to finish the renormalization procedure. We could then

in principle calculate the effect of inflationary nonminimally coupled massive scalars on the propagation of dynamical gravitons at one loop by solving the quantum corrected linearized Einstein equation for the graviton field by employing the Schwinger-Keldysh formalism [42]. This was done for the minimally coupled massless scalar in [31] [24] where no significant effects were found. One could also use the self-energy to calculate the corrections to the force of gravity from e.g. a point mass. This was done in [30] where corrections to the gravitational potentials were found. We will now give an outline of their calculation and discuss their results.

We start with the linearized Schwinger-Keldysh effective field equation, which is obtained by replacing $^{\mu\nu}\Sigma^{\rho\sigma}(x;x')$ in the linearized Einstein field equation with its retarded version $^{\mu\nu}\Sigma^{\rho\sigma}_{ret}(x;x')$ and reads

$$\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) - \int \mathrm{d}^4x' \left[^{\mu\nu}\Sigma^{\rho\sigma}_{ret}\right](x;x')\kappa h_{\rho\sigma}(x') = \frac{\kappa^2}{2}T^{\mu\nu}_{\mathrm{lin}}(x),\tag{6.2}$$

where $\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x)$ is obtained by expanding the Einstein equation $\sqrt{-g}(G_{\mu\nu} + \Lambda g_{\mu\nu})$ to first order in $\kappa h_{\mu\nu}$. To study how inflationary scalars affect the propagation of dynamical gravitons one must set the energy-momentum tensor to 0. Since we only know the self-energy at one loop order ($\propto \kappa^2$) we expand the graviton field and self-energy to order κ^2

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \kappa^2 h^{(1)}_{\mu\nu} + \mathcal{O}(\kappa^4),\tag{6.3}$$

$$\left[^{\mu\nu}\Sigma^{\rho\sigma}_{ren}\right]_{\mathrm{ret}}(x;x') = \kappa^2 \left[^{\mu\nu}\Sigma^{\rho\sigma}_{(1)}\right](x;x') + \mathcal{O}(\kappa^2).\tag{6.4}$$

Using these expansions and the fact that $h^{(0)}_{\mu\nu}$ solves the linearized Einstein equation with solution

$$h^{(0)}_{\mu\nu} = \epsilon_{\mu\nu}(\mathbf{k})u_0(\eta,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},\tag{6.5}$$

with

$$u_0(\eta,\mathbf{k}) = \frac{H}{\sqrt{2k^3}}\left[1 - \frac{ik}{Ha}\right]e^{-ik\eta},\tag{6.6}$$

and the polarization tensor which has the usual transverse, traceless form with vanishing temporal components

$$\epsilon_{\mu 0}(\mathbf{k}) = 0, \quad k_i\epsilon_{ij}(\mathbf{k}) = 0, \quad \epsilon_{ii}(\mathbf{k}) = 0,\tag{6.7}$$

(6.2) becomes

$$\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma}\kappa h^{(1)}_{\rho\sigma}(x) - \int \mathrm{d}^4x' \left[^{\mu\nu}\Sigma^{\rho\sigma}_{(1)}\right](x;x')\kappa h^{(0)}_{\rho\sigma}(x') = 0.\tag{6.8}$$

Solving this equation, no significant late-time effects were found [31][24]. For the nonminimally coupled massive scalar significant effects may be found due to the additional parameters m and ξ.

For quantum corrections to the force of gravity in the presence of a static point mass M on de Sitter the linearized energy-momentum tensor takes the form

$$T^{\mu\nu}_{\mathrm{lin}} = -a\delta^\mu_0\delta^\nu_0 M\delta^3(\mathbf{x}),\tag{6.9}$$

and the quantum corrected linear Einstein equation (6.2) becomes

$$\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) - \int \mathrm{d}^4x' \left[^{\mu\nu}\Sigma^{\rho\sigma}_{ret}\right](x;x')\kappa h_{\rho\sigma}(x') = -a\frac{\kappa^2}{2}\delta^\mu_0\delta^\nu_0 M\delta^3(\mathbf{x}).\tag{6.10}$$

The symmetries of the system imply a solution of the form,

$$h_{00}(x) = f_1(\eta, r), \tag{6.11}$$

$$h_{0i}(x) = \partial_i f_2(\eta, r), \tag{6.12}$$

$$h_{ij}(x) = \delta_{ij} f_3(\eta, r) + \partial_i \partial_j f_4(\eta, r), \tag{6.13}$$

where $r \equiv ||\mathbf{x}||$. Choosing the longitudinal (Newtonian) gauge $f_2 = 0$ and $f_4 = 0$, the classical solutions to the 0th order equation $\sqrt{-g}\mathcal{D}^{\mu\nu\rho\sigma} h_{\rho\sigma}^{(0)}(x) = \frac{\kappa}{2} T_{\text{lin}}^{\mu\nu}$ are

$$f_1^{(0)}(x) = \frac{2GM}{ar} \equiv -2\phi^{(0)}, \tag{6.14}$$

$$f_3^{(0)}(x) = \frac{2GM}{ar} \equiv -2\psi^{(0)}, \tag{6.15}$$

which define the 0th order potentials $\phi^{(0)}$ and $\psi^{(0)}$ in the longitudinal gauge. The one loop contributions to the potentials are given by $\phi^{(1)} = -f_1^{(1)}/2$ and $\psi^{(1)} = -f_3^{(1)}/2$. The resulting quantum corrected potentials $\phi_{\text{dS}} = \phi^{(0)} + \phi^{(1)}$ and $\psi_{\text{dS}} = \psi^{(0)} + \psi^{(1)}$ at late times ($a \gg 1$) calculated by Park, Prokopec and Woodard [30] are

$$\phi_{\text{dS}}(x) = -\frac{GM}{ar}\left\{1 + \frac{\hbar}{20\pi c^3}\frac{G}{(ar)^2} - \frac{\hbar G H^2}{30\pi c^5}\ln(a) + \mathcal{O}\left(\frac{1}{a^3}\right)\right\}, \tag{6.16}$$

$$\psi_{\text{dS}}(x) = -\frac{GM}{ar}\left\{1 - \frac{\hbar}{60\pi c^3}\frac{G}{(ar)^2} - \frac{\hbar G H^2}{30\pi c^5}\ln(a) + \mathcal{O}\left(\frac{1}{a^3}\right)\right\}. \tag{6.17}$$

Note the term $\propto \ln a$, which grows in time and will have a significant effect at sufficiently late times. Moreover, since they contribute equally to both potentials one can interpret them as a renormalization of the mass term or Newton's gravitational constant. Of course their scalar propagator was of the form $i\Delta(x; x') = A(y(x; x')) + k\ln(aa')$, where the last term breaks de Sitter invariance. In our case the analysis is de Sitter invariant; however, a correction term $\propto \ln a$ may arise from the conformal anomaly, though usually small, as is the case for massless fermions coupling to a light nearly minimally coupled scalar field [11]. Moreover, we have additional terms, namely the mass and nonminimal coupling; thus, we might find effects in terms of those parameters.

Appendix A

de Sitter Identities

The invariant distance function $y(x; x')$ (3.15) is given by

$$\bar{y}(x; x') = aa' H^2 \Delta x^2 = aa' H^2 \big[-(\eta - \eta')^2 + ||\mathbf{x} - \mathbf{x}'||^2 \big], \tag{A.1}$$

and is related to the geodesic distance $\ell(x; x')$ in de Sitter space by

$$\bar{y}(x; x') = 4 \sin^2 \big[H\ell(x; x')/2 \big]. \tag{A.2}$$

For the propagator and any function of the invariant distance we add a pole prescription to $\bar{y}$ and define

$$y(x; x') = aa' H^2 \Delta x^2 = aa' H^2 \big[-(|\eta - \eta'| - i\varepsilon)^2 + ||\mathbf{x} - \mathbf{x}'||^2 \big]. \tag{A.3}$$

The following de Sitter invariant identities have been used extensively in this thesis. In particular in the derivation of the TT-correlator. For a more extensive list of identities we refer to [25].

$$\partial_\rho y \partial^\rho y = \partial'_\rho y \partial'^\rho y = H^2 (4y - y^2) \tag{A.4}$$

$$(\partial_\rho y)(\partial^\rho \partial'^\nu y) = H^2 (2 - y) \partial'^\nu y \tag{A.5}$$

$$(\partial'_\sigma y)(\partial'^\mu \partial'^\sigma y) = H^2 (2 - y) \partial^\mu y \tag{A.6}$$

$$(\partial^\mu \partial'_\sigma y)(\partial'^\sigma \partial^\rho y) = 4H^4 g^{\mu\rho} - H^2 (\partial^\mu y)(\partial^\rho y) \tag{A.7}$$

$$(\partial'^\mu \partial_\sigma y)(\partial^\sigma \partial'^\rho y) = 4H^4 g'^{\mu\rho} - H^2 (\partial'^\mu y)(\partial'^\rho y) \tag{A.8}$$

$$\nabla_\rho \partial^\mu y = H^2 (2 - y) \delta_\rho^\mu \tag{A.9}$$

$$\nabla'_\rho \partial'^\mu y = H^2 (2 - y) \delta_\rho^\mu \tag{A.10}$$

$$\nabla_\mu \partial^\mu y \equiv \Box y = D H^2 (2 - y) \tag{A.11}$$

$$\nabla'_\mu \partial'^\mu y \equiv \Box' y = D H^2 (2 - y) \tag{A.12}$$

In tensor structures, such as the tensor structures above, one can safely take $y \equiv \bar{y}$, but in functions of y such as the propagator one needs the ε-prescription! Without this pole prescription the propagator (3.62) is not well defined.

Appendix B

The TT-correlator on de Sitter

B.1 The propagator

The scalar propagator (3.61) for a real massive scalar field on de Sitter space is given by the time ordered Chernikov-Tagirov propagator,

$$
\begin{aligned}
i\Delta(x;x') &= \langle \mathcal{T}[\hat{\phi}(x)\hat{\phi}(x')]\rangle \\
&= \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D-1}{2}+\nu\right)\Gamma\left(\frac{D-1}{2}-\nu\right)}{\Gamma\left(\frac{D}{2}\right)} \\
&\quad \times\, _2F_1\left(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4}\right) \\
&\equiv \frac{H^{D-2}}{(4\pi)^{D/2}} G(y),
\end{aligned}
\tag{B.1}
$$

where $\mathcal{T}$ denotes time ordering, $_2F_1$ is the Gauss hypergeometric function,

$$
\nu = \sqrt{\left(\frac{D-1}{2}\right)^2 - \frac{m^2}{H^2} - D(D-1)\xi},
\tag{B.2}
$$

$$
y(x;x') = aa'H^2\Delta x^2 = aa'H^2\big[-(|\eta-\eta'|-i\varepsilon)^2 + ||\mathbf{x}-\mathbf{x}'||^2\big]
\tag{B.3}
$$

and we have assumed a positive mass-squared, $m^2 + \xi R > 0$. If the effective mass is not strictly positive or if $\epsilon \neq 0$ the propagator develops de Sitter breaking contributions due to the particle creation in the deep infrared [20, 21] and thus becomes much more complicated.

It is convenient to define derivatives of the propagator function,

$$
\frac{d^n}{dy^n}G(y) \equiv \frac{d^n}{dz^n}G(y) = \left(-\frac{1}{4}\right)^n G_n(y),
\tag{B.4}
$$

where $z = 1 - y/4$ and

$$
G_n(y) = \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\,_2F_1\left(a+n, b+n; c+n; 1-\frac{y}{4}\right), \qquad G_0(y) \equiv G(y).
\tag{B.5}
$$
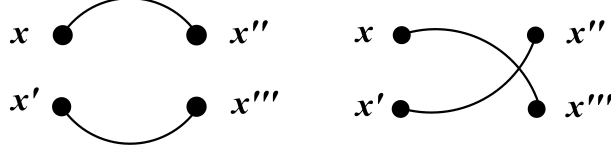
Figure B.1: Diagrammatic representation of pointsplitting $\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle$. The points x and x'' are split into x, x' and x'', x'''. The operator $\tau_{\mu\nu}(x; x')$ acts on the points x and x' and the operator $\tau_{\rho\sigma}(x''; x''')$ on the points x'' and x'''. Note that the left diagram can be obtained by interchanging $\rho \leftrightarrow \sigma$ and $x'' \leftrightarrow x'''$, explaining the factor two in (B.11)

.

B.2   TT-correlator

The TT-correlator (4.23) is computed by the method of pointsplitting. A diagrammatic representation is shown in Fig. (B.1). We define the following operator,

$$\tau_{\mu\nu}(x; x') = \tau^{(1)}_{\mu\nu}(x; x') + \tau^{(2)}_{\mu\nu}(x; x') + \tau^{(3)}_{\mu\nu}(x; x'), \tag{B.6}$$

with

$$
\begin{aligned}
\tau^{(1)}_{\mu\nu}(x; x') &= \left[(1 - 2\xi)\, \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{2}\,(1 - 4\xi)\, g_{\mu\nu}(x)g^{\alpha\beta}(x) \right] \partial_\alpha \partial'_\beta \\
&\equiv A^{\alpha\beta}_{\mu\nu}(x)\partial_\alpha\partial'_\beta, \tag{B.7} \\
\tau^{(2)}_{\mu\nu}(x; x') &= \xi G_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)m^2 = -\frac{1}{2}\left[(D-1)(D-2)H^2\xi + m^2 \right] g_{\mu\nu} \\
&\equiv B(x)g_{\mu\nu}, \tag{B.8} \\
\tau^{(3)}_{\mu\nu}(x; x') &= \xi \left[g_{\mu\nu}(x)g^{\alpha\beta}(x) - \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} \right] \left(\nabla_\alpha \partial_\beta + \nabla'_\alpha \partial'_\beta \right) \\
&\equiv C^{\alpha\beta}_{\mu\nu}(x) \left(\nabla_\alpha \partial_\beta + \nabla'_\alpha \partial'_\beta \right). \tag{B.9}
\end{aligned}
$$

It is useful to define the connected part of an expectation value of four operators as

$$\left[\langle \hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(x'')\hat{\phi}(x''')\rangle - \langle \hat{\phi}(x)\hat{\phi}(x')\rangle\langle \hat{\phi}(x'')\hat{\phi}(x''')\rangle \right]\big|_{x'\to x,\, x'''\to x''}. \tag{B.10}$$

Using the operator (B.6) and (B.10) we can calculate the one-loop contribution to the connected part of the TT-correlator by using a Wick contraction as follows,

$$
\begin{aligned}
\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle &= \tau_{\mu\nu}(x; x')\tau_{\rho\sigma}(x''; x''')\, \langle T[\hat{\phi}(x)\hat{\phi}(x')\hat{\phi}(x'')\hat{\phi}(x''')]\rangle\Big|_{x'\to x, x'''\to x''} \\
&= 2\tau_{\mu\nu}(x; x')\tau_{\rho\sigma}(x''; x''')\, \langle T[\hat{\phi}(x)\hat{\phi}(x'')]\rangle\langle T[\hat{\phi}(x')\hat{\phi}(x''')]\rangle\Big|_{x'\to x, x'''\to x''} \\
&= 2\tau_{\mu\nu}(x; x')\tau_{\rho\sigma}(x''; x''')\, i\Delta(x; x'')i\Delta(x'; x''')\big|_{x'\to x, x'''\to x''}. \tag{B.11}
\end{aligned}
$$

To organize the calculation we now define,

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(ij)} = 2\tau^{(i)}_{\mu\nu}(x; x')\tau^{(j)}_{\rho\sigma}(x''; x''')\, i\Delta(x; x'')i\Delta(x'; x''')\big|_{x'\to x, x'''\to x''}, \tag{B.12}$$

with $i, j \in \{1, 2, 3\}$. We can now write (B.11) as

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle = \sum_{i,j}\langle \hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\rangle^{(ij)}. \tag{B.13}$$

In what follows we calculate every term on the right hand side of (B.13) separately.

B.2.1 The (11) contribution

Upon plugging (B.1) into (B.12) for $i = 1$ and $j = 1$ we find,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(11)} = \frac{2H^{2(D-2)}}{(4\pi)^D}A_{\mu\nu}^{\alpha\beta}(x)A_{\rho\sigma}^{\gamma\delta}(x'')\partial_\alpha\partial'_\beta\partial'''_\gamma\partial'''_\delta$$

$$\times\left[G(y(x;x''))G(y(x';x'''))\right]|_{x'\to x,x'''\to x''}$$

$$= \frac{2H^{2(D-2)}}{(4\pi)^D}A_{\mu\nu}^{\alpha\beta}(x)A_{\rho\sigma}^{\gamma\delta}(x'')\partial_\alpha\partial''_\gamma G(y(x;x''))$$

$$\times\partial'_\beta\partial'''_\delta G(y(x';x'''))|_{x'\to x,x'''\to x''}$$

$$= \frac{2H^{2(D-2)}}{(4\pi)^D}A_{\mu\nu}^{\alpha\beta}(x)A_{\rho\sigma}^{\gamma\delta}(x'')\partial_\alpha\partial''_\gamma G\partial_\beta\partial''_\delta G \tag{B.14}$$

$$= \frac{2H^{2(D-2)}}{(4\pi)^D}A_{\mu\nu}^{\alpha\beta}(x)A_{\rho\sigma}^{\gamma\delta}(x'')\left[\partial_\alpha y\partial_\beta y\partial''_\gamma y\partial''_\delta y\left(\frac{d^2G}{dy^2}\right)^2\right. \tag{B.15}$$

$$\left.+2\partial_{(\alpha}y\partial_{\beta)}\partial''_{(\gamma}y\partial''_{\delta)}y\frac{dG}{dy}\frac{d^2G}{dy^2}+\partial_\alpha{}_)\partial''_{(\gamma}y\partial''_{\delta)}\partial_{(\beta}y\left(\frac{dG}{dy}\right)^2\right].$$

Upon contracting with $A_{\mu\nu}^{\alpha\beta}(x)$ and $A_{\rho\sigma}^{\gamma\delta}(x'')$ defined in (B.7) we obtain,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(11)} = \frac{H^{2D-4}}{(4\pi)^D}\left\{\left[\partial_{(\mu}y\partial_{\nu)}\partial''_{(\rho}y\partial''_{\sigma)}y\right]\left[4(1-2\xi)^2\frac{dG}{dy}\frac{d^2G}{dy^2}\right]\right.$$

$$+\left[\partial_{\mu)}\partial''_{(\rho}y\partial''_{\sigma)}\partial_{(\nu}y\right]\left[2(1-2\xi)^2\left(\frac{dG}{dy}\right)^2\right]$$

$$+\left[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y\right]\left[2(1-2\xi)^2\left(\frac{d^2G}{dy^2}\right)^2\right]$$

$$-H^2\left[\partial_\mu y\partial_\nu yg''_{\rho\sigma}+g_{\mu\nu}\partial''_\rho y\partial''_\sigma y\right](1-2\xi)(1-4\xi)$$

$$\times\left[(4y-y^2)\left(\frac{d^2G}{dy^2}\right)^2+2(2-y)\frac{dG}{dy}\frac{d^2G}{dy^2}-\left(\frac{dG}{dy}\right)^2\right]$$

$$+H^4\left[g_{\mu\nu}g''_{\rho\sigma}\right](1-4\xi)$$

$$\times\left[\frac{1}{2}(1-4\xi)(4y-y^2)^2\left(\frac{d^2G}{dy^2}\right)^2+(1-4\xi)(4y-y^2)(2-y)\frac{dG}{dy}\frac{d^2G}{dy^2}\right.$$

$$\left.\left.+\left[(1-4\xi)\left(2D-\frac{1}{2}(4y-y^2)\right)-8(1-2\xi)\right]\left(\frac{dG}{dy}\right)^2\right]\right\} \tag{B.16}$$

In the next subsections we calculate (B.12) for the other values of i and j.

B.2.2 The (12)+(21) contribution

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(12)} = \frac{2H^{2(D-2)}}{(4\pi)^D} A_{\mu\nu}^{\alpha\beta}(x) B g''_{\rho\sigma} \partial_\alpha \partial'_\beta \Big[G(y(x;x'')) G(y(x';x''')) \Big]\big|_{x'\to x, x'''\to x''}$$

$$= \frac{2H^{2(D-2)}}{(4\pi)^D} A_{\mu\nu}^{\alpha\beta}(x) B g''_{\rho\sigma} \partial_\alpha G \partial_\beta G$$

$$= 2\frac{H^{2(D-2)}}{(4\pi)^D} \left(\frac{dG}{dy}\right)^2 \Big\{ -\frac{H^2}{2}(1-4\xi)B(4y-y^2)g_{\mu\nu}g''_{\rho\sigma}$$

$$+ (1-2\xi)B\partial_{(\mu}y\partial_{\nu)}y g''_{\rho\sigma} \Big\}. \quad \text{(B.17)}$$

The (21) term is simply computed from the (12) term by the substitutions, $\mu\nu \leftrightarrow \rho\sigma$ and $x \leftrightarrow x''$. We find

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(12)+(21)} = \frac{H^{2D-4}}{(4\pi)^D}\left(\frac{dG}{dy}\right)^2 \Big[(D-1)(D-2)\xi + \frac{m^2}{H^2}\Big] \quad \text{(B.18)}$$

$$\times \Big\{ -H^2\big[g_{\mu\nu}\partial''_{(\rho}y\partial''_{\sigma)}y + \partial_{(\mu}y\partial_{\nu)}y g''_{\rho\sigma}\big](1-2\xi) + H^4\big[g_{\mu\nu}g''_{\rho\sigma}\big](1-4\xi)(4y-y^2) \Big\}.$$

B.2.3 The (13)+(31) contribution

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(13)} = \frac{2H^{2(D-2)}}{(4\pi)^D} A_{\mu\nu}^{\alpha\beta}(x) C_{\rho\sigma}^{\gamma\delta}(x'') \partial_\alpha \partial'_\beta (\nabla''_\gamma \partial''_\gamma + \nabla'''_\gamma \partial'''_\gamma)$$

$$\times \Big[G(y(x;x'')) G(y(x';x''')) \Big]\big|_{x'\to x, x'''\to x''}$$

$$= \frac{4H^{2(D-2)}}{(4\pi)^D} A_{\mu\nu}^{\alpha\beta}(x) C_{\rho\sigma}^{\gamma\delta}(x'') \Big[\partial_\alpha \nabla''_\gamma \partial''_\delta G \partial_\beta G \Big]$$

$$= \frac{H^{2(D-2)}}{(4\pi)^D} A_{\mu\nu}^{\alpha\beta}(x) C_{\rho\sigma}^{\gamma\delta}(x'') \Big[-4H^2 \frac{dG}{dy}\left(\frac{dG}{dy} - \frac{d^2G}{dy^2}(2-y)\right)\partial_\alpha y \partial_\beta y g''_{\gamma\delta}$$

$$+ 8\frac{dG}{dy}\frac{d^2G}{dy^2}\partial''_\gamma y \partial_\alpha \partial''_\delta y \partial_\beta y + 4\frac{dG}{dy}\frac{d^3G}{dy^2}\partial_\alpha y \partial_\beta y \partial''_\gamma y \partial''_\delta y \Big] \quad \text{(B.19)}$$

Next we work out $A^{\alpha\beta}_{\mu\nu}(x)C^{\gamma\delta}_{\rho\sigma}(x'')$ and add to it the (31) contribution by exchanging $\mu\nu \leftrightarrow \rho\sigma$ and $x \leftrightarrow x''$ in the (13) contribution. We find

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(13)+(31)} = \frac{H^{2D-4}}{(4\pi)^D}\frac{dG}{dy}\left\{[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y]\left[-8\xi(1-2\xi)\frac{d^3}{dy^3}\right]\right. \quad \text{(B.20)}$$

$$+ [\partial_{(\mu}y\partial_{\nu)}\partial''_{(\rho}y\partial''_{\sigma)}y]\left[-16\xi(1-2\xi)\frac{d^2}{dy^2}\right]$$

$$+H^2[\partial_\mu y\partial_\nu y g''_{\rho\sigma}+g_{\mu\nu}\partial''_\rho y\partial''_\sigma y]\left[4\xi(1-2\xi)\left((4y-y^2)\frac{d^3}{dy^3}\right.\right.$$

$$\left.+(D+1)(2-y)\frac{d^2}{dy^2}-(D-1)\frac{d}{dy}\right)$$

$$\left.+2\xi(1-4\xi)\left((4y-y^2)\frac{d^3}{dy^3}+2(2-y)\frac{d^2}{dy^2}\right)\right]$$

$$+H^4[g_{\mu\nu}g''_{\rho\sigma}]\left[-4\xi(1-4\xi)(4y-y^2)\left((4y-y^2)\frac{d^3}{dy^3}\right.\right.$$

$$\left.\left.\left.+(D+1)(2-y)\frac{d^2}{dy^2}-(D-1)\frac{d}{dy}\right)\right]\right\}G(y).$$

B.2.4   The (22) contribution

This one is trivial, we find

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(22)} = \frac{2H^{2(D-2)}}{(4\pi)^D}B^2 g_{\mu\nu}g''_{\rho\sigma}G^2$$

$$= \frac{H^{2D-4}}{(4\pi)^D}\left[(D-1)(D-2)\xi+\frac{m^2}{H^2}\right]^2\left[H^4 g_{\mu\nu}g''_{\rho\sigma}\right]\frac{G^2}{2}. \quad \text{(B.21)}$$

B.2.5   The (23)+(32) contribution

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(23)} = \frac{2H^{2(D-2)}}{(4\pi)^D}Bg_{\mu\nu}C^{\gamma\delta}_{\rho\sigma}(x'')(\nabla''_\gamma\partial''_\delta + \nabla'''_\gamma\partial'''_\delta)$$

$$\times G(y(x;x''))G(y(x';x'''))|_{x'\to x,x'''\to x''}$$

$$= \frac{4H^{2(D-2)}}{(4\pi)^D}GBg_{\mu\nu}C^{\gamma\delta}_{\rho\sigma}(x'')\nabla''_\gamma\partial''_\delta G$$

$$= \frac{4H^{2(D-2)}}{(4\pi)^D}G\left\{H^2\xi B\left[\frac{dG}{dy}(2-y)(D-1)+\frac{d^2G}{dy^2}(4y-y^2)\right]g_{\mu\nu}g''_{\rho\sigma}\right.$$

$$\left.-\frac{d^2G}{dy^2}\xi Bg_{\mu\nu}\partial''_{(\rho}y\partial''_{\sigma)}y\right\}. \quad \text{(B.22)}$$

To this we need to add the (32) term, which is computed by $\mu\nu \leftrightarrow \rho\sigma$ and exchanging $x \leftrightarrow x''$. We find

$$
\begin{aligned}
\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(23)+(32)} &= \frac{H^{2D-4}}{(4\pi)^D}\left[(D-1)(D-2)\xi + \frac{m^2}{H^2}\right]\xi \\
&\times G\Bigg\{ H^2\Big[g_{\mu\nu}\partial''_{(\rho}y\partial''_{\sigma)}y + \partial_{(\mu}y\partial_{\nu)}yg''_{\rho\sigma}\Big]2\frac{d^2}{dy^2} \\
&\quad -4H^4\big[g_{\mu\nu}g''_{\rho\sigma}\big]\left[(4y-y^2)\frac{d^2}{dy^2} + (D-1)(2-y)\frac{d}{dy}\right]\Bigg\}G(y)\,.
\end{aligned}
\tag{B.23}
$$

B.2.6   The (33) contribution

$$
\begin{aligned}
\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(33)} &= \frac{2H^{2(D-2)}}{(4\pi)^D}C^{\alpha\beta}_{\mu\nu}(x)C^{\gamma\delta}_{\rho\sigma}(x'')(\nabla_\alpha\partial_\beta + \nabla'_\alpha\partial'_\beta)(\nabla''_\gamma\partial''_\delta + \nabla'''_\gamma\partial'''_\delta) \\
&\qquad\qquad \times G(y(x;x''))G(y(x';x'''))|_{x'\to x,x'''\to x''} \\
&= \frac{4H^{2(D-2)}}{(4\pi)^D}C^{\alpha\beta}_{\mu\nu}(x)C^{\gamma\delta}_{\rho\sigma}(x'')\Big[G\nabla_\alpha\partial_\beta\nabla''_\gamma\partial''_\delta G + \nabla_\alpha\partial_\beta G\nabla''_\gamma\partial''_\delta G\Big] \\
&= \frac{4H^{2(D-2)}}{(4\pi)^D}C^{\alpha\beta}_{\mu\nu}(x)C^{\gamma\delta}_{\rho\sigma}(x'')\Bigg\{ H^4(2-y)\bigg[-G\frac{dG}{dy} + (2-y)\Big(\Big(\frac{dG}{dy}\Big)^2 + G\frac{d^2G}{dy^2}\Big)\bigg]g_{\alpha\beta}g''_{\gamma\delta} \\
&\quad + H^2\bigg[-2G\frac{d^2G}{dy^2} + (2-y)\Big(G\frac{d^3G}{dy^3} + \frac{dG}{dy}\frac{d^2G}{dy^2}\Big)\bigg](g_{\alpha\beta}\partial''_\gamma y\partial''_\delta y + \partial_\alpha y\partial_\beta yg''_{\gamma\delta}) \\
&\quad + 2G\frac{d^2G}{dy^2}\partial_\alpha\partial''_\gamma y\partial''_\delta\partial_\beta y + 4G\frac{d^3G}{dy^3}\partial''_\gamma y\partial_\alpha\partial''_\delta y\partial_\beta y + G\frac{d^4G}{dy^4} + \Big(\frac{d^2G}{d^2y}\Big)^2\partial_\alpha y\partial_\beta y\partial''_\gamma y\partial''_\delta y\Bigg\}\,.
\end{aligned}
$$

This evaluates to,

$$
\begin{aligned}
\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(33)} &= \frac{H^{2D-4}}{(4\pi)^D}(4\xi^2)\Bigg\{ \big[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y\big]\Big(\frac{d^2G}{dy^2}\Big)^2 \\
&\quad -H^2\big[\partial_\mu y\partial_\nu yg''_{\rho\sigma} + g_{\mu\nu}\partial''_\rho y\partial''_\sigma y\big]\bigg[(4y-y^2)\Big(\frac{d^2G}{dy^2}\Big)^2 + (D-1)(2-y)\frac{dG}{dy}\frac{d^2G}{dy^2}\bigg] \\
&\quad +H^4\big[g_{\mu\nu}g''_{\rho\sigma}\big]\bigg[(4y-y^2)^2\Big(\frac{d^2G}{dy^2}\Big)^2 + 2(D-1)(2-y)(4y-y^2)\frac{dG}{dy}\frac{d^2G}{dy^2} \\
&\qquad\qquad\qquad\qquad + (D-1)^2(2-y)^2\Big(\frac{dG}{dy}\Big)^2\bigg] \\
&\quad +G\bigg[\big[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y\big]\frac{d^4}{dy^4} + \big[\partial_{(\mu}y\partial_{\nu)}\partial''_{(\rho}y\partial''_{\sigma)}y\big]4\frac{d^3}{dy^3} + \big[\partial_{(\mu}\partial''_{(\rho}y\partial''_{\sigma)}\partial_{\nu)}y\big]2\frac{d^2}{dy^2} \\
&\quad -H^2\big[\partial_\mu y\partial_\nu yg''_{\rho\sigma} + g_{\mu\nu}\partial''_\rho y\partial''_\sigma y\big]\Big((4y-y^2)\frac{d^4}{dy^4} + (D+3)(2-y)\frac{d^3}{dy^3} - 2D\frac{d^2}{dy^2}\Big) \\
&\quad +H^4\big[g_{\mu\nu}g''_{\rho\sigma}\big]\Big((4y-y^2)^2\frac{d^4}{dy^4} + 2(D+1)(2-y)(4y-y^2)\frac{d^3}{dy^3} \\
&\qquad\qquad + \Big(4(D^2-3) - (D^2+2D-1)(4y-y^2)\Big)\frac{d^2}{dy^2} - (D-1)^2(2-y)\frac{d}{dy}\Big)\bigg]G(y)\Bigg\}
\end{aligned}
\tag{B.24}
$$

B.3 Full energy-momentum tensor correlator

In this section we construct the full energy-momentum tensor two point function. In order to do that, it is useful to first reduce different components by making use of the hypergeometric differential equation for $G(y)$ and its derivatives,

$$\left[(4y-y^2)\frac{d^2}{dy^2} + D(2-y)\frac{d}{dy} - M^2\right]G\big(y(x;x'')\big) = 0$$

$$\left[(4y-y^2)\frac{d^3}{dy^3} + (D+2)(2-y)\frac{d^2}{dy^2} - \big[M^2 + D\big]\frac{d}{dy}\right]G\big(y(x;x'')\big) = 0$$

$$\left[(4y-y^2)\frac{d^4}{dy^4} + (D+4)(2-y)\frac{d^3}{dy^3} - \big[M^2 + 2(D+1)\big]\frac{d^2}{dy^2}\right]G\big(y(x;x'')\big) = 0, \quad \text{(B.25)}$$

where

$$M^2 \equiv \frac{m^2}{H^2} + D(D-1)\xi. \tag{B.26}$$

From (B.16) we get,

$$\begin{aligned}
\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(11)} &= \frac{H^{2D-4}}{(4\pi)^D}\bigg\{\big[\partial_\mu y\partial_\nu y\partial_\rho'' y\partial_\sigma'' y\big]\Big[2(1-2\xi)^2\Big(\frac{d^2G}{dy^2}\Big)^2\Big] \\
&\quad + \big[\partial_{(\mu} y\partial_{\nu)}\partial_{(\rho}'' y\partial_{\sigma)}'' y\big]\Big[4(1-2\xi)^2\frac{dG}{dy}\frac{d^2G}{dy^2}\Big] \\
&\quad + \big[\partial_{\mu)}\partial_{(\rho}'' y\partial_{\sigma)}'' \partial_{(\nu)} y\big]\Big[2(1-2\xi)^2\Big(\frac{dG}{dy}\Big)^2\Big] \\
&\quad + H^2\big[\partial_\mu y\partial_\nu y g_{\rho\sigma}'' + g_{\mu\nu}\partial_\rho'' y\partial_\sigma'' y\big](1-2\xi)(1-4\xi) \\
&\quad \times\Big[(D-2)(2-y)\frac{dG}{dy}\frac{d^2G}{dy^2} - M^2G\frac{d^2G}{dy^2} + \Big(\frac{dG}{dy}\Big)^2\Big] \\
&\quad + H^4\big[g_{\mu\nu}g_{\rho\sigma}''\big]\Big[\frac{1}{2}(1-4\xi)^2\Big((D-1)\big(4D-(D-1)(4y-y^2)\big)\Big(\frac{dG}{dy}\Big)^2 \\
&\quad - 2(D-1)(2-y)M^2\frac{dG}{dy}G + M^4G^2\Big) - 8(1-2\xi)(1-4\xi)\Big(\frac{dG}{dy}\Big)^2\Big]\bigg\}.
\end{aligned} \tag{B.27}$$

From (B.18) we get,

$$\begin{aligned}
\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(12)+(21)} &= \frac{H^{2D-4}}{(4\pi)^D}\Big(\frac{dG}{dy}\Big)^2\big[M^2 - 2(D-1)\xi\big] \tag{B.28} \\
&\quad \times\Big\{-H^2\big[g_{\mu\nu}\partial_{(\rho}'' y\partial_{\sigma)}'' y + \partial_{(\mu} y\partial_{\nu)} y g_{\rho\sigma}''\big](1-2\xi) + H^4\big[g_{\mu\nu}g_{\rho\sigma}''\big](1-4\xi)(4y-y^2)\Big\}.
\end{aligned}$$

From (B.20) we get,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(13)+(31)} = \frac{H^{2D-4}}{(4\pi)^D}\Bigg\{[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y]\Big[-8\xi(1-2\xi)\frac{d^3G}{dy^3}\frac{dG}{dy}\Big] \quad\text{(B.29)}$$

$$+ [\partial_{(\mu}y\partial_{\nu)}\partial''_{(\rho}y\partial''_{\sigma)}y]\Big[-16\xi(1-2\xi)\frac{d^2G}{dy^2}\frac{dG}{dy}\Big]$$

$$+H^2[\partial_\mu y\partial_\nu yg''_{\rho\sigma}+g_{\mu\nu}\partial''_\rho y\partial''_\sigma y]\Big[\big[-2(D+2)\xi+8(D+1)\xi^2\big](2-y)\frac{d^2G}{dy^2}\frac{dG}{dy}$$

$$+\big[2(D+2+3M^2)\xi-8(D+1+2M^2)\xi^2\big]\Big(\frac{dG}{dy}\Big)^2\Big]$$

$$+H^4[g_{\mu\nu}g''_{\rho\sigma}]4\xi(1-4\xi)\Big[\big[-4D+(D-1-M^2)(4y-y^2)\big]\Big(\frac{dG}{dy}\Big)^2$$

$$+(2-y)M^2\frac{dG}{dy}G\Big]\Bigg\}.$$

The (22) contribution (B.21) remains unchanged,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(22)} = \frac{H^{2D-4}}{(4\pi)^D}\big[H^4g_{\mu\nu}g''_{\rho\sigma}\big]\frac{\big[M^2-2(D-1)\xi\big]^2}{2}G^2. \quad\text{(B.30)}$$

Next, from (B.23) we get,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(23)+(32)} = \frac{H^{2D-4}}{(4\pi)^D}\Bigg\{H^2\big[g_{\mu\nu}\partial''_{(\rho}y\partial''_{\sigma)}y+\partial_{(\mu}y\partial_{\nu)}yg''_{\rho\sigma}\big]\Big[2M^2\xi\frac{d^2G}{dy^2}G\Big]$$

$$+\big[H^4g_{\mu\nu}g''_{\rho\sigma}\big]\Big[4M^2\xi(2-y)\frac{dG}{dy}G-4M^4\xi G^2\Big]\Bigg\}. \quad\text{(B.31)}$$

Finally, the (33) contribution (B.24) can be written as,

$$\langle\mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle^{(33)} = \frac{H^{2D-4}}{(4\pi)^D}(4\xi^2)\Bigg\{[\partial_\mu y\partial_\nu y\partial''_\rho y\partial''_\sigma y]\Big(\Big(\frac{d^2G}{dy^2}\Big)^2+\frac{d^4G}{dy^4}G\Big) \quad\text{(B.32)}$$

$$+[\partial_{(\mu}y\partial_{\nu)}\partial''_{(\rho}y\partial''_{\sigma)}y]4G\frac{d^3G}{dy^3}+[\partial_{(\mu)}\partial''_{(\rho}y\partial''_{\sigma)}y\partial_{(\nu}y]2G\frac{d^2G}{dy^2}$$

$$+H^2[\partial_\mu y\partial_\nu yg''_{\rho\sigma}+g_{\mu\nu}\partial''_\rho y\partial''_\sigma y]\Big[(2-y)\Big(\frac{d^3G}{dy^3}G+\frac{dG}{dy}\frac{d^2G}{dy^2}\Big)-2(M^2+1)\frac{d^2G}{dy^2}G\Big]$$

$$+[H^4g_{\mu\nu}g''_{\rho\sigma}]\Big[4\frac{d^2G}{dy^2}G+(2-y)^2\Big(\frac{dG}{dy}\Big)^2+(2-y)(D-1-4M^2)\frac{dG}{dy}G$$

$$+M^2(2M^2-1)G^2\Big]\Bigg\}$$

When all of the above contributions are combined one gets,

$$\langle \mathcal{T}\{\hat{T}_{\mu\nu}(x)\hat{T}_{\rho\sigma}(x'')\}\rangle = \frac{H^{2D-4}}{(4\pi)^D}\Bigg\{ \left[\partial_\mu y\partial_\nu y\partial_\rho'' y\partial_\sigma'' y\right]\left[4\xi^2\frac{d^4G}{dy^4}G - 8\xi(1-2\xi)\frac{d^3G}{dy^3}\frac{dG}{dy}\right.$$

$$\left.+ 2(1-4\xi+6\xi^2)\left(\frac{d^2G}{dy^2}\right)^2\right] \quad\text{(B.33)}$$

$$+\left[\partial_{(\mu}y\partial_{\nu)}\partial_{(\rho}'' y\partial_{\sigma)}'' y\right]\left[16\xi^2\frac{d^3G}{dy^3}G + 4(1-8\xi+12\xi^2)\frac{d^2G}{dy^2}\frac{dG}{dy}\right]$$

$$+\left[\partial_{(\mu)}\partial_{(\rho}'' y\partial_{\sigma)}'' \partial_{\nu)} y\right]\left[8\xi^2\frac{d^2G}{dy^2}G + 2(1-2\xi)^2\left(\frac{dG}{dy}\right)^2\right]$$

$$+H^2\left[\partial_\mu y\partial_\nu y g_{\rho\sigma}'' + g_{\mu\nu}\partial_\rho'' y\partial_\sigma'' y\right]\left[(2-y)\left[4\xi^2\right]\frac{d^3G}{dy^3}G\right.$$

$$+(2-y)\left[D-2-8(D-1)\xi+4(4D-1)\xi^2\right]\frac{d^2G}{dy^2}\frac{dG}{dy}$$

$$+\left[(1-M^2)+4(D-1+2M^2)\xi-4(3D-1+4M^2)\xi^2\right]\left(\frac{dG}{dy}\right)^2$$

$$\left.+\left[-M^2+8M^2\xi-4(D+1+4M^2)\xi^2\right]\frac{d^2G}{dy^2}G\right]$$

$$+\left[H^4 g_{\mu\nu}g_{\rho\sigma}''\right]\left[16\xi^2\frac{d^2G}{dy^2}G + \left[2(D^2-D-4)-16(D^2-3)\xi+16(2D^2+2D-3)\xi^2\right.\right.$$

$$+\frac{4y-y^2}{2}\left(-(D-1)^2+2M^2+4((2D-1)(D-1)-4M^2)\xi\right.$$

$$\left.\left.-8(2D^2-2D+1-4M^2)\xi^2\right)\right]\left(\frac{dG}{dy}\right)^2$$

$$+(2-y)\left[-(D-1)M^2+8DM^2\xi-4(D-1+4(D+1)M^2)\xi^2\right]\frac{dG}{dy}G$$

$$\left.+\left[M^4-2M^2((D-1)+4M^2)\xi+2((D-1)^2+2(2D-3)M^2+8M^4)\xi^2\right]G^2\right]\Bigg\}.$$

Note that in the minimal coupling limit, when $\xi = 0$, the TT correlator is much simpler since the most difficult terms to renormalize drop out.

Appendix C

Extracting d'Alembertians

To keep the computational part of this thesis selfcontained we outline the method of extracting d'Alembertians and list the identities needed to fully renormalize the TT-correlator. For the full list of identities we refer to [37][38][39].

Extracting d'Alembertians from powers of $y/4$ reduces the degree of divergence by 2. Acting with a d'Alembertian on a nonsingular function $F(y)$ one obtains

$$\frac{\Box}{H^2}F(y) = (4y - y^2)F''(y) + D(2-y)F'(y), \tag{C.1}$$

where a nonsingular function $F(y)$ is defined as a function which, when expanded in powers of y does not contain the power $y^{1-\frac{D}{2}}$. Using (C.1) and rearranging one establishes the relation

$$\left(\frac{y}{4}\right)^{-\alpha} = -\frac{1}{(\alpha-1)(D/2-\alpha)}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-\alpha} + \frac{D-\alpha}{D/2-\alpha}\left(\frac{y}{4}\right)^{1-\alpha}, \tag{C.2}$$

where $\alpha \neq D/2$. When the d'Alembertian acts on a singular function $y^{1-D/2}$ we obtain the relation

$$\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-\frac{D}{2}} = \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D}i\delta^D(x-x') + \frac{D(D-2)}{4}\left(\frac{y}{4}\right)^{1-\frac{D}{2}}. \tag{C.3}$$

The derivation of relation (C.3) is given in chapter 5 starting at equation (5.22). We now list a few examples for specific values of α

$$\left(\frac{y}{4}\right)^{-D} = \frac{2}{D(D-1)}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{1-D} \tag{C.4}$$

$$\left(\frac{y}{4}\right)^{1-D} = \frac{2}{(D-2)^2}\frac{\Box}{H^2}\left(\frac{y}{4}\right)^{2-D} - \frac{2}{D-2}\left(\frac{y}{4}\right)^{2-D} \tag{C.5}$$

$$\left(\frac{y}{4}\right)^{2-D} = \frac{2}{(D-3)(D-4)}\frac{\Box}{H^2}\left[\left(\frac{y}{4}\right)^{3-D} - \left(\frac{y}{4}\right)^{1-\frac{D}{2}}\right] + \frac{D(D-2)}{2(D-3)(D-4)}\left(\frac{y}{4}\right)^{1-\frac{D}{2}}$$
$$- \frac{4}{D-4}\left(\frac{y}{4}\right)^{3-D} + \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)}\frac{i\delta^D(x-x')}{(aH)^D}. \tag{C.6}$$

The other terms can be generated similarly. Note that we added 0 to (C.6) by making use of (C.3) in order to isolate the divergence onto a delta function term. The expressions (C.6) appear

to have many divergences for $D = 4$; however, all divergences cancel except for the last term when expanding the powers of y around $D = 4$. To renormalize one needs to subtract off the delta function term in (C.6) by introducing appropriate counterterms.

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