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Poisson structures and convexity theorems

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Abstract

The proofs of Kostant's nonlinear convexity theorem and Van den Ban's more general convexity theorem remain in the fields of Lie theory and rely on induction. In 1991 Lu and Ratiu discovered an alternative proof to Kostant's nonlinear convexity theorem using a symplectic approach. In 2006 Foth and Otto used a similar symplectic approach for an alternative proof of Van den Ban's convexity theorem. In this thesis we study the prominent part that specific Poisson structures play in these symplectic approaches. For a thorough understanding we include many results concerning Poisson structures on Lie groups, examine the construction of the Lu-Evens Poisson structure and include decompositions of semisimple Lie algebras and groups.

Contents

1. Introduction

Historical context

In 1973 Bertram Kostant proved a theorem which later became known as *Kostant's nonlinear convexity theorem* [13], which shows that for the Iwasawa decomposition of a semisimple Lie group the projection of a certain Lie group action orbit is the convex hull of a discrete group orbit. In 1986 Van den Ban proved a more general version of this theorem [20]. Both proofs remain in the field of Lie theory and are based on induction.

In 1982 Atiyah [1] and independently Guillemin and Sternberg [8] proved convexity theorems concerning the images of a symplectic manifold under a moment map. This theorem allowed Lu and Ratiu to prove Kostant's nonlinear convexity theorem in a symplectic manner for the case of a complex semisimple Lie group [15]. They defined a Poisson structure to identify the Lie group action orbits as the symplectic leaves of this Poisson structure. Upon applying the convexity theorem of Atiyah-Guillemin-Sternberg (AGS) to these newly found symplectic leaves, one readily gets Kostant's nonlinear convexity theorem for the case of a complex semisimple Lie group. A result by Duistermaat [4] allowed them to prove the theorem in the general case.

This symplectic approach inspired Foth and Otto to use the same approach for proving Van den Ban's convexity theorem [6]. For this they largely relied upon the work of Evens and Lu [5], who developed a way of constructing Poisson manifolds as subsets of the variety of Lagrangian subalgebras of a double Lie algebra.

Outline

The present thesis explores the construction and use of Poisson structures for the purpose of proving the previously mentioned convexity theorems in a symplectic framework. In order for us to do this, the first sections deal with quickly covering many of the mathematical subjects which already mentioned without explanation in the above paragraphs and also fix notation. For example we will look at Poisson structures and the properties that make them such useful tools. Also we will review the Cartan decomposition and from there the Iwasawa decomposition, since both Kostant's and Van den Ban's convexity theorems concern the Iwasawa decomposition. We also briefly cover the concept of a moment map, as it plays an important role in applying the AGS theorem. The following chapters will then build upon this to follow the proofs of first Kostant's and then (only partially) Van den Ban's convexity theorem which employ this symplectic framework. Along the way we will also illustrate theorems (Theorem 2.5 and Theorem 2.6) and a proof to Drinfeld's theorem (Theorem 3.23) that were not readily available in the literature.

2. Multivectors

In this section we review the definitions of tensors, the wedge product, multivectors and multivector fields in order to fix notation and avoid confusion, since one can find different definitions in literature.

2.1. Spaces of alternating maps and the wedge product

Let V be an n dimensional real vector space. Then V^* denotes the dual space of V, i.e. the space of all linear maps $\alpha : V \to \mathbb{R}$. The vector space V^* is also n dimensional and there exists a natural isomorphism $\Psi : V \to (V^*)^*$ defined by $\Psi(v)(\alpha) = \alpha(v)$ for any $v \in V, \alpha \in V^*$.

Now we define the space of k-multilinear maps $T^kV^* = \{\eta : V^k \to \mathbb{R}\}$, then T^kV^* is a linear space. Observe that $T^1(V) = V^*$. We now define $\eta \in T^kV^*$ to be alternating if

$$\eta(v_1, ..., v_i, ..., v_j, ..., v_k) = -\eta(v_1, ..., v_j, ..., v_i, ..., v_k)$$

for any $v_1, ..., v_k \in V$. If $i \neq j$ and $v_i = v_j$, we then observe by the above that

$$\eta(v_1, ..., v_i, ..., v_i, ..., v_k) = 0 \tag{1}$$

for an alternating k-multilinear map η. We define the space $\wedge^k(V^*) \subset T^kV^*$ as the set of alternating maps. It is easy to show that $\wedge^k(V^*)$ is a linear subspace of T^kV^*. If $k > \dim V$ then every k-tuple $(v_1, ..., v_k)$ of elements of V has a linear dependency. Without loss of generality we assume that v_k can be expressed as the sum

$$v_k = \sum\nolimits_{i=1}^{k-1} c_i v_i$$

with $c_i \in \mathbb{R}$. Let $\eta \in \wedge^k(V^*)$, then we observe by (1)

$$\eta(v_1, ..., v_k) = \sum\nolimits_{i=1}^{k-1} c_i \eta(v_1, ..., v_i, ..., v_i) = 0$$

and therefore $\eta = 0$. We conclude that $\wedge^k(V^*) = 0$ for any $k > \dim V$, and as such define the space of alternating multilinear maps

$$\wedge(V^*) = \bigoplus\nolimits_{k=1}^{n} \wedge^k(V^*).$$

We now define the map $\mathrm{Alt} : T^kV^* \to T^kV^*$ by

$$\mathrm{Alt}\,\eta(v_1, ..., v_k) = \frac{1}{k!} \sum\nolimits_{\sigma \in S_k} \operatorname{sgn}\sigma\, \eta(v_{\sigma(1)}, ..., v_{\sigma(k)})$$

for $\eta \in T^kV^*$, $v_j \in V$ and S_k the set of permutations of the set $\{1, ..., k\}$.

Lemma 2.1 ([14, Lemma 12.3]). *Take any $\eta \in T^kV^*$, then*

(a) $\mathrm{Alt}\,\eta \in \wedge^k(V^*)$,

(b) $\mathrm{Alt}\,\eta = \eta$ *if and only if* $\eta \in \wedge^k(V^*)$.

We now take $\eta \in \wedge^k(V^*)$ and $\omega \in \wedge^l(V^*)$. We then define $\eta \otimes \omega : V^{k+l} \to \mathbb{R}$ by

$$(\eta \otimes \omega)(v_1, ..., v_{k+l}) = \eta(v_1, ..., v_k)\omega(v_{k+1}, ..., v_{k+l})$$

and so $\eta \otimes \omega \in T^{k+l}V^*$. We then define the *wedge product* $\cdot \wedge \cdot : \wedge^k(V^*) \times \wedge^l(V^*) \to \wedge^{k+l}(V^*)$ by

$$\eta \wedge \omega = \frac{(k+l)!}{k!l!}\,\mathrm{Alt}(\eta \otimes \omega),$$

which can also be expressed as

$$(\eta \wedge \omega)(v_1, ..., v_{k+l}) = \frac{1}{k!l!}\sum\nolimits_{\sigma \in S_{k+l}} \mathrm{sgn}\,\sigma\,\eta(v_{\sigma(1)}, ..., v_{\sigma(k)})\omega(v_{\sigma(k+1)}, ..., v_{\sigma(l)})$$

when applied to $v_j \in V$. The wedge product clearly defines a product map on $\wedge(V^*)$, which we find to be bilinear, associative and anticommutative ($\eta \wedge \omega = (-1)^{kl}\omega \wedge \eta$ for $\eta \in \wedge^k(V^*)$ and $\omega \in \wedge^l(V^*)$) by [14, Proposition 12.8]. By the same proposition we have

$$(\alpha_1 \wedge ... \wedge \alpha_k)(v_1, ..., v_k) = \det(\alpha_i(v_j)).$$

for $\alpha_j \in V^*, v_j \in V$. The above equations will prove to be useful when doing calculations involving a wedge product.

Now let $\alpha \in \wedge^k(V^*)$ and $v \in V$, we then define the *interior product* of α with v, denoted as $v \lrcorner \alpha$, by

$$(v \lrcorner \alpha)(w_1, ..., w_{k-1}) = \alpha(v, w_1, ..., w_{k-1}), \quad w_1, ..., w_{k-1} \in V.$$

Since V^* is an n dimensional vector space we can also look at k-multilinear maps on V^* and repeat the same process as above. We then define $T^kV = \{A : (V^*)^k \to \mathbb{R}\}$, and $\wedge^k(V) \subset T^kV$ the space of alternating maps. We now notice that $\wedge^1(V) = T^1(V) = (V^*)^* \cong V$. This is why elements of $\wedge^k(V)$ are called *k-vectors*, or more generally *multivectors*. The wedge product on the spaces of multivectors can be defined in the same way as before, and so for $A \in \wedge^k(V), B \in \wedge^l(V)$ and $\alpha_j \in V^*$ we find

$$(A \wedge B)(\alpha_1, ..., \alpha_{k+l}) = \frac{1}{k!l!}\sum\nolimits_{\sigma \in S_{k+l}} \mathrm{sgn}\,\sigma\,A(\alpha_{\sigma(1)}, ..., \alpha_{\sigma(k)})B(\alpha_{\sigma(k+1)}, ..., \alpha_{\sigma(l)})$$

Now take $v_j \in V$, then in literature we often see a k-vector A expressed as $v_1 \wedge ... \wedge v_k$, while in fact we then have $A = \Psi(v_1) \wedge ... \wedge \Psi(v_k)$. However to avoid superfluous notation, as we see more often, the isomorphism is supressed as we identify V and $(V^*)^*$.

Let V and W be vector spaces and $f : W \to V$ a linear map. We find that f induces a map $f^k : \wedge^k(W) \to \wedge^k(V)$ defined by the equation

$$(f^k(A))(\alpha_1, ..., \alpha_k) = A(f^*\alpha_1, ..., f^*\alpha_k), \quad A \in \wedge^k(W),\ \alpha_1, ..., \alpha_k \in V^*,$$

where $f^*: V^* \to W^*$ is the dual map of f. We call f^k the *k-multilinear extension of f*, and from now on we will also denote it by f without fear of confusion.

Let $A \in \wedge^k(V)$ and $\alpha \in V^*$, we then define the *interior product* of A with α, denoted as $\alpha \lrcorner A$, in the same way as before, namely

$$(\alpha \lrcorner A)(\beta_1, ..., \beta_{k-1}) = A(\alpha, \beta_1, ..., \beta_{k-1}), \quad \beta_1, ..., \beta_{k-1} \in V^*$$

such that $\alpha \lrcorner A \in \wedge^{k-1}(V)$. If $A \in \wedge^2(V)$ and $\alpha \in V^*$ we find that $\alpha \lrcorner A \in \wedge^1(V) = V$. Therefore A defines a map

$$A^{\#} : V^* \to V : \alpha \mapsto \alpha \lrcorner A.$$

If we calculate its dual map $(A^{\#})^* : V^* \to V$ we find

$$\beta((A^{\#})^*(\alpha)) = \alpha(A^{\#}(\beta)) = -A(\alpha, \beta) = -\beta(A^{\#}(\alpha)), \quad \forall \alpha, \beta \in V,$$

and therefore $(A^{\#})^* = -A^{\#}$. It is then also clear that any map $\phi : V^* \to V$ with $\phi^* = -\phi$ defines an element $B \in \wedge^2(V)$ such that $B^{\#} = \phi$.

Lemma 2.2. *Let V and W be vector spaces and $\iota : W \to V$ an injective linear map, then*

$$\iota(\wedge^2(W)) = \{A \in \wedge^2(V) : \operatorname{im} A^{\#} \subset \operatorname{im} \iota\}. \tag{2}$$

Proof. Take any $B \in \wedge^2(W)$ and $\alpha, \beta \in V^*$ then

$$\begin{aligned} \beta(\iota(B)^{\#}(\alpha)) &= \beta(\alpha \lrcorner \iota(B)) = \iota(B)(\alpha, \beta) = B(\iota^*\alpha, \iota^*\beta) = (\iota^*\beta)((\iota^*\alpha) \lrcorner B) \\ &= \beta(\iota(B^{\#}(\iota^*\alpha))) \end{aligned}$$

and therefore $\iota(B)^{\#} \subset \operatorname{im} \iota$. For the converse inclusion we take any $A \in \wedge^2(V)$ such that $\operatorname{im} A^{\#} \subset \operatorname{im} \iota$. Since ι is injective there exists a linear map $\psi : V^* \to W$ such that the following diagram commutes.

$$\begin{tikzcd}
W \arrow[r, "\iota"] & V \\
V^* \arrow[u, "\psi"] \arrow[ur, "A^{\#}"'] &
\end{tikzcd}$$

Then also $A^{\#} = -(A^{\#})^* = -\psi^* \circ \iota^*$ and we can therefore extend the above diagram to the following commuting diagram.

$$\begin{tikzcd}
W \arrow[r, "\iota"] & V \\
V^* \arrow[u, "\psi"] \arrow[ur, "A^{\#}"'] \arrow[r, "\iota^*"'] & W^* \arrow[u, "-\psi^*"']
\end{tikzcd}$$

Because ι is injective we know that ι^* is surjective and since the above diagram commutes we find $\operatorname{im} -\psi^* \subset \operatorname{im} \iota$. Now since ι is injective there exists a linear map $\phi : W^* \to W$ such that we can again expand the commuting diagram.

$$\begin{tikzcd} W \arrow[r, "\iota"] & V \\ V^* \arrow[u, "\psi"] \arrow[ur, "A^{\#}"'] \arrow[r, "\iota^*"'] & W^* \arrow[u, "-\psi^*"'] \arrow[ul, "\phi"'] \end{tikzcd} \tag{3}$$

We then observe that

$$\phi^* \circ \iota^* = (\iota \circ \phi)^* = (-\psi^*)^* = -\psi = -\phi \circ \iota^*,$$

and since ι^* is surjective we find that $\phi^* = -\phi$. This makes it possible to define $B \in \wedge^2(W)$ by taking $B^{\#} = \phi$. By diagram (3) we then find for any $\alpha, \beta \in V^*$ that

$$\iota(B)(\alpha, \beta) = (\iota^*\beta)(\phi(\iota^*\alpha)) = \beta(\iota(\phi(\iota^*\alpha))) = \beta(A^{\#}(\alpha)) = A(\alpha, \beta),$$

and therefore $A \in \iota(\wedge^2(W))$. We have shown both inclusions and therefore conclude (2). $\square$

2.2. Multivector fields and the Schouten bracket

If M is a smooth manifold we can define the smooth vector bundle

$$\wedge^k(M) = \bigcup_{m \in M} \wedge^k(T_m M)$$

and define $\mathfrak{X}_k(M)$ as the set of smooth sections of $\wedge^k(M)$. We note this to be very similar to the definition of diffential forms and also $\mathfrak{X}_0(M) = C^\infty(M)$ and $\mathfrak{X}_1(M) = \mathfrak{X}(M)$, the set of vector fields on M. This is why elements of $\mathfrak{X}_k(M)$ are called *k-vector fields*, or more generally *multivector fields*. We also observe that $\mathfrak{X}_k(M) = \{0\}$ for $k > \dim M$, and we then define $\mathfrak{X}_*(M)$ as the direct sum of all $\mathfrak{X}_k(M)$. We call $A \in \mathfrak{X}_*(M)$ *homogeneous* if there exists a k such that $A \in \mathfrak{X}_k(M)$, and we then call k the *degree of* A, also denoted by $\deg A$.

We can now also define the wedge product $\cdot \wedge \cdot : \mathfrak{X}_k(M) \times \mathfrak{X}_l(M) \to \mathfrak{X}_{k+l}(M)$ by taking the wedge product pointwise in each tangent space, such that for $A \in \mathfrak{X}_k(M)$ and $B \in \mathfrak{X}_l(M)$ in any $m \in M$ we get

$$(A \wedge B)(m) = A(m) \wedge B(m) \in \wedge^{k+l}(T_m M).$$

Therefore if $X_1, ..., X_k \in \mathfrak{X}(M)$ then $X_1 \wedge ... \wedge X_k$ is a k-vector field on M.

Let $X \in \mathfrak{X}(M)$ and let φ_t denote its flow. If $A, B \in \mathfrak{X}_*(M)$ we calculate for any $m \in M$

the Lie derivative of the wedge product

$$\begin{aligned}
(\pounds_X(A\wedge B))(m) &= \frac{d}{dt}\Big|_{t=0}(T_m\varphi_t)^{-1}(A\wedge B)(\varphi_t(m))\\
&= \frac{d}{dt}\Big|_{t=0}(T_m\varphi_t)^{-1}A(\varphi_t(m))\wedge(T_m\varphi_t)^{-1}B(\varphi_t(m))\\
&= \frac{d}{dt}\Big|_{t=0}(T_m\varphi_t)^{-1}A(\varphi_t(m))\wedge B(m)+A(m)\wedge(T_m\varphi_t)^{-1}B(\varphi_t(m))\\
&= (\pounds_X A)(m)\wedge B(m)+A(m)\wedge(\pounds_X B)(m)\\
&= ((\pounds_X A)\wedge B+A\wedge(\pounds_X B))(m)
\end{aligned}$$

and conclude that the Lie derivative $\pounds_X$ is a derivation relative to the wedge product. Since the Lie derivative of vector fields results in the Jacobi-Lie bracket, one might wonder if there exists something similar for multivector fields. The answer is yes and it is called the Schouten-Nijenhuis bracket, or Schouten bracket for short.

Theorem 2.3 (Schouten bracket theorem, [17, Theorem 10.6.1])**.** *There is a unique bilinear operation* $[\,\cdot\,,\,\cdot\,]:\mathfrak{X}_*(M)\times\mathfrak{X}_*(M)\to\mathfrak{X}_*(M)$ *natural with respect to the restriction to open sets, called the* Schouten bracket*, that satisfies the following properties for any homogeneous* $A,B,C\in\mathfrak{X}_*(M)$*:*

(a) It is a biderivation of degree -1*, that is, it is bilinear,*

$$\deg[A,B]=\deg A+\deg B-1,$$

and

$$[A,B\wedge C]=[A,B]\wedge C+(-1)^{(\deg A+1)\deg B}B\wedge[A,C].$$

(b) It is determined on $C^\infty(M)$ *and* $\mathfrak{X}(M)$ *by*

a) $[f,g]=0$*, for all* $f,g\in C^\infty(M)$*;*

b) $[X,f]=X(f)$*, for all* $f\in C^\infty(M), X\in\mathfrak{X}(M)$*;*

c) $[X,Y]$ *for all* $X,Y\in\mathfrak{X}(M)$ *is the usual Jacobi-Lie bracket of vector fields.*

(c) $[A,B]=(-1)^{\deg A\deg B}[B,A]$.

In addition, the Schouten bracket satisfies the graded Jacobi identity

$$(-1)^{\deg A\deg C}[[A,B],C]+(-1)^{\deg B\deg A}[[B,C],A]+(-1)^{\deg C\deg B}[[C,A],B]=0.$$

Definition 2.4. If A and B respectively are multivector fields on the smooth manifolds M and N, and $f:M\to N$ is smooth, we then say that A and B are *f-related*, denoted as $A\sim_f B$, if and only if

$$T_mf(A(m))=B(f(m)),\quad\forall m\in M.$$

It is a well known fact that for vector fields f-relatedness extends to the Lie bracket. This means that for vector fields X_j and Y_j on respectively M and N such that $X_j \sim_f Y_j$ for $j = 1, 2$, we find that $[X_1, X_2] \sim_f [Y_1, Y_2]$. The proof of this result is not naturally extended to the Schouten bracket of multivector fields. However if f is a submersion or a immmersion f-relatedness does extend to the Schouten bracket as stated in the following propositions.

Proposition 2.5. *Let M and N be smooth manifolds and $f : M \to N$ a submersion. If A_j and B_j are multivector fields on M and N respectively such that $A_j \sim_f B_j$ for $j = 1, 2$, then the Schouten brackets of the multivector fields are f-related, i.e.*

$$[A_1, A_2] \sim_f [B_1, B_2] \tag{4}$$

Proof. Take any $z \in M$. By the submersion theorem there exist open neighborhoods U of z and V of $f(z)$ with local coordinates $(x_1, ..., x_m)$ at z and $(\tilde{x}_1, ..., \tilde{x}_n)$ at $f(z)$ such that f becomes the projection

$$f(x_1, ..., x_n, x_{n+1}, ..., x_m) = (x_1, ..., x_n). \tag{5}$$

If we denote $\partial_j = \partial/\partial x_j$ and $\tilde{\partial}_j = \partial/\partial \tilde{x}_j$, we then find for any $x \in U$ that

$$T_x f(\partial_j) = \begin{cases} \tilde{\partial}_j & \text{if } 1 \leq j \leq n \\ 0 & \text{if } n < j \leq m \end{cases} \tag{6}$$

For practical purposes we denote $\partial_{i_1 \dots i_p} = \partial_{i_1} \wedge ... \wedge \partial_{i_p}$ for $i_1 < ... < i_p$, and then for $x \in U$

$$T_x f(\partial_{i_1 \dots i_p}) = \begin{cases} \tilde{\partial}_{i_1, \dots, i_p} & \text{if } 1 \leq i_p \leq n \\ 0 & \text{if } n < i_p \leq m \end{cases} \tag{7}$$

There exist smooth functions $A_1^{i_1 \dots i_p}$ and $A_2^{j_1 \dots j_q}$ on U such that

$$A_1 = \sum\nolimits_{i_1 < \dots < i_p} A_1^{i_1 \dots i_p} \partial_{i_1 \dots i_p} \quad \text{and} \quad A_2 = \sum\nolimits_{j_1 < \dots < j_q} A_2^{j_1 \dots j_q} \partial_{j_1 \dots j_q}.$$

Because $A_1 \sim_f B_1$ and $A_2 \sim_f B_2$ we find by (6)

$$B_1 = \sum\nolimits_{\substack{i_1 < \dots < i_p \\ i_p \leq n}} B_1^{i_1 \dots i_p} \tilde{\partial}_{i_1 \dots i_p} \quad \text{and} \quad B_2 = \sum\nolimits_{\substack{j_1 < \dots < j_q \\ j_q \leq n}} B_2^{j_1 \dots j_p} \tilde{\partial}_{j_1 \dots j_q},$$

where $B_1^{i_1 \dots i_p}$ and $B_2^{j_1 \dots j_p}$ are functions on V such that

$$B_1^{i_1 \dots i_p}(f(x)) = A_1^{i_1 \dots i_p}(x) \quad \text{and} \quad B_2^{j_1 \dots j_p}(f(x)) = A_2^{j_1 \dots j_q}(x), \quad \forall x \in U. \tag{8}$$

We note that the left hand sides of the above equations are independent of $x_{n+1}, ..., x_m$ by (5) and therefore for $i_p, j_q \leq n$ we conclude

$$\partial_\ell A_1^{i_1 \dots i_p} = 0 \quad \text{and} \quad \partial_\ell A_2^{j_1 \dots j_q} = 0, \quad \forall \ell > n. \tag{9}$$

We also note by (8) for $i_p, j_q \leq n$ that

$$\partial_\ell A_1^{i_1\ldots i_p}(x) = \tilde{\partial}_\ell B_1^{i_1\ldots i_p}(f(x)) \quad \text{and} \quad \partial_\ell A_2^{j_1\ldots j_q}(x) = \tilde{\partial}_\ell B_2^{j_1\ldots j_q}(f(x)), \quad \forall \ell \leq n,$$

by the chain rule since $\partial_k f_\ell(x) = \partial_k(x_\ell) = \delta_{k\ell}$ for $k \leq n$. It is now interesting to split both multivector fields into the following parts

$$A_1^e = \sum\nolimits_{\substack{i_1<\ldots<i_p \\ i_p\leq n}} A_1^{i_1\ldots i_p}\partial_{i_1\ldots i_p}, \qquad A_1^o = A_1 - A_1^e,$$
$$A_2^e = \sum\nolimits_{\substack{j_1<\ldots<j_q \\ j_q\leq n}} A_2^{j_1\ldots j_q}\partial_{j_1\ldots j_q}, \qquad A_2^o = A_2 - A_2^e,$$

since then the Schouten bracket splits into the parts

$$[A_1, A_2] = [A_1^e, A_2^e] + [A_1^e, A_2^o] + [A_1^o, A_2].$$

We now apply [17, eq. (10.6.12)] to calculate each part when mapped by $T_x f$ for $x \in U$,

$$\begin{aligned}
T_x f[A_1^e, A_2^e](x) = T_x f \sum_{\substack{i_1<\ldots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q\leq n}} \Big(&\sum_{k=1}^{p} (-1)^{k+1} A_1^{i_1\ldots i_p}(x)\partial_{i_k} A_2^{j_1\ldots j_q}(x)\partial^{i_k}_{i_1\ldots.i_p j_1\ldots j_q} \\
&- \sum_{l=1}^{q} (-1)^{l+p}\partial_{j_l} A_1^{i_1\ldots i_p}(x) A_2^{j_1\ldots j_q}(x)\partial^{j_l}_{i_1\ldots.i_p j_1\ldots j_q}\Big) \\
= \sum_{\substack{i_1<\ldots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q\leq n}} \Big(&\sum_{k=1}^{p} (-1)^{k+1} B_1^{i_1\ldots i_p}(f(x))\tilde{\partial}_{i_k} B_2^{j_1\ldots j_q}(f(x))\tilde{\partial}^{i_k}_{i_1\ldots.i_p j_1\ldots j_q} \\
&- \sum_{l=1}^{q} (-1)^{l+p}\tilde{\partial}_{j_l} B_1^{i_1\ldots i_p}(f(x)) B_2^{j_1\ldots j_q}(f(x))\tilde{\partial}^{j_l}_{i_1\ldots.i_p j_1\ldots j_q}\Big) \\
= [B_1, B_2](f(x))&
\end{aligned}$$

where $\partial^{i_k}_{i_1\ldots.i_p j_1\ldots j_q}$ means that ∂_{i_k} is omitted from $\partial_{i_1\ldots.i_p j_1\ldots j_q}$. For the second part we find

$$\begin{aligned}
T_x f[A_1^e, A_2^o](x) = T_x f \sum_{\substack{i_1<\ldots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q> n}} \Big(&\sum_{k=1}^{p} (-1)^{k+1} A_1^{i_1\ldots i_p}(x)\partial_{i_k} A_2^{j_1\ldots j_q}(x)\partial^{i_k}_{i_1\ldots.i_p j_1\ldots j_q} \\
&- \sum_{l=1}^{q} (-1)^{l+p}\partial_{j_l} A_1^{i_1\ldots i_p}(x) A_2^{j_1\ldots j_q}(x)\partial^{j_l}_{i_1\ldots.i_p j_1\ldots j_q}\Big) \\
= \sum_{\substack{i_1<\ldots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q> n}} (-1)^{q+p+1}&\partial_{j_q} A_1^{i_1\ldots i_p}(x) A_2^{j_1\ldots j_q}(x) T_x f(\partial_{i_1\ldots.i_p j_1\ldots j_{q-1}}) = 0
\end{aligned}$$

by (7) and (9). The third part also becomes zero,

$$T_xf[A_1^e, A_2^e](x) = T_xf \sum_{\substack{i_1<\ldots<i_p \\ i_p>n}} \sum_{j_1<\ldots<j_q} \Big(\sum_{k=1}^{p} (-1)^{k+1} A_1^{i_1\ldots i_p}(x)\partial_{i_k} A_2^{j_1\ldots j_q}(x)\partial^{i_k}_{i_1\ldots.i_pj_1\ldots j_q}$$

$$- \sum_{l=1}^{q} (-1)^{l+p} \partial_{j_l} A_1^{i_1\ldots i_p}(x) A_2^{j_1\ldots j_q}(x)\partial^{j_l}_{i_1\ldots.i_pj_1\ldots j_q} \Big)$$

$$= \sum_{\substack{i_1<\ldots<i_p \\ i_p>n}} \sum_{j_1<\ldots<j_q} (-1)^{p+1} A_1^{i_1\ldots i_p}(x)\partial_{i_p} A_2^{j_1\ldots j_q}(x) T_xf(\partial_{i_1\ldots.i_{p_1}j_1\ldots j_q}) = 0$$

by (7) and (9). This proves in local coordinates for any $z \in M$ that

$$T_zf([A_1, A_2](z)) = [B_1, B_2](f(z))$$

and therefore we conclude $[A_1, A_2] \sim_f [B_1, B_2]$. $\square$

Proposition 2.6. *Let M and N be smooth manifolds and $f : N \to M$ an immersion. If A_j and B_j are multivector fields on N and M respectively such that $A_j \sim_f B_j$ for $j = 1, 2$, then the Schouten brackets of the multivector fields are f-related, i.e.*

$$[A_1, A_2] \sim_f [B_1, B_2] \tag{10}$$

Proof. Take any $z \in N$. There exist open neighborhoods U of z and V of $f(z)$ with local coordinates $(x_1, ..., x_n)$ at z and $(\tilde{x}_1, ..., \tilde{x}_m)$ at $f(z)$ such that f becomes the inclusion map

$$f(x_1, ..., x_n) = (x_1, ..., x_n, 0, ..., 0). \tag{11}$$

We adopt the notation ∂_j, $\partial_{i_1\ldots i_p}$ and $\partial^{i_k}_{i_1\ldots i_pj_1\ldots j_q}$ as defined in the proof of Proposition 2.5 and we then note for any $x \in U$ that

$$T_xf(\partial_{i_1\ldots i_p}) = \tilde{\partial}_{i_1\ldots i_p}. \tag{12}$$

There exist smooth functions $A_1^{i_1\ldots i_p}$ and $A_2^{j_1\ldots j_q}$ on U with all $1 \le i_k, j_l \le n$ such that

$$A_1 = \sum\nolimits_{\substack{i_1<\ldots<i_p \\ i_p\le n}} A_1^{i_1\ldots i_p}\partial_{i_1\ldots i_p} \quad \text{and} \quad A_2 = \sum\nolimits_{\substack{j_1<\ldots<j_q \\ j_q\le n}} A_2^{j_1\ldots j_q}\partial_{j_1\ldots j_q}.$$

There also exist smooth functions $B_1^{i_1\ldots i_p}$ and $B_2^{j_1\ldots j_q}$ on V with all $1 \le i_k, j_l \le m$ such that

$$B_1 = \sum\nolimits_{i_1<\ldots<i_p} B_1^{i_1\ldots i_p}\tilde{\partial}_{i_1\ldots i_p} \quad \text{and} \quad B_2 = \sum\nolimits_{j_1<\ldots<j_q} B_2^{j_1\ldots j_p}\tilde{\partial}_{j_1\ldots j_q},$$

Because $A_1 \sim_f B_1$ and $A_2 \sim_f B_2$ we find by (7) for any $x \in U$,

$$B_1^{i_1\ldots i_p}(f(x)) = \begin{cases} A^{i_1\ldots i_p}(x) & \text{if } 1 \le i_p \le n \\ 0 & \text{if } n < i_p \le m \end{cases} \tag{13}$$

$$B_2^{j_1\ldots j_p}(f(x)) = \begin{cases} A_2^{j_1\ldots j_q}(x) & \text{if } 1 \le j_q \le n \\ 0 & \text{if } n < j_q \le m \end{cases} \tag{14}$$

We also note by (13) and (14) for $i_p, j_q \leq n$ that

$$\partial_\ell A_1^{i_1\dots i_p}(x) = \tilde{\partial}_\ell B_1^{i_1\dots i_p}(f(x)) \quad \text{and} \quad \partial_\ell A_2^{j_1\dots j_q}(x) = \tilde{\partial}_\ell B_2^{j_1\dots j_q}(f(x)), \quad \forall \ell \leq n, \tag{15}$$

by the chain rule since $\partial_k f_\ell(x) = \delta_{k\ell}$. We also note by (13) and (14) for $i_p, j_q > n$ that

$$\tilde{\partial}_\ell B_1^{i_1\dots i_p}(f(x)) = 0 \quad \text{and} \quad \tilde{\partial}_\ell B_2^{j_1\dots j_q}(f(x)) = 0, \quad \forall \ell \leq n, \tag{16}$$

using the chain rule again. We split both multivector fields into the parts

$$\begin{aligned} B_1^e &= \sum\nolimits_{\substack{i_1<\dots<i_p \\ i_p\leq n}} B_1^{i_1\dots i_p} \tilde{\partial}_{i_1\dots i_p}, \qquad & B_1^o &= B_1 - B_1^e, \\ B_2^e &= \sum\nolimits_{\substack{j_1<\dots<j_q \\ j_q\leq n}} B_2^{j_1\dots j_q} \tilde{\partial}_{j_1\dots j_q}, \qquad & B_2^o &= B_2 - B_2^e, \end{aligned}$$

since then the Schouten bracket splits into the parts

$$[B_1, B_2] = [B_1^e, B_2^e] + [B_1^e, B_2^o] + [B_1^o, B_2^e] + [B_1^o, B_2^o].$$

We now apply [17, eq. (10.6.12)] to calculate $[A_1, A_2](x)$ when mapped by $T_x f$ for $x \in U$,

$$\begin{aligned} T_x f[A_1, A_2](x) = T_x f &\sum_{\substack{i_1<\dots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\dots<j_q \\ j_q\leq n}} \Big(\sum_{k=1}^{p} (-1)^{k+1} A_1^{i_1\dots i_p}(x) \partial_{i_k} A_2^{j_1\dots j_q}(x) \partial^{i_k}_{i_1\dots.i_p j_1\dots j_q} \\ &\qquad - \sum_{l=1}^{q} (-1)^{l+p} \partial_{j_l} A_1^{i_1\dots i_p}(x) A_2^{j_1\dots j_q}(x) \partial^{j_l}_{i_1\dots.i_p j_1\dots j_q} \Big) \\ = &\sum_{\substack{i_1<\dots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\dots<j_q \\ j_q\leq n}} \Big(\sum_{k=1}^{p} (-1)^{k+1} B_1^{i_1\dots i_p}(f(x)) \tilde{\partial}_{i_k} B_2^{j_1\dots j_q}(f(x)) \tilde{\partial}^{i_k}_{i_1\dots.i_p j_1\dots j_q} \\ &\qquad - \sum_{l=1}^{q} (-1)^{l+p} \tilde{\partial}_{j_l} B_1^{i_1\dots i_p}(f(x)) B_2^{j_1\dots j_q}(f(x)) \tilde{\partial}^{j_l}_{i_1\dots.i_p j_1\dots j_q} \Big) \\ = &[B_1^e, B_2^e](f(x)) \end{aligned}$$

by (12) and (15). We apply [17, eq. (10.6.12)] to calculate the remaining Schouten brackets that make up $[B_1, B_2](f(x))$, and see by (13), (14) and (16) that the underlined terms are zero.

$$\begin{aligned} [B_1^e, B_2^o](f(x)) = &\sum_{\substack{i_1<\dots<i_p \\ i_p\leq n}} \sum_{\substack{j_1<\dots<j_q \\ j_q> n}} \Big(\sum_{k=1}^{p} (-1)^{k+1} B_1^{i_1\dots i_p}(f(x)) \underline{\tilde{\partial}_{i_k} B_2^{j_1\dots j_q}(f(x))} \tilde{\partial}^{i_k}_{i_1\dots.i_p j_1\dots j_q} \\ &\qquad - \sum_{l=1}^{q} (-1)^{l+p} \tilde{\partial}_{j_l} B_1^{i_1\dots i_p}(f(x)) \underline{B_2^{j_1\dots j_q}(f(x))} \tilde{\partial}^{j_l}_{i_1\dots.i_p j_1\dots j_q} \Big) = 0, \end{aligned}$$

$$[B_1^o, B_2^e](f(x)) = \sum_{\substack{i_1<\ldots<i_p \\ i_p>n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q\leq n}} \Big(\sum_{k=1}^{p} (-1)^{k+1} \underline{B_1^{i_1\ldots i_p}(f(x))} \tilde{\partial}_{i_k} B_2^{j_1\ldots j_q}(f(x)) \tilde{\partial}^{i_k}_{i_1\ldots .i_p j_1\ldots j_q}$$
$$- \sum_{l=1}^{q} (-1)^{l+p} \underline{\tilde{\partial}_{j_l} B_1^{i_1\ldots i_p}(f(x))} B_2^{j_1\ldots j_q}(f(x)) \tilde{\partial}^{j_l}_{i_1\ldots .i_p j_1\ldots j_q} \Big) = 0,$$

$$[B_1^o, B_2^o](f(x)) = \sum_{\substack{i_1<\ldots<i_p \\ i_p>n}} \sum_{\substack{j_1<\ldots<j_q \\ j_q>n}} \Big(\sum_{k=1}^{p} (-1)^{k+1} \underline{B_1^{i_1\ldots i_p}(f(x))} \tilde{\partial}_{i_k} B_2^{j_1\ldots j_q}(f(x)) \tilde{\partial}^{i_k}_{i_1\ldots .i_p j_1\ldots j_q}$$
$$- \sum_{l=1}^{q} (-1)^{l+p} \tilde{\partial}_{j_l} B_1^{i_1\ldots i_p}(f(x)) \underline{B_2^{j_1\ldots j_q}(f(x))} \tilde{\partial}^{j_l}_{i_1\ldots .i_p j_1\ldots j_q} \Big) = 0.$$

This proves in local coordinates for any $z \in N$ that

$$T_z f([A_1, A_2](z)) = [B_1, B_2](f(z))$$

and therefore we conclude $[A_1, A_2] \sim_f [B_1, B_2]$. $\square$

3. Poisson structures

In this section we look at Poisson structures on manifolds and in particular Lie groups, how they are defined and which characteristics make them useful for proving Kostant's and Van den Ban's convexity theorems.

3.1. Poisson manifolds

Definition 3.1. A *Poisson structure* (or Poisson bracket) on a manifold M is a bilinear operation $\{,\}$ on $C^\infty(M)$, the smooth functions on M, such that

1. $(M, \{,\})$ is a Lie algebra,
2. if $h \in C^\infty(M)$ then $\{\cdot, h\}$ is a derivation, i.e. for all $f, g \in C^\infty(M)$ we have

$$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$

A *Poisson manifold* is a manifold equipped with a Poisson structure, denoted as $(M, \{,\})$.

Due to the derivation property of the bracket, it is possible to define Hamiltonian vector fields on a Poisson manifold.

Theorem 3.2 ([17, Proposition 10.2.1]). *For $h \in C^\infty(M)$ there exists a unique vector field X_h on M such that*

$$X_h(g) = \{g, h\}, \quad \forall g \in C^\infty(M).$$

We call X_h the Hamiltonian vector field *of h.*

The mapping $h \mapsto X_h$ is a Lie algebra anti-homomorphism from $C^\infty(M) \to \mathfrak{X}(M)$ [17, Proposition 10.2.2], i.e. $X_{\{g,h\}} = -[X_g, X_h]$.

Even though the Poisson structure is a bracket of functions it can be shown that due to its derivation property the bracket $\{f, g\}$ at a point $m \in M$ only depends on f by its differential $df_m \in T_z^*M$ [17]. Therefore there exists a mapping

$$\pi_m : T_m^*M \times T_m^*M \to \mathbb{R} : (df_m, dg_m) \mapsto \{f, g\}(m).$$

Since the Poisson bracket is antisymmetric we note that π_m is an alternating map and therefore $\pi_m \in \wedge^2(T_mM)$. This allows us to come up with a different definition for a Poisson structure on a manifold.

Definition 3.3. Let $\pi \in \mathfrak{X}_2(M)$, a bivector field on M, such that the bracket

$$\{f, g\}(m) = \pi(m)(df(m), dg(m)), \qquad m \in M, \quad f, g \in C^\infty(M) \tag{17}$$

is a Poisson bracket. We then say that π *defines a Poisson structure*. A Poisson manifold can then also be denoted by (M, π).

We remark that a 2-vector field is often called a *bivector field*. Let π be a bivector field on a smooth manifold M and define a bracket $\{,\}$ on M by (17). It is clear that this bracket is bilinear and antisymmetric and for any $f, g, h \in C^\infty(M)$ we observe by the product rule that

$$\begin{aligned}\{fg, h\}(m) &= \pi(m)(d(fg)(m), dh(m)) = \pi(m)((f\,dg + df\,g)(m), h(m)) \\ &= f(m)\pi(m)(dg(m), dh(m)) + \pi(m)(df(m), dg(m))g(m) \\ &= f(m)\{g, h\}(m) + \{f, h\}(m)g(m)\end{aligned}$$

and therefore $\{\cdot, h\}$ is a derivation. Therefore $\{,\}$ is a Poisson bracket if and only if the Jacobi identity holds for the bracket. We mention without proof that this is the case if and only if the Schouten bracket of π with itself is zero [17, p.357], which then gives the following theorem.

Theorem 3.4. *A bivector field $\pi \in \mathfrak{X}_2(M)$ defines a Poisson structure on a smooth manifold M if and only if the Schouten bracket of π with itself is zero, i.e.* $[\pi, \pi] = 0$.

We note that the Schouten bracket has become a useful tool for checking if bivector fields define Poisson structures by the above theorem.

Let (M, π) be a Poisson manifold. We define the map $\pi^\# : T^*M \to TM$ such that the equation

$$\pi(m)(\alpha, \alpha') = \alpha(\pi^\#(m)(\alpha')), \quad \alpha, \alpha' \in T_m^*M$$

holds for all $m \in M$. We call this the vector bundle mapping associated to π, and for any $h \in C^\infty(M)$ we easily see $\pi^\#(m)(dh(m)) = X_h(m)$. We now also define the subset

$$\pi^\#(T^*M) = \bigcup\nolimits_{m \in M} \pi^\#(T_m^*M) \subset TM$$

as the *characteristic distribution* of the Poisson structure. We define the *rank* of π at $m \in M$ as the dimension of $\pi^\#(T_m^*M)$.

Assume that M is a Poisson manifold such that the characteristic distribution is equal to TM. Then $\pi^\#$ defines an isomorphism $T_m^*M \to T_mM$ for every $m \in M$ and then it is possible to define a 2-form Ω on M by

$$\Omega(m)(X, X') = \pi\big((\pi^\#(m))^{-1}(X), (\pi^\#(m))^{-1}(X')\big) = \big((\pi^\#(m))^{-1}(X)\big)(X')$$

for $X, X' \in T_mM$. The Pauli-Jost theorem [10, Theorem 1] [19] then proves that Ω is closed ($d\Omega = 0$) as a consequence of the Poisson structure's Jacobi identity and therefore defines a symplectic structure on M. This already alludes to how Poisson structures are related to symplectic structures.

Definition 3.5. Let (N, π_N) and (M, π_M) both be Poisson manifolds, and $f : N \to M$ a smooth map. We call f a *Poisson map* if and only if π_N and π_M are f-related.

Let (M, π_M) be a Poisson manifold and $i : N \to M$ an injectively immersed submanifold. We call i a *Poisson immersion* if for any $n \in N$ we find

$$\pi_M^\#(i(n))(T_{i(n)}^*M) \subset \operatorname{im} T_n i, \tag{18}$$

which in other words means that for any $n \in N$ the characteristic distribution of π at $i(n)$ lies within the image of $T_n i : T_nN \to T_{i(n)}M$.

Proposition 3.6. *Let (M, π_M) be a Poisson manifold, N a smooth manifold and $i : N \to M$ a Poisson immersion. There exists an* induced Poisson structure *on N such that i is a Poisson map.*

Proof. Since i is an immersion we know that $T_n i : T_n N \to T_{i(n)} M$ is an injective map for any $n \in N$. By (18) we apply Lemma 2.2 to find there exists a $B_n \in \wedge^2(T_n N)$ such that

$$T_n i(B_n) = \pi_M(i(n)) \tag{19}$$

If we now define the bivector field $\pi_N \in \mathfrak{X}_2(N)$ by $\pi_N(n) = B_n$ we find that π_N and π_M are i-related by (19). By Proposition 2.6 we then find for any $n \in N$ that $[\pi_N, \pi_N](n) = 0$ since $[\pi_M, \pi_M](i(n)) = 0$ and $T_n i$ is injective. We conclude that π_N defines a Poisson structure on N and that i is a Poisson map. □

Definition 3.7. Let (M, π_M) be a Poisson manifold and let N be a submanifold of M. We call N a *Poisson submanifold* if the inclusion map is a Poisson immersion.

Lemma 3.8. *Let (N, π_N) be a Poisson submanifold of (M, π_M) with inclusion map i. The Hamiltonian vector fields of $f \in C^\infty(M)$ and $f \circ i \in C^\infty(N)$ are i-related. Moreover, for $n \in N$, let a be the integral curve of $X_{f\circ i}$ such that $a(0) = n$ and b be the integral curve of X_f such that $b(0) = i(n)$, then $b = i \circ a$.*

Proof. We recall that π_N is the induced Poisson structure on N such that it is i-related with π_M by Proposition 3.6 and therefore we find for any $n \in N$ and $\alpha, \beta \in T^*_{i(n)} M$

$$\begin{aligned}\beta(\pi_M^\#(i(n))(\alpha) &= \pi_M(i(n))(\alpha, \beta) = T_n i(\pi_N(n))(\alpha, \beta) = \pi_N(n)((T_n i)^*\alpha, (T_n i)^*\beta) \\ &= ((T_n i)^*\beta)(\pi_N(n)((T_n i)^*\alpha)) = \beta(T_n i(\pi_N^\#(n)((T_n i)^*\alpha)))\end{aligned}$$

and as such $\pi_M^\#(i(n)) = T_n i \circ \pi_N^\#(n) \circ (T_n i)^*$. We also recall $d(f \circ i)(n) = df(i(n)) \circ T_n i = (T_n i)^* df(i(n))$ for any $n \in N$ and we are then able to calculate

$$\begin{aligned}T_n i(X_{f\circ i}(n)) &= T_n i(\pi_N^\#(n)(d(f \circ i)(n))) = T_n i(\pi_N^\#(n)((T_n i)^* df(i(n)))) \\ &= \pi_M^\#(i(n))(df(i(n))) = X_f(i(n))\end{aligned}$$

which shows us that the Hamiltonian vector fields $X_{f\circ i}$ and X_f are i-related.

We define the curve $c = i \circ a$. It is clear that $c(0) = i(n) = b(0)$, and for any $t \in \mathbb{R}$ we observe

$$c'(t) = T_{c(t)} i(a'(t)) = T_{i(a(t))} i(X_{f\circ i}(a(t))) = X_f(i(a(t))) = X_f(c(t))$$

and therefore c is an integral curve of X_f. By the uniqueness of integral curves we then conclude $b = c = i \circ a$. □

Definition 3.9. Let (M, π) be a Poisson manifold and $m_1, m_2 \in M$. If there exists a path between m_1 and m_2 that is piecewise constructed of integral curves of Hamiltonian vector fields, then we say that m_1 and m_2 are *in the same symplectic leaf*. This is easily seen to be an equivalence relation and we call the equivalence class containing $m \in M$ the *symplectic leaf through* m.

Theorem 3.10 (Symplectic Stratification Theorem, [17, Proposition 10.4.4])**.** *Let (M, π) be a finite dimensional Poisson manifold. Then M is the disjoint union of its symplectic leaves. Each symplectic leaf in M is an injectively immersed Poisson submanifold, and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through a point m equals the rank of the Poisson structure at that point, and the tangent space to the leaf at m equals $\pi^{\#}(T_m^* M)$.*

The above theorem tells us how Poisson structures on a manifold can be used to identify submanifolds as symplectic, which will prove to be the main purpose the Poisson structures used in sections 6 and 7.

Proposition 3.11. *Let (M, π) be a Poisson manifold with $i : N \to M$ a Poisson submanifold. Let $n \in N$, then the symplectic leaf through $i(n)$ in M is a subset of $i(N)$.*

Proof. Take any m in the symplectic leaf through $i(n)$. By Definition 3.9 there exists a path from $i(n)$ to m which is piecewise constructed of the integral curves of Hamiltonian vector fields. Therefore there exist points $m_j \in M$, functions $f_j \in C^\infty(M)$ and scalars $T_j \in \mathbb{R}$ with $1 \leq j \leq k$ such the integral curves b_j of X_{f_j} take on the values $b_j(0) = m_{j-1}$ and $b_j(T_j) = m_j$, where $m_0 = i(n)$ and $m_k = m$.

Assume that $m_{j-1} \in i(N)$, then there exists some $n' \in N$ such that $i(n') = m_{j-1}$. We define by a_j the integral curve of the Hamiltonian vector field of the function $f_j \circ i \in C^\infty(N)$ such that $a_j(0) = n'$. By Lemma 3.8 we then find that $m_j = b_j(T_j) = i(a_j(T_j)) \in i(N)$.

Since $m_0 = i(n) \in i(N)$ we may now conclude by induction that $m = m_k \in i(N)$, and we have therefore shown that the symplectic leaf through $i(n)$ is a subset of $i(N)$. $\square$

3.2. Poisson Lie groups and the double Lie algebra

Lie groups are a specific example of manifolds and therefore the notion of Poisson structures makes sense on a Lie group. In this section we look at multiplicative multivector fields on Lie groups and how they can be utilized in defining a Poisson structures on a Lie group.

Definition 3.12. A multivector field Π on a Lie group G is called *multiplicative* if

$$\Pi(g_1 g_2) = r_{g_2}\Pi(g_1) + l_{g_1}\Pi(g_2), \quad g_1, g_2 \in G \tag{20}$$

where r_{g_2} and l_{g_1} are the differential mappings for the right multiplication by g_2 and left multiplication by g_1 respectively.

For a multiplicative multivector field Π we easily observe that it is zero in the identity of the Lie group G, as inserting $g_1 = g_2 = e$ into (20) gives us $\Pi(e) = 0$. The intrinsic derivative of Π then exists at the identity (see Appendix A) and it is defined as

$$d_e\Pi : \mathfrak{g} \to \wedge^2(\mathfrak{g}) : X \mapsto \pounds_{\overline{X}}\Pi(e)$$

where $\overline{X}$ is any vector field on G such that $\overline{X}(e) = X$. The following lemma gives some interesting properties of multiplicative multivector fields.

Lemma 3.13. *Assume that G is connected. Then,*

(a) a multivector field Π is multiplicative if and only if $\Pi(e) = 0$ and the Lie derivative $\pounds_V \Pi$ is left invariant whenever V is a left invariant vector field on G;

(b) the Schouten bracket of a left invariant multivector field and a multiplicative multivector field is left invariant;

(c) the Schouten bracket of two multiplicative multivector fields is again multiplicative;

(d) a multiplicative multivector field is identically zero on G if and only if its intrinsic derivative at e is zero.

Proof. Parts (a), (c) and (d) are from [15, Lemma 4.6] and we therefore refer the reader to the original article for their proofs. We included (b) in this lemma, and we therefore show its proof. Take Π_1 a left invariant multivector field and Π_2 a multiplicative multivector field on G. We may assume that $\Pi_1 = V_1 \wedge ... \wedge V_k$ where all V_j are left invariant vector fields. By the Schouten bracket product rule [17, eq. (10.6.9)] we find

$$[\Pi_1, \Pi_2] = \sum\nolimits_{j=1}^{k} (-1)^{j-1} V_1 \wedge \cdots \wedge \check{V}_j \wedge \cdots \wedge V_k \wedge \pounds_{V_j} \Pi_2$$

which immediately shows that the Schouten bracket is left invariant since each $\pounds_{V_j}\Pi_2$ is left invariant by (a). □

Let Π be a multiplicative bivector field on a Lie group G. The dual map of the intrinsic derivative $d_e\Pi : \mathfrak{g} \to \wedge^2(\mathfrak{g})$ defines on $\mathfrak{g}^*$ the antisymmetric bracket $[\cdot,\cdot]_\Pi : \wedge^2(\mathfrak{g}^*) \to \mathfrak{g}^*$, i.e.

$$[\xi, \eta]_\Pi(X) = d_e\Pi(X)(\xi, \eta), \quad X \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*.$$

If $\xi \in \mathfrak{g}^*$ we denote the corresponding left invariant 1-form on G by ξ^l, and similarly we denote by X^l the left invariant vector field on G corresponding to $X \in \mathfrak{g}$. We can now extend the bracket $[,]_\Pi$ to the left invariant 1-forms, by defining $[\xi^l, \eta^l]_\Pi = ([\xi, \eta]_\Pi)^l$. For $X \in \mathfrak{g}$ we then find by Lemma 3.13 (a) that $\pounds_{X^l}\Pi$ is left invariant and then

$$\pounds_{X^l}\Pi(\xi^l, \eta^l)(e) = d_e\Pi(X)(\xi, \eta) = X([\xi, \eta]_\Pi) = X^l([\xi^l, \eta^l]_\Pi)(e)$$

and therefore by left invariance we find $\pounds_{X^l}\Pi(\xi^l, \eta^l) = X^l([\xi^l, \eta^l]_\Pi)$.

Lemma 3.14. *If G is a connected Lie group, λ a left invariant bivector field and Π a multiplicative bivector field on G, then*

$$[\lambda, \Pi](\xi_1^l, \xi_2^l, \xi_3^l) = -\sum\nolimits_{\tau \in A_3} \lambda(\xi_{\tau(1)}^l, [\xi_{\tau(2)}^l, \xi_{\tau(3)}^l]_\Pi) \tag{21}$$

for any left invariant 1-forms ξ_j^l on G, and A_3 the set of even permutations.

Proof. We may assume that $\lambda = X^l \wedge Y^l$ for some $X, Y \in \mathfrak{b}$. We then find that

$$[\lambda, \Pi] = [X^l, \Pi] \wedge Y^l - X^l \wedge [Y^l, \Pi] = \pounds_{X^l}\Pi \wedge Y^l - X^l \wedge \pounds_{Y^l}\Pi$$

We can then calculate

$$\begin{aligned}
[\lambda, \Pi](\xi_1^l, \xi_2^l, \xi_3^l) &= \frac{1}{2}\sum_{\tau \in S_3} \mathrm{sgn}(\tau)\left(\pounds_{X^l}\Pi(\xi^l_{\tau(1)}, \xi^l_{\tau(2)})Y^l(\xi^l_{\tau(3)}) - X^l(\xi^l_{\tau(1)})\pounds_{Y^l}\Pi(\xi^l_{\tau(2)}, \xi^l_{\tau(3)})\right) \\
&= \frac{1}{2}\sum_{\tau \in S_3} \mathrm{sgn}(\tau)\left(X^l([\xi^l_{\tau(1)}, \xi^l_{\tau(2)}]_\Pi)Y^l(\xi^l_{\tau(3)}) - X^l(\xi^l_{\tau(1)})Y^l([\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi)\right) \\
&= \sum_{\tau \in A_3} \left(X^l([\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi)Y^l(\xi^l_{\tau(1)}) - X^l(\xi^l_{\tau(1)})Y^l([\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi)\right) \\
&= -\sum_{\tau \in A_3} (X^l \wedge Y^l)(\xi^l_{\tau(1)}, [\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi) = -\sum_{\tau \in A_3} \lambda(\xi^l_{\tau(1)}, [\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi)
\end{aligned}$$

which completes our proof. □

The above lemma now allows us to prove the following lemma.

Lemma 3.15. *[15, Lemma 4.7] On a connected Lie group G, a multiplicative bivector field Π defines a Poisson structure if and only if the dual map of its intrinsic derivative at e defines a Lie algebra structure on $\mathfrak{g}^*$.*

Proof. We recall that Π defines a Poisson structure if and only if the Schouten bracket $[\Pi, \Pi]$ equals zero. Since $[\Pi, \Pi]$ is multiplicative by Lemma 3.13 we find by the same lemma that $[\Pi, \Pi]$ equals zero if and only if the intrinsic derivative $d_e[\Pi, \Pi]$ equals zero. The bracket $[,]_\Pi$ defined by the dual map of $d_e\Pi$ is necessarily antisymmetric and therefore defines a Lie algebra structure on $\mathfrak{g}^*$ if and only if the Jacobi identity holds. We therefore prove the lemma by showing that this Jacobi identity for $[,]_\Pi$ is equivalent to $d_e[\Pi, \Pi] = 0$.

We take any $X \in \mathfrak{g}$ and recall that then $\pounds_{X^l}\Pi$ is left invariant. For $\xi_j \in \mathfrak{g}^*$ we then find by Lemma 3.14,

$$\begin{aligned}
d_e[\Pi, \Pi](X)(\xi_1, \xi_2, \xi_3) &= \pounds_{X^l}[\Pi, \Pi](\xi_1^l, \xi_2^l, \xi_3^l)(e) \\
&= 2[\pounds_{X^l}\Pi, \Pi](\xi_1^l, \xi_2^l, \xi_3^l)(e) \\
&= -2\sum\nolimits_{\tau \in A_3} \pounds_{X^l}\Pi(\xi^l_{\tau(1)}, [\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi)(e) \\
&= -2\sum\nolimits_{\tau \in A_3} X^l([\xi^l_{\tau(1)}, [\xi^l_{\tau(2)}, \xi^l_{\tau(3)}]_\Pi]_\Pi)(e) \\
&= -2\sum\nolimits_{\tau \in A_3} X([\xi_{\tau(1)}, [\xi_{\tau(2)}, \xi_{\tau(3)}]_\Pi]_\Pi).
\end{aligned}$$

The above equation holds for any $X \in \mathfrak{g}$, and hence the Jacobi identity for $[,]_\Pi$ is equivalent to $d_e[\Pi, \Pi] = 0$. □

The above lemma will prove to be a useful tool when showing that a bivector field on a Lie group defines a Poisson structure. One such case is when we take any $R \in \wedge^2(\mathfrak{g})$ and then define the bivector field π on the connected Lie group G by

$$\pi(g) = r_g R - l_g R, \quad g \in G. \tag{22}$$

We then easily check that π is multiplicative by calculating π for $g_1, g_2 \in G$ and using that l_{g_1} and r_{g_2} commute.

$$\begin{aligned}\pi(g_1 g_2) &= r_{g_1 g_2} R - l_{g_1 g_2} R = r_{g_2} r_{g_1} R - l_{g_1} l_{g_2} R \\ &= r_{g_2}(r_{g_1} R - l_{g_1} R) + l_{g_1}(r_{g_2} R - l_{g_2} R) = r_{g_2}\pi(g_1) + l_{g_1}\pi(g_2)\end{aligned}$$

The intrinsic derivative of π at the identity is explicitly given as

$$d_e\pi(X) = \frac{d}{dt}\Big|_{t=0} r_{\exp tX}^{-1}\pi(\exp tX) = \frac{d}{dt}\Big|_{t=0} R - \mathrm{Ad}_{\exp tX} R = -\operatorname{ad}_X R \tag{23}$$

and the dual map of $d_e\pi$ defines a bracket $[,]_\pi : \wedge^2(\mathfrak{g}^*) \to \mathfrak{g}^*$.

Lemma 3.16. *The bracket $[,]_\pi$ on $\mathfrak{g}^*$ defined by the dual map of the intrinsic derivative at the identity of π as defined in (22) is given by*

$$[\xi, \eta]_\pi = \operatorname{ad}^*_{\xi \lrcorner R} \eta - \operatorname{ad}^*_{\eta \lrcorner R} \xi$$

for $\xi, \eta \in \mathfrak{g}^*$, where ad^*_X denotes the dual map of ad_X with respect to the canonical pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Proof. We show this with a calculation as for any $X \in \mathfrak{g}$ we find

$$\begin{aligned}[\xi, \eta]_\pi(X) &= d_e\pi(X)(\xi, \eta) = (-\operatorname{ad}_X R)(\xi, \eta) = -R(\operatorname{ad}^*_X \xi, \eta) - R(\xi, \operatorname{ad}^*_X \eta) \\ &= (\operatorname{ad}^*_X \xi)(\eta \lrcorner R) - (\operatorname{ad}^*_X \eta)(\xi \lrcorner R) = \xi(\operatorname{ad}_X(\eta \lrcorner R)) - \eta(\operatorname{ad}_X(\xi \lrcorner R)) \\ &= -\xi(\operatorname{ad}_{\eta \lrcorner R} X) + \eta(\operatorname{ad}_{\xi \lrcorner R} X) = (\operatorname{ad}^*_{\xi \lrcorner R} \eta - \operatorname{ad}^*_{\eta \lrcorner R} \xi)(X)\end{aligned}$$

which shows the desired equation. $\square$

We define the element $[R, R] \in \wedge^3(\mathfrak{g})$ as

$$[R, R](\xi_1, \xi_2, \xi_3) = \sum\nolimits_{\tau \in A_3} \xi_{\tau(1)}([\xi_{\tau(2)} \lrcorner R, \xi_{\tau(3)} \lrcorner R]) \tag{24}$$

for $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$. The notation used is reminiscent of the Schouten bracket, which will be explained by Lemma 3.18.

Lemma 3.17. *The bivector field π as in (22) defines a Poisson structure on G if and only if $[R, R]$ as defined in (24) is ad-invariant.*

Proof. We start by calculating $\operatorname{ad}_X[R, R]$ for any $X \in \mathfrak{g}$ when applied to $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$.

$$\begin{aligned}&(\operatorname{ad}_X[R, R])(\xi_1, \xi_2, \xi_3) \\ &\quad= [R, R](\operatorname{ad}^*_X \xi_1, \xi_2, \xi_3) + [R, R](\xi_1, \operatorname{ad}^*_X \xi_2, \xi_3) + [R, R](\xi_1, \operatorname{ad}^*_X \xi_2, \xi_3) \\ &\quad= \sum\nolimits_{\tau \in A_3} [R, R](\operatorname{ad}^*_X \xi_{\tau(1)}, \xi_{\tau(2)}, \xi_{\tau(3)}) \\ &\quad= \sum\nolimits_{\tau \in A_3} (\operatorname{ad}^*_X \xi_{\tau(1)})([\xi_{\tau(2)} \lrcorner R, \xi_{\tau(3)} \lrcorner R]) \\ &\qquad\quad + \xi_{\tau(3)}([(\operatorname{ad}^*_X \xi_{\tau(1)}) \lrcorner R, \xi_{\tau(2)} \lrcorner R]) + \xi_{\tau(2)}([\xi_{\tau(3)} \lrcorner R, (\operatorname{ad}^*_X \xi_{\tau(1)}) \lrcorner R]) \\ &\quad= \sum\nolimits_{\tau \in A_3} \xi_{\tau(1)}([X, [\xi_{\tau(2)} \lrcorner R, \xi_{\tau(3)} \lrcorner R]]) \\ &\qquad\quad + \xi_{\tau(1)}([(\operatorname{ad}^*_X \xi_{\tau(2)}) \lrcorner R, \xi_{\tau(3)} \lrcorner R]) + \xi_{\tau(1)}([\xi_{\tau(2)} \lrcorner R, (\operatorname{ad}^*_X \xi_{\tau(3)}) \lrcorner R])\end{aligned}$$

In order to rewrite the above we first take any $X \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$ and calculate

$$\begin{aligned}\eta(\operatorname{ad}_X^* \xi) \lrcorner R) &= R(\operatorname{ad}_X^* \xi, \eta) = \operatorname{ad}_X R(\xi, \eta) - R(\xi, \operatorname{ad}_X^* \eta) = [\xi, \eta]_\pi(X) - (\operatorname{ad}_X^* \eta)(\xi \lrcorner R) \\ &= (\operatorname{ad}_\xi \eta)(X) - \eta(\operatorname{ad}_X(\xi \lrcorner R)) = \eta(\operatorname{ad}_\xi^* X) - \eta([X, \xi \lrcorner R])\end{aligned}$$

which then gives us $(\operatorname{ad}_X^* \xi) \lrcorner R = \operatorname{ad}_\xi^* X - [X, \xi \lrcorner R]$. We note that we defined the map $\operatorname{ad}_\xi : \mathfrak{g}^* \to \mathfrak{g}^* : \eta \mapsto [\xi, \eta]_\pi$, even though $\mathfrak{g}^*$ is not necessarily a Lie algebra and ad is mostly reserved for the adjoint representation of a Lie algebra. Nonetheless the terminology is useful at the moment and we use the newly found identity to continue our calculation.

$$\begin{aligned}
&(\operatorname{ad}_X[R, R])(\xi_1, \xi_2, \xi_3) \\
&\quad= \textstyle\sum_{\tau \in A_3} \xi_{\tau(1)}\big([[X, \xi_{\tau(2)} \lrcorner R], \xi_{\tau(3)} \lrcorner R] + [\xi_{\tau(2)} \lrcorner R, [X, \xi_{\tau(3)} \lrcorner R]]\big) \\
&\qquad+ \xi_{\tau(1)}\big([\operatorname{ad}_{\xi_{\tau(2)}}^* X - [X, \xi_{\tau(2)} \lrcorner R], \xi_{\tau(3)} \lrcorner R] + [\xi_{\tau(2)} \lrcorner R, \operatorname{ad}_{\xi_{\tau(3)}}^* X - [X, \xi_{\tau(3)} \lrcorner R]]\big) \\
&\quad= \textstyle\sum_{\tau \in A_3} \xi_{\tau(1)}\big([\operatorname{ad}_{\xi_{\tau(2)}}^* X, \xi_{\tau(3)} \lrcorner R] + [\xi_{\tau(2)} \lrcorner R, \operatorname{ad}_{\xi_{\tau(3)}}^* X]\big) \\
&\quad= \textstyle\sum_{\tau \in A_3} -(\operatorname{ad}_{\xi_{\tau(3)} \lrcorner R}^* \xi_{\tau(1)})(\operatorname{ad}_{\xi_{\tau(2)}}^* X) + (\operatorname{ad}_{\xi_{\tau(2)} \lrcorner R}^* \xi_{\tau(1)})(\operatorname{ad}_{\xi_{\tau(3)}}^* X) \\
&\quad= \textstyle\sum_{\tau \in A_3} (-\operatorname{ad}_{\xi_{\tau(3)} \lrcorner R}^* \xi_{\tau(1)} + \operatorname{ad}_{\xi_{\tau(1)} \lrcorner R}^* \xi_{\tau(3)})(\operatorname{ad}_{\xi_{\tau(2)}}^* X) \\
&\quad= \textstyle\sum_{\tau \in A_3} [\xi_{\tau(1)}, \xi_{\tau(3)}]_\pi(\operatorname{ad}_{\xi_{\tau(2)}}^* X) = \sum_{\tau \in A_3} [\xi_{\tau(2)}, [\xi_{\tau(1)}, \xi_{\tau(3)}]_\pi]_\pi(X)
\end{aligned}$$

We therefore see that $[R, R]$ is ad-invariant if and only if the Jacobi identity holds for the bracket $[,]_\pi$. According to Lemma 3.15 the latter statement is equivalent to π defining a Poisson structure on G. □

We call any $R \in \wedge^2(\mathfrak{g})$ such that $[R, R]$ is ad-invariant a *classical r-matrix*. The above lemma has thus given a method for defining a Poisson structure on a connected Lie group using a classical r-matrix.

Lemma 3.18. *Let $R \in \wedge^2(\mathfrak{g})$ and R^l and R^r respectively be the left and right invariant multivector fields on G such that $R^l(e) = R^r(e) = R$. Then the Schouten brackets of the multivector fields with themselves are*

$$[R^l, R^l](g) = -2l_g[R, R], \tag{25}$$
$$[R^r, R^r](g) = 2r_g[R, R], \tag{26}$$

for any $g \in G$ and with $[R, R] \in \wedge^3(\mathfrak{g})$ as defined in (24).

Proof. We start with the case of the left invariant multivector field. Take $X_{i,j} \in \mathfrak{g}$ such that $R = \sum_j X_{1,j} \wedge X_{2,j}$. If we take any $\xi \in \mathfrak{g}^*$, we find that

$$\xi \lrcorner R = \sum\nolimits_j \sum\nolimits_{\rho \in S_2} \operatorname{sgn}(\rho)\, \xi(X_{\rho(1),j})\, X_{\rho(2),j}$$

Also $R^l = \sum_j X_{1,j}^l \wedge X_{2,j}^l$ and therefore by [17, eq. (10.6.10)] we observe

$$[R^l, R^l] = -\sum\nolimits_{j,k} \sum\nolimits_{\rho, \rho' \in S_2} \operatorname{sgn}(\rho\rho')[X_{\rho(1),j}^l, X_{\rho'(1),k}^l] \wedge X_{\rho(2),j}^l \wedge X_{\rho'(2),k}^l.$$

The mapping $X \mapsto X^l$ where $X^l(e) = X$ from $\mathfrak{g}$ to the left invariant vector fields on G is a Lie algebra homomorphism and therefore

$$[R^l, R^l] = -\sum\nolimits_{j,k} \sum\nolimits_{\rho,\rho' \in S_2} \operatorname{sgn}(\rho\rho')[X_{\rho(1),j}, X_{\rho'(1),k}]^l \wedge X^l_{\rho(2),j} \wedge X^l_{\rho'(2),k}$$

from which we conclude that $[R^l, R^l]$ is left invariant. If we then apply the above equation when taken in the identity to any $\xi_1, \xi_2, \xi_3 \in \mathfrak{g}^*$ we find

$$\begin{aligned}
&[R^l, R^l](e)(\xi_1, \xi_2, \xi_3) \\
&= -\sum_{j,k} \sum_{\rho,\rho' \in S_2} \sum_{\tau \in S_3} \operatorname{sgn}(\rho\rho') \operatorname{sgn}(\tau)\xi_{\tau(1)}([X_{\rho(1),j}, X_{\rho'(1),k}])\xi_{\tau(2)}(X_{\rho(2),j})\xi_{\tau(3)}(X_{\rho'(2),k}) \\
&= -\sum_{j,k} \sum_{\rho,\rho' \in S_2} \sum_{\tau \in S_3} \operatorname{sgn}(\tau)\xi_{\tau(1)}([\operatorname{sgn}(\rho)\xi_{\tau(2)}(X_{\rho(2),j})X_{\rho(1),j}, \operatorname{sgn}(\rho')\xi_{\tau(3)}(X_{\rho'(2),k})X_{\rho'(1),k}]) \\
&= -\sum\nolimits_{\tau \in S_3} \operatorname{sgn}(\tau)\xi_{\tau(1)}([\xi_{\tau(2)} \lrcorner R, \xi_{\tau(3)} \lrcorner R]) = -2\sum\nolimits_{\tau \in A_3} \xi_{\tau(1)}([\xi_{\tau(2)} \lrcorner R, \xi_{\tau(3)} \lrcorner R]) \\
&= -2[R, R](\xi_1, \xi_2, \xi_3)
\end{aligned}$$

and therefore $[R^l, R^l](e) = -2[R, R]$. Since $[R^l, R^l]$ is left invariant we conclude (25).

For the case of the right invariant multivector field we can repeat the above process, but we note that the mapping $X \mapsto X^r$ where $X^r(e) = X$ from $\mathfrak{g}$ to the right invariant vector fields on G is a Lie algebra anti-homomorphism, which makes that we pick up an extra minus along the way, such that we find (26). $\square$

Definition 3.19. A *Poisson Lie group* is a Lie group U equipped with a Poisson structure π_U such that the multiplication map $\mu : U \times U \to U$ is a Poisson map, where $U \times U$ is equipped with the Poisson structure $\pi_U \oplus \pi_U$.

From this definition it is easy to discern that for any $u_1, u_2 \in U$ we find

$$\pi_U(u_1u_2) = r_{u_2}\pi_U(u_1) + l_{u_1}\pi_U(u_2) \tag{27}$$

and thus (U, π_U) is a Poisson Lie group if and only if π_U is multiplicative. The Lie algebra $\mathfrak{u}$ of U, is also the Lie algebra of the identity connected component of U. The bivector field π_U still defines a Poisson structure on this identity component and we therefore conclude by Lemma 3.15 that the dual map of the intrinsic derivative $d_e\pi_U$ defines a Lie algebra structure on $\mathfrak{u}^*$, which is often expressed as

$$[\xi, \eta]_{\mathfrak{u}^*}(X) = d_e\pi_U(X)(\xi, \eta) = \frac{d}{dt}\bigg|_{t=0} \left(r^{-1}_{\exp(tX)}\pi_U(\exp(tX))\right)(\xi, \eta), \quad X \in \mathfrak{u},\ \xi, \eta \in \mathfrak{u}^*.$$

If we now look at the direct sum of vector spaces $\mathfrak{u} \oplus \mathfrak{u}^*$, we notice that there exists a natural scalar product defined by

$$\langle X + \xi, Y + \eta \rangle = \eta(X) + \xi(Y), \quad X, Y \in \mathfrak{u},\ \xi, \eta \in \mathfrak{u}^*, \tag{28}$$

and that $\langle , \rangle$ is symmetric and nondegenerate. On the vector space $\mathfrak{u} \oplus \mathfrak{u}^*$ it is now possible to uniquely define a Lie algebra structure such that both $\mathfrak{u}$ and $\mathfrak{u}^*$ are subalgebras and that $\langle , \rangle$

becomes ad-invariant with respect to this new Lie algebra structure [16, Theorem 1.12]. We call this the *double Lie algebra of* $\mathfrak{u}$, denote it by $\mathfrak{d} = \mathfrak{u} \bowtie \mathfrak{u}^*$ and denote the corresponding Lie bracket as $[,]_{\mathfrak{d}}$. Due to the ad-invariance of the pairing this Lie bracket is explicititly given by

$$\begin{cases} [X,Y]_{\mathfrak{d}} &= [X,Y]_{\mathfrak{u}} \\ [X,\xi]_{\mathfrak{d}} &= \operatorname{ad}^\vee_X \xi - \operatorname{ad}^\vee_\xi X \\ [\xi,\eta]_{\mathfrak{d}} &= [\xi,\eta]_{\mathfrak{u}^*} \end{cases} \qquad X,Y \in \mathfrak{u},\ \xi,\eta \in \mathfrak{u}^*,$$

where $\operatorname{ad}^\vee_X = (-\operatorname{ad}_X)^*$ and $\operatorname{ad}^\vee_\xi = (-\operatorname{ad}_\xi)^*$ with respect to the canonical pairing between $\mathfrak{u}$ and $\mathfrak{u}^*$.

Interestingly we can now also define an action of U on $\mathfrak{d}$ given by

$$u \cdot (X+\xi) = \operatorname{Ad}_u X + (\operatorname{Ad}^\vee_u \xi) \lrcorner (r_u^{-1} \pi_U(u)) + \operatorname{Ad}^\vee_u \xi, \quad u \in U, X \in \mathfrak{u}, \xi \in \mathfrak{u}^* \tag{29}$$

where $\operatorname{Ad}^\vee_u = (\operatorname{Ad}_{u^{-1}})^*$ with respect to the canonical pairing between $\mathfrak{u}^*$ and $\mathfrak{u}$. For any $Z \in \mathfrak{u}$ and $\xi, \eta \in \mathfrak{u}^*$ we observe

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \eta\big((\operatorname{Ad}^\vee_{\exp(tZ)} \xi) \lrcorner (r^{-1}_{\exp(tZ)} \pi_U(\exp(tZ)))\big) \\ &\qquad = \frac{d}{dt}\Big|_{t=0} (r^{-1}_{\exp(tZ)} \pi_U(\exp(tZ)))(\operatorname{Ad}^\vee_{\exp(tZ)} \xi, \eta) \\ &\qquad = \frac{d}{dt}\Big|_{t=0} (r^{-1}_{\exp(tZ)} \pi_U(\exp(tZ)))(\xi,\eta) + \pi_U(e)(\operatorname{ad}^\vee_Z \xi, \eta) \\ &\qquad = [\xi,\eta]_{\mathfrak{u}^*}(Z) = (\operatorname{ad}_\xi \eta)(Z) = \eta(-\operatorname{ad}^\vee_\xi Z). \end{aligned}$$

Using the above we find for any $X, Z \in \mathfrak{u}$ and $\xi \in \mathfrak{u}^*$,

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0} \exp(tZ) \cdot (X+\xi) \\ &\qquad = \frac{d}{dt}\Big|_{t=0} \operatorname{Ad}_{\exp(tZ)} X + (\operatorname{Ad}^\vee_{\exp(tZ)} \xi) \lrcorner (r^{-1}_{\exp(tZ)} \pi_U(\exp(tZ))) + \operatorname{Ad}^\vee_{\exp(tZ)} \xi \\ &\qquad = \operatorname{ad}_Z X - \operatorname{ad}^\vee_\xi Z + \operatorname{ad}^\vee_Z \xi = [Z, X+\xi]_{\mathfrak{d}}, \end{aligned}$$

and we thus observe that the action defined in (29) corresponds with the adjoint action of U as a subgroup of the adjoint group of $\mathfrak{d}$.

Definition 3.20. A triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ together with a nondegenerate ad-invariant symmetric bilinear form $\langle , \rangle$ on $\mathfrak{g}$ is called a *Manin triple* if $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic subalgebras of $\mathfrak{g}$ with respect to $\langle , \rangle$ such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is a direct sum of vector spaces.

We see immediately that $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}^*)$ is a Manin triple. Interestingly, if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple, there exists an identification of $\mathfrak{g}_-$ with $\mathfrak{g}_+^*$ by mapping $X_- \mapsto \langle X_-, \cdot \rangle$, since $\langle , \rangle$ is nondegenerate.

3.3. Drinfeld's theorem

The following section addresses a theorem by Drinfeld [3] which finds a correspondence between Poisson homeneous spaces and the variety of Lagrangian subalgebras of $\mathfrak{d}$. Although Drinfeld himself describes the proof of this theorem "simple and more or less straightforward" and therefore does not give it in the original publication, we have found otherwise and therefore include our version of the proof in this section. We note that contrary to our earlier expectations, Drinfeld's theorem is not necessary in defining the Lu-Evens Poisson structure. However, despite its inapplicability, we still would like give an exposition of this work as it expands upon the literature used in the present thesis.

Let (U, π_U) be a Poisson Lie group with Lie algebra $\mathfrak{u}$, $\mathfrak{d}$ be the double Lie algebra of $\mathfrak{u}$ and the pairing $\langle , \rangle$ as defined in (28).

Definition 3.21. A subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ is called *Lagrangian* if $\dim \mathfrak{l} = \dim \mathfrak{u}$ and $\langle a, b \rangle = 0$ for all $a, b \in \mathfrak{l}$. We denote the set of Lagrangian subalgebras of $\mathfrak{d}$ by $\mathcal{L}(\mathfrak{d})$.

We note immediately that $\mathfrak{u}$ and $\mathfrak{u}^*$ are both Lagrangian subalgebras. As all Lagrangian subalgebras are n dimensional by definition it is clear that $\mathcal{L}(\mathfrak{d})$ is a subset of $\mathrm{Gr}(n, \mathfrak{d})$, the Grassmannian of n dimensional subspaces of $\mathfrak{d}$. The adjoint group D of $\mathfrak{d}$ is a Lie group that acts on $\mathrm{Gr}(n, \mathfrak{d})$. Because elements of D as Lie algebra homomorphisms they also preserve Lie subalgebras. Finally since $\langle , \rangle$ is ad-invariant we find that D preserves $\mathcal{L}(\mathfrak{d})$. We remember that U acts on $\mathfrak{d}$ by (29), and therefore can be immersed into D as a Lie subgroup. Therefore U also acts on $\mathcal{L}(\mathfrak{d})$.

Definition 3.22. Let (U, π_U) be a Poisson Lie group and (M, π) be a Poisson manifold. We call M a (U, π_U)-*homogeneous Poisson space* if it is equipped with a transitive action $\sigma : U \times M \to M : (u, m) \mapsto um$ which is a Poisson map, where $U \times M$ is equipped with the Poisson structure $\pi_U \oplus \pi$.

Since σ is a Poisson map, we find for any $u \in U$ and $m \in M$ that

$$\pi(um) = T_m\sigma_u(\pi(m)) + T_u\sigma_m(\pi_U(u)), \tag{30}$$

where $T_m\sigma_u$ and $T_u\sigma_m$ are respectively the differentials of the maps $\sigma_u : M \to M : m \mapsto um$ and $\sigma_m : U \to M : u \mapsto um$.

As the action is transitive, we see that σ_m is a submersion and also that the entire Poisson structure on M is determined by $\pi(m) \in \wedge^2(T_mM)$ for a fixed $m \in M$. Now let U_m be the stabilizer subgroup of m in U, with Lie algebra $\mathfrak{u}_m$. Then $\sigma_m : U \to M$ factors through to a diffeomorphism $\overline{\sigma}_m : U/U_m \to M : uU_m \mapsto um$ [22, Proposition 15.5]. Through this diffeomorphism we can identify $\mathfrak{u}/\mathfrak{u}_m \cong T_mM$ by the map

$$\psi_m \equiv T_{eU_m}(\overline{\sigma}_m) : X + \mathfrak{u}_m \mapsto \frac{d}{dt}\Big|_{t=0} \exp(tX)m.$$

Then $\psi_m^{-1}\pi(m) \in \wedge^2(\mathfrak{u}/\mathfrak{u}_m)$ and we note that $T_e\sigma_m = \psi_m \circ \rho_m$ by the chain rule, where $\rho_m : \mathfrak{u} \to \mathfrak{u}/\mathfrak{u}_m$ is the canonical projection. Also we find that the dual map of ρ_m has image

$$\mathrm{im}\, \rho_m^* = \mathfrak{u}_m^\perp = \{\eta \in \mathfrak{u}^* : \eta|_{\mathfrak{u}_m} = 0\}$$

and it therefore defines an isomorphism

$$\chi_m : \mathfrak{u}_m^{\perp} \to (\mathfrak{u}/\mathfrak{u}_m)^*,$$

such that $\rho_m^* \circ \chi_m = \mathrm{id}_{\mathfrak{u}_m^{\perp}}$. We now define a subspace $\mathfrak{l}_m \subset \mathfrak{d}$ depending on m by

$$\mathfrak{l}_m = \{X + \xi : X \in \mathfrak{u}, \xi \in \mathfrak{u}_m^{\perp}, (\chi_m \xi) \lrcorner (\psi_m^{-1} \pi(m)) = X + \mathfrak{u}_m\}. \tag{31}$$

Theorem 3.23 (Drinfeld [3])**.**

1. *$\mathfrak{l}_m$ is a Lagrangian subalgebra of $\mathfrak{d}$ for all $m \in M$.*
2. *For all $m \in M$ and $u \in U$,*

$$\mathfrak{l}_m \cap \mathfrak{u} = \mathfrak{u}_m, \tag{32}$$
$$u \cdot \mathfrak{l}_m = \mathfrak{l}_{um}. \tag{33}$$

3. *Let M be a U-homogeneous space. The existence of a (U, π_U)-homogeneous Poisson structure π on M is equivalent to the existence of a U-equivariant map*

$$\mathrm{P} : M \to \mathcal{L}(\mathfrak{d}) : m \mapsto \mathfrak{l}_m$$

such that $\mathfrak{l}_m \cap \mathfrak{u} = \mathfrak{u}_m$ for all $m \in M$.

We call the map P the *Drinfeld map* and $\mathfrak{l}_m$ the *Lagrangian subalgebra* $\mathfrak{d}$ *associated to* (M, π) *at the point* m. Before we prove the above theorem we prove some useful lemmas first.

Lemma 3.24. *For a fixed $m \in M$ there exists a left invariant bivector field λ on U such that $\rho_m \lambda(e) = \psi_m^{-1} \pi(m)$. Then $\varpi := \lambda + \pi_U$ and π are σ_m-related.*

Proof. We know that $\psi_m^{-1}\pi(m) \in \wedge^2(\mathfrak{u}/\mathfrak{u}_m)$, and so we know there exist $X_j, Y_j \in \mathfrak{u}$ such that $\psi_m^{-1}\pi(m) = \sum_j (X_j + \mathfrak{u}_m) \wedge (Y_j + \mathfrak{u}_m)$. We now define $\lambda(e) = \sum_j X_j \wedge Y_j \in \wedge^2(\mathfrak{u})$, such that $\rho_m \lambda(e) = \psi_m^{-1}\pi(m)$. We make λ a left invariant bivector field on U, i.e. $\lambda(u) = l_u \lambda(e)$. We then observe for any $u' \in U$ that

$$(\sigma_m \circ l_u)(u') = uu'm = \sigma_u(u'm) = (\sigma_u \circ \sigma_m)(u')$$

and by differentiating the above identity for u' at e we can calculate

$$T_u \sigma_m \lambda(u) = T_u \sigma_m l_u \lambda(e) = T_m \sigma_u T_u \sigma_m \lambda(e) = T_m \sigma_u \psi_m \rho_m \lambda(e) = T_m \sigma_u \pi(m). \tag{34}$$

By substituting (34) into equation (30) we get

$$\pi(\sigma_m(u)) = T_u \sigma_m(\lambda(u) + \pi_U(u)) = T_u \sigma_m(\varpi(u)), \tag{35}$$

which shows that ϖ and π are σ_m-related. $\square$

An interesting consequence of this lemma is when we use that σ_m is a submersion and apply Proposition 2.5 to find that $[\pi,\pi]$ and $[\varpi,\varpi]=[\lambda,\lambda]+2[\lambda,\pi_U]$ are σ_m-related.

Lemma 3.25. *If $\xi_1,\xi_2,\xi_3\in\mathfrak{u}^*$, then*

$$[\lambda,\lambda](e)(\xi_1,\xi_2,\xi_3)=-2\sum\nolimits_{\tau\in A_3}\xi_{\tau(1)}([\xi_{\tau(2)}\lrcorner\lambda(e),\xi_{\tau(3)}\lrcorner\lambda(e)]),\tag{36}$$

$$[\lambda,\pi_U](e)(\xi_1,\xi_2,\xi_3)=-\sum\nolimits_{\tau\in A_3}\lambda(e)(\xi_{\tau(1)},[\xi_{\tau(2)},\xi_{\tau(3)}]_{\mathfrak{u}^*}).\tag{37}$$

Proof. We apply Lemma 3.18 to find $[\lambda,\lambda](e)=[\lambda(e),\lambda(e)]\in\wedge^3(\mathfrak{g})$. Then we immediately find (36) by using (24).

Take U_e as the identity connected component of U, and define ξ_j^l as the left invariant 1-form on U_e corresponding to ξ_j for $j=1,2,3$. Since π_U is multiplicative we can apply Lemma 3.14 to find

$$[\lambda,\pi_U](\xi_1^l,\xi_2^l,\xi_3^l)=-\sum\nolimits_{\tau\in A_3}\lambda(\xi_{\tau(1)}^l,[\xi_{\tau(2)}^l,\xi_{\tau(3)}^l]_{\pi_U})$$

If we recall that $[,]_{\mathfrak{u}*}$ is the bracket defined by π_U, then taking the above equation in e immediately delivers us (37). $\square$

On the other hand if we now take $X_j+\xi_j\in\mathfrak{d}$ for $j=1,2,3$, we then find

$$\begin{aligned}
&\langle X_1+\xi_1,[X_2+\xi_2,X_3+\xi_3]_{\mathfrak{d}}\rangle\\
&\quad=\langle X_1,[\xi_2,\xi_3]_{\mathfrak{d}}+[X_2,\xi_3]_{\mathfrak{d}}+[\xi_2,X_3]_{\mathfrak{d}}\rangle+\langle\xi_1,[X_2,X_3]_{\mathfrak{d}}+[X_2,\xi_3]_{\mathfrak{d}}+[\xi_2,X_3]_{\mathfrak{d}}\rangle\\
&\quad=\langle X_1,[\xi_2,\xi_3]_{\mathfrak{u}^*}\rangle+\langle[X_1,X_2]_{\mathfrak{u}},\xi_3\rangle-\langle\xi_2,[X_1,X_3]_{\mathfrak{u}}\rangle\\
&\qquad+\langle\xi_1,[X_2,X_3]_{\mathfrak{u}}\rangle+\langle X_2,[\xi_3,\xi_1]_{\mathfrak{u}^*}\rangle+\langle[\xi_1,\xi_2]_{\mathfrak{u}^*},X_3\rangle\\
&\quad=\sum\nolimits_{\tau\in A_3}\langle X_{\tau(1)},[\xi_{\tau(2)},\xi_{\tau(3)}]_{\mathfrak{u}^*}\rangle+\langle\xi_{\tau(1)},[X_{\tau(2)},X_{\tau(3)}]_{\mathfrak{u}}\rangle
\end{aligned}\tag{38}$$

All the previous results now combine in the result:

Lemma 3.26. *Let $\xi_j\in\mathfrak{u}_m^\perp$ for $j=1,2,3$, then*

$$(\psi_m^{-1}[\pi,\pi](m))(\chi_m\xi_1,\chi_m\xi_2,\chi_m\xi_3)=-2\langle\xi_1\lrcorner\lambda(e)+\xi_1,[\xi_2\lrcorner\lambda(e)+\xi_2,\xi_3\lrcorner\lambda(e)+\xi_3]_{\mathfrak{d}}\rangle.$$

Proof. We had already noted by Lemma 3.24 that

$$\begin{aligned}
(\psi_m^{-1}[\pi,\pi](m))(\chi_m\xi_1,\chi_m\xi_2,\chi_m\xi_3)&=(\psi_m^{-1}[\pi,\pi](\sigma_m(e)))(\chi_m\xi_1,\chi_m\xi_2,\chi_m\xi_3)\\
&=(\psi_m^{-1}T_e\sigma_m[\varpi,\varpi](e))(\chi_m\xi_1,\chi_m\xi_2,\chi_m\xi_3)\\
&=([\lambda,\lambda](e)+2[\lambda,\pi_U](e))(\rho_m^*\chi_m\xi_1,\rho_m^*\chi_m\xi_2,\rho_m^*\chi_m\xi_3)\\
&=[\lambda,\lambda](e)(\xi_1,\xi_2,\xi_3)+2[\lambda,\pi_U](e)(\xi_1,\xi_2,\xi_3)
\end{aligned}$$

We can then apply Lemma 3.25, and (38) to find

$$\begin{aligned}
&(\psi_m^{-1}[\pi,\pi](m))(\xi_1,\xi_2,\xi_3)=[\lambda,\lambda](e)(\xi_1,\xi_2,\xi_3)+2[\lambda,\pi_U](e)(\xi_1,\xi_2,\xi_3)\\
&\quad=-2\sum\nolimits_{\tau\in A_3}\xi_{\tau(1)}([\xi_{\tau(2)}\lrcorner\lambda(e),\xi_{\tau(3)}\lrcorner\lambda(e)]_{\mathfrak{u}})+\lambda(e)(\xi_{\tau(1)},[\xi_{\tau(2)},\xi_{\tau(3)}]_{\mathfrak{u}^*})\\
&\quad=-2\sum\nolimits_{\tau\in A_3}\langle\xi_{\tau(1)},[\xi_{\tau(2)}\lrcorner\lambda(e),\xi_{\tau(3)}\lrcorner\lambda(e)]_{\mathfrak{u}}\rangle+\langle\xi_{\tau(1)}\lrcorner\lambda(e),[\xi_{\tau(2)},\xi_{\tau(3)}]_{\mathfrak{u}^*}\rangle\\
&\quad=-2\langle\xi_1\lrcorner\lambda(e)+\xi_1,[\xi_2\lrcorner\lambda(e)+\xi_2,\xi_3\lrcorner\lambda(e)+\xi_3]_{\mathfrak{d}}\rangle
\end{aligned}$$

which completes our proof. $\square$

The next lemma gives a relatively result from linear algebra, which we mention without proof.

Lemma 3.27 ([5, Lemma 2.24])**.** *Let V be an n dimensional vector space and let V^* be its dual space. On the direct sum vector space $V \oplus V^*$, consider the symmetric product defined by*

$$\langle v_1 + \lambda_1, v_2 + \lambda_2 \rangle = \lambda_1(v_2) + \lambda_2(v_1), \quad v_1, v_2 \in V; \lambda_1, \lambda_2 \in V^*.$$

Let V_0 be any subspace of V. For $\alpha \in \wedge^2(V/V_0)$, define

$$W_\alpha = \{v + \lambda : v \in V, \lambda \in V_0^\perp, \lambda \lrcorner \alpha = v + V_0\}$$

Then $\alpha \mapsto W_\alpha$ is a one-to-one correspondence between elements in $\wedge^2(V/V_0)$ and maximal isotropic subspaces W of $V \oplus V^*$ such that $W \cap V = V_0$.

Proof of Drinfeld's theorem (Theorem 3.23). First we prove part 2, as we will use its results in proving part 1. It is clear that $\mathfrak{u}_m \subset \mathfrak{l}_m$. Also if $X \in \mathfrak{l}_m \cap \mathfrak{u}$, then $X + \mathfrak{u}_m = 0 \lrcorner (\psi_m^{-1}\pi(m)) = 0$ and therefore $X \in \mathfrak{u}_m$, from which we conclude (32).

Take any $X \in \mathfrak{u}_m$, then $\exp(tX)m = m$ for all $t \in \mathbb{R}$. From this we then find that for any $u \in U$ and $t \in \mathbb{R}$ that

$$\exp(t(u \cdot X))um = \exp(t \operatorname{Ad}_u X)um = u \exp(tX)u^{-1}um = u \exp(tX)m = um$$

and so we find that $u \cdot X \in \mathfrak{u}_{um}$ and therefore $u \cdot \mathfrak{u}_m \subset \mathfrak{u}_{um}$. Since now also $\mathfrak{u}_{um} = u \cdot (u^{-1} \cdot \mathfrak{u}_{um}) \subset u \cdot \mathfrak{u}_{u^{-1}um} = u \cdot \mathfrak{u}_m$ we conclude that $u \cdot \mathfrak{u}_m = \mathfrak{u}_{um}$, or in other words $\operatorname{Ad}_u \mathfrak{u}_m = \mathfrak{u}_{um}$. From this we can immediately also conclude that if $\xi \in \mathfrak{u}_m^\perp$ that then $\operatorname{Ad}_u^\vee \xi \in \mathfrak{u}_{um}^\perp$.

If we now take any $u \in U$ and $X + \xi \in \mathfrak{l}_m$, we then find that $u \cdot (X + \xi) \in \mathfrak{l}_{um}$ if and only if

$$(\chi_{um} \operatorname{Ad}_u^\vee \xi) \lrcorner (\psi_{um}^{-1}\pi(um)) = \operatorname{Ad}_u X + (\operatorname{Ad}_u^\vee \xi) \lrcorner (r_{u^{-1}}\pi_U(u)) + \mathfrak{u}_{um}. \tag{39}$$

In order to show this we first note that $\sigma_m \circ r_u = \sigma_{um}$ and then we apply Lemma 3.24 to calculate

$$\begin{aligned}
\psi_{um}^{-1}\pi(um) &= \psi_{um}^{-1}\pi(\sigma_m(u)) = \psi_{um}^{-1}T_u\sigma_m(\varpi(u)) = \psi_{um}^{-1}T_u\sigma_m r_u r_u^{-1}\varpi(u) \\
&= \psi_{um}^{-1}T_e\sigma_{um}r_u^{-1}\varpi(u) = \rho_{um}r_u^{-1}(l_u\lambda(e) + \pi_U(u)) \\
&= \rho_{um}(\operatorname{Ad}_u \varpi(e) + r_u^{-1}\pi_U(u))
\end{aligned}$$

We apply this to find

$$\begin{aligned}
(\chi_{um} \operatorname{Ad}_u^\vee \xi) \lrcorner (\psi_{um}^{-1}\pi(um)) &= (\chi_{um} \operatorname{Ad}_u^\vee \xi) \lrcorner (\rho_{um}(\operatorname{Ad}_u \lambda(e) + r_u^{-1}\pi_U(u))) \\
&= \rho_{um}\big((\rho_{um}^*\chi_{um} \operatorname{Ad}_u^\vee \xi) \lrcorner (\operatorname{Ad}_u \lambda(e) + r_u^{-1}\pi_U(u))\big) \\
&= (\operatorname{Ad}_u^\vee \xi) \lrcorner (\operatorname{Ad}_u \lambda(e) + r_u^{-1}\pi_U(u)) + \mathfrak{u}_{um} \\
&= \operatorname{Ad}_u(\xi \lrcorner \lambda(e)) + (\operatorname{Ad}_u^\vee \xi) \lrcorner (r_{u^{-1}}\pi_U(u)) + \mathfrak{u}_{um}.
\end{aligned} \tag{40}$$

We then observe that since $X + \xi \in \mathfrak{l}_m$ we get

$$\begin{aligned}\xi \lrcorner \lambda(e) + \mathfrak{u}_m &= \rho_m((\rho_m^* \chi_m \xi) \lrcorner \lambda(e)) = (\chi_m \xi) \lrcorner (\rho_m \lambda(e)) = (\chi_m \xi) \lrcorner (\psi_m^{-1} \pi(m)) \\ &= X + \mathfrak{u}_m.\end{aligned}$$

Therefore $X - \xi \lrcorner \lambda(e) \in \mathfrak{u}_m$ which means that $\mathrm{Ad}_u(X - \xi \lrcorner \lambda(e)) \in \mathfrak{u}_{um}$. This combined with (40) proves that equation (39) holds, and therefore $u \cdot \mathfrak{l}_m \subset \mathfrak{l}_{um}$. We can easily conclude from this result that (33) holds by looking at $u^{-1} \cdot \mathfrak{l}_{um}$.

For part 1 it is clear by Lemma 3.27 that $\mathfrak{l}_m$ is a maximal isotropic subspace of $\mathfrak{d}$. Take any $a, b \in \mathfrak{l}_m$ and $Z \in \mathfrak{u}_m$. We then find

$$\langle Z, [a, b]_{\mathfrak{d}} \rangle = \langle [Z, a]_{\mathfrak{d}}, b \rangle = \frac{d}{dt}\Big|_{t=0} \langle \exp(tz) \cdot a, b \rangle$$

Since $Z \in \mathfrak{u}_m$ we observe that $\exp(tZ)m = m$ for any $t \in \mathbb{R}$ and thus using (33) we find

$$\exp(tZ) \cdot a \in \exp(tZ) \cdot \mathfrak{l}_m = \mathfrak{l}_{\exp(tZ)m} = \mathfrak{l}_m, \forall t \in \mathbb{R}.$$

Using that $\mathfrak{l}_m$ is isotropic we therefore find that

$$\langle z, [a, b]_{\mathfrak{d}} \rangle = \frac{d}{dt}\Big|_{t=0} \langle \exp(tz) \cdot a, b \rangle = \frac{d}{dt}\Big|_{t=0} 0 = 0 \tag{41}$$

Now take any $X_j + \xi_j \in \mathfrak{l}_m$ for $j = 1, 2, 3$. We now apply Lemma 3.24 and find that there exists a left invariant bivector field λ on U such that

$$\begin{aligned}X_j + \mathfrak{u}_m &= (\chi_m \xi_j) \lrcorner (\psi_m^{-1} \pi(m)) = (\chi_m \xi_j) \lrcorner (\rho_m \lambda(e)) = \rho_m((\rho_m^* \chi_m \xi_j) \lrcorner (\lambda(e))) \\ &= \xi_j \lrcorner \lambda(e) + \mathfrak{u}_m\end{aligned}$$

and so there exists $Z_j \in \mathfrak{u}_m$ such that $X_j = Z_j + \xi_j \lrcorner \lambda(e)$. We note that $\xi_j \lrcorner \lambda(e) + \xi_j \in \mathfrak{l}_m$ since $z_j \in \mathfrak{l}_m$ by part 2. We then find by repeatedly applying (41) that

$$\begin{aligned}\langle X_1 + \xi_1, &[X_2 + \xi_2, X_3 + \xi_3]_{\mathfrak{d}} \rangle \\ &= \langle Z_j + \xi_1 \lrcorner \lambda(e) + \xi_1, [Z_2 + \xi_2 \lrcorner \lambda(e) + \xi_2, Z_3 + \xi_3 \lrcorner \lambda(e) + \xi_3]_{\mathfrak{d}} \rangle \\ &= \langle \xi_1 \lrcorner \lambda(e) + \xi_1, [Z_2 + \xi_2 \lrcorner \lambda(e) + \xi_2, Z_3 + \xi_3 \lrcorner \lambda(e) + \xi_3]_{\mathfrak{d}} \rangle \\ &= \langle [\xi_1 \lrcorner \lambda(e) + \xi_1, Z_2 + \xi_2 \lrcorner \lambda(e) + \xi_2]_{\mathfrak{d}}, \xi_3 \lrcorner \lambda(e) + \xi_3 \rangle \\ &= \langle [\xi_3 \lrcorner \lambda(e) + \xi_3, \xi_1 \lrcorner \lambda(e) + \xi_1]_{\mathfrak{d}}, Z_2 + \xi_2 \lrcorner \lambda(e) + \xi_2 \rangle \\ &= \langle [\xi_3 \lrcorner \lambda(e) + \xi_3, \xi_1 \lrcorner \lambda(e) + \xi_1]_{\mathfrak{d}}, \xi_2 \lrcorner \lambda(e) + \xi_2 \rangle \\ &= \langle \xi_1 \lrcorner \lambda(e) + \xi_1, [\xi_2 \lrcorner \lambda(e) + \xi_2, \xi_3 \lrcorner \lambda(e) + \xi_3]_{\mathfrak{d}} \rangle\end{aligned}$$

We can then apply Lemma 3.26 again and that $[\pi, \pi] = 0$ since π defines a Poisson structure to find that

$$\langle X_1 + \xi_1, [X_2 + \xi_2, X_3 + \xi_3]_{\mathfrak{d}} \rangle = (\psi_m^{-1} [\pi, \pi](m))(\xi_1, \xi_2, \xi_3) = 0$$

Since $\mathfrak{l}_m$ is maximal isotropic we then find that $[X_2 + \xi_2, X_3 + \xi_3] \in \mathfrak{l}_m$, from which we conclude that $\mathfrak{l}_m$ is a subalgebra of $\mathfrak{d}$.

For proving one direction of part 3 we notice that a (U, π_U)-homogeneous Poisson structure π on M defines a Drinfeld map as a direct consequence of parts 1 and 2 of the theorem.

For the converse we want to show that the Drinfeld map P induces a (U, π_U)-homogeneous Poisson structure on M. Pick any $m \in M$. Because $\mathfrak{l}_m \subset \mathfrak{d}$ is maximal isotropic and because $\mathfrak{l}_m \cap \mathfrak{u} = \mathfrak{u}_m$, Lemma 3.27 shows that there is a unique element $\pi(m) \in \wedge^2(\mathfrak{u}/\mathfrak{u}_m)$ such that (31) holds. Define a bivector field π on M by (30). This is well defined because the Drinfeld map is U-equivariant, which gives (33). By Lemma 3.24 there exists a left invariant bivector field λ on U such that $\rho_m \lambda(e) = \psi_m^{-1}\pi(m)$, and as such $\xi_j \lrcorner \psi_m^{-1}\pi(m) = \xi_j \lrcorner \lambda(e) + \mathfrak{u}_m$ for $\xi_j \in \mathfrak{u}_m^{\perp}$. Therefore $\xi_j \lrcorner \lambda(e) + \xi_j \in \mathfrak{l}_m$, while Lemma 3.26 gives us

$$(\psi_m^{-1}[\pi, \pi](m))(\chi_m \xi_1, \chi_m \xi_2, \chi_m \xi_3) = -2\langle \xi_1 \lrcorner \lambda(e) + \xi_1, [\xi_2 \lrcorner \lambda(e) + \xi_2, \xi_3 \lrcorner \lambda(e) + \xi_3] \rangle$$

and since $\mathfrak{l}_m$ is a Lagrangian subalgebra we then find $[\pi, \pi] = 0$ and therefore that π is Poisson. It is (U, π_U)-homogeneous because (33) holds by definition. $\square$

3.4. Lu-Evens Poisson structure

In this section we discuss a Poisson structure on the set of Lagrangian subalgebras of a double Lie algebra found by Evens and Lu in [5]. In section 7 we will see this structure play an important part in proving Van den Ban's convexity theorem.

As in the previous section, let (U, π_U) be a Poisson Lie group with Lie algebra $\mathfrak{u}$ and let $\mathfrak{d} = \mathfrak{u} \bowtie \mathfrak{u}^*$ be the double Lie algebra. We note $\mathfrak{d}^* = (\mathfrak{u} \oplus \mathfrak{u}^*)^* = \mathfrak{u}^* \oplus (\mathfrak{u}^*)^* \cong \mathfrak{u}^* \oplus \mathfrak{u}$, and thus we denote elements of $\mathfrak{d}^*$ as $\xi + X$ with $\xi \in \mathfrak{u}^*$ and $X \in \mathfrak{u}$. There exists a natural isomorphism between $\mathfrak{d}$ and its dual defined by the nondegenerate pairing $\langle , \rangle$ which is given by

$$\# : \mathfrak{d}^* \to \mathfrak{d} : \xi + X \mapsto X + \xi.$$

We now define $R \in \wedge^2(\mathfrak{d})$ by

$$R(\xi + X, \eta + Y) = \eta(X) - \xi(Y), \quad X, Y \in \mathfrak{u}, \xi, \eta \in \mathfrak{u}^* \tag{42}$$

and we then easily observe $(\xi + X) \lrcorner R = X - \xi$. We recall from section 3.3 that the Lie group D is defined as the adjoint group of $\mathfrak{d}$ and that U is a Lie subgroup of D.

Lemma 3.28. *R as defined in (42) is a classical r-matrix and*

$$[R, R](f_1, f_2, f_3) = -\langle \# f_1, [\# f_2, \# f_3]_{\mathfrak{d}} \rangle \tag{43}$$

for $f_1, f_2, f_3 \in \mathfrak{d}^*$.

Proof. We first calculate $[R, R]$ explicitly. Take any $f_j = \xi_j + X_j \in \mathfrak{d}^*$, then

$$\begin{aligned}
&[R, R](f_1, f_2, f_3) \\
&\quad = \sum\nolimits_{\tau \in A_3} (\xi_{\tau(1)} + X_{\tau(1)})([(\xi_{\tau(2)} + X_{\tau(2)}) \lrcorner R, (\xi_{\tau(3)} + X_{\tau(3)}) \lrcorner R]_{\mathfrak{d}}) \\
&\quad = \sum\nolimits_{\tau \in A_3} \langle X_{\tau(1)} + \xi_{\tau(1)}, [X_{\tau(2)} - \xi_{\tau(2)}, X_{\tau(3)} - \xi_{\tau(3)}]_{\mathfrak{d}} \rangle.
\end{aligned}$$

By using the symmetry and the ad-invariance of $\langle , \rangle$ and the isotropy of $\mathfrak{u}$ and $\mathfrak{u}^*$ we then find

$$\begin{aligned}
&[R, R](f_1, f_2, f_3) \\
&\quad = \langle X_1 + \xi_1, [X_2 - \xi_2, X_3 - \xi_3]_{\mathfrak{d}} \rangle + \langle X_1 - \xi_1, [X_2 + \xi_2, X_3 - \xi_3]_{\mathfrak{d}} \rangle \\
&\qquad + \langle X_1 - \xi_1, [X_2 - \xi_2, X_3 + \xi_3]_{\mathfrak{d}} \rangle \\
&\quad = \langle X_1 + \xi_1, [X_2 - \xi_2, X_3 - \xi_3]_{\mathfrak{d}} \rangle + \langle X_1 - \xi_1, 2[X_2, X_3]_{\mathfrak{d}} - 2[\xi_2, \xi_3]_{\mathfrak{d}} \rangle \\
&\quad = \langle X_1 + \xi_1, [X_2 - \xi_2, X_3 - \xi_3]_{\mathfrak{d}} \rangle - \langle X_1, 2[\xi_2, \xi_3]_{\mathfrak{d}} \rangle - \langle \xi_1, 2[X_2, X_3]_{\mathfrak{d}} \rangle \\
&\quad = \langle X_1 + \xi_1, [X_2 - \xi_2, X_3 - \xi_3]_{\mathfrak{d}} - 2[\xi_2, \xi_3]_{\mathfrak{d}} - 2[X_2, X_3]_{\mathfrak{d}} \rangle \\
&\quad = -\langle X_1 + \xi_1, [X_2 + \xi_2, X_3 + \xi_3]_{\mathfrak{d}} \rangle = -\langle \# f_1, [\# f_2, \# f_3]_{\mathfrak{d}} \rangle
\end{aligned}$$

and thus we have proven (43). Now we ought to check that $[R, R]$ is ad-invariant, with respect to the adjoint representation of D on $\mathfrak{d}$, which we denote by $\overline{\mathrm{Ad}}$ to distinguish it from the adjoint representation of U on $\mathfrak{u}$. For any $d \in D$ we observe

$$\begin{aligned}
f_1(\overline{\mathrm{Ad}}_d \# f_2) &= \langle \# f_1, \overline{\mathrm{Ad}}_d \# f_2 \rangle = \langle \overline{\mathrm{Ad}}_{d^{-1}} \# f_1, \# f_2 \rangle = f_2(\overline{\mathrm{Ad}}_{d^{-1}} \# f_1) \\
&= \overline{\mathrm{Ad}}_d^\vee f_2(\# f_1) = \langle \# \overline{\mathrm{Ad}}_d^\vee f_2, \# f_1 \rangle = f_1(\# \overline{\mathrm{Ad}}_d^\vee f_2)
\end{aligned}$$

and thus $\# \circ \overline{\mathrm{Ad}}_d^\vee = \overline{\mathrm{Ad}}_d \circ \#$. We apply this to (43) and see

$$\begin{aligned}
[R, R](\overline{\mathrm{Ad}}_d^\vee f_1, \overline{\mathrm{Ad}}_d^\vee f_2, \overline{\mathrm{Ad}}_d^\vee f_3) &= -\langle \# \overline{\mathrm{Ad}}_d^\vee f_1, [\# \overline{\mathrm{Ad}}_d^\vee f_2, \# \overline{\mathrm{Ad}}_d^\vee f_3]_{\mathfrak{d}} \rangle \\
&= -\langle \overline{\mathrm{Ad}}_d \# f_1, [\overline{\mathrm{Ad}}_d \# f_2, \overline{\mathrm{Ad}}_d \# f_3]_{\mathfrak{d}} \rangle \\
&= -\langle \overline{\mathrm{Ad}}_d \# f_1, \overline{\mathrm{Ad}}_d [\# f_2, \# f_3]_{\mathfrak{d}} \rangle \\
&= -\langle \# f_1, [\# f_2, \# f_3]_{\mathfrak{d}} \rangle \\
&= [R, R](f_1, f_2, f_3)
\end{aligned}$$

which is an equivalent statement of $[R, R]$ being ad-invariant. $\square$

Since R is a classical r-matrix we can conclude by Lemma 3.17 that the bivector field

$$\pi_-(d) = \frac{1}{2}(r_d R - l_d R), \quad d \in D, \tag{44}$$

defines a Poisson structure on D and thus (D, π_-) is a Poisson Lie group.

Definition 3.29. Let (G, π) be a Poisson Lie group with a Lie subgroup H. We call H a *Poisson Lie subgroup* of (G, π) if the inclusion map is a Poisson immersion.

Proposition 3.30. *(U, π_U) is a Poisson Lie subgroup of (D, π_-).*

Proof. In order to prove that the inclusion $i : U \to D$ map is a Poisson immersion, we need to show for any $u \in U$,

$$\pi_-^\#(i(u))(T^*_{i(u)} D) \subset \operatorname{im} T_u i.$$

Since i is a Lie group homomorphism we find $i \circ l_u = l_{i(u)} \circ i$ for any $u \in U$, and therefore

$$T_u i \circ l_u = T_e(i \circ l_u) = (l_{i(u)} \circ i) = l_{i(u)} \circ T_e i,$$

from which we conclude that $\operatorname{im} T_u i = l_{i(u)} \operatorname{im} T_e i$. Also $T^*_{i(u)} D = (l^{-1}_{i(u)})^* T^*_e D = (l^{-1}_{i(u)})^* \mathfrak{d}^*$ and therefore we ought to prove

$$\big(l^{-1}_{i(u)} \pi_-(i(u))\big)^{\#}(\mathfrak{d}^*) = l^{-1}_{i(u)}\big(\pi^{\#}_-(i(u))((l^{-1}_{i(u)})^* \mathfrak{d}^*)\big) \subset \operatorname{im} T_e i. \tag{45}$$

We therefore want to explicitly calculate $l^{-1}_{i(u)} \pi_-(i(u)) \in \wedge^2(\mathfrak{d})$. For any $\xi + X, \eta + Y \in \mathfrak{d}^*$ we get

$$\begin{aligned} 2(l^{-1}_{i(u)} \pi_-(i(u)))(\xi + X, \eta + Y) &= (l^{-1}_{i(u)} r_{i(u)} R - R)(\xi + X, \eta + Y) \\ &= \overline{\mathrm{Ad}}^{-1}_{i(u)} R(\xi + X, \eta + Y) - R(\xi + X, \eta + Y) \end{aligned} \tag{46}$$

where $\overline{\mathrm{Ad}}$ denotes the adjoint representation of D on $\mathfrak{d}$. In the proof of Lemma 3.28 we showed the identity $\overline{\mathrm{Ad}}^{\vee}_d = \#^{-1} \circ \overline{\mathrm{Ad}}_d \circ \#$ for any $d \in D$. Also since the action of U on $\mathfrak{d}$ by (29) we find that $\overline{\mathrm{Ad}}_{i(u)} a = u\dot{u}$. If $\xi + X \in \mathfrak{d}^*$ we then find

$$\begin{aligned} \overline{\mathrm{Ad}}^{\vee}_{i(u)}(\xi + X) &= \#^{-1}(\overline{\mathrm{Ad}}_{i(u)}(X + \xi)) = \#^{-1}(u \cdot (X + \xi)) \\ &= \mathrm{Ad}^{\vee}_u \xi + \mathrm{Ad}_u X + (\mathrm{Ad}^{\vee}_u \xi) \lrcorner (r^{-1}_u \pi_U(u)). \end{aligned}$$

Using the above we compute

$$\begin{aligned} &\overline{\mathrm{Ad}}^{-1}_{i(u)} R(\xi + X, \eta + Y) \\ &= R(\mathrm{Ad}^{\vee}_u \xi + \mathrm{Ad}_u X + (\mathrm{Ad}^{\vee}_u \xi) \lrcorner (r^{-1}_u \pi_U(u)), \mathrm{Ad}^{\vee}_u \eta + \mathrm{Ad}_u Y + (\mathrm{Ad}^{\vee}_u \eta) \lrcorner (r^{-1}_u \pi_U(u))) \\ &= (r^{-1}_u \pi_U(u))(\mathrm{Ad}^{\vee}_u \xi, \mathrm{Ad}^{\vee}_u \eta) - (r^{-1}_u \pi_U(u))(\mathrm{Ad}^{\vee}_u \eta, \mathrm{Ad}^{\vee}_u \xi) \\ &\quad + (\mathrm{Ad}^{\vee}_u \eta)(\mathrm{Ad}_u X) - (\mathrm{Ad}^{\vee}_u \xi)(\mathrm{Ad}_u Y) \\ &= 2(\mathrm{Ad}^{-1}_u r^{-1}_u \pi_U(u))(\xi, \eta) + \eta(X) - \xi(Y) \\ &= 2(l^{-1}_u \pi_U(u))(\xi, \eta) + R(\xi + X, \eta + Y). \end{aligned}$$

We note that $T_e i : \mathfrak{u} \to \mathfrak{d}$ and therefore if $\xi + X \in \mathfrak{d}$ then $(T_e i)^*(\xi + X) = \xi$. By combining the above with (46) we observe

$$\begin{aligned} &(l^{-1}_{i(u)} \pi_-(i(u)))(\xi + X, \eta + Y) = (l^{-1}_u \pi_U(u))(\xi, \eta) \\ &= (l^{-1}_u \pi_U(u))((T_e i)^*(\xi + X), (T_e i)^*(\eta + Y)) = (T_e i(l^{-1}_u \pi_U(u)))(\xi + X, \eta + Y) \end{aligned}$$

and therefore $(l^{-1}_{i(u)} \pi_-(i(u)) = T_e i(l^{-1}_u \pi_U(u))$. By Lemma 2.2 we conclude that (45) holds, and therefore i is a Poisson immersion. Therefore U is a Poisson Lie subgroup of (D, π_-), but we still need to check that the induced Poisson structure on U by π_- is exactly π_U by showing that π_U and π_- are i-related. We recall $i \circ l_u = l_{i(u)} \circ i$ and observe

$$\pi_-(i(u)) = l_{i(u)}(T_e i(l^{-1}_u \pi_U(u))) = T_u(l_{i(u)} \circ i \circ l^{-1}_u)(\pi_U(u)) = T_u i(\pi_U(u)),$$

which completes the proof. □

Pick any $\mathfrak{l} \in \mathcal{L}(\mathfrak{d})$. The orbit $D \cdot \mathfrak{l}$ by the adjoint action of D on $\mathcal{L}(\mathfrak{d})$ is a smooth manifold, as it is identified with $D/D_{\mathfrak{l}}$ by [22, Proposition 15.5], where $D_{\mathfrak{l}}$ is the stabilizer subgroup of $\mathfrak{l}$ in D. We specifically name the action

$$\alpha : D \times D \cdot \mathfrak{l} \to D \cdot \mathfrak{l} : (d, \mathfrak{l}') \mapsto d \cdot \mathfrak{l}'$$

and we note that α is transitive. From α we define the maps $\alpha_d : D \cdot \mathfrak{l} \to D \cdot \mathfrak{l} : \mathfrak{l}' \mapsto d \cdot \mathfrak{l}'$ and $\alpha_{\mathfrak{l}'} : D \to D \cdot \mathfrak{l} : d \mapsto d \cdot \mathfrak{l}'$. Because α is transitive we observe that $\alpha_{\mathfrak{l}}$ is a submersion. We define a bivector field Π on $D \cdot \mathfrak{l}$ by

$$\Pi(\mathfrak{l}') = \tfrac{1}{2} T_e \alpha_{\mathfrak{l}'}(R), \quad \mathfrak{l}' \in D \cdot \mathfrak{l}. \tag{47}$$

Proposition 3.31. Π *defines a Poisson structure on* $D \cdot \mathfrak{l}$.

Proof. For any $d \in D$ we observe that $\alpha_{d \cdot \mathfrak{l}} = \alpha_{\mathfrak{l}} \circ r_d$ and therefore

$$2\Pi(\alpha_{\mathfrak{l}}(d)) = 2\Pi(d \cdot \mathfrak{l}) = T_e \alpha_{d \cdot \mathfrak{l}}(R) = T_e(\alpha_{\mathfrak{l}} \circ r_d)(R) = T_d \alpha_{\mathfrak{l}}(r_d R) = T_d \alpha_{\mathfrak{l}}(R^r(d))$$

from which we conclude that 2Π and R^r, the right invariant multivector field on D defined by R, are $\alpha_{\mathfrak{l}}$-related. By Proposition 2.5 and Lemma 3.18 we then find that

$$4[\Pi, \Pi](d \cdot \mathfrak{l}) = [2\Pi, 2\Pi](\alpha_{\mathfrak{l}}(d)) = T_d \alpha_{\mathfrak{l}}(2 r_d [R, R]) = 2 T_e \alpha_{d \cdot \mathfrak{l}}([R, R]) \tag{48}$$

for any $d \in D$.

Take any $\mathfrak{l}' \in D \cdot \mathfrak{l}$ and $v \in T^*_{\mathfrak{l}'}(D \cdot \mathfrak{l})$. Then $f := v \circ T_e \alpha_{\mathfrak{l}'} \in \mathfrak{d}^*$, and since α describes the adjoint action we find that $\ker T_e \alpha_{\mathfrak{l}'} = \mathfrak{d}_{\mathfrak{l}'} = \{a \in \mathfrak{d} : [a, \mathfrak{l}']_{\mathfrak{d}} \subset \mathfrak{l}'\}$, the normalizer subalgebra of $\mathfrak{l}'$ in $\mathfrak{d}$. Therefore $f \in \mathfrak{d}_{\mathfrak{l}'}^{\perp}$. We know $\mathfrak{l}' \subset \mathfrak{d}_{\mathfrak{l}'}$ since $\mathfrak{l}'$ a subalgebra and then $f \in \mathfrak{l}'^{\perp}$. Because $\mathfrak{l}'$ is isotropic we find $\#(\mathfrak{l}'^{\perp}) \subset \mathfrak{l}'$ and as such $\#(f) \in \mathfrak{l}'$.

To determine $[\Pi, \Pi](\mathfrak{l}')$ we apply it to $v_j \in T^*_{\mathfrak{l}'}(D \cdot \mathfrak{l})$ for $j = 1, 2, 3$. By (48) this equals applying $[R, R]$ to $f_j := v_j \circ T_e \alpha_{\mathfrak{l}'}$ up to a scalar. By the previous paragraph all $\#(f_j) \in \mathfrak{l}'$ and by (43) we then find

$$[\Pi, \Pi](\mathfrak{l}')(v_1, v_2, v_3) = \tfrac{1}{2}[R, R](f_1, f_2, f_3) = -\tfrac{1}{2}\langle \# f_1, [\# f_2, \# f_3]_{\mathfrak{d}} \rangle = 0$$

since $\mathfrak{l}'$ is a Lagrangian subalgebra. Therefore $[\Pi, \Pi]$ equals zero everywhere, which completes our proof. $\square$

The following lemma is not needed in defining the Lu-Evens Poisson structure, but will be used in the proof of Lemma 7.6.

Lemma 3.32. *Let* $\mathfrak{l}' \in D \cdot \mathfrak{l}$ *such that* $\mathfrak{l}'$ *is its own normalizer, i.e.* $\mathfrak{d}_{\mathfrak{l}'} = \mathfrak{l}'$. *Then there exists a linear isomorphism* $\phi : \mathfrak{l}' \to T^*_{\mathfrak{l}'}(D \cdot \mathfrak{l})$ *such that for any* $a, b \in \mathfrak{l}'$ *we find*

$$\Pi(\mathfrak{l}')(\phi(a), \phi(b)) = \langle \mathrm{pr}_{\mathfrak{u}}\, a, b \rangle,$$

where $\mathrm{pr}_{\mathfrak{u}} : \mathfrak{d} \to \mathfrak{u}$ *is the projection along* $\mathfrak{u}^*$. *Furthermore,*

$$(T_e \alpha_{\mathfrak{l}'}) * \circ \phi = \chi,$$

where $\chi : \mathfrak{l}' \to \mathfrak{l}'^{\perp} : a \mapsto \langle a, \cdot \rangle$.

Proof. We note that the map $T_e\alpha_{\mathfrak{l}'} : \mathfrak{d} \to T_{\mathfrak{l}'}(D \cdot \mathfrak{l})$ has kernel $\mathfrak{d}_{\mathfrak{l}'} = \mathfrak{l}'$, and therefore there exists a linear isomorphism $\psi : \mathfrak{d}/\mathfrak{l}' \to T_{\mathfrak{l}'}(D \cdot \mathfrak{l})$ such that $T_e\alpha_{\mathfrak{l}'} = \psi \circ \rho$, where $\rho : \mathfrak{d} \to \mathfrak{d}/\mathfrak{l}'$ the canonical projection. We note that $\psi^* : T^*_{\mathfrak{l}'}(D \cdot \mathfrak{l}) \to (\mathfrak{d}/\mathfrak{l}')^*$ also is an isomorphism while $\rho^* : (\mathfrak{d}/\mathfrak{l}')^* \to \mathfrak{d}^*$ clearly has image $\mathfrak{l}'^{\perp}$ and therefore defines and defines an isomorphism $\overline{\rho^*} : (\mathfrak{d}/\mathfrak{l}')^* \to \mathfrak{l}'^{\perp}$. We finally notice that $\chi : \mathfrak{l}' \to \mathfrak{l}'^{\perp} : X + \xi \mapsto \langle X + \xi, \cdot \rangle = \xi + X$ is an isomorphism since $\mathfrak{l}'$ is Lagrangian. This allows us to define the isomorphism $\phi = (\psi^*)^{-1} \circ (\overline{\rho^*})^{-1} \circ \chi$, for which we notice

$$(T_e\alpha_{\mathfrak{l}'})^* \circ \phi = \rho^* \circ \psi^* \circ \phi = \rho^* \circ (\overline{\rho^*})^{-1} \circ \chi = \overline{\rho^*} \circ (\overline{\rho^*})^{-1} \circ \chi = \chi.$$

By combining the above with (47) we calculate for any $X + \xi, Y + \eta \in \mathfrak{l}'$,

$$\begin{aligned}
\Pi(\mathfrak{l}')(\phi(X+\xi), \phi(Y+\eta)) &= \tfrac{1}{2} T_e\alpha_{\mathfrak{l}'}(R)(\phi(X+\xi), \phi(Y+\eta)) \\
&= \tfrac{1}{2} R(\chi(X+\xi), \chi(Y+\eta)) = \tfrac{1}{2} R(\xi + X, \eta + Y) = \tfrac{1}{2}(\eta + Y)((\xi + X) \lrcorner R) \\
&= \tfrac{1}{2}\langle X - \xi, Y + \eta\rangle = \tfrac{1}{2}\langle (X - \xi) + (X + \xi), Y + \eta\rangle = \langle X, Y + \eta\rangle \\
&= \langle \mathrm{pr}_{\mathfrak{u}}(X + \xi), Y + \eta\rangle,
\end{aligned}$$

which completes the proof. □

Proposition 3.33. *$(D \cdot \mathfrak{l}, \Pi)$ is a (D, π_-)-homogeneous space.*

Proof. We already observed that the action α is transitive. By the definition of Π we find

$$\Pi(d \cdot \mathfrak{l}') = \tfrac{1}{2} T_e\alpha_{d\cdot\mathfrak{l}'}(R) = T_d\alpha_{\mathfrak{l}'}(\tfrac{1}{2} r_d R) = T_d\alpha_{\mathfrak{l}'}(\tfrac{1}{2} l_d R + \pi_-(d))$$

and since $\alpha_{\mathfrak{l}'} \circ l_d = \alpha_d \circ \alpha_{\mathfrak{l}'}$ this gives us

$$\Pi(d \cdot \mathfrak{l}') = T_{\mathfrak{l}'}\alpha_d(T_e\alpha_{\mathfrak{l}'}(\tfrac{1}{2}R)) + T_d\alpha_{\mathfrak{l}'}(\pi_-(d)) = T_{\mathfrak{l}'}\alpha_d(\Pi(\mathfrak{l}')) + T_d\alpha_{\mathfrak{l}'}(\pi_-(d))$$

and thus we conclude that α is a Poisson map. □

We recall that the inclusion map $i : U \to D$ is Poisson, and if we define the action $\tilde{\alpha} : U \times D \cdot \mathfrak{l} \to D \cdot \mathfrak{l}$ by

$$\tilde{\alpha}(u, \mathfrak{l}') = \alpha(i(u), \mathfrak{l}'), \qquad u \in U, \quad \mathfrak{l}' \in D \cdot \mathfrak{l}.$$

We then define $U \cdot \mathfrak{l} \subset D \cdot \mathfrak{l}$ as the orbit of $\mathfrak{l}$ by the action $\tilde{\alpha}$, which is a smooth manifold as it is identified with $U/U_{\mathfrak{l}}$ by [22, Proposition 15.5], where $U_{\mathfrak{l}}$ is the stabilizer subgroup of $\mathfrak{l}$ in U. If we denote by $\tilde{\alpha}_{\mathfrak{l}'} : U \to U \cdot \mathfrak{l} : u \mapsto \tilde{\alpha}(u, \mathfrak{l}')$ and by $i_{\mathfrak{l}}$ the inclusion map $U \cdot \mathfrak{l} \to D \cdot \mathfrak{l}$ we observe the following diagram to be commutative.

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d, "\tilde{\alpha}_{\mathfrak{l}'}"'] & D \arrow[d, "\alpha_{\mathfrak{l}'}"] \\
U \cdot \mathfrak{l} \arrow[r, "i_{\mathfrak{l}}"'] & D \cdot \mathfrak{l}
\end{tikzcd} \tag{49}$$

Proposition 3.34. *$U \cdot \mathfrak{l}$ is a Poisson submanifold of* $(D \cdot \mathfrak{l}, \Pi)$.

Proof. Take any $\mathfrak{l}' \in U \cdot \mathfrak{l}$ and any $v, w \in T^*_{\mathfrak{l}'}(D \cdot \mathfrak{l})$. We observe

$$\Pi(\mathfrak{l}')(v,w) = \tfrac{1}{2}(T_e\alpha_{\mathfrak{l}'}(R))(v,w) = \tfrac{1}{2}(R)((T_e\alpha_{\mathfrak{l}'})^*v, (T_e\alpha_{\mathfrak{l}'})^*w) = w(\tfrac{1}{2}T_e\alpha_{\mathfrak{l}'}(((T_e\alpha_{\mathfrak{l}'})^*v)\lrcorner R))$$

and we therefore conclude that $\Pi^\#(\mathfrak{l}')(v) = \frac{1}{2}T_e\alpha_{\mathfrak{l}'}(((T_e\alpha_{\mathfrak{l}'})^*v)\lrcorner R))$. We recall from the proof of Proposition 3.31 that $\xi + X := (T_e\alpha_{\mathfrak{l}'})^*v \in \mathfrak{d}^*$ and $X + \xi = \#(\xi + X) \in \mathfrak{l}' \subset \ker T_e\alpha_{\mathfrak{l}'}$. We also recall that $(\xi + X)\lrcorner R = X - \xi$ and therefore

$$\Pi^\#(\mathfrak{l}')(v) = \tfrac{1}{2}T_e\alpha_{\mathfrak{l}'}((\xi + X)\lrcorner R) = \tfrac{1}{2}T_e\alpha_{\mathfrak{l}'}((X - \xi) + (X + \xi)) = T_e\alpha_{\mathfrak{l}'}(X).$$

We recall $T_e i : \mathfrak{u} \to \mathfrak{d} : X \mapsto X$ and therefore by diagram (49) we observe

$$\Pi^\#(\mathfrak{l}')(v) = T_e\alpha_{\mathfrak{l}'}(T_e iX) = T_e(\alpha_{\mathfrak{l}'} \circ i)(X) = T_e(i_{\mathfrak{l}} \circ \tilde{\alpha}_{\mathfrak{l}'})(X) = T_{\mathfrak{l}'}i_{\mathfrak{l}}(T_e\tilde{\alpha}_{\mathfrak{l}'}(X)) \in \operatorname{im} T_{\mathfrak{l}'}i_{\mathfrak{l}}$$

and therefore $i_{\mathfrak{l}} : U \cdot \mathfrak{l} \to D \cdot \mathfrak{l}$ is a Poisson immersion since $i_{\mathfrak{l}}(\mathfrak{l}') = \mathfrak{l}'$, from which we conclude that $U \cdot \mathfrak{l}$ is a Poisson submanifold of $(D \cdot \mathfrak{l}, \Pi)$. $\square$

By the above proposition we know that there exists an induced Poisson structure on $U \cdot \mathfrak{l}$ defined by a bivector field Π', that Π' and Π are $i_{\mathfrak{l}}$-related and that therefore $i_{\mathfrak{l}}$ is a Poisson map from the Poisson manifold $(U \cdot \mathfrak{l}, \Pi')$ to the $(D \cdot \mathfrak{l}, \Pi)$.

Lemma 3.35. *Let (M, π_M) and (N, π_N) be Poisson manifolds and let $f : M \to N$ be a Poisson map. Let $M_0 \subset M$ and $N_0 \subset N$ be Poisson submanifolds such that $f(M_0) \subset N_0$, then the induced map $f_0 : M_0 \to N_0$ is a Poisson map.*

Proof. If we denote the inclusion maps i_M and i_N we observe that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ M_0 \arrow[u, "i_M"] \arrow[r, "f_0"] & N_0 \arrow[u, "i_N"'] \end{tikzcd} \tag{50}$$

Let $\pi_{M_0} \in \mathfrak{X}_2(M_0)$ and $\pi_{N_0} \in \mathfrak{X}_2(N_0)$ define the induced Poisson structures on M_0 and N_0 respectively. Since π_{M_0} and π_M are i_M-related while π_M and π_N are f-related we find that π_{M_0} and π_N are $(f \circ i_M)$-related. By diagram (50) this means that π_{M_0} and π_N are $(i_N \circ f_0)$-related and therefore

$$\pi_N(i_N(f_0(m))) = T_m(i_N \circ f_0)(\pi_{M_0}(m)) = T_{f_0(m)}i_N(T_m f_0(\pi_{M_0}(m)))$$

for any $m \in M_0$. On the other hand since i_N is a Poisson immersion we know $T_{f_0(m}$ is injective while

$$\pi_N(i_N(f_0(m))) = T_{f_0(m)}i_N(\pi_{N_0}(f_0(m)))$$

and therefore $\pi_{N_0}(f_0(m)) = T_m f_0(\pi_{M_0}(m))$ for any $m \in M_0$. We conclude that f_0 is a Poisson map since π_{M_0} and π_{N_0} are f_0-related. $\square$

Corollary 3.36. *$(U \cdot \mathfrak{l}, \Pi')$ is a (U, π_U)-homogeneous space.*

Proof. By Proposition 3.30 (U, π_U) is a Poisson Lie subgroup of (D, π_-) with inclusion map i. Its orbit $(U \cdot \mathfrak{l}, \Pi')$ is a Poisson submanifold of $(D \cdot \mathfrak{l}, \Pi)$ with inclusion map $i_\mathfrak{l}$ by Proposition 3.34. It is clear that $(U \times U \cdot \mathfrak{l}, \pi_U \oplus \Pi')$ is a Poisson submanifold of $(D \times D \cdot \mathfrak{l}, \pi_- \oplus \Pi)$ by the map

$$i \times i_\mathfrak{l} : U \times U \cdot \mathfrak{l} \to D \times D \cdot \mathfrak{l} : (u, \mathfrak{l}') \mapsto (i(u), i_\mathfrak{l}(\mathfrak{l}')).$$

Because $(D \cdot \mathfrak{l}, \Pi)$ is a (D, π_-)-homogeneous space by Proposition 3.33 we know that the action map α is a Poisson map. By the definition of $U \cdot \mathfrak{l}$ we observe that $\alpha(U \times U \cdot \mathfrak{l}) \subset U \cdot \mathfrak{l}$. By Lemma 3.35 we conclude that the induced action map

$$\alpha_0 : U \times U \cdot \mathfrak{l} \to U \cdot \mathfrak{l}$$

is a Poisson map. Since α_0 clearly is transitive we conclude that $(U \cdot \mathfrak{l}, \Pi')$ is a (U, π_U)-homogeneous space. $\square$

Since $(U \cdot \mathfrak{l}, \Pi')$ is a (U, π_U)-homogeneous space we recall from Drinfeld's theorem (Theorem 3.23) that there exists a Drinfeld map $\mathrm{P} : U \cdot \mathfrak{l} \to \mathcal{L}(\mathfrak{d})$, which maps any $\mathfrak{l}' \in U \cdot \mathfrak{l}$ to its associated Lagrangian subalgebra of $\mathfrak{d}$.

Theorem 3.37 ([5, Theorem 2.21])**.** *For any $\mathfrak{l} \in \mathcal{L}(\mathfrak{d})$, the Lagrangian subalgebra of $\mathfrak{d}$ associated to $(U \cdot \mathfrak{l}, \Pi')$ at $\mathfrak{l}$ equals*

$$\mathfrak{u}_\mathfrak{l} \oplus (\mathfrak{u} \oplus \mathfrak{u}_\mathfrak{l}^\perp) \cap \mathfrak{l}, \tag{51}$$

where $\mathfrak{u}_\mathfrak{l}$ is the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{u}$.

Proof. We recall that the Lagrangian subalgebra of $\mathfrak{d}$ associated to $(U \cdot \mathfrak{l}, \Pi')$ at $\mathfrak{l}$, which we will denote by $\mathfrak{l}'$, is defined as

$$\mathfrak{l}' = \{X + \xi : X \in \mathfrak{u}, \xi \in \mathfrak{u}_\mathfrak{l}^\perp, (\chi_\mathfrak{l}\xi) \lrcorner (\psi_\mathfrak{l}^{-1}\Pi'(\mathfrak{l})) = X + \mathfrak{u}_\mathfrak{l}\},$$

where the maps $\psi_\mathfrak{l} : \mathfrak{u}/\mathfrak{u}_\mathfrak{l} \to T_\mathfrak{l}(U \cdot \mathfrak{l})$ and $\chi_\mathfrak{l} : \mathfrak{u}_\mathfrak{l}^\perp \to (\mathfrak{u}/\mathfrak{u}_\mathfrak{l})^*$ are isomorphisms. We will prove that $\mathfrak{l}'$ equals (51).

Take any $X + \xi \in \mathfrak{l}'$, then $\xi \in \mathfrak{u}_\mathfrak{l}^\perp$, and thus there exists $Y \in \mathfrak{u}$ such that $\xi + Y \in \mathfrak{d}_\mathfrak{l}^\perp$. We know by the proof of Proposition 3.34 that there exists some $v \in T_\mathfrak{l}^*(D \cdot \mathfrak{l})$ such that $(T_e\alpha_\mathfrak{l})^* v = \xi + Y$ and $\Pi^\#(\mathfrak{l})(v) = T_e\alpha_\mathfrak{l}(Y)$. Since $\mathfrak{l}$ lies in both $D \cdot \mathfrak{l}$ and $U \cdot \mathfrak{l}$ we find that $i_\mathfrak{l}(\mathfrak{l}) = \mathfrak{l}$. Since Π and Π' are $i_\mathfrak{l}$-related we also find

$$\begin{aligned}\Pi^\#(\mathfrak{l})(v) = v \lrcorner \Pi(\mathfrak{l}) &= v \lrcorner (T_\mathfrak{l} i_\mathfrak{l} \Pi'(\mathfrak{l})) = v \lrcorner ((T_\mathfrak{l} i_\mathfrak{l} \circ \psi_\mathfrak{l} \circ \psi_\mathfrak{l}^{-1})\Pi'(\mathfrak{l})) \\ &= (T_\mathfrak{l} i_\mathfrak{l} \circ \psi_\mathfrak{l})((T_\mathfrak{l} i_\mathfrak{l} \circ \psi_\mathfrak{l})^* v) \lrcorner (\psi_\mathfrak{l}^{-1}\Pi'(\mathfrak{l}))\end{aligned} \tag{52}$$

By differentiating the maps in diagram (49) and extending it with $\mathfrak{u}/\mathfrak{u}_\mathfrak{l}$ we get the following commuting diagram.

$$\begin{tikzcd}
\mathfrak{u}/\mathfrak{u}_\mathfrak{l} \arrow[dr, "\psi_\mathfrak{l}"'] & \mathfrak{u} \arrow[l, "\rho_\mathfrak{l}"'] \arrow[r, "T_e i"] \arrow[d, "T_e\tilde{\alpha}_\mathfrak{l}"] & \mathfrak{d} \arrow[d, "T_e\alpha_\mathfrak{l}"] \\
& T_\mathfrak{l}(U \cdot \mathfrak{l}) \arrow[r, "T_\mathfrak{l} i_\mathfrak{l}"'] & T_\mathfrak{l}(D \cdot \mathfrak{l})
\end{tikzcd} \tag{53}$$

We recall that $\chi_{\mathfrak{l}} \circ \rho_{\mathfrak{l}}^* = \mathrm{id}_{(\mathfrak{u}/\mathfrak{u}_{\mathfrak{l}})^*}$ and we therefore conclude by the above diagram,

$$\begin{aligned}(T_{\mathfrak{l}} i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}})^* &= \chi_{\mathfrak{l}} \circ \rho_{\mathfrak{l}}^* \circ (T_{\mathfrak{l}} i_{\mathfrak{l}} \psi_{\mathfrak{l}})^* = \chi_{\mathfrak{l}} \circ (T_{\mathfrak{l}} \circ i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}} \circ \rho_{\mathfrak{l}})^* = \chi_{\mathfrak{l}} \circ (T_e \alpha_{\mathfrak{l}} \circ T_e i)^* \\ &= \chi_{\mathfrak{l}} \circ (T_e i)^* \circ (T_e \alpha_{\mathfrak{l}})^*\end{aligned}$$

and therefore,

$$(T_{\mathfrak{l}} i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}})^* v = \chi_{\mathfrak{l}} (T_e i)^* (T_e \alpha_{\mathfrak{l}})^* v = \chi_{\mathfrak{l}} (T_e i)^* (\xi + Y) = \chi_{\mathfrak{l}} \xi.$$

By combining the above equation with (52) and by using diagram (53) again we observe,

$$\begin{aligned}\Pi^{\#}(\mathfrak{l})(v) &= (T_{\mathfrak{l}} i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}})\big((\chi_{\mathfrak{l}} \xi) \lrcorner (\psi_{\mathfrak{l}}^{-1} \Pi'(\mathfrak{l}))\big) = (T_{\mathfrak{l}} i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}})(X + \mathfrak{u}_{\mathfrak{l}}) = (T_{\mathfrak{l}} i_{\mathfrak{l}} \circ \psi_{\mathfrak{l}})(\rho_{\mathfrak{l}}(X)) \\ &= T_e \alpha_{\mathfrak{l}} (T_e i(X)) = T_e \alpha_{\mathfrak{l}}(X).\end{aligned}$$

Therefore $X - Y \in \ker T_e \alpha_{\mathfrak{l}} = \mathfrak{d}_{\mathfrak{l}}$ and thus $X - Y \in \mathfrak{u} \cap \mathfrak{d}_{\mathfrak{l}} = \mathfrak{u}_{\mathfrak{l}}$.

It is clear that $Y + \xi \in \mathfrak{u} \oplus \mathfrak{u}_{\mathfrak{l}}^{\perp}$. Since $\mathfrak{l}$ is Lagrangian and a subset of $\mathfrak{d}_{\mathfrak{l}}$ we find

$$Y + \xi = \#(\xi + Y) \subset \#(\mathfrak{d}_{\mathfrak{l}}^{\perp}) \subset \#(\mathfrak{l}^{\perp}) = \mathfrak{l},$$

from which we finally conclude

$$X + \xi = (X - Y) + (Y + \xi) \in \mathfrak{u}_{\mathfrak{l}} \oplus (\mathfrak{u} \oplus \mathfrak{u}_{\mathfrak{l}}^{\perp}) \cap \mathfrak{l},$$

and therefore $\mathfrak{l}'$ is a subset of (51). We easily observe that (51) is isotropic since

$$\langle \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{\mathfrak{l}} \rangle = 0, \qquad \langle \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u} \oplus \mathfrak{u}_{\mathfrak{l}}^{\perp} \rangle = 0, \qquad \langle \mathfrak{l}, \mathfrak{l} \rangle = 0,$$

and since $\mathfrak{l}'$ is maximal isotropic we conclude that $\mathfrak{l}'$ equals (51). □

Let U^* be a connected and simply connected Lie group with Lie algebra $\mathfrak{u}^*$. Since there is a symmetry between $\mathfrak{u}$ and $\mathfrak{u}^*$ in the definition of $\mathfrak{d}$, we expect that there also exists a symmetry between U and U^* in D. Lemma 3.31 shows us that the Poisson structure π_- restricted to U is a Poisson structure π_U. In the same way we can restrict π_- to U^* and call it π_{U^*}. It must be a multiplicative bivector field and by (23) we calculate the intrinsic derivative

$$\begin{aligned}d_e \pi_{U^*}(\xi)(X, Y) &= (-\tfrac{1}{2} \overline{\mathrm{ad}}_{\xi} R)(X, Y) = -\tfrac{1}{2}(R([\xi, X]_{\mathfrak{d}}, Y) + R(X, [\xi, Y]_{\mathfrak{d}})) \\ &= -\tfrac{1}{2}((\mathrm{ad}_X^{\vee} \xi)(Y) - (\mathrm{ad}_Y^{\vee} \xi)(X)) = \xi(-[X, Y]_{\mathfrak{u}}),\end{aligned}$$

for any $X, Y \in \mathfrak{u}$ and $\xi \in \mathfrak{u}^*$. We therefore conclude by Lemma 3.15 that π_{U^*} defines a Poisson structure on U^*. Clearly (U^*, π_{U^*}) is a Poisson Lie subgroup of (D, π_-). The orbit $U^* \cdot \mathfrak{l}$ is a Poisson submanifold of $(D \cdot \mathfrak{l}, \Pi)$ and a (U^*, π_{U^*})-homogeneous space, but we omit the proofs here as they are very similar the proofs of Proposition 3.34 and Corollary 3.36. Since $U \cdot \mathfrak{l}$ and $U^* \cdot \mathfrak{l}$ are both Poisson submanifolds of $D \cdot \mathfrak{l}$ we observe by Proposition 3.11 that $(U \cdot \mathfrak{l}) \cap (U^* \cdot \mathfrak{l})$ contains the symplectic leaf through $\mathfrak{l}$.

4. Moment maps and convexity theorems

Let G be a Lie group with Lie algebra $\mathfrak{g}$ which acts on a Poisson manifold (M, π) by the smooth left action $\sigma : G \times M \to M : (g, m) \mapsto gm$. For every $g \in G$ we can then define the diffeomorphism $\sigma_g : M \to M : m \mapsto gm$. We say that the action σ leaves π invariant if π is σ_g-related to itself for every $g \in G$, and the action then describes a symmetry of (M, π).

By this action we can also define for $X \in \mathfrak{g}$ a vector field σ_X on M called the *infinitesimal generator* by

$$\sigma_X(m) = \frac{d}{dt}\Big|_{t=0} \sigma(\exp(tX), m)$$

The mapping $X \mapsto \sigma_X$ is a Lie algebra anti-homomorphism [22, Lemma 15.1], i.e. $\sigma_{[X,Y]} = -[\sigma_X, \sigma_Y]$.

Definition 4.1. A map $J : M \to \mathfrak{g}^*$ defines functions $J_X \in C^\infty(M)$ for $X \in \mathfrak{g}$ by $J_X(m) = J(m)(X)$. We call J a *moment map* for the action σ if the Hamiltonian vector field of J_X equals the infinitesimal generator of X, i.e. $\pi^\#(dJ_X) = \sigma_X$. We then also call σ a *Hamiltonian action*.

If we define $\mathfrak{X}_H(M)$ as the set of Hamiltonian vector fields on M, then by the above definition $\sigma_X \in \mathfrak{X}_H(M)$ for any $X \in \mathfrak{g}$ and a Hamiltonian action σ. Then also the diagram below commutes.

$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "X \mapsto J_X"'] \arrow[dr, "X \mapsto \sigma_X"] & \\ C^\infty(M) \arrow[rr, "h \mapsto X_h"'] & & \mathfrak{X}_H(M) \end{tikzcd}$$

Interestingly the mapping $X \mapsto J_X$ for a moment map J is a Lie algebra homomorphism, i.e. $J_{[X,Y]} = \{J_X, J_Y\}$, since the other two maps in the diagram are anti-homomorphisms.

There then exist the following convexity theorems relating to moment maps.

Theorem 4.2 (Atiyah [1]-Guillemin-Sternberg [8])**.** *If $J : M \to \mathfrak{t}^*$ is a moment map for a Hamiltonian torus T action on a compact symplectic manifold M, then the image $J(M)$ is the convex hull of $J(M^T)$, where M^T denotes the fixed point set of the action.*

Theorem 4.3 (Duistermaat [4])**.** *If $J : M \to \mathfrak{t}^*$ is a moment map for a Hamiltonian torus action on a compact symplectic manifold M and if Q is the fixed point set (or a connected component of it) of an anti-symplectic involution of M leaving J invariant, then $J(Q) = J(M)$ and it is the convex hull of $J(M^T \cap Q)$.*

Theorem 4.4 (Hilgert-Neeb-Plank [9, Theorem 4.1 *(i),(v)*])**.** *If $J : M \to \mathfrak{t}^*$ is a moment map for a Hamiltonian torus T action on a connected symplectic manifold M, such that J is proper, i.e. J is a closed mapping and $J^{-1}(Z)$ is compact for every $Z \in \mathfrak{t}^*$. Then $J(M)$ is a closed, locally polyhedral, convex set. More precisely, for each $m \in M$ there exists a neighborhood $U_{J(m)} \subset \mathfrak{t}^*$ of $J(m)$ such that $J(M) \cap U_{J(m)} = (J(m) + \Gamma_{J(m)}) \cap U_{J(m)}$, where $\Gamma_{J(m)} = \mathfrak{t}_m^\perp + C_m$. Here, $\mathfrak{t}_m$ denotes the Lie algebra of the stabilizer T_m of m, and $C_m \subset \mathfrak{t}_m^*$ denotes the cone which is spanned by the weights of the linearized action of T_m. The cone $\Gamma_{J(m)}$ is independent of the choice of preimage point of $J(m)$.*

We recall from the previous section that a Poisson structure foliates a manifold into symplectic leaves. Since the above convexity theorems only apply to symplectic manifolds, we discern the reason to why Poisson structures are particularly useful in proving Kostant's and Van den Ban's convexity theorems. Poisson structures are a means identifying particular submanifolds as symplectic, upon which the above convexity theorems then may be applied.

5. Decompositions of semisimple Lie algebras

Both Kostant's and Van den Ban's convexity theorems concern the Iwasawa decomposition of a semisimple Lie group. This section covers certain aspects of the structure of semisimple Lie algebras and semisimple Lie groups needed for the purpose of the present thesis, starting with the Cartan decomposition.

5.1. Cartan decomposition

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $B_{\mathfrak{g}}$ be its Killing form. We then call an automorphism τ on $\mathfrak{g}$ an *involution* if $\tau^2 = \mathrm{id}_{\mathfrak{g}}$, the identity map on $\mathfrak{g}$.

Lemma 5.1. *Let V be a real or complex vector space and τ an endomorphism of V such that $\tau^2 = \mathrm{id}_V$. Then τ only has eigenvalues 1 and -1, and V is a direct sum of the $(+1)$- and (-1)-eigenspaces.*

Proof. Let $(\cdot,\cdot)$ be an inner product on V. The bilinear form (or sesquilinear form if V is complex) $\langle\cdot,\cdot\rangle$ defined by

$$\langle X, Y\rangle = (X, Y) + (\tau X, \tau Y)$$

is easily shown to be an inner product of V. Since $\tau^2 = \mathrm{id}_V$ it is clear that $\langle \tau X, \tau Y\rangle = \langle X, Y\rangle$, and therefore τ is orthogonal if V is real and unitary if V is complex with respect to $\langle\cdot,\cdot\rangle$. Also if λ is an eigenvalue of τ and $v \neq 0$ such that $\tau v = \lambda v$, then $\lambda^2 v = \tau^2 v = v$ and therefore $\lambda = \pm 1$, while

$$V = \ker(\tau - \mathrm{id}_V) \oplus \ker(\tau + \mathrm{id}_V),$$

the direct sum of eigenspaces. □

Definition 5.2. We call an involution θ on $\mathfrak{g}$ a *Cartan involution* if and only if the symmetric bilinear form

$$B_\theta(X, Y) = -B_{\mathfrak{g}}(X, \theta Y), \quad X, Y \in \mathfrak{g} \tag{54}$$

is positive definite.

Since θ is an involution, its only possible eigenvalues are 1 and -1 by Lemma 5.1 and thus has corresponding eigenspaces, $\mathfrak{k}$ and $\mathfrak{p}$ respectively. We find the following inclusions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \tag{55}$$

due to θ being an automorphism, and we note that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$ while $\mathfrak{p}$ is not. We also note that if $X \in \mathfrak{k}$ then $\mathrm{ad}\, X$ preserves the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$. If $Y \in \mathfrak{p}$ then $\mathrm{ad}\, Y$ maps $\mathfrak{k}$ and $\mathfrak{p}$ into one another. Interestingly we then find that $\mathrm{ad}\, X \,\mathrm{ad}\, Y$ also maps $\mathfrak{k}$ and $\mathfrak{p}$ into one another, and as such

$$B_{\mathfrak{g}}(X, Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}\, X \,\mathrm{ad}\, Y) = \mathrm{Tr}_{\mathfrak{k}}(\mathrm{ad}\, X \,\mathrm{ad}\, Y) + \mathrm{Tr}_{\mathfrak{p}}(\mathrm{ad}\, X \,\mathrm{ad}\, Y) = 0.$$

This also gives $B_\theta(X,Y) = -B_\mathfrak{g}(X,\theta Y) = B_\mathfrak{g}(X,Y) = 0$, and we therefore conclude that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to both $B_\mathfrak{g}$ and B_θ. We recall that B_θ is positive definitie since θ is a Cartan involution and find as a result

$$\begin{aligned} B_\mathfrak{g}(X_1,X_2) = -B_\theta(X_1,\theta X_2) = -B_\theta(X_1,X_2) < 0, \quad \forall X_1,X_2 \in \mathfrak{k}, \\ B_\mathfrak{g}(Y_1,Y_2) = -B_\theta(Y_1,\theta Y_2) = B_\theta(Y_1,Y_2) > 0, \quad \forall Y_1,Y_2 \in \mathfrak{p}, \end{aligned}$$

from which we conclude that $B_\mathfrak{g}$ is negative (positive) definite on $\mathfrak{k}$ (on $\mathfrak{p}$).

Definition 5.3. We call a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a *Cartan decomposition* if and only if the subspaces bracket according to (55) and the Killing form $B_\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.

We have previously found that a Cartan involution leads to a Cartan decomposition, however one can also easily show the converse. Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We define the map $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(X+Y) = X - Y \qquad X \in \mathfrak{k},\, Y \in \mathfrak{p}.$$

It is clear that $\theta^2 = \mathrm{id}_\mathfrak{g}$ while θ is an automorphism as a consequence of (55) and therefore θ is an involution. It is easily checked that B_θ as defined in (54) is positive definite on both $\mathfrak{k}$ and $\mathfrak{p}$, and therefore θ is a Cartan involution.

5.2. Cartan decomposition in the complex case

If $\mathfrak{g}$ is a complex semisimple Lie algebra, we can regard it as a real vector space $\mathfrak{g}^\mathbb{R}$ by only allowing multiplication by real scalars. The Lie bracket of $\mathfrak{g}$ still defines a Lie bracket on $\mathfrak{g}^\mathbb{R}$ and therefore $\mathfrak{g}^\mathbb{R}$ is a real Lie algebra. If $\{v_1,...,v_n\}$ is a basis of $\mathfrak{g}$ we find that $\{v_1,...,v_n,iv_1,...,iv_n\}$ is a basis of $\mathfrak{g}^\mathbb{R}$. The Killing form $B_{\mathfrak{g}^\mathbb{R}}$ on $\mathfrak{g}^\mathbb{R}$ then equals $2\,\mathrm{Re}\,B_\mathfrak{g}$. The Killing form of $\mathfrak{g}$ is nondegenerate by the semisimplicity of $\mathfrak{g}$ and from this we conclude that $B_{\mathfrak{g}^\mathbb{R}}$ is also nondegenerate and therefore $\mathfrak{g}^\mathbb{R}$ is also semisimple. We call a $\mathfrak{u}$ a *real form* of $\mathfrak{g}$ if $\mathfrak{u}$ is a subalgebra of $\mathfrak{g}^\mathbb{R}$ such that $\mathfrak{g}^\mathbb{R} = \mathfrak{u} \oplus i\mathfrak{u}$ as vector spaces. If $\mathfrak{u}$ is also a compact subalgebra of $\mathfrak{g}^\mathbb{R}$, i.e. the Killing form $B_{\mathfrak{g}^\mathbb{R}}$ is negative definite on $\mathfrak{u}$, then $\mathfrak{u}$ is a *compact real form* of $\mathfrak{g}$. We mention without proof that every complex semisimple Lie algebra has a compact real form [11, Theorem 6.11].

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$, then

$$B_{\mathfrak{g}^\mathbb{R}}(iX,iY) = 2\,\mathrm{Re}\,B_\mathfrak{g}(iX,iY) = -2\,\mathrm{Re}\,B_\mathfrak{g}(X,X) = -B_{\mathfrak{g}^\mathbb{R}}(X,X) > 0, \quad \forall X,Y \in \mathfrak{u}.$$

and therefore $B_{\mathfrak{g}^\mathbb{R}}$ is positive definite on $i\mathfrak{u}$. Clearly $\mathfrak{k} = \mathfrak{u}$ and $\mathfrak{p} = i\mathfrak{u}$ bracket according to (55) and therefore $\mathfrak{g}^\mathbb{R} = \mathfrak{u} \oplus i\mathfrak{u}$ is a Cartan decomposition. We then also observe that the corresponding Cartan involution is complex conjugation with respect to $\mathfrak{u}$, the proof of which we omit. In light of previous observations, this means that any complex semisimple Lie algebra has a Cartan decomposition. It requires more work to prove, but this is also true for any real semisimple Lie algebra (see [11, Corollary 6.18]).

Moreover, if $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ regarded as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then $\mathfrak{k}$ is a compact real form and $\mathfrak{p} = i\mathfrak{k}$.

Conversely let $\mathfrak{g}$ be real semisimple Lie algebra. We define a complex vector space $\mathfrak{g}^{\mathbb{C}}$ by the real tensor product

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$$

which we call the *complexification* of $\mathfrak{g}$. We identify $\mathfrak{g}$ with the subspace $\mathfrak{g} \otimes 1 \subset \mathfrak{g}^{\mathbb{C}}$, and multiplication by a complex scalar is defined on $\mathfrak{g}^{\mathbb{C}}$ as $\lambda(X \otimes \mu) = X \otimes \lambda\mu$. The complexification is a complex Lie algebra as the bracket of $\mathfrak{g}$ extends to $\mathfrak{g}^{\mathbb{C}}$. It is clear that $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra, $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{g} \oplus i\mathfrak{g}$ as real vector spaces and thus $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$. We note that $B_{\mathfrak{g}^{\mathbb{C}}}$ restricted to $\mathfrak{g} \times \mathfrak{g}$ is equal to $B_{\mathfrak{g}}$ and therefore $\mathfrak{g}^{\mathbb{C}}$ is semisimple since $B_{\mathfrak{g}^{\mathbb{C}}}$ must be nondegenerate.

Now assume there exists a Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a subalgebra of $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$ by (55). Since $B_{(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}}$ is the real part of the complex linear extension of $2B_{\mathfrak{g}}$, and since $\mathfrak{u} \oplus i\mathfrak{u} = (\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$ we find that $\mathfrak{u}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}}$.

5.3. Global Cartan decomposition

Up until now we have only been concerned with the Cartan decomposition on the level of semisimple Lie algebras. In order to lift the decomposition to the group level we first define the following.

Definition 5.4. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We call G a *semisimple Lie group* if and only if $\mathfrak{g}$ is semisimple.

By the previous section there exists a Cartan decomposition of the Lie algebra of a semisimple Lie group. The next theorem gives an analogous result of the Cartan decomposition for the group.

Theorem 5.5 ([11, Theorem 6.31])**.** *Let G be a real semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then there exists a unique Lie group automorphism Θ of G with differential θ. Moreover,*

(a) $\Theta^2 = \mathrm{id}_G$,

(b) the subgroup of G fixed by Θ is K,

(c) the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism onto,

(d) K is closed,

(e) K contains the center Z of G,

(f) K is compact if and only if Z is finite.

The group automorphism Θ is called the *global Cartan involution*, while the diffeomorphism described in *(c)* is called the *global Cartan decomposition*. We note that in the literature the symbol θ usually denotes both the Cartan involution and the global Cartan involution, as the context of use generally makes it clear which of the two it signifies. In this thesis we adopt this convention.

If we now define $P = \exp \mathfrak{p}$, the global Cartan decomposition is often referred to as $G = KP$. Note though that P is not a subgroup of G as $\mathfrak{p}$ is not a subalgebra of $\mathfrak{g}$. However P is a closed submanifold of G and diffeomorphic to $\mathfrak{p}$ through $\exp : \mathfrak{p} \to P$. We also note that for $X \in \mathfrak{k}, Y \in \mathfrak{p}$ we get

$$\exp(X)\exp(Y)\exp(-X) = \exp(\mathrm{Ad}_{\exp X} Y) = \exp(e^{\mathrm{ad}_X} Y) \in \exp \mathfrak{p} = P$$

since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. Since K is the analytic subgroup of G with Lie algebra $\mathfrak{k}$ we then find that K acts on P by conjugation, and as such we can define a K-orbit $\mathcal{O}_p = \{kpk^{-1} : k \in K\}$ in P.

5.4. Iwasawa decomposition

Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let θ be the corresponding Cartan involution which in turn defines a positive definite symmetric bilinear form B_θ as in (54). Take any $H \in \mathfrak{p}$ and any $X, Y \in \mathfrak{g}$, then,

$$\begin{aligned} B_\theta(\mathrm{ad}(H)X, Y) &= -B([H, X], \theta Y) = B([X, H], \theta Y) = B(X, [H, \theta Y]) \\ &= B(X, \theta([\theta H, Y])) = -B(X, \theta([H, Y])) = B_\theta(X, \mathrm{ad}(H)Y) \end{aligned}$$

and we therefore find that $\mathrm{ad}(H)$ is self-adjoint with respect to B_θ. If we now choose $\mathfrak{a}$ as a maximal abelian subalgebra of $\mathfrak{p}$, then $\{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ is a commuting set of endomorphisms of $\mathfrak{g}$, and thus all of its elements preserve one another's eigenspaces. Because all $\mathrm{ad}(H)$ are self-adjoint the eigenvalues are real for any simultaneous eigenvector X of all $\mathrm{ad}\, H$ we can therefore find a functional $\alpha : \mathfrak{a} \to \mathbb{R}$ such that $\alpha(H)$ is the eigenvalue of $\mathrm{ad}(H)$ corresponding to X. The simultaneous eigenspace corresponding to $\alpha \in \mathfrak{a}^*$ is then defined as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}\}.$$

We now call α a *restricted root* of $\mathfrak{g}$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. We denote the set of these roots as $\Delta(\mathfrak{g}, \mathfrak{a})$. The corresponding space $\mathfrak{g}_\alpha$ is then called a *restricted root space*. The following theorems characterize the restricted root spaces further.

Lemma 5.6. *For $\alpha, \beta \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that $\alpha + \beta \neq 0$ we find that $B_\mathfrak{g}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.*

Theorem 5.7 ([11, Proposition 6.40])**.** *The restricted roots and the restricted root spaces have the following properties:*

(a) $\mathfrak{g}$ *is the orthogonal direct sum* $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha$ *with respect to* B_θ

(b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(c) $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ *and hence* $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ *implies* $-\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$

(d) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ *orthogonally with respect to* B_θ*, where* $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$*, the centralizer of* $\mathfrak{a}$ *in* $\mathfrak{k}$

If we now choose a positive system $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{a})$ we can define the set

$$\mathfrak{n} = \bigoplus\nolimits_{\alpha \in \Delta^+} \mathfrak{g}_\alpha. \tag{56}$$

One way of choosing a positive system is by a *lexicographic ordering*. Here we choose an ordered basis $\{H_1, ..., H_k\}$ of $\mathfrak{a}$ and then call a restricted root α positive if there exists a j such that $\alpha(H_j) > 0$ and $\alpha(H_i) = 0$ for all $i < j$.

Theorem 5.7 allows us to characterize $\mathfrak{n}$. For example it is quite obvious that $\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$ by *(b)*. Also since Δ^+ is finite and *(b)* we see that $\mathfrak{n}$ is nilpotent. We also observe that $[\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n}$ because of *(b)* combined with the fact that $\mathfrak{a} \subset \mathfrak{g}_0$ by *(d)*. We therefore observe that $\mathfrak{a} \oplus \mathfrak{n}$ also is a subalgebra of $\mathfrak{g}$, and because $\mathfrak{a}$ is abelian we note that $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] \subset \mathfrak{n}$, making $\mathfrak{a} \oplus \mathfrak{n}$ solvable. We can now find another

Proposition 5.8 (Lie algebra Iwasawa decomposition)**.** *A real semisimple Lie algebra* $\mathfrak{g}$ *is a direct sum of the Lie subalgebras* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *as vector spaces, with* $\mathfrak{k}$, $\mathfrak{a}$ *and* $\mathfrak{n}$ *as previously defined.*

Proof. We know that $\mathfrak{a} \cap \mathfrak{n} = 0$ by the definition of $\mathfrak{n}$ and since $\mathfrak{a} \subset \mathfrak{g}_0$. Therefore the directness of the sum depends on $\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$ being zero, and thus we take any $X \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$. Since $\theta X = X$ we find that $X \in \theta\mathfrak{a} \oplus \theta\mathfrak{n}$, but since $\theta\mathfrak{a} = \mathfrak{a}$ and $\mathfrak{n} \cap \theta\mathfrak{n} = 0$ we also find that $X \in \mathfrak{a} \subset \mathfrak{p}$ and so $X = -\theta X = -X$ which gives $X = 0$.

We now take any $X \in \mathfrak{g}$, then by Theorem 5.7 we find there exist $H \in \mathfrak{a}$, $Y \in \mathfrak{m}$ and $X_\alpha \in \mathfrak{g}_\alpha$ such that

$$\begin{aligned} X &= H + Y + \sum\nolimits_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})} X_\alpha \\ &= H + Y + \sum\nolimits_{\alpha \in \Delta^+} X_\alpha + X_{-\alpha} \\ &= H + Y + \sum\nolimits_{\alpha \in \Delta^+} X_\alpha + X_{-\alpha} + \theta X_{-\alpha} - \theta X_{-\alpha} \\ &= \left(Y + \sum\nolimits_{\alpha \in \Delta^+} X_{-\alpha} + \theta X_{-\alpha}\right) + H + \left(\sum\nolimits_{\alpha \in \Delta^+} X_\alpha - \theta X_{-\alpha}\right) \end{aligned}$$

We therefore find that $X \in \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, which completes the proof. ☐

Much as the Cartan decomposition this result can be lifted to the level of a Lie group, as specified in next the theorem which we mention without proof.

Theorem 5.9 (Iwasawa decomposition, [11, Proposition 6.46])**.** *Let* G *be a real semisimple Lie group, let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ *be an Iwasawa decomposition of the Lie algebra* $\mathfrak{g}$ *of* G*, and let* K*,* A *and* N *be the analytic subgroups of* G *with Lie algebras* $\mathfrak{k}$*,* $\mathfrak{a}$ *and* $\mathfrak{n}$*. Then the multiplication map* $K \times A \times N \to G : (k, a, n) \mapsto kan$ *is a diffeomorphism onto. The groups* A *and* N *are simply connected.*

The diffeomorphism defined in the above theorem is called the *Iwasawa decomposition*, and it is often used to define a projection such as

$$\rho_A : G = KAN \to A : g = kan \mapsto a.$$

which will play a part in things to come. Another interesting thing to remember is that since $\mathfrak{a}$ is abelian that $A = \exp(\mathfrak{a})$. This is particularly useful when we realize that $\{e\} \times \mathfrak{a} \subset K \times \mathfrak{p}$ is a closed submanifold and is diffeomorphically mapped by the global Cartan decomposition to $A = e\exp(\mathfrak{a}) \subset K\exp(\mathfrak{p}) = G$. Therefore A is closed in G and $\exp : \mathfrak{a} \to A$ is a diffeomorphism. We define $\log : A \to \mathfrak{a}$ as the inverse of this diffeomorphism. We mention without proof that $\exp : \mathfrak{n} \to N$ also is a diffeomorphism [21, Lemma 17.13].

6. Kostant's nonlinear convexity theorem

Let G be a semisimple Lie group with Cartan decomposition KP and a corresponding Iwasawa decomposition $G = KAN$, such that $A \subset P$ and then $\rho_A : G \to A$ is corresponding the middle projection. We recall that $\exp : \mathfrak{a} \to A$ is a diffeomorphism with inverse $\log$. We have already previously noted that for any $a \in A$ the K-orbit $\mathcal{O}_a = \{kak^{-1} : k \in K\}$ runs through P. We also recall that the Weyl group of A in K is defined as $W = W(K, A) = N_K(A)/Z_K(A)$, where $N_K(A)$ and $Z_K(A)$ are respectively the normalizer and the centralizer of A in K. We also recall that W is a finite group. We are now ready to state Kostant's nonlinear convexity theorem:

Theorem 6.1 (Kostant [13, Theorem 4.1]). *Fix $a \in A$, then image of the K-orbit $\mathcal{O}_a$ under the middle projection diffeomorphically mapped to $\mathfrak{a}$ equals the convex hull of the Weyl group orbit $W \cdot a$ mapped into $\mathfrak{a}$, i.e.*

$$\log(\rho_A(\mathcal{O}_a)) = \mathrm{conv}[\log(W \cdot a)]$$

The strategy for proving the above theorem in a symplectic framework used by Lu and Ratiu [15] is to find a Poisson structure on AN such that its symplectic leaves coincide with the K-orbits of a K-action on AN such that there exists a K-equivariant map $AN \to P$. Since the K-orbits are then proven to be symplectic manifolds, we apply Theorem 4.2 (and Theorem 4.3 depending on the case) for a suitable Hamiltonian action and moment map defined on the K-orbits. It is then a small step to prove Kostant's nonlinear convexity theorem using convexity theorems from section 4. We follow their approach, which first proves Theorem 6.1 in the case where G is a complex semisimple Lie group and then uses this result to also prove the theorem in the real case.

6.1. Complex case

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. We can then also regard $\mathfrak{g}$ as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ which has a Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$. We have previously observed that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and that $\mathfrak{p} = i\mathfrak{k}$. We can also choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, which leads to the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$. We now define $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$, which we remember to be a solvable subalgebra of $\mathfrak{g}$, and $B = AN$, which is a connected subgroup of G since A and N are analytic subgroups.

Lemma 6.2. *The spaces $(\mathfrak{g}, \mathfrak{k}, \mathfrak{b})$ together with $\langle , \rangle = \operatorname{Im} B_{\mathfrak{g}}$, the imaginary part of the Killing form of $\mathfrak{g}$, is a Manin triple.*

Proof. By the Lie algebra Iwasawa decompositon we see that indeed the decomposition of vector spaces holds and both $\mathfrak{k}$ and $\mathfrak{b}$ are obviously also subalgebras of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple we know that its Killing form is nondegenerate which ensures that $\langle , \rangle$ is a nondegenerate ad-invariant symmetric bilinear form. We now only need to show that both $\mathfrak{k}$ and $\mathfrak{b}$ are isotropic with respect to $\langle , \rangle$.

First of all recall that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$ and therefore $B_{\mathfrak{g}}$ is the complex linear extension of $B_{\mathfrak{k}}$, which is real on $\mathfrak{k}$, and we therefore find that $\langle \mathfrak{k}, \mathfrak{k} \rangle = 0$. By the same argument we find that $B_{\mathfrak{g}}$ is real on $\mathfrak{a} \subset \mathfrak{p} = i\mathfrak{k}$, and therefore $\langle \mathfrak{a}, \mathfrak{a} \rangle = 0$. Then by Lemma 5.6 we find that $\langle \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n} \rangle = 0$ and therefore $\mathfrak{b}$ is also isotropic. □

As a consequence we know $\mathfrak{k} \mapsto \mathfrak{b}^* : X \mapsto \langle X, \cdot \rangle$ to be an isomorphism, and therefore $\mathfrak{k} \cong \mathfrak{b}^*$. For $\xi \in \mathfrak{b}^*$ we denote the corresponding element in $\mathfrak{k}$ by $\xi_{\mathfrak{k}}$.

Definition 6.3. On the subgroup $B = AN$, define a bivector field π by

$$(r_b^{-1}\pi(b))(\xi, \eta) = \langle \rho_{\mathfrak{k}}(\mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}), \rho_{\mathfrak{b}}(\mathrm{Ad}_b^{-1} \eta_{\mathfrak{k}}) \rangle, \quad \xi, \eta \in \mathfrak{b}^*, b \in B \tag{57}$$

The next lemmas will all be angled towards proving Theorem 6.7 which states that π defines a Poisson structure. There is another way of proving this [16, Theorem 4.3], but it requires theory regarding double Lie groups and dressing transformations which isn't necessary to address for the purpose of this thesis. However due to the mentioned publication this Poisson structure is often referred to in literature as the Lu-Weinstein Poisson structure.

Lemma 6.4 ([15, Lemma 4.3])**.** *π is antisymmetric.*

Proof. We simply calculate this using the symmetry and ad-invariance of $\langle , \rangle$ and the isotropy of $\mathfrak{k}$ and $\mathfrak{b}$. Take any $b \in B$ and $\xi, \eta \in \mathfrak{k}$.

$$\begin{aligned}
&(r_b^{-1}\pi(b))(\xi, \eta) + (r_b^{-1}\pi(b))(\eta, \xi) \\
&\qquad = \langle \rho_{\mathfrak{k}}(\mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}), \rho_{\mathfrak{b}}(\mathrm{Ad}_b^{-1} \eta_{\mathfrak{k}}) \rangle + \langle \rho_{\mathfrak{b}}(\mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}), \rho_{\mathfrak{k}}(\mathrm{Ad}_b^{-1} \eta_{\mathfrak{k}}) \rangle \\
&\qquad = \langle (\rho_{\mathfrak{k}} + \rho_{\mathfrak{b}})(\mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}), (\rho_{\mathfrak{k}} + \rho_{\mathfrak{b}})(\mathrm{Ad}_b^{-1} \eta_{\mathfrak{k}}) \rangle \\
&\qquad = \langle \mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}, \mathrm{Ad}_b^{-1} \eta_{\mathfrak{k}} \rangle = \langle \xi_{\mathfrak{k}}, \eta_{\mathfrak{k}} \rangle = 0.
\end{aligned}$$

This then proves that π is indeed a bivector field. □

Lemma 6.5 ([15, Lemma 4.4])**.** *π is multiplicative.*

Proof. We take any $\xi \in \mathfrak{b}^*$, $X \in \mathfrak{b}$ and $b \in B$, and then first observe that

$$\langle (\mathrm{Ad}_b^* \xi)_{\mathfrak{k}}, X \rangle = (\mathrm{Ad}_b^* \xi)(X) = \xi(\mathrm{Ad}_b X) = \langle \xi_{\mathfrak{k}}, \mathrm{Ad}_b X \rangle = \langle \mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}, X \rangle = \langle \rho_{\mathfrak{k}} \mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}}, X \rangle$$

such that $\rho_{\mathfrak{k}} \mathrm{Ad}_b^{-1} \xi_{\mathfrak{k}} = (\mathrm{Ad}_b^* \xi)_{\mathfrak{k}}$. Now take any $\xi, \eta \in \mathfrak{b}^*$, we can then calculate that

$$\begin{aligned}
r_{b_1 b_2}^{-1}\pi(b_1 b_2)(\xi, \eta) &= \langle \rho_{\mathfrak{k}}(\mathrm{Ad}_{b_1 b_2}^{-1} \xi_{\mathfrak{k}}), \rho_{\mathfrak{b}}(\mathrm{Ad}_{b_1 b_2}^{-1} \eta_{\mathfrak{k}}) \rangle = \langle \mathrm{Ad}_{b_2}^{-1} \mathrm{Ad}_{b_1}^{-1} \xi_{\mathfrak{k}}, \rho_{\mathfrak{b}} \mathrm{Ad}_{b_2}^{-1} \mathrm{Ad}_{b_1}^{-1} \eta_{\mathfrak{k}} \rangle \\
&= \langle \mathrm{Ad}_{b_1}^{-1} \xi_{\mathfrak{k}}, \mathrm{Ad}_{b_2} \rho_{\mathfrak{b}} \mathrm{Ad}_{b_2}^{-1} (\rho_{\mathfrak{b}} + \rho_{\mathfrak{k}}) \mathrm{Ad}_{b_1}^{-1} \eta_{\mathfrak{k}} \rangle \\
&= \langle \mathrm{Ad}_{b_1}^{-1} \xi_{\mathfrak{k}}, \mathrm{Ad}_{b_2} \mathrm{Ad}_{b_2}^{-1} \rho_{\mathfrak{b}} \mathrm{Ad}_{b_1}^{-1} \eta_{\mathfrak{k}} \rangle \\
&\qquad + \langle \rho_{\mathfrak{k}} \mathrm{Ad}_{b_1}^{-1} \xi_{\mathfrak{k}}, \mathrm{Ad}_{b_2} \rho_{\mathfrak{b}} \mathrm{Ad}_{b_2}^{-1} \rho_{\mathfrak{k}} \mathrm{Ad}_{b_1}^{-1} \eta_{\mathfrak{k}} \rangle \\
&= r_{b_1}^{-1}\pi(b_1)(\xi, \eta) + \langle \mathrm{Ad}_{b_2}^{-1} (\mathrm{Ad}_{b_1}^* \xi)_{\mathfrak{k}}, \rho_{\mathfrak{b}} \mathrm{Ad}_{b_2}^{-1} (\mathrm{Ad}_{b_1}^* \eta)_{\mathfrak{k}} \rangle \\
&= r_{b_1}^{-1} r_{b_2}^{-1} r_{b_2} \pi(b_1)(\xi, \eta) + r_{b_2}^{-1}\pi(b_2)(\mathrm{Ad}_{b_1}^* \xi, \mathrm{Ad}_{b_1}^* \eta) \\
&= r_{b_1 b_2}^{-1} r_{b_2} \pi(b_1)(\xi, \eta) + \mathrm{Ad}_{b_1} r_{b_2}^{-1}\pi(b_2)(\xi, \eta) \\
&= r_{b_1 b_2}^{-1} r_{b_2} \pi(b_1)(\xi, \eta) + r_{b_1 b_2}^{-1} l_{b_1} \pi(b_2)(\xi, \eta)
\end{aligned}$$

We conclude that π is multiplicative. □

We remember that the dual map of the intrinsic derivative of a multiplicative bivector field π at e defines a bracket $[,]_\pi$ on $\mathfrak{b}^*$ and we will see that it defines a Lie algebra structure according to the next proposition.

Proposition 6.6 ([15, Proposition 4.8])**.** *The dual map of the intrinsic derivative of π at e coincides with the Lie bracket map on $\mathfrak{k}$.*

Proof. We take any $X \in \mathfrak{b}$ and $\xi, \eta \in \mathfrak{b}^*$ and calculate

$$\begin{aligned}
\langle([\xi,\eta]_\pi)_\mathfrak{k}, X\rangle &= [\xi,\eta]_\pi(X) = d_e\pi(X)(\xi,\eta) = \frac{d}{dt}\Big|_{t=0}(r^{-1}_{\exp(tX)}\pi(\exp(tX)))(\xi,\eta) \\
&= \frac{d}{dt}\Big|_{t=0}\langle\rho_\mathfrak{k}(\mathrm{Ad}^{-1}_{\exp(tX)}\xi_\mathfrak{k}), \rho_\mathfrak{b}(\mathrm{Ad}^{-1}_{\exp(tX)}\eta_\mathfrak{k})\rangle = \langle\rho_\mathfrak{k}\xi_\mathfrak{k}, \rho_\mathfrak{b}(-\operatorname{ad}_X\eta_\mathfrak{k})\rangle \\
&= \langle\xi_\mathfrak{k}, -[X,\eta_\mathfrak{k}]\rangle = \langle\xi_\mathfrak{k}, [\eta_\mathfrak{k}, X]\rangle = \langle[\xi_\mathfrak{k},\eta_\mathfrak{k}], X\rangle
\end{aligned}$$

and therefore $([\xi,\eta]_\pi)_\mathfrak{k} = [\xi_\mathfrak{k}, \eta_\mathfrak{k}]$. This shows us that $[,]_\pi$ defines a Lie algebra structure on $\mathfrak{b}^*$ and then isomorphism $\mathfrak{b}^* \to \mathfrak{k} : \xi \mapsto \xi_\mathfrak{k}$ is a Lie algebra homomorphism. ☐

The next theorem is an immediate consequence of applying Lemma 3.15 to Lemma 6.5 and Proposition 6.6.

Theorem 6.7 ([15, Theorem 4.9])**.** *The bivector field π as in Definition 3.12 defines a Poisson structure on B.*

Since we now have a Poisson structure on B, we want to utilize it by finding that the K-orbits of P correspond to the symplectic leaves of this structure. In order to do this we first have to find a diffeomorphism which identifies B and P. We recall the Cartan decomposition $G = KP$ and the Iwasawa decomposition $G = KB$, and therefore we find the two diffeomorphisms

$$\begin{aligned}
\psi_I &: B \to K\backslash G : b \mapsto Kb \\
\psi_C &: P \to K\backslash G : p \mapsto Kp
\end{aligned}$$

which gives us the diffeomorphism $\psi = \psi_I^{-1} \circ \psi_C : P \to B, p \mapsto b$ if $p = kb$ by the Iwasawa decomposition. This means that $\psi = \rho_B\big|_P$, where $\rho_B : G \to B$ is defined as the projection by the Iwasawa decomposition. Also if we define the left K-actions

$$\begin{aligned}
K \times K\backslash G \to K\backslash G &: (k, Kg) \mapsto Kgk^{-1} \\
\sigma^B : K \times B \to B &: (k, b) \mapsto \rho_B(bk^{-1}) \\
\sigma^P : K \times P \to P &: (k, p) \mapsto kpk^{-1}
\end{aligned}$$

one can then easily check that ψ, ψ_I and ψ_C are intertwining maps for these actions. In the case of σ^B this then also illustrates that σ^B indeed defines a left K-action. Therefore the K-orbits on P for the adjoint action σ^P are diffeomorphically mapped to the K-orbits of the action σ^B on B. We will show that the symplectic leaves of π coincide with the latter by showing that characteristic bundle of π is spanned precisely by the infinitesimal generators of σ^B. We recall the definition of the infinitesimal generator and calculate it for $X \in \mathfrak{k}$ in $b \in B$ to explicitly be

$$\begin{aligned}\sigma_X^B(b) &= \frac{d}{dt}\Big|_{t=0} \rho_B(b\exp(-tX)) = \frac{d}{dt}\Big|_{t=0} (r_b \circ \rho_B \circ r_b^{-1})(l_b(\exp(-tX))) \\ &= r_b \rho_{\mathfrak{b}} \operatorname{Ad}_b(-X) = r_b(-\rho_{\mathfrak{b}} \operatorname{Ad}_b X)\end{aligned}$$

since r_b and ρ_B commute for any $b \in B$.

Theorem 6.8 ([15, Theorem 4.11]). *For any $\xi \in \mathfrak{b}^*$, we have $\pi^\#(\xi^l) = \sigma^B_{\xi_{\mathfrak{k}}}$. Therefore the symplectic leaves of π in B are exactly the orbits of the K-action σ^B in B.*

Proof. Take any $\eta \in \mathfrak{b}^*$, and let η^r be the right invariant 1-form on B defined by η. We then calculate the pairing of η^r with $\pi^\#(\xi^l)$ at a point $b \in B$ is given by

$$\begin{aligned}(\eta^r, \pi^\#(\xi^l))(b) &= \pi(b)(\eta^r(b), \xi^l(b)) = \pi(b)(r^*_{b^{-1}}\eta, l^*_{b^{-1}}\xi) = \pi(b)(r^*_{b^{-1}}\eta, r^*_{b^{-1}} r^*_b l^*_{b^{-1}}\xi) \\ &= (r_b^{-1}\pi(b))(\eta, \operatorname{Ad}^*_{b^{-1}}\xi) = \langle \rho_{\mathfrak{k}} \operatorname{Ad}_b^{-1} \eta_{\mathfrak{k}}, \rho_{\mathfrak{b}} \operatorname{Ad}_b^{-1}(\operatorname{Ad}^*_{b^{-1}}\xi)_{\mathfrak{k}}\rangle \\ &= \langle \rho_{\mathfrak{k}} \operatorname{Ad}_b^{-1} \eta_{\mathfrak{k}}, \rho_{\mathfrak{b}} \operatorname{Ad}_b^{-1} \rho_{\mathfrak{k}} \operatorname{Ad}_b \xi_{\mathfrak{k}}\rangle = \langle \rho_{\mathfrak{k}} \operatorname{Ad}_b^{-1} \eta_{\mathfrak{k}}, \rho_{\mathfrak{b}} \operatorname{Ad}_b^{-1}(1-\rho_{\mathfrak{b}}) \operatorname{Ad}_b \xi_{\mathfrak{k}}\rangle \\ &= \langle \rho_{\mathfrak{k}} \operatorname{Ad}_b^{-1} \eta_{\mathfrak{k}}, \rho_{\mathfrak{b}} \xi_{\mathfrak{k}}\rangle - \langle \rho_{\mathfrak{k}} \operatorname{Ad}_b^{-1} \eta_{\mathfrak{k}}, \rho_{\mathfrak{b}} \operatorname{Ad}_b^{-1} \rho_{\mathfrak{b}} \operatorname{Ad}_b \xi_{\mathfrak{k}}\rangle \\ &= \langle \eta_{\mathfrak{k}}, -\operatorname{Ad}_b \rho_{\mathfrak{b}} \operatorname{Ad}_b^{-1} \rho_{\mathfrak{b}} \operatorname{Ad}_b \xi_{\mathfrak{k}}\rangle = \eta(-\rho_{\mathfrak{b}} \operatorname{Ad}_b \xi_{\mathfrak{k}}) = \eta(r_b^{-1}\sigma^B_{\xi_{\mathfrak{k}}}(b)) \\ &= \eta^r(b)(\sigma^B_{\xi_{\mathfrak{k}}}(b)) = (\eta^r, \sigma^B_{\xi_{\mathfrak{k}}})(b).\end{aligned}$$

We therefore conclude that $\pi^\#(\xi^l) = \sigma^B_{\xi_{\mathfrak{k}}}$ for all $\xi \in \mathfrak{b}^*$. □

Let T be the connected subgroup of K with Lie algebra $\mathfrak{t} = i\mathfrak{a}$, such that $\mathfrak{t}$ is maximal abelian in $\mathfrak{k}$. Then T is a maximal torus of K and we can identify $\mathfrak{a}$ with $\mathfrak{t}^*$ through $\langle , \rangle$ by Lemma 5.6.

Theorem 6.9 ([15, Theorem 4.13]). *The restriction of σ^B to T leaves the Poisson structure π on B invariant and the map*

$$J := \log \circ \rho_A : B = AN \to \mathfrak{a} : an \mapsto \log(a)$$

is a moment map for this T-action.

Proof. Since $[\mathfrak{t}, \mathfrak{a}] = i[\mathfrak{a}, \mathfrak{a}] = 0$ we find that T and A commute. From the fact that $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$ we find that $\mathfrak{n}$ is an ideal of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Therefore N is a normal subgroup of TAN. From this we find that for $t \in T$ and $b = an \in B = AN$ that $tnt^{-1} \in N$ and so

$$J(\sigma^B(t, b)) = \log(\rho_A(\rho_B(bt^{-1}))) = \log(\rho_A(tant^{-1})) = \log(\rho_A(a(tnt^{-1}))) = \log(a) = J(b)$$

which gives us that J is invariant under T. For $X \in \mathfrak{t}$, let $J_X \in C^\infty(B)$ be the X-component of J, i.e. $J_X(b) = \langle J(b), X\rangle$ for $b \in B$. Interestingly since N is a normal subgroup of TAN we find for any $b = an, b' = a'n' \in B$

$$\begin{aligned}J(bb') &= J(ana'n')) = \log(\rho_A(aa'(a'^{-1}na')n')) = \log(aa') = \log(a) + \log(a') \\ &= J(b) + J(b')\end{aligned}$$

Now if $X \in \mathfrak{t}$, there exists $\xi = \langle X, \cdot \rangle \in \mathfrak{b}^*$ such that $\xi_{\mathfrak{k}} = X$. We also take $Y \in \mathfrak{b}$ and Y^l as its corresponding left invariant vector field on B and then calculate

$$\begin{aligned}(dJ_X, Y^l)(b) &= \frac{d}{dt}\Big|_{t=0} J_X(b\exp(tY)) = \frac{d}{dt}\Big|_{t=0} \langle J(b\exp(tY)), X\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle J(b) + J(\exp(tY)), X\rangle = \frac{d}{dt}\Big|_{t=0} \langle \log(\rho_A(\exp(tY))), X\rangle \\ &= \langle \rho_{\mathfrak{a}} Y, X\rangle = \langle \rho_{\mathfrak{a}} Y, \rho_{\mathfrak{t}} X\rangle = \langle Y, X\rangle = \xi(Y) = (\xi^l, Y^l)(b)\end{aligned}$$

which shows that dJ_X equals the left invariant 1-form ξ^l. Therefore by Theorem 6.8 we find that $\pi^\#(dJ_X) = \sigma^B_X$ which proves that J is a moment map. □

Proof of Kostant's theorem (Theorem 6.1) in the complex case. We have already shown that there exists an equivariant diffeomorphism $\psi = \rho_B|_P : P \to B$ with respect to the K-actions on both manifolds. If we take $\mathcal{O}_a$ as the K-orbit through $a \in A$ by σ^P, then $\mathcal{O}'_a = \rho_B(\mathcal{O}_a)$ gives the K-orbit through a in B. We know this K-orbit to be a symplectic leaf of the Poisson structure π by Theorem 6.8 and it is necessarily compact since K is compact. The T-action on B defines a T-action on $\mathcal{O}'_a$ and therefore J also defines a moment map on $\mathcal{O}'_a$. We can then apply Theorem 4.2 to find that

$$J(\mathcal{O}'_a) = \operatorname{conv}[J({\mathcal{O}'}^T_a)]$$

where ${\mathcal{O}'}^T_a$ is the fixed point set of the action. However since $\rho_B|_P$ is an intertwining map we find that ${\mathcal{O}'}^T_a = \rho_B(\mathcal{O}^T_a)$ and therefore we can restate the previous equation as

$$\begin{aligned}\log(\rho_A(\mathcal{O}_a)) &= \log(\rho_A(\rho_B(\mathcal{O}_a))) = J(\mathcal{O}'_a) = \operatorname{conv}[J({\mathcal{O}'}^T_a)] = \operatorname{conv}[\log(\rho_A(\rho_B(\mathcal{O}^T_a)))] \\ &= \operatorname{conv}[\log(\rho_A(\mathcal{O}^T_a))].\end{aligned}$$

It is a well known fact that the fixed point set of the K-orbit $\mathcal{O}_a$ is exactly the Weyl group orbit $W \cdot a$, and as such we find

$$\log(\rho_A(\mathcal{O}_a)) = \operatorname{conv}[\log(\rho_A(W \cdot a))] = \operatorname{conv}[\log(W \cdot a)]$$

which then proves Kostant's nonlinear convexity theorem in the complex case. □

6.2. Real case

As we have proven Kostant's theorem (Theorem 6.1) for a complex semisimple Lie group we will now use this result to also prove the real case.

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$. By Theorem 5.5 (e) the center of G lies in K and because conjugation by any element of the center is the identity mapping on G we find that the K-orbits are unchanged when the center is factorized from G. We therefore assume without loss of generality that the center of G is trivial. Therefore the center of $\mathfrak{g}$ is also trivial, and therefore $\mathfrak{g}$ is isomorphic to $\operatorname{ad}(\mathfrak{g})$, the set of inner derivations of $\mathfrak{g}$. Because $\mathfrak{g}$ is semisimple every derivation of $\mathfrak{g}$ is an inner derivation, and as such $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der}(\mathfrak{g})$ is a

Lie algebra isomorphism. It is well known that $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$, the set of automorphisms of $\mathfrak{g}$. Since G is connected and has trivial center we see that it is isomorphic to the identity connected component of $\mathrm{Aut}(\mathfrak{g})$.

We define $\mathfrak{g}_1 = \mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$, and as such $\mathfrak{g}_1^{\mathbb{R}} = \mathfrak{g} \oplus i\mathfrak{g}$. We define the complex Lie group $G_1 = \mathrm{Aut}(\mathfrak{g}_1)_I$ and we regard it as a complexification of G. We recall that the complexification of a real semisimple Lie algebra is semisimple and thus $\mathfrak{g}_1$ is semisimple. Since G_1 is connected and has Lie algebra $\mathfrak{g}_1$ it is a complex semisimple Lie group.

On $\mathfrak{g}_1$ we define the map

$$\tau : \mathfrak{g}_1 \to \mathfrak{g}_1 : X + iY \mapsto X - iY, \quad X, Y \in \mathfrak{g}.$$

We note that τ is a conjugation of $\mathfrak{g}_1$ (and thus complex antilinear) and a involution of $\mathfrak{g}_1^{\mathbb{R}}$. Since $\mathrm{ad} : \mathfrak{g}_1 \to \mathrm{Der}(\mathfrak{g}_1)$ is a Lie algebra isomorphism and

$$\mathrm{ad}(\tau X)Y = [\tau X, Y] = \tau[X, \tau^{-1}Y] = \tau(\mathrm{ad}(X)(\tau^{-1}Y)) = (\tau\,\mathrm{ad}(X)\tau^{-1})(Y)$$

for any $X, Y \in \mathfrak{g}_1$, we find that τ defines a map

$$C_\tau : \mathrm{Der}(\mathfrak{g}_1) \to \mathrm{Der}(\mathfrak{g}_1) : \varphi \mapsto \tau\varphi\tau^{-1},$$

such that $\mathrm{ad} \circ \tau = C_\tau \circ \mathrm{ad}$. Let $e^{\cdot} : \mathrm{Der}(\mathfrak{g}_1) \to G_1 = \mathrm{Aut}(\mathfrak{g}_1)_I$ be the exponential map, then clearly $e^{C_\tau(\varphi)} = e^{\tau\varphi\tau^{-1}} = \tau e^{\varphi}\tau^{-1}$. Interestingly if $\psi \in \mathrm{Aut}(\mathfrak{g}_1)$ it must be complex linear, and since τ and τ^{-1} are complex antilinear we find that $\tau\psi\tau^{-1}$ is again complex linear since its differential must at any point by the chain rule. As such $\tau\psi\tau^{-1}$ is an automorphism of $\mathfrak{g}_1$ and therefore

$$\zeta_\tau : G_1 \to G_1 : \psi \mapsto \tau\psi\tau^{-1}$$

is a Lie group automorphism and with tangent mapping at the identity equal to C_τ. We therefore say that τ lifts to a group automorphism on G_1, which we from now on also denote by τ.

We define G_1^τ as the fixed point set of τ in G_1. We see that G_1^τ is a subgroup: take any $g, g' \in G_1$, then $\tau(g^{-1}g') = \tau(g)^{-1}\tau(g') = g^{-1}g'$ and therefore $g^{-1}g' \in G_1^\tau$. We also observe that G_1^τ is closed: define $\varphi : G_1 \to G_1 : g \mapsto g^{-1}\tau(g)$, then φ obviously continuous and $G_1^\tau = \varphi^{-1}(\{e\})$. From this we conclude that G_1^τ is a closed subgroup, hence a Lie group. Now X is an element of the Lie algebra of G_1^τ if and only if $\exp(tX) = \tau(\exp(tX))$ for all $t \in \mathbb{R}$. The latter is equivalent to $\tau X = X$ which is true if and only if $X \in \mathfrak{g}$. From this we conclude that the connected identity component of G_1^τ equals G.

We observed earlier that there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We also recall that $\mathfrak{k}_1 = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact real form of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1$. Therefore $\mathfrak{g}_1^{\mathbb{R}} = \mathfrak{k}_1 + i\mathfrak{k}_1$ is a Cartan decomposition of $\mathfrak{g}_1^{\mathbb{R}}$, and as such we define $\mathfrak{p}_1 = i\mathfrak{k}_1 = i\mathfrak{k} \oplus \mathfrak{p}$. It is clear that $\mathfrak{k}_1$ and $\mathfrak{p}_1$ are invariant under τ and the fixed point sets are $\mathfrak{k}_1^\tau = \mathfrak{k}$ and $\mathfrak{p}_1^\tau = \mathfrak{p}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let $\mathfrak{a}'$ be any maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Take any $X \in \mathfrak{a}' \cap \mathfrak{p}$, then $[X, \mathfrak{a}] = 0$ while $X \in \mathfrak{p}$ and therefore $X \in \mathfrak{a}$. We then find that $\mathfrak{a} = \mathfrak{a}' \cap \mathfrak{p}$ since the other inclusion is by definition of $\mathfrak{a}'$.

Now take any $X \in \mathfrak{a}'$, with its Cartan decomposition $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, i.e. $X_{\mathfrak{k}} \in \mathfrak{k}, X_{\mathfrak{p}} \in \mathfrak{p}$. If we take any $H \in \mathfrak{a} \subset \mathfrak{p}$ then $[H, X_{\mathfrak{p}}] \in \mathfrak{k}$ and $[H, X_{\mathfrak{k}}] \in \mathfrak{p}$, while also $[H, Z] = 0$, and so we conclude that $[H, X_{\mathfrak{p}}] = [H, X_{\mathfrak{k}}] = 0$. This especially means that $X_{\mathfrak{p}} \in \mathfrak{a}$ and so $X_{\mathfrak{k}} = X - X_{\mathfrak{p}} \in \mathfrak{a}'$. From this we conclude that $\mathfrak{a}' = \mathfrak{a} + \mathfrak{a}' \cap \mathfrak{k}$.

We define $\mathfrak{a}_1 = \mathfrak{a} + i(\mathfrak{a}' \cap \mathfrak{k}) \subset \mathfrak{p}_1$, such that it is maximal abelian in $\mathfrak{p}_1$. Clearly the fixed point set $\mathfrak{a}_1^\tau = \mathfrak{a}$. We choose a basis $X_1, ..., X_n$ of $\mathfrak{a}_1$ such that $X_1, ..., X_m$ is a basis for $\mathfrak{a}$. From this ordered basis define the positive systems of $\Delta^+ \subset \Delta(\mathfrak{g}, \mathfrak{a})$ and $\Delta_1^+ \subset \Delta(\mathfrak{g}_1, \mathfrak{a}_1)$ using a lexicographic ordering. We can then define $\mathfrak{n}$ and $\mathfrak{n}_1$ as in (56) such that the following Iwasawa decompositions exist by Proposition 5.8:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \qquad \mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1.$$

Since $\mathfrak{g}_1$, $\mathfrak{k}_1$ and $\mathfrak{a}_1$ invariant under τ while $\tau^2 = \mathrm{id}_{\mathfrak{g}_1}$, we find $\mathfrak{n}_1$ is invariant under τ. Its fixed point set $\mathfrak{n}_1^\tau$ equals $\mathfrak{n}$ as $\mathfrak{n} = \mathfrak{g} \cap \mathfrak{n}_1$.

By Theorem 5.5 there exist the global Cartan decompositions

$$G = KP = K \exp \mathfrak{p}, \qquad G_1 = K_1 P_1 = K_1 \exp \mathfrak{p}_1,$$

while by Theorem 5.9 there exist the Iwasawa decompositions

$$G = KAN, \qquad G_1 = K_1 A_1 N_1.$$

Lemma 6.10. *Let $\mathfrak{v}$ be a τ-invariant subspace of $\mathfrak{g}_1$ such that $\exp : \mathfrak{g}_1 \to G_1$ bijectively maps $\mathfrak{v}$ to $V \subset G_1$. The subset V is then τ-invariant and if $\mathfrak{v}^\tau$ and V^τ are the fixed point sets of τ in $\mathfrak{v}$ and V respectively, then $V^\tau = \exp(\mathfrak{v}^\tau)$.*

Proof. Since τ is a group automorphism of G_1 with differential $\tau : \mathfrak{g}_1 \to \mathfrak{g}_1$, we observe that $\tau \circ \exp = \exp \circ \tau$. Now take any $g \in V$, then there exists a element $X \in \mathfrak{v}$ such that $g = \exp X$. Because $\mathfrak{v}$ is τ-invariant we then observe

$$\tau(g) = \tau(\exp X) = \exp \tau(X) \in \exp(\mathfrak{v}) = V,$$

and therefore V is τ-invariant.

Assume that $X \in \mathfrak{v}^\tau$, then $\tau(\exp X) = \exp \tau X = \exp X$ and therefore $\exp(\mathfrak{v}^\tau) \subset V^\tau$. Now assume $g \in V^\tau$, then there exists $Y \in \mathfrak{v}$ such that $g = \exp Y$. Also $\exp(\tau Y) = \tau(\exp Y) = \exp Y$, and since $\exp : \mathfrak{v} \to V$ is bijective we find that $\tau Y = Y$ and therefore $Y \in \mathfrak{v}^\tau$, showing the other inclusion such that we can conclude that $V^\tau = \exp(\mathfrak{v}^\tau)$. □

Corollary 6.11. *P_1, A_1 and N_1 are τ-invariant, and their fixed point sets under τ are P, A and N respectively.*

Proof. This follows from $\mathfrak{p}_1^\tau = \mathfrak{p}$, $\mathfrak{a}_1^\tau = \mathfrak{a}$ and $\mathfrak{n}_1^\tau = \mathfrak{n}$ and that $\exp$ defines diffeomorphisms $\mathfrak{p}_1 \to P_1$, $\mathfrak{a}_1 \to A_1$, $\mathfrak{n}_1 \to N_1$, $\mathfrak{p} \to P$, $\mathfrak{a} \to A$ and $\mathfrak{n} \to N$ by Lemma 6.10. □

Since P_1 is τ-invariant we can conclude that K_1 is also τ-invariant by the global Cartan decomposition. We then prove $(K_1^\tau)_e$ to be K in the same way that we proved that $(G_1^\tau)_e$ equals G.

We fix $a \in A$, and denote by $\mathcal{O}_a$ the K-orbit in P through a and by $\mathcal{O}_{a,1}$ the K_1-orbit in P_1 through a. The orbit $\mathcal{O}_{a,1}$ is τ-invariant because $\tau(a) = a$ and K_1 is τ-invariant. Also

$$\begin{aligned}\mathcal{O}_a &= \{kak^{-1} : k \in K\} = \{kak^{-1} : k \in (K_1^\tau)_e\} = (\{kak^{-1} : k \in K_1^\tau\})_a \\ &= ((\{kak^{-1} : k \in K_1\})^\tau)_a = (\mathcal{O}_{a,1}^\tau)_a\end{aligned}$$

and therefore $\mathcal{O}_a$ is the a-connected component of $\mathcal{O}_{a,1}^\tau$.

Proposition 6.12 ([15, Proposition 6.1])**.** *Let π be the Poisson structure on $B_1 = A_1N_1$ as defined in Definition 6.3. Then τ restricted to B_1 is anti-Poisson with respect to π, namely, $\tau_*\pi = -\pi$ where $\tau_*\pi$ denotes the pushforward of π by τ.*

Proof. Lemma 6.5 says that π is multiplicative on B_1. Since τ is a group automorphism, we know that

$$\tau \circ l_g = l_{\tau(g)} \circ \tau \qquad\qquad \tau \circ r_g = r_{\tau(g)} \circ \tau$$

which gives a similar property for the tangent maps by differentiating the above and applying the chain rule. We use this to show that the pushforward $\tau_*\pi$, defined by $(\tau_*\pi)(b_1) = T_{\tau(b_1)}\tau(\pi(\tau(b_1)))$ since τ is an involution, is also multiplicative:

$$\begin{aligned}(\tau_*\pi)(b_1b_2) &= T_{\tau(b_1b_2)}\tau(\pi(\tau(b_1b_2))) = T_{\tau(b_1b_2)}\tau(\pi(\tau(b_1)\tau(b_2))) \\ &= T_{\tau(b_1b_2)}\tau(l_{\tau(b_1)}\pi(\tau(b_2)) + r_{\tau(b_2)}\pi(\tau(b_1))) \\ &= T_{\tau(b_1)\tau(b_2)}\tau\left(l_{\tau(b_1)}\pi(\tau(b_2))\right) + T_{\tau(b_1)\tau(b_2)}\tau\left(r_{\tau(b_2)}\pi(\tau(b_1))\right) \\ &= l_{\tau(\tau(b_1))}T_{\tau(b_2)}\left(\pi(\tau(b_2))\right) + r_{\tau(\tau(b_2))}T_{\tau(b_1)}\tau\left(\pi(\tau(b_1))\right) \\ &= l_{b_1}T_{\tau(b_2)}\tau\left(\pi(\tau(b_2))\right) + r_{b_2}T_{\tau(b_1)}\tau\left(\pi(\tau(b_1))\right) \\ &= l_{b_1}(\tau_*\pi)(b_2) + r_{b_2}(\tau_*\pi)(b_1)\end{aligned}$$

Now $\tau_*\pi + \pi$ is also a multiplicative bivector field and by Lemma 3.13 (d) it is identically zero if its intrinsic derivative at the identity is zero. Therefore we prove $\tau_*\pi = -\pi$ by showing that $d_e(\tau_*\pi) = -d_e\pi$.

We recall that the dual of $\mathfrak{b}_1$ is identified with $\mathfrak{k}_1$ through $\langle , \rangle = \operatorname{Im} B_{\mathfrak{g}_1}$. For $\xi \in \mathfrak{b}_1^*$ we define $\xi_{\mathfrak{k}_1} \in \mathfrak{k}_1$ such that $\xi = \langle \xi_{\mathfrak{k}_1}, \cdot\rangle$. Proposition 6.6 tells us that the dual map of $d_e\pi$ coincides with the Lie map on $\mathfrak{k}_1$

$$d_e\pi(X)(\xi, \eta) = \langle [\xi_{\mathfrak{k}_1}, \eta_{\mathfrak{k}_1}], X\rangle, \quad X \in \mathfrak{b}_1, \xi, \eta \in \mathfrak{b}_1^*.$$

On the other hand if we calculate the intrinsic derivative of $\tau_*\pi$ at e we find

$$\begin{aligned}d_e(\tau_*\pi)(X) &= \frac{d}{dt}\Big|_{t=0} r^{-1}_{\exp(tX)}T_{\tau(\exp(tX))}\tau(\pi(\tau(\exp(tX)))) \\ &= \frac{d}{dt}\Big|_{t=0} T_{\exp(tX)}\tau(r^{-1}_{\exp(t\tau(X))}\pi(\exp(t\tau(X)))) = \tau(d_e\pi(\tau(X)))\end{aligned}$$

and therefore for any $X \in \mathfrak{b}_1, \xi, \eta \in \mathfrak{b}_1^*$,

$$d_e(\tau_*\pi)(X)(\xi, \eta) = d_e\pi(\tau(X))(\tau^*\xi, \tau^*\eta) = \langle [(\tau^*\xi)_{\mathfrak{k}_1}, (\tau^*\xi)_{\mathfrak{k}_1}], \tau(X)\rangle.$$

The Killing form $B_{\mathfrak{g}_1}$ is the complex linear extension of $B_{\mathfrak{g}}$ and thus it is real on $\mathfrak{g} \times \mathfrak{g}$. It follows that for any $X = X' + iX''$ and $Y = Y' + iY''$ in $\mathfrak{g}_1$ with $X', X'', Y', Y'' \in \mathfrak{g}$, we have

$$\begin{aligned}
\langle X, \tau(Y) \rangle &= \operatorname{Im} B_{\mathfrak{g}_1}(X' + iX'', Y' - iY'') \\
&= \operatorname{Im}[B_{\mathfrak{g}_1}(X', Y') - iB_{\mathfrak{g}_1}(X', Y'') + iB_{\mathfrak{g}_1}(X'', Y') + B_{\mathfrak{g}_1}(X'', Y'')] \\
&= \operatorname{Im}[-B_{\mathfrak{g}_1}(X', Y') - iB_{\mathfrak{g}_1}(X', Y'') + iB_{\mathfrak{g}_1}(X'', Y') - B_{\mathfrak{g}_1}(X'', Y'')] \\
&= \operatorname{Im} B_{\mathfrak{g}_1}(-X' + iX'', Y' + iY'') = \langle -\tau(X), Y \rangle
\end{aligned}$$

and as such for $\xi \in \mathfrak{b}_1^*$ and $Y \in \mathfrak{b}_1$,

$$\langle (\tau^*\xi)_{\mathfrak{k}_1}, Y \rangle = (\tau^*\xi)(Y) = \xi(\tau(Y)) = \langle \xi_{\mathfrak{k}_1}, \tau(Y) \rangle = \langle -\tau(\xi_{\mathfrak{k}_1}), Y \rangle.$$

Using the above results we show

$$d_e(\tau_*\pi)(X)(\xi, \eta) = \langle -\tau([-\tau(\xi_{\mathfrak{k}_1}), -\tau(\xi_{\mathfrak{k}_1})]), X \rangle = -\langle \tau^2([\xi_{\mathfrak{k}_1}, \xi_{\mathfrak{k}_1}]), X \rangle = -d_e\pi(X)(\xi, \eta),$$

which completes our proof. □

Proof of Kostant's Theorem (Theorem 6.1) in the real case. By Theorem 6.8 the symplectic leaves of π_1 in B_1 are the K_1-orbits. We have also previously observed that the projection $\rho_{B_1}|_{P_1} : P_1 \to B_1$ is an K_1-equivariant map. We fix $a \in A$, then the K_1-orbit in B_1 through a is given by $\mathcal{O}'_{a,1} = \rho_{B_1}(\mathcal{O}_{a,1})$. The orbit $\mathcal{O}'_{a,1}$ is a compact symplectic manifold because K_1 is compact by Theorem 5.5 (f), and since τ is anti-Poisson with respect to π on B_1 by Proposition 6.12 we find that τ is anti-symplectic on $\mathcal{O}'_{a,1}$.

We also note that the K-orbit through a in B, denoted by $\mathcal{O}'_a$, equals $\rho_{B_1}(\mathcal{O}_a)$. Since K_1 and B_1 are τ-invariant we find that ρ_{B_1} and τ commute, and therefore ρ_{B_1} maps τ fixed point sets to one another. Since $\rho_{B_1} : P_1 \to B_1$ a diffeomorphism and $a \in B_1$ we find

$$\mathcal{O}'_a = \rho_{B_1}(\mathcal{O}_a) = \rho_{B_1}((\mathcal{O}^{\tau}_{a,1})_a) = (\rho_{B_1}(\mathcal{O}^{\tau}_{a,1}))_a = (\rho_{B_1}(\mathcal{O}_{a,1})^{\tau})_a = (\mathcal{O}'^{\tau}_{a,1})_a.$$

such that $\mathcal{O}'_a$ is a connected component of the fixed point set of $\mathcal{O}'_{a,1}$ under τ.

Define $\mathfrak{a}_0$ as the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{a}_1$ with respect to the Killing form $B_{\mathfrak{g}_1}$, and $\operatorname{pr}_{\mathfrak{a}} : \mathfrak{a}_1 \to \mathfrak{a}$ as the projection along $\mathfrak{a}_0$. Since A_1 is abelian we can also uniquely decompose $A_1 = AA_0$ for $A_0 = \exp \mathfrak{a}_0$, and define the projection $\rho_A : B_1 = AA_0N_1 \to A$.

If T_1 is the maximal torus of K_1 with Lie algebra $i\mathfrak{a}_1$, then we know by Theorem 6.9 that its action leaves π invariant on B_1 and that

$$J : B_1 = A_1N_1 \to \mathfrak{a}_1 : b_1 = a_1n_1 \mapsto \log a_1,$$

is a moment map for the T_1-action. If we define T as the subtorus of T with Lie algebra $\mathfrak{t} = i\mathfrak{a} \subset i\mathfrak{a}_1 = \mathfrak{t}_1$, we then define the map $J_T := \operatorname{pr}_{\mathfrak{a}} \circ J : B_1 \to \mathfrak{a}$. For any $b_1 \in B_1$ and $X \in \mathfrak{t}$ we then observe

$$J_{T,X}(b_1) := \langle J_T(b_1), X \rangle = \langle \operatorname{pr}_{\mathfrak{a}} J(b_1), X \rangle = \langle J(b_1), \operatorname{pr}_{\mathfrak{t}} X \rangle = \langle J(b_1), X \rangle = J_X(b_1),$$

and as such J_T is a moment map for the T-action, since J is a moment map for the T_1 action while $T \subset T_1$. For any $b_1 = aa_0n_1 \in B_1$ we find

$$J_T(b_1) = J_T(aa_0n_1) = \iota_{\mathfrak{a}} \log(aa_0) = \iota_{\mathfrak{a}}(\log(a) + \log(a_0)) = \log(a)$$

which gives $J_T = \log \circ \rho_A$. We note A_0 to be τ-invariant as A and A_1 are τ-invariant, which makes it clear that J_T is τ-invariant.

We restrict the T-action and the moment map J_T to the symplectic leaf $\mathcal{O}'_{a,1}$ such that we can apply Theorem 4.3 to find

$$J_T(\mathcal{O}'_a) = \mathrm{conv}[J_T(\mathcal{O}'^T_{a,1} \cap \mathcal{O}'_a)] = \mathrm{conv}[J_T(\mathcal{O}'^T_a)]$$

Since $\rho_A \circ \rho_{B_1} = \rho_A$ and because ρ_B is a T-equivariant map we find that the above is equivalent to

$$\log(\rho_A(\mathcal{O}_a)) = \mathrm{conv}[\log(\rho_A(\mathcal{O}^T_a))]$$

We now take any $p = \exp Y \in P$ and assume that it is fixed by T, then for all $X \in \mathfrak{a}$ and $t \in \mathbb{R}$ we find that

$$\exp Y = \exp tiX \exp Y \exp(-tiX) = \exp(\mathrm{Ad}(\exp tiX)Y) = \exp(e^{ti\,\mathrm{ad}\,X}Y)$$

and taking the $\log$ and derivative of t at zero we find that $[X, Y] = 0$ for all $X \in \mathfrak{a}$. Since $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$ we then find that $Y \in \mathfrak{a}$ and so $p \in A$. Therefore $\mathcal{O}^T_a$ equals $\mathcal{O}_a \cap A$, which is well known to be the orbit of the relative Weyl group $W = W(K, A)$ through a, and as

$$\log(\rho_A(\mathcal{O}_a)) = \mathrm{conv}[\log(\rho_A(\mathcal{O}^T_a))] = \mathrm{conv}[\log(\rho_A(W \cdot a))] = \mathrm{conv}[\log(W \cdot a)]$$

which completes our proof of Kostant's nonlinear convexity theorem (Theorem 6.1) in the real case. $\square$

7. Van den Ban's convexity theorem

Let G be a real semisimple Lie group of finite center with Lie algebra $\mathfrak{g}$ and let τ be a involution of G i.e. a Lie group automorphism such that $\tau^2 = \mathrm{id}_G$. Denote by G^τ the fixed point set of G under τ, and let H be an open subgroup of G^τ. There exists a τ-invariant maximal compact subgroup K of G^τ and the associated global Cartan involution θ commutes with τ. We denote the differential mappings of the involutions also by τ and θ, which are involutions of $\mathfrak{g}$. Since $\mathfrak{h}$, the Lie algebra of H, is also the Lie algebra of G^τ, we find that $\mathfrak{h}$ is the $(+1)$-eigenspace of τ. We denote by $\mathfrak{q}$ the (-1)-eigenspace of τ, and by Lemma 5.1 we then know that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. The Cartan involution θ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as the usual eigenspace decomposition, and $\mathfrak{k}$ is the Lie algebra of K. Since θ and τ commute they have simultaneous eigenspaces and therefore $\mathfrak{g}$ decomposes into the subspaces

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{q}).$$

We define $\mathfrak{a}^{-\tau}$ as a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$, and we choose maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ such that it contains $\mathfrak{a}^{-\tau}$. Let $X \in \mathfrak{a}$ and $Y \in \mathfrak{a}^{-\tau}$, then $[X, \tau Y] = -[X, Y] = 0$ and as such $[\tau X, Y] = \tau[X, \tau Y] = 0$. Because $X - \tau X$ lies in $\mathfrak{p} \cap \mathfrak{q}$ and since it commutes with any $Y \in \mathfrak{a}^{-\tau}$ we find that $X - \tau X$ is an element of $\mathfrak{a}^{-\tau}$ by the definition of $\mathfrak{a}^{-\tau}$. We conclude that

$$\tau X = X - (X - \tau X) \in \mathfrak{a}$$

and therefore $\mathfrak{a}$ is τ-invariant. For any $X \in \mathfrak{a}$ we then observe

$$X = \tfrac{1}{2}(X + \tau X) + \tfrac{1}{2}(X - \tau X) \in (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{q})$$

and as such we find the direct sum of vector spaces $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{q})$, since the converse inclusion is obvious. Clearly $\mathfrak{a}^{-\tau} \subset \mathfrak{a} \cap \mathfrak{q}$, while $\mathfrak{a} \cap \mathfrak{q}$ is abelian and a subset of $\mathfrak{p} \cap \mathfrak{q}$. By the maximality of $\mathfrak{a}^{-\tau}$ we then conclude $\mathfrak{a}^{-\tau} = \mathfrak{a} \cap \mathfrak{q}$. We define $\mathfrak{a}^\tau = \mathfrak{a} \cap \mathfrak{h}$ such that $\mathfrak{a} = \mathfrak{a}^\tau \oplus \mathfrak{a}^{-\tau}$ and then we denote by $\mathrm{pr}_{\mathfrak{a}^{-\tau}} : \mathfrak{a} \to \mathfrak{a}^{-\tau}$ the projection along $\mathfrak{a}^\tau$.

We decompose $\mathfrak{g}$ with respect to both $\mathfrak{a}^{-\tau}$ and $\mathfrak{a}$ and denote the corresponding sets of restricted roots by $\Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta(\mathfrak{g}, \mathfrak{a})$. We choose an ordered basis $H_1, ..., H_n$ of $\mathfrak{a}$ such that $H_1, ..., H_m$ is a basis of $\mathfrak{a}^{-\tau}$ and then pick the positive systems $\Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta^+(\mathfrak{g}, \mathfrak{a})$ by a lexicographic ordering.

Lemma 7.1. *The positive systems $\Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta^+(\mathfrak{g}, \mathfrak{a})$ are compatible in the following sense,*

$$\Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}) = \{\alpha|_{\mathfrak{a}^{-\tau}} : \alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a}), \alpha|_{\mathfrak{a}^{-\tau}} \neq 0\}. \tag{58}$$

Proof. Let $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ such that $\alpha|_{\mathfrak{a}^{-\tau}} \neq 0$, then for any nonzero $X \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{a}^{-\tau}$ we observe

$$\alpha|_{\mathfrak{a}^{-\tau}}(H)X = \alpha(H)X = [H, X],$$

and therefore $\alpha|_{\mathfrak{a}^{-\tau}} \in \Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$. Since α is a positive root and nonzero on $\mathfrak{a}^{-\tau}$ there exists a $H_j \in \mathfrak{a}^{-\tau}$ such that $\alpha(H_j) > 0$ while $\alpha(H_i) = 0$ if $i < j$, and therefore $\alpha|_{\mathfrak{a}^{-\tau}} \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$.

For the converse inclusion let $\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and choose a nonzero $X \in \mathfrak{g}_\beta$. By Theorem 5.7 there exists a decomposition

$$X = X_0 + \sum\nolimits_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})} X_\alpha + X_{-\alpha} \tag{59}$$

and then for any $H \in \mathfrak{a}^{-\tau}$ we find

$$\beta(H)X = [H, X] = \sum\nolimits_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a})} \alpha(H)(X_\alpha - X_{-\alpha}). \tag{60}$$

Due to the directness of the root space decomposition we find that all nonzero X_0, X_α and $X_{-\alpha}$ are linearly independent and therefore by combining (59) and (60) we find

$$\beta(H)X_0 = 0, \quad \beta(H)X_\alpha = \alpha(H)X_\alpha, \quad \beta(H)X_{-\alpha} = -\alpha(H)X_{-\alpha}.$$

for all $H \in \mathfrak{a}^{-\tau}$. Hence we conclude $X_0 = 0$. Now assume there exists $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ such that $X_{-\alpha} \neq 0$, then $\alpha|_{\mathfrak{a}^{-\tau}} = -\beta \notin \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$, which contradicts our earlier observation and therefore $X_{-\alpha} = 0$ for all $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$. Now since $X \neq 0$ there must exist some $\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ such that $X_\alpha \neq 0$. We then find that $\beta(H) = \alpha(H)$ for all $H \in \mathfrak{a}^{-\tau}$ such that $\alpha|_{\mathfrak{a}^{-\tau}} = \beta$ and we therefore conclude (58). $\square$

Using $\Delta^+(\mathfrak{g}, \mathfrak{a})$ we define $\mathfrak{n}$ by (56) which then results by Proposition 5.8 in the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

We then define the spaces

$$\mathfrak{n}_0 = \bigoplus\nolimits_{\alpha\in\Delta^+(\mathfrak{g},\mathfrak{a}),\alpha|_{\mathfrak{a}^{-\tau}}=0} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_1 = \bigoplus\nolimits_{\beta\in\Delta^+(\mathfrak{g},\mathfrak{a}^{-\tau})} \mathfrak{g}_\beta, \tag{61}$$

such that we can make the decomposition $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$. By Theorem 5.9 there exists the Iwasawa decomposition $G = KAN$, where A and N are the analytic subgroups of G generated by $\mathfrak{a}$ and $\mathfrak{n}$ respectively, and we denote by $\rho_A : G \to A$ the corresponding middle projection. We denote by $A^{-\tau}$ the analytic subgroup generated by $\mathfrak{a}^{-\tau}$.

We note that $\theta \circ \tau$ is an involution since θ and τ commute. Since $\mathfrak{a}^{-\tau} \subset \mathfrak{p} \cap \mathfrak{q}$ we find that $(\theta \circ \tau)|_{\mathfrak{a}^{-\tau}} = \mathrm{id}_{\mathfrak{a}^{-\tau}}$. Therefore if $\beta \in \Delta(\mathfrak{g}, \mathfrak{a}^{-\tau})$ we observe that

$$(\theta \circ \tau)(\mathfrak{g}_\beta) = \mathfrak{g}_{(\theta\circ\tau)^*\beta} = \mathfrak{g}_\beta$$

and thus $\mathfrak{g}_\beta$ and $\mathfrak{n}_1$ are $(\theta \circ \tau)$-invariant. The involution $\theta \circ \tau$ decomposes $\mathfrak{g}_\beta$ into $(+1)$- and (-1)-eigenspaces, which we denote by $(\mathfrak{g}_\beta)_+$ and $(\mathfrak{g}_\beta)_-$ respectively. We then define

$$\Delta^+_- = \{\beta \in \Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau}) : (\mathfrak{g}_\beta)_- \neq 0\}$$

and the closed cone

$$\Gamma(\Delta^+_-) = \sum\nolimits_{\beta\in\Delta^+_-} \mathbb{R}_+ H_\beta$$

where $H_\beta \in \mathfrak{a}^{-\tau}$ such that H_β is orthogonal to $\ker\beta$ with respect to the Killing form $B_\mathfrak{g}$ and $\beta(H_\beta) = 1$. Denote by $W_{K\cap H}$ the Weyl group

$$W_{K\cap H} = W_{K\cap H}(\mathfrak{a}^{-\tau}) = N_{K\cap H}(\mathfrak{a}^{-\tau})/Z_{K\cap H}(\mathfrak{a}^{-\tau}).$$

We are now ready to state Van den Ban's convexity theorem:

Theorem 7.2 (Van den Ban [20, Theorem 1.1])**.** *Let G be a real semisimple Lie group of finite center with an involution τ and let H be a open subgroup of G^τ. Fix $a \in A^{-\tau}$, then*

$$(\mathrm{pr}_{\mathfrak{a}^{-\tau}} \circ \log \circ \rho_A)(aH) = \mathrm{conv}[W_{K\cap H} \cdot \log a] + \Gamma(\Delta^+_-). \tag{62}$$

In the next section we follow a part of the proof of the above convexity theorem by Foth and Otto in the publication [6] for the case of a complex semisimple Lie group. Its approach is acknowledged to be inspired by the method used by Lu and Ratiu to prove Kostant's nonlinear convexity theorem. A Lu-Evens Poisson structure is defined on an H-orbit in $K\backslash G$, which is then used to foliate the H-orbit into its symplectic leaves. A suitable Hamiltonian action and moment map are then defined on these symplectic leaves such that Theorem 4.4 can be applied to prove Van den Ban's convexity theorem. Contrary to the previous sections, we will only concern ourselves with the part of the proof which is concerned with Poisson structures, as the focus of this thesis lies there.

7.1. Complex case

Let G be a simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. As in section 6.1 we can regard $\mathfrak{g}$ as a real Lie algebra $\mathfrak{g}^\mathbb{R}$ with Cartan decomposition $\mathfrak{k} \oplus \mathfrak{p}$. We denote by θ both the corresponding Cartan and global Cartan involutions.

Let τ be an anti-holomorphic involution on G such that it commutes with θ, and also denote the corresponding complex antilinear involution on $\mathfrak{g}$ by τ. It decomposes $\mathfrak{g}^\mathbb{R}$ into a $(+1)$- and (-1) eigenspace, which we denote by $\mathfrak{h}$ and $\mathfrak{q}$ respectively.

The analytic subgroup K generated by $\mathfrak{k}$ is also the fixed point set G^θ. Because $K = G^\theta$ and since τ and θ commute we observer that K is τ-invariant. The analytic subgroup H generated by $\mathfrak{h}$ is then also the identity connected component of G^τ.

As before we fix a maximal abelian subalgebra $\mathfrak{a}^{-\tau}$ of $\mathfrak{p}\cap\mathfrak{q}$. We recall that if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}^{-\tau} \subset \mathfrak{a}$, that then $\mathfrak{a} = \mathfrak{a}^\tau \oplus \mathfrak{a}^{-\tau}$ where $\mathfrak{a}^\tau = \mathfrak{a}\cap\mathfrak{h}$, while also $\mathfrak{a}^{-\tau} = \mathfrak{a}\cap\mathfrak{q}$. By choosing an ordered basis $H_1, ..., H_n$ is a basis $\mathfrak{a}$ such that $H_1, ..., H_m$ is a basis of $\mathfrak{a}^{-\tau}$ we can pick the positive systems $\Delta^+(\mathfrak{g}, \mathfrak{a}^{-\tau})$ and $\Delta^+(\mathfrak{g}, \mathfrak{a})$ by a lexicographic ordering such that (58) holds. We define $\mathfrak{n}$ by (56) such that by Proposition 5.8 we get the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ which gives the group Iwasawa decomposition $G = KAN$ by Theorem 5.9. We recall from Lemma 6.2 that $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a}\oplus\mathfrak{n}, \langle,\rangle = \mathrm{Im}\, B_\mathfrak{g})$ is a Manin triple, and therefore $\mathfrak{k}$ is a Lagrangian subalgebra of $\mathfrak{g}$. We recall from section 3.3 that G acts on $\mathcal{L}(\mathfrak{g})$, the set of Lagrangian subalgebras of $\mathfrak{g}$, by the adjoint action.

Lemma 7.3. *There exists a manifold structure on the G-orbit through $\mathfrak{k}$ in $\mathcal{L}(\mathfrak{g})$, such that the map*

$$\kappa : K\backslash G \to G \cdot \mathfrak{k} : Kb \mapsto \mathrm{Ad}_b^{-1}(\mathfrak{k}) \tag{63}$$

is a diffeomorphism.

Proof. We define the smooth right action

$$\varsigma : G \times G \cdot \mathfrak{k} \to G \cdot \mathfrak{k} : (g, \mathfrak{l}) \mapsto g^{-1} \cdot \mathfrak{l} = \mathrm{Ad}_g^{-1}(\mathfrak{l}),$$

which is clearly transitive. We observe that the stabilizer is given by

$$G_{\mathfrak{k}} = \{g \in G : \mathrm{Ad}_g^{-1}(\mathfrak{k}) = \mathfrak{k}\} = N_G(\mathfrak{k}),$$

where $N_G(\mathfrak{k})$ is the normalizer subgroup of $\mathfrak{k}$ in G. Then there exists a manifold structure on $G \cdot \mathfrak{k}$ such that the map $\kappa \equiv \overline{\varsigma_{\mathfrak{k}}} : N_G(\mathfrak{k})\backslash G \to G \cdot \mathfrak{k}$ is a diffeomorphism the equivalent of [22, Proposition 15.5] for smooth right actions. It is clear for any $k \in K$ and $X \in \mathfrak{k}$ that $\mathrm{Ad}_k X \in \mathfrak{k}$, and thus $K \subset N_G(\mathfrak{k})$.

Now take any $g \in N_G(\mathfrak{k})$. Since $G = K \exp \mathfrak{p}$ and $\mathfrak{p} = \mathrm{Ad}(K)\mathfrak{a}$ by [11, Theorem 6.51] we find that there exist $k, k' \in K$ and $X \in \mathfrak{a}$ such that

$$g = k' \exp(\mathrm{Ad}_k X) = k'k \exp X k^{-1}.$$

Since $\mathrm{Ad}_g(\mathfrak{k}) \subset \mathfrak{k}$ we find that $e^{\mathrm{ad}\, X}(\mathfrak{k}) = \mathrm{Ad}_{\exp X}(\mathfrak{k}) \subset \mathfrak{k}$. Now take any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$. If $Y \in \mathfrak{g}_\alpha$, then $\theta Y \in \mathfrak{g}_{-\alpha}$ and as such $Y + \theta Y \in \mathfrak{g}^\theta = \mathfrak{k}$. We then also observe that

$$e^{\alpha(X)}Y + e^{-\alpha(X)}\theta Y = e^{\mathrm{ad}\, X}(Y + \theta Y) \in \mathfrak{k},$$

and therefore

$$e^{\alpha(X)}Y + e^{-\alpha(X)}\theta Y = \theta(e^{\alpha(X)}Y + e^{-\alpha(X)}\theta Y) = e^{-\alpha(X)}Y + e^{\alpha(X)}\theta Y.$$

The above gives us $\alpha(X) = 0$ since Y and θY independent. We thus find $\alpha(X) = 0$ for all $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ and as such conclude that $X = 0$, hence $g \in K$. Therefore $N_G(\mathfrak{k}) \subset K$, which completes our proof. □

For any $a \in A$, we use the above lemma to identify the left coset Ka with the Lagrangian subalgebra $\mathfrak{l} := \mathrm{Ad}_a^{-1}(\mathfrak{k})$ of $\mathfrak{g}$. As H is a subgroup of G we find that KaH is an immersed submanifold of $K\backslash G$ diffeomorphic to $H \cdot \mathfrak{l}$ by the same lemma. In order to define a Lu-Evens Poisson structure on the orbits we must determine what to take for $\mathfrak{d} = \mathfrak{u} \bowtie \mathfrak{u}^*$, and the obvious choice seems to be $\mathfrak{d} = \mathfrak{g}$, $\mathfrak{u} = \mathfrak{h}$ with the nondegenerate ad-invariant symmetric bilinear form $\langle , \rangle = \mathrm{Im}\, B_{\mathfrak{g}}$. We still must find a subalgebra of $\mathfrak{g}$ which can be identified as $\mathfrak{u}^*$.

We recall that $\mathfrak{p} = i\mathfrak{k}$ and since τ is complex antilinear we similarly find that

$$\tau(iX) = -i\tau(X) = -iX, \quad \forall X \in \mathfrak{h}, \qquad \tau(iY) = -i\tau(Y) = iY, \quad \forall Y \in \mathfrak{q},$$

and as such we conclude that $\mathfrak{q} = i\mathfrak{h}$. Moreover, $\mathfrak{h}$ is a real form of $\mathfrak{g}$. We also like to note that if $X \in \mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then iX belongs to $\mathfrak{p}$ and necessarily commutes with every element of $\mathfrak{a}$. Since $\mathfrak{a}$ is maximal abelian we find that $iX \in \mathfrak{a}$. It is quite obvious that $i\mathfrak{a} \subset i\mathfrak{p} = \mathfrak{k}$ centralizes $\mathfrak{a}$, and we therefore find that $\mathfrak{m} = i\mathfrak{a}$. If we define $\mathfrak{c}$ as $\mathfrak{a} \oplus i\mathfrak{a}$ we find by Theorem 5.7 the decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus\nolimits_{\alpha \in \Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha. \tag{64}$$

Clearly $\mathfrak{c}$ is maximal abelian in $\mathfrak{g}$, and since $\mathfrak{a} \subset \mathfrak{p}$ we find that the adjoint action of $\mathfrak{c}$ on $\mathfrak{g}$ is diagonalizable. By [11, Corollary 2.13] we conclude that $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$. We note that $\mathfrak{g}$ can be regarded as the complexification of $\mathfrak{h}$ since $\mathfrak{h}$ is a real form of $\mathfrak{g}$. It is clear that

$$\mathfrak{c}^\tau := \mathfrak{c} \cap \mathfrak{h} = (\mathfrak{a} \cap \mathfrak{h}) \oplus i(\mathfrak{a} \cap \mathfrak{q}) = i\mathfrak{a}^{-\tau} \oplus \mathfrak{a}^\tau$$

is a real form of $\mathfrak{a} \oplus i\mathfrak{a}$ and therefore $\mathfrak{c}$ can be considered as the complexification of $\mathfrak{c}^\tau$. We therefore conclude that $\mathfrak{c}^\tau$ is a Cartan subalgebra of $\mathfrak{h}$, which is clearly θ-stable. Since $\mathfrak{a}^{-\tau}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$ we find that $i\mathfrak{a}^{-\tau}$ is maximal abelian in $\mathfrak{k} \cap \mathfrak{h}$, and therefore the compact dimension of $\mathfrak{c}^\tau$ is maximal. There exists a decomposition

$$\mathfrak{g} = (\mathfrak{a} \oplus i\mathfrak{a}) \oplus \bigoplus\nolimits_{\beta \in \Delta(\mathfrak{g},\mathfrak{c})} \mathfrak{g}_\beta \tag{65}$$

as complex vector spaces since $\mathfrak{a} \oplus i\mathfrak{a}$ is a Cartan subalgebra. By [11, Proposition 6.70] there exists no $\beta \in \Delta(\mathfrak{g}, \mathfrak{c})$ such that β maps all of $i\mathfrak{a}^{-\tau}$ to zero, or equivalently on $\mathfrak{a}^{-\tau}$ since β complex linear.

We note for $\beta \in \Delta(\mathfrak{g}, \mathfrak{c})$ that $\beta|_{\mathfrak{a}} \in \Delta(\mathfrak{g}, \mathfrak{a})$ is a root if $\beta|_{\mathfrak{a}} \neq 0$. Conversely if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ there must exist $\beta \in \Delta(\mathfrak{g}, \mathfrak{c})$ such that $\beta|_{\mathfrak{a}} = \alpha$ by (64) and (65). We therefore conclude that there exist no $\alpha \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that $\alpha|_{\mathfrak{a}^{-\tau}} = 0$. Interestingly we observe for the spaces defined in (61) that $\mathfrak{n}_0 = \emptyset$ and $\mathfrak{n}_1 = \mathfrak{n}$, from which we conclude that $(\theta \circ \tau)(\mathfrak{n}) = \mathfrak{n}$. We find that $\tau\mathfrak{n} = \theta\mathfrak{n}$ since θ is an involution and then $\tau\mathfrak{n} \cap \mathfrak{n} = \{0\}$ by Theorem 5.7 (c). Also since $\mathfrak{n} = \mathfrak{n}_1$, we observe

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus\nolimits_{\beta \in \Delta(\mathfrak{g},\mathfrak{a}^{-\tau})} \mathfrak{g}_\beta,$$

from which we conclude that $\mathfrak{c}$ is the centralizer of $\mathfrak{a}^{-\tau}$. We denote by $\mathfrak{c}^{-\tau} = \mathfrak{c} \cap \mathfrak{q}$ the fixed point set of $\mathfrak{c}$ under $-\tau$.

Lemma 7.4. $(\mathfrak{g}, \mathfrak{h}, \mathfrak{c}^{-\tau} \oplus \mathfrak{n})$ *equipped with* $\langle , \rangle = \operatorname{Im} B_{\mathfrak{g}}$*, the imaginary part of the Killing form of* $\mathfrak{g}$*, is a Manin triple.*

Proof. Since $\mathfrak{h}$ is a real form of $\mathfrak{g}$ we observe that $B_{\mathfrak{g}}$ is a complex linear extension of $B_{\mathfrak{h}}$, which is real on $\mathfrak{h}$. We conclude that $\mathfrak{h}$ is isotropic with respect to $\operatorname{Im} B_{\mathfrak{g}}$. Because $\mathfrak{c}^{-\tau} \subset \mathfrak{q} = i\mathfrak{h}$ we conclude by similar reasoning that $\langle \mathfrak{c}^{-\tau}, \mathfrak{c}^{-\tau} \rangle = 0$ and by Lemma 5.6 we find that $\langle \mathfrak{n}, \mathfrak{c}^{-\tau} \oplus \mathfrak{n} \rangle = 0$ since $\mathfrak{c}^{-\tau} \subset \mathfrak{a} \oplus \mathfrak{m} = \mathfrak{g}_0$. Therefore $\mathfrak{c}^{-\tau} \oplus \mathfrak{n}$ is also isotropic with respect to $\operatorname{Im} B_{\mathfrak{g}}$.

We know by Theorem 5.7 that

$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \mathfrak{n} \oplus \theta\mathfrak{n} = \mathfrak{c} \oplus \mathfrak{n} \oplus \tau\mathfrak{n} \tag{66}$$

is a direct sum of vector spaces. Therefore if $X \in \mathfrak{g}$ there exist $H \in \mathfrak{c}$ and $Y, Y' \in \mathfrak{n}$ such that

$$X = H + Y + \tau Y' = (\tfrac{1}{2}(H + \tau H) + Y' + \tau Y') + \tfrac{1}{2}(H - \tau H) + (Y - Y')$$

from which we observe that $X \in \mathfrak{h} \oplus \mathfrak{c}^{-\tau} \oplus \mathfrak{n}$. Since the right hand side of (66) is a direct sum of vector spaces we know that $\mathfrak{c}^{-\tau} \cap \mathfrak{n} = \{0\}$.

Take any $X \in \mathfrak{h} \cap (\mathfrak{c}^{-\tau} \oplus \mathfrak{n})$. Since $\tau X = X$ we observe that $X \in \mathfrak{c}^{-\tau} \oplus \tau\mathfrak{n}$. Because $\tau\mathfrak{n} \cap \mathfrak{n} = \{0\}$ and $\mathfrak{c}^{-\tau} \cap \mathfrak{n} = \{0\}$, we then find

$$X \in (\mathfrak{c}^{-\tau} \oplus \mathfrak{n}) \cap (c^{-\tau} \oplus \tau\mathfrak{n}) = \mathfrak{c}^{-\tau}.$$

Then $X = \tau X = -X$ and therefore X must be zero. We conclude that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{c}^{-\tau} \oplus \mathfrak{n}$$

is a direct sum of vector spaces, which completes our proof. $\square$

By the above lemma we observe that $\mathfrak{c}^{-\tau} \oplus \mathfrak{n}$ is isomorphic to the dual of $\mathfrak{h}$ through $\langle , \rangle$. It is now appropriate to define $\mathfrak{h}^* = \mathfrak{c}^{-\tau} \oplus \mathfrak{n}$ and we then find $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ such that we can regard $\mathfrak{g}$ as the double Lie algebra of $\mathfrak{h}$. It is now possible to use the results found in section 3.4. First we define a Poisson structure π_- on G by (44) with R as defined in (42). We then use Lemma 3.30 to find a Poisson structure π_H on H such that (H, π_H) is a Poisson Lie subgroup of (G, π_-).

Let $C^{-\tau}$ and H^* be the analytic subgroups generated by $\mathfrak{c}^{-\tau}$ and $\mathfrak{h}^*$ respectively. Since $\mathfrak{c}^{-\tau} \subset \mathfrak{g}_0$ we know $[c^{-\tau}, \mathfrak{n}] \subset \mathfrak{n}$ and therefore $C^{-\tau}$ normalizes N. Then $C^{-\tau}N$ is a subgroup of G which is connected and is clearly also a subgroup of H^*. The Lie algebra of $C^{-\tau}N$ clearly contains both $\mathfrak{c}^{-\tau}$ and $\mathfrak{n}$, and therefore also $\mathfrak{h}^*$, hence $H^* \subset C^{-\tau}N$. We conclude that $H^* = C^{-\tau}N$. There also exists a Poisson structure π_{H^*} on H^* such that (H^*, π_{H^*}) is a Poisson Lie subgroup of (G, π_-). We define

$$\mathfrak{l} \equiv \mathrm{Ad}_a^{-1}(\mathfrak{k}) = \kappa^{-1}(Ka). \tag{67}$$

We define a Poisson structure Π on the orbit $G \cdot \mathfrak{l}$ such that $(H \cdot \mathfrak{l}, \Pi_H)$ and $(H^* \cdot \mathfrak{l}, \Pi_{H*})$ are Poisson submanifolds, where Π_H and Π_{H^*} are the Poisson structures induced by Π. Let $i_{\mathfrak{l}} : H \cdot \mathfrak{l} \to G \cdot \mathfrak{l}$ denote the inclusion map, which is a Poisson immersion. We also recall that $(H \cdot \mathfrak{l}, \Pi_H)$ is a (H, π_H)-homogeneous space. Through the diffeomorphism κ defined in Lemma 7.3 we then find that $P_a = \kappa(H \cdot \mathfrak{l}) \subset K\backslash G$ and $P_a^* = \kappa(H \cdot \mathfrak{l}) \subset K\backslash G$ are both Poisson submanifolds of $K\backslash G$, and then M_a, the symplectic leaf through Ka, must lie in $P_a \cap P_a^*$ by Proposition 3.11. If $\Pi_0 \in \mathfrak{X}_2(P_a)$ is the bivector field which defines the Poisson structure on P_a, then Π_H and Π_0 are κ-related. We also note that as sets $P_a = KaH$ and $P_a^* = KaH^*$.

For the rest of the section we fix $a \in A^{-\tau}$.

Lemma 7.5 ([6, Lemma 3.2]). *The Poisson manifold P_a is regular and equals the union of A^τ-translates of M_a, i.e. each $p \in P_a$ can be written $p = ma'$ with unique $a' \in A^\tau$, $m \in M_a$.*

Proof. Take any $c \in C^\tau$, and any $X \in \mathfrak{g}$, then $\tau(\mathrm{Ad}_c X) = \mathrm{Ad}_{\tau(c)} \tau X = \mathrm{Ad}_c \tau X$. Therefore if $X \in \mathfrak{h}$ then $\mathrm{Ad}_c X \in \mathfrak{h}$ and if $X \in \mathfrak{c}^{-\tau}$ then $\mathrm{Ad}_c X \in \mathfrak{c}^{-\tau}$. Since $\mathfrak{c}^\tau \subset \mathfrak{g}_0$ we know that $[\mathfrak{c}^\tau, \mathfrak{n}] \subset \mathfrak{n}$, and since C^τ is the analytic subgroup of $\mathfrak{c}^\tau$ we find that Ad_c leaves $\mathfrak{n}$ invariant, and thus Ad_c leaves both $\mathfrak{h}$ and $\mathfrak{h}^*$ invariant. We then find that Ad_c leaves R as defined in (42) invariant. We recall that the Poisson structure π_- defined on G equals zero at g if $\mathrm{Ad}_g R = R$, and we therefore conclude $\pi_-(c) = 0$ for any $c \in C^\tau$.

Since the Poisson structure π_H on H is induced by π_- we find that $\pi_H(c) = 0$ for any $c \in C^\tau$. The orbit $P_a = KaH$ is a (H, π_H)-homogeneous space and the corresponding left action of H on P_a is given by

$$\sigma : H \times P_a \to P_a : (h_1, Kah) \mapsto Kahh_1^{-1}.$$

We define the maps $\sigma_p : H \to P_a : h \mapsto \sigma(h, p)$ and $\sigma_h : P_a \to P_a : p \mapsto \sigma(h, p)$, such the (H, π_H)-homogeneity of P_a gives

$$\Pi_0(\sigma_h(p)) = \Pi_0(\sigma(p, h)) = T_e\sigma_h(\Pi_0(p)) + T_e\sigma_p(\pi_H(h)), \qquad p \in P_a, \quad h \in H.$$

For any $c \in C^\tau$ we recall that $\pi_H(c)$ equals zero and thus we conclude by the above that $\sigma_c : P_a \to P_a$ is a Poisson diffeomorphism. For any $a_1 \in A^\tau \subset C^\tau$ we then know that the above map with $c = a_1^{-1}$ is also a Poisson diffeomorphism and therefore maps M_a, the symplectic leaf through Ka, diffeomorphically to M_{aa_1}, the symplectic leaf through Kaa_1. Let $a_1 \neq a_2 \in A^\tau$, then there exists no $c' \in C^{-\tau}$ such that $a_1 c' = a_2$. Therefore $a_1 C^{-\tau} \cap a_2 C^{-\tau} = \emptyset$ which in turn gives $P^*_{aa_1} \cap P^*_{aa_2} = \emptyset$ by uniqueness of the Iwasawa decomposition. Since $M_{aa_j} \subset P^*_{aa_j}$ for both $j = 1, 2$ we conclude $M_{aa_1} \neq M_{aa_2}$. We define the map

$$\gamma : M_a \times A^\tau \to P_a : (Kah, a') \mapsto Kaha', \tag{68}$$

and we note by the previous observations that it must be injective.

Let $p = Kah \in P_a$ and let M_p be the symplectic leaf through p. We know by the previous observations that the codimension of M_p in P_a is at least $\dim(\mathfrak{a}^\tau)$. Through the diffeomorphism κ defined in Lemma 7.3 the point p is identified with $\mathfrak{l}' := \mathrm{Ad}_{ah}^{-1} \mathfrak{k}$, and since $\mathfrak{l}' \in H \cdot \mathfrak{l}$ we find

$$\begin{aligned}\dim(P_a) &= \dim(H \cdot \mathfrak{l}) = \dim(H \cdot \mathfrak{l}') = \dim(H/H_{\mathfrak{l}'}) = \dim(\mathfrak{h}/\mathfrak{h}_{\mathfrak{l}'}) = \dim((\mathfrak{h}/\mathfrak{h}_{\mathfrak{l}'})^*) \\ &= \dim(\mathfrak{h}_{\mathfrak{l}'}^\perp).\end{aligned}$$

We know by Theorem 3.10 that the dimension of M_p equals the rank of $\Pi'^{\#}(\mathfrak{l}')$, and therefore

$$\mathrm{codim}(M_p) = \dim(P_a) - \dim(M_p) = \dim(\mathfrak{h}_{\mathfrak{l}'}^\perp) - \mathrm{rank}(\Pi'^{\#}(\mathfrak{l}')). \tag{69}$$

We also notice that since $\mathfrak{k}$ is its own normalizer algebra in $\mathfrak{g}$, so is $\mathfrak{l}'$ and therefore $\mathfrak{h}_{\mathfrak{l}'} \subset \mathfrak{l}'$. We then notice by Theorem 3.37 that the Lagrangian subalgebra associated to $\mathfrak{l}'$ is a subset of $\mathfrak{l}'$:

$$\mathfrak{h}_{\mathfrak{l}} \oplus (\mathfrak{h} \oplus \mathfrak{h}_{\mathfrak{l}'}^\perp) \cap \mathfrak{l}' \subset \mathfrak{l}'$$

Since both subalgebras are Lagrangian and therefore of equal dimension we conclude that

$$\mathfrak{l}' = \{X + \xi : X \in \mathfrak{h}, \xi \in \mathfrak{h}_{\mathfrak{l}'}^{\perp}, (\chi_{\mathfrak{l}'}\xi) \lrcorner (\psi_{\mathfrak{l}'}^{-1}\Pi'(\mathfrak{l}')) = X + \mathfrak{h}_{\mathfrak{l}'}\},$$

where $\psi_{\mathfrak{l}'}$ is the map identifying $T_{\mathfrak{l}'}(H \cdot \mathfrak{l}') \cong \mathfrak{h}/\mathfrak{h}_{\mathfrak{l}'}$ and $\chi_{\mathfrak{l}'}$ is the map identifying $\mathfrak{h}_{\mathfrak{l}'}^{\perp} \cong (\mathfrak{h}/\mathfrak{h}_{\mathfrak{l}'})^*$. Interestingly we then find that

$$\begin{aligned}\operatorname{codim}(M_p) &= \dim(\{\xi \in \mathfrak{h}_{\mathfrak{l}'}^{\perp} : (\chi_{\mathfrak{l}'}\xi) \lrcorner (\psi_{\mathfrak{l}'}^{-1}\Pi'(\mathfrak{l}')) = 0\}) \\ &= \dim(\{X + \xi \in \mathfrak{l}' : X = 0\}) = \dim(\mathfrak{l}' \cap \mathfrak{h}^*).\end{aligned}$$

Since $\mathfrak{k}$ is compact we find that $\mathfrak{l}'$ also is a compact subalgebra, while the maximal compact subspace of $\mathfrak{h}^*$ equals $i\mathfrak{a}^{\tau}$, and therefore $\operatorname{codim}(M_p) \leq \dim(\mathfrak{a}^{\tau})$. Combined with our previous observation we conclude that $\operatorname{codim}(M_p) = \dim(\mathfrak{a}^{\tau})$ and therefore M_pA^{τ} has the same dimension as P_a. We thus infer that M_pA^{τ} is an open subset of P_a. Since $\operatorname{codim}(M_p) = \dim(\mathfrak{a}^{\tau})$ for any $p \in P_a$ we conclude by (69) that the rank of the Poisson structure is constant in all of P_a, and therefore P_a is a regular Poisson manifold.

Let $\Sigma(P_a)$ be the set of symplectic leaves of P_a and let $M_p \in \Sigma(P_a)$ be the symplectic leaf through $p \in P_a$. If $a' \in A^{\tau}$ we note by previous observations $M_pa' = M_{pa'}$ and thus A^{τ} acts on $\Sigma(P_a)$. Because of this action, $\Sigma(P_a)$ can be represented as a disjoint union of its A^{τ}-orbits. For each such orbit we choose a representative M', and we define $\Sigma' \subset \Sigma(P_a)$ as the set of representatives. Since P_a is the union of all symplectic leaves we then find

$$P_a = \bigsqcup_{M' \in \Sigma'} M'A^{\tau},$$

which is a disjoint union of open nonempty subsets. Since P_a is connected we conclude that Σ' consists of at most one element and therefore $P_a = M_aA^{\tau}$, which shows that γ as defined in (68) is surjective and thus bijective. □

We define the torus $T = \exp(i\mathfrak{a}^{-\tau}) \subset H$. For $t \in T$ and $Kah \in P_a$ we observe that $Kaht^{-1} \in KaH = P_a$ and we therefore observe that the map

$$\sigma_T : T \times P_a \to P_a : (p, t) \mapsto pt^{-1}$$

is a left action. We note that since T and A^{τ} commute, T also acts on M_a by the above lemma. The next lemma illustrates that this action is Hamiltonian and defines its corresponding moment map.

Lemma 7.6 ([6, Lemma 3.3])**.** *The action of T on M_a is Hamiltonian with a moment map $J = -\operatorname{pr}_{\mathfrak{a}^{-\tau}} \circ \log \circ \rho_A$. Here, $\mathfrak{t}^*$ is identified with $\mathfrak{a}^{-\tau}$ via $\langle , \rangle$. Moreover, the moment map is proper.*

Proof. We will first prove that σ_T is a Hamiltonian action on P_a and with J the corresponding map. Choose $Z \in \mathfrak{t} = i\mathfrak{a}^{-\tau}$ and any $p = Kah \in P_a$. We will prove that the Hamiltonian vector field of $J_Z \in C^{\infty}(P_a)$ equals the infinitesimal generator $(\sigma_T)_Z(p)$.

By the Iwasawa decomposition there exists $k \in K$ and $b \in AN$ such that $ah = kb$, and therefore $p = Kah = Kb$. For $X \in \mathfrak{h}$ we then define the path $c(t) \in P_a$ by

$$c(t) = p\exp(tX) = Kb\exp(tX) = K\exp(t\operatorname{Ad}_b X)b$$

such that $c(0)=p$ and $c'(0)\in T_pP_a$. We then observe that

$$dJ_Z(p)(c'(0)) = \frac{d}{dt}\Big|_{t=0} J_Z(c(t)) = \frac{d}{dt}\Big|_{t=0}\langle J(c(t)), Z\rangle$$

while we also note

$$\begin{aligned} J(c(t)) &= (-\operatorname{pr}_{\mathfrak{a}^{-\tau}}\circ\log\circ\rho_A)(K\exp(t\operatorname{Ad}_b X)b) = -(\operatorname{pr}_{\mathfrak{a}^{-\tau}}\circ\log)(\rho_A(\exp(t\operatorname{Ad}_b X))\rho_A(b)) \\ &= -(\operatorname{pr}_{\mathfrak{a}^{-\tau}}(\log(\rho_A(\exp(t\operatorname{Ad}_b X))) + \log(\rho_A(b))). \end{aligned}$$

Because $\mathfrak{t}\subset\mathfrak{k}\cap\mathfrak{h}$ and $\mathfrak{a}^{\tau}\subset\mathfrak{h}$ while both $\mathfrak{k}$ and $\mathfrak{h}$ are isotropic subspaces with respect to $\langle,\rangle$ we observe $\langle\mathfrak{k}\oplus\mathfrak{a}^{\tau},\mathfrak{t}\rangle=0$. By Lemma 5.6 we also observe

$$\langle\mathfrak{n},\mathfrak{t}\rangle = \operatorname{Im} B_{\mathfrak{g}}(\mathfrak{n}, i\mathfrak{a}^{-\tau}) = i\operatorname{Re} B_{\mathfrak{g}}(\mathfrak{n},\mathfrak{a}^{-\tau}) = 0,$$

and therefore $\mathfrak{t}$ and $\mathfrak{k}\oplus\mathfrak{a}^{\tau}\oplus\mathfrak{n}$ are orthogonal with respect to $\langle,\rangle$. We then find

$$\begin{aligned} dJ_Z(p)(c'(0)) &= -\frac{d}{dt}\Big|_{t=0}\langle(\operatorname{pr}_{\mathfrak{a}^{-\tau}}(\log(\rho_A(\exp(t\operatorname{Ad}_b X))) + \log(\rho_A(b))), Z\rangle \\ &= -\langle\operatorname{pr}_{\mathfrak{a}^{-\tau}}(\operatorname{pr}_{\mathfrak{a}}(\operatorname{Ad}_b X)), Z\rangle = -\langle\operatorname{Ad}_b X, Z\rangle = -\langle\operatorname{Ad}_b^{-1} Z, X\rangle. \end{aligned}$$

If we define $\mathfrak{l}' := \kappa^{-1}(p) = \operatorname{Ad}_b^{-1}(\mathfrak{k})$, then there exists an isomorphism $\phi:\mathfrak{l}'\to T^*_{\mathfrak{l}'}(G\cdot\mathfrak{l})$ by Lemma 3.32. We note that then $T_p(i_{\mathfrak{l}}\circ\kappa^{-1})(c'(0))\in T_{\mathfrak{l}'}(G\cdot\mathfrak{l})$, which we can calculate explicitly to be,

$$\begin{aligned} T_p(i_{\mathfrak{l}}\circ\kappa^{-1})(c'(0)) &= \frac{d}{dt}\Big|_{t=0} i_{\mathfrak{l}}(\kappa^{-1}(Kb\exp(tX))) = \frac{d}{dt}\Big|_{t=0} i_{\mathfrak{l}}(\operatorname{Ad}^{-1}_{b\exp(tX)}(\mathfrak{k})) \\ &= \frac{d}{dt}\Big|_{t=0}\operatorname{Ad}^{-1}_{\exp(tX)}\operatorname{Ad}_b^{-1}(\mathfrak{k}) = \frac{d}{dt}\Big|_{t=0}\exp(-tX)\cdot\mathfrak{l}' = -T_e\alpha_{\mathfrak{l}'}(X) \end{aligned} \tag{70}$$

Since $Z\in\mathfrak{k}$ we find that $\operatorname{Ad}_b^{-1} Z\in\mathfrak{l}'$ and we then find by Lemma 3.32,

$$\begin{aligned} \phi(\operatorname{Ad}_b^{-1} Z)(T_p(i_{\mathfrak{l}}\circ\kappa^{-1})(c'(0))) &= -\phi(\operatorname{Ad}_b^{-1} Z)(T_e\alpha_{\mathfrak{l}'}(X)) = -(T_e\alpha_{\mathfrak{l}'})^*(\phi(\operatorname{Ad}_b^{-1} Z))(X) \\ &= -\chi(\operatorname{Ad}_b^{-1} Z)(X) = -\langle\operatorname{Ad}_b^{-1} Z, X\rangle. \end{aligned}$$

and therefore

$$dJ_Z(p) = (T_p(i_{\mathfrak{l}}\circ\kappa^{-1}))^*\phi(\operatorname{Ad}_b^{-1} Z).$$

If we denote by Π_0 the Poisson structure on P_a we find that Π_0 and Π_H are κ^{-1}-related while we recall that Π_H and Π are $i_{\mathfrak{l}}$-related, and thus Π_0 and Π are $(i_{\mathfrak{l}}\circ\kappa^{-1})$-related. Take any $Y+\eta\in\mathfrak{l}'$, we then find using Lemma 3.32 that

$$\begin{aligned} &\Big(\phi^*\big(T_p(i_{\mathfrak{l}}\circ\kappa^{-1})(\Pi_0^{\#}(p)(dJ_Z(p)))\big)\Big)(Y+\eta) \\ &= \Pi_0(p)\big((T_p(i_{\mathfrak{l}}\circ\kappa^{-1}))^*\phi(\operatorname{Ad}_b^{-1} Z), (T_p(i_{\mathfrak{l}}\circ\kappa^{-1}))^*\phi(Y+\eta)\big) \\ &= \big(T_p(i_{\mathfrak{l}}\circ\kappa^{-1})\Pi_0(p)\big)(\phi(\operatorname{Ad}_b^{-1} Z), \phi(Y+\eta)) \\ &= \Pi(i_{\mathfrak{l}}(\kappa^{-1}(p)))(\phi(\operatorname{Ad}_b^{-1} Z), \phi(Y+\eta)) \\ &= \Pi(\mathfrak{l}')(\phi(\operatorname{Ad}_b^{-1} Z), \phi(Y+\eta)) = \langle\operatorname{pr}_{\mathfrak{h}}\operatorname{Ad}_b^{-1} Z, Y+\eta\rangle. \end{aligned} \tag{71}$$

Since $\mathfrak{t} \subset \mathfrak{g}_0 = \{X \in \mathfrak{g} : [X, \mathfrak{a}] = 0\}$ we observe that $[\mathfrak{a}, \mathfrak{t}] = 0$ and $[\mathfrak{n}, \mathfrak{t}] \subset \mathfrak{n}$ by Theorem 5.7 (b). Therefore if $H \in \mathfrak{a}$ and $Y \in \mathfrak{n}$ we find

$$\mathrm{Ad}_{\exp H} Z = e^{\mathrm{ad}_H} Z = Z \quad \text{and} \quad \mathrm{Ad}_{\exp Y} Z = e^{\mathrm{ad}_Y} Z = Z + \sum\nolimits_{k=1}^{\infty} (\mathrm{ad}_Y)^k Z \in Z + \mathfrak{n}.$$

Because A and N are analytic subgroups while $b \in AN$ we observe that $\mathrm{Ad}_b^{-1} Z \in Z + \mathfrak{n}$. Since $\mathfrak{n} \subset \mathfrak{h}^*$ we then find that $\mathrm{pr}_{\mathfrak{h}}(\mathrm{Ad}_b^{-1} Z) = Z$. We use (70) with $X = -Z$ to quickly calculate

$$\begin{aligned}
\phi^*(T_p(i_{\mathfrak{l}} \circ \kappa^{-1})((\sigma_T)_Z(p))))(Y + \eta) &= (\phi(Y + \eta)) \frac{d}{dt}\Big|_{t=0} i_{\mathfrak{l}}(\kappa^{-1}(Kb\exp(-tZ))) \\
&= (\phi(Y + \eta))(T_e \alpha_{\mathfrak{l}'}(Z)) = ((T_e \alpha_{\mathfrak{l}'})^* \circ \phi)(Y + \eta)(Z) \\
&= \chi(Y + \eta)(Z) = \langle Z, Y + \eta \rangle,
\end{aligned}$$

and by combining the above with (71) and $\mathrm{pr}_{\mathfrak{h}}(\mathrm{Ad}_b^{-1} Z) = Z$ we observe

$$\Pi_0^{\#}(p)(dJ_Z(p)) = (\sigma_T)_Z(p),$$

since ϕ^*, $T_{\mathfrak{l}'} i_{\mathfrak{l}}$ and $T_p \kappa^{-1}$ are all injective. We conclude that σ_T is a Hamiltonian action on P_a with J as its moment map. Since M_a is a symplectic leaf of P_a and since T also defines an action on M_a, the restriction of J to M_a is then the moment map for the Hamiltonian action of T on M_a.

For the proof that J is a proper map we refer to the original article. $\square$

We notice that the left hand side of (62) is exactly $-J(aH) = -J(KaH) = -J(P_a)$. Take any $p \in P_a$, then by Lemma 7.5 there exist $a' \in A^\tau$, $m \in M_a$ such that $p = ma'$. Since $\log(a') \in \mathfrak{a}^\tau$ we find

$$\begin{aligned}
J(p) &= (-\mathrm{pr}_{\mathfrak{a}^{-\tau}} \circ \log \circ \rho_A)(ma') = -(\mathrm{pr}_{\mathfrak{a}^{-\tau}} \circ \log)(\rho_A(m)a') \\
&= -\mathrm{pr}_{\mathfrak{a}^{-\tau}}(\log(\rho_A(m)) + \log(a')) = -\mathrm{pr}_{\mathfrak{a}^{-\tau}}(\log(\rho_A(m))) = J(m).
\end{aligned}$$

As such $J(aH)$ equals $J(M_a)$, and since M_a is a connected symplectic manifold, one can use Theorem 4.4 to find $J(aH)$. Foth and Otto calculate that the convex polyhedral set $-J(M_a) \subset \mathfrak{a}^{-\tau}$ is exactly as described on the right hand side of (62), by closely studying the local cones mentioned in Theorem 4.4. This is a quite extensive process which is unrelated to Poisson structures and we therefore choose not to study it in the present thesis.

7.2. Real case

In the previous section we looked at a part of the proof of Van den Ban's convexity theorem in the case that G is a complex semisimple Lie group. In the unpublished article [18], Otto studies a generalization of Duistermaat's theorem (Theorem 4.3) for noncompact manifolds, and one of the generalization's mentioned applications is Van den Ban's convexity theorem in the case that G is a real semisimple Lie group. The strategy is very similar to the one used in section 6.2. On a complexification of G, for which one knows by the previous result that Van den Ban's convexity theorem holds, one uses that G is the fixed point set an involution upon which the new generalization of Duistermaat's theorem [18, Corollary 3.2] is applied. This eventually allows one to prove Van den Ban's convexity theorem in the real case.

We would like to note however that the article referred to in this section has not appeared in a peer reviewed journal. Furthermore, we have not attempted to check the details of the article ourselves for a lack of time.

8. Outlook

In 2016 Bălibanu and Van den Ban published an article [2] in which they prove a generalization of Van den Ban's convexity theorem (Theorem 7.2). One might not unreasonably expect that there also exists a proof for this latest generalization which employs the symplectic approach. This was our original motivation for studying the alternative proofs of Kostant's and Van den Ban's theorems. However due to a lack of time we were not able to explore if a similar approach also works for this new convexity theorem.

A. Intrinsic derivative

In section 3.2 we define on a Lie group G for a k-vector field Π which is zero in the identity the *intrinsic derivative* at the identity by

$$d_e\Pi : \mathfrak{g} \mapsto \wedge^k(\mathfrak{g}) : X \mapsto (\pounds_{\overline{X}}\Pi)(e) \tag{72}$$

where $\overline{X}$ is any vector field on G such that $\overline{X}(e) = X$. It is not straightforward that this is well defined and therefore we would like to shine a light on the matter in this appendix. For this we follow the approach from [7, p.149-150].

A.1. Definition

Let M be a smooth manifold and let E_1 and E_2 be vector bundles over M. For any $m \in M$ there exists an open neighborhood U of m such that we can trivialize open subbundles $V_j = (E_j)_U \subset E_j$ by the maps

$$\tau_1 : V_1 \to U \times \mathbb{R}^k, \qquad \tau_2 : V_2 \to U \times \mathbb{R}^l,$$

which are both isomorphisms of vector bundles on U. If $\varphi : E \to F$ is a vector bundle map, then $\varphi^\tau := \tau_2^{-1} \circ \varphi \circ \tau_1$ describes a map from $U \times \mathbb{R}^k$ to $U \times \mathbb{R}^l$. We can also regard φ^τ as a map $U \to \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l)$. The derivative $T_m\varphi^\tau$ at m then defines a mapping $T_mM \to T_{\varphi^\tau(m)} \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l) \cong \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^l)$.

For m we define $K^\tau_m = \ker \varphi^\tau(m) \subset \mathbb{R}^k$ and $L^\tau_m = \operatorname{coker} \varphi^\tau(m) = \mathbb{R}^l / \operatorname{im} \varphi^\tau(m)$. We denote the corresponding inclusion map $\iota^\tau_m : K^\tau_m \to \mathbb{R}^k$ and the projection map $\rho^\tau_m : \mathbb{R}^l \to L^\tau_m$. We define the *intrinsic derivative* of φ^τ to be the linear map $d_m\varphi^\tau : T_mM \to \operatorname{Hom}(K^\tau_m, L^\tau_m)$ at m given by

$$d_m\varphi^\tau(v) = \rho^\tau_m \circ T_m\varphi^\tau(v) \circ \iota^\tau_m, \quad \forall v \in T_mM.$$

The definition of the intrinsic derivative is clearly dependent on the choice of the trivializations τ used for E_1 and E_2. We note that τ_1 defines a bijective linear map

$$A^\tau(m') : (E_1)_{m'} \to \mathbb{R}^k$$

for every $m' \in U$, where $(E_1)_{m'}$ is the fiber of E_1 at m. In the same way τ_2 defines $B^\tau(m') : (E_2)_{m'} \to \mathbb{R}^l$ for all $m' \in U$. Using this notation we observe that

$$\varphi^\tau(m') = B^\tau(m') \circ \varphi(m') \circ A^\tau(m')^{-1}, \quad \forall m' \in U. \tag{73}$$

If σ_1 and σ_2 are also trivializations of E_1 and E_2 respectively we find that the maps

$$\begin{aligned} A^{\sigma\tau}(m') &= A^\sigma(m') \circ (A^\tau(m'))^{-1} \in \mathrm{GL}(\mathbb{R}^k) \\ B^{\sigma\tau}(m') &= B^\sigma(m') \circ (B^\tau(m'))^{-1} \in \mathrm{GL}(\mathbb{R}^l) \end{aligned}$$

are smooth maps from U to $\mathrm{GL}(\mathbb{R}^k)$ and $\mathrm{GL}(\mathbb{R}^l)$, and describe the transformation from one pair of trivializations to the other. As in the case of τ, there exists a map $\varphi^\sigma : U \to$

$\mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$, which at m has kernel K_m^σ with ι_m^σ as inclusion map and cokernel L_m^σ with ρ_m^σ as projection map. We note by (73) that

$$\varphi^\sigma(m') = B^{\sigma\tau}(m')\varphi^\tau(m')A^{\sigma\tau}(m')^{-1} \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$$

for any $m' \in U$. Clearly $A^{\sigma\tau}(m)K_m^\tau = K_m^\sigma$ and as such we define $\overline{A^{\sigma\tau}}(m) : K_m^\tau \to K_m^\sigma$ such that the diagram

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "A^{\sigma\tau}(m)"] & \mathbb{R}^k \\
K_m^\tau \arrow[u, "\iota_m^\sigma"] \arrow[r, "\overline{A^{\sigma\tau}}(m)"] & K_m^\sigma \arrow[u, "\iota_m^\tau"']
\end{tikzcd}$$

commutes. Similarly we see

$$B^{\sigma\tau}(m)\ker\rho_m^\tau = B^{\sigma\tau}(m)\,\mathrm{im}\,\varphi^\tau(m) = \mathrm{im}\,\varphi^\sigma(m) = \ker\rho_m^\sigma$$

and define $\overline{B^{\sigma\tau}}(m) : L_m^\tau \to L_m^\sigma$ such that the diagram

$$\begin{tikzcd}
\mathbb{R}^l \arrow[r, "B^{\sigma\tau}(m)"] \arrow[d, "\rho_m^\tau"'] & \mathbb{R}^l \arrow[d, "\rho_m^\sigma"] \\
L_m^\tau \arrow[r, "\overline{B^{\sigma\tau}}(m)"] & L_m^\sigma
\end{tikzcd}$$

commutes. Now there exists a natural isomorphism

$$\Upsilon : \mathrm{Hom}(K_m^\tau, L_m^\tau) \to \mathrm{Hom}(K_m^\sigma, L_m^\sigma) : C \mapsto \overline{B^{\sigma\tau}}(m)C\overline{A^{\sigma\tau}}(m)^{-1}$$

defined by the trivializations. If the intrinsic derivative is independent of the choice of trivialization, then we should find that $\Upsilon(d_m\varphi^\tau(v)) = d_m\varphi^\sigma(v)$ for any $v \in T_mM$. We can simply calculate the derivative of φ^σ in m using the Leibniz rule, and thus $T_m\varphi^\sigma(v)$ equals

$$T_mB^{\sigma\tau}(v)\varphi^\tau(m)A^{\sigma\tau}(m)^{-1} + B^{\sigma\tau}(m)T_m\varphi^\tau(v)A^{\sigma\tau}(m)^{-1} + B^{\sigma\tau}(m)\varphi^\tau(m)T_m((A^{\sigma\tau})^{-1})(v)$$

for any $v \in T_mM$. We observe

$$\begin{aligned}
\varphi^\tau(m)A^{\sigma\tau}(m)^{-1} \circ \iota_m^\sigma &= (\varphi^\tau(m) \circ \iota_m^\tau)\overline{A^{\sigma\tau}}(m)^{-1} = 0, \\
\rho_m^\sigma \circ B^{\sigma\tau}(m)\varphi^\tau(m) &= \overline{B^{\sigma\tau}}(m)(\rho_m^\tau \circ \varphi^\tau(m)) = 0,
\end{aligned}$$

since $\ker\varphi^\tau(m) = K_m^\tau = \mathrm{im}\,\iota_m^\tau$ and $\mathrm{im}\,\varphi^\tau(m) = \ker\rho_m^\tau$. The intrinsic derivative of φ^σ at m then is

$$\begin{aligned}
d_m\varphi^\sigma(v) &= \rho_m^\sigma \circ B^{\sigma\tau}(m)T_m\varphi^\tau(v)A^{\sigma\tau}(m)^{-1} \circ \iota_m^\sigma \\
&= \overline{B^{\sigma\tau}}(m)(\rho_m^\tau \circ T_m\varphi^\tau(v) \circ \iota_m^\tau)\overline{A^{\sigma\tau}}(m)^{-1} = \Upsilon(d_m\varphi^\tau(v))
\end{aligned}$$

for any $v \in T_mM$, and thus we see that the intrinsic derivative is independent of the choice of trivialization. If we now define $K_m = \ker \varphi(m)$ and $L_m = \operatorname{coker} \varphi(m)$, it is clear that for any pair of trivializations τ on U there naturally exist isomorphisms

$$\overline{A^\tau}(m) : K_m \to K_m^\tau, \qquad \overline{B^\tau}(m) : L_m \to L_m^\tau,$$

such that if σ is another pair of trivializations on U,

$$\overline{A^\sigma}(m) = \overline{A^{\sigma\tau}}(m) \circ \overline{A^\tau}(m), \qquad \overline{B^\sigma}(m) = \overline{B^{\sigma\tau}}(m) \circ \overline{B^\tau}(m).$$

We then find there exists a unique linear map

$$d_m\varphi : T_mM \to \operatorname{Hom}(K_m, L_m),$$

such that for any pair of trivializations τ on U,

$$d_m\varphi^\tau = \overline{B^\tau}(m) d_m\varphi \overline{A^\tau}(m)^{-1}.$$

We then call $d_m\varphi$ the intrinsic derivative of φ at m.

A.2. Special case

Let G be a Lie group, then $\wedge^k(G)$ is a vector bundle over G and $G \times \mathbb{R}$ is a product bundle over G. We can trivialize $\wedge^k(G)$ by the map

$$\wedge^k(G) \to G \times \wedge^k(\mathfrak{g}) : (g, A) \mapsto (g, r_g^{-1}A)$$

since the differential of right multiplication by g^{-1} defines a isomorphism $T_gG \to \mathfrak{g}$. We define

$$\wedge^k(\mathfrak{g}) \to \operatorname{Hom}(\mathbb{R}, \wedge^k(\mathfrak{g})) : A \mapsto (t \mapsto tA)$$

which is an isomorphism since $\psi \mapsto \psi(1)$ is its inverse. Now if Π is a k-vector field on G we can also see it as a vector bundle map $G \times \mathbb{R} \to G \times \wedge^k(\mathfrak{g})$ and therefore we are able to calculate its intrinsic derivative. Recall that $d_g\Pi(v)$, the intrinsic derivative of Π at $g \in G$ applied to some $v \in T_gG$, describes a linear map from the kernel to the cokernel of $t \mapsto t\Pi(g)$. However the kernel of $(t \mapsto tA)$ is nonzero if and only if $A = 0$ and therefore it only makes sense to define the intrinsic derivative at a point g_0 if $\Pi(g_0) = 0$. In this case the kernel is $\mathbb{R}$ and the cokernel is $\wedge^k(\mathfrak{g})$ and because of this we can identify $d_{g_0}\Pi$ as a mapping $T_{g_0}G \to \wedge^k(\mathfrak{g})$. In this thesis we only deal with the case that Π is zero at the identity and since $T_eG = \mathfrak{g}$ we find that $d_e\Pi : \mathfrak{g} \to \wedge^k(\mathfrak{g})$. Also because of the trivilization of $\wedge^2(G)$ used we find that $d_e\Pi$ is defined by

$$d_e\Pi : \mathfrak{g} \mapsto \wedge^k(\mathfrak{g}) : X \mapsto \frac{d}{dt}\bigg|_{t=0} r_{\exp tX}^{-1}\Pi(\exp tX)$$

which coincides with the manner in which the intrinsic derivative is calculated in this thesis. However this still does not imply that this definition of the intrinsic derivative coincides with the one given by (72).

If we take $\overline{X}$ as any vector field on G such that $\overline{X}(e) = X \in \mathfrak{g}$, we define $\gamma(t)$ as its integral path with $\gamma(0) = e$ and its flow by ψ_t. If we now pick any trivialization of $\wedge^2(G)$ to $G \times \wedge^2(\mathfrak{g})$, then we find that $T_{\gamma(t)}\psi_t \in \mathrm{GL}(\wedge^2(\mathfrak{g}))$ and we therefore may apply the Leibniz rule when calculating the Lie derivative

$$\begin{aligned}(\pounds_{\overline{X}}\Pi)(e) &= \frac{d}{dt}\Big|_{t=0}(T_{\gamma(t)}\psi_t)^{-1}\Pi(\gamma(t)) = \frac{d}{dt}\Big|_{t=0}(T_e\psi_0)^{-1}\Pi(\gamma(t)) + (T_{\gamma(t)}\psi_t)^{-1}\Pi(e) \\ &= \frac{d}{dt}\Big|_{t=0}\Pi(\gamma(t)) = d_e\Pi(\gamma'(0)) = d_e\Pi(X)\end{aligned}$$

Now since the Lie derivative and the intrinsic derivative are both independent of the choice of trivialization, we see that (72) holds in general.

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