Flag manifolds and the Matsuki correspondence for semisimple Lie groups

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Abstract

The original proof of Matsuki duality relies heavily on algebraic methods ([Mat79],[Mat82]). In 1992, Mirkovic, Uzawa and Vilonen gave a geometric proof of Matsuki duality for a flag manifold associated with a Borel subgroup, and in the real case for such a manifold associated with a minimal parabolic subgroup ([Mir92]). In 2002, Bremigan and Lorch extended this result to flag manifolds associated with general parabolic subgroups ([BL02]). The goal of this thesis is to analyze the geometric proof of Bremigan and Lorch. To make our examination as self contained as possible, a lot of details have been added to the original proof including general results from the structure theory of semisimple Lie algebras and groups. The orbits of $\mathrm{SL}(\mathbb{C}, 2)$ and $\mathrm{SL}(\mathbb{R}, 2)$ on their respective flag manifolds will be studied as examples of Matsuki correspondence.

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1 Introduction

The purpose of this thesis is to investigate Matsuki correspondence between orbits on real or complex flag manifolds. The correspondence is named after Toshihiko Matsuki who established it in the real case for minimal parabolic groups in 1977 ([Mat79]). In 1982 he extended the results to non-minimal parabolic groups ([Mat82]). Both results rely heavily on algebraic methods. The Matsuki correspondence turns out to be useful in a number of situations, specifically in geometry and representation theory (see [Sch82, Section 4.5] and [BL02, Section 1]).

In 1992, Mirkovic, Uzawa and Vilonen gave a geometric proof of Matsuki duality for a flag manifold associated with a Borel subgroup, and in the real case for such a manifold associated with a minimal parabolic subgroup ([Mir92]). This proof is extended to flag manifolds associated with general parabolic subgroups of G by Bremigan and Lorch in 2002. The proof makes use of the moment-norm technique. This technique has turned out to be useful for studying group actions and orbit correspondences in multiple geometric settings (see [Mir92] and [BL02, Section 1]).

Before giving an outline of the structure of this thesis, let us present a concise version of Matsuki correspondence for a complex semisimple Lie group G. Let θ and σ be two commuting complex conjugations on G of which θ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ (the Lie algebra $\mathfrak{g}$ viewed as a real Lie algebra). Let K denote the fixed point group of $\sigma\theta$ and let G_0 denote the fixed point group of σ. Let Q be a parabolic subgroup of G and let X denote the variety of parabolic subgroups of G that are conjugate to Q. There is a natural bijection $G/Q \to X$ and via this bijection we see that X is a smooth manifold. Both G_0 and K act by conjugation on X. Matsuki correspondence is a one-to-one correspondence between the G_0-orbits and K-orbits in X (see Theorems 57 and 58 for the exact formulation). A similar correspondence exists for real semisimple Lie groups and we will discuss the proofs for both cases in this paper (see Theorems 68 and 69).

The setup of this thesis is as follows. First, we shall revisit the required basic theory needed to properly understand and investigate the Matsuki correspondence. Concepts which we shall investigate will include (but will not be limited to) flag manifolds (Section 1.2), parabolic subgroups and their Levi decompositions (Section 1.1 and Section 2.2), involutions and complex conjugations on semisimple Lie groups (Section 2.1) and Kähler structure on the flag manifold (Section 2.5).

Secondly, we will make the required preparations to apply the moment-norm technique. We will introduce certain smooth functions $f^{\pm} : X \to \mathbb{R}$. For these functions we shall investigate the integral curves of their gradients, the critical points of $f^{\pm}$ and the Hessians of $f^{\pm}$ at these points (Sections 3.1 - 3.3). These data will be used to establish a useful stratification of X (Section 4.2).

Finally we shall prove the complex version of Matsuki correspondence for complex semisimple Lie groups (Section 4.3). After this, the real version for complex semisimple Lie groups and the real version for real semisimple Lie groups will be derived from the complex case for complex groups (Sections 5.1 and 5.3).

To guide the intuition on Matsuki correspondence, we will examine two examples. We shall investigate Matsuki correspondence for the semisimple Lie groups $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{R})$ (Sections 1.2 and 6.1 respectively). Thorough computations to support these results are postponed to Sections 6.2 and 6.3.

1.1 Basic structure theory

In this section we will revisit some basic properties of the framework in which we shall investigate Matsuki correspondence.

A **Lie algebra** is a vector space $\mathfrak{g}$ equipped with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is anti-symmetric and satisfies the Jacobi identity. Concretely, for $X, Y, Z \in \mathfrak{g}$, we require $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. The algebra is called real or complex, depending on whether the vector space $\mathfrak{g}$ is real or complex. We will often write $\mathrm{ad}(X)$ for the map $\mathfrak{g} \to \mathfrak{g}$ given by $X \mapsto [X, Y]$.

Closely related to a Lie algebra is the notion of a Lie group. A **Lie group** is a smooth manifold equipped with the structure of a group such that the product map $(x, y) \mapsto xy$ and the inversion map $x \mapsto x^{-1}$ are smooth. Let $e \in G$ denote the identity element of the group G, and let T_eG denote the tangent space of the manifold G at the point e. A natural map to consider on a group is the conjugation map $C_x : G \to G$ given by $y \mapsto xyx^{-1}$. A natural operation to consider on smooth manifolds is differentiation which brings us to define the map $\mathrm{Ad} : G \to \mathrm{GL}(T_e G)$ as $\mathrm{Ad}(x) = T_e C_x$. By applying differentiation at $e \in G$ we get a linear map $T_e\mathrm{Ad} : T_e G \to \mathrm{End}(T_e G)$. It turns out that the vector space $T_e G$ combined with the map $T_e\mathrm{Ad}$ is a Lie algebra (see [vdB10, Cor 4.14]) such that $\mathrm{ad} = T_e\mathrm{Ad}$.

It is one of the fundamentals of Lie theory that every Lie algebra can be realized as the tangent space at the identity of a simply connected Lie group. If two simply connected Lie groups have isomorphic Lie algebras, then these groups are isomorphic as well. We now adopt the notation $\mathfrak{g} = T_e G$ and $T_e\mathrm{Ad} = \mathrm{ad}$. Via the map Ad, there is a natural action of G on its Lie algebra $\mathfrak{g}$ called the adjoint action. For $x \in G$, the map $\mathrm{Ad}(x)$ is a Lie algebra automorphism of $\mathfrak{g}$ i.e. $\mathrm{Ad}(x)[X, Y] = [\mathrm{Ad}(x)X, \mathrm{Ad}(x)Y]$ for $X, Y \in \mathfrak{g}$.

An algebra is called **simple** if it is not abelian (i.e. $[,]$ is non-trivial on $\mathfrak{g}$) and $\mathfrak{g}$ has no proper nonzero ideals. An algebra is called semisimple if it is a direct sum of simple ideals. A Lie group is called **semisimple** if it is connected and if its associated Lie algebra is semisimple (see [Kna02, Page 61]). An algebra is called **reductive** if for every ideal $\mathfrak{a}$ in $\mathfrak{g}$, there is a corresponding ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. Every reductive Lie algebra is the direct sum of a semisimple Lie algebra and an abelian Lie algebra (see [Kna02, Cor. 1.53]). For the purposes of this thesis, we will define a **reductive group** to be a Lie group such that its associated Lie algebra is reductive (see [Kna02, Page 384] for a broader discussion of reductive groups).

Let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{R}$ and let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The Killing form B of $\mathfrak{g}$ is the map $\mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, defined by $B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. The following lemma contains a property of the Killing form which we will use often throughout this thesis.

Lemma 1. *The Killing form is invariant under any automorphism of $\mathfrak{g}$. Specifically, the Killing form is invariant under the adjoint action of G on $\mathfrak{g}$.*

Proof. Let φ be an automorphism of $\mathfrak{g}$. It follows from $\varphi[X, \varphi^{-1}Y] = [\varphi X, Y]$ that $\mathrm{ad}(X)(\varphi X) = \varphi \circ \mathrm{ad}(X) \circ \varphi^{-1}$. By using the cyclic property of the trace, we get the following result.

$$\begin{aligned} B(\varphi X, \varphi Y) &= \mathrm{Tr}(\mathrm{ad}(\varphi X)\mathrm{ad}(\varphi Y)) \\ &= \mathrm{Tr}(\varphi \mathrm{ad}(X)\varphi^{-1}\varphi \mathrm{ad}(Y)\varphi^{-1}) \\ &= \mathrm{Tr}(\varphi^{-1}\varphi \mathrm{ad}(X)\mathrm{ad}(Y)) = B(X, Y) \end{aligned}$$

We conclude that the Killing form is invariant under any automorphism of $\mathfrak{g}$. $\square$

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathrm{Int}(\mathfrak{g})$ denote the analytic subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. The group $\mathrm{Int}(\mathfrak{g})$ equals $\mathrm{Ad}(G)_e$ where subscript e denotes the connected component of the identity (see [Kna02, Section I.11]). Notice that for a semisimple group G, we have $\mathrm{Int}(\mathfrak{g}) = \mathrm{Ad}(G)$ since G is connected.

Let $\mathfrak{g}$ be a complex Lie algebra. A Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is called a real form of $\mathfrak{g}$ if $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus i\mathfrak{g}_0$. When we forget about the complex structure of a complex Lie algebra $\mathfrak{g}$, we will denote this with $\mathfrak{g}^{\mathbb{R}}$ i.e. $\mathfrak{g}^{\mathbb{R}} \simeq \mathfrak{g}_0 \oplus_{\mathbb{R}} i\mathfrak{g}_0$ seen as real vector spaces and for a real form $\mathfrak{g}_0$ of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a real or complex Lie algebra. An automorphism σ of $\mathfrak{g}$ that satisfies $\sigma^2 = \mathrm{Id}_{\mathfrak{g}}$ is called an **involution**. For a complex Lie algebra this means in particular that σ is complex linear. In both the real and complex case, the eigenvalues of any involution σ of $\mathfrak{g}$ are ± 1. This yields a decomposition of $\mathfrak{g}$ as $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_\pm$ denotes the ± 1 eigenspace of σ.

Assume now that $\mathfrak{g}$ is a complex Lie algebra. By a **conjugation** of $\mathfrak{g}$ we mean an involution τ of $\mathfrak{g}^{\mathbb{R}}$ such that $\tau(\lambda X) = \overline{\lambda}\tau(X)$ for all $X \in \mathfrak{g}$ and all $\lambda \in \mathbb{C}$. It follows from this that the ± 1 eigenspaces of $\mathfrak{g}^{\mathbb{R}}$ are related to each other by $\mathfrak{g}_- = i\mathfrak{g}_+$. This implies that $\mathfrak{g}_+$ is a real form of $\mathfrak{g}$ with associated conjugation map τ. Conversely, let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$. For $X, Y \in \mathfrak{g}_0$, we define the map $\sigma : \mathfrak{g} \to \mathfrak{g}$ to be $\sigma(X + iY) = X - iY$. Then σ is a complex conjugation of $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ in the above sense.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. An involution θ is called a **Cartan involution** if the bilinear form $B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $(X, Y) \mapsto -B(X, \theta Y)$ is a positive definite inner product. The form B_θ is called the **Cartan inner product** on $\mathfrak{g}$ induced by θ, and $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are orthogonal with respect to this inner product (see [vdB15, Lemma 15.5]).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A conjugation of $\mathfrak{g}$ induces an involution of $\mathfrak{g}^{\mathbb{R}}$. A conjugation on $\mathfrak{g}$ is called Cartan if the induced involution of $\mathfrak{g}^{\mathbb{R}}$ is Cartan.

Let G be a real or complex Lie group. An **involution** on G is an automorphism σ of G such that $\sigma^2 = \mathrm{Id}_G$. By the chain rule it follows that the tangent map $T_e\sigma$ of an involution σ is an involution of $\mathfrak{g}$. The involutions σ and $T_e\sigma$ are usually denoted by the same symbol.

Let G be a complex Lie group. Then $G^{\mathbb{R}}$ denotes G viewed as a real Lie group (thus, we forget about the structure of a complex manifold). A

conjugation on G is an involution σ of $G^{\mathbb{R}}$ such that $T_e\sigma$ is a conjugation of $\mathfrak{g}$. The following lemma contains the relation between complex conjugations and the Killing form on a complex Lie algebra.

Lemma 2. *Let σ be a complex conjugation of a complex Lie algebra $\mathfrak{g}$ and let B denote the Killing form on $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, we get $B(X,Y) = \overline{B(\sigma X, \sigma Y)}$.*

Proof. Recall that $\mathfrak{g}^\sigma$ is a real form of $\mathfrak{g}$ and let $X_1, X_2 \in \mathfrak{g}^\sigma$. It follows from $B(X_1, X_2) = B_{\mathfrak{g}^\sigma}(X_1, X_2)$ that $B(X_1, X_2) \in \mathbb{R}$, hence $\overline{B(\sigma X_1, \sigma X_2)} = B(X_1, X_2)$. Let $Y_1, Y_2 \in \mathfrak{g}^\sigma$, then we can do the following:

$$\begin{aligned} B(\sigma(X_1 + iX_2), \sigma(Y_1 + iY_2)) &= B(X_1 - iX_2), Y_1 - iY_2) \\ &= B(X_1, Y_1) - B(X_2, Y_2) \\ &- i(B(X_1, Y_2) + B(X_2, Y_1)) \\ &= \overline{B(X_1 + iX_2, Y_1 + iY_2)} \end{aligned}$$

□

Let us return to the situation of a real semisimple Lie algebra. For a Cartan involution, we use the notation $\mathfrak{k} = \mathfrak{g}_+$ and $\mathfrak{p} = \mathfrak{g}_-$ and the associated decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the **Cartan decomposition** of $\mathfrak{g}$. The following lemma expresses how the Cartan involution extends from a Lie algebra to a Lie group:

Lemma 3. *Let G be a real semisimple Lie group and let θ be a Cartan involution on its associated Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ and let K denote the connected Lie subgroup of G associated to the algebra $\mathfrak{k}$. Then:*

(a) there exists a unique Lie group automorphism $\boldsymbol{\theta}$ of G with differential θ,
(b) $\boldsymbol{\theta}^2 = 1$,
(c) the subgroup of G fixed by $\boldsymbol{\theta}$ equals K,
(d) K is closed.

Proof. See [Kna02, Th. 6.31]. □

The automorphism $\boldsymbol{\theta}$ of G is called the **global Cartan involution** associated with θ. From now on we shall denote it with θ to emphasize the relation with the involution on Lie algebra level. Cartan involutions will be studied in greater detail in Section 2.1.

Let G be a semisimple Lie group, either complex or real, and let $\mathfrak{g}$ be its associated Lie algebra. A subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ that is closed under $\mathrm{ad}(X)$ for all $X \in \mathfrak{t}$. An element $X \in \mathfrak{g}$ is said to be a **semisimple element** if $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ diagonalizes over the algebraic closure of the base field. A **torus** in $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{g}$ consisting of semisimple elements. A torus $\mathfrak{t}$ in $\mathfrak{g}$ is called maximal if there is no torus in $\mathfrak{g}$ properly containing $\mathfrak{t}$. A maximal torus in $\mathfrak{g}$ is called a **Cartan subalgebra** of $\mathfrak{g}$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then the centralizer of $\mathfrak{t}$ in $\mathfrak{g}$ equals $\mathfrak{t}$ (see [vdB10, Lemma 31.4]). Define T as $Z_G(\mathfrak{t})$, then T is a closed subgroup of G with Lie algebra $\mathfrak{t}$ and T is called a **Cartan subgroup** of G.

A Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ is called **split** if for all $X \in \mathfrak{t}$, the map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ diagonalizes over the base field. Notice that in a complex semisimple Lie algebra, every Cartan subalgebra is split. A compact real semisimple Lie algebra will not contain a split Cartan subalgebra, and a non compact real semisimple Lie algebra will contain at least one Cartan subalgebra that is not split, and there need not be a split Cartan subalgebra at all.

From now on, $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\mathfrak{t}$ be a Cartan subalgebra in $\mathfrak{g}$. A **root** of $\mathfrak{t}$ in $\mathfrak{g}$ is a non-zero element of $\mathfrak{t}^*$ such that the subspace $\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : [H, Y] = \alpha(H)Y, H \in \mathfrak{t}\}$ of $\mathfrak{g}$ is non-zero. The set of roots with respect to $\mathfrak{t}$ is denoted $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$. The complement of $\cup_{\alpha\in\Delta}\mathrm{Ker}(\alpha)$, is a finite union of convex, connected components which are called Weyl chambers. Let $\mathcal{C}$ denote such a chamber. By construction, a root is either negative or positive on $\mathcal{C}$. The roots that are positive on a specific choice of Weyl chamber $\mathcal{C}$, are denoted $\Delta^+ = \Delta(\mathfrak{g}, \mathfrak{t}, \mathcal{C})$ (see [vdB10, Section 31]).

Let $\mathfrak{g}$ be a Lie algebra (complex or real). Then $\mathfrak{g}$ is called a **solvable Lie algebra** if there is a sequence of subalgebras $\mathfrak{g}_j \subset \mathfrak{g}$ satisfying $0 = \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_k = \mathfrak{g}$ such that $\mathfrak{g}_{j-1}$ is an ideal in $\mathfrak{g}_j$ and such that $\mathfrak{g}_j/\mathfrak{g}_{j-1}$ is abelian. Let $\mathfrak{g}$ be the semisimple Lie algebra associated to G. All Borel subalgebras are conjugate under $\mathrm{Int}(\mathfrak{g})$. A Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called a **Borel subalgebra** in $\mathfrak{g}$ if it is a maximally solvable subalgebra of $\mathfrak{g}$. An example of a Borel subalgebra is $\mathfrak{t} \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ where Δ^+ denotes a choice of positive roots.

A subalgebra of a complex semisimple Lie algebra is called **complex parabolic** if it contains a Borel subalgebra. A subalgebra $\mathfrak{q}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called **real parabolic** if $\mathfrak{q}_\mathbb{C}$ is complex parabolic in $\mathfrak{g}_\mathbb{C}$

where the subscript $\mathbb{C}$ denotes the complexification. Let G be a semisimple Lie group, either complex or real, and let $\mathfrak{g}$ be its associated Lie algebra. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. It is known that $\mathfrak{q}$ equals its own normalizer i.e. $N_{\mathfrak{g}}(\mathfrak{q}) = \mathfrak{q}$. Define $Q = N_G(\mathfrak{q})$, then Q is a closed subgroup of G with Lie algebra $\mathfrak{q}$. The subgroup Q of G is called a **parabolic subgroup** of G.

Lemma 4. *Let G be a complex semisimple Lie group and let Q be a parabolic subgroup of G. Then Q equals its own normalizer in G i.e. $N_G(Q) = Q$. Furthermore, Q is connected.*

Proof. See [Hum75, Section 23.1]. $\square$

Parabolic subgroups of real semisimple Lie groups need not be connected. For example, $\mathrm{SL}(2, \mathbb{R})$ is a real semisimple Lie group and the subgroup of upper triangular matrices is parabolic, but this subgroup is not connected.

Lemma 5. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{s}$ be a reductive Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{t} \subset \mathfrak{s}$ be an abelian subalgebra. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then it is also a Cartan subalgebra of $\mathfrak{s}$.*

Proof. Let $X \in \mathfrak{t}$. Since $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g}$, it follows that X is semisimple i.e. $\mathrm{ad}(X)$ diagonalizes over the base field. Since $\mathrm{ad}(X)$ is a Lie algebra-homomorphism, it leaves $\mathfrak{s}$ invariant. Hence, we see that $\mathrm{ad}_{\mathfrak{s}}(X) = \mathrm{ad}(X)|_{\mathfrak{s}}$ diagonalizes. Hence, the elements of $\mathfrak{t}$ are semisimple with respect to $\mathfrak{s}$.

We still need to prove that $\mathfrak{t}$ is maximal in $\mathfrak{s}$. Assume that $\mathfrak{t}$ is not maximal and that it is contained in a Cartan subalgebra $\mathfrak{t}'$ in $\mathfrak{s}$. By using the self-centralizing property of Cartan algebras, we obtain the following result:

$$\mathfrak{t}' = Z_{\mathfrak{s}}(\mathfrak{t}') \subset Z_{\mathfrak{g}}(\mathfrak{t}') \overset{A}{\subset} Z_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$$

At inclusion A we used $\mathfrak{t} \subset \mathfrak{t}'$. We obtain $\mathfrak{t}' = \mathfrak{t}$ which proves that $\mathfrak{t}$ is maximal with respect to $\mathfrak{s}$. $\square$

Lemma 6. *Let $\mathfrak{t}$ be a Cartan subalgebra in a semisimple Lie algebra $\mathfrak{g}$. For every $S \in \mathfrak{t}$ the centralizer of S in $\mathfrak{g}$ is a reductive algebra.*

Proof. This is a corollary of the theorem in [Hum11, p. 26]. $\square$

Lemma 7. *Let $\mathfrak{g}$ be complex semisimple Lie algebra. Let σ and θ be commuting complex conjugations of $\mathfrak{g}$ and let θ be Cartan. Let $\mathfrak{s}_0 \subset \mathfrak{g}$ be a σ- and θ-stable torus in $\mathfrak{g}$. Then there exists a σ- and θ-stable Cartan subalgebra in $\mathfrak{g}$ containing $\mathfrak{s}_0$.*

Proof. Define $\mathfrak{u} = \mathfrak{g}^\theta$ and let $\mathfrak{s}_0^\pm = \mathfrak{s}_0 \cap \mathfrak{g}^{\pm\sigma}$. Define $\mathfrak{v} = Z_\mathfrak{u}(\mathfrak{s}_0^+)$. Notice that $\mathfrak{v}$ is σ-stable and notice that $\mathfrak{s}_0^\pm \subset \mathfrak{v}^{\pm\sigma}$. We may extend $\mathfrak{s}_0^-$ to a maximal abelian subspace of $\mathfrak{v}^{-\sigma}$ and denote this subspace by $\mathfrak{s}^-$. Let $\mathfrak{s}$ be a maximal abelian subspace of $\mathfrak{v}$ containing $\mathfrak{s}^-$. We will show that $\mathfrak{s}$ is σ-stable.

Let $X \in \mathfrak{s}$. Notice that $X - \sigma X \in \mathfrak{v}^{-\sigma}$ centralizes $\mathfrak{s}^-$. By maximality of $\mathfrak{s}^-$ in $\mathfrak{v}^{-\sigma}$, we see that $X - \sigma X \in \mathfrak{s}^-$ which implies $X - \sigma X \in \mathfrak{s}$. Since $X \in \mathfrak{s}$, this implies that $\sigma X = X - (X - \sigma X) \in \mathfrak{s}$. Hence, $\mathfrak{s}$ is σ-stable. Finally, $\mathfrak{s}_\mathbb{C} = \mathfrak{s} \oplus \sigma\mathfrak{s}$ is a θ, σ-stable Cartan subalgebra in $\mathfrak{u}_\mathbb{C} = \mathfrak{g}$. □

1.2 A restricted version of Matsuki correspondence with examples

In this section we shall present a rudimentary version of Matsuki correspondence. We shall demonstrate Matsuki correspondence for the Lie group $\mathrm{SL}(2, \mathbb{C})$.

Let G be a complex semisimple Lie group and let $Q \subset G$ be a parabolic subgroup. Define X to be the set of parabolic subgroups of G that are conjugate to Q. Notice that G has a transitive action on X. Let $x \in X$ and let G_x denote the subgroup of elements of G that leave x invariant. Since the G-action is transitive, we see that $X \simeq G/G_x$. Through this identification we obtain X as a complex manifold. A parabolic subgroup equals its own normalizer (see Lemma 4) and we obtain $G_x = x$ from which we see $X \simeq G/x$. A manifold of this form is called a flag manifold. By picking $x = Q$, we obtain $X \simeq G/Q$.

The following theorem contains an illustrative version of Matsuki correspondence for flag manifolds.

Theorem 8. *Let G be a (connected) complex semisimple Lie group and let $Q \subset G$ be a parabolic subgroup. Let θ and σ be two commuting complex conjugations on G such that θ is Cartan. We use the notation $G_0 = G^\sigma$ and $K = G^{\theta\sigma}$ for the points in G fixed by σ and $\theta\sigma$ respectively. There is a one-to-one correspondence between G_0- and K-orbits in $X \simeq G/Q$ called*

Matsuki correspondence. Two such orbits are in correspondence when their intersection contains precisely one K_0-orbit (we use the notation $K_0 = K^\sigma$).

The requirement that we need to find two commuting conjugations is not very restrictive. We will investigate this in Section 2.1.

Let G be the complex semisimple Lie group $\mathrm{SL}(2, \mathbb{C})$. There is a natural G-action on $\mathbb{C}^2$ by multiplication. Recall that for an n-dimensional vector space V, the flag variety $\mathcal{F}(V)$ is defined to be the set of sequences of subspaces $\{0\} = V_0 \subset V_1 \subset \dots V_{n-1} \subset V_n = V$ such that $\dim(V_{i+1}/V_i) = 1$ for $i \in \{0, \dots, n-1\}$.

The natural action of $\mathrm{SL}(V)$ on $\mathcal{F}(V)$ is transitive. It follows that $\mathcal{F}(V)$ carries the unique structure of a smooth manifold such that for every $F \in \mathcal{F}(V)$, the map $g \mapsto gF$ between $\mathrm{SL}(V)$ and $\mathcal{F}(V)$ is a submersion.

In particular, $\mathcal{F}(\mathbb{C}^2)$ consists of all sequences $0 \subset V_1 \subset \mathbb{C}^2$ where V_1 is a one-dimensional subspace of $\mathbb{C}^2$. Clearly a choice of flag in $\mathbb{C}^2$ is equivalent to a choice of one-dimensional subspace in $\mathbb{C}^2$, hence $\mathcal{F}(\mathbb{C}^2) \simeq \mathbb{P}^1(\mathbb{C})$.

The map $z \mapsto [z : 1] := \mathbb{C} \cdot (z, 1)$ extends to a biholomorphic diffeomorphism from the Riemann sphere $\widehat{\mathbb{C}}$ onto the projective space $\mathbb{P}^1(\mathbb{C})$. For

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C})$$

and $z \in \mathbb{C}$, we get the following $\mathrm{SL}(2, \mathbb{C})$-action on $\mathbb{P}^1(\mathbb{C})$:

$$g[z : 1] = [az + b : cz + d] = [(az + b)(cz + d)^{-1} : 1]$$

It follows from this formula that the action of $\mathrm{SL}(2, \mathbb{C})$ on $\widehat{\mathbb{C}}$ is given by:

$$g \cdot z = \frac{a \cdot z + b}{c \cdot z + d} \text{ for } z \in \widehat{\mathbb{C}} \tag{1}$$

This action is called the action by fractional linear transformation.

Let $P = \mathrm{SL}(2, \mathbb{C})_\infty$ be the stabilizer of $\infty \in \widehat{\mathbb{C}}$. It is not hard to establish that P is given by the upper triangular matrices in G (notice that P equals the stabilizer of $[1 : 0] \in \mathbb{P}^1(\mathbb{C})$). Since the action is transitive, it follows that $G/P \simeq \widehat{\mathbb{C}}$. (It is known that every proper parabolic subgroup of $\mathrm{SL}(2, \mathbb{C})$ can be realized as a stabilizer of a flag in $\mathcal{F}(\mathbb{C}^2)$, hence we can realize it as a stabilizer of the action on $\widehat{\mathbb{C}}$.)

Define the following complex conjugations on G:

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{pmatrix} \tag{2}$$

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*-1} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} \tag{3}$$

It is straightforward to check that both conjugations commute and it is well known that θ is Cartan in G. The precise form of the fixed point sets and their orbits in $\widehat{\mathbb{C}}$ are given in Table 1. The intersections of G^σ- and $G^{\sigma\theta}$-orbits are given in Table 2. Computing these orbits is postponed till Section 5.2.

Table 1: The fixed point sets with their orbits in $\widehat{\mathbb{C}}$

G^σ	$SU(1,1)$	Orbits: • $D_0 = \{z \in \mathbb{C} : \lvert z\rvert < 1\}$ • $\partial D_0 = \{z \in \mathbb{C} : \lvert z\rvert = 1\}$ • $D_\infty = \{z \in \widehat{\mathbb{C}} : \lvert z\rvert > 1\}$
$G^{\sigma\theta}$	$\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^*\right\}$	Orbits: • $\{0\}$ • $\{\infty\}$ • $\mathbb{C}^*$
$G^{\sigma\theta} \cap G^\sigma$	$\left\{\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} : t \in \mathbb{R}\right\}$	Orbits: • $\{0\}$ • $\{\infty\}$ • $\{z \in \mathbb{C} : \lvert z\rvert = r\}$ for all $r \in \mathbb{R}_+$

Table 2: Intersections of G^σ- and $G^{\sigma\theta}$-orbits

G^σ-orbits $\rightarrow$

$G^{\sigma\theta}$-orbits $\downarrow$

$\cap$	D_0	∂D_0	D_∞
$\{0\}$	$\{0\}$	$\emptyset$	$\emptyset$
$\{\infty\}$	$\emptyset$	$\emptyset$	$\{\infty\}$
$\mathbb{C}^*$	$D_0 \backslash \{0\}$	∂D_0	$\mathbb{C} \setminus D$

The intersections in table 2 that consist of precisely one $G^{\sigma\theta} \cap G^\sigma$-orbit are marked red. Notice that every row and every column contains one and only one red marked intersection. Hence, the following orbits are in correspondence:

- $D_0 \longleftrightarrow \{0\}$
- $\partial D_0 \longleftrightarrow \mathbb{C}^*$
- $D_\infty \longleftrightarrow \{\infty\}$

2 Preliminaries

2.1 Involutions and conjugations on a semisimple Lie group G

Recall that X was defined as the variety of subgroups in G conjugate to a parabolic subgroup Q. There is a natural action of G on X. Matsuki correspondence is a correspondence between orbits in X of subgroups of G that are fixed by certain conjugations and involutions of G. In this section we will study these involutions and conjugations.

Let $\mathfrak{g}$ be a complex Lie algebra. A real form of $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ where we view $\mathfrak{g}$ as a real linear space. Using this decomposition, we can define a complex conjugation σ in the following way: let $X + iY \in \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, then $\sigma(X + iY) = X - iY$. For the other way around, let σ be a complex conjugation on $\mathfrak{g}$. Notice that $\mathfrak{g}^{-\sigma} = i\mathfrak{g}^{\sigma}$ so that $\mathfrak{g} = \mathfrak{g}^{\sigma} \oplus i\mathfrak{g}^{\sigma}$. From this it follows that $\mathfrak{g}^{\sigma}$ is a real form of $\mathfrak{g}$ (remember that superscript σ denotes the fixed point set under σ). We see that we may identify real forms of $\mathfrak{g}$ with conjugations on $\mathfrak{g}$. The real form need not be unique and there is a type of real form which is of special interest: a compact real form i.e. a real form $\mathfrak{g}_u$ such that the Killing form of $\mathfrak{g}$ is strictly negative definite when restricted to $\mathfrak{g}_u$ (there are many equivalent ways to characterize compactness of a Lie algebra). There exists a compact real form for any complex semisimple Lie algebra and this form is unique up to conjugation by $\mathrm{Int}\ \mathfrak{g}^{\mathbb{R}}$ [Kna02, Th. 6.11, Cor. 6.20].

Let τ denote the conjugation associated to a compact real form $\mathfrak{g}_u$. It is known that τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ [Kna02, Th. 6.14]. We will use this to show that every real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan involution. For this, we require the following result (the proof is obtained from [Kna02, Th. 6.16]):

Theorem 9. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution and let σ be an involution on $\mathfrak{g}_0$. Then there exists a $\varphi \in \mathrm{Int}\ \mathfrak{g}_0$ such that $\varphi\theta\varphi^{-1}$ commutes with σ.*

Proof. Set $\omega = \sigma\theta$, this is an automorphism of $\mathfrak{g}_0$. Its inverse is easily seen to be $\omega^{-1} = \theta\sigma$. We will first show that ω is symmetric with respect to the

Cartan inner product B_θ (let B be the Killing form of $\mathfrak{g}_0$).

$$\begin{aligned} B_\theta(\omega X, Y) &= -B(\omega X, \theta Y) \\ &\overset{A}{=} -B(X, \omega^{-1}\theta Y) \\ &= -B(X, \theta\sigma\theta Y) = -B(X, \theta\omega Y) = B_\theta(X, \omega Y) \end{aligned}$$

At A we used Lemma 1. We conclude that ω is symmetric with respect to B_θ. A B_θ-symmetric automorphism of $\mathfrak{g}_0$ has real non zero eigenvalues, hence $\rho := \omega^2$ is positive definite. The operator ρ interacts with θ in the following way:

$$\rho\theta = \sigma\theta\sigma\theta\theta = \theta\theta\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}$$

Specifically, we obtain $\theta\rho\theta^{-1} = \rho^{-1}$. We will extend this to $\theta\rho^r\theta^{-1} = \rho^{-r}$ for $r \in \mathbb{R}$.

Let Λ denote the set of eigenvalues of ρ. Since ρ is a positive definite symmetric operator with respect to the inner product B_θ, we see that $\Lambda \subset]0, \infty[$ and that Λ is a finite set. For each $\lambda \in \Lambda$, let $P(\lambda) \in \mathrm{End}(\mathfrak{g}_0)$ be the orthogonal projection with image equal to the eigenspace $\mathrm{Ker}(\rho - \lambda I)$ where I denotes the identity map on $\mathfrak{g}_0$. Notice that $P(\lambda)^2 = P(\lambda)$ and that $P(\lambda)P(\mu) = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$. We can write I and ρ as follows:

$$I = \sum_{\lambda\in\Lambda} P(\lambda), \qquad \rho = \sum_{\lambda\in\Lambda} \lambda P(\lambda)$$

For $r \in \mathbb{R}$ we define the endomorphism $\rho^r \in \mathrm{End}(\mathfrak{g}_0)$ by

$$\rho^r = \sum_{\lambda\in\Lambda} \lambda^r P(\lambda).$$

By using the above mentioned properties, notice that $\rho^1 = \rho$, $\rho^0 = I$ and ρ^r is invertible by ρ^{-r}. Furthermore, for every $k \in \mathbb{Z}$ we have

$$(\rho^r)^k = \rho^{rk}.$$

Finally, it follows from $\theta\rho\theta^{-1} = \rho^{-1}$ that Λ is invariant under the map $\lambda \mapsto \lambda^{-1}$ and that $\theta P(\lambda)\theta^{-1} = P(\lambda^{-1})$ for all $\lambda \in \Lambda$. From this we obtain the following:

$$\theta\rho^r\theta^{-1} = \sum_{\lambda\in\Lambda} \lambda^r P(\lambda^{-1}) = \sum_{\lambda\in\Lambda} (\lambda^{-1})^r P((\lambda^{-1})^{-1}) = \sum_{\lambda\in\Lambda} \lambda^{-r} P(\lambda) = \rho^{-r}$$

Set $\varphi = \rho^{\frac{1}{4}}$ which is the element of $\mathrm{Int}(\mathfrak{g}_0)$ for which we can prove the desired result.

$$\begin{aligned}
(\varphi\theta\varphi^{-1})\sigma &= \rho^{\frac{1}{4}}\theta\rho^{-\frac{1}{4}}\sigma \\
&= \rho^{\frac{1}{2}}\theta\sigma \\
&= \rho^{-\frac{1}{2}}\rho\omega^{-1} \\
&= \rho^{-\frac{1}{2}}\omega = \omega\rho^{-\frac{1}{2}} = \sigma\theta\rho^{-\frac{1}{2}} = \sigma\rho^{\frac{1}{4}}\theta\rho^{-\frac{1}{4}} = \sigma(\varphi\theta\varphi^{-1})
\end{aligned}$$

□

Let $\mathfrak{g}$ be a complex Lie algebra. In order to prove that every real semisimple Lie algebra has a Cartan involution, we need to relate the Killing form of $\mathfrak{g}$ to the Killing form of $\mathfrak{g}^{\mathbb{R}}$. We will do this in the following two lemmas.

Lemma 10. *The Lie algebra $\mathfrak{g}$ is semisimple over $\mathbb{C}$ if and only if $\mathfrak{g}^{\mathbb{R}}$ is semisimple over $\mathbb{R}$.*

Proof. See [Kna02, Remark 1.58]. □

Lemma 11. *Let $B_{\mathfrak{g}}$ denote the Killing form of $\mathfrak{g}$ and let $B_{\mathfrak{g}^{\mathbb{R}}}$ denote the Killing form of $\mathfrak{g}^{\mathbb{R}}$. The following identity holds:*

$$B_{\mathfrak{g}^{\mathbb{R}}} = 2\mathrm{Re}B_{\mathfrak{g}}$$

Proof. Notice $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ and let $V_1, \dots, V_n$ be a basis of $\mathfrak{g}_0$ over $\mathbb{R}$. By this choice of basis, we see $\mathrm{End}(\mathfrak{g}_0) \simeq \mathrm{M_n}(\mathbb{R}) \subset \mathrm{M_n}(\mathbb{C}) \simeq \mathrm{End}(\mathfrak{g})$. Notice that $V_1, \dots, V_n, iV_1, \dots, iV_n$ is a basis of $\mathfrak{g}^{\mathbb{R}}$ from which we obtain $\mathrm{End}(\mathfrak{g}^{\mathbb{R}}) \simeq \mathrm{M_{2n}}(\mathbb{R})$. Let $X \in \mathfrak{g}$. There is an embedding $\mathrm{ad}(\mathfrak{g}) \hookrightarrow \mathrm{M_n}(\mathbb{C}) \hookrightarrow \mathrm{M_{2n}}(\mathbb{R})$ by the following identification:

$$\mathrm{ad}(X) \mapsto \begin{pmatrix} \mathrm{Re}\ \mathrm{ad}(X) & -\mathrm{Im}\ \mathrm{ad}(X) \\ \mathrm{Im}\ \mathrm{ad}(X) & \mathrm{Re}\ \mathrm{ad}(X) \end{pmatrix}$$

Let j denote the above mapping. Using the linearity of the trace and the fact that the real part of a sum equals the sum of its real parts, we get the following:

$$\begin{aligned}
\mathrm{Tr}\ (j(\mathrm{ad}(X))) &= \mathrm{Tr}\ \begin{pmatrix} \mathrm{Re}\ \mathrm{ad}(X) & -\mathrm{Im}\ \mathrm{ad}(X) \\ \mathrm{Im}\ \mathrm{ad}(X) & \mathrm{Re}\ \mathrm{ad}(X) \end{pmatrix} \\
&= \mathrm{Tr}\ (2\ \mathrm{Re}\ \mathrm{ad}(X)) \\
&= 2\ \mathrm{Re}\ \mathrm{Tr}\ (\mathrm{ad}(X))
\end{aligned}$$

For $X, Y \in \mathfrak{g}$, the Killing form takes on the following expression: $B_{\mathfrak{g}}(X, Y) = \text{Tr }(\text{ad}X\text{ad}Y)$. Using this expression, we are able to finish the proof:

$$B_{\mathfrak{g}^{\mathbb{R}}}(X, Y) = \text{Tr }(j(\text{ad}(X)\text{ad}(Y)) = 2\text{ Re Tr }(\text{ad}(X)\text{ad}(Y)) = 2\text{Re } B_{\mathfrak{g}}(X, Y)$$

□

Let σ be the complex conjugation of $\mathfrak{g}$ with respect to a given real form $\mathfrak{g}_0$, and τ conjugation with respect to a compact real form $\mathfrak{g}_u$ which exists by earlier remarks in this section. Using Theorem 9, we will prove that every real semisimple Lie algebra has a Cartan involution. Let us first make the following remark. Let $\varphi \in \text{Int } \mathfrak{g}_0$, then $\varphi\tau\varphi^{-1}$ is a complex conjugation whose set of fixed points equals $\varphi(\mathfrak{g}_u)$. By Equation 1 we obtain that $\varphi(\mathfrak{g}_u)$ is a compact subalgebra of $\mathfrak{g}$, hence $\varphi\tau\varphi^{-1}$ is the conjugation of a compact real form of $\mathfrak{g}$.

The following proof is due to [Kna02, Th. 6.18].

Theorem 12. *Every real semisimple Lie algebra $\mathfrak{g}_0$ has a Cartan involution.*

Proof. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let σ be the complex conjugation associated to $\mathfrak{g}_0$. There exists a compact real form $\mathfrak{g}_u$ for $\mathfrak{g}$ and let τ be the complex conjugation associated to $\mathfrak{g}_u$. Both σ and τ are involutions on $\mathfrak{g}^{\mathbb{R}}$ and τ is a Cartan involution with of this algebra. By Theorem 9, there exists a $\varphi \in \text{Int } \mathfrak{g}^{\mathbb{R}}$ such that $\varphi\tau\varphi^{-1}$ commutes with σ. The involution $\varphi\tau\varphi^{-1}$ is the conjugation with respect to $\varphi(\mathfrak{g}_u)$. By the remarks above, $\varphi(\mathfrak{g}_u)$ is a compact real form for $\mathfrak{g}$ with associated involution $\varphi\tau\varphi^{-1}$. Set $\theta = \varphi\tau\varphi^{-1}$, we will prove that this map restricts to a Cartan involution on $\mathfrak{g}_0$.

Using Lemma 11 we get the following identity between the Cartan inner products on $\mathfrak{g}^{\mathbb{R}}$ and $\mathfrak{g}$:

$$(B_{\mathfrak{g}^{\mathbb{R}}})_\theta(X, Y) = -B_{\mathfrak{g}^{\mathbb{R}}}(X, \theta Y) = -2\text{Re } B_{\mathfrak{g}}(X, \theta Y) \tag{4}$$

Since θ represents conjugation of a compact real form, we know that it is also a Cartan involution on $\mathfrak{g}^{\mathbb{R}}$. Hence, the left part of the above expression is positive when $X = Y$. Now, let $X \in \mathfrak{g}^\sigma$, i.e. $\sigma X = X$. We know by construction that σ commutes with θ and we get $\sigma\theta X = \theta\sigma X = \theta X$. It follows that θ restricts to an involution on $\mathfrak{g}_0$. Let $X, Y \in \mathfrak{g}_0$, then we get the following result for the Cartan inner products:

$$B_\theta(X, Y) = -B_{\mathfrak{g}_0}(X, \theta Y) \stackrel{A}{=} -B_{\mathfrak{g}}(X, \theta Y) = \frac{1}{2}(B_{\mathfrak{g}^{\mathbb{R}}})_\theta(X, Y)$$

At equality A, we used that the Killing form of $\mathfrak{g}_0$, is the restriction of the Killing form on $\mathfrak{g}$. The expression on the right is positive definite because of Equation 4. We conclude that θ is a Cartan involution on $\mathfrak{g}_0$. $\square$

When we assume the existence of a Cartan conjugations on a complex semisimple Lie algebras, we can prove the existence of a Cartan involution on a real semisimple Lie algebra. It will be useful later on to prove the other way around, i.e. when we assume the presence of a Cartan involution on a real algebra, then we can construct a Cartan conjugation on its complexification.

Corollary 13. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let θ_0 be a Cartan involution on $\mathfrak{g}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let $(\theta_0)_{\mathbb{C}}$ denote the complex linear extension of θ_0 to $\mathfrak{g}$. Let σ be the complex conjugation associated to the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Define $\theta = (\theta_0)_{\mathbb{C}}\sigma$. Then θ is a Cartan conjugation on $\mathfrak{g}$ that commutes with σ.*

Proof. Since $(\theta_0)_{\mathbb{C}}$ is complex linear on $\mathfrak{g}$ and since σ is complex anti linear on $\mathfrak{g}$, it is immediate that θ is complex anti linear. We should check that θ is an involution of $\mathfrak{g}^{\mathbb{R}}$. By complex linearity, $(\theta_0)_{\mathbb{C}}$ preservers both $\mathfrak{g}_0$ and $i\mathfrak{g}_0$. On the first of these spaces, σ equals $\mathrm{Id}_{\mathfrak{g}}$ and on the second σ equals $-\mathrm{Id}_{\mathfrak{g}}$. This implies that $(\theta_0)_{\mathbb{C}}$ and σ commute. Clearly, θ is an automorphism of $\mathfrak{g}^{\mathbb{R}}$. On $\mathfrak{g}_0$, σ equals $\mathrm{Id}_{\mathfrak{g}_0}$ which clearly commutes with $(\theta_0)_{\mathbb{C}} = \theta_0$. On $i\mathfrak{g}_0$, σ equals $-\mathrm{Id}_{\mathfrak{g}_0}$ which commutes with $(\theta_0)_{\mathbb{C}} = -i\theta_0$. We conclude that $(\theta_0)_{\mathbb{C}}$ and σ commute from which we obtain $\theta^2 = (\theta_0)_{\mathbb{C}}\sigma(\theta_0)_{\mathbb{C}}\sigma = (\theta_0)_{\mathbb{C}}^2\sigma^2 = \mathrm{Id}_{\mathfrak{g}}$. Hence, $\theta^2 = \mathrm{Id}_g$ and we see that θ is an involution on $\mathfrak{g}^{\mathbb{R}}$. In order to prove that θ is a Cartan, we need to show that θ is a Cartan involution on $\mathfrak{g}^{\mathbb{R}}$

The Killing form $B_{\mathfrak{g}}$ is the complex linear extension of $B_{\mathfrak{g}_0}$ hence $B_{\mathfrak{g}} = (B_{\mathfrak{g}_0})_{\mathbb{C}}$. We need to check that B_θ is positive definite on $\mathfrak{g} \times \mathfrak{g}$. Let $X, Y \in \mathfrak{g}_0$ such that $X + iY \neq 0$. Notice that $\theta(X + iY) = (\theta_0)_{\mathbb{C}}\sigma(X + iY) = (\theta_0)_{\mathbb{C}}X - (\theta_0)_{\mathbb{C}}(Y) = \theta_0(X) - i\theta_0(Y)$. For now, we will adopt the notation $B_\theta(X + iY, X + iY) = -B_{\mathfrak{g}^{\mathbb{R}}}(X + iY, \theta(X + iY))$.

$$\begin{aligned}
B_\theta(X &+ iY, X + iY) \\
&= B_\theta(X, X) + B_\theta(X, iY) + B_\theta(iY, X) + B_\theta(iY, iY) \\
&= -B_{\mathfrak{g}^{\mathbb{R}}}(X, \theta X) - B_{\mathfrak{g}^{\mathbb{R}}}(X, \theta(iY)) - B_{\mathfrak{g}^{\mathbb{R}}}(iY, \theta X) - B_{\mathfrak{g}^{\mathbb{R}}}(iY, \theta(iY)) \\
&= -B_{\mathfrak{g}_0}(X, \theta_0 X) + iB_{\mathfrak{g}_0}(X, \theta_0 Y) - iB_{\mathfrak{g}_0}(Y, \theta_0 X) - (i)(-i)B_{\mathfrak{g}_0}(Y, \theta_0 Y)
\end{aligned}$$

The terms on the far left and far right equal $-B_{\mathfrak{g}_0}(X, \theta_0 X)$ and $-B_{\mathfrak{g}_0}(Y, \theta_0 Y)$ respectively. Both terms are positive since θ_0 is a Cartan involution on $\mathfrak{g}_0$.

Since $B_{\mathfrak{g}_0}$ is symmetric and since $B_{\mathfrak{g}_0}$ respects the decomposition of $\mathfrak{g}_0$ into $\mathfrak{g}_0^{\theta_0} \oplus \mathfrak{g}_0^{\theta_0}$, the two terms in the middle cancel each other. We conclude that B_θ is positive definite on $\mathfrak{g}^{\mathbb{R}} \times \mathfrak{g}^{\mathbb{R}}$ which implies that θ is Cartan on $\mathfrak{g}$.

Finally, we need to show that θ and σ commute. Here we use the fact that $(\theta_0)_{\mathbb{C}}$ and σ commute:

$$\theta\sigma = (\theta_0)_{\mathbb{C}}\sigma\sigma = (\theta_0)_{\mathbb{C}} = \sigma\sigma(\theta_0)_{\mathbb{C}} = \sigma((\theta_0)_{\mathbb{C}}\sigma) = \sigma\ \theta$$

□

2.2 Levi decomposition of parabolic groups

Let G be complex semisimple Lie group and let $\mathfrak{g}$ be its associated Lie algebra. In this section we will examine the Levi decomposition of a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ and its parabolic subgroup $Q \subset G$. We will introduce some notation and study some general properties of parabolic algebras.

Lemma 14. *Let θ be a Cartan conjugation on $\mathfrak{g}$. There exists a θ-stable Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$.*

Proof. Let $\mathfrak{g} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$ where $\mathfrak{g}_u = \mathfrak{g}^\theta$. Let $\mathfrak{t}_0 \subset \mathfrak{g}_u$ be a Cartan subalgebra in $\mathfrak{g}_u$. Define $\mathfrak{t} = \mathfrak{t}_0 \oplus i\mathfrak{t}_0$. Then $\mathfrak{t}$ is θ-stable, maximal and abelian and it consists of semisimple elements since $\mathfrak{t}_0$ consists of semisimple elements. We conclude that $\mathfrak{t}$ is a θ-stable Cartan subalgebra in $\mathfrak{g}$. □

Let $\mathfrak{t}$ be a θ-stable Cartan subalgebra in $\mathfrak{g}$ and let $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ denote the set of roots of $\mathfrak{g}$ relative to $\mathfrak{t}$. Let $\Delta^+ = \Delta^+(\mathfrak{g}, \mathfrak{t})$ be a choice of positive roots. For $\alpha \in \Delta$, define the following subspace of $\mathfrak{g}$:

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} : [H, Y] = \alpha(H)Y, H \in \mathfrak{t}\}$$

The Borel subalgebra $\mathfrak{b}$ associated to the positive system Δ^+ is a subalgebra characterized as follows:

$$\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n} \quad \text{with } \mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

A subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called parabolic if it contains an Int $(\mathfrak{g})$-conjugate of $\mathfrak{b}$.

We will characterize all parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}$. Let $\Pi \subset \Delta^+$ be the system of simple roots in Δ^+ and let Π' be a subset of Π. Define $\Gamma = \Delta^+ \cup \{\alpha \in \Delta : \alpha \in \mathrm{Span}_{\mathbb{Z}}(\Pi')\}$ and note that $\Gamma \subset \Delta$. Then $\mathfrak{q}_{\Pi'} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ is a parabolic subalgebra and it turns out that every parabolic subalgebra containing $\mathfrak{b}$ is of this form [Kna02, Lemma 7.76].

Let $\mathfrak{q}$ denote the parabolic subalgebra $\mathfrak{q}_{\Pi'}$ defined above. Define the following subalgebras of $\mathfrak{q}$:

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Gamma \cap -\Gamma} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_{\mathfrak{q}} = \bigoplus_{\alpha \in \Gamma, \alpha \notin -\Gamma} \mathfrak{g}_\alpha \tag{5}$$

It follows from this definition that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{q}}$. Notice that $\mathfrak{l}$ is reductive and $\mathfrak{n}_{\mathfrak{q}}$ is a maximal solvable ideal of $\mathfrak{q}$. A decomposition of this type is called a **Levi decomposition**(see [Kna02, Cor 5.94]).

Let $Q = N_G(\mathfrak{q})$ be the parabolic subgroup associated to $\mathfrak{q}$ and let $T \subset Q$ be the Cartan subgroup in Q associated to $\mathfrak{t}$. Let L and N_Q denote the Lie subgroups of Q associated to $\mathfrak{l}$ and $\mathfrak{n}_{\mathfrak{q}}$. We can decompose Q as the semidirect product $L \ltimes N_Q$. In general, the Levi component L is not unique, but there is only one Levi component L such that $T \subset L$ [Spr98, Cor. 8.4.4].

Define $\tilde{\mathfrak{n}}_{\mathfrak{q}} = \bigoplus_{\alpha \in \Gamma, \alpha \notin -\Gamma} \mathfrak{g}_{-\alpha}$ and $\tilde{\mathfrak{q}} = \mathfrak{l} \oplus \tilde{\mathfrak{n}}_{\mathfrak{q}}$. The following two equalities will be useful later on and follow directly from definition:

$$\mathfrak{l} = \mathfrak{q} \cap \tilde{\mathfrak{q}} \quad \text{and} \quad \mathfrak{g} = \tilde{\mathfrak{n}}_{\mathfrak{q}} \oplus \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{q}} \tag{6}$$

As a map $\mathfrak{g} \to \mathfrak{g}$, θ is skew linear. Since $\mathfrak{t}$ is $\theta-$stable, this induces a skew linear automorphism on $\mathfrak{t}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}, \mathbb{C})$ by $\lambda \mapsto \lambda \circ \theta^{-1}$ hence $\lambda \mapsto \overline{\lambda} \circ \theta$ is a complex linear map. For $\alpha \in \Delta$, we adopt the notation $\theta(\alpha) = \overline{\alpha \circ \theta}$.

Lemma 15. *Let $\alpha \in \Delta$. Then $\theta(\alpha) = -\alpha$.*

Proof. Let $\mathfrak{t}_{\mathbb{R}}$ denote the real torus in $\mathfrak{t}$ i.e. $\{H \in \mathfrak{t} : \alpha(H) \in \mathbb{R}, \forall \alpha \in \Delta\}$. From the remarks above, it is clear that $\theta(\alpha) \in \mathfrak{t}^*$. It suffices to prove the result for $H \in \mathfrak{t}_{\mathbb{R}}$.

$$\theta\alpha(H) = \overline{\alpha(\theta(H))} = \alpha(-H) = -\alpha(H)$$

□

Lemma 16. *Let $\varphi : \mathfrak{g} \to \mathfrak{g}$ be a skew linear automorphism such that φ leaves $\mathfrak{t}$ invariant. Then $\varphi(\Delta) = \Delta$ and $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\varphi\alpha}$ for all $\alpha \in s\Delta$, where $\varphi\alpha = \overline{\alpha} \circ \varphi^{-1}$.*

Proof. Let $H \in \mathfrak{t}$, $\alpha \in \Delta$ and $X \in \mathfrak{g}_\alpha$.

$$\begin{aligned} [H, \varphi(X)] &= \varphi([\varphi^{-1}H, X]) \\ &= \varphi(\alpha(\varphi^{-1}H)X) \\ &= \overline{\alpha(\varphi^{-1}H)}\varphi(X) = \varphi(\alpha)(H)\varphi(X) \end{aligned}$$

It follows that $\varphi(\alpha) \in \Delta$ and $\varphi(X) \in \mathfrak{g}_{\varphi\alpha}$ □

The following two corollaries follow from Equation 6, Lemma 15 and Lemma 16.

Corollary 17. *Let $\alpha \in \Delta$. Then $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. In general: if $\mathfrak{t}$ is stable under the conjugation δ, then $\delta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\delta(\alpha)}$.*

Corollary 18. *The Cartan conjugation θ sends $\mathfrak{n}_\mathfrak{q}$ to $\tilde{\mathfrak{n}}_\mathfrak{q}$ and $\tilde{\mathfrak{n}}_\mathfrak{q}$ to $\mathfrak{n}_\mathfrak{q}$. We obtain $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q})$.*

2.3 Realizing the flag manifold as an adjoint orbit

Let X be the variety of parabolic subgroups of G that are conjugate to Q. In order to prove the Matsuki correspondence, it is pivotal that we obtain a description X as an adjoint orbit of G_u in $\mathfrak{g}_u$. The goal of this section is to obtain this description (Theorem 22).

Let σ be a complex conjugation on G and let θ be a Cartan conjugation on G that commutes with σ. Define $G_u = G^\theta$ and let $\mathfrak{g}_u$ be its Lie algebra. Let $T \subset Q$ be a θ-stable Cartan subgroup. Let $\mathfrak{t}$ be the Lie algebra of T and define $\mathfrak{t}_u = \mathfrak{t} \cap \mathfrak{g}_u$.

Lemma 19. *Let $Q = L \ltimes N_Q$ be the unique Levi decomposition of Q such that $T \subset L$. Then*

$$X \simeq G/Q \simeq G_u/(G_u \cap L) \tag{7}$$

Proof. Since Q equals the normalizer of Q in G, we can associate X with G/Q (see Section 1.2 for details). Let $G \simeq G_u \exp(i\mathfrak{t}_u)N = G_u T_\mathbb{R} N$ be the Iwasawa decomposition of G. This yields the following decomposition on the parabolic group: $Q = (Q \cap G_u)(T_\mathbb{R} \cap Q)(N \cap Q) = (Q \cap G_u)T_\mathbb{R}N$. We obtain $X \simeq G/Q \simeq G_u/(G_u \cap Q)$. We are left with showing that $G_u \cap L \simeq G_u \cap Q$ holds.

Since $L \subset Q$, $G_u \cap L \subset G_u \cap Q$ is immediate. The other inclusion follows from the observation that L is the largest θ-stable Lie subgroup in Q (this follows from the fact that $\mathfrak{l}$ is the largest θ-stable Lie subalgebra of $\mathfrak{q}$, which follows from Corollary 18). Hence $G_u \cap Q \subset G_u \cap L$ and we conclude that $G_u \cap Q = K \cap L$ which proves the lemma. $\square$

There is a continuous map $G_u \to G_u/(G_u \cap L)$ given by $g \mapsto g(G_u \cap L)$. Since G_u is compact, it follows that X is compact as well. Also, since G_u acts transitively on G_u, it follows that G_u acts transitively on X as well. All elements of X are G_u-conjugate, but not all parabolic subgroups of G are G-conjugate let alone G_u-conjugate.

Lemma 20. *There exists an element $\Upsilon \in \mathfrak{t}_u$ such that $\alpha(\Upsilon) = 0$ for $\alpha \in \Delta(\mathfrak{l}, \mathfrak{t})$, and $\alpha(i\Upsilon) < 0$ for $\alpha \in \Delta(\mathfrak{n}_\mathfrak{q}, \mathfrak{t})$.*

Proof. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a simple system such that $\{\alpha_1, \ldots, \alpha_k\}$ is associated to Q in the sense of Section 2.2. Let $Z_1, \ldots, Z_n$ be the dual basis of $\alpha_1, \ldots, \alpha_n$ in $\mathfrak{t}^{**} \simeq \mathfrak{t}$. Notice the following:

$$\alpha_i(\theta Z_j) = \overline{(\theta\alpha_i)(Z_j)} \overset{A}{=} \overline{-\alpha_i(Z_j)} \overset{B}{=} -\alpha_i(Z_j) = \alpha_i(-Z_j)$$

At equality A we used Lemma 15 and at equality B we used that $\alpha_i(Z_j) = \delta_{ij}$ by definition where δ_{ij} denotes the Kronecker-delta. The above holds for all $\alpha_i \in \Pi$ and since $\mathfrak{t} \simeq \mathfrak{t}_u \oplus i\mathfrak{t}_u$, we get $Z_j \in i\mathfrak{t}_u$. Define $\Upsilon = \sum_{j=1}^{k} iZ_j$. It is immediate that Υ satisfies the claim. $\square$

Combining the results of Lemmas 20 and 15, we see that $\alpha \in \Delta(\tilde{\mathfrak{n}}_\mathfrak{q}, \mathfrak{t})$ implies $\alpha(i\Upsilon) > 0$. By Equation 6 we get that $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ implies $\alpha(\Upsilon, \mathfrak{t}) \in \mathbb{R}$. We will use this later on, specifically in Section 3.3. The following lemma contain a useful properties of Υ_x.

Lemma 21. *Let Υ be as in Lemma 20 and let $Z_G(\Upsilon)$ denote the centralizer of Υ in G. It holds that $Z_\mathfrak{g}(\Upsilon) = \mathfrak{l}$ and $Z_G(\Upsilon) = L$.*

Proof. By Lemma 20 and Equation 6, we see that the $Z_\mathfrak{g}(\Upsilon) = \mathfrak{l}$. Hence, on subgroup level, we obtain the identity $Z_G(\Upsilon)_e = L_e$ where the subscript e denotes the connected component of the identity.

It is known that L is connected (see [Spr98, Cor. 8.4.4]) and that $Z_G(\Upsilon)$ is connected (see [Spr98, Th. 6.4.7] and notice $Z_G(\Upsilon) = Z_G(\exp \mathbb{C}\Upsilon))$. Combining this yields $Z_G(\Upsilon) = L$. $\square$

Theorem 22. *Let $f : X \to \mathrm{Ad}(G_u)\Upsilon$ be given by $f(gQg^{-1}) = \mathrm{Ad}(g)\Upsilon$ for $g \in G_u$. The map f is a diffeomorphism between X and the adjoint G_u-orbit of Υ in $\mathfrak{g}_u$.*

Proof. Denote the G-stabilizer of Υ by $Z_G(\Upsilon)$ and denote the $\mathfrak{g}$-stabilizer of Υ by $Z_{\mathfrak{g}}(\Upsilon)$. By Lemma 19, the map f is well defined if $Z_{G_u}(\Upsilon)$ equals $G_u \cap L$. From Lemma 21, we obtain $Z_G(\Upsilon) = L$ which implies $Z_{G_u}(\Upsilon) = L \cap G_u$ by restriction. Hence, f is well defined and it follows immediately that f is one-to-one. Since f is surjective by construction, it follows that f is bijective. $\square$

By the theorem above, there exists a $g \in G_u/(G_u \cap L)$ such that $Q_x = gQg^{-1}$. We can find a Levi decomposition for Q_x by setting $L_x = gLg^{-1}$ and $N_x = gN_{\mathfrak{q}}g^{-1}$. Notice that the Levi factor L_x is well defined since $g \in G_u$ is determined modulo $G_u \cap L$. Similarly, on the algebra level we may set $\mathfrak{l}_x := \mathrm{Ad}(g)\mathfrak{l}$, $\mathfrak{n}_x = \mathrm{Ad}(g)\mathfrak{n}_{\mathfrak{q}}$ and $\tilde{\mathfrak{n}}_x = \mathrm{Ad}(g)\tilde{\mathfrak{n}_{\mathfrak{q}}}$. With regard to Equation 6, we may write:

$$\mathfrak{g} = \tilde{\mathfrak{n}}_x \oplus \mathfrak{l}_x \oplus \mathfrak{n}_x \tag{8}$$

This decomposition does not intrinsically depend on x, but on the choice of Levi decomposition of Q i.e. on the initial choice of Cartan subgroup T and Cartan conjugation θ. Continuing in this notation, Theorem 22 gives a correspondence between $x \in X$ and $\Upsilon_x = \mathrm{Ad}(g)\Upsilon$ in the G_u-orbit of Υ in $\mathfrak{g}_u$. Let $\mathcal{O}$ denote this orbit, hence $\mathcal{O} = \mathrm{Ad}(G_u)\Upsilon$. Let $\mathfrak{q}_x$ be the Lie algebra of Q_x. We get the following identifications:

$$\mathfrak{g}/\mathfrak{q}_x \simeq T_xX \simeq T_{\Upsilon_x}\mathcal{O} \simeq \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x) \tag{9}$$

The following lemma will be useful later on.

Lemma 23. *Let $\mathfrak{s}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{s} \subset \mathfrak{q}_x$. Then $\mathfrak{s} \subset \mathfrak{l}_x$ and $\Upsilon_x \in \mathfrak{s}$.*

Proof. By assumption, we get $\theta\mathfrak{s} \subset \theta\mathfrak{q}_x$. Since $\mathfrak{s}$ is θ-stable, we see that $\mathfrak{s} = \mathfrak{s} \cap \theta\mathfrak{s} \subset \mathfrak{q}_x \cap \theta\mathfrak{q}_x$. By Corollary 18, we obtain $\mathfrak{s} \subset \mathfrak{l}_x$.
By combining Lemmas 21 and 6, we deduce that $\mathfrak{l}_x$ is reductive. Since $\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{l}_x$ is reductive, we obtain that $\mathfrak{s}$ is a Cartan subalgebra in $\mathfrak{l}_x$. By using Lemma 21 once more, we see that Υ_x belongs to the center of $\mathfrak{l}_x$, hence $\Upsilon_x \in \mathfrak{s}$. $\square$

2.4 The tangent space of X

Using the action of G_u, we will describe the tangent space of X at any $x \in X$ (Lemma 24).

We will use $T_xX \simeq \mathfrak{g}/\mathfrak{q}_x$ as in Equation 9. Let $\iota : \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x) \to \mathfrak{g}/\mathfrak{q}_x$ be the inclusion map induced by the inclusion $\mathfrak{g}_u \subset \mathfrak{g}$ (this is well-defined since $\mathfrak{g}_u \cap \mathfrak{l}_x \subset \mathfrak{q}_x$). Note that ι is a linear isomorphism. For $Z \in \mathfrak{g}$, we will denote the decomposition with respect to Equation 8 as $Z = Z_{\tilde{\mathfrak{n}}_x} + Z_{\mathfrak{l}_x} + Z_{\mathfrak{n}_x}$.

Lemma 24. *Let $I_x : \mathfrak{g}/\mathfrak{q}_x \to \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ be defined by*

$$I_x(Z + \mathfrak{q}_x) = (Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) + (\mathfrak{g}_u \cap \mathfrak{l}_x), \quad Z \in \mathfrak{g}$$

Then I_x is the inverse of ι.

Proof. We will first show that I_x is well defined. Let $Z_1 + \mathfrak{q}_x = Z_2 + \mathfrak{q}_x$. Since $\mathfrak{q}_x = \mathfrak{l}_x \oplus \mathfrak{n}_x$, Equation 8 gives us that Z_1 and Z_2 have identical $\tilde{\mathfrak{n}}_x$-component. Since I_x depends solely on this component, we get $I_x(Z_1 + \mathfrak{q}_x) = I_x(Z_2 + \mathfrak{q}_x)$. Notice that $(Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x})$ is θ-invariant, which implies $(Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) \in \mathfrak{g}_u$ and we conclude that I_x is well defined.

Next, we will show that I_x is the inverse of ι. Since $\mathfrak{q}_x = \mathfrak{l} \oplus \mathfrak{n}_x$, Equation 8 tells us that $Z - (Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) \in \mathfrak{q}_x$. Hence, the element $Z \in \mathfrak{g}/\mathfrak{q}_x$ can be represented by $(Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) + \mathfrak{q}_x$. Using this, we can show that I_x is a right inverse for ι:

$$\iota(I_x(Z + \mathfrak{q}_x)) = \iota((Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) + (\mathfrak{g}_u \cap \mathfrak{l}_x)) = (Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) + \mathfrak{q}_x = Z$$

To show that I_x is a left inverse for ι, observe that $Z \in \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ implies that $\theta(Z) = Z$. From Corollary 18 we obtain that this implies $\theta Z_{\tilde{\mathfrak{n}}_x} = Z_{\mathfrak{n}_x}$. Using Equation 8 once more, we show that I_x is a left inverse of ι:

$$\begin{aligned} I_x(\iota(Z + (\mathfrak{g}_u \cap \mathfrak{l}_x))) &= I_x(\iota(Z_{\tilde{\mathfrak{n}}_x} + Z_{\mathfrak{n}_x} + (\mathfrak{g}_u \cap \mathfrak{l}_x))) \\ &= I_x(Z_{\tilde{\mathfrak{n}}_x} + \mathfrak{q}_x) \\ &= (Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}) + (\mathfrak{g}_u \cap \mathfrak{l}_x) = Z + (\mathfrak{g}_u \cap \mathfrak{l}_x) \end{aligned}$$

□

2.5 Vector fields and Kähler structure on X

In this section we will describe a Kähler structure on X. Connected to this subject, we will introduce and study the almost complex structure on X induced by the complex structure of G (Lemma 25), and finally we will construct a specific Kähler form on X (Theorem 28).

Let M be a smooth manifold. A Kähler structure on M is an interplay between four different but compatible structures: symplectic, Riemannian, almost complex and Hermitian structure. All structures vary smoothly with respect to $x \in M$.
Symplectic structure: a 2-form ω on M that is both closed and non-degenerate. The 2-form ω is called the symplectic form.
Riemannian structure: a form g on M that is positive definite, symmetric and bilinear. The form g is called the Riemannian metric.
Almost complex structure: a complex structure $J_x : T_xM \to T_xM$, $J_x^2 = -\mathrm{Id}_M$ on each tangent space T_xM, depending smoothly on $x \in M$.

Now assume that M is a complex manifold i.e. around each $x \in M$ there are complex coordinates in $\mathbb{C}^n$ such that the transition maps between charts are biholomorphic. Multiplication by i in the tangent space gives M a natural almost complex structure, denote it with J.
Hermitian structure: a form h on M that is positive definite, skew symmetric and sesquilinear. The sesquilinearity should be considered with respect to J i.e. for u and v smooth vector fields on M, we get $h(u_x, J_xv_x) = \bar{i}h(u_x, v_x)$. The form h is called the Hermitian form.

Using the above structures, we can define a Kähler structure on a complex manifold M:
Kähler structure: a Hermitian form h on M such that its imaginary part is a closed 2-form. It follows from the non-degeneracy of h, that its imaginary part is non-degenerate as well and we see that the imaginary part is in fact a symplectic form. For this reason, Im h is denoted with ω which is called the Kähler form.

Kähler structures on M can be defined on manifolds that do not have an a priori Hermitian structure. In order to do this, we need a compatibility condition on three different structures: symplectic, Riemannian and almost complex structure. Let M be a complex, smooth manifold. We call the above structures compatible on T_xM, if they satisfy the following equations (let u

and v be smooth vector fields on M):

$$g_x(u_x, v_x) = \omega_x(J_x u_x, v_x), \qquad \omega_x(u_x, v_x) = g_x(u_x, J_x v_x) \tag{10}$$

The triplet (g, ω, J) is called compatible if it is compatible for all $x \in M$. For a compatible triplet it is straightforward to verify that $g + i\omega$ is a Hermitian form. The notation $h = g + i\omega$ is used in both approaches of Kähler structure (for more details on Kähler structure, see [Hec, Section 3.1]).

Since G/Q is a complex manifold, there is a complex structure induced on $G_u/(G_u \cap L_x)$. A complex structure induces an almost complex structure J, hence by Lemma 19 there is an almost complex structure on X. The following lemma shows what the almost complex structure on X looks like.

Lemma 25. *The almost complex structure J_x at $T_{\Upsilon_x}(\mathcal{O}) \simeq \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ induced by isomorphism with $\mathfrak{g}/\mathfrak{q}_x$, is given by*

$$Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x} + (\mathfrak{g} \cap \mathfrak{l}_x) \overset{J_x}{\mapsto} i(Z_{\tilde{\mathfrak{n}}_x} - \theta Z_{\tilde{\mathfrak{n}}_x}) + (\mathfrak{g} \cap \mathfrak{l}_x)$$

Proof. The almost complex structure J_x on $\mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ is compatible with the complex structure on $\mathfrak{g}/\mathfrak{q}_x$. This means $J_x(I_x(Z + \mathfrak{q}_x)) = I_x(iZ + \mathfrak{q}_x)$. The term on the right evaluates to $iZ_{\tilde{\mathfrak{n}}_x} + \theta i Z_{\tilde{\mathfrak{n}}_x} + (\mathfrak{g}_u \cap \mathfrak{l}_x) = i(Z_{\tilde{\mathfrak{n}}_x} - \theta Z_{\tilde{\mathfrak{n}}_x}) + (\mathfrak{g}_u \cap \mathfrak{l}_x)$ from which the desired result follows. □

The above result is part of a bigger picture. From Equation 8, we obtain $\tilde{\mathfrak{n}}_x \simeq \mathfrak{g}/\mathfrak{q}_x$ as complex linear spaces. By Lemma 19 we get $\mathfrak{q}/\mathfrak{q}_x \simeq \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ as complex linear spaces. This induces $\tilde{\mathfrak{n}}_x \simeq \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$ as complex linear spaces.

For $Z \in \mathfrak{g}$, let ξ_Z denote the vector field on X induced by Z, i.e. $\xi_Z(y) = \partial_t|_{t=0} \exp(tZ)y$ for $y \in X$. By Equation 9 and Lemma 24, we get the following for $Z \in \mathfrak{g}$:

$$\xi_Z(x) = \xi_{(Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x})}(x) \tag{11}$$

By Lemma 25 we obtain

$$J_x \xi_Z(x) = \xi_{i(Z_{\tilde{\mathfrak{n}}_x} - \theta Z_{\tilde{\mathfrak{n}}_x})}(x) \tag{12}$$

For brevity of notation, we will often write ξ_Z instead of $\xi_Z(y)$ when it is clear from context whether we mean the vector field ξ_Z of X, or the vector $\xi_Z(y) \in T_y X$ tangent to X in y. The following lemma will be useful later on.

Lemma 26. *Let $g \in G_u$, $Z \in \mathfrak{g}_u$ and $x \in X$. Then $T_x l_g \xi_Z(x) = \xi_{\mathrm{Ad}(g)Z}(gx)$ where l_x denotes the left action of G_u on X.*

Proof.

$$\begin{aligned}
T_x l_g \xi_Z(x) &= T_x l_g \, \partial_t|_{t=0} \exp(tZ)x \\
&= \partial_t|_{t=0}\, g \exp(tZ)x = \partial_t|_{t=0} \exp(t\mathrm{Ad}(g)Z)gx = \xi_{\mathrm{Ad}(g)Z}(gx)
\end{aligned}$$

□

At this point, the (almost) complex structure is the only part of the triple present at X. It turns out that the coadjoint orbits of G have a G-invariant symplectic structure (see [Aud04, Page 60/61]). Inspired by the construction of this symplectic structure, we define the following form on X (B is the Killing form on $\mathfrak{g}$).

$$\omega_x(\xi_Z(x), \xi_W(x)) = -B(\Upsilon_x, [Z, W]) \quad x \in X, \quad Z, W \in \mathfrak{g}_u \tag{13}$$

On the left, Z and W are elements of $T_x X \simeq \mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$. Hence, we should check that the right hand part of the above definition is independent of choice of representative in $\mathfrak{g}_u \cap \mathfrak{l}_x$. From the Ad-invariance of the Killing form, we obtain $-B(\Upsilon_x, [Z, W]) = -B([\Upsilon_x, Z], W) = B([\Upsilon_x, W], Z)$. Since $Z_G(\Upsilon_x) = \mathfrak{l}_x$, any component of Z and W in $\mathfrak{l}_x$ is send to zero. Hence, $\omega_x(\xi_Z(x), \xi_W(x))$ with $Z, W \in \mathfrak{g}_u$, does not depend on the choice of representative of in $\mathfrak{g}_u/(\mathfrak{g}_u \cap \mathfrak{l}_x)$.

We will show that the Equation 13 defines a symplectic form on X, the following lemma will be useful for this:

Lemma 27. *The form defined in Equation 13 is symplectic and G_u-invariant.*

Proof. Since B is bilinear and since $[,]$ is bilinear and skew-symmetric, it follows that ω is linear and skew-symmetric, hence is a well-defined 2-form on X. Next, since G is semisimple, the Killing form is non-degenerate from which it follows that ω_x is non-degenerate (if $[Z, W] = 0$ for all $W \in \mathfrak{g}_u$, then $\mathrm{ad}(Z) = 0$ as a map $\mathfrak{g}_u \to \mathfrak{g}_u$). For G_u-invariance, notice that G_u acts on X by left multiplication:

$$\begin{aligned}
g^{-1} \cdot \omega_x(\xi_Z(x), \xi_W(x)) &= \omega_{gx}(T_e l_g \xi_Z(x), T_e l_g \xi_W(x)) \\
&\overset{A}{=} -B(\Upsilon_{gx}, [\mathrm{Ad}(g)Z, \mathrm{Ad}(g)W]) \\
&= -B(\mathrm{Ad}(g)\Upsilon_x, \mathrm{Ad}(g)[Z, W]) \\
&\overset{B}{=} -B(\Upsilon_x, [Z, W]) = \omega_x(\xi_Z(x), \xi_W(x))
\end{aligned}$$

At Equality A we use Lemma 26, at Equality B we use the fact that the Killing form is invariant under the adjoint action of G. We conclude that ω is G_u-invariant. For the proof that ω is closed, we refer to [vdB16, Prop. 4.4]. $\square$

Using the symplectic form from Equation 13, we can construct a Kähler form on X.

Theorem 28. *There is a G_u-invariant Kähler form on X given by*

$$\langle \xi_Z, \xi_W \rangle_x = -2iB(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}]) \tag{14}$$

Here, $x \in X$ and $Z, W \in \mathfrak{g}$ induce left invariant vector fields ξ_Z, ξ_W on X.

Proof. Following the remarks above, the natural construction for a Kähler form is

$$h(\xi_Z, \xi_W) = \langle \xi_Z, \xi_W \rangle = \omega(J\xi_Z, \xi_W) + i\omega(\xi_Z, \xi_W) = g(\xi_Z, \xi_W) + i\omega(\xi_Z, \xi_W) \tag{15}$$

Here we defined g to be $(\xi_Z, \xi_W) \mapsto \omega(J\xi_Z, \xi_W)$, hence for $x \in X$ we get $g_x(\xi_Z(x), \xi_W(z)) = \omega(J_x\xi_Z(x), \xi_W)$. We will prove that ω is real and then we will show that g is positive definite which turns h in a Kähler form (and g in a Riemannian metric). After some work, we will end up with the result from Equation 28.

$$\begin{aligned} \omega_x(\xi_Z, \xi_W) &= \omega_x(\xi_{Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}}, \xi_{W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x}}) \\ &= -B(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x}]) \end{aligned}$$

Recall that B is compatible with the Cartan conjugation θ i.e. the eigenspaces of θ are B-orthogonal to each other. Since $\Upsilon_x \in \mathfrak{t}_u = \mathfrak{t} \cap \mathfrak{g}_u$, we get $\theta(\Upsilon_x) = \Upsilon_x$ and hence, we may ignore elements in the minus eigenspace of θ in the right argument of B. We obtain the following expression:

$$\omega_x(\xi_Z, \xi_W) = -B(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] + [\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) \tag{16}$$

One of the definitions of a Kähler structure, is that ω is the imaginary part of a Hermitian structure h. By Equation 15 this means that we want ω to be real. We can check that ω is real, by checking if it is J-invariant:

$$\begin{aligned} \omega_x(J_x\xi_Z, J_x\xi_W) &= \omega_x(\xi_{iZ_{\tilde{\mathfrak{n}}_x} - i\theta Z_{\tilde{\mathfrak{n}}_x}}, \xi_{iW_{\tilde{\mathfrak{n}}_x} - i\theta W_{\tilde{\mathfrak{n}}_x}}) \\ &= -B(\Upsilon_x, [iZ_{\tilde{\mathfrak{n}}_x}, -i\theta W_{\tilde{\mathfrak{n}}_x}] + [i\theta Z_{\tilde{\mathfrak{n}}_x}, -iW_{\tilde{\mathfrak{n}}_x}]) \\ &= -B(\Upsilon_x, -i^2[Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] - i^2[\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) = \omega(\xi_Z, \xi_W) \end{aligned}$$

Next, lets turn our attention to g from Equation 14. Clearly g is bilinear, symmetric and it depends smoothly on x since the assignment $x \mapsto \Upsilon_x$ is smooth (this follows from the fact that $g \mapsto \mathrm{Ad}(g)$ is smooth). In order to prove that g is a Riemannian metric, it suffices to prove that g is positive definite.

$$\begin{aligned} g_x(\xi_Z, \xi_W) &= \omega(J_x\xi_Z, \xi_W) \\ &= -B(\Upsilon_x, [iZ_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] + [-i\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) \\ &= -B(i\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] - [\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) \end{aligned} \tag{17}$$

Set $W = Z$ and take the root space decomposition of: $Z_{\tilde{\mathfrak{n}}_x} = \sum_{\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{t})} Z_\alpha$. We get the following expression for $g(\xi_Z, \xi_Z)$:

$$\begin{aligned} g_x(\xi_Z, \xi_Z) &= -B(i\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta Z_{\tilde{\mathfrak{n}}_x}] - [\theta Z_{\tilde{\mathfrak{n}}_x}, Z_{\tilde{\mathfrak{n}}_x}]) \\ &= -B(i\Upsilon_x, 2[Z_{\tilde{\mathfrak{n}}_x}, \theta Z_{\tilde{\mathfrak{n}}_x}]) \\ &= -2B(i\Upsilon_x, [\sum Z_\alpha, \sum \theta Z_\alpha]) \\ &= -2B([i\Upsilon_x, \sum Z_\alpha], \sum \theta Z_\alpha]) \\ &= -2B(\sum \alpha(i\Upsilon_x) Z_\alpha, \sum \theta Z_\alpha]) \overset{(1.)}{=} -2\sum \alpha(i\Upsilon_x) B(Z_\alpha, \theta Z_\alpha]) \end{aligned}$$

In step (1.) we used that the individual root spaces Z_α are B-orthogonal to each other. By Lemma 20, $\alpha(i\Upsilon_x) < 0$ and remember that B is positive definite on the negative eigenspace of θ. We conclude that the above expression is positive for $Z \neq 0$ and we conclude that g is positive definite (since ω is skew-symmetric, it follows that h is positive definite as well).

Finally, let us show that $h(\xi_Z, \xi_W) = \langle \xi_Z, \xi_W \rangle$, satisfies Equation 14. We combine Equation 17 and Equation 16 to obtain:

$$\begin{aligned} h_x(\xi_Z, \xi_W) &= g_x(J_x\xi_Z, \xi_W) + i\omega(\xi_Z, \xi_W) \\ &= -iB(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] - [\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) \\ &+ -iB(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}] + [\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}]) \\ &= -2iB(\Upsilon_x, [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}]) \end{aligned}$$

$\square$

3 Properties of $f^{\pm}$

3.1 The functions $f^{\pm}$ and their critical points

In this section, we shall define real-valued functions $f^{\pm}$ on X using the moment map with respect to the symplectic form ω (defined in section 2.5). We shall investigate some of its properties including critical points i.e. the points where $\partial_x f^{\pm}(x) = 0$.

Let $Z \in \mathfrak{g}_u$ and let ξ_Z be its associated vector field on X. A moment map is a map $m_{G_u} : X \to \mathfrak{g}_u^*$ such that $dm_{G_u}(V)(Z) = \omega(V, \xi_Z)$ holds. A moment map is called G_u-equivariant if for $g \in G_u$, $m_{G_u}(g^{-1}x)(Z) = m_{G_u}(x)(\mathrm{Ad}(g)Z)$. A G_u-equivariant moment map is unique up to addition with a constant linear form in $\{\lambda \in \mathfrak{g}_u^* : \lambda|_{[\mathfrak{g}_u, \mathfrak{g}_u]} = 0\}$ (the annihilator of $[\mathfrak{g}_u, \mathfrak{g}_u]$). Since $\mathfrak{g}_u$ is semisimple, the annihilator equals the zero ideal and the G_u-equivariant moment map is unique (see [Aud04, Page 75]). The following lemma asserts the existence of the G_u-equivariant moment map.

Lemma 29. *The G_u-equivariant moment map is given by $m_{G_u}(x)(Z) = B(\Upsilon_x, Z)$ with $\Upsilon_x \in \mathfrak{t}_u$ as in Section 2.3 and B the Killing form on $\mathfrak{g}$.*

Proof. Since G_u acts transitively on X, it suffices to prove $dm_{G_u}(V)(Z) = \omega(V, \xi_Z)$ for $V = \xi_W$ with $W \in \mathfrak{g}_u$.

$$\begin{aligned}
(dm_{G_u})_x(\xi_W)(Z) &= \partial_t|_{t=0} B(\exp(tW)\Upsilon_x, Z) \\
&\overset{A}{=} B(\partial_t|_{t=0} \mathrm{Ad}(\exp(tW))\Upsilon_x, Z) \\
&= B(\mathrm{ad}(W)\Upsilon_x, Z) \\
&= B([W, \Upsilon_x], Z) = -B(\Upsilon_x, [W, Z]) = \omega(\xi_W, \xi_Z)
\end{aligned}$$

At Equation A we used the linearity of the Killing form, we end up with the expression for ω from Equation 13. We conclude that m_{G_u} is a moment map. Let $g \in G_u$, the following computation shows that m_{G_u} is G_u-equivariant.

$$\begin{aligned}
m_{G_u}(gx)(Z) &= B(\Upsilon_{gx}, Z) \\
&= B(\mathrm{Ad}(g)\Upsilon_x, Z) = B(\Upsilon_x, \mathrm{Ad}(g)^{-1}Z) = m_{G_u}(x)(\mathrm{Ad}(g^{-1}Z)
\end{aligned}$$

$\square$

Since the Killing form is non-degenerate ($\mathfrak{g}_u \subset \mathfrak{g}$ is semisimple), it gives us an identification between $\mathfrak{g}_u$ and $\mathfrak{g}_u^*$ via $Z \mapsto B(Z, *)$. From this, we see that $\mu_{G_u} : X \to \mathfrak{g}_u$ given by $\mu_{G_u}(x) = \Upsilon_x$, is in fact the dual map of the moment map m_{G_u}. Notice that this is precisely the function f from Theorem 22. The differential of the dual map equals the map I_x from Lemma 24.

On $\mathfrak{g}_u$, the Killing form B is negative definite. This makes the map $Z \mapsto -B(Z,Z)$ into a norm-squared function and we will denote it by $||.||_B^2$. Through the identification between $\mathfrak{g}_u$ and $\mathfrak{g}_u^*$, this norm is transferred to $\mathfrak{g}_u^*$ as well. Using the notation $\Upsilon_x = \mathrm{Ad}(g)\Upsilon$ for some $g \in G_u$ (see Section 2.3), we get the following:

$$\begin{aligned} ||m_{G_u}(x)||_B^2 &= ||\mu_{G_u}(x)||_B^2 \\ &= -B(\Upsilon_x, \Upsilon_x) = -B(\mathrm{Ad}(g)\Upsilon, \mathrm{Ad}(g)\Upsilon) = -B(\Upsilon, \Upsilon) \end{aligned} \quad (18)$$

Here we used that the Killing form is invariant under the adjoint action. Notice that the norm of the moment map and its dual are constant on X.

Let σ be a complex conjugation on G that commutes with θ. Let $K = G^{\theta\sigma}$ and let $K_0 = K^\sigma$ be the real form of this subgroup with respect to σ. Let $\mathfrak{k}$ and $\mathfrak{k}_0$ denote their respective Lie algebras. Then $m_{K_0}(x)(Z) = B(\Upsilon_x, Z)$ for $Z \in \mathfrak{k}_0$ denotes a moment map with respect to the K_0-action on X. Thus the moment map for K_0 is obtained from the map for G_u by restriction. Hence, $\mu_{K_0}(x)$ is the projection of Υ_x onto $\mathfrak{k}_0$ in the decomposition $\mathfrak{g}_u = \mathfrak{k}_0 \oplus \mathfrak{g}_u^{-\sigma}$. Through this observation, we arrive at the definition of $f^\pm$:

$$f^\pm : X \to \mathbb{R}, \qquad f^\pm(x) = 2||Pr_{\pm\sigma}\mu_{G_u}(x)||_B^2 \quad (19)$$

In this equation, $Pr_{\pm\sigma}$ is the projection onto $\mathfrak{g}_u^{\pm\sigma}$ in $\mathfrak{g}_u = \mathfrak{g}_u^\sigma \oplus \mathfrak{g}_u^{-\sigma}$, hence the plus and minus superscript of f are associated to the plus and minus eigenspace of σ in $\mathfrak{g}_u$. Notice that $f^+(x) = 2||\mu_{K_0}(x)||_B^2$. We will now investigate some properties of $f^\pm$.

Lemma 30. *The functions $f^\pm$ are K_0-invariant and their sum is constant.*

Proof. Notice that the projection onto the $\pm$-eigenspace of σ, can be written in the following way: $Pr_{\pm\sigma}\mu_{G_u}(x) = \frac{1}{2}(\mu_{G_u}(x) \pm \sigma\mu_{G_u}(x))$ (this formula also explains the factor 2 in Equation 19). If we substitute this formula in the definition of $f^\pm$, we get the following:

$$\begin{aligned} f^\pm(x) &= -2B(\frac{1}{2}(\mu_{G_u}(x) \pm \sigma\mu_{G_u}(x)), \frac{1}{2}(\mu_{G_u}(x) \pm \sigma\mu_{G_u}(x))) \\ &= -B(\Upsilon_x, \Upsilon_x) \mp B(\Upsilon_x, \sigma\Upsilon_x) \end{aligned} \quad (20)$$

Since the Killing form is invariant under the adjoint action, we get $B(\Upsilon_x, \Upsilon_x) = B(\Upsilon, \Upsilon)$ (see Equation 18). Hence $f^+(x) + f^-(x) = -2B(\Upsilon, \Upsilon)$ which is constant. For K_0-invariance, let $k \in K_0$. Notice the following:

$$B(\mathrm{Ad}(k)\Upsilon_x, \sigma \mathrm{Ad}(k)\Upsilon_x) = B(\mathrm{Ad}(k)\Upsilon_x, \mathrm{Ad}(k)\sigma\Upsilon_x) = B(\Upsilon_x, \sigma\Upsilon_x)$$

Combining this result with Equation 20 yields the K_0-invariance of $f^\pm$. $\square$

Lemma 31. *For $Z \in \mathfrak{g}_u$ and $x \in X$, we get*

$$\xi_Z(f^+)(x) = -2B(Z, [\mu_{G_u}(x), \sigma\mu_{G_u}(x)])$$

Here $\xi_Z(f^+)$ denotes the directional derivative of f^+ with respect to the vector field ξ_Z.

Proof. If h denotes a differentiable function on $\mathbb{R}^2$, then $\partial_t|_{t=0}\, h(t,t) = \partial_t|_{t=0}\, h(t,0) + \partial_t|_{t=0}\, h(0,t)$. We can use a similar trick here as well:

$$\begin{aligned}
-\xi_Z(f^+)(x) &= -\,\partial_t|_{t=0}\, f^+(\exp(tZ)x) && (21)\\
&= -\,\partial_t|_{t=0}\, B(\Upsilon,\Upsilon) - B(\mathrm{Ad}(\exp(tZ)\Upsilon_x, \sigma\mathrm{Ad}(\exp(tZ)\Upsilon_x)\\
&= B([Z,\Upsilon_x], \sigma\Upsilon_x) + B(\Upsilon_x, \sigma[Z,\Upsilon_x])\\
&= B([Z,\Upsilon_x], \sigma\Upsilon_x) + B(\Upsilon_x, [\sigma Z, \sigma\Upsilon_x])\\
&= B(Z, [\Upsilon_x, \sigma\Upsilon_x]) - B(\sigma Z, [\Upsilon_x, \sigma\Upsilon_x]) && (22)
\end{aligned}$$

In the last line we used that the Killing form is associative with respect to the Lie bracket (i.e. $B([X,Y],Z) = B(X,[Y,Z])$. Next, we will use that σ is its own inverse and that B is invariant under σ on $\mathfrak{g}_u$ (notice that $B_{\mathfrak{g}_u}$ is real and use Lemma 2)to get a nice expression for the right hand side of the last line above:

$$\begin{aligned}
B(\sigma Z, [\Upsilon_x, \sigma\Upsilon_x]) &= B(\sigma\sigma Z, \sigma[\Upsilon_x, \sigma\Upsilon_x])\\
&= B(Z, [\sigma\Upsilon_x, \Upsilon_x]) = -B(Z, [\Upsilon_x, \sigma\Upsilon_x])
\end{aligned}$$

Combining this equation with Equation 22 yields the following result:

$$\begin{aligned}
-\xi_Z(f^+)(x) &= B(Z, [\Upsilon_x, \sigma\Upsilon_x]) - B(\sigma Z, [\Upsilon_x, \sigma\Upsilon_x])\\
&= B(Z, [\Upsilon_x, \sigma\Upsilon_x]) + B(Z, [\Upsilon_x, \sigma\Upsilon_x]) = 2B(Z, [\Upsilon_x, \sigma\Upsilon_x])
\end{aligned}$$

$\square$

Theorem 32. *Let $x \in X$. The following statements are equivalent.*
(a) x is a critical point for f^+.
(b) x is a critical point for f^-.
(c) $[\Upsilon_x, \sigma\Upsilon_x] = 0$.
(d) $\sigma(\Upsilon_x) \in \mathfrak{l}_x$.
(e) $\mu_{K_0}(x) \in \mathfrak{q}_x$.

Proof. **(a) $\Leftrightarrow$ (b)** Follows immediately from Lemma 30.
(a) $\Leftrightarrow$ (c) For a differentiable function, a point x is critical if and only if its derivative equals zero. Since the Killing form is non-degenerate, the result follows from Lemma 31.
(c) $\Leftrightarrow$ (d) It follows from Lemma 20 that $\mathfrak{l}$ is the centralizer of Υ. Hence, $\mathfrak{l}_x$ is the centralizer of Υ_x from which the result follows.
(d) $\Rightarrow$ (e) Assume (d). Since $\Upsilon \in \mathfrak{t}_u \subset \mathfrak{q}$, it follows that $\Upsilon_x \in \mathfrak{q}_x$. By (d), we get $\sigma(\Upsilon_x) \in \mathfrak{l}_x \subset \mathfrak{q}_x$. Hence $2\mu_{K_0}(x) = 2Pr_\sigma \Upsilon_x = \Upsilon_x + \sigma\Upsilon_x \in \mathfrak{q}_x$.
(d) $\Leftarrow$ (e) By Section 2.3 there exists a $g \in G_u$ such that $\Upsilon_x = \mathrm{Ad}(g)\Upsilon$. Notice that $\theta\mathrm{Ad}(g)\Upsilon = \mathrm{Ad}(\theta g)\theta\Upsilon = \mathrm{Ad}(g)\Upsilon$ since $\Upsilon \in \mathfrak{t}_u$. This implies $\theta\mu_{K_0}(x) = \mu_{K_0}(x)$. Assume (e) i.e. $\sigma(\mu_{K_0}(x)) \in \mathfrak{q}_x$. Since θ and σ commute, we get $\sigma(\mu_{K_0}(x)) \in \mathfrak{q}_x \cap \theta(\mathfrak{q}_x)$. Corollary 18 implies that $\sigma(\mu_{K_0}(x)) \in \mathfrak{l}_x$. □

Let us place a small remark here. The manifold X is compact (see Section 2.3) and the functions $f^\pm : X \to \mathbb{R}$ are continuous. This implies that $f^\pm$ will attain a maximum and a minimum. This implies that $f^\pm$ will have at least one critical value.

Theorem 33. *Let $x \in X$. The following statements are equivalent.*
(a) x is a critical point of $f^\pm$.
(b) $\mathfrak{q}_x$ contains a σ- and θ-stable Cartan subalgebra of $\mathfrak{g}$.

Proof. **(a) $\Rightarrow$ (b)** Assume (a). Write $\Upsilon_x = \Upsilon_\sigma + \Upsilon_{-\sigma}$ with respect to the decomposition $\Upsilon_x \in \mathfrak{t}_u = \mathfrak{g}_u^\sigma \oplus \mathfrak{g}_u^{-\sigma}$. By combining the decomposition with Theorem 32, we see that $0 = [\Upsilon_x, \sigma\Upsilon_x] = [\Upsilon_\sigma + \Upsilon_{-\sigma}, \Upsilon_\sigma - \Upsilon_{-\sigma}] = -2[\Upsilon_\sigma, \Upsilon_{-\sigma}]$ and we obtain that Υ_σ and $\Upsilon_{-\sigma}$ are commuting elements.

The following arguments show that that $\mathrm{ad}(\Upsilon_{\pm\sigma})$ is diagonalizable with the respect to B_θ; let $Z \in \mathfrak{g}_u$ and $U, V \in \mathfrak{g}$:

$$\begin{aligned}
B_\theta(\mathrm{ad}(Z)U, V) &= -B(\mathrm{ad}(Z)U, \theta V) \\
&= +B(U, \mathrm{ad}(Z)\theta V) \\
&= +B(U, \theta(\mathrm{ad}(\theta Z)V)) \\
&= +B(U, \theta(\mathrm{ad}(Z)V)) = -B_\theta(U, \mathrm{ad}(Z)V)
\end{aligned}$$

Hence $\mathrm{ad}(Z)^* = -\mathrm{ad}(Z)$. We conclude that $\mathrm{ad}(Z)$ is diagonalizable and specifically that $\mathrm{ad}(\Upsilon_{\pm\sigma})$ are diagonalizable, hence $\Upsilon_{\pm\sigma}$ are semisimple elements. Define $\mathfrak{s}_0 = \mathrm{Span}\{\Upsilon_x, \sigma\Upsilon_x\}$. Then $\mathfrak{s}_0$ is a torus in $\mathfrak{g}$. By Lemma 7 there exists a σ- and θ-stable Cartan subalgebra $\mathfrak{s}$ in $\mathfrak{g}$ containing $\mathfrak{s}_0$. We have $\mathfrak{s} = Z_{\mathfrak{g}}(\mathfrak{s}) \subset Z_{\mathfrak{g}}(\Upsilon_x) = \mathfrak{l}_x \subset \mathfrak{q}_x$ which proves (b).
(a) $\Leftarrow$ (b) Assume (b) and let $\mathfrak{s}$ be a σ- and θ-stable Cartan subalgebra of $\mathfrak{g}$. From Lemma 23 we obtain $\Upsilon_x \in \mathfrak{s} \subset \mathfrak{l}_x$. By using that $\mathfrak{s}$ is σ-stable, we see that $\sigma\Upsilon_x \subset \sigma\mathfrak{s} = \mathfrak{s}$. Hence, Υ_x and $\sigma\Upsilon_x$ are both elements of the commutative subalgebra $\mathfrak{s}$ and we get $[\Upsilon_x, \sigma\Upsilon_x] = 0$. Theorem 32 now tells us that x is a critical point of $f^\pm$. □

Notice that statement (b) of the above theorem is independent of the choice of Υ_x. This implies that (a) is independent as well which is quite remarkable since $f^\pm$ does depend on the choice of Υ_x. From now on, we will denote the Cartan subalgebra of Theorem 33 (b) with $\mathfrak{s}$. Using this notation, we can lift the result of Lemma 20 to the Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$:

Lemma 34. *Let $x \in X$ be a critical point of $f^\pm$ and let Υ_x as in Theorem 32. Let $\mathfrak{s}$ be a σ- and θ-stable Cartan subalgebra of $\mathfrak{q}_x$. Then $\alpha(\Upsilon_x) = 0$ for $\alpha \in \Delta(\mathfrak{l}_x, \mathfrak{s})$ and $\alpha(i\Upsilon_x) < 0$ for $\alpha \in \Delta(\mathfrak{n}_x, \mathfrak{s})$*

Proof. Let $g \in G_u$ such that $Q_x = gQg^{-1}$. There is a g-induced canonical mapping from $\Delta(\mathfrak{g}, \mathfrak{t})$ to $\Delta(\mathfrak{g}, \mathrm{Ad}(g)\mathfrak{t})$ given by $\alpha \mapsto g.\alpha = \mathrm{Ad}(g^{-1})^*\alpha$, hence $g.\alpha(Z) = \alpha(\mathrm{Ad}(g^{-1})Z)$. From Lemma 20, it is immediate that $\alpha(\Upsilon_x) =$ for $\alpha \in \Delta(\mathfrak{l}_x, \mathrm{Ad}(g)\mathfrak{t})$ and $\alpha(i\Upsilon_x) < 0$ for $\alpha \in \Delta(\mathfrak{n}_x, \mathrm{Ad}(g)\mathfrak{t})$.

Both $\mathrm{Ad}(g)\mathfrak{t}$ and $\mathfrak{s}$ are θ-stable Cartan subalgebras of $\mathfrak{l}_x$, hence they are conjugate by some $l \in L_x \cap G_u$ (see [Kna02, Prop. 7.35]). By the same canonical mapping described above, which is this time induced by l, we can associate the roots of $\Delta(\mathfrak{q}_x, \mathrm{Ad}(g)\mathfrak{t})$ to $\Delta(\mathfrak{q}_x, \mathfrak{s})$ by $\alpha \mapsto l.\alpha$. Since L_x is the centralizer of Υ_x (see Lemma 21 for a similar statement on the group level), it follows that $\mathrm{Ad}(l^{-1})\Upsilon_x = \Upsilon_x$ which implies $\alpha(\Upsilon_x) = 0$ for $\alpha \in \Delta(\mathfrak{l}_x, \mathfrak{s})$ and $\alpha(i\Upsilon) < 0$ for $\alpha \in \Delta(\mathfrak{n}_x, \mathfrak{s})$. □

3.2 The tangent and normal bundles of X and the gradients $\nabla f^\pm$

In this section, we will study a decomposition of T_xX for $x \in X$ a critical point of f^+. We need this decomposition in the next section in order to study the Hessian of f^+ at x. We will study the integral curve of $\nabla f^+(x)$ and show that it is tangent to the G_0-orbit and $K-$orbit of x in X.

Define $V = T_x X$ and $V^1 = T_x(K_0 x)$ (subscript zero stands for the fixed points with respect to σ, hence $K_0 = K^\sigma$). By taking the orthogonal complement with respect to the Riemannian form g from Equation 17 in V, we define the two more vector subspaces of V: $V_2 = V_1^\perp \cap T_x(G_0 x)$ and $V_3 = V_1^\perp \cap T_x(Kx)$. We will prove that $V = V^1 \oplus V^2 \oplus V^3$ as an orthogonal direct sum.

Let us first study how σ acts on $\tilde{\mathfrak{n}}_x \subset \mathfrak{g}$. We have $\tilde{\mathfrak{n}}_x = \oplus \mathfrak{g}_\alpha$ for $\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ where $\mathfrak{s}$ is a σ- and θ-stable Cartan subalgebra in $\mathfrak{q}_x$ (see Theorem 33). We want to know what happens to α when σ is applied. Since $\mathfrak{g} = \tilde{\mathfrak{n}}_x \oplus \mathfrak{n}_x \oplus \mathfrak{l}_x$ (see Equation 6), there are three possibilities which we will list now:

$$\Delta_1 = \{\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s}) : \sigma\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})\}$$
$$\Delta_2 = \{\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s}) : \sigma\alpha \in \Delta(\mathfrak{n}_x, \mathfrak{s})\}$$
$$\Delta_3 = \{\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s}) : \sigma\alpha \in \Delta(\mathfrak{l}_x, \mathfrak{s})\}$$

In this partition of $\Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$, we can decompose $\tilde{\mathfrak{n}}_x$ into the following subspaces:

$$\sum_{\alpha \in \Delta_1} \mathfrak{g}_\alpha = \tilde{\mathfrak{n}}_x^\sigma \oplus \tilde{\mathfrak{n}}_x^{-\sigma}, \quad \sum_{\alpha \in \Delta_2} \mathfrak{g}_\alpha = \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta}, \quad \sum_{\alpha \in \Delta_3} \mathfrak{g}_\alpha = \tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x \tag{23}$$

Hence, we obtain the following expression:

$$\tilde{\mathfrak{n}}_x = \tilde{\mathfrak{n}}_x^\sigma \oplus \tilde{\mathfrak{n}}_x^{-\sigma} \oplus \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x) \tag{24}$$

A similar expression can be found for $\mathfrak{n}_x$. Recall that for non zero $Z_\alpha \in \mathfrak{g}_\alpha$ and non zero $Z_\beta \in \mathfrak{g}_\beta$, $B(Z_\alpha, Z_\beta)$ is non zero if and only if $\beta = -\alpha$ (see [Kna02, Prop. 2.17]). By Lemma 15, this implies $\beta = \theta(\alpha)$. From this we obtain that the decomposition in Equation 24 is B-orthogonal.

Theorem 35. *Let $x \in X$ be a critical point of $f^\pm$. The following identities hold:*

$$\begin{array}{rllll}
T_x(G_0x) \simeq \tilde{\mathfrak{n}}_x^\sigma \oplus & & \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus & \tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus & (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x) \\
T_x(Kx) \simeq \tilde{\mathfrak{n}}_x^\sigma \oplus & \tilde{\mathfrak{n}}_x^{-\sigma} \oplus & \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus & & (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x) \\
T_x(K_0x) \simeq \tilde{\mathfrak{n}}_x^\sigma \oplus & & \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus & & (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x)
\end{array}$$

Proof. Let H denote the groups K_0, G_0 or K and let $\mathfrak{h}$ denote the Lie algebra of H. We can write $T_x(Hx) \simeq T_x(H/(H \cap Q_x)) \simeq \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{q}_x) \simeq Pr_{\tilde{\mathfrak{n}}_x}\mathfrak{h}$ where we use $\mathfrak{g} = \tilde{\mathfrak{n}}_x \oplus \mathfrak{l}_x \oplus \mathfrak{n}_x$. It follows that we may use projection onto $\tilde{\mathfrak{n}}_x$ to find an expression for the tangent space.

We begin with the isomorphism for $T_x(G_0x)$. We can plug Equation 24 and its counterpart for $\mathfrak{n}_x$, into the expression $\mathfrak{g} = \tilde{\mathfrak{n}}_x \oplus \mathfrak{l}_x \oplus \mathfrak{n}_x$ (Equation 6):

$$\mathfrak{g} = (\tilde{\mathfrak{n}}_x^\sigma \oplus \mathfrak{n}_x^\sigma)\bigoplus(\tilde{\mathfrak{n}}_x^{-\sigma} \oplus \mathfrak{n}_x^{-\sigma})\bigoplus(\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta})\bigoplus(\tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta})\bigoplus M$$

The subspace M on the right is defined as $(\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x) \oplus \mathfrak{l}_x \oplus (\mathfrak{n}_x \cap \mathfrak{l}_x)$. Notice that the operator $1+\sigma$ sends the negative eigenspace of σ to zero, hence projects onto the positive eigenspace. Using this, we get the following expression for $\mathfrak{g}_0$:

$$\mathfrak{g}_0 = (\tilde{\mathfrak{n}}_x^\sigma \oplus \mathfrak{n}_x^\sigma) \oplus (1+\theta)(\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta}) \oplus (1-\theta)(\tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus \mathfrak{n}_x^{-\sigma\theta}) \oplus Pr_{\mathfrak{g}_0}M$$

Since σ and θ commute, they work in almost the same way on $Z \in \tilde{\mathfrak{n}}_x^{\pm\sigma\theta}$: $\sigma Z = \sigma(\pm\sigma\theta Z) = \pm\theta Z$ (the $\mathfrak{n}_x$ variant of this result follows in the same way). The image of $1+\theta$ on $(\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{-\sigma\theta})$, contains the points $X + \theta X$ for $X \in \tilde{\mathfrak{n}}_x^{\sigma\theta}$. This means that using projection to $\tilde{\mathfrak{n}}_x$ on this image, we at least get $\tilde{\mathfrak{n}}_x^{\sigma\theta}$. Since the image of $1+\theta$ is contained in $(\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta})$, we conclude that $Pr_{\tilde{\mathfrak{n}}_x}((1+\theta)(\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta})) = \tilde{\mathfrak{n}}_x^{\sigma\theta}$. An analogous result for $(1-\theta)$ holds and we see that after projection to $\tilde{\mathfrak{n}}_x$, we get the following:

$$Pr_{\tilde{\mathfrak{n}}_x}(\mathfrak{g}_0) = \tilde{\mathfrak{n}}_x^\sigma \oplus \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus Pr_{\tilde{\mathfrak{n}}_x}Pr_{\mathfrak{g}_0}M$$

We are left to show that $Pr_{\tilde{\mathfrak{n}}_x}Pr_{\mathfrak{g}_0}M = \tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x$. Notice that $M = \mathfrak{l}_x + \sigma\mathfrak{l}_x$, hence $Pr_{\mathfrak{g}_0}M = (1+\sigma)M = M$. From the decomposition $M = (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x) \oplus \mathfrak{l}_x \oplus (\mathfrak{n}_x \cap \mathfrak{l}_x)$, it follows that $Pr_{\tilde{\mathfrak{n}}_x}M = (\tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x)$.

The second isomorphism follows analogously. For the third isomorphism we can find an expression for $\mathfrak{k}_0$ by taking the intersection between $\mathfrak{g}_0$ and $\mathfrak{k}$:

$$\mathfrak{g}_0 \cap \mathfrak{g}^{\sigma\theta} = \mathfrak{k}_0 = (1+\theta)\tilde{\mathfrak{n}}_x^\sigma \oplus (1+\theta)\tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus Pr_{\mathfrak{k}_0}(M)$$

In the same way as above, we use projection onto $\tilde{\mathfrak{n}}_x$ and the first two terms will give $\tilde{\mathfrak{n}}_x^\sigma \oplus \tilde{\mathfrak{n}}_x^{\sigma\theta}$. Notice $Pr_{\mathfrak{k}_0}(M) = (1+\sigma)(1+\sigma\theta)M = (1+\sigma)M + (1+\sigma)\theta M$. Notice that $\theta M = M$ since $\mathfrak{l}_x$ is θ-stable (see Corollary 18). By using $(1+\sigma)M = M$, we obtain $Pr_{\mathfrak{k}_0}(M) = \tilde{\mathfrak{n}}_x \cap \sigma\mathfrak{l}_x$ from which the desired result follows. $\square$

Corollary 36. *From Theorem 35, we obtain the following identities (here, x is a critical point of $f^{\pm}$):*

$$T_x(G_0x) + T_x(Kx) = T_xX$$

$$T_x(G_0x) \cap T_x(Kx) = T_x(K_0x)$$

We are interested in finding the relation between the tangent spaces described above, and the integral curve for the gradient of f^+. For this, we require an expression for $\nabla f^+(x)$. This expression can be found in the following lemma. Let ξ_Z be the vector field on X associated to $Z \in \mathfrak{g}$ and let J denote the almost complex structure on X as in Section 2.5.

Lemma 37. *Let $x \in X$ (not necessarily a critical point of f^+). Then $\nabla f^+(x) = -2J\xi_{\sigma\mu_{G_u}(x)}(x) = -4J\xi_{\mu_{K_0}}(x)$.*

Proof. Let $Z \in \mathfrak{g}_u$ and let g denote the Riemannian metric from Equation 17. From $\xi_Z f^+(x) = g_x(\nabla f^+(x), \xi_Z(x))$, we see that it suffices to show $\xi_Z f^+(x) = g_x(-2J\xi_{\sigma\mu_{G_u}(x)}(x), \xi_Z(x))$ for all $Z \in \mathfrak{g}_u$. From Lemma 31 we get $\xi_Z f^+(x) = -2B(Z, [\mu_{G_u}(x), \sigma\mu_{G_u}(x)])$. On the other hand, using Equation 10 and Equation 13, we get the following:

$$\begin{aligned} g_x(-2J\xi_{\sigma\mu_{G_u}(x)}(x), \xi_Z(x)) &= \omega_x(-2J^2\xi_{\sigma\mu_{G_u}(x)}(x), \xi_Z(x)) \\ &= 2\omega_x(\xi_{\sigma\mu_{G_u}(x)}(x), \xi_Z(x)) \\ &= -2B(\mu_{G_u}(x), [\sigma\mu_{G_u}(x), Z]) \\ &= -2B([\mu_{G_u}(x), \sigma\mu_{G_u}(x)], Z) \end{aligned}$$

Since this holds for all $Z \in \mathfrak{g}_u$, the left equation follows. For the right hand side of the equation, notice that $\mu_{K_0}(x) = \frac{1}{2}(\mu_{G_u}(x) + \sigma\mu_{G_u}(x))$ (see Section 3.1). By linearity of the map $Z \mapsto \xi_Z$, we get the following:

$$-4J\xi_{\mu_{K_0}(x)} = -2J(\xi_{\mu_{G_u}(x)} + \xi_{\sigma\mu_{G_u}(x)})$$

Since $\mu_{G_u}(x) \in \mathfrak{q}_x$, it follows from Equation 12 that $\xi_{\mu_{G_u}}(x) = 0$ which implies the right hand part and concludes the proof. $\square$

By Lemma 30, $f^+ + f^-$ is constant. Applying the gradient operator on this expression yields $0 = \nabla f^+ + \nabla f^-$, hence $\nabla f^- = -\nabla f^+$. This is a useful result which allows us to prove the following statement for $f^{\pm}$ by proving it for f^+.

Lemma 38. *Let $x \in X$, then $\nabla f^{\pm}(x)$ is tangent to the G_0-orbit and K-orbit of x in X.*

Proof. By Lemma 37, we have $\nabla f^+(x) = -2J\xi_{\sigma\mu_{G_u}(x)}(x) = 2\xi_{\sigma(i\mu_{G_u}(x))}(x)$ and it suffices to prove that $\xi_{\sigma(i\mu_{G_u}(x))}(x)$ is an element of the tangent space of both orbits i.e. $\sigma(i\mu_{G_u}(x)) \in (\mathfrak{g}_0 + \mathfrak{q}_x) \cap (\mathfrak{k} + \mathfrak{q}_x)$. Notice that $i\mu_{G_u}(x) \in \mathfrak{q}_x$ via the construction of $\mu_{G_u}(x) = \Upsilon_x$. From this, we obtain the following two expressions:

$$\sigma(i\mu_{G_u}(x)) + \mathfrak{q}_x = i\mu_{G_u}(x) + \sigma(i\mu_{G_u}(x)) + \mathfrak{q}_x = \mathfrak{g}^{\sigma} + \mathfrak{q}_x = \mathfrak{g}_0 + \mathfrak{q}_x$$

$$\begin{aligned} \sigma(i\mu_{G_u}(x)) + \mathfrak{q}_x &= \sigma(i\mu_{G_u}(x)) - i\mu_{G_u}(x) + \mathfrak{q}_x \\ &= \sigma(i\mu_{G_u}(x)) + \theta(i\mu_{G_u}(x)) + \mathfrak{q}_x \in \mathfrak{g}^{\sigma\theta} + \mathfrak{q}_x = \mathfrak{k} + \mathfrak{q}_x \end{aligned}$$

Hence we obtain $\sigma(i\mu_{G_u}(x)) \in (\mathfrak{g}_0 + \mathfrak{q}_x) \cap (\mathfrak{k} + \mathfrak{q}_x)$ which concludes the proof for f^+. The f^- result follows from the remarks above. $\square$

Using the above lemma, we can find the reverse implication of the second equation of Corollary 36:

Corollary 39. *The point $x \in X$ is a critical point of $f^{\pm}$ if and only if $T_x(G_0x) \cap T_x(Kx) = T_x(K_0x)$.*

Proof. By the remark above, we only need to prove the implication '$\Leftarrow$'. Assume $T_x(G_0x) \cap T_x(Kx) = T_x(K_0x)$. By Lemma 38, we get $\nabla f^+(x) \in T_x(K_0x)$. From Lemma 30, we obtain that f^+ is constant on K_0-orbits, hence $\nabla f^+(x) = 0$ which means that x is a critical point of f^+. $\square$

With the following general result, we can investigate orbits in X containing integral curves of $\nabla f^{\pm}$.

Theorem 40. *Let M be a smooth manifold and H a Lie group acting smoothly from the left on M. Let ξ be a vector field on M such that for all $x \in M$, it holds that $\xi(x) \in T_x(Hx)$. Then every integral curve of ξ is contained in a single H-orbit of M.*

Proof. Let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \to M$ denote an integral curve for ξ. Let $t_0 \in I$ and fix $x_0 \in M$ such that $\gamma(t_0) \in Hx_0$. Let H_{x_0} be the stabilizer of x_0 in H and define $N \simeq H/H_{x_0}$. Consider the following map:

$$\varphi : N \to M, \quad hH_{x_0} \mapsto hx_0$$

The map φ is a smooth and injective immersion. Let $y \in N$. By definition $T_{\varphi(y)}(Hx_0)$ equals the image of $T_y\varphi$. Define the section $\overline{\xi} : N \to TN$ by $T_y\varphi \cdot \overline{\xi(y)} = \xi(\varphi(y))$ for all $y \in N$. This section is well defined since the image of ξ is contained in the image of φ by assumption. We get the following commuting diagram:

$$\begin{array}{ccc} N & \xrightarrow{\varphi} & M \\ \downarrow{\scriptstyle\overline{\xi}} & & \downarrow{\scriptstyle\xi} \\ TN & \xrightarrow{T\varphi} & TM \end{array}$$

The section $\overline{\xi}$ is a smooth vector field of N in the point $\gamma(t_0)$. There exists an integral curve $\overline{\gamma} :]t_0-\delta, t+\delta[\to N$ of $\overline{\xi}$ with $\delta > 0$, such that $\varphi\circ\overline{\gamma}(t_0) = \gamma(t_0)$. It follows that $\varphi \circ \overline{\gamma}$ is an integral curve of ξ in M. By decreasing δ, we may assume that $]t_0-\delta, t_0+\delta[\subset I$ and by uniqueness of integral curves this implies $\gamma = \varphi \circ \overline{\gamma}$ on $]t_0 - \delta, t_0 + \delta[$. Since the image of φ lies inside Hx_0, this implies that $\gamma(]t_0 - \delta, t_0 + \delta[) \subset Hx_0$.

Let $\mathcal{O}$ denote an H-orbit on M and define $I_{\mathcal{O}} = \{t \in I : \gamma(t) \in \mathcal{O}\}$. Since H-orbits on M cover M, we obtain

$$I = \bigcup_{\mathcal{O}\in H\backslash M} I_{\mathcal{O}}$$

where $H\backslash M$ denotes the set of H-orbits of M. Since the H-orbits of M are mutually disjoint, we see that the sets $I_{\mathcal{O}}$ are mutually disjoint. It follows form the above computations that I_{Hx_0} is open in I. Since the sets $I_{\mathcal{O}}$ are mutually disjoint, we find that

$$I_{Hx_0} = I\backslash \bigcup_{\mathcal{O}\neq Hx_0} I_{\mathcal{O}}.$$

Since I_{Hx_0} is the complement of an open set in I, we obtain that I_{Hx_0} is closed in I. By connectedness of I we obtain $I_{Hx_0} = I$. It follows that $\gamma(I) \subset Hx_0$ which concludes the proof. $\square$

The next corollary follows from Theorem 40 and Lemma 38.

Corollary 41. *Each integral curve of ∇f^+ is contained in a single G_0^0-orbit and in a single K^0-orbit*

Notice that Corollary 36 implies that V is the orthogonal direct sum of V^1 and $V^2 \oplus V^3$. We will end this section with a description of V^2 and V^3 and we will prove that they are orthogonal, thereby proving $V = V^1 \oplus V^2 \oplus V^3$ as orthogonal direct sum.

From Lemma 34, we obtain that $\mathrm{ad}(i\Upsilon_x)$ acts on $\mathfrak{g}_\alpha$ with $\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$, with a non zero eigenvalue. It follows that $\mathrm{ad}(i\Upsilon_x)$ acts on $\tilde{\mathfrak{n}}_x$ as an invertible linear transformation. This action preserves the decomposition of $\Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ into Δ_1, Δ_2 and Δ_3 or equivalently it preserves the decomposition of $\tilde{\mathfrak{n}}_x$ into the associated root spaces from Equation 23. This allows us to define the following subspaces:

$$\mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma\theta}) \subset \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta} \quad \text{and} \quad \mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma}) \subset \tilde{\mathfrak{n}}_x^{\sigma} \oplus \tilde{\mathfrak{n}}_x^{-\sigma} \tag{25}$$

We will investigate these subspaces in Theorem 43 but it will be useful to prove the following lemma first.

Lemma 42. *Let $Z, W \in \mathfrak{g}$ and let $B_{\mathfrak{g}^\mathbb{R}}$ be the Killing form of the algebra $\mathfrak{g}^\mathbb{R}$. Let g be the Riemannian form of Equation 17. The following holds*

$$g_x(\xi_Z(x), \theta_W(x)) = B_{\mathfrak{g}^\mathbb{R},\theta}([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], W_{\tilde{\mathfrak{n}}_x}) = B_{\mathfrak{g}^\mathbb{R},\theta}(\mathrm{ad}(i\Upsilon_x)Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x})$$

Proof. Using Equation 17, we get the following:

$$\begin{aligned} g_x(\xi_Z, \xi_W) &= B(i\Upsilon_x, [\theta Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}] - [Z_{\tilde{\mathfrak{n}}_x}, \theta W_{\tilde{\mathfrak{n}}_x}]) \\ &= B([i\Upsilon_x, \theta Z_{\tilde{\mathfrak{n}}_x}], W_{\tilde{\mathfrak{n}}_x}]) - B([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) \\ &= B(\theta[\theta i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], W_{\tilde{\mathfrak{n}}_x}]) - B([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) \\ &\overset{(1.)}{=} -B(\theta[i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], W_{\tilde{\mathfrak{n}}_x}]) - B([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) \\ &= -2\mathrm{Re}\ B([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) \end{aligned} \tag{26}$$

In step (1.), we used that $\theta i\Upsilon_x = -i\theta\Upsilon_x = -i\Upsilon_x$ since θ is a complex conjugation and Υ_x is θ-invariant by construction. In the last step we used Lemma 2. Next, by applying Lemma 11, we get the following:

$$\begin{aligned} -2\mathrm{Re}\ B([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) &= -B_{\mathfrak{g}^\mathbb{R}}([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], \theta W_{\tilde{\mathfrak{n}}_x}]) \\ &= B_{\mathfrak{g}^\mathbb{R},\theta}([i\Upsilon_x, Z_{\tilde{\mathfrak{n}}_x}], W_{\tilde{\mathfrak{n}}_x}]) \\ &= B_{\mathfrak{g}^\mathbb{R},\theta}(\mathrm{ad}(i\Upsilon_x)Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}) \end{aligned}$$

□

Theorem 43. *The map $Z \mapsto \xi_Z(x)$ maps*
1. $\mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma\theta})$ *isomorphically onto* V^2
2. $\mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma})$ *isomorphically onto* V^3
Furthermore, $V^2 \perp V^3$ *with respect to the Riemannian form* g *from Equation 17.*

Proof. By Theorem 35, we see that $\dim V_x^2 = \dim \tilde{\mathfrak{n}}_x^{\sigma\theta}$ and $\dim V_x^3 = \dim \tilde{\mathfrak{n}}_x^{-\sigma}$. Hence, we need only prove that that $\mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma\theta})$ is mapped into V^2 and $\mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma})$ into V^3.

Let $Z \in \mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma\theta})$. Then $Z \in \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta}$ hence $\xi_Z(x) \in T_x(G_0x)$. It follows from Lemma 42 that $g_x(\xi_Z(x), \xi_W(x)) = 0$ for all $W \in \tilde{\mathfrak{n}}_x^{\sigma} \oplus \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus (\tilde{\mathfrak{n}}_x \cap \mathfrak{l}_x)$. Since $W \mapsto \xi_W(x)$ maps the latter space onto $T_x(K_0x)$ (see Theorem 35), it follows that $\xi_Z(X) \in T_x(K_0x)^{\perp}$ i.e. $\xi_Z(x) \in V^2$. The second inclusion follows by application of analogous arguments.

We are left to show $V^2 \perp V^3$. Let $z \in V_x^2$ and $w \in V_x^3$. By the first part of this theorem, there exists $Z \in \mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma\theta})$ and $W \in \mathrm{ad}(i\Upsilon_x)^{-1}(\tilde{\mathfrak{n}}_x^{-\sigma})$ such that $z = \xi_Z(x)$ and $w = \xi_W(x)$. By Lemma 42 we obtain $g_x(v, w) = B_{\mathfrak{g}^{\mathbb{R}},\theta}(\mathrm{ad}(i\Upsilon_x)Z_{\tilde{\mathfrak{n}}_x}, W_{\tilde{\mathfrak{n}}_x}) = 0$ since $\tilde{\mathfrak{n}}_x^{-\sigma\theta}$ is perpendicular to $\tilde{\mathfrak{n}}_x^{\sigma} \oplus \tilde{\mathfrak{n}}_x^{-\sigma}$ with respect to g_x. This concludes the proof. □

3.3 The Hessian of $f^{\pm}$ at critical points

In this section we will study the behavior of the Hessian of $f^{\pm}$ at critical points of $f^{\pm}$.

In the context of a Riemannian manifold M, the Hessian of a smooth function $f : M \to \mathbb{R}$ can be defined as $Hess(f)(X, Y) = X(Yf) - df(\nabla_X Y)$ where X, Y are vector fields on M and ∇_X is the covariant derivative in the direction of X. We are only interested in the Hessian at critical points i.e. at $x \in M$ such that $df_x = 0$. From the equality $X(Y(f))(x) = Y(X(f))(x) - df(x)[X, Y]$, we obtain $Hess(f)_x(X, Y) = X(Yf)(x)$.

Lemma 44. *Let* $x \in X$ *be a critical point of* $f^{\pm}$*. For* $W, Z \in \mathfrak{g}_u$*, the following expression holds:*

$$\mathrm{Hess}_x f^+(\xi_W, \xi_Z) = B([\sigma\Upsilon_x, Z - \sigma Z], [\Upsilon_x, W - \sigma W])$$

Proof. From Equation 22, we obtain $\xi_Z(f^+)(x) = -B(Z - \sigma Z, [\Upsilon_x, \sigma\Upsilon_x])$. The next step is to apply ξ_W to this expression:

$$\begin{aligned}(\xi_W \xi_Z f^+)(x) &= \partial_t|_{t=0}\, \xi_Z(f^+)(\exp(tW)x) \\ &= -\,\partial_t|_{t=0}\, B(Z - \sigma Z, [\mathrm{Ad}(e^{tW})\Upsilon_x, \sigma \mathrm{Ad}(e^{tW})\Upsilon_x]) \\ &= -B(Z - \sigma Z, [[W, \Upsilon_x], \sigma\Upsilon_x] + [\Upsilon_x, \sigma[W, \Upsilon_x]])\end{aligned}$$

In the last line we used the same trick as in the proof of Lemma 31. By the Jacobi identity we obtain the following expression for $[\Upsilon_x, \sigma[W, \Upsilon_x]]$:

$$[\Upsilon_x, \sigma[W, \Upsilon_x]] = [\Upsilon_x, [\sigma W, \sigma\Upsilon_x]] = -[\sigma\Upsilon_x, [\Upsilon_x, \sigma W]] - [\sigma W, [\sigma\Upsilon_x, \Upsilon_x]]$$

In view of Theorem 32 we obtain $[\sigma\Upsilon_x, \Upsilon_x] = 0$. Plugging this into the above equation, we get the following:

$$\begin{aligned}(\xi_W \xi_Z f^+)(x) &= -B(Z - \sigma Z, [[W, \Upsilon_x], \sigma\Upsilon_x] - [\sigma\Upsilon_x, [\Upsilon_x, \sigma W]]) \\ &= -B(Z - \sigma Z, [[W, \Upsilon_x], \sigma\Upsilon_x] + [[\Upsilon_x, \sigma W], \sigma\Upsilon_x]) \\ &= -B(Z - \sigma Z, [[W, \Upsilon_x], \sigma\Upsilon_x] - [[\sigma W, \Upsilon_x], \sigma\Upsilon_x]) \\ &= -B(Z - \sigma Z, [[W - \sigma W, \Upsilon_x], \sigma\Upsilon_x])\end{aligned}$$

We arrive at the desired result by using the associativity of the Killing form and the skew-symmetry of the Lie brackets once again:

$$\begin{aligned}(\xi_W \xi_Z f^+)(x) &= -B(Z - \sigma Z, [[W - \sigma W, \Upsilon_x], \sigma\Upsilon_x]) \\ &= -B(Z - \sigma Z, -[\sigma\Upsilon_x, [W - \sigma W, \Upsilon_x]]) \\ &= B([Z - \sigma Z, \sigma\Upsilon_x], [W - \sigma W, \Upsilon_x]) \\ &= B([\sigma\Upsilon_x, Z - \sigma Z], [\Upsilon_x, W - \sigma W])\end{aligned}$$

□

Lemma 45. *Let $x \in X$ be a critical point of f^+ and let $\alpha, \beta \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ with $Z \in \mathfrak{g}_\alpha$ and $W \in \mathfrak{g}_\beta$. We get the following expression for the Hessian at this point:*

$$\mathrm{Hess}_x(f^+)(\xi_Z(x), \xi_W(x)) = -4\alpha(i\Upsilon_x) \cdot \sigma\alpha(i\Upsilon_x) \cdot \mathrm{Re}(B(Z, \theta W - \sigma W - \sigma\theta W)) \tag{27}$$

Proof. Define $\tilde{W} = (W + \theta W) - \sigma(W + \theta W)$. By Equation 11 and Lemma 44, we get the following expression for the Hessian:

$$\mathrm{Hess}_x(f^+)(\xi_Z, \xi_W) = B([\sigma\Upsilon_x, (Z + \theta Z) - \sigma(Z + \theta Z)], [\Upsilon_x, \tilde{W}])$$

Notice that $[\Upsilon_x, \tilde{W}]$ is invariant under θ since both Υ_x and $\tilde{W}$ are invariant under θ. By using linearity of the Killing form and the brackets, and Lemma 2, we obtain the following expression:

$$\begin{aligned}
\mathrm{Hess}_x(f^+)(\xi_Z, \xi_W) &= +B([\sigma\Upsilon_x, \theta Z - \sigma\theta Z], [\Upsilon_x, \tilde{W}]) \\
&+ B([\sigma\Upsilon_x, Z - \sigma Z], [\Upsilon_x, \tilde{W}]) \\
&= 2\mathrm{Re}B([\sigma\Upsilon_x, Z - \sigma Z], [\Upsilon_x, \tilde{W}]) \\
&= -2\mathrm{Re}B([\Upsilon_x, [\sigma\Upsilon_x, Z - \sigma Z]], \tilde{W}) \\
&= -2\mathrm{Re}B([\Upsilon_x, [\sigma\Upsilon_x, Z]], \tilde{W}) \\
&+ 2\mathrm{Re}B([\Upsilon_x, [\sigma\Upsilon_x, \sigma Z]], \tilde{W}) \qquad (28)
\end{aligned}$$

Applying Lemma 2 to the second term, yields the following:

$$\begin{aligned}
2\mathrm{Re}B([\Upsilon_x, [\sigma\Upsilon_x, \sigma Z]], \tilde{W}) &= 2\mathrm{Re}\overline{B([\Upsilon_x, [\sigma\Upsilon_x, \sigma Z]], \tilde{W})} \\
&= 2\mathrm{Re}B(\sigma[\Upsilon_x, [\sigma\Upsilon_x, \sigma Z]], \sigma\tilde{W}) \\
&= 2\mathrm{Re}B([\sigma\Upsilon_x, \sigma[\sigma\Upsilon_x, \sigma Z]], -\tilde{W}) \\
&= -2\mathrm{Re}B([\sigma\Upsilon_x, [\Upsilon_x, Z]], \tilde{W}) \qquad (29)
\end{aligned}$$

Combining $[\sigma\Upsilon_x, \Upsilon_x] = 0$ from Theorem 32 with the Jacobi identity, we obtain $[\sigma\Upsilon_x, [\Upsilon_x, Z]] = [\Upsilon_x, [\sigma\Upsilon_x, Z]]$. Combining this with Equations 28 and 29 yields the following identity:

$$\mathrm{Hess}_x(f^+)(\xi_Z, \xi_W) = -4\mathrm{Re}(B([\Upsilon_x, [\sigma\Upsilon_x, Z]], \tilde{W}))$$

We can manipulate this identity until we arrive at the desired result.

$$\begin{aligned}
-4\mathrm{Re}(B([\Upsilon_x, [\sigma\Upsilon_x, Z]], \tilde{W})) &= -4\mathrm{Re}(B(-i^2[\Upsilon_x, [\sigma\Upsilon_x, Z]], \tilde{W})) \\
&= -4\mathrm{Re}(B(-[i\Upsilon_x, i[\sigma\Upsilon_x, Z]], \tilde{W})) \\
&= -4\mathrm{Re}(B([i\Upsilon_x, [\sigma i\Upsilon_x, Z]], \tilde{W})) \\
&= -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}(B(Z, \tilde{W}))
\end{aligned}$$

In the last step of the above computation we used that $\alpha(i\Upsilon_x)$ is real by Lemma 34. Finally, notice that $B(Z, W) = 0$ since W is not in $\mathfrak{g}_{-\alpha}$. □

Theorem 46. *The vector space V, V^1, V^2 and V^3 where defined in Section 3.2. Let $x \in X$ be a critical point of $f^{\pm}$. Then:*
(a) $\mathrm{Hess}_x(f^+)|_{V \times V^1} = 0$*,*
(b) $\mathrm{Hess}_x(f^+)|_{V^2 \times V^3} = 0$*,*
(c) $\mathrm{Hess}_x(f^+)|_{V^2 \times V^2}$ *is negative definite,*
(d) $\mathrm{Hess}_x(f^+)|_{V^3 \times V^3}$ *is positive definite.*

Proof. **(a)** Let $z \in V$ and $w \in V^1$. Then there exist $Z \in \mathfrak{g}_u$ and $W \in \mathfrak{k}_0$ such that $z = \xi_Z(x)$ and $w = \xi_W(x)$. We get $\mathrm{Hess}_x(f^+)(z, w) = \xi_Z \xi_W f^+(x)$. But f^+ is K_0-invariant because of Lemma 30 hence $\xi_W(f^+)(x) = 0$ from which the result follows.

(b) Let $w \in V^2$ and $z \in V^3$. By Theorem 43 and Equation 25 there exist $W \in \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta}$ and $Z \in \tilde{\mathfrak{n}}_x^{\sigma} \oplus \tilde{\mathfrak{n}}_x^{-\sigma}$ such that $w = \xi_W(x)$ and $z = \xi_Z(x)$. By Equation 11 we may represent w and v as $\xi_{W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x}}$ and $\xi_{Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}}$ respectively. We may now apply Lemma 44 for $W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x}$ and $Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}$ since these are elements of $\mathfrak{g}_u$.

Notice that $W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x} \in \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta} \oplus \mathfrak{n}_x^{\sigma\theta}$ and that $Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x} \in \tilde{\mathfrak{n}}_x^{\sigma} \oplus \tilde{\mathfrak{n}}_x^{-\sigma} \oplus \mathfrak{n}_x^{\sigma} \oplus \mathfrak{n}_x^{\sigma}$. By the remark under Equation 24, these spaces are B-orthogonal. Finally, since $\mathrm{ad}(i\Upsilon_x)$ and $\mathrm{ad}(\Upsilon_x)$ leave this decomposition into root spaces fixed, it follows that $\mathrm{Hess}_x(f^+)(\xi_{W_{\tilde{\mathfrak{n}}_x} + \theta W_{\tilde{\mathfrak{n}}_x}}, \xi_{Z_{\tilde{\mathfrak{n}}_x} + \theta Z_{\tilde{\mathfrak{n}}_x}}) = 0$ by Lemma 44.

(c) Let $z \in V^2$ and $z \neq 0$. Then there exists a $Z \in \tilde{\mathfrak{n}}_x^{\sigma\theta} \oplus \tilde{\mathfrak{n}}_x^{-\sigma\theta}$ with $\mathrm{ad}(i\Upsilon_x)Z \in \tilde{\mathfrak{n}}_x^{-\sigma\theta}$ such that $z = \xi_Z(x)$ and $Z \neq 0$. Recall that we defined Δ_2 to be the $\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ such that $\sigma(\alpha) \in \Delta(\mathfrak{n}_x, \mathfrak{s})$. Alternatively, we could define Δ_2 to be the $\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ such that $\theta\sigma(\alpha) \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$. Hence Δ_2 splits into orbits of the group $\{1, \theta\sigma\}$. We will call two elements $\alpha, \beta \in \Delta_2$ conjugate when $\alpha = \beta$ or $\alpha = \theta\sigma\beta$. Equivalence of elements of Δ_2 is denoted by $\alpha \sim \beta$.

For $\alpha, \beta \in \Delta$, define $H_{\alpha\beta}(Z) = \mathrm{Hess}_x(f^+)(\xi_{Z_\alpha}(x), \xi_{Z_\beta}(x))$ where Z_α denotes the projection of Z onto $\mathfrak{g}_{u,\alpha}$. By linearity we get the following expression for the Hessian:

$$\mathrm{Hess}_x(f^+)(\xi_Z(x), \xi_Z(x)) = \sum_{\alpha,\beta \in \Delta_2} H_{\alpha\beta}(Z) \tag{30}$$

Claim 1 If $\alpha, \beta \in \Delta_2$ and $\alpha \not\sim \beta$, then $H_{\alpha\beta}(Z) = 0$.
Claim 2 If $\alpha \in \Delta_2$ and $Z_\alpha \neq 0$ or $Z_{\sigma\theta\alpha} \neq 0$, then $\displaystyle\sum_{\beta,\gamma \in \Delta_2 : \beta \sim \alpha, \gamma \sim \alpha} H_{\beta\gamma}(Z) < 0$.

Since Z is non zero, there will be an $\alpha \in \Delta_2$ such that either Z_α or $Z_{\sigma\theta\alpha}$ is nonzero. Combining both claims with Equation 30 shows $\mathrm{Hess}_x(f^+)(z, z) < 0$. The claims are proven in Lemma 47.

(d) The proof is similar to that of part (c). Let $z \in V^3$ and $z \neq 0$. Then there exists a $Z \in \tilde{\mathfrak{n}}_x^\sigma \oplus \tilde{\mathfrak{n}}_x^{-\sigma}$ with $\mathrm{ad}(i\Upsilon_x)Z \in \tilde{\mathfrak{n}}_x^{-\sigma}$ such that $z = \xi_Z(x)$ and $Z \neq 0$. Recall that Δ_1 was defined as the $\alpha \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ such that $\sigma(\alpha) \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$. We will call two elements $\alpha, \beta \in \Delta_1$ conjugate when $\alpha = \beta$ or $\alpha = \sigma\beta$. Equivalence of elements of Δ_1 is denoted by $\alpha \sim \beta$.

Claim 3 If $\alpha, \beta \in \Delta_3$ and $\alpha \not\sim \beta$, then $H_{\alpha\beta}(Z) = 0$.
Claim 4 If $\alpha \in \Delta_3$ and $Z_\alpha \neq 0$ or $Z_{\sigma\alpha} \neq 0$, then $\displaystyle\sum_{\beta,\gamma \in \Delta_3 : \beta \sim \alpha, \gamma \sim \alpha} H_{\beta\gamma}(Z) > 0$.

Since Z is non zero, there will be an $\alpha \in \Delta_1$ such that either Z_α or $Z_{\delta\alpha}$ is nonzero. Combining both claims with Equation 30 shows $\mathrm{Hess}_x(f^+)(z,z) > 0$. The claims are proven in Lemma 47. □

Lemma 47. *The claims in Theorem 46 (c) and (d) are valid.*

Proof. **Claim 1 and 2** We use the same notation as in Theorem 46 (c). Let $\alpha, \beta \in \Delta_2$. Notice that $\sigma\theta\beta \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$, hence $B(Z_\alpha, \sigma\theta Z_\beta) = 0$. Hence, by Lemma 45 we obtain the following:

$$H_{\alpha\beta}(Z) = -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\beta - \sigma Z_\beta) \tag{31}$$

Claim 1 Let $\alpha, \beta \in \Delta_2$ and $\alpha \not\sim \beta$. From $\beta \not\sim \alpha$, we obtain $\beta \neq \alpha$ and $\sigma\theta\beta \neq \alpha$. By Lemma 15 and applying θ on both sides, we get $\theta\beta \neq -\alpha$ and $\sigma\beta \neq -\alpha$. We arrive at $B(Z_\alpha, \theta Z_\beta - \sigma Z_\beta) = 0$. It follows by Equation 31 that $H_{\alpha\beta}(Z) = 0$.

Claim 2 Let $\alpha \in \Delta_2$ and $Z_\alpha \neq 0$ or $Z_{\theta\sigma\alpha} \neq 0$. We need to check two cases: $\alpha = \sigma\theta\alpha$ and $\alpha \neq \sigma\theta\alpha$.
Case 1 Assume $\alpha = \sigma\theta\alpha$. By using Equation 31 we get the following:

$$\begin{aligned}\sum_{\beta,\gamma \in \Delta_2 : \beta \sim \alpha, \gamma \sim \alpha} H_{\beta\gamma}(Z) &= H_{\alpha\alpha}(Z) \\ &= -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \sigma Z_\alpha)\end{aligned}$$

By Corollary 34, we see that $\alpha(i\Upsilon_x) > 0$ since $\alpha \in \Delta_2 \subset \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ and $\sigma(\alpha)(i\Upsilon_x) < 0$ since $\sigma(\alpha) \in \Delta(\mathfrak{n}_x, \mathfrak{s})$. Hence, we need to show that $\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \sigma Z_\alpha) < 0$.

It follows from $\mathrm{ad}(i\Upsilon_x)Z \in \tilde{\mathfrak{n}}_x^{-\sigma\theta}$ and $\alpha = \sigma\theta\alpha$ that $Z_\alpha \in \tilde{\mathfrak{n}}_x^{-\sigma\theta}$. From $\sigma\theta Z_\alpha = -Z_\alpha$ we obtain $\sigma Z_\alpha = -\theta Z_\alpha$. We get the following:

$$B(Z_\alpha, \theta Z_\alpha - \sigma Z_\alpha) = B(Z_\alpha, \theta Z_\alpha + \theta Z_\alpha) = 2B(Z_\alpha, \theta Z_\alpha) < 0$$

Case 2 Assume $\alpha \neq \sigma\theta\alpha$. Using Equation 31, we get the following:

$$\begin{aligned} \sum_{\beta,\gamma\in\Delta_2:\beta\sim\alpha,\gamma\sim\alpha} H_{\beta\gamma}(Z) &= H_{\alpha\alpha}(Z) + 2H_{\sigma\theta\alpha,\alpha}(Z) + H_{\sigma\theta\alpha,\sigma\theta\alpha}(Z) \qquad (32) \\ &= -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \sigma Z_\alpha) \\ &- 8\sigma\theta(\alpha)(i\Upsilon_x)\theta(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_{\sigma\theta\alpha}, \theta Z_\alpha - \sigma Z_\alpha) \\ &- 4\sigma\theta(\alpha)(i\Upsilon_x)\theta(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_{\sigma\theta\alpha}, \theta Z_{\sigma\theta\alpha} - \sigma Z_{\sigma\theta\alpha}) \end{aligned}$$

By Corollary 17, we see that the θ-terms in the outside of the Killing form cancel each other. By Lemma 34, we see that $-\sigma(\alpha(i\Upsilon_x))\alpha(i\Upsilon_x) > 0$. We will show that the Killing form expressions in the above equation, are all non positive.

Since $\alpha \neq \sigma\theta\alpha = -\sigma\alpha$, we observe that $B(Z_\alpha, -\sigma Z_\alpha) = 0$ and $B(Z_{\sigma\theta\alpha}, -\sigma Z_{\sigma\theta\alpha}) = 0$. Hence, the first Killing form of the above expression equals $B(Z_\alpha, \theta Z_\alpha$ which is non positive. Similarly, the third Killing form equals $B(Z_{\sigma\theta\alpha}), \sigma Z_{\sigma\theta\alpha})$ which is non positive.

We are left with the second Killing form from Equation 32. Notice $\mathrm{ad}(i\Upsilon_x)(Z_\alpha + Z_{\sigma\theta\alpha}) \in \tilde{\mathfrak{n}}_x^{-\sigma\theta}$. This implies $\sigma\theta[\alpha(i\Upsilon_x)Z_\alpha - \sigma\alpha(i\Upsilon_x)Z_{\sigma\theta\alpha}] = -\alpha(i\Upsilon_x)Z_\alpha + \sigma\alpha(i\Upsilon_x)Z_{\sigma\theta\alpha}$ which implies $\alpha(i\Upsilon_x)\sigma\theta Z_\alpha = \sigma\alpha(i\Upsilon_x)Z_{\sigma\theta\alpha}$. Specifically, we obtain:

$$Z_{\sigma\theta\alpha} = \frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\sigma\theta Z_\alpha$$

Since $\alpha \in \Delta_2$ and by Lemma 34, we see that $\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)} < 0$. Applying this identity to the second Killing form of the equation above yields the following:

$$\mathrm{Re}B(Z_{\sigma\theta\alpha}, \theta Z_\alpha - \sigma Z_\alpha) = \frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}B(\sigma\theta Z_\alpha, \theta Z_\alpha - \sigma Z_\alpha)$$

By Lemma 2 we see that $B(\sigma\theta Z_\alpha, \theta Z_\alpha) = \overline{B(\sigma Z_\alpha, Z_\alpha)} = 0$. By using the same lemma once more, we obtain the following:

$$\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}B(\sigma\theta Z_\alpha, -\sigma Z_\alpha) = -\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}\overline{B(\theta Z_\alpha, Z_\alpha)} \leq 0$$

Finally, notice that by the assumptions on Z_α and $Z_{\sigma\theta\alpha}$, at least one of the Killing form expressions below Equation 32 will be non zero. Hence, the sum at Equation 32 is (strictly) negative.
Claim 3 Let $\alpha, \beta \in \Delta_3$ and $\alpha \not\sim \beta$. We again obtain Lemma 45, but for different roots. Notice that $\sigma\beta \in \Delta_3$ hence $B(Z_\alpha, -\sigma Z_\beta) = 0$. From $\beta \not\sim \alpha$, we obtain $\beta \neq \alpha$ and $\sigma\beta \neq \alpha$. By Lemma 15 and applying θ on both sides, we get $\theta\beta \neq -\alpha$ and $\sigma\theta\beta \neq -\alpha$. We arrive at $B(Z_\alpha, \theta Z_\beta - \sigma Z_\beta) = 0$.

Claim 3 and 4 We use the same notation as in Theorem 46 (d). Let $\alpha, \beta \in \Delta_1$. Notice that $\sigma\beta \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$, hence $B(Z_\alpha, \sigma Z_\beta) = 0$. Hence, by Lemma 45 we obtain the following:

$$H_{\alpha\beta}(Z) = -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\beta - \sigma\theta Z_\beta) \tag{33}$$

Claim 3 Let $\alpha, \beta \in \Delta_1$ and $\alpha \not\sim \beta$. From $\beta \not\sim \alpha$, we obtain $\beta \neq \alpha$ and $\sigma\beta \neq \alpha$. By Lemma 15 and applying θ on both sides, we get $\theta\beta \neq -\alpha$ and $\theta\sigma\beta \neq -\alpha$. We arrive at $B(Z_\alpha, \theta Z_\beta - \theta\sigma Z_\beta) = 0$. It follows by Equation 33 that $H_{\alpha\beta}(Z) = 0$.

Claim 4 Let $\alpha \in \Delta_1$ and $Z_\alpha \neq 0$ or $Z_{\sigma\alpha} \neq 0$. We need to check two cases: $\alpha = \sigma\alpha$ and $\alpha \neq \sigma\alpha$.
Case 1 Assume $\alpha = \sigma\alpha$. By using Equation 33 we get the following:

$$\begin{aligned} \sum_{\beta,\gamma\in\Delta_1:\beta\sim\alpha,\gamma\sim\alpha} H_{\beta\gamma}(Z) &= H_{\alpha\alpha}(Z) \\ &= -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \theta\sigma Z_\alpha) \end{aligned}$$

By Corollary 34, we see that $\alpha(i\Upsilon_x) > 0$ since $\alpha \in \Delta_1 \subset \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$ and $\sigma(\alpha)(i\Upsilon_x) > 0$ since $\sigma(\alpha) \in \Delta(\tilde{\mathfrak{n}}_x, \mathfrak{s})$. Hence, we need to show that $\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \sigma\theta Z_\alpha) < 0$.

It follows from $\mathrm{ad}(i\Upsilon_x)Z \in \tilde{\mathfrak{n}}_x^{-\sigma}$ and $\alpha = \sigma\alpha$ that $Z_\alpha \in \tilde{\mathfrak{n}}_x^{-\sigma}$. From $\sigma Z_\alpha = -Z_\alpha$ we obtain $\sigma Z_\alpha = -Z_\alpha$. We get the following:

$$B(Z_\alpha, \theta Z_\alpha - \theta\sigma Z_\alpha) = B(Z_\alpha, \theta Z_\alpha + \theta Z_\alpha) = 2B(Z_\alpha, \theta Z_\alpha) < 0$$

Case 2 Assume $\alpha \neq \sigma\alpha$. Using Equation 31, we get the following:

$$\begin{aligned}\sum_{\beta,\gamma\in\Delta_1:\beta\sim\alpha,\gamma\sim\alpha} H_{\beta\gamma}(Z) &= H_{\alpha\alpha}(Z) + 2H_{\sigma\alpha,\alpha}(Z) + H_{\sigma\alpha,\sigma\alpha}(Z) \qquad (34)\\ &= -4\alpha(i\Upsilon_x)\sigma(\alpha)(i\Upsilon_x)\mathrm{Re}B(Z_\alpha, \theta Z_\alpha - \sigma\theta Z_\alpha)\\ &- 8\sigma(\alpha)(i\Upsilon_x)\alpha(i\Upsilon_x)\mathrm{Re}B(Z_{\sigma\alpha}, \theta Z_\alpha - \sigma\theta Z_\alpha)\\ &- 4\sigma(\alpha)(i\Upsilon_x)\alpha(i\Upsilon_x)\mathrm{Re}B(Z_{\sigma\alpha}, \theta Z_{\sigma\alpha} - \sigma\theta Z_{\sigma\alpha})\end{aligned}$$

By Lemma 34, we see that $-\sigma(\alpha(i\Upsilon_x))\alpha(i\Upsilon_x) < 0$. We will show that the Killing form expressions in the above equation, are all non positive.

Since $\alpha \neq \sigma\alpha$, we get $-\alpha \neq \theta\sigma\alpha$ and we observe that $B(Z_\alpha, -\theta\sigma Z_\alpha) = 0$ and $B(Z_{\sigma\theta\alpha}, -\sigma\theta Z_{\sigma\theta\alpha}) = 0$. Hence, the first Killing form of the above expression equals $B(Z_\alpha, \theta Z_\alpha$ which is non positive. Similarly, the third Killing form equals $B(Z_{\sigma\theta\alpha}), \sigma Z_{\sigma\theta\alpha})$ which is non positive.

We are left with the second Killing form from Equation 34. Notice $\mathrm{ad}(i\Upsilon_x)(Z_\alpha + Z_{\sigma\alpha}) \in \tilde{\mathfrak{n}}_x^{-\sigma}$. This implies $\sigma[\alpha(i\Upsilon_x)Z_\alpha - \sigma\alpha(i\Upsilon_x)Z_{\sigma\alpha}] = -\alpha(i\Upsilon_x)Z_\alpha + \sigma\alpha(i\Upsilon_x)Z_{\sigma\alpha}$ which implies $\alpha(i\Upsilon_x)\sigma Z_\alpha = -\sigma(\alpha)(i\Upsilon_x)Z_{\sigma\alpha}$. Specifically, we obtain:

$$Z_{\sigma\alpha} = -\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\sigma Z_\alpha$$

Since $\alpha \in \Delta_1$ and by Lemma 34, we see that $-\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)} < 0$. Applying this identity to the second Killing form of the equation above yields the following:

$$\mathrm{Re}B(Z_{\sigma\alpha}, \theta Z_\alpha - \theta\sigma Z_\alpha) = -\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}B(\sigma Z_\alpha, \theta Z_\alpha - \theta\sigma Z_\alpha)$$

From $\sigma\alpha \neq \alpha$, we see that $\theta\sigma\alpha \neq -\alpha$. We obtain $B(\sigma Z_\alpha, \theta Z_\alpha) = 0$. By using the same lemma once more, we obtain the following:

$$-\frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}B(\sigma Z_\alpha, -\theta\sigma Z_\alpha) = \frac{\alpha(i\Upsilon_x)}{\sigma\alpha(i\Upsilon_x)}\mathrm{Re}\overline{B(Z_\alpha, \theta Z_\alpha)} \leq 0$$

Finally, notice that by the assumptions on Z_α and $Z_{\sigma\alpha}$, at least one of the Killing form expressions below Equation 34 will be non zero. Hence, the sum at Equation 34 is (strictly) positive. $\square$

4 Matsuki correspondence for a complex group

4.1 Limit points of the integral curve $\gamma(t,x)$

Let $\gamma(t,x)$ be the integral curve for ∇f^+ through the point $x \in X$ and let $\mathcal{C}$ denote the set of critical points of $f^\pm$. In this section, we will prove that $\lim_{t\to\pm\infty} \gamma(t,x)$ exists and lies inside $\mathcal{C}$ (Theorem 50). The proof for the most part is technical and relies heavily on a version of Łojasiewicz's inequality (Lemma 49 (c)).

Recall that superscript 0 denotes the connected component of the identity.

Lemma 48. *$\mathcal{C}$ contains finitely many K_0^0-orbits.*

Proof. Let N_c denote the vector subspace of T_cX consisting of vectors that are orthogonal to $T_c(K_0^0c)$ in T_cX. By the first part of Corollary 36, $N_c \subset V_c^2 \oplus V_c^3$. By the second part of this corollary, $V_c^2 \oplus V_c^3 \subset N_c$. By Theorem 46, the Hessian of f^+ at $c \in C$ is non degenerate. Hence, there is a transverse slice S to $K_0^0 \cdot c$ through c, with a neighborhood U of c in S on which c is the only critical point of $f^+|_U$.

Since f^+ is K_0-invariant (Lemma 30), the set K_0^0c consists of critical points of f^+ (see the remark below the proof). By construction, the set K_0^0U has only K_0^0c as critical points of f^+, hence the K_0^0-orbits are topologically isolated. By adding open subsets complementary to $K_0^0\mathcal{C}$, we can extend $K_0^0\mathcal{C}$ to a cover of X. Since X is compact and since the individual orbits K_0^0c in $\mathcal{C}$ are disjoint, it follows that there can only be finitely many of these K_0^0c orbits. ☐

In the above proof, we use that f^+ is K_0-invariant in order to prove that all elements of K_0^0c are critical points of f^+. This result does not extend to K_0c. The reason for this is that since K_0^0 is connected, the Hessian of f^+ cannot become degenerate on K_0^0c. For a different connected component V in K_0c, we know that f^+ attains the same value as on K_0^0c because of K_0-invariance of f^+. We do not know anything about the Hessian of f^+ on V and this Hessian could be degenerate, indicating that points in V are not necessarily critical points of f^+.

Let $x \in X$, not necessarily a critical point. Let $\gamma(t,x)$ be the integral curve through x of ∇f^+. Since X is compact, these integral curves are complete.

We will call a point $y \in X$ a limit point of $\gamma(t,x)$ if given any neighborhood U of y, there exists an increasing, unbounded sequence $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $\gamma(t_n,x) \in U$ for all $n \in \mathbb{N}$. If $\lim_{t\to\infty} \gamma(t,x)$ exists, the only limit point of $\gamma(t,x)$ will be the point $\lim_{t\to\infty} \gamma(t,x)$.

Lemma 49. *(a) $\mathcal{L}$ is nonempty and lies in a single K_0^0-orbit in $\mathcal{C}$.*
(b) Let $c \in \mathcal{C}$ such that $\mathcal{L} \subset K_0^0 c$. Let $U \subset X$ be an open neighborhood of $K_0^0 c$. Then there exists $t_0 \geq 0$ such that for $t \geq t_0$, $\gamma(t,x) \in U$.
(c) There exists a neighborhood V of $K_0^0 c$, together with constants $m > 0$ and $\frac{3}{4} \leq r < 1$, such that the following holds for $z \in V$:

$$||\nabla f^{-}(z)|| \geq m|f^{-}(z) - f(K_0^0 c)|^r$$

Here, f^- is adjusted by a constant such that f^- equals 0 on $K_0^0 c$. The norm on the left side, is the Kähler norm associated to the Kähler form of Equation 14.

Proof. **(a)** Notice that $\{\gamma(n,x)\}_{n\in\mathbb{N}}$ is a sequence in X. By compactness of X, there exists a convergent subsequence and we see that $\mathcal{L}$ is nonempty. Next, we will prove that $\mathcal{L}$ is connected, the proof of which comes from [Kir84, 2.10].

Assume that $\mathcal{L} \subset U \cup V$ with U, V disjoint open sets in X. Since $U \cup V$ is open, $Y = X \setminus (U \cup V)$ is a closed subset of a compact manifold X, hence is compact. Let $y \in Y$ and V_y an open neighborhood of y in Y. Since y is not a limit point of $\gamma(t,x)$, there exists a $t_y \in \mathbb{R}$ such that for $t \geq t_y$, $\gamma(t,x) \notin V_y$ (formally, we may only state that $\gamma(t,x) \in V_y$ finitely often, by making t_y bigger, we get that $\gamma(t,v_y)$ does not lie in V_y at all). The collection of sets $\{V_y\}_{y\in Y}$ form a cover of Y, hence we may choose a finite subcover which we shall index by a finite subset $I \subset Y$. Let $T = \mathrm{Max}_{y\in I} t_y$. For $t \geq T$, we get $\gamma(t,x) \notin V_y$ for all $y \in I$, hence $\gamma(t,x) \notin Y$. By $Y = X \setminus (U \cap V)$, it follows that $\gamma(t,x) \in (U \cap V)$. Finally, since $[T,\infty)$ is connected, $\gamma([T,\infty))$ is connected in $U \cap V$ which means that $\gamma([T,\infty))$ either lies in U or in V (there is no connected component shared by both opens). Hence, $\mathcal{L}$ must lies in either U or V and we see that $\mathcal{L}$ cannot be written as the union of two disjoint non-empty open subsets. We conclude that $\mathcal{L}$ is connected.

Let $c \in \mathcal{L}$. Then there exists a monotonically increasing sequence t_n such that $\lim_{n\to\infty} \gamma(t_n,x) = c$. We will prove that $c \in \mathcal{C}$. First, observe the following:

$$\begin{aligned}\partial_t f^+(\gamma(t,x)) &= df^+(\gamma(t,x))\gamma'(t,x)\\ &= df^+(\gamma(t,x))\nabla f^+\gamma(t,x)) = ||\nabla f^+(\gamma(t,x))||^2 \geq 0\end{aligned}$$

It follows that $f^+ \circ \gamma$ is monotonically increasing. Since X is compact it follows that $f^+ \circ \gamma$ has an upper bound which implies that $\lim_{t\to\infty} f^+(\gamma(t,x))$ exists. This implies that $\lim_{t\to\infty} ||\nabla f^+(\gamma(t,x))|| = 0$. Hence, by continuity of ∇f^+, we see $\lim_{n\to\infty} ||\nabla f^+(\gamma(t_n,x))|| = ||\nabla f^+(c)|| = 0$. This implies $c \in \mathcal{C}$.

Finally, since $\mathcal{L}$ is connected it should lie in a connected component of $\mathcal{C}$, hence $\mathcal{L}$ is contained in a single K_0^0-orbit by Lemma 48.

(b) Let $c \in \mathcal{C}$ such that $\mathcal{L} \subset K_0^0 c$ (existence is guaranteed by part (a)). Assume that the statement is false i.e. there exists an open neighborhood U of $K_0^0 c$ such that $t \geq t_0$ implies $\gamma(t,x) \notin U$ for all $t_0 \geq 0$. Hence $\{\gamma(n,x)\}_{n\in\mathbb{N}} \subset X \setminus U$. Since U is open, $X \setminus U$ is closed in X, hence $X \setminus U$ is compact and $\{\gamma(n,x)\}_{n\in\mathbb{N}}$ has a convergent subsequence. Via $\mathcal{L} \subset K_0^0 c \subset U$, all limit points of $\gamma(t,x)$ should be inside U which is a contradiction. We conclude that for every open neighborhood U of $K_0^0 c$, there exists a $t_0 \geq 0$ such that $t \geq t_0$ implies $\gamma(t,x) \in U$.

(c) The desired result is an adaptation of Łojasiewicz's inequality, first proved by Łojasiewicz in [Loj65, p.62]. We will apply the version presented on [HH, p.26] for real analytic manifolds. Using Łojasiewicz's inequality, there exists a neighborhood U_c of c, together with constants $m > 0$ and $0 < r < 1$, such that for all $z \in U_c$, we get the the following:

$$||\nabla f^-(z)|| \geq m|f^-(z)|^r \tag{35}$$

Since f^- and the Kähler norm are K_0-invariant (Lemma 30 and Theorem 28 respectively), the same result holds for the K_0^0-translates of U_c. Define $V = (\bigcup_{k\in K_0^0} kU_c) \cup \{z \in X : f^-(z) \in (-1,1)\}$. Equation 35 holds on V and notice that $|f^-(z)| < 1$ for $z \in V$. This makes $t \mapsto |f^-(z)|^t$ monotonically decreasing, and hence we may choose r to be larger such that $r \in [\frac{3}{4}, 1)$. Finally, notice that $K_0^0 c$ is contained in V since f^- equals 0 on this subset, which concludes the proof. $\square$

Define $\pi^\pm(x) = \lim_{t\to\pm\infty} \gamma(t,x)$. Notice that the minus sign in $\pi^-(x)$ is not referring to f^- directly, but it does so indirectly. If we let $\gamma_\pm(t,x)$ denote the integral curves to $\pm\nabla f^+$ (notice $-\nabla f^+ = \nabla f^-$), we can equivalently define $\pi^\pm(x)$ as $\lim_{t\to\infty}\gamma_\pm(t,x)$. For now, we will stick to $\gamma(t,x)$ as integral curve of ∇f^+.

Theorem 50. *Let $x \in X$. The limits $\pi^\pm(x)$ exist and belong to $\mathcal{C}$.*

Proof. Let $c \in \mathcal{C}$ such that $L \subset K_0^0 c$ and let $s \in \mathbb{R}$. Since c is a limit point, the distance traveled along $\gamma(s,x)$ towards c, will at least be the distance between c and $\gamma(s,x)$. Hence, the distance between $\gamma(s,x)$ and c will always be smaller then $\int_s^\infty ||\gamma'(t,x)||dt$. We will show the convergence of $\pi^+(x)$, by showing that the above integral tends to zero for $s \to \infty$.

Let V be the neighborhood of $K_0^0 c$ of Lemma 49 (c). By part (b) of the same Lemma and by rescaling $\mathbb{R}$, we may assume that $\gamma(t,x) \in V$ for $t \geq s \geq 0$. Define $H(t) = f^-(\gamma(t,x))$ where f^- is adjusted by a constant such that it equals zero on $K_0^0 c$. By the chain rule, we get the following:

$$\begin{aligned} H'(t) &= g_x(\nabla f^-(\gamma(t,x)), \gamma'(t,x)) \\ &= g_x(-\nabla f^+(\gamma(t,x)), \nabla f^+(\gamma(t,x))) \\ &= -||\nabla f^+(\gamma(t,x))||^2 = -||\gamma'(t,x)||^2 \end{aligned} \tag{36}$$

It follows that $|H'(t)|^{\frac{1}{2}} = ||\gamma'(t,x)||$. Observe that $H(t) \geq 0$ since it is defined as the norm of a projection (see Section 3.1), and $H'(t) \leq 0$ because of Equation 36. By the inequality of Lemma 49 (c), we get for $t \geq s$: $|H'(t)|^{\frac{1}{2}} \geq m|H(t)|^r$. Squaring both sides yields

$$|H'(t)| = -H'(t) \geq m^2 H(t)^{2r} = m^2 H(t)^{2-\epsilon} \quad \text{where we define } \epsilon = 2(1-r)$$

Notice that $0 < \epsilon \leq \frac{1}{2}$. This inequality can be rewritten as $\partial_t H(t)^{\epsilon-1} \geq m^2(1-\epsilon)$ since $H(t)$ is nonnegative. Integrating both sides yields the following:

$$H(t)^{\epsilon-1} \geq m^2(1-\epsilon)t \quad \text{hence} \quad H(t) \leq M t^{\frac{-1}{1-\epsilon}} \tag{37}$$

Here, $M = (m^2(1-\epsilon))^{\frac{-1}{1-\epsilon}}$ but since ϵ is a constant, we should just regard it as a constant. With that said, notice that M is finite for all possible values of ϵ.

Define the auxiliary function $F(t) = t^{1+\epsilon}$. Using Hölder's inequality, we can find an upper limit for the integral we are interested in:

$$\begin{aligned}
\int_s^\infty ||\gamma'(t,x)||dt &= \int_s^\infty \sqrt{-H(t)}dt \\
&= \int_s^\infty \frac{\sqrt{-H(t)F(t)}}{F(t)}dt \\
&\leq \left(\int_s^\infty -H'(t)F(t)dt\right)^{\frac{1}{2}} \cdot \left(\int_s^\infty \frac{1}{F(t)}dt\right)^{\frac{1}{2}} \qquad (38)
\end{aligned}$$

We can evaluate the integral on the right straight away: $\int_s^\infty F(t)^{-1}dt = \frac{s^{-\epsilon}}{\epsilon}$. This is finite since $\epsilon > 0$. For the integral on the left, we will use integration by parts:

$$\int_s^\infty -H'(t)F(t)dt \;=\; -\left.H(t)F(t)\right|_s^\infty + \int_s^\infty H(t)F'(t)dt \qquad (39)$$

Combining Equation 37 with the definition of $F(t) = t^{1+\epsilon}$,we see that $H(t)F(t) = O(t^{\frac{\epsilon^2-1}{1-\epsilon}})$ where we use big O notation. Since $\epsilon < \frac{1}{2}$ implies $1-\epsilon > 0$ and $\epsilon^2 - 1 < 0$, we see that $H(t)F(t) \overset{t\to\infty}{\longrightarrow} 0$. Using the right part of Equation 37 once more, we see $H(s)F(s) = O(s^{\frac{-\epsilon^2}{1-\epsilon}})$. With identical arguments, we see $\int_s^\infty H(t)F'(t)dt = O(s^{\frac{-\epsilon^2}{1-\epsilon}})$. Hence, the integrals on the right hand side of Equation 39 are finite which justifies our use of integration by parts. Plugging these results into Equation 38, we get the following:

$$\int_s^\infty ||\gamma'(t,x)||dt = O(s^{\frac{-\epsilon}{2-2\epsilon}}) \quad \text{hence} \quad \int_s^\infty ||\gamma'(t,x)||dt \overset{s\to\infty}{\longrightarrow} 0 \qquad (40)$$

We conclude that $\pi^+(x) = \lim_{t\to\infty}\gamma(t,x)$ converges and by Lemma 49 (a), the limit point lies inside $\mathcal{C}$. The same arguments hold for π^- when we interchange f^- with f^+ and γ with $\gamma_{\nabla f^-}$. □

Theorem 50 allows us the see $\pi^\pm$ as a map i.e. $\pi^\pm : X \to \mathcal{C}$. In the following sections we will make use of this interpretation. More specifically, the inverse images of $\pi^\pm$ of "K_0^0-like" orbits $\mathcal{C}$, will correspond with "G_0-like" and "K-like" orbits in X. Two orbits in X are called Matsuki dual if they correspond to the same orbit in $\mathcal{C}$.

4.2 The stratifications S_c^+ and S_c^-

In this section we will study two stratifications of X, indexed by the K_0^0-orbits in $\mathcal{C}$. That is, we will define two decompositions of X into smooth (connected) submanifolds. It turns out that the strata accompanying the orbit $K_0^0 c$, are precisely the G_0^0- and K^0-orbits of $c \in \mathcal{C}$. At the core of the proof lies the analysis by Kirwan of real, minimally non-degenerate functions on a compact Riemannian space (see [Kir84, Ch. 10]).

Define the following two subsets of X:

$$S_c^+ = \{x \in X : \pi^+(x) \in K_0^0 c\} \qquad S_c^- = \{x \in X : \pi^-(x) \in K_0^0 c\}$$

Notice that the sets $S_c^{\pm}$ depend on c through their K_0^0-orbit in X. Furthermore, if $c, c' \in \mathcal{C}$ belong to different K_0^0-orbits then the associated sets S_c^+ and $S_{c'}^+$ are disjoint. This statement is also valid with superscripts $-$ instead of $+$. Let $S_{\mathcal{C}}^+$ and $S_{\mathcal{C}}^-$ denote the collection of the sets S_c^+, respectively S_c^-, for $c \in \mathcal{C}$. Then Lemma 51 below shows that $S_{\mathcal{C}}^+$ and $S_{\mathcal{C}}^-$ form (Morse) stratifications of X. Before we prove this lemma, we should first introduce two submanifolds $\Sigma_c^{\pm}$ of X containing $K_0^0 c$.
We start by selecting an open, relatively compact neighborhood Ω^+ of e in G_0^0. Then $K_0^0 \Omega^+$ is open and relatively compact as well, so we just as well assume from the start that Ω^+ is left K_0^0-invariant. Let $c \in \mathcal{C}$ and let G_{0c}^0 denote the associated isotropy subgroup of G_0^0. The image Ω_c^+ of Ω^+ in G/G_{0c}^0 is open and relatively compact. Moreover, the map $G_0^0 \to X$, $g \mapsto gc$ induces an injective immersion $j_c : G_0^0/G_{0c}^0 \to X$. It follows that $\Sigma_c^+ := j_c(\Omega_c^+)$ is a locally closed submanifold of X containing $K_0^0 c$ as a compact submanifold. By application of Theorem 46 we see that for every $x \in K_0^0 c$ the Hessian of of f^+ is negative semidefinite on the tangent space

$$T_x(\Sigma_c^+) = T_x(G_0^0 x)$$

and that $T_x(\Sigma_c^+)$ is a maximal subspace of $T_x X$ with this property. By shrinking Ω^+ if necessary we may arrange that f^+ takes on its maximum value in Σ_c^+ precisely on $K_0^0 c$.
Similarly, we can construct a locally closed submanifold Σ_c^- of X on which f^+ takes on its minimum precisely on $K_0^0 c$, and such that for each $x \in K_0^0 c$, the subspace $T_x \Sigma_c^-$ is maximal in $T_x X$ with respect to the condition that the Hessian of f^+ at x is positive semidefinite on $T_x \Sigma_c^-$.
The next lemma shows that $S_{\mathcal{C}}^+$ and $S_{\mathcal{C}}^-$ form (Morse) stratifications of X.

Lemma 51. *If $c \in \mathcal{C}$, then the following assertions are valid.*
(a) The strata S_c^+ and S_c^- are smooth (connected) submanifolds of X.
(b) There exists a K_0^0-stable neighborhood U_c of c in X such that $\Sigma_c^+ \cap U_c = S_c^+ \cap U_c$ and $\Sigma_c^- \cap U_c = S_c^- \cap U_c$.

Proof. **(a)** The submanifolds $\Sigma_c^\pm$ satisfy the conditions of of Theorem 10.4 of [Kir84]. Part of the proof of this theorem requires an adaptation of the Riemannian metric on X such that the associated gradient of f^+ is tangent to both Σ_c^+ and Σ_c^-, see [Kir84, Lemma 10.5]. In our situation this is automatic since by Lemma 38, the gradient of f^+ is tangent to both the G_0- and K-orbit through c. Anyway, from Theorem 10.4 of [Kir84] it follows that S_c^+ and S_c^- are (locally closed) smooth submanifolds.

For connectedness: Let $x, y \in S_c^+$ and let γ_x and γ_y denote the integral curves of ∇f^+ such that $\gamma_x(0) = x$ and $\gamma_y(0) = y$ respectively. Then $\lim_{t\to\infty} \gamma_x(t) \in K_0^0 c$ and $\lim_{t\to\infty} \gamma_y(t) \in K_0^0 c$. By reparameterizing γ_x and γ_y, we may assume that there exists a value $t \in \mathbb{R}$ such that both $\gamma_x(t) \in K_0^0 c$ and $\gamma_y(t) \in K_0^0 c$. Notice that $K_0^0 c$ is path-connected since it is a connected component of a manifold, hence there is a path μ in $K_0^0 c$ of finite length connecting $\gamma_x(t)$ and $\gamma_y(t)$. Notice that $K_0^0 c \subset S_c^+$. By combining the paths γ_x, μ and γ_y appropriately, we can construct a path between x and y that lies inside S_c^+. This implies that S_c^+ is connected.

(b) This statement also follows from the theorem of Kirwan. □

Before continuing we remark that in the present situation f^+ is actually a smooth function with a non-degenerate critical manifold in the sense of Bott, i.e. the set $\mathcal{C}$ of critical points is a compact smooth submanifold of X and the Hessian of f_+ at any point $c \in \mathcal{C}$ has null space equal to $T_c\mathcal{C}$. According to [Kir84, Remark 10.18], this implies that the integral curves $t \mapsto \gamma_c(t)$ of f^+ have limits for $t \to \pm\infty$. This observation could in fact be used to replace the present proof of Theorem 50.

For the following five lemmas, the proof for S_c^+ is analogous to that for S_c^-. Therefore we will omit the proof of the latter.

Lemma 52. *Let $c \in \mathcal{C}$, then $S_c^+ \subset G_0^0 c$ and $S_c^- \subset K^0 c$.*

Proof. Let $x \in S_c^+$. Then $\pi^+(x) = uc$ for some $u \in K_0^0$. Let U_c denote the open neighborhood of Lemma 51 (b). By Lemma 49 (b), there is a large $t \in \mathbb{R}$

such that $\gamma(t,x) \in U_c$. Via $\pi^+(\gamma(t,x)) = \pi^+(x)$ it is clear that $\gamma(t,x) \in S_c^+$ and by using Lemma 51 (b) once more, we see that $\gamma(t,x) \in \Sigma_c^+$. Hence, there exists a $g_1 \in G_0^0$ such that $\gamma(t,x) = g_1 c$. By Corollary 41, x and $\gamma(t,x)$ lie in the same G_0^0 orbit, hence there exists a $g_2 \in G_0^0$ such that $\gamma(t,x) = g_2 x$. Combining the two expressions for $\gamma(t,x)$, we see that $x = g_2^{-1} g_1 c \in G_0^0 c$. We conclude $S_c^+ \subset G_0^0 c$. □

Lemma 53. *Let $c \in \mathcal{C}$, then $S_c^+ \subset G_0^0 c$ and $S_c^- \subset K^0 c$ as open subsets.*

Proof. We want to prove that S_c^+ is open in $G_0^0 c$ and we will do this by proving that every element $x \in S_c^+$ has an open neighborhood in $G_0^0 c$ which is contained in S_c^+.

By Lemma 51 (b), there exists an open K_0^0-stable neighborhood U_c of c such that $U_c \cap \Sigma_c^+ = U_c \cap S_c^+$.
Since $c \in U_c$ and since it is K_0^0-stable, U_c is an open neighborhood of $K_0^0 c$. By Lemma 49 (b), there is a large $t \in \mathbb{R}$ such that $\gamma(t,x) \in U_c$. We consider the flow map $\varphi : y \mapsto \gamma(t,y)$, which is a diffeomorphism of X. It follows from Theorem 40 that φ preserves each orbit $G_0^0 y$. This orbit has a unique manifold structure for which the map $g \mapsto gy$ is a submersion from G_0^0 onto $G_0^0 y$. For this manifold structure, the inclusion map $G_0^0 y \to X$ is an injective immersion. A priori it is not clear that $G_0^0 y$ is a locally closed submanifold. It follows from the proof of Theorem 40 that the flow map φ restricts to the orbit as a diffeomorphism $\varphi_y : G_0^0 y \to G_0^0 y$ for the specified manifold structure.

On the other hand, it follows from the definition of S_c^+ that this locally closed submanifold is invariant under φ. From $\varphi(x) = \gamma(t,x) \in U_c$ it follows by continuity that there exists an open neighborhood V_x of x in X such that $\varphi(V_x)$ is an open subset of U_c. It now follows that

$$\varphi(V_x \cap S_c^+) = \varphi(V_x) \cap S_c^+ = \varphi(V_x) \cap U_c \cap S_c^+ = \varphi(V_x) \cap \Sigma_c^+.$$

The latter set is an open neighborhood of $\varphi(x)$ in $G_0^0 c$. It follows that

$$V_x \cap S_c^+ = \varphi^{-1}(\varphi(V_x) \cap \Sigma_c^+) = \varphi_c^{-1}(\varphi(V_x) \cap \Sigma_c^+)$$

is an open neighborhood of x in $G_0^0 c$ which is contained in S_c^+. □

Lemma 54. *Let $c \in \mathcal{C}$. Then $S_c^+ = G_0^0 c$ and $S_c^- = K^0 c$.*

Proof. By Lemma 52, $S_c^+ \subset G_0^0 c$. Let $x \in G_0^0 c$. Then there exists a $c' \in \mathcal{C}$ such that $x \in S_{c'}^+$. By Lemma 52 we get $S_{c'}^+ \subset G_0^0 c'$ so $x \in G_0^0 c'$. It follows that $G_0^0 c = G_0^0 c'$, hence $S_{c'}^+ \subset G_0^0 c$. We thus see that $G_0^0 c$ equals the union of all the strata which it meets.

Since different strata are disjoint, it follows that $G_0^0 c$ is a disjoint union of strata. By Lemma 53, we know that the strata are open and since G_0^0 is connected, this implies that there can only by one stratum that meets $G_0^0 c$, hence $S_c^+ = G_0^0 c$. $\square$

Lemma 55. *Every G_0^0-orbit in X meets $\mathcal{C}$ and every K^0-orbits meets $\mathcal{C}$.*

Proof. Let $x \in X$. Then $c = \pi_+(x) \in \mathcal{C}$ and $x \in S_c^+$. From Lemma 54 we obtain $x \in G_0^0 c$ from which we obtain $G_0^0 x \subset G_0^0 c$. It follows that $c \in \mathcal{C} \cap G_0^0 x$. $\square$

Lemma 56. *For $c \in \mathcal{C}$, $G_0^0 c \cap \mathcal{C} = K_0^0 c$ and $K^0 c \cap \mathcal{C} = K_0^0 c$.*

Proof. Let $c' \in G_0^0 c \cap \mathcal{C}$. then $c' \in S_c^+ \cap \mathcal{C}$ by Lemma 51(b). Then $\pi^+(c') = c'$ since $c' \in \mathcal{C}$. Since $c' \in S_c^+$, it follows that $\pi^+(c') \in K_0^0 c$ and we obtain $G_0^0 c \cap \mathcal{C} \subset K_0^0 c$. For the other way around, observe that $K_0^0 c \subset G_0^0 c$ by $K_0^0 \subset G_0^0$, and notice that $K_0^0 c \subset \mathcal{C}$ by Lemma 48. $\square$

4.3 Matsuki correspondence for complex semisimple groups

In this section, we shall formulate and prove Matsuki correspondence in the complex case. The basic idea is to look at inverse images of the maps $\pi^\pm : X \to \mathcal{C}$. For a K_0^0-orbit in $\mathcal{C}$, these images turn out to be G_0^0- and K^0-orbits in X. Orbits in X that stem from the same orbit in $\mathcal{C}$ are called Matsuki dual.

Let G be a complex semisimple (connected) group with Cartan conjugation θ and complex conjugation σ commuting with θ. Define $G_u = G^\theta$ and $G_0 = G^\sigma$, and let $K_0 = G_u \cap G_0$. Set $K = (K_0)_\mathbb{C} = K^{\sigma\theta}$. We will denote the connected component of the identity with a superscript 0. Let G_0' be a group such that $G_0^0 \subset G_0' \subset G_0$ and let $K_0' = G_0' \cap K_0$ be the maximal compact subgroup of G_0'. Let K' be the complexification of K_0', i.e. the unique group such that $K^0 \subset K' \subset K$ and $K' \cap K_0 = K_0'$. Let Q be any parabolic subgroup of G and let $X = G/Q$.

The Matsuki correspondence can be described in the following way:

Theorem 57. *(a) There is a bijection between the following sets:*

$$\begin{array}{rcl} \{G'_0\text{-}orbits\ in\ X\} & \longleftrightarrow & \{K'_0\text{-}orbits\ in\ \mathcal{C}\} \\ \beta^+ & \longmapsto & \beta^+ \cap \mathcal{C} = \pi^+(\beta^+) \\ (\pi^+)^{-1}(\beta) = G'_0 c & \longleftarrow & K'_0 c = \beta \end{array}$$

For a G'_0-orbit β^+, $\beta = \beta^+ \cap \mathcal{C}$ is the unique K'_0-orbit in β^+ on which $f^+|_{\beta^+}$ assumes a maximum value.
(b) There is a bijection between the following sets:

$$\begin{array}{rcl} \{K'\text{-}orbits\ in\ X\} & \longleftrightarrow & \{K'_0\text{-}orbits\ in\ \mathcal{C}\} \\ \beta^- & \longmapsto & \beta^- \cap \mathcal{C} = \pi^-(\beta^-) \\ (\pi^-)^{-1}(\beta) = K' c & \longleftarrow & K'_0 c = \beta \end{array}$$

For a K'-orbit β^-, $\beta = \beta^- \cap \mathcal{C}$ is the unique K'_0-orbit in β^- on which $f^+|_{\beta^-}$ assumes a minimal value.
(c) The above defined mappings yield a bijection between the following sets:

$$\{G'_0\text{-}orbits\ in\ X\} \longleftrightarrow \{K'\text{-}orbits\ in\ X\}$$

Proof. **(a)** We will first prove the statement for $G'_0 = G^0_0$. Let β^+ be a G^0_0-orbit in X. By Lemma 55, β^+ meets $\mathcal{C}$ which implies that we may also view β^+ as a G^0_0-orbit of an element c of $\mathcal{C}$ i.e. $\beta^+ = G^0_0 c$. By Lemma 54, $\pi^+(\beta^+) = K^0_0 c$ and by Lemma 56 $\beta^+ \cap \mathcal{C} = K^0_0 c$. Notice that the $c \in \mathcal{C}$ is unique up to K^0_0 which makes $\beta^+ \mapsto \beta^+ \cap \mathcal{C}$ into an injection.

For the other way around, let β be a K^0_0-orbit in $\mathcal{C}$ i.e. $\beta = K^0_0 c$. We get $(\pi^+)^{-1}(\beta) = S^+_c$ from the definition of S^+_c. By Lemma 54 we see $(\pi^+)^{-1}(\beta) = G^0_0 c$. Since the individual strata S^+_c are disjoint, we see that the map $\beta \mapsto (\pi^+)^{-1}(\beta) = \beta^+$ is injective.

These maps are each others inverses and since they are injective, we conclude that both maps are bijective. For the general case, take G'_0 and K'_0 as above. From $K'_0 = K^0_0 K'_0 = \bigcup_{k \in K'_0} K^0_0 k$, we obtain K'_0 as a union of $K^0_0 k$ (co)sets. We can find a similar decomposition for G'_0 by noticing that in Iwasawa decomposition, all the disconnected components will be in the maximal compact group, hence in K_0'. Hence, we find that $G_0' = \bigcup_{k\in K_0'} G_0^0 k$. Using Lemma 56, we find the following:

$$(G_0'c)\cap\mathcal{C} = \left(\bigcup_{k\in K_0'} G_0^0 kc\right)\cap\mathcal{C} = \bigcup_{k\in K_0'}(G_0^0 kc\cap\mathcal{C}) = \bigcup_{k\in K_0'} K_0^0 kc = K_0'c$$

This proves that $\longmapsto$ is well defined and injective. The $\longleftarrow$ follows from using the same decomposition as above, and the fact that the inverse image of a union, is the union of the inverse images.

For any integral curve of ∇f^+ in β^+, the maximum value of f^+ is reached when $t\to\infty$. Since $\pi^+(\beta^+) = \beta^+\cap\mathcal{C} = \beta$, it is clear that f^+ attains its maximum on β.

(b),(c) The proof of (b) is analogous to that of (a). Statement (c) follows directly from (a) en (b). □

Theorem 57 contains the essence of the Matsuki correspondence. We call a G_0'- and a K'-orbit dual, if they are associated to the same K_0'-orbit through the above theorem. There are two things which we should point out. The first is that the c used in the above theorem is not unique. If we have an element $x\in X$ and we want to find the unique critical K_0'-orbit in $G_0'x$, then $c=\pi^+(x)$ is the logical choice, yielding $K_0'c$ as critical orbit. But for $y\in G_0'x$ with $y\notin G_0^0x$, we get $\pi^+(x)\neq\pi^+(y)$ even though they are in the same G_0'-orbit and represent the same critical K_0'-orbit.

Secondly, we should point out that not every K_0'-orbit of X is a critical orbit for f^+, despite f^+ being K_0-invariant. Also, $G_0'x\cap K_0'x$ can contain multiple K_0'-orbits without the two orbits being each others dual ().

Let α be a K_0'-orbit in X. Then α^+ is the G_0'-orbit in X associated to α by Theorem 57. Similarly, α^- is the associated K'-orbit in X. The following theorem contains more properties of the Matsuki correspondence.

Theorem 58. *(a) Let α^+ and β^- be G_0'- and K'-orbits respectively. The following statements are equivalent.*

(i) α^+ *and* β^- *are in duality.*
(ii) $\alpha^+\cap\beta^-\cap\mathcal{C}\neq\emptyset$
(iii) $\alpha^+\cap\beta^-$ *contains exactly one* $K_0'-$*orbit.*
(iv) $\alpha^+\cap\beta^-\neq\emptyset$ *and* f^+ *is constant on* $\alpha^+\cap\beta^-$.

(b) Let α and β be two K_0'-orbits in $\mathcal{C}$.

$$\alpha^+ \subset Cl(\beta^+) \Leftrightarrow \alpha^- \cap \beta^+ \neq \emptyset \Leftrightarrow \beta^- \subset Cl(\alpha^-)$$

Here, Cl denotes the topological closure in X. If $\alpha^+ \subset Cl(\beta^+)$ and $\alpha \neq \beta$, then $f^+(\alpha) < f^+(\beta)$.
(c) Let α be a K_0'-orbit of $\mathcal{C}$. The flow yields the following two continuous mappings: $\gamma : (-\infty, \infty] \times \alpha^+ \to \alpha^+$ and $\gamma : [-\infty, \infty) \times \alpha^- \to \alpha^-$. Thus the orbits of K_0' in $\mathcal{C}$ are strong deformation retracts of the corresponding G_0'- and K'-orbits in X via the gradient flow of f^+. In particular π^+ and π^- are continuous on any G_0'- respectively K'-orbit.

Proof. **(a) (i) $\Leftrightarrow$ (ii)** By Theorem 57, both α^+ and β^- intersect $\mathcal{C}$ in a single K_0'-orbit. In duality, this is the same K_0'-orbit and the equivalence of (i) and (ii) follows directly.
(i) $\Rightarrow$ (iii) By Theorem 57 (a), the maximum value of f^+ in α^+, is uniquely attained in the K_0'-orbit $\alpha^+ \cap \mathcal{C}$. By Theorem 57, the minimum value of f^+ in β^-, is uniquely attained in the K_0'-orbit $\beta^- \cap \mathcal{C}$. Assume that α^+ and β^- are dual i.e. $\alpha^+ \cap \mathcal{C} = \beta^- \cap \mathcal{C}$. Define $m = f^+(\alpha^+ \cap \mathcal{C})$, hence m is the maximum value of f^+ on α^+ and the minimum value of f^+ on β^-.

Clearly the K_0'-orbit $\alpha^+ \cap \mathcal{C}$ is inside $\alpha^+ \cap \beta^-$. Assume that there is a second K_0'-orbit in $\alpha^+ \cap \beta^-$ and denote this orbit with γ. Then $f^+(\gamma) < m$ since m is maximum value of f^+ in α^+ uniquely attained at the K_0'-orbit $\alpha^+ \cap \mathcal{C}$. Similarly, $f^+(\gamma) > m$ since m is the minimum value of f^+ in β^- uniquely attained at the K_0'-orbit $\beta^- \cap \mathcal{C}$. This is a contradiction from which we conclude that $\alpha^+ \cap \beta^-$ contains exactly one K_0'-orbit.
(iii) $\Rightarrow$ (iv) Since $K_0' \subset K_0$, f is K_0'-invariant (Lemma 30) which implies that it is constant on every K_0'-orbit.
(ii) $\Leftarrow$ (iv) We will prove the contrapositive statement i.e. "not (ii)" should imply "not (iv)". Let $\alpha^+ \cap \beta^- \cap \mathcal{C} = \emptyset$. If $\alpha^+ \cap \beta^- = \emptyset$, then there is nothing left to prove. Let $x \in \alpha^+ \cap \beta^-$. By Corollary 41, $\gamma(\mathbb{R}, x)$ is contained within one single G_0- and one single K-orbit. Since $\gamma(\mathbb{R}, x)$ is connected (it is clearly path-connected), we know that it lies in a G_0^0- and a K^0-orbit. This implies that $\gamma(\mathbb{R}, x) \subset \alpha^+$ and $\gamma(\mathbb{R}, x) \subset \beta^-$. By assumption $\gamma(\mathbb{R}, x) \cap \mathcal{C} = \emptyset$. Notice that the function $t \mapsto f^+(\gamma(t, x))$ is monotonically increasing since its derivative is positive. Hence, f^+ is not constant on $\alpha^+ \cap \beta^-$.

(b) We will prove the first equivalence, the second equivalence follows analogously. Let α and β be K_0'-orbits of $\mathcal{C}$.

'$\Rightarrow$' Assume $\alpha^+ \subset Cl(\beta^+)$ and let $x \in \alpha^+$. Since G_0 and G_0', and K and K_0' contain a similar neighborhood of the identity, $T_x(G_0x) = T_x(G_0'x)$ and $T_x(Kx) = T_x(K_0'x)$ hold. From Corollary 36 it follows that $G_0'K_0'x$ contains an open neighborhood U_x of x. Since $x \in Cl(\beta^+)$ there exists an element $y \in \beta^+$ which lies inside U_x. Since $y \in U_x \subset G_0'K_0'x$, we can write $y = gk \cdot x$ with $g \in G_0'$ and $k \in K_0'$ which is equivalent to $g^{-1}y = kx$. By definition, $g^{-1}y \in \beta^+$ and $kx \in \alpha^-$ and we obtain $\alpha^- \cap \beta^+ \neq \emptyset$.
'$\Leftarrow$' Suppose $x \in \alpha^- \cap \beta^+$. By Corollary 41, $\gamma(\mathbb{R}, x)$ is contained in β^+ and we obtain that $\pi^-(x) \in Cl(\beta^+)$. From $x \in \alpha^-$, it follows that $K_0'\pi^-(x)$ is the critical orbit of both α^+ and α^- (this follows from α^- and α^+ being dual orbits). We obtain $\alpha^+ = G_0'\pi^-(x) \subset G_0'Cl(\beta^+)$ and since β^+ is a G_0'-orbit in X, we get $\alpha^+ \subset Cl(\beta^+)$ which completes the proof.

For the final part of (b), let $\alpha^+ \subset Cl(\beta^+)$ and $\alpha \neq \beta$. From the above equivalence, we get that $\alpha^- \cap \beta^+ \neq \emptyset$ and let $x \in \alpha^- \cap \beta^+$. Since $\alpha \neq \beta$, we know that α^- and β^+ are not dual since they have a different critical K_0'-orbit. By part (ii) from part (a), we see that x cannot be a critical point of f^+. By Theorem 57, f^+ restricted to β^+ assumes a maximum on β and f^+ restricted to α^- assumes a minimum on α. Hence, $f^+(x) < f^+(\beta)$ and $f^+(x) > f^+(\alpha)$.
(c) We will only prove the statement for α^+ since the proof for π^- is analogous. Since γ is smooth, the only thing we need te prove is the continuity at infinity: for all $\epsilon > 0$, there must be a neighborhood $U \subset \alpha^+$ of x and a $t_0 \in \mathbb{R}$ such that for all $x' \in U$ and $t \geq t_0$, we get $L(\pi^+(x), \gamma(t, x')) < \epsilon$ where L denotes the distance on X.

By Lemma 49 (c), there is a neighborhood V of α such that Łojasiewicz's inequality holds. Let $m > 0$ be as in Lemma 59 below. Put $U = \{y \in X : f^+(y) > f^+(\alpha)$. Then $\alpha^+ \cap U \subset V$. By continuity of f^+, the set U is open in X which implies that $U \cap V$ is an open neighborhood of α in X such that $U \cap V \cap \alpha^+ = U \cap \alpha^+$. Replacing V by $V \cap U$ we see that we may assume that $V \cap \alpha^+ = \{y \in \alpha^+ : f^+(y) > f^+(\alpha) - m\}$. Notice that if $y \in V \cap \alpha^+$, then $\gamma(\mathbb{R}_+, y) \subset V \cap \alpha^+$ since f^+ can only increase along γ (hence $f^+(\gamma(t, x)) > f^+(\alpha) - m$ for $t > 0$). This is precisely the condition for which Equation 40 is valid. Hence, by the estimates given in the proof of Equation 40, there exists a $s_0 \in \mathbb{R}$ such that for all $y \in V \cap \alpha^+$, we get $\int_s^\infty ||\gamma'(t, y)||dt < \frac{\epsilon}{3}$.

Let $x \in \alpha^+$ be fixed and let $t_0 \in \mathbb{R}$ such that $\gamma(t_0, x) \in V \cap \alpha^+$. Let U_x be an open neighborhood of x such that for all $x' \in U_x$, it holds that

$\gamma(t_0, x') \in V \cap \alpha^+$ and such that $L(\gamma(s_0+t_0, x), \gamma(s_0+t_0, x')) < \frac{\epsilon}{3}$. Now, for $t \geq s_0 + t_0$, the following holds by application of the triangle inequality:

$$\begin{aligned} L(\pi^+(x), \gamma(t, x')) &\leq L(\pi^+(x), \gamma(s_0+t_0, x)) \\ &+ L(\gamma(s_0+t_0, x), \gamma(s_0+t_0), x') \\ &+ L(\gamma(s_0+t_0, x'), \gamma(t, x')) \end{aligned}$$

The first and the third term in equation, are smaller then $\int_{s_0}^{\infty} ||\gamma'(t, y)||dt$ which is smaller then $\frac{\epsilon}{3}$. The second term is smaller than $\frac{\epsilon}{3}$ by construction of U_x. We conclude that γ is continuous on $(-\infty, \infty] \times \alpha^+$. $\square$

Lemma 59. *Let α be a K_0'-orbit of $\mathcal{C}$ and let V be an open neighborhood of α on X. There exists a positive real number m such that for all $y \in \alpha^+$, the inequality $f^+(y) > f^+(\alpha) - m$ implies that $y \in V$.*

Proof. Assume that the statement is false. This means that there exists a sequence $\{y_i\} \subset \alpha^+$ such that $y_i \notin V$ and such that $\lim_{i\to\infty} f^+(y_i) = f(\alpha)$. Since $\{y_i\}$ is a sequence in X which is compact, we may assume without loss of generality, that the sequence converges to some value y. Since f^+ is continuous, $f^+(y) = f^+(\alpha)$. If $y \in V$, then V cannot be an open set since $y_i \notin V$. If $y \in \alpha^+$ and not $y \in V$, then we contradict Theorem 57 (a) by having f^+ attain a maximum value outside of α.
The only possibility left is that y belongs to the border of α^+. Let β^+ be the G_0'-orbit of y. Every element of β can be written as ry with $r \in G_0'$, and every element of β lies at the border of α^+ since ry_i is a sequence in α^+ that converges to ry. Hence, the entire orbit β^+ is inside the border of α^+. By Theorem 58 (b), we get $f^+(\beta) < f^+(\alpha)$. Since $f^+(\beta)$ is the maximum value of f^+ in β^+, we get $f^+(y) \leq f^+(\beta) < f^+(\alpha)$ which contradicts $\lim_{i\to\infty} f^+(y_i) = f^+(\alpha)$. $\square$

5 Matsuki correspondence for a real group

5.1 Real Matsuki correspondence for complex groups

In Chapter 4, we have proven Matsuki correspondence for complex semisimple Lie groups. For these groups there is in fact a different variant of Matsuki correspondence which requires three commuting complex conjugations, one of which is a Cartan. This result is called "real Matsuki correspondence" even though the group G for which this result holds is complex. This version can be used to prove Matsuki correspondence for real groups which is the reason for the name. We will explore this in section 5.3. The present section is dedicated to proving the "real Matsuki correspondence" for complex semisimple groups G (formulated in Theorem 68 and 69).

As mentioned above we require an extra conjugation on G that commutes with θ and σ, let τ denote such a conjugation and let $Q < G$ be parabolic such that $\tau(Q) = Q$. Notice that τ on $X \simeq G/Q$ is well defined and let X^τ denote the fixed points under τ. It is known that τ-stable parabolic subgroups Q that are G-conjugate, are in fact G^τ conjugate, hence $X^\tau \simeq (G/Q)^\tau \simeq G^\tau/Q^\tau$ (see [Bor91, Th. 20.9]). The quotient $(G/Q)^\tau$ does not change when we replace G with a simply connected cover of G, from which we obtain that X^τ is a connected since G^τ is connected (see [Ste68, Th. 8.2]). Since G_u^τ has a continuous embedding into X^τ and since G_u^τ is compact, we see that a G_u^τ-orbit in X^τ is closed. It follows from $\mathfrak{q}^\tau + \mathfrak{g}_u^\tau = \mathfrak{g}^\tau$ that the G_u^τ-orbit of the identity is also open, hence G_u^τ acts transitive on X^τ.

Let $T \subset Q$ be a τ, σ-stable Cartan subalgebra (for existence we refer to Theorem 73 in Section 5.2). The radical of $\mathfrak{q}$ is unique and since Q is τ-stable, we see that $\tau\mathfrak{n}_\mathfrak{q} = \mathfrak{n}_{\tau\mathfrak{q}} = \mathfrak{n}_\mathfrak{q}$. Since $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q})$ (see Corollary 18) and since θ and τ commute, we obtain that $\tau\mathfrak{l} = \mathfrak{l}$. It follows that the Levi decomposition of Q is τ-stable. This implies that $\Upsilon - \tau\Upsilon$ satisfies the conditions of Lemma 34 and we may assume without loss of generality that $\Upsilon \in \mathfrak{t}_u^{-\tau}$.

In Theorem 22 we identified X with the G_u-orbit of Υ in $\mathfrak{g}_u$. Clearly the same argument hold for X^τ which we can now identify with the G_u^τ-orbit of Υ in $\mathfrak{g}_u^\tau$. Let $\mathcal{O}^{-\tau}$ denote this orbit (this is analogous to the definition of $\mathcal{O}$ in Section 2.3). We can sum up the above results as follows:

$$X^\tau \simeq (G/Q)^\tau \simeq (G_u/(G_u \cap L))^\tau \simeq G_u^\tau/(G_u^\tau \cap L) \simeq G_u^\tau\Upsilon \simeq \mathcal{O}^{-\tau} \qquad (41)$$

It should be clear by now that X^τ and X share a lot of the same properties.

The above sequence of isomorphism summarizes that the analogy holds for all arguments from Section 2.3. The following lemmas, can be seen as the τ-analogues of Sections 2.5 to 3.3.

Lemma 60. *The Kähler form of Theorem 28 is τ-sesquilinear i.e. for $x \in X$ and $Z, W \in \mathfrak{g}$, we have $\langle \xi_{\tau Z}, \xi_{\tau W} \rangle_{\tau x} = \overline{\langle \xi_Z, \xi_W \rangle_x}$.*

Proof. Let $Q_x = gQg^{-1}$ with some $g \in G_u$. From $\Upsilon_x = \mathrm{Ad}(g)\Upsilon$ we get $\tau(\Upsilon_x) = \mathrm{Ad}(\tau g)\tau\Upsilon$. Since $\Upsilon \in \mathfrak{t}_u^{-\tau}$ we get $\tau\Upsilon = -\Upsilon$ and we obtain $\tau(\Upsilon_x) = -\mathrm{Ad}(\tau g)\Upsilon = -\Upsilon_{\tau x}$.

Recall that $\mathfrak{n}_x$ is defined as $\mathrm{Ad}(g)\mathfrak{n}_\mathfrak{q}$ (see Section 2.3) and $\tau\mathfrak{n}_\mathfrak{q} = \mathfrak{n}_\mathfrak{q}$. It follows that $\tilde{\mathfrak{n}}_{\tau x} = \tau\tilde{\mathfrak{n}}_x$. When we plug both of these formulas into Equation 14, we obtain the following:

$$\begin{aligned} \langle \xi_{\tau Z}, \xi_{\tau W} \rangle_{\tau x} &= -2iB(\Upsilon_{\tau x}, [(\tau Z)_{\tilde{\mathfrak{n}}_{\tau x}}, \theta(\tau W)_{\tilde{\mathfrak{n}}_{\tau x}}]) \\ &= -2iB(-\tau\Upsilon_x, [\tau(Z_{\tilde{\mathfrak{n}}_x}), \theta\tau(W_{\tilde{\mathfrak{n}}_x})]) \\ &= 2iB(\tau\Upsilon_x, \tau[(Z_{\tilde{\mathfrak{n}}_x}), \theta(W_{\tilde{\mathfrak{n}}_x})]) \\ &= 2i\overline{B(\Upsilon_x, [(Z_{\tilde{\mathfrak{n}}_x}), \theta(W_{\tilde{\mathfrak{n}}_x})])} = \overline{\langle \xi_Z, \xi_W \rangle_x} \end{aligned}$$

Here we used Lemma 2 and that τ and θ commute. □

Lemma 61. *The function f^+ on X is τ-invariant.*

Proof. It follows from $\Upsilon_{\tau x} = -\tau\Upsilon_x$ (see proof 60) that $\mu_{G_u}(\tau x) = -\tau\mu_{G_u}(x)$ (for definition, see Section 3.1). Since τ commutes with θ and σ, we see that projection onto $K_0 = G^{\sigma\theta} \cap G^\sigma$ is τ-equivariant. The result now follows from Lemma 2:

$$f^+(\tau x) = ||\mu_{K_0}(\tau x)||_B^2 = ||-\tau\mu_{K_0}(x)||_B^2 = ||\mu_{K_0}(x)||_B^2 = f^+(x)$$

□

Lemma 62. *The vector field ∇f^+ is τ-invariant.*

Proof. It follows from Lemma 60 that the Riemannian metric g is τ-invariant (it is the real part of a τ-sesquilinear form). By Lemma 61 df^+ is a τ-invariant 1-form from which the result follows (see also Lemma 37). □

Corollary 63. *(a) Let $x \in X^\tau$. Then $\nabla f^+(x)$ is tangent to X^τ and in particular: $\nabla(f^+|_{X^\tau}) = (\nabla f^+(x))_{X^\tau}$ as vector fields on X^τ.*
(b) The maps $\pi^\pm : X \to \mathcal{C}$ are τ-equivariant. This yields that $\pi^\pm : X^\tau \to \mathcal{C}^\tau$ is a well defined, surjective map.

Lemma 64. *Let $x \in X^\tau$. The following are equivalent:*
(a) x is a critical point of f^+.
(b) x is a critical point of $f^+|_{X^\tau}$.
(c) $\mathfrak{q}_x$ contains a τ, σ, θ-stable Cartan algebra.

Proof. **(a) $\Leftrightarrow$ (b)** The statement (a) implies (b) is trivial, so assume (b). Then $\xi_Z f^+(x) = 0$ for all $Z \in \mathfrak{g}_u^\tau$. Hence, we need to prove that $\xi_Z f^+(x) = 0$ for all $Z \in \mathfrak{g}_u^{-\tau}$.

By Lemma 31 we obtain $-\frac{1}{2}\xi_Z f^+(x) = B(Z, [\Upsilon_x, \sigma\Upsilon_x]) = B(\tau Z, \tau[\Upsilon_x, \sigma\Upsilon_x])$. In the last step we used Lemma 2 and the observation that $B(Z, [\Upsilon_x, \sigma\Upsilon_x])$ is real since $-\frac{1}{2}\xi_Z f^+(x)$ is real. We have $\tau Z = -Z$ by definition from which we get $\xi_Z f^+(x) = B(-Z, \tau[\Upsilon_x, \sigma\Upsilon_x])$.

From $\Upsilon_x = g\Upsilon g^{-1}$ for a $g \in G_u^\tau$, we get the following:

$$\begin{aligned}
\tau[\Upsilon_x, \theta\Upsilon_x] &= [\tau\Upsilon_x, \tau\theta\Upsilon_x] \\
&= [\mathrm{Ad}(\tau g)\tau\Upsilon, \theta\mathrm{Ad}(\tau g)\tau\Upsilon] \\
&= [-\mathrm{Ad}(g)\Upsilon, \theta(-\mathrm{Ad}(g)\Upsilon)] = [\Upsilon_x, \theta\Upsilon_x]
\end{aligned}$$

Hence, we obtain $-\frac{1}{2}\xi_Z f^+(x) - B(Z, [\Upsilon_x, \sigma\Upsilon_x])$ which together with $-\frac{1}{2}\xi_Z f^+(x) = B(Z, [\Upsilon_x, \sigma\Upsilon_x])$ implies $\xi_Z f^+(x) = 0$ for all $Z \in \mathfrak{g}_u^{-\tau}$. We conclude that $\xi_Z f^+(x) = 0$ holds for all $Z \in \mathfrak{g}_u^{-\tau}$, hence x is a critical point of f^+.
(a) $\Leftarrow$ (c) This follows from Theorem 32.
(a) $\Rightarrow$ (c) Let $x \in X^\tau$ be a critical point of f^+. Define the subgroup $W = Z_G(\Upsilon_{x,\sigma}) \cap Z_G(\Upsilon_{x,-\sigma})$. It follows as a corollary from Lemma 21, that the subgroup W is inside Q_x (the connected Lie subgroup of G associated to $\mathfrak{q}_x$). The elements Υ_x and $\sigma\Upsilon_x$ are semisimple (see the proof of Theorem 33). By [Hum11, Section 2.2] it follows that W is a reductive group. It follows from Theorem 73 (the proof of which is postponed to Section 5.2) that $\mathfrak{q}_x$ contains a τ, σ, θ-stable Cartan subalgebra. $\square$

Lemma 65. *For $x \in X^\tau$, $\nabla f^\pm(x)$ is tangent to $G_0^\tau x$ and $K^\tau x$.*

Proof. Since $i\Upsilon_x \in \mathfrak{q}_x$, we see that $\sigma(i\Upsilon_x) + \mathfrak{q}_x = i\Upsilon_x + \sigma(i\Upsilon_x) + \mathfrak{q}_x$. Notice the the right hand part lies in $\mathfrak{g}_0$. From $\tau\Upsilon_x = -\Upsilon_x$ we get $i\Upsilon_x \in \mathfrak{g}^\tau$. Since σ and τ commute we see that $i\Upsilon_x + \sigma(i\Upsilon_x)$ is fixed by τ and since $\mathfrak{q}_x$ is $\tau-$stable (Lemma 64), we see that $i\Upsilon_x + \sigma(i\Upsilon_x) + \mathfrak{q}_x \in \mathfrak{g}_0^\tau$. Since $\nabla f^+(x)$ is tangent to the G_0-orbit of x in X and since $\nabla f^+(x)$ is tangent to X^τ by Corollary 63, we get that $\nabla f^+(x)$ is tangent to the G_0^τ-orbit of x in X^τ. A similar argument can be made for the K^τ-orbit. $\square$

In the following lemma, we will use the same notation as in Section 3.2.

Lemma 66. *Let $x \in \mathcal{C}^\tau$ which is the set of critical points of $f^+|_{X^\tau}$. The following statements hold:*
(a) $V^\tau = T_x(X^\tau)$,
(b) $V^{1\tau} = T_x(K_0^\tau x)$,
(c) $V^{2\tau}$ *is the orthogonal complement to* $V^{1\tau}$ *in* $T_x(G_0^\tau x)$,
(d) $V^{3\tau}$ *is the orthogonal complement to* $V^{1\tau}$ *in* $T_x(K^\tau x)$,
(e) $T_x(G_0^\tau) + T_x(K^\tau) = T_x(X^\tau)$,
(f) $T_x(G_0^\tau) \cap T_x(K^\tau) = T_x(K_0^\tau)$,
(g) $\mathrm{Hess}_x(f^+)|_{V^\tau \times V^{1\tau}} = 0$,
(h) $\mathrm{Hess}_x(f^+)|_{V^{2\tau} \times V^{3\tau}} = 0$,
(i) $\mathrm{Hess}_x(f^+)|_{V^{2\tau} \times V^{2\tau}}$ *is negative definite,*
(j) $\mathrm{Hess}_x(f^+)|_{V^{3\tau}, V^{3\tau}}$ *is positive definite.*

Proof. It follows from Equation 41 that for $U \subset G$, we obtain $(TU)^\tau \simeq TU^\tau$. From this rule, statement (a) - (f) become a direct result of statements in Section 3.2. Similarly, statement (g) - (j) follows from Theorem 46 with the observation that $\mathrm{Hess}_x(f^+|_{X^\tau}) = \mathrm{Hess}_x(f^+)|_{T_xX^\tau \times T_xX^\tau}$. ☐

The above proposition tells us that for $x \in \mathcal{C}^\tau$, $Hess f^+(x)$ is non degenerate on the normal bundle to the K_0^τ-orbit of x in X^τ. Hence, the topological arguments from Lemma 48 are still valid which gives is the following corollary:

Corollary 67. *$\mathcal{C}^\tau$ contains finitely many K_0^τ−orbits.*

It should be clear by now that a lot of the arguments above, follow directly from taking the fixed points under τ of results obtained in Section 2.3 to 3.3. The arguments from Chapter 4 apply to the present context as well and we see that the proof for "real Matsuki correspondence" or "τ-fixed point version of Matsuki correspondence", is analogous to that of the complex case. Hence, if we take the notation from Section 4.3, we get the following result:

Theorem 68. *(a) There is a bijection between the following sets:*

$$\begin{array}{rcl} \{{G'_0}^\tau\text{-orbits in } X^\tau\} & \longleftrightarrow & \{{K'_0}^\tau\text{-orbits in } \mathcal{C}^\tau\} \\ \beta^+ & \longmapsto & \beta^+ \cap \mathcal{C}^\tau = \pi^+(\beta^+) \\ (\pi^+)^{-1}(\beta) = {G_0^\tau}' c & \longleftarrow & {K_0^\tau}' c = \beta \end{array}$$

For a ${G_0'}^\tau$-orbit β^+, $\beta = \beta^+ \cap \mathcal{C}^\tau$ is the unique ${K_0'}^\tau$-orbit in β^+ on which $f^+|_{\beta^+}$ assumes a maximum value.
(b) There is a bijection between the following sets:

$$\begin{array}{rcl} \{{K'}^\tau\text{-orbits in } X^\tau\} & \longleftrightarrow & \{{K_0'}^\tau\text{-orbits in } \mathcal{C}^\tau\} \\ \beta^- & \longmapsto & \beta^- \cap \mathcal{C} = \pi^-(\beta^-) \\ (\pi^-)^{-1}(\beta) = {K'}^\tau c & \longleftarrow & {K_0'}^\tau c = \beta \end{array}$$

For a ${K'}^\tau$-orbit β^-, $\beta = \beta^- \cap \mathcal{C}^\tau$ is the unique ${K_0'}^\tau$-orbit in β^- on which $f^+|_{\beta^-}$ assumes a minimal value.
(c) The above defined mappings yield a bijection between the following sets:

$$\{{G_0'}^\tau\text{-orbits in } X^\tau\} \longleftrightarrow \{{K'}^\tau\text{-orbits in } X^\tau\}$$

Theorem 69. *(a) Let α^+ and β^- be ${G_0'}^\tau$- and ${K'}^\tau$-orbits respectively. The following are equivalent:*

(i) α^+ and β^- are in duality.
(ii) $\alpha^+ \cap \beta^- \cap \mathcal{C}^\tau \neq \emptyset$
(iii) $\alpha^+ \cap \beta^-$ contains exactly one ${K_0'}^\tau-$orbit.
(iv) $\alpha^+ \cap \beta^- \neq \emptyset$ and f^+ is constant on $\alpha^+ \cap \beta^-$.

(b) Let α and β be two ${K_0'}^\tau$-orbits in $\mathcal{C}^\tau$.

$$\alpha^+ \subset Cl(\beta^+) \Leftrightarrow \alpha^- \cap \beta^+ \neq \emptyset \Leftrightarrow \beta^- \subset Cl(\alpha^-)$$

Here, Cl denotes the topological closure in X^τ. If $\alpha^+ \subset Cl(\beta^+)$ and $\alpha \neq \beta$, then $f^+(\alpha) < f^+(\beta)$.
(c) Let α be a ${K_0'}^\tau$-orbit of $\mathcal{C}^\tau$. The flow yields the following two continuous mappings: $\gamma : (-\infty, \infty] \times \alpha^+ \to \alpha^+$ and $\gamma : [-\infty, \infty) \times \alpha^- \to \alpha^-$. Thus the orbits of K_0' in $\mathcal{C}^\tau$, are strong deformation retracts of the corresponding ${G_0'}^\tau$- and ${K'}^\tau$-orbits in X^τ via the gradient flow of f^+. In particular, restricting π^+ and π^- to any ${G_0'}^\tau$- respectively ${K'}^\tau$-orbit is continuous.

5.2 A τ, σ, θ-stable Cartan subalgebra in a complex reductive group

In this section, we will complete the proof of implication $(a) \Rightarrow (c)$ of Lemma 64. We will prove that on a complex reductive group with commuting complex conjugations τ, σ, θ with θ being a Cartan, there exists a τ, σ, θ-stable

Cartan subalgebra (Theorem 73). In order to prove this, we will first show that a τ, σ, θ-stable Cartan subalgebra exists in a complex semisimple Lie algebra under certain conditions (Theorem 72). Using this result, we will prove the statement on Lie algebra level which implies the result on Lie group level. We will use the following notation: $\tau' = \tau\theta$, $\sigma' = \sigma\theta$ and $\mathfrak{g}_u = \mathfrak{g}^\theta$.

In order to prove Theorem 72, we require the following two lemmas.

Lemma 70. *The following statements are equivalent:*
(a) Every $(\sigma', \tau')-$*stable split Cartan subalgebra in* $\mathfrak{g}$ *is trivial.*
(b) $\mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'} = 0.$

Proof. **(a) ⇒ (b)** Assume (a) and assume that (b) is false. Then the space $\mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'}$ is non-trivial and since θ commutes with both σ' and τ', we see that the space is θ-stable. This yields the following decomposition:

$$\begin{aligned} \mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'} &= \mathfrak{n} \cap \mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'} \oplus i\mathfrak{g}_u \cap \mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'} \\ &= \mathfrak{g}_u^{-\sigma} \cap \mathfrak{g}_u^{-\tau} \oplus i(\mathfrak{g}_u^{-\sigma} \cap \mathfrak{g}_u^{-\tau}) \end{aligned}$$

We obtain $\mathfrak{g}_u^{-\sigma} \cap \mathfrak{g}_u^{-\tau} \neq 0$. Let $X \in \mathfrak{g}_u^{-\sigma} \cap \mathfrak{g}_u^{-\tau}$ be non zero. Then $\mathbb{C}X$ is a non-trivial (σ', θ')stable Cartan subalgebra in $\mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'}$ which contradicts (a).

(a) ⇐ (b) We will prove this by contraposition. Assume that (a) is false and let $\mathfrak{t} \subset \mathfrak{g}$ be a non-trivial $(\sigma', \tau')-$stable split Cartan subalgebra in $\mathfrak{g}$. There is a non zero $X \in \mathfrak{t}$ and by assumption $X \in \mathfrak{g}^{-\sigma'} \cap \mathfrak{g}^{-\tau'}$ i.e. (b) is false. □

Lemma 71. *Let* $\mathfrak{u}$ *be a compact Lie algebra and let* σ *be a conjugation on this algebra. Let* $\mathfrak{t}^{-\sigma}$ *be a maximal abelian subspace of* $\mathfrak{u}^{-\sigma}$ *and let* $\mathfrak{t}$ *be a maximal abelian subalgebra in* $\mathfrak{u}$ *containing* $\mathfrak{t}^{-\sigma}$*. Then* $\mathfrak{t}$ *is* σ*-stable.*

Proof. Let $X \in \mathfrak{t}$ be non zero and notice $X - \sigma X \in \mathfrak{t}^{-\sigma}$ by maximality of $\mathfrak{t}^{-\sigma}$ in $\mathfrak{u}^{-\sigma}$. It follows from $\sigma(X) = X - (X - \sigma X) \in \mathfrak{t} + \mathfrak{t}^{-\sigma} = \mathfrak{t}$ that $\sigma(\mathfrak{t}) \subset \mathfrak{t}$, hence $\mathfrak{t}$ is σ-stable. □

Theorem 72. *Let* $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\tau} = 0$*. Then there exists a Cartan subalgebra* $\mathfrak{t} \subset \mathfrak{g}$ *that is* τ, σ, θ*-stable.*

Proof. Since the conjugations commute, $\mathfrak{u} = \mathfrak{g}^\theta$ is a τ, σ-stable subalgebra of $\mathfrak{g}^\mathbb{R}$. On $\mathfrak{u}$, the involutions τ' and σ' reduce to τ and σ and we obtain the following decomposition of $\mathfrak{u}$:

$$\mathfrak{u} = \mathfrak{u}^\sigma \cap \mathfrak{u}^\tau \oplus \mathfrak{u}^\sigma \cap \mathfrak{u}^{-\tau} \oplus \mathfrak{u}^{-\sigma} \cap \mathfrak{n}^\tau$$

Notice that $\mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{-\tau} = 0$ by assumption. Choose a maximal abelian subspace $\mathfrak{t}^{-\sigma}$ in $\mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{\tau}$ and a maximal abelian subalgebra $\mathfrak{t}^{-\tau}$ in $\mathfrak{u}^{\sigma} \cap \mathfrak{u}^{-\tau}$. Notice that $[\mathfrak{u}^{\sigma}, \mathfrak{u}^{-\sigma}] \subset \mathfrak{u}^{-\sigma}$ and that $[\mathfrak{u}^{\tau}, \mathfrak{u}^{-\tau}] \subset \mathfrak{u}^{-\tau}$. We obtain the following:

$$[\mathfrak{u}^{\sigma} \cap \mathfrak{u}^{-\tau}, \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{\tau}] \subset \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{-\tau} = 0$$

This result implies that $\mathfrak{t}^{-\sigma} \oplus \mathfrak{t}^{-\tau}$ is maximal abelian in $\mathfrak{u}^{\sigma} \cap \mathfrak{u}^{-\tau} \oplus \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{\tau} = \mathfrak{u}^{-\sigma\tau}$. Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{n}$ containing $\mathfrak{t}^{-\sigma} \oplus \mathfrak{t}^{-\tau}$. We will prove that $\mathfrak{t}$ is σ- and τ-stable.

Let $X \in \mathfrak{t}$. Then $X - \sigma X$ commutes with $\mathfrak{t}^{-\sigma}$ and belongs to $\mathfrak{u}^{-\sigma} = \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{\tau} \oplus \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{-\tau} = \mathfrak{u}^{-\sigma} \cap \mathfrak{u}^{\tau}$. By maximality of $\mathfrak{t}^{-\sigma}$ in $\mathfrak{u}^{-\sigma}$, it follows that $X - \sigma X \in \mathfrak{t}^{-\sigma}$ which implies $X - \sigma X \in \mathfrak{t}$. From $X \in \mathfrak{t}$ we obtain that $\sigma X = X - (X - \sigma X) \in \mathfrak{t}$ which proves that $\mathfrak{t}$ is σ-stable. That $\mathfrak{t}$ is τ-stable follows in an analogous fashion.

We have obtained that $\mathfrak{t}$ is σ- and τ-stable Cartan subalgebra. This implies that $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \oplus \sigma\mathfrak{t}$ is a τ, σ, θ-stable Cartan subalgebra of $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}$. □

Theorem 73. *Let W be a complex reductive group with three commuting complex conjugations τ, σ, θ where θ Cartan. Then W contains a τ, σ, θ-stable Cartan subalgebra.*

Proof. Let A be a maximal σ', τ'-stable split Cartan subalgebra i.e. for $a \in A$, $\sigma'(a) = \tau'(a) = a^{-1}$. It follows from [Hel01, 11.3] that such a Cartan subalgebra exists and that we may assume that it is θ-stable as well. Let $H = Z_W(A)$ be the center of A in W which is τ, σ, θ-stable. Moving over to Lie algebra level, W being reductive allows us the write the Lie algebra $\mathfrak{h}$ of H, as a direct product of the Lie algebra $\mathfrak{a}$ of A, and a direct product of simple Lie algebras $\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$. Since $\mathfrak{a}$ is τ, σ, θ-stable, it suffices to find a maximal τ, σ, θ-stable Cartan subalgebra in $\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$. Notice that $\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$ has no non trivial σ', τ'-stable split Cartan subalgebra by construction, hence Lemma 70 and Theorem 72 yield the desired Cartan subalgebra. □

5.3 Matsuki correspondence for real semisimple Lie groups

The complex Matsuki correspondence is about a complex (connected) semisimple Lie group G, equipped with commuting conjugations σ,θ of which θ is also Cartan. This gives rise to a real form G^σ (previously denoted G_0) and a complex subgroup $G^{\sigma\theta}$ (previously denoted K). Let Q be a parabolic subgroup of G. The complex Matsuki correspondence relates G^σ- and $G^{\sigma\theta}$-orbits in G/Q via $G^\sigma \cap G^{\sigma\theta}$-orbits in G/Q.

For real Matsuki correspondence there is one additional complex conjugation τ which commutes with both θ and σ and such that the parabolic subgroup Q is τ-invariant. Then G^τ is a real form of G and G^τ/Q^τ a real flag manifold. The real Matsuki correspondence relates $(G^\tau)^\sigma$- and $(G^\tau)^{\sigma\theta}$-orbits on G^τ/Q^τ via $(G^\tau)^\sigma \cap (G^\tau)^{\sigma\theta}$-orbits in G^τ/Q^τ.

Notice that in both cases the starting point is a complex Lie group G. In this section will describe the real Matsuki correspondence in terms of a real (connected) semisimple Lie group G_0, two commuting involutions σ_0 and θ_0 where θ_0 is a Cartan involution on G_0 and a parabolic subgroup Q_0 of G_0. Let $\mathfrak{g}_0$ be the Lie algebra of G_0, and let $\mathfrak{q}_0$ be the Lie algebra of Q_0. Notice that $Q_0 = N_{G_0}(\mathfrak{q}_0)$.

Let G_0' be a connected real semisimple Lie group covering G_0 via $\varphi : G_0' \to G_0$. Define $\mathfrak{q}_0'$ as $\varphi_*^{-1}(\mathfrak{q}_0)$ where φ_* denotes the differential map of φ. Then $Q_0' = N_{G_0'}(\mathfrak{q}_0')$ is a parabolic subgroup of G_0' and observe $Q_0' = \varphi^{-1}(Q_0)$. From this result we get an induced diffeomorphism $\overline{\varphi} : G_0'/Q_0' \to G_0/Q_0$. Hence, we are free to choose a specific covering group. Let $\mathfrak{g} = (\mathfrak{g}_0)_\mathbb{C}$ and let G be the simply connected complex semisimple Lie group G with Lie algebra $\mathfrak{g}$. Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the complex conjugation of $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{g}^\tau$.

Define $\theta = (\theta_0)_\mathbb{C}\tau$. By Corollary 13, θ is a complex conjugation of $\mathfrak{g}$. Let $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{q}_0$ be the decomposition of $\mathfrak{g}_0$ with respect to σ_0 i.e. $h_0 = \mathfrak{g}_0^{\sigma_0}$ and $q_0 = \mathfrak{g}_0^{-\sigma_0}$. Define $\sigma = (\sigma_0\theta_0)_\mathbb{C}\tau$, which is an conjugation of $\mathfrak{g}$ by construction.

Lemma 74. *The conjugations σ and θ commute.*

Proof. By Corollary 13, τ commutes with θ and it is clear that $(\sigma_0)_\mathbb{C}$ and θ commute. Notice that θ_0 leaves the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{g}^\tau \oplus g^{-\tau}$ fixed (same argument as for Corollary 13) and since τ and $(\sigma_0)_\mathbb{C}$ are linear, it follows that they commute. Hence, all elements that arise in the definition of σ and θ commute from which the result follows. □

Lemma 75. *The conjugation σ on $\mathfrak{g}$ defines a real form $\mathfrak{g}^\sigma$ on $\mathfrak{g}$ called the Flensted-Jensen real form.*

Proof. Out of the definition of σ, we see that $\mathfrak{g}^\sigma = \mathfrak{g}_0^{\sigma_0\theta_0} \oplus i\mathfrak{g}_0^{-\sigma_0\theta_0}$. Using the decompositions of $\mathfrak{g}_0$ associated to σ_0 and θ_0, we get the following:

$$\mathfrak{g}_0^d = \mathfrak{g}_0^{\sigma_0\theta_0} \oplus \mathfrak{g}^{-\sigma_0\theta_0} = (\mathfrak{k}_0 \cap \mathfrak{h}_0) \oplus (\mathfrak{p}_0 \cap \mathfrak{q}_0) \bigoplus i((\mathfrak{k}_0 \cap \mathfrak{q}_0) \oplus (\mathfrak{p}_0 \cap \mathfrak{h}_0)) \quad (42)$$

It is clear from the above expression, that the orthogonal complement of $\mathfrak{g}^\sigma$ is $i\mathfrak{g}^\sigma$. $\square$

Since σ and θ commute, the Lie subalgebra of $\mathfrak{g}^\sigma$ given by $\mathfrak{g}^\theta \cap \mathfrak{g}^\sigma$, is compact and is maximal as such. From Equation 42 we obtain $\mathfrak{g}^\sigma \cap \mathfrak{g}^\theta = (\mathfrak{k}_0 \cap \mathfrak{h}_0) \bigoplus i(\mathfrak{p}_0 \cap \mathfrak{h}_0)$. Let $\mathfrak{h}$ denote $(\mathfrak{h}_0)_\mathbb{C}$, then the maximally compact subalgebra of $\mathfrak{g}_0^d$ is given by $\mathfrak{h} \cap \mathfrak{g}^\sigma$. Notice that $\mathfrak{h} = (\mathfrak{g}^{\theta_0})_\mathbb{C} = \mathfrak{g}^{(\theta_0)_\mathbb{C}}$. Since σ_0 and θ_0 commute with each other, $(\sigma_0)_\mathbb{C}$ and $(\theta_0)_\mathbb{C}$ commute as well. This implies $\sigma\theta = (\sigma_0)_\mathbb{C}$ and hence $h = \mathfrak{g}^{\sigma\theta}$.

So at this point we have a complex (connected) semisimple Lie group G and complex commuting conjugations σ, θ, τ on G where θ is Cartan. Notice that $Q = N_G(\mathfrak{q}_0)$ is a parabolic subgroup of G that is τ-stable and such that $Q^\tau = Q_0$. By Theorem 68, there is a duality between the $G^\sigma \cap G^\tau$- and $G^{\sigma\theta} \cap G^\tau$-orbits in $X^\tau \simeq G^\tau/Q^\tau \simeq G_0/Q_0$. Two orbits are dual when their intersection equals exactly one $G^{\sigma\theta} \cap G^\sigma \cap G^\tau$-orbit in X^τ.

Let us examine the Lie algebras of the subgroups that come out of these intersections. The first intersection equals $\mathfrak{g}^\sigma \cap \mathfrak{g}^\tau = \mathfrak{g}_0^{\sigma_0\theta_0} \oplus i\mathfrak{g}_0^{-\sigma_0\theta_0} \cap \mathfrak{g}_0 = \mathfrak{g}_0^{\sigma_0\theta_0}$ and the second intersection equals $\mathfrak{g}^{\sigma\theta} \cap \mathfrak{g}^\tau = \mathfrak{h} \cap \mathfrak{g}_0 = \mathfrak{h}_0$. The third intersection equals $\mathfrak{g}^{\sigma\theta} \oplus \mathfrak{g}^\sigma \oplus \mathfrak{g}^\tau$. By intersecting with $\mathfrak{g}^\tau$, we are restricted to the case that τ equals the identity. Hence the intersection equals $\mathfrak{g}_0^{\sigma_0\theta_0} \cap \mathfrak{g}_0^{\sigma_0}$. We end up with correspondence between $G_0^{\sigma_0\theta_0}$- and $G_0^{\theta_0}$-orbits in G_0/Q_0 via $G_0^{\theta_0} \cap G_0^{\sigma_0\theta_0}$-orbits of G_0/Q_0. Hence, we have established Matsuki correspondence for real (connected) semisimple Lie groups.

6 Real and complex Matsuki correspondence for $G = \mathrm{SL}(2, \mathbb{C})$

To illustrate Matsuki correspondence, we have studied Matsuki correspondence for the complex semisimple Lie group $\mathrm{SL}(2, \mathbb{C})$ in Section 1.2. In Section 6.1 we will examine Matsuki correspondence for the real semisimple Lie group $\mathrm{SL}(2, \mathbb{R})$. The calculations of the fixed point groups and orbits for both groups, can be found in Sections 6.2 and 6.3.

6.1 Matsuki correspondence for $\mathrm{SL}(2, \mathbb{C})$

In this section we shall describe Matsuki correspondence for the real case (Theorem 68).

Let the notation be as in section 1.2. For the real version of Matsuki correspondence we require a third conjugation. Let τ be the complex conjugation associated to the real form $\mathrm{SL}(2, \mathbb{R})$ in $\mathrm{SL}(2, \mathbb{C})$, hence:

$$\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}$$

Let the subscript 0 denote the fixed point set with respect to τ: $G^\tau = G_0 = \mathrm{SL}(2, \mathbb{R})$. Analogous to Section 1.2 we get an action of G_0 on $X = G/P$ where P denotes the stabilizer of ∞ in $\widehat{\mathbb{C}}$. It follows from Equation 41 that we may restrict this G_0 action to $X^\tau \simeq G_0/P_0$ and it is straightforward to see $X^\tau = \widehat{\mathbb{R}}$. The action of G_0 onto X^τ is given by the restriction of the action described in Equation 1.

By Theorem 68 there is the one-to-one correspondence between G_0^σ- and $G_0^{\sigma\theta}$-orbits in $G_0/P_0 \simeq \widehat{\mathbb{R}}$. Two orbits are dual when their intersection equals precisely one $G_0^{\sigma\theta} \cap G^\sigma$-orbit. Table 3 will give an overview of the fixed point sets and their orbits in $\widehat{\mathbb{R}}$, Table 4 contains the intersections of G_0^σ- and $G_0^{\sigma\theta}$-orbits.

Table 3: The fixed point groups with their orbits in $\widehat{\mathbb{R}}$

G_0^σ	$\left\{\pm\begin{pmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{pmatrix} : t\in\mathbb{R}\right\}$	Orbits: • $\widehat{\mathbb{R}}\setminus[-1,1]$ • $(-1,1)$ • $\{+1\}$ • $\{-1\}$
$G_0^{\sigma\theta}$	$\left\{\begin{pmatrix} r & 0\\ 0 & r^{-1}\end{pmatrix} : r\in\mathbb{R}^*\right\}$	Orbits: • $\{0\}$ • $\{\infty\}$ • $\mathbb{R}_+$ • $\mathbb{R}_-$
$G_0^{\sigma\theta}\cap G^\sigma$	$\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$	Orbits: $\{r\}$ for every $r\in\widehat{\mathbb{R}}$

Table 4: Intersections of G_0^σ- and $G_0^{\sigma\theta}$-orbits

G_0^σ-orbits $\rightarrow$

$G_0^{\sigma\theta}$-orbits $\downarrow$

$\cap$	$\widehat{\mathbb{R}}\setminus[-1,1]$	$(-1,1)$	$\{+1\}$	$\{-1\}$
$\{0\}$	$\emptyset$	$\{0\}$	$\emptyset$	$\emptyset$
$\{\infty\}$	$\{\infty\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbb{R}_+$	$[1,\infty)$	$(0,1)$	$\{1\}$	$\emptyset$
$\mathbb{R}_-$	$(-\infty,-1]$	$(-1,0)$	$\emptyset$	$\{-1\}$

The intersections which consist of precisely one $G_0^{\sigma\theta}$-orbit are marked red. Notice that there are multiple intersections which consist of more than one $G_0^{\sigma\theta}$-orbit. We get the following duality:

- $\widehat{\mathbb{R}}\setminus[-1,1]\longleftrightarrow\{\infty\}$
- $(-1,1)\longleftrightarrow\{0\}$
- $\{+1\}\longleftrightarrow\mathbb{R}_+$
- $\{-1\}\longleftrightarrow\mathbb{R}_-$

6.2 Calculating the fixed points

We will calculate the groups of fixed points used in the previous two sections. For the fixed points, the real case follows directly from the complex case by intersection with $G^\tau = \mathrm{SL}(2, \mathbb{R})$. For this reason we will only calculate the fixed points in the complex case.

Let's start by calculating G^σ. By Equation 2 this group consists of the matrices in G that satisfy the following equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & \overline{c} \\ \overline{b} & \overline{a} \end{pmatrix}$$

We obtain $a = \overline{d}$ and $b = \overline{c}$. Using this we can express the group of fixed points in the following way:

$$G^\sigma = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \text{ for } a, b \in \mathbb{C} \right\} = SU(1,1)$$

Using Equation 3, we can do an analogous calculation for $G^{\sigma\theta}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

Hence, we obtain $a = \overline{d}$, $b = -b$ and $c = -c$. We get $b = c = 0$ and since we are working in G, we get $d = a^{-1}$. Hence we get the following expression for $G^{\sigma\theta}$:

$$G^{\sigma\theta} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^* \right\}$$

Finally, an expression for $G^{\sigma\theta} \cap G^\sigma$ can be found by taking the intersection of the expressions above:

$$G^{\sigma\theta} \cap G^\sigma = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} : a \in \mathbb{C}^* \text{ such that } \overline{a} = a^{-1} \right\} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} : t \in \mathbb{R} \right\}$$

6.3 Calculating the orbits

In this section we will calculate the relevant orbits in X and X^τ that were used in Sections 1.2 and 6.1 respectively.

Notice that the subgroups G_0^σ, $G_0^{\sigma\theta}$ and $G_0^{\sigma\theta} \cap G_0^\sigma$ are equal to the subgroups G^σ, $G^{\sigma\theta}$ and $G^{\sigma\theta} \cap G^\sigma$ intersected with G_0. The orbits of the intersection of the subgroups, equal the intersections of the orbits of subgroups since the conjugations commute. Hence, the fixed point groups of the real case are obtained by intersecting the fixed point groups of the complex case with G_0. The orbits are obtained by taking the intersection of the orbits in the complex case with the orbit of G_0 in X^τ which equals $\widehat{\mathbb{R}}$.

Notice that the amount of orbits may change when we move from the complex to the real case and that connectedness of the orbit may be lost. The G^σ-orbits will turn out to be the complicated ones. We will first describe the $G^{\sigma\theta}$- and $G^{\sigma\theta} \cap G^\sigma$-orbits.

Let $z \in \widehat{\mathbb{C}}$. We will first calculate the $G^{\sigma\theta}$-orbits. The action of $g \in G^{\sigma\theta}$ on z is described by Equation 1 and combining this with the results from Section 6.2 we get the following:

$$gz = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} z = \frac{az + 0}{0z + \frac{1}{a}} = a^2 z$$

It is immediate that for $z \in \mathbb{C}^*$, the orbit equals $\mathbb{C}^*$. It is also clear that $z = 0$ and $z = \infty$ are stabilized by $G^{\sigma\theta}$ and we obtain the orbits described in Table 1. The $G^{\sigma\theta} \cap G^\sigma$-orbits follow in analogous fashion.

We will tackle the G^σ-orbits in $\widehat{\mathbb{C}}$ in several lemmas.

Lemma 76. *Every G^σ-orbit in $\widehat{\mathbb{C}}$ is invariant under rotation i.e. invariant under multiplication by e^{it} with $t \in \mathbb{R}$.*

Proof. Let $a, b \in \mathbb{C}$ satisfy $|a|^2 - |b|^2 = 1$ and let a be described in polar coordinates as ρe^{it}. Using Equation 1, we can do the following calculation:

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} z = \frac{az + b}{\overline{b}z + \overline{a}} = \frac{\rho e^{it} z + b}{\overline{b}z + \overline{\rho e^{it}}} = \frac{\rho z + be^{-it}}{\overline{b}e^{it}z + \rho} \cdot \frac{e^{it}}{e^{-it}} = \frac{\rho z + be^{-it}}{\overline{be^{-it}}z + \rho} \cdot e^{2it}$$

By Section 6.2, the matrix on the left is in G^σ. Define $\tilde{b} = be^{-it}$ and notice that $|a|^2 - |\tilde{b}|^2 = |a|^2 - |b|^2 = 1$. Hence, we can describe the quotient on the right hand side as an element of the G^σ-orbit of z:

$$\frac{\rho z + be^{-it}}{\overline{be^{-it}}z + \rho} = \begin{pmatrix} a & \tilde{b} \\ \overline{\tilde{b}} & \overline{a} \end{pmatrix} z$$

This implies that every rotation of an element of a G^σ-orbit, is inside the same G^σ-orbit. $\square$

By letting b and z have the same angle with respect to a real line, we obtain some nice results:

Lemma 77. *Let $b \in \mathbb{C}$ be such that $\rho^2 - |b|^2 = 1$ and let $b = ye^{it}$ be its polar decomposition. Let $z = re^{it} \in \mathbb{C}$ for some $r \in \mathbb{R}_+ \cup \{0\}$. The following formula holds:*

$$f(r, y) := \left| \begin{pmatrix} \rho & b \\ \bar{b} & \rho \end{pmatrix} z \right| = \frac{\sqrt{1+y^2}r + y}{yr + \sqrt{1+y^2}} \tag{43}$$

Proof. We obtain the following by applying Equation 1:

$$\left| \begin{pmatrix} \rho & b \\ \bar{b} & \rho \end{pmatrix} z \right| = \left| \frac{\rho re^{it} + ye^{it}}{ye^{it}re^{-it} + \rho} \right| = \frac{|\rho re^{it} + ye^{it}|}{|yr + \rho|} = \frac{\rho r + y}{yr + \rho}$$

Plugging in $\rho = \sqrt{1+y^2}$ gives the desired result. □

Lemma 78. *The function $f(r, y)$ has the following properties; let $y, r \geq 0$.*
(a) For a fixed r, $f(r, y)$ is continuous in the second parameter for $y \in [0, \infty)$.
(b) For a fixed r, $\lim_{y\to\infty} f(r, y) = 1$ and $f(r, 0) = r$.
(c) For $0 \leq r < 1$, $f(r, y)$ is monotonically increasing in y. For $r = 1$, $f(r, y)$ is constant in y. For $r > 1$, $f(r, y)$ is monotonically decreasing.

Proof. **(a)** Since $yr + \sqrt{1+y^2} \geq 1$, the continuity follows from composition of continuous functions.
(b) Notice that $\lim_{y\to\infty} \frac{\sqrt{1+y^2}}{y} = 1$. Multiplying both the numerator and denominator by $\frac{1}{y}$ yields the result.

(c) We require the y-derivative of $f(r,y)$:

$$\begin{aligned}
\partial_y f(r,y) &= \partial_y \frac{1}{yr+\sqrt{1+y^2}}(\sqrt{1+y^2}r+y) \\
&= -\frac{r+\frac{y}{\sqrt{1+y^2}}}{(yr+\sqrt{1+y^2})^2}(\sqrt{1+y^2}r+y) \\
&+ \frac{1}{yr+\sqrt{1+y^2}}(1+\frac{ry}{\sqrt{1+y^2}}) \\
&= \frac{1}{(yr+\sqrt{1+y^2})^2}\Bigg(-(r+\frac{y}{\sqrt{1+y^2}})\cdot(\sqrt{1+y^2}r+y) \\
&+ (yr+\sqrt{1+y^2})\cdot(1+\frac{ry}{\sqrt{1+y^2}})\Bigg) \\
&= \frac{1}{(ry+\sqrt{1+y^2})^2}\cdot\frac{1-r^2}{\sqrt{1+y^2}}
\end{aligned}$$

The term on the left is positive, the sign of the term on the right depends on $1-r^2$. For $0 \leq r < 1$ we get $1-r^2 > 0$, for $r = 1$ we get $1-r^2 = 0$ and for $r > 1$ we get $1-r^2 < 0$. This implies the desired result. □

Theorem 79. *The G^σ-orbits in $\widehat{\mathbb{C}}$ are listed in Table 1.*

Proof. Let $z \in \mathbb{C}$ with $|z| = 1$. By Lemma 76, all elements of length 1 lie inside this orbit. By the following reasoning, the orbit can't be larger then this circle:

$$\left|\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} e^{it}\right| = \left|\frac{ae^{it}+b}{\overline{b}e^{it}+\overline{a}}\right| = \left|\frac{ae^{it/2}+be^{-it/2}}{\overline{b}e^{it/2}+\overline{a}e^{-it/2}}\right| = \left|\frac{\tilde{a}+\tilde{b}}{\overline{\tilde{b}}+\overline{\tilde{a}}}\right| = 1$$

Here we use $\tilde{a} = ae^{it/2}$ and $\tilde{b} = be^{-it/2}$. Hence, the G^σ-orbit of any element of the unit circle, equals the unit circle.

Let $z = 0$ and let $c \in \mathbb{R}$ such that $0 < c < 1$. By Lemma 78 and the Intermediate Value Theorem, we know that there will be a $g \in G^\sigma$ such that $|gz| = c$. By Lemma 76 this implies that the G^σ-orbit of z will at least contain the open unit disk. Notice that the boundary of the disk already a full G^σ-orbit, hence it cannot be contained in the same orbit as the open unit disk.

Since $SU(1,1)$ is connected (it is isomorphic to $SU(2)$) its orbit through 0 is connected as well (the action is continuous), which implies that any $|z| > 1$ cannot be part of this orbit. Hence, the orbit of any element of the unit disk, equals the unit disk.

Finally, let $z = \infty$. We get the following:

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} z = \frac{a}{\bar{b}}$$

It follows from $|a|^2 - |b|^2 = 1$ that $|b| < |a|$. For $b = 0$, z is fixed. For $b \neq 0$, we see that $|\frac{a}{\bar{b}}| > 1$. For these values we can use Lemmas 78 and 76 in the same way as above. We conclude that the G^σ-orbit of an element $z \in \widehat{\mathbb{C}}$ with $|z| > 1$, equals the $z \in \widehat{\mathbb{C}}$ with $|z| > 1$. $\square$

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