Utrecht University

Bachelor thesis

The Black-Litterman model

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Abstract

The aim of this thesis is to review the Black-Litterman model and some of its predecessors. After reviewing the models, we will try to compare the effectiveness of them by running a MATLAB simulation.
In the first part, we will reconstruct and comment on the Markowitz model. These comments show that although the theoretical approach is solid, the model does not survive empirical testing. The thesis then continues to review the Capital Asset Pricing Model and its flaws. The harsh assumptions simplify the model too much, lowering the quality of the model. In the next part the thesis reviews the Black-Litterman model and it flaws. There are some fundamental issues that are needed to simplify the model, but that may make this model unfit for real world use. These flaws however, have a more general source such that these flaws are applicable to all models that are based on mean-variance analysis. We then compare these derived models by using a MATLAB simulation for 19 stocks.

Contents

1 Introduction

Portfolio diversification is not a concept of the last 50 years. In 'The Merchant of Venice' , written by William Shakespear, the merchant Antonio says:

> My ventures are not in one bottom trusted, Nor to one place;
> nor is my whole estate Upon the fortune of this present year;
> Therefore, my merchandise makes me not sad.
>
> Act 1, scene 1.

Apparently, Shakespear understood covariance at an intuitive level. [Markowitz, 1999]

Although there was no theoretical framework to support the approach of diversification, investment companies held a wide variety of stocks in order to reduce risks. Markowitz' model filled this gap by developing "a theory of investment that covered the effects of diversification when risks are correlated, distinguished between efficient and inefficient portfolios, and analyzed risk-return trade-offs on the portfolio as a whole" [Markowitz, 1999]. Investment companies did no longer need to have a very diversified portfolio in order to minimize risk, but an efficient portfolio could be created by the using of the mean-variance theory. Although it took some time for Markowitz' ideas to be universally accepted, his ideas were further developed as well by universities as by banks like Goldman Sachs where the Black-Litterman model was developed.

Robert Litterman and Fischer Black published their paper in 1991. There has been quite some research discussing this model ever since. In 1998, other researchers at Goldman Sachs provided a paper that focused on the application of the Black-Litterman model in practice [Bevan and Winkelmann, 1998]. Satchell and Scowcroft (2000) attempted to demystify the Black-Litterman model, but instead introduced a new non-Bayesian expression of the model.

In this thesis we will conduct some quantitative research to see how the model holds against the other models. In order to do this, we first reconstruct the Markowitz model and the Capital Asset Pricing Model in order to get a good understanding of the quality of this research.

2 Markowitz' model

The paper introducing the Markowitz Model was published in 1952 by Harry Markowitz. It describes a model for selecting a portfolio for one time period (so the investor buys all his assets at time $t = 0$ and sells at time $t = 1$). The key concept is that investors should consider the risk and return of a security not by itself, but as a part of the total portfolio.

2.1 Derivation of the model for 3 securities

Before we start deriving the model, we should discuss the main assumptions made. The first main assumption is that the (rational) investor should maximize his discounted expected returns. The second main assumption is that an investor thinks of expected returns as desirable and thinks of variance of return as undesirable. Investors are risk-averse, this means that an investor prefers to have as little risk as possible. Since a higher variance leads to a higher spread of the possible returns, variance can be seen as risk. The first assumption is pretty intuitive and won't spark too much of a debate. The second one however, does. It would imply that a diversified portfolio is never preferable over a non-diversified portfolio. This interferes with the common sense that a diversified portfolio is less risky and thus better.

Now to derive the model:
Assume we have a portfolio with N different assets. Let R_i denote the return on the i^{th} security. Let μ_i be $\mathbb{E}[R_i]$ which is the expected return on the i^{th} asset. Finally, let σ_{ij} be the covariance between R_i and R_j and X_i the percentage of the portfolio that is in security i. The total yield can now be given by:

$$R = \sum_{i=1}^{N} R_i X_i$$

Note that we assume R_i and thus R (which is a linear combination of R_i) to be random variables and $\sum_{i=1}^{N} X_i = 1$ since X_i are percentages. Markowitz excluded short sales (that is, selling borrowed securities hoping the price goes down. The investor has to repay the number of securities at some given date), so $X_i \geq 0$. Since the total yield is a weighted sum of all the available securities we see that the expected return $\mathbb{E}[R]$ from the portfolio is

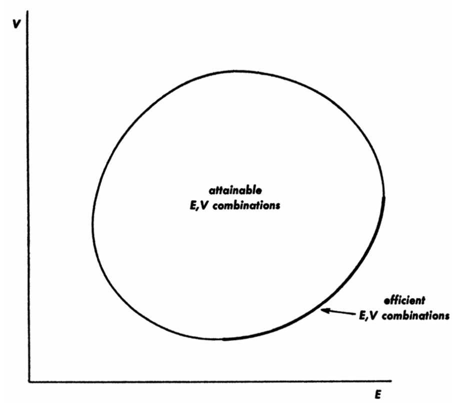
$$E = \sum_{i=1}^{N} X_i \mu_i$$

and the variance is given by

$$V = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} X_i X_j$$

We assume that the investors has some probability beliefs about the possible states of the world in period $t = 1$. He therefore can now choose a combination of expected returns and variance according to his taste. Since one of our assumptions was that investors like expected returns (E) and dislike variance (V), we introduce the so called E-V rule: An investor would want a portfolio with minimum V for some constant E and maximum E for some constant V. This can be explained graphically by the following figure:

Figure 1: Efficient E-V [Markowitz, 1952]



We consider the three security case, we can now write our model as:

$$
\begin{aligned}
E &= \sum_{i=1}^{3} X_i \mu_i && (1) \\
V &= \sum_{i=1}^{3} \sum_{j=1}^{3} X_i X_j \sigma_{ij} \\
1 &= \sum_{i=1}^{3} X_i && (2) \\
X_i &\geq 0 \, \forall i
\end{aligned}
$$

Because of (2) we get

$$X_3 = 1 - X_1 - X_2$$

We can now write E and V as functions of X_1, X_2 so our system of equations can be rewritten as:

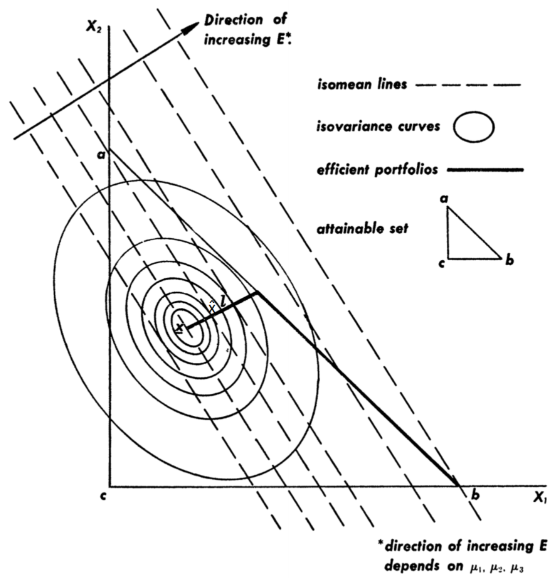
$$E = E(X_1, X_2)$$
$$V = V(X_1, X_2)$$
$$X_1 \geq 0, X_2 \geq 0, 1 - X_1 - X_2 \geq 0$$

We can now construct a two-dimensional figure to compare portfolio choices. The isomean curve is defined to be the set of all portfolios for a constant mean, so we keep expected returns constant and we vary variance. Likewise, the isovariance is defined to be the set of all portfolios for a constant variance, so we keep the variance constant and vary the expected returns. The set of isomean curves is a set of parallel linear lines. We can show this by writing (1) as

$$X_2 = \frac{E - \mu_3}{\mu_2 - \mu_3} - \frac{\mu_1 - \mu_3}{\mu_2 - \mu_3} X_1$$

We do this by substituting $X_3 = 1 - X_1 - X_2$ and solving for X_2. The intercept of this line will be $\frac{E-\mu_3}{\mu_2-\mu_3}$ and the slope $\frac{\mu_1-\mu_3}{\mu_2-\mu_3}$. If E is changed only the intercept of the line will change, not the slope, so we have a family of parallel lines. We can show that the isovariances lines form a family of concentric ellipses in a similar, but slightly more complicated way. The center of these concentric ellipses, is the minimum for V. We will call this point **X**. Variance will increase if an ellipse lies further away from this point **X**. This centric point **X** doesn't necessarily have to fall within the region for a feasible solution. We call **X** efficient if it is in the feasible region. Let us label the related expected return and variance of the point **X** as **E** and **V**. Since there is no other point in the feasible set with a lower variance than **X** (with the same or greater E) and there is no portfolio with greater E with the same or lower V this is the most desirable portfolio for a fixed V or fixed E. If we now fix E (so we pick one of the isomean lines) we can find the lowest V related to that line by finding the point at which the line is tangent to the isovariance curve. We will call this point $\hat{\mathbf{X}}$. Doing this for all isomean lines we can define a straight line as shown in the figure below.

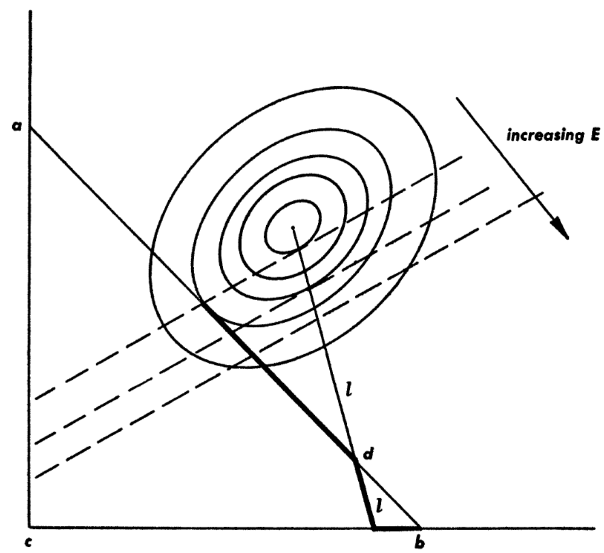
Figure 2: Isomean and isovariance curves with **X** inside the feasible area [Markowitz, 1952]



We will now consider a situation where **X** lies outside the feasible set. We can still construct the line that is most efficient though.

As soon as the most efficient line is in the feasible area it becomes part of the solution.

Figure 3: Isomean and isovariance curves with **X** outside the feasible area [Markowitz, 1952]



2.2 Derivation of the model for n securities

We now have considered a framework for 3 securities. We will now set up the framework for n securities, so that we can use the model more generally. Since we want to minimize the variance of our portfolio, depending on a fixed expected return, we want to minimize:

$$Var(C_1) = Var(C_0 + \text{return on portfolio}) = Var(r^T\theta) = \theta^T\Sigma\theta$$

where C_1 is the value of the portfolio at $t = 1$, C_1 is equal to our initial investment C_0 plus the return we have on our portfolio. We can also write this in vector notation by multiplying the returnvector r with the investment vector θ which is the vector describing the investment for different assets in our portfolio. The variance of the returnvector is given by Σ. Using the rule $Var(ab) = a^2var(b)$ we see that $Var(r^T\theta) = \theta^T\Sigma\theta$. We have to make the assumption that Σ is positive definite (and thus invertible). This is usually the case for real life data, so this assumption isn't too harsh. There are two constraints though. First of all, since we are trying to minimize the variance for a given expected return,we should fix expected returns. Second, we can only invest current capital, so the sum of all assets in $t = 0$ equals our starting capital. These constraints can be summarized as:

$$\mu^T\theta = \mu_p \text{ and } \bar{1}^T\theta = C_0$$

So we have fixed the expected returns to the expected returns for our portfolio p, and we have put a constraint on what we can spend. To simplify the earlier expression we can write:

$$\text{Min } \left\{\theta^T \Sigma \theta | A^T \theta = B\right\} \tag{3}$$

with

$$A = \begin{pmatrix} \mu & \mathbf{1} \end{pmatrix} \; B = \begin{pmatrix} \mu_p \\ C_0 \end{pmatrix}$$

We can solve this system using Lagrangian optimization. We find the following first order conditions by differentiating with respect to θ and with respect to the Lagrangian multiplier λ_0.

$$\text{first order conditions:} \begin{cases} 2\Sigma\theta + A\lambda_0 = 0 & (4) \\ A^T\theta = B \text{ with } \lambda_0 = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} & (5) \end{cases}$$

Solving equation (4) for θ gives

$$\theta = \Sigma^{-1} A \lambda$$

where $\lambda = -\frac{1}{2}\lambda_0$ so that the second equation of (4) becomes

$$A^T \Sigma^{-1} A \lambda = B \Longrightarrow \lambda = (A^T \Sigma^{-1} A)^{-1} B = H^{-1} B$$

Where $H = (A^T \Sigma^{-1} A)$ and $H^T = (A^T \Sigma^{-1} A)^T$

Rewriting this leads to $H^T = A^T (\Sigma^{-1})^T A = A^T \Sigma^{-1} A = H$ thus H is a symmetric 2×2-matrix. Putting these values into our minimization problem we can find the minimum variance under these conditions.

$$\theta^T \Sigma \theta = \theta^T \Sigma \Sigma^{-1} A \lambda = \theta^T A \lambda = (A^T \theta)^T H^{-1} B = B^T H^{-1} B$$

We have shown that H is a symmetric 2×2-matrix, so lets write H as:

$$H \equiv \begin{pmatrix} a & b \\ b & c \end{pmatrix} \Longrightarrow H^{-1} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$$

Let $d \equiv det(H) = ac - b^2$. Since $H = (A^T \Sigma^{-1} A)$ we can show that
$a = \mu^T \Sigma^{-1} \mu$
$b = \mu^T \Sigma^{-1} \mathbf{1} = \mathbf{1^T} \Sigma^{-1} \mu$
$c = \mathbf{1^T} \Sigma^{-1} \mathbf{1}$
Since we have assumed that the covariance matrix is positive definite, the inverse covariance matrix is also positive definite. This leads to $\theta^T \Sigma^{-1} x > 0$ for all non-zero $N \times 1$-vectors x. From this we can deduce that $a > 0$, $c > 0$ and

$$(b\mu - a\mathbf{1})^T)\Sigma^{-1}(b\mu - a\mathbf{1}) = bba - abb - abb + aac = a(ac - b^2) = ad > 0$$

and since $a > 0$ it must be that $d > 0$. Using our definition for H we can rewrite the variance as:

$$Var(R_p) = \frac{1}{d}\begin{pmatrix}\mu_p & C_0\end{pmatrix}\begin{pmatrix}c & -b \\ -b & a\end{pmatrix}\begin{pmatrix}\mu_p \\ C_0\end{pmatrix}$$
$$\sigma_p^2 = \frac{1}{d}(c\mu_p^2 - 2bC_0\mu_p + aC_0) \tag{6}$$

In order to get this into the desired form, we need to take a few steps. First notice that $a = \frac{d-b^2}{c}$. Next, divide the left side by $\frac{1}{c}$ and the right side by $\frac{c}{c^2}$ this leads to:

$$\sigma_p^2 = \frac{c\mu_p^2 - 2bC_0\mu_p + dC_0^2/c + b^2C_0^2/c}{d}$$
$$\frac{\sigma_p^2}{1/c} = \frac{\mu_p^2 - 2bC_0\mu_p/c + dC_0^2/c^2 + b^2C_0^2/c^2}{d/c^2}$$
$$= \frac{(\mu_p - bC_0/c)^2}{d/c^2} + C_0^2$$

Moving the first term from the right hand side to the left hand side, and dividing the entire equation by C_0^2 leads to the hyperbola:

$$\frac{\sigma_p^2}{C_0^2/c} - \frac{(\mu_p - bC_0/c)^2}{dC_o^2/c^2} = 1$$

We can find the center of this hyperbola by asking ourselves: for what values of σ_p, μ_p is the following equation equal to zero?

$$\frac{\sigma_p}{C_0^2/c} - \frac{(\mu_p - bC_0/c)^2}{dC_0^2/c^2}$$

We find the center of the hyperbola to be $(0, \frac{b}{c}C_0)$. We can find the slope of the asymptotes by taking the square root of the denominator of the second term divided by the denominator of the first term. So we find: $\pm\sqrt{\frac{dC_0^2/c^2}{C_0^2/c}} = \pm\sqrt{\frac{d}{c}}$ Combining this with the center of the hyperbola, the formula for the asymptotes are given by:

$$\mu_p = \frac{b}{c}C_0 \pm \sqrt{\frac{d}{c}}\sigma_p$$

Since we've already found the relation between the minimum variance and the corresponding returns, we also want to know the right combination of assets, namely θ. We have

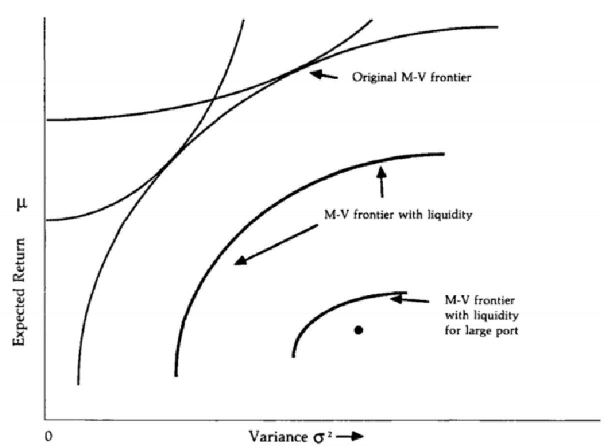
$$\theta = \Sigma^{-1}A\lambda = \Sigma^{-1}AH^{-1}B = \frac{c\mu_p - bC_0}{d}\Sigma^{-1}\mu + \frac{aC_0 - b\mu_p}{d}\Sigma^{-1}\bar{1}$$
$$= \frac{1}{d}\Sigma^{-1}((a\bar{1} - b\mu)C_0 + (c\mu - b\bar{1})\mu_p) \tag{7}$$

So for any return, μ_p, we want, we can now find the minimum variance that corresponds with this return by using (6) and the corresponding portfolio by using (7).

2.3 Critique on the Markowitz' model

Jobson and Korkie showed that in a majority of the cases equal weighting of securities outperforms Markowitz optimization. This could be due to a number of factors. Firstly, Markowitz omits certain factors which make securities preferable. One of these factors is liquidity. When the total value of a portfolio is large, as for banks or investment companies, a 1 % change in the portfolio could represent a substantial change in the total value of the firm. Since the model assigns proportions of the total investment wealth to the assets, the proportion is likely to change thus liquidity is a desirable feature. The following figure describes the change of the mean-variance frontier (the line on which the efficient portfolios are located) due to the addition of liquidity. Accounting for liquidity has a negative effect on the Sharpe ratio, that is $\frac{\text{expected returns}}{\text{variance}}$. We can see this by noticing that the Mean-variance frontier has shifted downwards, implying more variance for fixed expected returns. This was to be expected since less liquidity brings in more risk (variance).

Figure 4: Mean-variance frontier with liquidity factor [Michaud, 1989]



One of the main problems of the Markowitz model is its inability to cope with erroneous predictions. A slight change in the predictions could lead to a completely different portfolio because of the change in expected values, variance and covariance of the securities.

3 Capital asset pricing model

The Capital asset pricing model was developed by Sharpe and Lintner in the sixties. It heavily relies on the mean-variance model of Markowitz which was discussed earlier. The zero-beta CAPM, developed by Fischer Black in 1972, was more robust against empirical testing and was thus important for the widespread acknowledgement of the capital asset pricing model.

3.1 Derivation of the model

We assume that investors think in probabilistic terms, so the desirability of an investment is dependent on two parameters, the expected value and the standard deviation. Thus we can represent the investors utility by

$$U = f(E_w, \sigma_w)$$

Where E_w is the expected future wealth and σ_w is the standard deviation from the expected wealth. We assume (just as in the Markowitz Model) that investors desire more expected future wealth, thus $\frac{dU}{dE_w} > 0$. Since most investors are risk-averse they prefer less risk given the level of E_w, so $\frac{dU}{d\sigma_w} < 0$. This indicates an upward sloping indifference curve. (An indifference curve is a set of combinations where the investor is equally well-off.) We can see this by considering the following: the investor dislikes variance, so if variance rises he must be compensated by more expected returns, so we expect an upward sloping curve. Since at high levels of risk the investor needs to be compensated with more expected return, this curve is convex. We define the total return on his investment at time t, to be

$$R \equiv \frac{W_t - W_{t-1}}{W_{t-1}}$$

Rewriting this gives

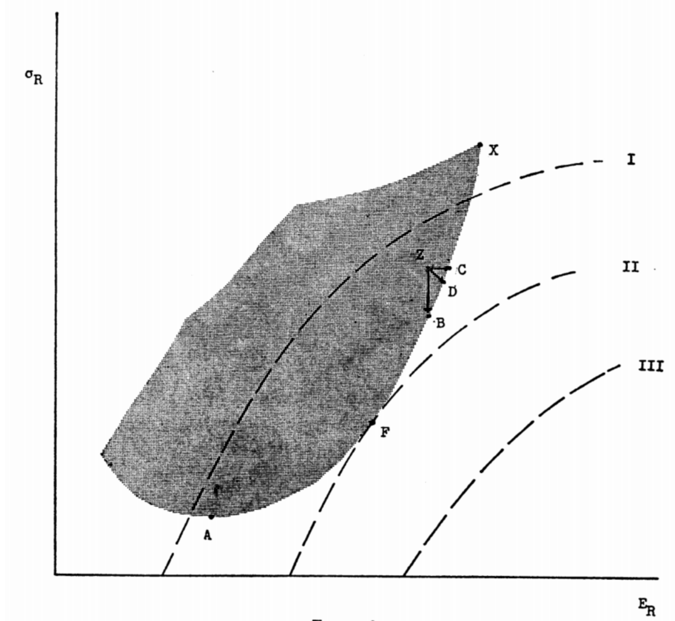
$$W_t = RW_{t-1} + W_{t-1}$$

We can now relate his utility function to expected returns instead of expected wealth

$$U = g(E_R, \sigma_R)$$

An indifference curve more to the right is preferred over an indifference curve to the left, since at equal variance levels, the expected return is higher for the curve to the right. So $III > II > I$. The shaded area represents the possible combinations for securities assuming no security is without any risk (even government bonds have some risk). The shape of this set is almost completely arbitrary, but notice that there is no riskless security since for every point in the set the variance is bigger than zero. The points on the right side of the shaded area are efficient since there is no point with lower variance for that given expected return. So the efficient set is, corresponding with Markowitz, the curve AFBDCX.

Figure 5: Indifference curves for a σ_R E_R graph [Sharpe, 1964]



Let there now be two investments, a and b. The investor invests a proportion α in a and the remainder in b, we call this portfolio c. The expected return will be:

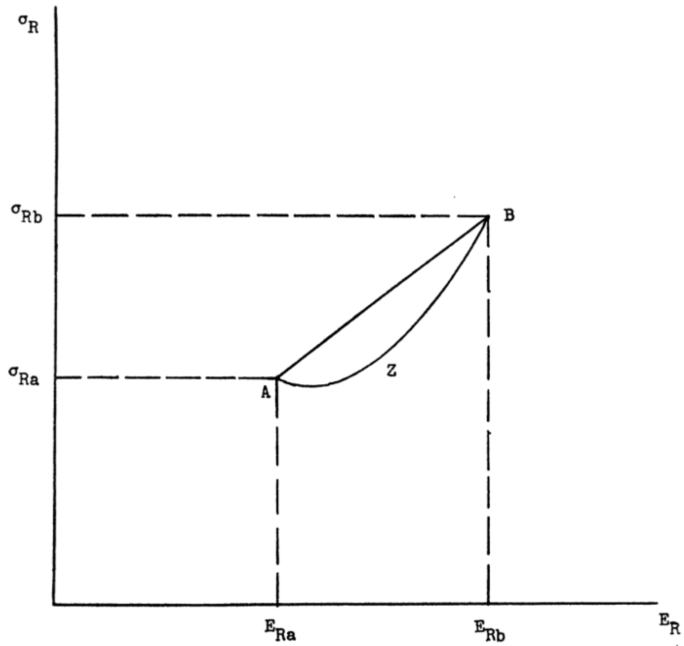
$$E_{Rc} = \alpha E_{Ra} + (1 - \alpha) E_{Rb}$$

The predicted variance will be

$$\sigma_{Rc} = \sqrt{\alpha^2 \sigma_{Ra}^2 + (1 - \alpha)^2 \sigma_{Rb}^2 + 2 r_{ab} \alpha (1 - \alpha) \sigma_{Ra} \sigma_{Rb}}$$

where r_{ab} is the correlation between a and b, usually this value is between 0 and +1. We can elaborate graphically.

Figure 6: A mixed portfolio in the σ_R, E_R-plane [Sharpe, 1964]



If a and b are perfectly correlated (+1), their combination must be on the straight line between A and B. The smaller the correlation, the more U-shaped the curve will be. The curve Z represents zero correlation.

Let's now include riskless securities, we will call this security P with $\sigma_p = 0$ and $E_{Rp} \equiv$ pure rate of interest. We invest a proportion α in P and the remainder in a risky security A. We obtain:

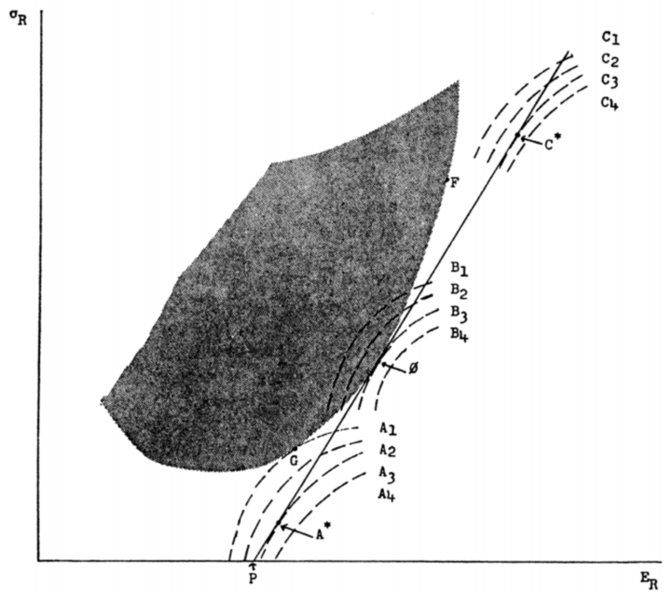
$$E_{Rc} = \alpha E_{Rp} + (1 - \alpha) E_{Ra}$$

And the standard deviation:

$$\sigma_{Rc} = \sqrt{\alpha^2 \sigma_{Rp}^2 + (1-\alpha)^2 \sigma_{Ra}^2 + 2 r_{pa} \alpha (1-\alpha) \sigma_{Rp} \sigma_{Ra}} = (1-\alpha)\sigma_{Ra}$$

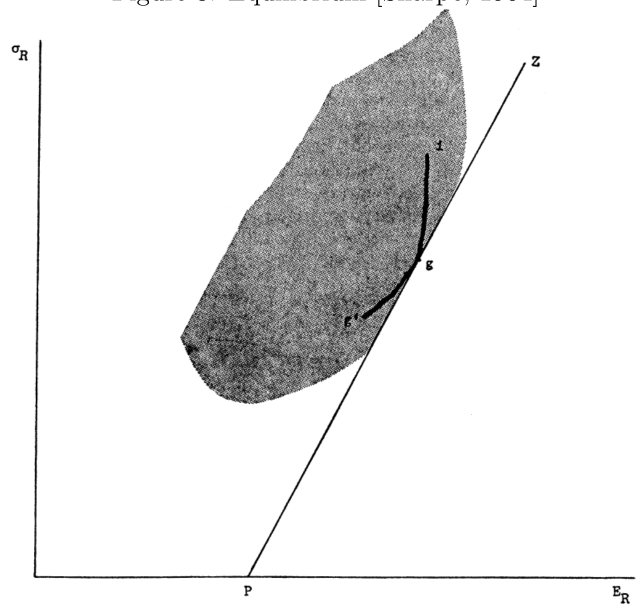
Since $\sigma_{Rp} = 0$. This implies that there is a linear relation, and thus all combination must be on a straight line starting at (E_{Rp},0). We can achieve investment points A and B by lending money at the pure rate of interest and investing that in respectively A and B. Points on the line that is tangent to the shaded area (the set of possible portfolios) are dominant to those investments however. Thus all the investments on the original curve, X to ϕ, are dominated by a combination of investing in ϕ and lending at the pure rate of interest. If we allow borrowing (thus a negative α) we can extend the lines PA, PB etc. since we can borrow money to invest more in A or B. The line $P\phi C$ is now the optimal investment curve.

Figure 7: Multiple investors [Sharpe, 1964]



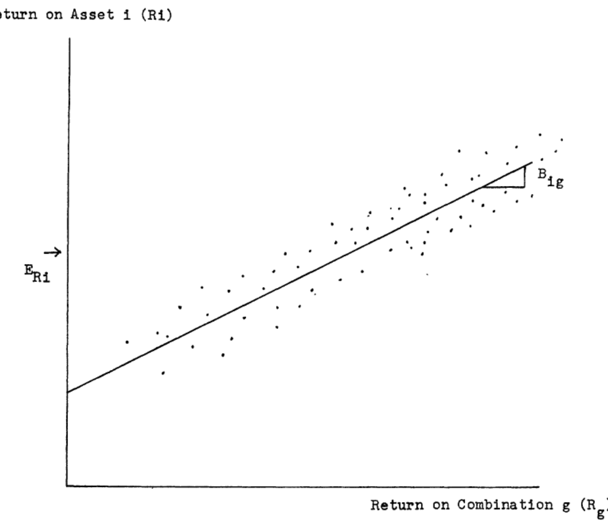
We now need two new assumptions. Firstly, the homogeneity of investors expectations, and secondly a common pure rate of interest. These are harsh assumptions but they are needed for simplicity. If there are multiple investors, with different preferences, we can see that they still all are in the market for ϕ. In the figure above, investor A lends some money at the pure rate of interest and invests the rest of his money in ϕ, investor B invests all his money in ϕ and investor C borrows money at the pure rate of interest to invest more in ϕ. It is intuitive that because of this preference for ϕ the price of the securities in ϕ and thus the attractiveness (since returns are dependent on the old prices). On the other hand, securities not in ϕ will become more attractive, since their value falls. So when will we reach equilibrium? This occurs when prices have shifted in such a way that there will be multiple efficient portfolios on the same straight line, as illustrated in figure 8. This indicates that they are perfectly correlated. Consider the line in figure 8. The point **1** is a single asset, where g is an efficient portfolio. The bold line connecting $\mathbf{1}, g$ and g' shows all the possible combinations of $\alpha\mathbf{1} + (1-\alpha)g$. Notice that at $\alpha = 0$ there will still be some of asset 1 in the portfolio since there is some of asset 1 in g. Since g is an efficient portfolio, the continuous line connecting points $\mathbf{1}$andg' through g must be tangent to the efficient portfolio line PZ

Figure 8: Equilibrium [Sharpe, 1964]



We will try to relate the expected return to elements of risk. We will consider a regression between the return on an asset i and the return on the portfolio g which could look like the figure 9. The scatter of the observations around their mean is due to the total risk, but a part of the scatter, due to the slope of the regression line, is due to a relationship with the return on g. Thus we can formulate a relationship between $R_i$andR_g, the returns on the single asset i and the portfolio g. We will call this factor β. This is the nondiversifiable part of risk, also called systematic risk. The so-called capital market line, given by $\bar{r} = r_f + \frac{r_g - r_f}{\sigma_g}\sigma$, with slope $\frac{r_g - r_f}{\sigma_g}$ is the line PZ .

Figure 9: A possible regression between the return on asset i and portfolio g [Sharpe, 1964]



Putting everything we have developed in this section together, we find a relationship between the return on an asset i, the pure rate of interest, the β-coefficient and the expected returns on other assets. We will now derive the CAPM formula.
We form a portfolio of asset i and portfolio g by making it a weighted combination of i and g with weights $(\alpha, 1-\alpha)$. We assume asset i is not efficient (as most single assets are). We find:

$$\begin{aligned} r(\alpha) &= \alpha r_i + (1-\alpha) r_g \\ &= \alpha(r_i - r_g) + r_g \end{aligned} \tag{8}$$

$$\begin{aligned} \sigma(\alpha) &= \sqrt{\alpha^2\sigma_i^2 + (1-\alpha)^2\sigma_g^2 + 2\alpha(1-\alpha)\sigma_{g,i}} \\ &= \sqrt{\alpha^2(\sigma_i^2 + \sigma_g^2 - 2\sigma_{g,i}) + 2\alpha(\sigma_{g,i} - \sigma_g^2) + \sigma_g^2} \end{aligned} \tag{9}$$

Note that for $\alpha = 0$, $(\sigma(0), r(0)) = (\sigma_g, r_g)$ and when $\alpha = 1$ $(\sigma(1), r(1)) = (\sigma_i, r_i)$. So the curves touch the capital market line, defined earlier, only at the point (σ_g, r_g). The other points are in the feasible region, including (σ_i, r_i). This means that the curve is tangent to the capital market line. In this common point, the slope of the capital market line is equal to the derivative of the portfolio. So we find :

$$\frac{dr(\alpha)}{d\sigma(\alpha)} = \frac{r_g - r_f}{\sigma_g} \tag{10}$$

Rewriting $\frac{dr(\alpha)}{d\sigma(\alpha)}$ as $\frac{dr(\alpha)/d\alpha}{d\sigma(\alpha)/d\alpha}$ and differentiating (8) and (9) gives:

$$\frac{dr(\alpha)/d\alpha}{d\sigma(\alpha)/d\alpha} = \frac{r_i - r_g}{(\sigma_{g,i} - \sigma_g^2)/\sigma_g} \tag{11}$$

Combining (11) with (10) we find:

$$\begin{aligned} \frac{r_i - r_g}{(\sigma_{g,i} - \sigma_g^2)/\sigma_g} &= \frac{r_g - r_f}{\sigma_g} \\ r_i - r_f &= \beta_i(r_g - r_f) \end{aligned}$$

Where $\beta = \sigma_{g,i}/\sigma_g$. To generalize this result to a portfolio p notice that:

$$\begin{aligned} r_p - r_f &= -rf + \sum_{i=1}^{n} \alpha_i r_i \\ &= \sum_{i=1}^{n} \alpha_i (r_i - r_f) \\ &= \sum_{i=1}^{n} \alpha_i \beta_i (r_g - r_f) \text{ (CAPM formula for a single asset } i) \\ &= (r_g - r_f) \sum_{i=1}^{n} \alpha_i \beta_i \end{aligned}$$

3.2 Critique on the Capital asset pricing model

This model is, of course, a simplified version of reality. Certain assumptions are made to prevent the model from getting too complex. These assumptions however, do lower the quality of the model. The main assumption, that here is homogeneity for the expectations of the investors is the harshest assumption. Besides assuming that every investor has the same expectations, the quality of these expectations are ignored and assumed to be correct. Behavioural finance could perhaps shed some light on the bias that these expectations have. Another big assumption is the rate of pure interest, which is constant. Besides that, it is risk-free to lend or borrow money for this rate. Fischer Black (1972) developed a version of the CAPM without risk-free borrowing/lending to tackle this assumption. Empirical testing on the original CAPM wasn't very succesful, the model developed by Fischer Black however had some succes. Empirical data suggests that the prediction the CAPM makes for the relation between beta and average return has a positive bias. As a result, estimates for high beta stocks are too high and estimates for low beta stocks are too low. The CAPM should thus be used as a theoretical framework to introduce portfolio theory.

4 The Black-Litterman model

The Black-Litterman model was developed by Fischer Black and Robert Litterman for Goldman Sachs in 1990 and published in 1992. The model is built on the Capital asset pricing model and the Markowitz mean-variance model. The portfolio is proportional to the market equilibrium portfolio plus a weighted sum of portfolios reflecting the investor's views.

4.1 Reconstructing the Black-Litterman model

The Black-Litterman model begins with a neutral equilibrium portfolio. General equilibrium theory states that if aggregate portfolio is at equilibrium, each sub-portfolio must be at equilibrium as well. Any utility function can be used, so we use the most common one, the quadratic utility function. We start deriving the model by optimizing the utility function of the investor:

$$U = w_m^T \Pi - (\frac{\delta}{2}) w_m^T \Sigma w_m \tag{12}$$

where:

U	is the investors utility function
w_m	is the vector of weights allocating the assets according to the market
Π	is the vector of excess returns for assets (returns - risk free returns)
δ	is the risk aversion parameter
Σ	is the covariance matrix of the excess returns

Since U is convex, it will have one global maximum. If we maximize this function with respect to the weights we get (note there are no constraints currently):

$$\frac{dU}{dw} = \Pi - \delta \Sigma w = 0 \Rightarrow \Pi = \delta \Sigma w \tag{13}$$

To deduce the value of δ we rewrite (13). We multiply both sides by w_m^T. Note that $w_m^T \Pi$ is the total excess return for that portfolio, and $w_m^T \Sigma w_m$ is the total variance of the market portfolio.

$$w_m^T \Pi = (r - r_f) = \delta w_m^T \Sigma w_m = \delta \sigma^2$$
$$\delta = (r - r_f)/\sigma^2$$

Where r is the return on the portfolio. Usually, $r - r_f > 0$. If this was violated, it would not make sense to invest in this portfolio since the investor's expectations state that the return for risk free assets would be higher.

The expected returns, μ are centered at the equilibrium values, so they are normally distributed with mean Π.

$$\mu = \Pi + \epsilon_1$$

Where $\epsilon_1 \sim N(0, \tau\Sigma)$, where τ is a scalar indicating the uncertainty about the CAPM prediction. The investor also has a number of views of his own, which we can describe in relative or absolute ways. We can embed these views in a matrix in the following way. Let there be 3 assets, asset A,B and C. Our first view is that asset A will outperform asset B by 3%-point with confidence level ω_1. Our second view is that asset C will return 2 % with confidence level ω_2. We define our link matrix, P, to be a $K \times N$-matrix where K is the number of views, and N is the number of assets. Our view matrix Q will be a $K \times 1$-matrix. Since Q is an estimate, it will have an error term. We assume that $\epsilon_2 \sim N(0, \Omega)$, thus Ω is the covariance (uncertainty) of our views. In this example we have the following matrices:

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; Q = \begin{pmatrix} 3 \\ 2 \end{pmatrix}; \Omega = \begin{pmatrix} \omega_{11} & 0 \\ 0 & \omega_{21} \end{pmatrix}$$

So we can write

$$P\mu = Q + \epsilon_2$$

Where μ is the vector of expected market returns. In statistics, the inverse of variance is precision. We can thus give the precision of our prediction by Ω^{-1} and the precision of the market by $(\tau\Sigma)^{-1}$.
We try to update our expected returns by taking a weighted average of what the market expects and our own views. We will call this new optimized return vector μ^*. We find the formula:

$$\mu^* = E(R) = [(\tau\Sigma)^{-1} + P^T\Omega P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega Q] \tag{14}$$

Notice that

$$(\tau\Sigma)^{-1}\Pi + P^T\Omega Q = (\tau\Sigma)^{-1}\Pi + P^T\Omega P\mu$$

so the last part of the equation is the sum of precision times return for the market and investor's prediction. The first part of the equation, is the inverse of the sum of the total variance. It is there to normalise the weights.

The new covariance matrix for $E(R)$ takes the additional variance of the investor's view into account. The new covariance matrix can be given by:

$$\Sigma_p = \Sigma + M^{-1} \text{ where } M = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]$$

Now that we have the new covariance matrix, we can calculate the new portfolio weights,w^*, updating our maximization function and differentiating with respect to w^*.

First we want to rewrite μ^* so that we can use it in a simpler form.
We follow the method of [Walters, 2014, Appendix D]

$$\begin{aligned}
\mu^* &= [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q] \\
&= [(\tau\Sigma)^{-1} + P^T\Omega^{-1}O]^{-1}(\tau\Sigma)^{-1}(\tau\Sigma)[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q] \\
&= [I + \tau\Sigma P^T\Omega^{-1}P]^{-1}[\Pi + \tau\Sigma P\Omega^{-1}Q] \\
&= [I + \tau\Sigma P^T\Omega^{-1}P]^{-1}[(I + \tau\Sigma P^T\Omega^{-1}P)\Pi + \tau\Sigma P^T\Omega^{-1}(Q - P\Pi)] \\
&= \Pi + (I + \tau\Sigma P^T\Omega P]^{-1}[\tau\Sigma P^T\Omega^{-1}](Q - P\Pi) \\
&= \Pi + (I + \tau\Sigma P^T\Omega P]^{-1}\tau\Sigma P^T\Omega^{-1}[(\Omega + P^T\tau\Sigma P)(\Omega + P^T\tau\Sigma P)^{-1}](Q - P\Pi) \\
&= \Pi + (I + \tau\Sigma P^T\Omega P]^{-1}(\tau\Sigma P^T + \tau\Sigma P^T\Omega^{-1}P^T\tau\Sigma P)(\Omega + P^T\tau\Sigma P)^{-1}(Q - P\Pi) \\
&= \Pi + (I + \tau\Sigma P^T\Omega P]^{-1}(I + \tau\sigma P^T\Omega^{-1}P)\tau\Sigma P^T(\Omega + P^T + \tau\Sigma P)^{-1}(Q - P\Pi) \\
&= \Pi + \tau\Sigma P^T(\Omega + \tau P\Sigma P^T)^{-1}(Q - P\Pi) \\
&= \Pi + \Sigma P^T(\frac{\Omega}{\tau} + P\Sigma P^T)^{-1}(Q - P\Pi)
\end{aligned}$$

Now we can optimize our portfolio using this simplified version of μ^*

$$\max\,(w^*)'\mu^* - \frac{\delta}{2}(w^*)'\Sigma_p w^*$$

We find the first order condition by differentiating with respect to w^*:

$$\mu^* - \delta\Sigma_p w^* = 0 \Rightarrow \mu* = \delta\Sigma_p w^*$$

Rewriting gives:

$$w^* = \frac{1}{\delta}\Sigma_p^{-1}\mu^*$$

We can now plug in the formula for μ^* and plug in $\Pi = \delta\Sigma w_m$ we find:

$$w^* = w_m + P^T(\frac{\Omega}{\tau} + P\Sigma P^T)^{-1}(\frac{Q}{\delta} - P\Sigma w_m) \tag{15}$$

4.2 Critique

The main issue of the Black-Litterman model is a problem of a general kind. As for all mean-variance optimization models, the Black-Litterman model is heavily reliant on a few simplifications and assumptions. First of all, there are the simplifications. Factors such as liquidity, marketability, taxes and transactions fees are excluded from the model. This is necessary in order to maintain a workable model, but it is obviously a bad representation of the real world. Second, there is another fundamental problem. One of the main assumptions that is made, is that standard deviation is equal to risk. This assumption does not take business risk, valuation risk, financing risk, modelling risk etc. into consideration. Even if this was not such a strong assumption, risk, or variance, is treated as a symmetrical factor in the Black-Litterman model, where variance is asymmetrical in theory.

5 Performance of the Black-Litterman model

In this section we will compare the Black-Litterman model and the Markowitz model for different values of τ. We will compare the results for the period: 1 january 2000 - 1 january 2015 and the post crisis period 1 january 2010 - 1 january 2015. The first period is displayed on the left side of the page, and the last period is displayed on the right side of the page. Instead of the market weights we have used Markowitz portfolios as a starting point for the Black-Litterman model.

We use 5 views for this model. Our first view is that asset 3M-company (MMM) will outperform Merck company (MRK) by 1 percent-point. Our second view is that Alcoa inc. (AA) will outperform Hewlett Packard (HPQ) by 1 percent-point. Our third view is that Caterpillar inc. (CAT) will return 2 percent. Our fourth view is that Cisco inc. (CSCO) will return -1 percent. Our last view is that E. I. du Pont de Nemours and Company will return 2 percent. These views are based on the average returns for the last 5 months. Our views vector Q and our confidence matrix Ω are given by:

$$Q = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \\ 2 \end{pmatrix} \quad \Omega = \begin{pmatrix} 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.275 \end{pmatrix}$$

The linkmatrix P is constructed as described in section 4.1. To compute δ we use a risk free rate of 1.35% per annum. This is the yearly interest paid for a 5-year U.S. Treasury bill. Since our returns our on a monthly basis, we use $r_f = 1.0135^{1/12} - 1$.

Our approach is as follows: We run the Markowitz script to generate 10 portfolios for different risk levels. The initial expected returns are the mean returns of the data. Once we have these portfolios we run the Black-Litterman script to "update" these portfolios to finally run our simulation script and plot the returns. Note that these returns are on monthly basis and not on yearly basis.

First we have calculated the efficient frontier and our 10 portfolios using the Markowitz script provided by [Agrawal,].

Figure 10: Efficient frontier

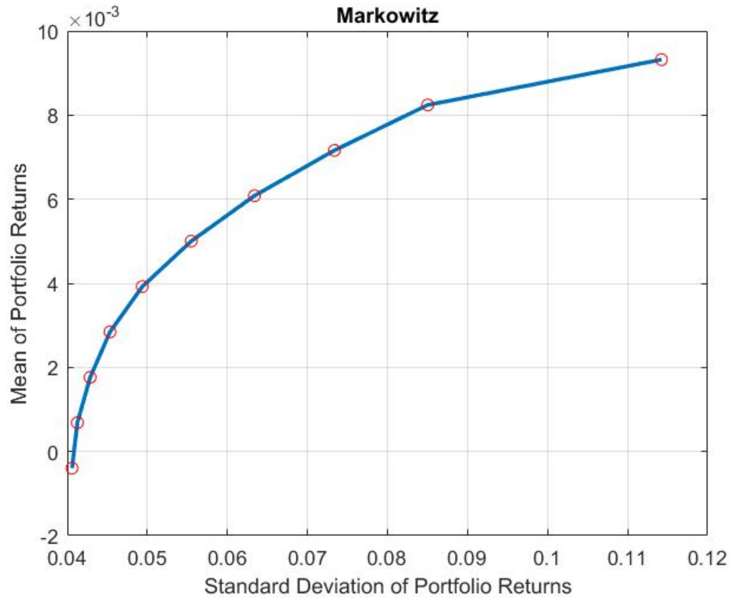


Figure 11: Black-Litterman versus Markowitz returns for $\tau = 0.05$

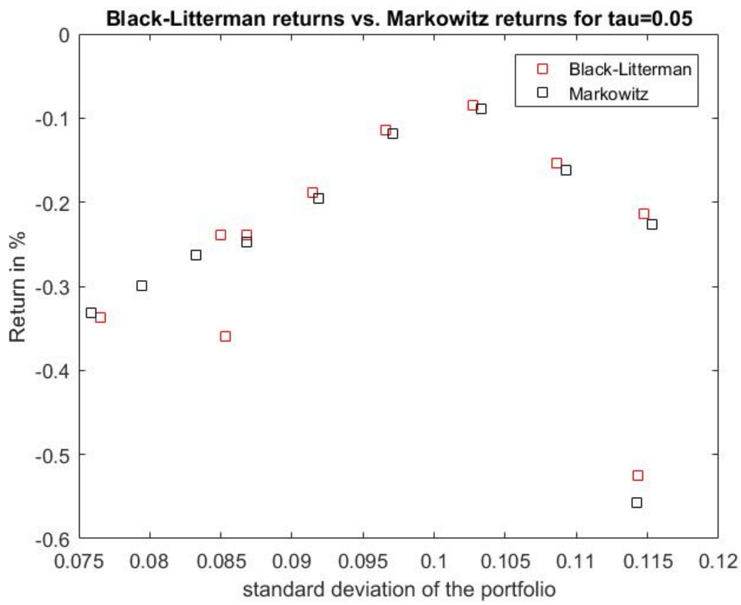


Figure 12: Black-Litterman versus Markowitz returns for $\tau = 0.05$ post crisis

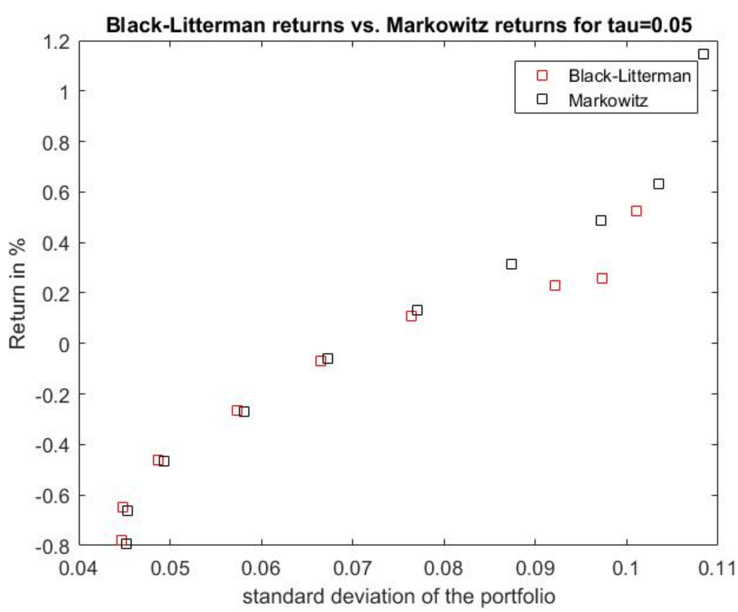


Figure 13: Black-Litterman versus Markowitz returns for $\tau = 0.25$

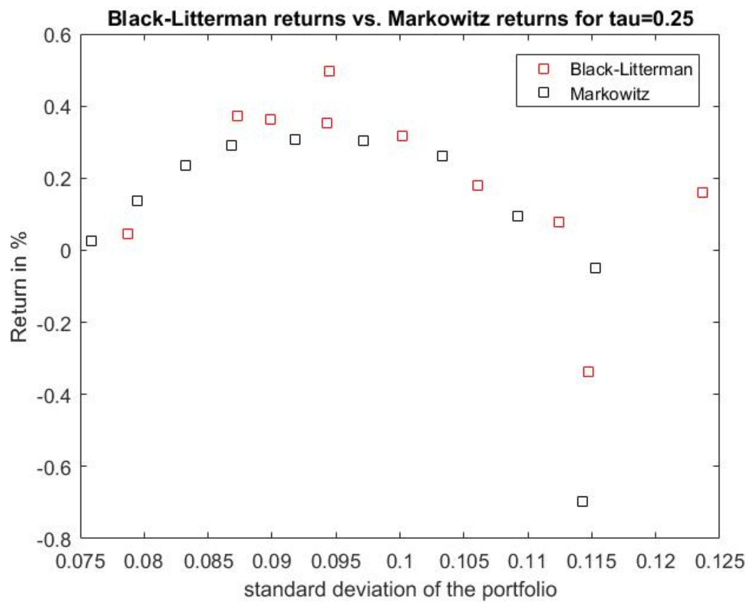


Figure 14: Black-Litterman versus Markowitz returns for $\tau = 0.25$ post crisis

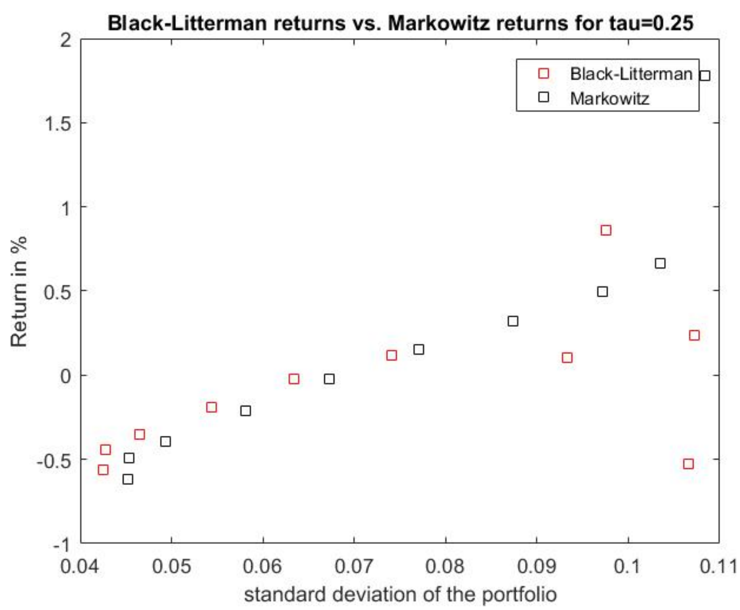


Figure 15: Black-Litterman versus Markowitz returns for $\tau = 0.50$

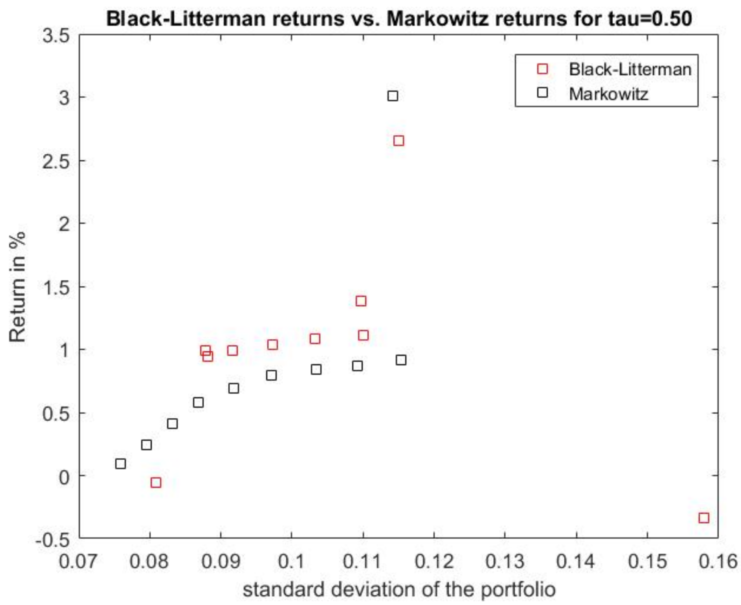


Figure 16: Black-Litterman versus Markowitz returns for $\tau = 0.50$ post crisis

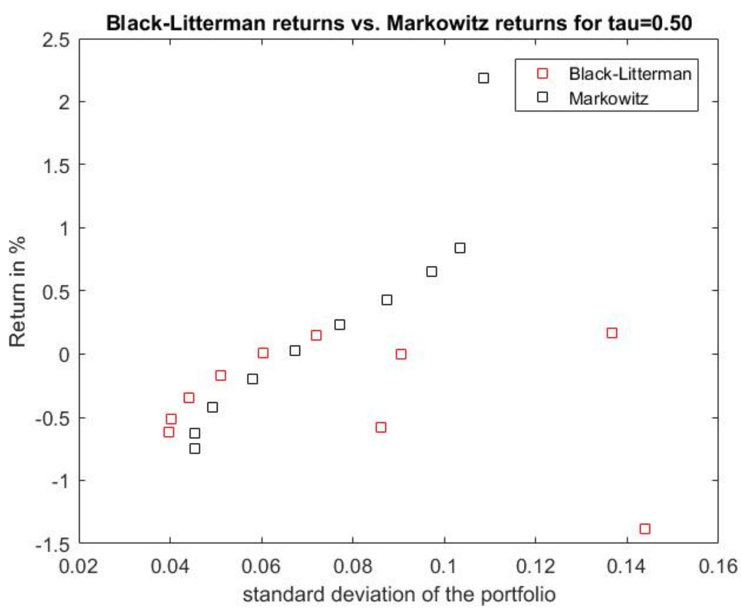


Figure 17: Black-Litterman versus Markowitz returns for $\tau = 0.75$

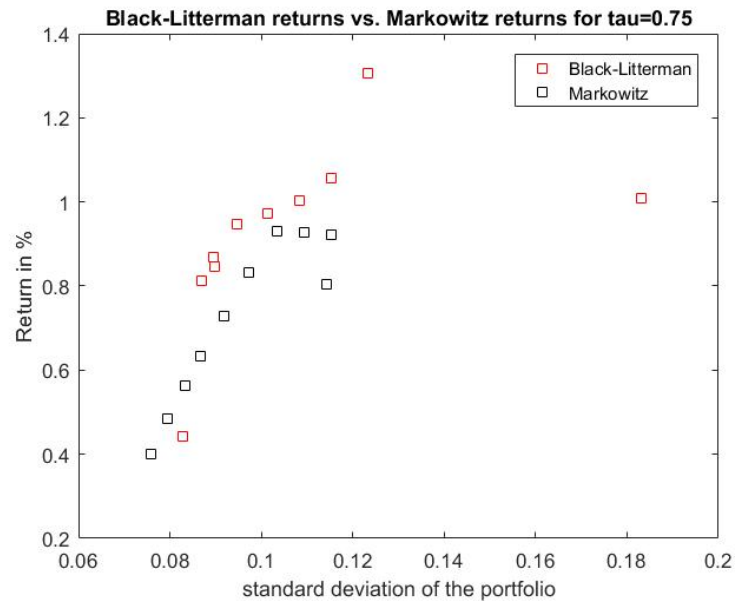


Figure 18: Black-Litterman versus Markowitz returns for $\tau = 0.75$ post crisis

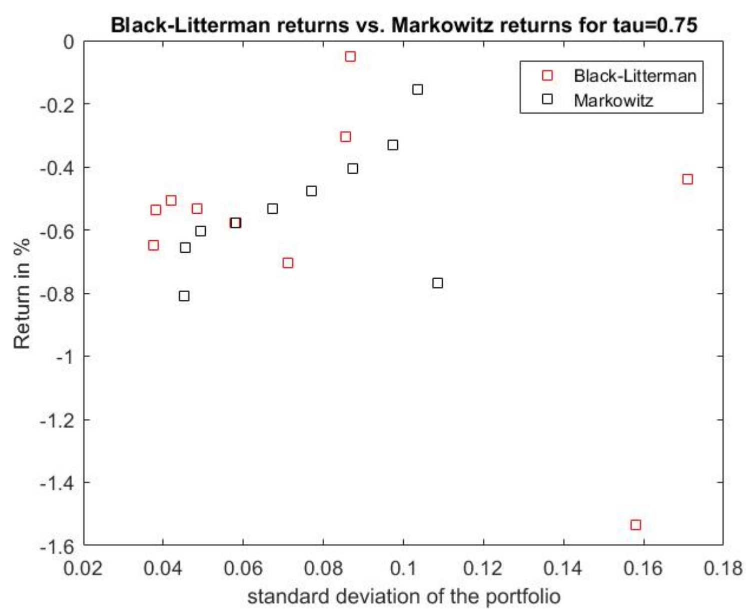


Figure 19: Black-Litterman versus Markowitz returns for $\tau = 1$

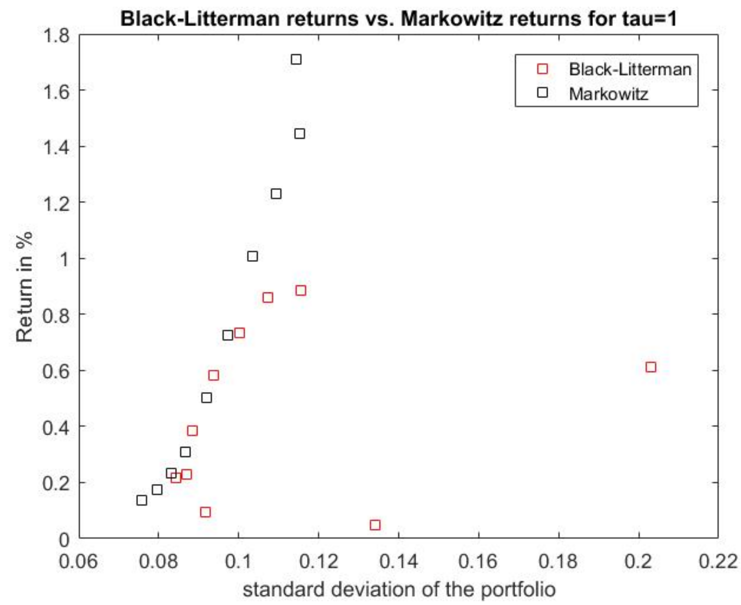
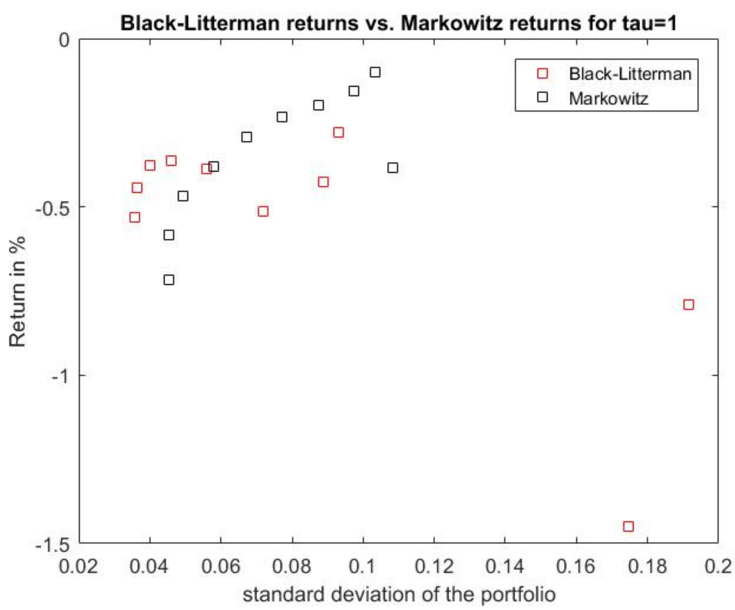


Figure 20: Black-Litterman versus Markowitz returns for $\tau = 1$ post crisis



So what should be the value of τ ? From the figures it seems that $\tau = 0.50$ or $\tau = 0.75$ looks like the best bet. This value of τ however, is very dependent on the quality of our views though, so we have to be tedious when drawing conclusions.

In overall, the Black-Litterman model gives slightly better returns than the Markowitz model. We also notice that the returns in the post crisis period are significantly lower. This could be due to "missing" the golden years that were prior to the crisis that were included in the other data. There is no significant difference between the performance of the Black-Litterman model in the post crisis period compared to the Markowitz model. There are however, a few notes to this simulation.

First of all, we have used the normal distribution centered around the average returns of the stock in the last 15 years. This evolution of stock prices doesn't necessarily have to be normal distributed. There are other models describing the evolution of stock prices in a better way. Second, our views vector, Q, is not based on financial expertise but solely on the returns in the last few months. Third, the confidence in our views, Ω, is also not based on any empirical evidence or knowledge about the specific sub market (for example: companies in manufacturing have been doing pretty good lately because of the low oil prices).

6 Appendix

This section includes the MATLAB files used for the simulation of section 5.
First the script to compute the Markowitz portfolio:

```
function  pwvt = markowitz
global pwvt Returns sigma Price ticker
c= yahoo;
ticker = {'MMM' 'AA' 'AXP' 'T' 'BAC' 'BA' 'CAT' 'CVX' 'CSCO' 'KO' 'DD' 'XOM' 'GE
for i=1:19
    Price.(ticker{i})= fetch(c,ticker(i),'close','Jan␣1␣00','Jan␣1␣15','m');
    temp = Price.(ticker{i});
    ClosePrice(:,i) = temp(:,2);
end
Returns = price2ret(ClosePrice);
sigma = cov(Returns);
p = Portfolio('name', 'Markowitz');
p = p.setAssetList(ticker);
disp(p);
p = p.estimateAssetMoments(Returns);
p = p.setDefaultConstraints;
[prisk, preturns] = p.plotFrontier;
pwvt = p.estimateFrontier;
hold on
plot(prisk,preturns,'ored');
```

Next we display the Black-Litterman and the simulation script

```
function [w_opt]=BlackLitterman(tau,Q,P,Omega)
global pwvt Returns sigma w_opt
w_opt = zeros(19,10);
for i= 1:10
mw = pwvt(:,i);
var = mw.' * sigma * mw;
riskfree = 1.0135^(1/12)-1;
for k=1:19
    returns(1,k) = mean(Returns(:,k));
end
delta = (returns*mw-riskfree)/var;
Pi=delta * sigma * mw;
ts= tau * sigma;
er=inv(inv(ts)+ P.'*Omega*P)*(inv(ts)*Pi+P.'*Omega*Q);
M = inv(ts)+P.'*inv(Omega)*P;
sigma2=sigma+inv(M);
w_opt(:,i)=inv(delta *sigma2)*er;
for j = 1:19
    if w_opt(j,i) < 0
        w_opt(j,i)= 0
    end
end

if sum(w_opt(:,i)) < 1
    k = randi(10,1);
    w_opt(k,i) = w_opt(k,i) + (1-sum(w_opt(:,i)));
end
end
```

```
function simulation
global Returns w_opt pwvt Price ticker
c= yahoo;
ticker = {'MMM' 'AA' 'AXP' 'T' 'BAC' 'BA' 'CAT' 'CVX' 'CSCO' 'KO' 'DD' 'XOM' 'GE
for i=1:19
    Price.(ticker{i})= fetch(c,ticker(i),'close','Jan 1 00','Jan 1 15','m');
    temp = Price.(ticker{i});
    ClosePrice(:,i) = temp(:,2);
end
for i=1:19
   variance(i,1) = var(Returns(:,i));
    average(i,1) = mean(Returns(:,i));
end
current_stock = zeros(19,1);
for i=1:19
current_stock(i,1) = Price.(ticker{i})(1,2) ;
end
stock_returns = zeros(19,1);
for i = 1:19
stock_returns (i,1) = normrnd(average(i),variance(i,1))*100;
end
for i=1:19
    stdstock(1,i) = std(Returns(:,i));
end

for i=1:10
    stdstockblack(1,i) = stdstock * w_opt(:,i);
end

for i=1:10
    stdstockmarko(1,i) =  stdstock * pwvt(:,i);
end

holderblack = stock_returns.' * w_opt;
holdermarko = stock_returns.' * pwvt;
for i=1:10
    profitblack(1,i)= mean(holderblack(:,i));
    profitmarko(1,i)= mean(holdermarko(:,i));
end

plot(stdstockblack,profitblack,'rs')
title('Black-Litterman returns vs. Markowitz returns for tau=1')
xlabel('standard deviation of the portfolio')
ylabel('Return in %')
hold on;
plot(stdstockmarko, profitmarko,'ks')
legend('Black-Litterman','Markowitz')
```

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