The Zeta function according to Riemann

Willem Pranger
3477029

February 8, 2013

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1.1 The goal of this paper

In this paper, we present the proof outlined by Riemann in his ground breaking paper from 1859. In this paper, Riemann proves several results about the so called Riemann zeta function and he gives an analytic representation of the prime-counting function $\pi(x)$. He closes his ten page paper with a strong approximation of $\pi(x)$, which was stronger than the best analytic approximation at that time. Riemann left a lot of the formal proofs to the reader, giving them only hints at the solution, but essentially his proof is incomplete. The formal completion of the proof comes form Hadamard and von Mangoldt. In this paper, we will thoroughly work out Riemann's original approach and we will be able to proof all his assertions except for one switching of summation and integration. The first proof of the legality of this action was given by von Mangoldt, but his proof is indirect and uses a totally different approach than suggested by Riemann. The only knowledge required in this paper, is some background in complex analysis.

1.2 Introducing the Riemann Zeta function

In 1737 Leonard Euler proved the following interesting identity ([L],441):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - \frac{1}{p^s}} \tag{1}$$

Here P denotes the set of prime numbers and the product on the right is called the Euler product. This indicates that there could be a useful link between prime numbers and the Riemann zeta function (Rzf), but to see this connection we will first have to investigate some properties of the Rzf.

First of all, we will have to prove that $\zeta(s)$ is a convergent and analytic complex function for $Re(s) > 1$. Denote the set of complex numbers s for which $Re(s) > 1$ by B. We'll start with convergence. Let β denote $Re(s)$:

$$|\zeta(s)| - 1 \leq \sum_{n=1}^{\infty} |n^{-s}| - 1 = \sum_{n=1}^{\infty} |n^{-\beta}| - 1 \leq \int_1^{\infty} x^{-\beta} dx = \left. \frac{x^{-\beta+1}}{1-\beta} \right|_{x=1}^{\infty} = \frac{1}{\beta - 1} < \infty$$

The Riemann-integral is larger than the summation because we can look at it as a Darboux integral, where the summation is can be seen as one of its lower Darboux sums. Since the lower Darboux sum will always be smaller than the integral, we deduce that our summation will be smaller than the integral. We conclude that $\zeta(s)$ is an absolutely convergent series, which means it also converges uniformly on B ([L],52). The analyticity of $\zeta(s)$ will follow from the following lemma which we will not proof.

Lemma *Let f_n be a converging sequence of analytic functions on an open set U which converges to f. If the convergence is uniform on every compact subset K of U, then f is analytic on U. ([L],156)*

Notice that n^{-s} is analytic for every $Re(s) > 1$ and n is a natural number. In the sequence $f_k(s) = \sum_{n=1}^{k} n^{-s}$ all elements are analytic which allows us to use the lemma. As a result, $\zeta(s)$ is an analytic function on the open set B.
Now look at $\log(|\zeta(s)|)$. If this is finite valued, we know that $|\zeta(s)|$ is non-zero. We can prove this

using the product formula:

$$\log(|\zeta(s)|) = -\sum_{p\in P}\log(|1-\frac{1}{p^s}|) \leq \sum_{p\in P}^{\leq N}\log(|1-\frac{1}{p^s}|) + \sum_{p\in P, p\geq N} p^{-s}$$

Here we used that for small values of x, $|\log(1-x)| \leq x$. The sum on the right is finite since it is smaller than the converging series $\sum n^{-s}$. We conclude that $\zeta(s) \neq 0$.

To get some concrete results for $\zeta(s)$, we'll have to look at its individual terms. The smart move is to create an integral which evaluates to something that looks like n^{-s}. That integral is $\int_0^\infty e^{-nx}x^{s-1}dx$. Substituting $\frac{t}{n}$ for x gives a jacobian which equals n^{-1} and results in our term multiplied with the gamma function:

$$\int_0^\infty e^{-nx}x^{s-1}dx = \int_0^\infty e^{-t}(\frac{t}{n})^{s-1}dt\cdot n^{-1} = \int_0^\infty e^{-t}t^{s-1}dt\cdot n^{-s} = \Gamma(s)n^{-s}$$

Using this identity and summing over n gives us the following expression:

$$\zeta(s)\cdot\Gamma(s) = \sum_{n=1}^\infty\int_0^\infty e^{-nx}x^{s-1}dx$$

For $Re(s) > 1$, both $\Gamma(s)$ and $\zeta(s)$ are finite which means that the sum converges. The integrals are taken over a domain in which the functions are non-negative and the integrals converge, which means they evaluate to positive reals. As a result, the summation is absolutely convergent since its value is the finite value $\zeta(s)\cdot\Gamma(s)$. Absolute convergence implies uniform convergence so we're allowed to switch the order of summation and integration ([L],100):

$$\zeta(s)\cdot\Gamma(s) = \int_0^\infty\sum_{n=1}^\infty e^{-nx}x^{s-1}dx = \int_0^\infty((\sum_{n=0}^\infty e^{-nx})-1)\cdot x^{s-1}dx = \int_0^\infty(\frac{1}{1-e^{-x}}-1)\cdot x^{s-1}dx$$

Here we use that $e^{-nx} < 1$ whenever $x > 0$. This entails that our summation is a geometric series which is convergent. The only point where this fails is $x = 0$, but $\{0\}$ is a set with measure zero which means it does not contribute to the integral and our move is still valid.
If we continue we get the following:

$$\zeta(s)\Gamma(s) = \int_0^\infty\frac{1-(1-e^{-x})}{1-e^{-x}}\cdot x^{s-1}dx = \int_0^\infty\frac{e^{-x}}{1-e^{-x}}\cdot x^{s-1}dx = \int_0^\infty\frac{x^{s-1}}{e^x-1}dx \qquad (2)$$

1.3 The Hankel Integral

We can evaluate the integral using contour integration. The contour K_ϵ that we will use is drawn in figure 1, the function that we will integrate is $\frac{x^{s-1}}{e^{-x}-1}$. The path can be split up in three parts: it starts at minus infinity, turns around zero and runs back to minus infinity. We will show that the integral over the turn around zero, which we will call C_ϵ, goes to zero as ϵ goes to zero. C_ϵ can be parameterized with $\epsilon e^{2\pi it}$. As a result we get the following expression inside our integral:

$$\frac{(\epsilon e^{2\pi it})^{s-1}}{e^{-\epsilon e^{w\pi it}}-1}\cdot 2\pi i\epsilon e^{2\pi it}$$

Taking the absolute value simplify's our expression to:

$$\frac{2\pi\epsilon^{Re(s)}}{|e^{-\epsilon\cos(2\pi t)+i\sin(2\pi t)}-1|}$$

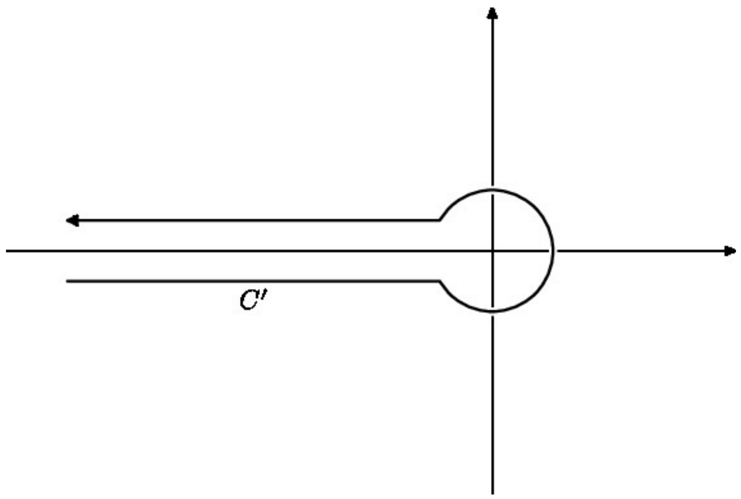

Figure 1: In this picture, $C' = K_\epsilon$. The radius of the circle is ϵ.

Using L'Hospital's rule for the limit $\epsilon \downarrow 0$, we can calculate the limit:

$$\lim_{\epsilon\downarrow 0} \frac{2\pi\epsilon^{Re(s)}}{|e^{-\epsilon\cos(2\pi t)i\sin(2\pi t)}-1|} = \lim_{\epsilon\downarrow 0} \frac{2\pi Re(s)\epsilon^{Re(s)-1}}{|e^{-\epsilon\cos(2\pi t)i\sin(2\pi t)}\cdot(-\cos(2\pi t)i\sin(2\pi t))|} = \frac{0}{|\cos(2\pi t)i\sin(2\pi t)|} = 0 \tag{3}$$

Here we use that $Re(s) > 1$. Notice this reasoning fails for a finite number of discrete values of t, but these values do not contribute to the integral since there measure is zero. Because $\epsilon e^{2\pi it}$ goes to 0 uniformly, we are allowed to swap the limit and the integral ([L], 100):

$$\lim_{\epsilon\downarrow 0}\int_0^1 \frac{2\pi\epsilon^{Re(s)}}{|e^{-\epsilon\cos(2\pi t)i\sin(2\pi t)}-1|}dt = \int_0^1 \lim_{\epsilon\downarrow 0}\frac{2\pi\epsilon^{Re(s)}}{|e^{-\epsilon\cos(2\pi t)i\sin(2\pi t)}-1|}dt = 0$$

This proves that the integral part over C_ϵ, does not contribute to the integral when $\epsilon \downarrow 0$.

Define the function $H(s)$ to be $\int_{K_0}\frac{x^{s-1}}{e^{-x}-1}dx$, i.e. the limit$\epsilon \downarrow 0$ of $\int_{K_\epsilon}\frac{x^{s-1}}{e^{-x}+1}dx$. This integral only exists out of two parts which differ by a constant. The constant arises from $x^{s-1} = e^{\log(x)(s-1)}$. The logarithm is define on $\mathbb{C}$ with the exception of one straight line. As is custom, the negative reals are left out. This means that $\log(-3)$ is ill defined since its imaginary part can either be πi or $-\pi i$. In other words, the logarithm has no unique analytic extension to all of $\mathbb{C}$. However, the limit $t \uparrow 0$ on $\log(-3+ti)$ is well defined and equal to $\log(3) - \pi i$. The same goes for $t \downarrow 0$ on $\log(-3+ti)$ which equals $\log(3) + \pi i$. This is not only true for 3, but for every negative real. As a consequence, we can rewrite $H(s)$ in the following way:

$$\begin{aligned} H(s) &= \int_{K_0}\frac{x^{s-1}}{e^{-x}-1}dx = \int_{-\infty}^0 \frac{e^{\log(x)(s-1)}}{e^{-x}-1}dx + \int_0^{-\infty}\frac{e^{\log(x)(s-1)}}{e^{-x}-1}dx \\ &= \int_\infty^0 \frac{e^{(\log(x)-\pi i)(s-1)}}{e^x-1}(-1)dx + \int_0^\infty \frac{e^{(\log(x)+\pi i)(s-1)}}{e^x-1}(-1)dx \end{aligned}$$

The integral running form $-\infty$ to 0 and back, should be considered as the horizontal path of K_ϵ, where we let ϵ tend to zero. Notice that we have eliminated the logarithmical problem since the logarithm is now working on positive reals. Since the logarithm is continuous, the limit taken from above and from below coincide. Writing $-1 = e^{\pi i}$, will allow us to find a compact expression for

$H(s)$:

$$\begin{aligned} H(s) &= e^{-\pi i(s-1)}e^{\pi i}\int_{\infty}^{0}\frac{e^{\log(x)(s-1)}}{e^x-1}dx + e^{\pi i(s-1)}e^{\pi i}\int_{0}^{\infty}\frac{e^{\log(x)(s-1)}}{e^x-1}dx \\ &= e^{-\pi i s}\int_{0}^{\infty}\frac{e^{\log(x)(s-1)}}{e^x-1}(-1)dx + e^{\pi i s}\int_{0}^{\infty}\frac{e^{\log(x)(s-1)}}{e^x-1}dx \\ &= (-e^{-\pi i s}+e^{\pi i s})\int_{0}^{\infty}\frac{e^{\log(x)(s-1)}}{e^x-1}dx = 2i\sin(\pi s)\zeta(s)\cdot\Gamma(s) \end{aligned}$$

We have found an relation between $H(s)$ and $\zeta(s)$. For the gamma function, there is an identity called Euler's reflection formula ([L],415): $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$. This allows us to write $\zeta(s)$ in terms of the other functions:

$$\zeta(s) = H(s)\Gamma(1-s)\frac{1}{2\pi i}$$

The gamma function is meromorphic on $\mathbb{C}$ with simple poles on the non-positive integers. If $H(s)$ is analytic on a larger domain than B, we can extend $\zeta(s)$ analytically. Before we can do that, we first have to extend $H(s)$ analytically to a domain in which the integral diverges. To clarify this, we'll take a look at the essence of analytic continuation first and then return to $H(s)$ later.

1.4 Analytic Continuation

Our goal is the following: let f be analytic on U and g be analytic on K, U and K open in $\mathbb{C}$. Assume $U \cap K$ is a non-empty open and $f(z) = g(z), \forall z \in U \cap K$. Then f and g can be extended to a unique analytic function h on $U \cup K$ for which $h(z) = f(z), \forall z \in U$ and $h(z) = g(z) \forall z \in K$. This is the essence of analytic continuation: an analytic function f on $U \subsetneq \mathbb{C}$ can uniquely be extended to an analytic function on a bigger domain. It is not always possible to extend an analytic function, but in this paper we will use it frequently. We will prove analytic continuation using two lemma's, which both are taken form ([L],62).

Lemma 1 *Let $f(z)$ be an non-constant analytic function on a neighborhood U of zero for which $f(0) = 0$. Then their exists a $s > 0$ such that $f(z)$ is non-zero for all $z \in U$ with $0 < |z| < s$*
Proof: Let f be as in lemma 1. This their is an $m > 0$ such that

$$f(z) = \sum_{k=m}^{\infty} a_k z^k = a_m z^m \cdot (1 + \sum_{k=1}^{\infty}\frac{a_{k+m}}{a_m}z^k) = a_m z^m \cdot (1 + h(z))$$

In this equation, $h(z)$ denotes an analytic function for which $h(0) = 0$: for small z, $|h(z)| < 1$. But for that same small z, $a_m z^m$ is non-zero which means $f(z) \neq 0$. This proves the lemma.

Lemma 2 *Let f and g denote two non-constant analytic functions on neighborhoods of zero U and V. Suppose $f(x) = g(x)$ for every x in a set of infinitely many points S, having 0 as a point of accumulation. Than f and g have the same coefficients in their power series expansion and denote the same function on both U and V.*

Proof: Define $h = f - g$. Then $h(x) = 0$ for all x in S. Since h is analytic, it is continuous such that follows that $h(0) = 0$. We can apply lemma 1 and notice that for every choice of $s > 0$, we can find an $x \in S$ such that $0 < |x| < s$ and $h(x) = 0$ since zero is a point of accumulation of S. Through this contradiction we conclude that h is the constant function zero, which means:

$$h(z) = f(z) - g(z) = \sum_{k\in\mathbb{N}} a_k z^k - \sum_{k\in\mathbb{N}} b_k z^k = \sum_{k\in\mathbb{N}}(a_k - b_k)z^k = 0$$

This implies $a_k = b_k$. We conclude $f = g$ and since both converge on their respective domain, they denote the same function on $U \cup V$ which concludes our proof.

Our lemma's only work on neighborhoods of zero, but this is not a problem. If the set S in lemma 2 would have z as a point of accumulation, we could translate S to $S - z$ and $f(t)$ to $f(t - z)$. We conclude that the only essential properties are the fact that we have infinitely many points S on which f and g coincide and that S has a point of accumulation.
Now our goal is clear as well: we want to find an infinite set of points with a point of accumulation on which g is an analytic function and $\zeta(s) = g(s)$. In this way we can extend the domain of $\zeta(s)$.

1.5 The functional equation

Lets take a closer look at $H(s)$. Notice that $x^{s-1}(e^{-x} - 1)^{-1}$ is analytic on for all $s \in \mathbb{C}$. Another useful observation is that the limit x goes to infinity, is zero for all values of s. Combining this, we can use a differentiation lemma which we will use without a proof([L],409):

Differentiation Lemma *Let I be an interval of real numbers. Let U be an open set in $\mathbb{C}$ and $f(t, z)$ be a continuous function on $I \times U$. Assume:*
(i) For each compact subset K of U, the integral $\int_I f(t, z)dt$ is uniformly convergent for all $z \in K$
(ii) For each $t \in I$, $f(z, t)$ is analytic over z.
Then $\int_I f(z, t)dt$ is analytic over z on U.

Using this lemma, we will conclude that $H(s)$ is analytic on entire $\mathbb{C}$, it is a so called entire function. At this point, we only have one expression for $H(s)$ and therefore for $\zeta(s)$. We can however find another analytic expression for $H(s)$ which through analytic continuation will hold for all $s \in \mathbb{C}$.

Take a look at the contour in figure 2 named F, the path taken clockwise. Since $Re(s) < 0$, the function disappears at infinity. $x^{s-1}(e^{-x} - 1)^{-1}$ goes to zero faster than x^{-1} when $|x| \to \infty$, which means that is becomes small, faster than the outer circle becomes big. This means that the outside circle will not contribute to the integral. The circle around zero will also give zero. As a result, we can express our original $H(s)$ in terms of the integral over F. We will use Cauchy's residue formula with a minus sign, since our path is taken clockwise.

$$\begin{aligned}
\int_F \frac{x^{s-1}}{e^{-x} - 1}dx &= -2\pi i \sum_{k=1}^{\infty} \left(-(-2\pi i k)^{s-1} + -(2\pi i k)^{s-1}\right) \\
&= (2\pi)^s \sum_{k=1}^{\infty} k^{s-1} e^{\frac{\pi}{2}i} \left(e^{-i\frac{\pi}{2}(s-1)} + e^{i\frac{\pi}{2}(s-1)}\right) \\
&= (2\pi)^s \zeta(1 - s)(-e^{-\frac{\pi}{2}s} + e^{\frac{\pi}{2}s}) = 2i(2\pi)^s \zeta(1 - s) \sin\left(\pi \frac{s}{2}\right) = H(s)
\end{aligned}$$

Here we use that the residue of $(e^{-x} - 1)^{-1}$ at $x = 0$ equals -1. Thanks to symmetry, the residue of $(e^{-x} - 1)^{-1}$ at $x = 2ik$ with $k \in \mathbb{Z}$ also equals -1. This equation is valid for $Re(s) < 0$. But $2i(2\pi)^s\zeta(1 - s) \sin\left(\pi \frac{s}{2}\right)$ denotes an analytic function when $Re(s) < 0$, which equals $H(s)$. Thanks to analytic continuation, they denote the same analytic function, which means the equality holds for all $s \in \mathbb{C}$.
The same reasoning holds for the equation $\zeta(s) = H(s)\Gamma(1 - s)\frac{1}{2\pi i}$, which will have to hold for all $s \in \mathbb{C}$ as well. This proves that $\zeta(s)$ is analytic on $\mathbb{C} \backslash \{1\}$.
In short, we have two descriptions for the same analytic function $H(s)$. If we combine both, we get:

$$\zeta(s) = H(s)\Gamma(1 - s)\frac{1}{2\pi i} = (2\pi)^s \frac{1}{\pi} \sin\left(\pi \frac{s}{2}\right)\Gamma(1 - s)\zeta(1 - s)$$

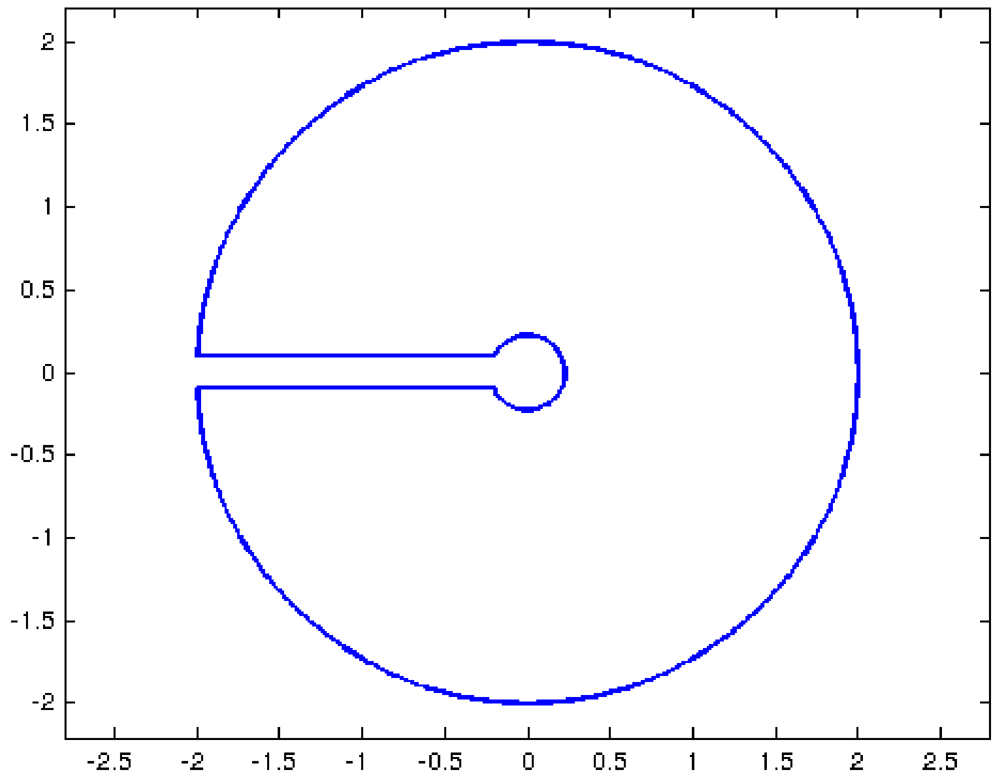

Figure 2: This path is taken to be clockwise. Notice that this way, part of the path becomes K_0.

Notice this is a functional equation for the zeta-function. We can rewrite this functional equation into something nicer:

$$\varepsilon(s) = \frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

This function satisfies the easy functional equation $\varepsilon(s) = \varepsilon(1-s)$ and it is a entire function. As it turns out, the zeroes of this function make it possible for us to construct an explicit function $\pi(x)$ that counts the amount of primes under x.

1.6 Linking prime-counting to $\log(\zeta(s))$

We want to construct an analytic prime-counting function $\pi(x)$. Such a function is not necessarily unique since all we want is a way to reconstruct the number of primes $\leq x$. For example, if $\pi(x)$ is such a function, $\pi(x) + 12$ would also qualify. Using the work we have done above for $\zeta(s)$ we will be able to find a useful expression for the number of primes beneath a value x.
Let $Re(s) > 1$. In this part of the complex plane the Euler product holds. Since the logarithm of a product is the summation of its logarithms, we can get a nice expression for the logarithm of the zeta-function:

$$\log(\zeta(s)) = \log(\prod_{p\in P}\frac{1}{1-\frac{1}{p^s}}) = -\sum_{p\in P}\log\left(1-\frac{1}{p^s}\right)$$

Since $\zeta(s)$ is non-zero for $Re(s) > 1$, this logarithm is well defined. For $|x| < 1$, $\log(1-x)$ is analytic. Since the real part of s is bigger than one, $|p^{-s}| < 1$, which means we can rewrite the above:

$$-\sum_{p\in P}\log\left(1-\frac{1}{p^s}\right)=\sum_{p\in P}(\frac{1}{p^s})+\frac{1}{2}\sum_{p\in P}(\frac{1}{p^s})^2+\frac{1}{3}\sum_{p\in P}(\frac{1}{p^s})^3+\cdots$$

Notice that we can write $p^{-k\cdot s}$ as the following integral: $s\int_{p^k}^{\infty}x^{-s-1}dx$. We want to plug this into our sum and switch the order of summation and integration. Therefore, we do the following to prove that our summations are absolutely convergent:

$$\sum_{p\in P}|(\frac{1}{p^s})|\leq\sum_{p=1}^{\infty}|(\frac{1}{p^s})|=\sum_{p=1}^{\infty}(\frac{1}{p^a})$$

Here, $Re(s)=a>1$. This implies that the summation on the right handside is convergent, which proves our original sum to be absolutly convergent. This makes the sum uniformly convergent ([L],52) which allows us the switch the order of integration and summation([L],100):

$$\begin{aligned}\frac{1}{s}\log(\zeta(s)) &= \frac{1}{s}\sum_{p\in P}(\frac{1}{p^s})+\frac{1}{2s}\sum_{p\in P}(\frac{1}{p^s})^2+\frac{1}{3s}\sum_{p\in P}(\frac{1}{p^s})^3+\cdots\\ &= \sum_{p\in P}\int_{p}^{\infty}x^{-s-1}dx+\frac{1}{2}\sum_{p\in P}\int_{p^2}^{\infty}x^{-s-1}dx+\frac{1}{3}\sum_{p\in P}\int_{p^3}^{\infty}x^{-s-1}dx+\cdots\\ &= \int_0^{\infty}x^{-s-1}\left(\sum_{p\in P}\mathbb{1}_{[p,\infty)}+\frac{1}{2}\sum_{p\in P}\mathbb{1}_{[p^2,\infty)}+\frac{1}{3}\sum_{p\in P}\mathbb{1}_{[p^3,\infty)}+\cdots\right)dx\end{aligned}$$

Here, $\mathbb{1}_{[a,b)}$ denotes the characteristic function of $[a,b)$ which is 1 on this subset and 0 elsewhere. Notice that $\mathbb{1}_{[p,\infty)}$ is zero for all primes under p, and 1 for all primes greater than or equal to p, the function 'counts' if it is larger than the prime number p. As a result, for all x, $\mathbb{1}_{[p,\infty)}(x)$ is 1 for every $p<x$ and 0 for all $p\geq x$ from which it follows that $\sum_{p\in P}\mathbb{1}_{[p,\infty)}(x)$ counts the number of primes less then x. For this reason we define $\pi(x)=\sum_{p\in P}\mathbb{1}_{[p,\infty)}(x)$. Notice that through the same reasoning, $\sum_{p\in P}\mathbb{1}_{[p^k,\infty)}(x)$ counts the number of kth powers of prime numbers between x. We can write this sum as $\pi(x^{\frac{1}{k}})$. Define $J(x)=\pi(x)+\frac{1}{2}\pi(x^{\frac{1}{2}})+\frac{1}{3}\pi(x^{\frac{1}{3}})+\cdots$ so we can rewrite the equation as follows:

$$\frac{1}{s}\log(\zeta(s))=\int_0^{\infty}x^{-s-1}J(x)dx \tag{4}$$

Notice that there is no necessity to start the integral at zero, we could also have started at 1 just as Riemann did in his original paper. It turns out that starting from zero is a bit easier. By using Möbius-inversion, we will be able to reconstruct $\pi(x)$ from $J(x)$. This means that a analytic expression for $J(x)$ will give us an analytic expression for $\pi(x)$. A first step in this direction is to rewrite the integral using Fouriertheory as we will do in the next section.

1.7 An expression for $J(x)$

Write $s=a+iy$ with $a,y\in\mathbb{R}$ and $a>1$. If we put this in the integral, we get:

$$\frac{1}{a+iy}\log(\zeta(a+iy))=\int_0^{\infty}x^{-a-iy-1}J(x)dx$$

We want to use the Fourier inversion formula for which we will first need to construct the Fourier transform of $\frac{1}{a+iy}\log(\zeta(a+iy))$. By substituting x by e^{λ}, we can get an integral which at least looks more like a Fourier transformation:

$$\frac{1}{a+iy}\log(\zeta(a+iy))=\int_{-\infty}^{\infty}e^{(-a-iy-1)\lambda}J(e^{\lambda})e^{\lambda}d\lambda=\int_{-\infty}^{\infty}e^{(-a-iy)\lambda}J(e^{\lambda})d\lambda$$

Notice that we can easily read off the Fourier transform since the definition of a transform $\widehat{\phi}$ of ϕ is $\phi(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\phi}(\lambda)e^{iy\lambda}d\lambda$. For this reason, define $\phi(\lambda) = 2\pi J(e^{\lambda})e^{-a\lambda}$. Hence, if we consider y as the free variable, we have constructed the Fourier transform since:

$$\frac{1}{a+iy}\log(\zeta(a+iy)) = \int_{-\infty}^{\infty}\phi(\lambda)e^{iy\lambda}d\lambda$$

Through the inversion formula we get:

$$\phi(\lambda) = 2\pi J(e^{\lambda})e^{-a\lambda} = \int_{-\infty}^{\infty}\frac{1}{a+iy}\log(\zeta(a+iy))e^{iy\lambda}dy$$

Notice that we pretended $\phi(\lambda)$ to be continuous which it, thanks to the $J(\lambda)$ term, is clearly not. Since we do not want to make our notation more complicated, we redefine $\phi(\lambda) = \frac{\phi(\lambda+0)+\phi(\lambda-0)}{2}$. Here, $\phi(\lambda+0) = \lim_{t\downarrow\lambda}\phi(t)$ and $\phi(\lambda-0) = \lim_{t\uparrow\lambda}\phi(t)$. Since $J(\lambda)$ is the only non-continuous part in $\phi(\lambda)$, this breaks down to redefining $J(\lambda) = \frac{J(\lambda+0)+J(\lambda-0)}{2}$. This way, we can still use the inversion formula as we did.

From this expression we can get an expression for $J(e^{\lambda})$. It is more convenient to rewrite $x = e^{\lambda}$ and notice that this time, since we do no longer integrate over x or λ, we do not get a Jacobian:

$$J(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{a+iy}\log(\zeta(a+iy))x^{a+iy}dy = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty}\frac{\log(\zeta(s))}{s}x^{s}(-i)ds \tag{5}$$

Notice that we redefined $J(x)$. Since $J(x)$ is defined to be a summation of $\frac{1}{k}\pi(x^{\frac{1}{k}})$ terms. Therefore, we implicitly redefined $\pi(x)$ as well. It comes down to the following: $\pi(x) = \frac{\pi(x+0)+\pi(x-0)}{2}$. All these new definitions do not change the fundamental properties of our functions, it merely shortens our notation when we work with Fourier transforms and inversion formula's.
Indirectly, we linked $\zeta(s)$ to $\pi(x)$. The basic idea is to write $\zeta(s)$ in terms of other well known functions and $\varepsilon(s)$. But before we can get a nice expression for these functions, we need some tools which we'll develop in chapter 2 and 3. We'll give an analytic construction of $\pi(x)$ in chapter 4.

1.8 Some concrete values of $\zeta(s)$.

Most of the results of the following chapters can be done without calculating any particular value of $\zeta(s)$. But some results take a nicer shape if we actually compute some values. Therefore, we will prove that $\zeta(s) < 0$ for $s \in (0,1)$ and $\zeta(0) = -\frac{1}{2}$.

For the first result, we look at $(1-2^{1-s})\zeta(s)$ which is analytic on $s \in (0,1)$. Notice that on this domain, $1-2^{1-s}$ is negative. Next, we manipulate this expression:

$$(1-2^{1-s})\zeta(s) = (1-2^{1-s})\sum_{n=1}^{\infty}n^{-s} = \sum_{n=1}^{\infty}\frac{1}{n^s} - \frac{2}{(2n)^s} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$$

This holds for $Re(s) > 1$. Since all the terms in the last sum are monotonically decreasing, the Leibniz-criterium proves that the sum is convergent, also for $Re(s) > 0$. Using analytic continuation, we can see that the equality holds for $Re(s) > 0$. The first term in this alternating sum is positive and since the terms are strictly decreasing; the sum has a positive value. Notice that using this sum, we can express $\zeta(s)$ as a converging sum on the interval $s \in (0,1)$:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\frac{1}{1-2^{s-1}}$$

Combining the above, we conclude that $\zeta(s)$ is negative on $s \in (0,1)$, which was our goal.

For the second property, we will prove an even stronger result. Notice that $\frac{x}{e^x-1}$ defines a holomorphic function in $x=0$. Let B_n be defined in the following way, using the power series in $x=0$; $\frac{x}{e^x-1} = \sum_n \frac{B_n}{n!}x^n$. These B_n are called the Bernoulli numbers and they show up in a variety of places, including the zeta function. Let n denote a non-negative integer, then we will prove the following: $\zeta(-n) = \frac{(-1)^n}{n+1}B_{n+1}$.

In paragraph 1.2, we proved $\zeta(s) = \frac{1}{2\pi i}H(s)\gamma(1-s)$ where $H(s)$ represents an integral on $\frac{x}{e^{-x}-1}$ over the contour K_0. The first thing we want to do is put the sum expression into the integral:

$$\begin{aligned}
\zeta(-n) = \frac{1}{2\pi i}H(-n)\Gamma(n+1) &= \frac{n!}{2\pi i}\int_{K_0}\frac{x^{-n-1}}{e^{-x}-1}dx = \frac{n!}{2\pi i}\int_{K_0}\frac{x}{e^{-x}-1}x^{-n-2}dx \\
&= \frac{n!}{2\pi i}\int_{K_0} -\sum_{k=1}^{\infty}\frac{B_k}{k!}(-x)^k x^{-n-2}dx \\
&= \frac{n!}{2\pi i}\left[-\sum_{k=1}^{\infty}(-1)^k\frac{B_k}{k!}\int_{|x|=\epsilon}x^{k-n-2}dx\right]
\end{aligned}$$

Here, we first used a variant of Cauchy's theorem([L],143) and the fact that K_0 and the disc around zero with radius ϵ are homologous. This implies that the integral taken over these contours, are the same. Secondly, notice that $\frac{x}{e^{-x}-1}$ is analytic on a neighborhood of zero and on this neighborhood, its power series converges absolutely. Therefore, we are allowed to swap summation and integration. The resulting integral is easily evaluated with the regular Cauchy's theorem and is in fact only non-zero when $k=n+1$. What remains is the following:

$$-\frac{n!}{2\pi i}\sum_{k=1}^{\infty}(-1)^k\frac{B_k}{k!}\int_{|x|=\epsilon}x^{k-n-2}dx = \frac{n!}{2\pi i}(-1)^{n+2}\frac{B_{n+1}}{(n+1)!}2\pi i = \tfrac{(-1)^n}{n+1}B_{n+1}$$

This is a nice result which allows us to concretely calculate $\zeta(s)$ in a region on which it's original expression diverged. Especially notice that $B_1 = \frac{-1}{2}$ which implies that $\zeta(0) = \frac{-1}{2}$. Notice that this result also proves the trivial zeroes to be zero, since odd and larger than one Bernoulli numbers are equal to zero.

This result allows us to calculate $\varepsilon(0)$ as well. Since $\varepsilon(s)$ is an entire function, it is continuous and it is enough to calculate the limit $\lim_{s\downarrow 0}\varepsilon(s)$. Remember that $\zeta(s)$ is analytic in a neighborhood of 0 and therefore $\lim_{s\downarrow 0}\zeta(s) = \zeta(0)$. Using this in combination with Euler's reflection formula for Γ, we can compute $\varepsilon(0)$:

$$\begin{aligned}
\lim_{s\downarrow 0}\varepsilon(s) &= \lim_{s\downarrow 0}\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \\
&= \lim_{s\downarrow 0}(s-1)\pi^{-\frac{s}{2}}\zeta(s)\cdot\lim_{s\downarrow 0}\frac{s}{2}\Gamma(\frac{s}{2}) \\
&= \frac{1}{2}\cdot\lim_{s\downarrow 0}\frac{s}{2}\frac{\pi}{\sin(\pi\frac{s}{2})}\frac{1}{\Gamma(1-\frac{s}{2})} = \frac{1}{2}\cdot 1\cdot\frac{1}{\Gamma(1)} = \frac{1}{2}
\end{aligned}$$

Here we used that $\lim_{s\downarrow 0}\frac{as}{\sin(as)} = 1$ with a denotes a non-zero complex number, which is easily proved using l'Hospitals rule. We will use $\varepsilon(0) = \frac{1}{2}$ later on.

2 A product representation of $\varepsilon(s)$

2.1 Jensen's theorem

We are working with $\zeta(s)$ inside a logarithm. We can express $\zeta(s)$ in terms of $\varepsilon(s)$ and other known functions. Since these are inside a logarithm, the most convenient expression would be a product. For example, we can write $\Pi(\frac{s}{2}) = \Gamma(\frac{s}{2}+1)$ as an infinite product, which inside a logarithm takes the form of a sum over logarithmic terms. Integrals over sums can often be evaluated as sums over integrals, which (most of the times) is easier. Therefore it would be nice if we can write $\varepsilon(s)$ as an infinite product. It turns out that we can, and in this chapter will prove the following equality:

$$\varepsilon(s) = \varepsilon(0) \cdot \prod_{\rho}(1 - \frac{s}{\rho}) \tag{6}$$

In this equation, ρ denotes the roots of ε. The product will turn out to be conditionally convergent, and therefore the product must be understood as a product in pairs of the form ρ and $1-\rho$. Using the functional equation, one can see these are precisely the non-trivial roots of $\zeta(s)$. For proving convergence we obviously need some sort of estimate to connect the modulus of the product with a certain number of roots. We'll use Jensen's theorem for this:

Jensen's theorem *Let $f(z)$ be a analytic function on a disc with radius R centered around 0, denoted $B(0,R)$. Suppose $f(0) \neq 0$ and that f is non-zero on the boundary of $B(0,R)$. Let $z_1, z_2, \cdots, z_n$ be the roots of f inside $B(0,R)$, counted with multiplicity. Then:*

$$\log\left|f(0) \cdot \frac{R}{z_1} \cdot \frac{R}{z_2} \cdots \frac{R}{z_n}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{it})|dt$$

Proof: First assume that f is non-zero on $B(0,R)$. Since $f(z)$ might be a negative real number and since we want to prove our assertion by taking a look at $\frac{\log(f(z))}{z}$, we construct $\log(f(z))$ to be $\log|f(0)| + \int_0^z \frac{f'(t)}{f(t)}dt$. Though it is not a useful way to calculate $\log(f(z))$, its useful for us since $\log(f(z))$ is now well defined. Using Cauchy's residue theorem, we can do the following:

$$\log(f(0)) = \frac{1}{2\pi i}\int_{|z|=R} \frac{\log(f(z))}{z}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\log(f(Re^{it}))}{Re^{it}} \cdot (Rie^{it}dt) = \frac{1}{2\pi}\int_0^{2\pi} \log(f(Re^{it}))dt$$

Writing $f(0)$ in polar coordinates, one can immediately see that the real part of $\log(f(0))$ equals $\log(|f(0)|)$. Since the real part of the integral, is the integral over the real part, we conclude:

$$Re[\log(f(0))] = \log(|f(0)|) = Re[\frac{1}{2\pi}\int_0^{2\pi} \log(f(Re^{it}))dt] = \frac{1}{2\pi}\int_0^{2\pi} \log(|f(Re^{it})|)dt$$

This proves Jensen's theorem for non-zero f. Now assume that f does have zeroes inside $B(0,R)$. Define:

$$F(z) = f(z) \cdot \frac{R^2 - \overline{z_1}z}{R(z-z_1)} \cdot \frac{R^2 - \overline{z_2}z}{R(z-z_2)} \cdots \frac{R^2 - \overline{z_n}z}{R(z-z_n)}$$

This function is analytic since $\lim_{z\to z_i} \frac{R^2-\overline{z_i}z}{R(z-z_i)} = \frac{-\overline{z_i}}{R}$, which can be seen using L'Hospitals rule. This function is non-zero since the zeroes of order k, which look like $(z-z_i)^k$, cancel against terms of the from $\left(\frac{R^2-\overline{z_1}z}{R(z-z_1)}\right)$, which appear k in F. This means we can apply the above reasoning and conclude:

$$\log(|F(0)|) = \frac{1}{2\pi}\int_0^{2\pi} \log(|F(Re^{it})|)dt$$

Notice that $\log(|F(0)|)$ is exactly $\log|f(0)\cdot\frac{R}{z_1}\cdot\frac{R}{z_2}\cdots\frac{R}{z_n}|$. To conclude Jensen's theorem, we only need to prove $|f(Re^{it})| = |F(Re^{it})|$. This can be done by setting $|z| = R$ and multiplying individual factors with $\frac{\overline{z}}{R}$, which has length one so it does not change the modulus:

$$|\frac{R^2 - \overline{z_i}z}{R(z - z_i)}| = |\frac{R^2 - \overline{z_i}z}{R(z - z_i)}\cdot\frac{\overline{z}}{R}| = |\frac{R^2\overline{z} - \overline{z_i}R^2}{R^2(z - z_i)}| = |\frac{\overline{z} - \overline{z_i}}{z - z_i}| = 1$$

This last result finishes the proof of Jensen's theorem.

2.2 Estimating the number of roots

As mentioned before, we need an indication of the amount of roots of $\varepsilon(s)$ to prove convergence of the product. Jensen's theorem gives us a way to connect the roots with an integral and finding an estimate for $|\varepsilon(s)|$ itself, will yield an estimate for the number of roots. We will therefore first prove the following estimate:

Lemma 2.1 *Let $R > 0$ be large enough, then $|\varepsilon(s)| \leq R^R$ holds for all $s \in \mathbb{C}$ satisfying the disc equation: $|s - \frac{1}{2}| \leq R$.*

It would help if we knew where the maximum of $|\varepsilon(s)|$ is on the disc. The maximum-modulus principle tells us that the maximum is at the boundary of the disc. Since $\varepsilon(s)$ is analytic in $s = \frac{1}{2}$, $\varepsilon(s)$ can be written in terms of a power series in $s = \frac{1}{2}$ which has the following form: $\varepsilon(s) = a_0 + a_2\cdot(s - \frac{1}{2})^2 + a_4\cdot(s - \frac{1}{2})^4\cdots$, where all the coefficients are positive. This indicates that the maximum is at $s = \frac{1}{2} + R$ and that $\varepsilon(s)$ is monotonically increasing on the positive real line. This means that we need to show $\varepsilon(\frac{1}{2} + R) \leq R^R$. Let N denote a natural number such that $R + \frac{1}{2} \leq 2N < R + \frac{1}{2} + 2$. We'll simply fill in the definition of our function:

$$\begin{aligned}\varepsilon(\frac{1}{2} + R) \leq \varepsilon(2N) = \frac{2N}{2}\cdot(2N-1)\cdot\Gamma(\frac{2N}{2})\pi^{-\frac{2N}{2}}\zeta(2N) &= N!(2N-1)\cdot\pi^{-N}\zeta(2N)\\ &\leq N^N 2N\pi^0\zeta(2N)\end{aligned}$$

Since $\zeta(s)$ is a monotonically decreasing function for real valued $s \geq 2$, $\zeta(2N) \leq \zeta(2)$. Plugging this in and remembering how we chose N leads to the following:

$$\varepsilon(\frac{1}{2} + R) \leq N^{N+1}2\zeta(2N) \leq N^{N+1}2\zeta(2) < 2\zeta(2)(\frac{R}{2} + \frac{1}{4} + 1)^{\frac{R}{2}+\frac{1}{4}+1} < R^R$$

The last inequality holds for large R, since for these values, $\frac{1}{4} + 1$ and $2\zeta(2)$ are small relative to R. This proves the assertion.

We will use this inequality and Jensen's theorem, to estimate the amount of roots of ε in the disc $|s - \frac{1}{2}| \leq 2R$. First observe that $\varepsilon(s)$ is analytic, so we are actually allowed to use Jensen's theorem. Secondly, if a root of $\varepsilon(s)$ lies on the boundary of the disc, there is a value T such that for all $\delta < T$ there are no roots on the boundary of $|s - \frac{1}{2}| \leq 2R + \delta$. (If this where not the case, analytic continuation would prove $\varepsilon(s) = 0$ for all $s \in \mathbb{C}$.) Taking the limit $\delta \to 0$ would still allow the theorem to be used.

Define $k(R)$ to be the number of zeroes of $\varepsilon(s)$ in $|s - \frac{1}{2}| \leq 2R$, counted with multiplicity. We'll prove $k(R) \leq 3R\log(R)$. Our disc is not centered at zero, which means we use the translated version of Jensen's theorem:

$$\begin{aligned}\log\left(|\varepsilon(\frac{1}{2})\cdot\frac{2R}{\rho_1 - \frac{1}{2}}\cdots\frac{2R}{\rho_{k(R)} - \frac{1}{2}}|\right) &= \log\varepsilon(\frac{1}{2}) + \sum_{|\rho - \frac{1}{2}| \leq 2R}\log(\frac{2R}{|\rho - \frac{1}{2}|})\\ &= \frac{1}{2\pi}\int_0^{2\pi}\log|\varepsilon(2Re^{it} - \frac{1}{2})|dt\end{aligned}$$

The integral on the right is smaller than $\frac{1}{2\pi}\int_0^{2\pi}\log|K|dt$, where K denotes the maximum of $|\varepsilon(2Re^{it}-\frac{1}{2})|$ on the boundary. Using the lemma 2.1, we get the following estimate:

$$\log\varepsilon(\frac{1}{2})+\sum_{|\rho-\frac{1}{2}|\leq 2R}\frac{2R}{|\rho-\frac{1}{2}|}\leq\frac{1}{2\pi}\int_0^{2\pi}\log|K|dt\leq\frac{1}{2\pi}\int_0^{2\pi}\log|(2R)^{2R}|dt=2R\cdot\log(2R)$$

Now focus on the roots ρ inside the disc with radius R instead of $2R$. For these roots, the terms inside the sum are larger than $\log(2)$: $\log(\frac{2R}{|\rho-\frac{1}{2}|})\geq\log(\frac{2R}{R})=\log(2)$. We gain the following inequality:

$$\begin{aligned}k(R)\cdot\log(2)\leq\sum_{|\rho-\frac{1}{2}|\leq R}\log(2)\leq\sum_{|\rho-\frac{1}{2}|\leq R}\log(\frac{2R}{|\rho-\frac{1}{2}|}) &\leq \sum_{|\rho-\frac{1}{2}|\leq 2R}\log(\frac{2R}{|\rho-\frac{1}{2}|})\\ &\leq 2R\log(2R)-\log(\varepsilon(\frac{1}{2}))\end{aligned}$$

Dividing both sides by $\log(2)$:

$$k(R)\leq\frac{2}{\log(2)}\cdot R\cdot\log(R)+2R-\frac{\log(\varepsilon(\frac{1}{2}))}{\log(2)}=2R\left(\frac{1}{\log(2)}\cdot\log(R)+1\right)-\frac{\log(\varepsilon(\frac{1}{2}))}{\log(2)}\leq 3R\log(R)$$

The last inequality holds, since $\frac{1}{\log(2)}<\frac{3}{2}$ implies that for large values of R, $\frac{1}{\log(2)}\log(R)+1<\frac{3}{2}\cdot\log(R)$ and since the difference between these two will only increase for bigger values of R, the constant term above can be neglected. We conclude that $k(R)\leq 3R\log(R)$ holds for large values of R.

2.3 Proving that $\prod_\rho(1-\frac{s}{\rho})$ converges

Let $\rho\in\mathbb{C}$ such that $\varepsilon(\rho)=0$. Since the functional equation $\varepsilon(s)=\varepsilon(1-s)$ holds, $\varepsilon(1-\rho)$ is also holds. The infinite product we are investigating, sums over all these roots ρ. But we can also use the symmetry to simplify our product, this means rewriting the product in the following way:

$$\prod_\rho(1-\frac{s}{\rho})=\prod_{Im(\rho)>0}(1-\frac{s}{\rho})(1-\frac{s}{1-\rho})=\prod_{Im(\rho)>0}(1-\frac{s}{\rho}-\frac{s}{1-\rho}+\frac{s^2}{\rho(1-\rho)})=\prod_{Im(\rho)>0}(1-\frac{s(1-s)}{\rho(1-\rho)})$$

In paragraph 1.7 we proved $\zeta(s)$ to be negative on $[0,1)$. This also implies that it is non-zero in this region. Hence, $Im(\rho)\neq 0$, a result we need for the above expression. The product is taken in increasing order of modulus.

Now remember that a general infinite product $\prod(1+a_n)$ converges to a non-zero value if $\sum\log(1+a_n)$ is a convergent sum. Here we assume that all of the $1+a_n$ are inside the domain of the logarithm. Since $\lim_{n\to\infty}\log(1+a_n)/a_n=0$ for $|a_n|\overset{n\to\infty}{\longrightarrow}\infty$, convergence of $\sum|a_n|$ proofs the convergence of $\sum\log(1+a_n)$. Therefore, it will be enough to prove the convergence of $\sum|\rho(1-\rho)|^{-1}$. Now using the reverse triangle inequality we can find a termwise estimate for the sum:

$$|\rho(1-\rho)|=|(\rho-\frac{1}{2})^2-\frac{1}{4}|\geq\left||\rho-\frac{1}{2}|^2-\frac{1}{4}\right|=|\rho-\frac{1}{2}|^2-\frac{1}{4}=|\rho-\frac{1}{2}|^2(1-\frac{1}{4|\rho-\frac{1}{2}|^2})$$

Since $|\rho-\frac{1}{2}|^2$ can be taken to be monotonically increasing, there is a constant $C>0$ such that $|\rho-\frac{1}{2}|^2\geq C$. (We know that the 'first' zero, has absolute value $>\frac{1}{2}$.) On can show that $C>\frac{1}{4}$. Since $\frac{d}{dx}(1-\frac{1}{4x})^{-1}=-(1-\frac{1}{4x})^{-2}\frac{1}{4x^2}$, the right part of our estimate is monotonically decreasing and therefore we find the following termwise estimate for our sum:

$$\sum_\rho|\rho(1-\rho)|^{-1}\leq\sum_\rho|\rho-\frac{1}{2}|^{-2}(1-\frac{1}{4|\rho-\frac{1}{2}|^2})^{-1}\leq\sum_\rho|\rho-\frac{1}{2}|^{-2}(1-\frac{1}{4C})^{-1}=(1-\frac{1}{4C})^{-1}\sum_\rho|\rho-\frac{1}{2}|^{-2}$$

This means that it suffices to prove the convergence of $\sum|\rho-\frac{1}{2}|^{-2}$. In the following lemma, we will prove an even stronger result.

Lemma 2.2 *Let $\epsilon > 0$, then $\sum_\rho |\rho - \frac{1}{2}|^{-(1+\epsilon)}$ converges.*

Proof: Number the roots ρ of $\varepsilon(s)$ in order of increasing $|\rho - \frac{1}{2}|$. Define R_n to be the positive real number satisfying $4 \cdot R_n \cdot \log(R_n) = n$. Now we use the estimate of the number of roots of the preceding paragraph. Let n be big enough to ensure that R_n is big enough for the estimate to work; the number of roots inside the disc $|s - \frac{1}{2}| \leq R_n$ is smaller than $3R_n \log(R_n)$. Since $4R_n \log(R_n) = n$, we know that there are at most $\frac{3}{4}n$ roots inside the disc. Specifically, the nth root is not inside $|s - \frac{1}{2}| \leq R_n$ which implies $|\rho_n - \frac{1}{2}| > R_n$. This inequality can be used to estimate the summation:

$$\sum_\rho |\rho - \frac{1}{2}|^{-(1+\epsilon)} \leq \sum \frac{1}{R_n^{1+\epsilon}} = \sum \frac{(4\log(R_n))^{1+\epsilon}}{n^{1+\epsilon}} \leq 4^{1+\epsilon} \cdot \sum \frac{\log(n)^{1+\epsilon}}{n^{1+\epsilon}}$$

The last inequality holds since $\log(n) = \log(R_n) + \log(4) + \log(\log(R_n)) \geq \log(R_n)$. For every value of $a > 0$, there is a value at which $n^a > \log(n)^a$ holds for every n above this value. Picking $a = \frac{\epsilon}{2(1+\epsilon)}$ will give us a nice inequality:

$$\sum_\rho |\rho - \frac{1}{2}|^{-(1+\epsilon)} \leq \sum \frac{\log(n)^{1+\epsilon}}{n^{1+\epsilon}} \leq \sum \frac{n^{\frac{\epsilon(1+\epsilon)}{2(1+\epsilon)}}}{n^{1+\epsilon}} = \sum \frac{1}{n^{1+\frac{\epsilon}{2}}} = \zeta(1 + \frac{\epsilon}{2})$$

We already proved that $\zeta(1 + \frac{\epsilon}{2})$ to be finite, which finishes our proof.
This proves the product to be convergent. Notice we have also proven that the sum is absolutely convergent since all terms are non-negative. This fact proves the sum to be uniformly convergent, but does not allow us to change the order in which we multiply inside $\prod_\rho(1 - \frac{s}{\rho})$. Before we started our proof of its convergence, we first coupled ρ and $1 - \rho$ together and proved the product to be convergent. This is an important fact that will reappear several times.

2.4 Linking $\prod_\rho(1 - \frac{s}{\rho})$ to $\varepsilon(s)$

The idea is as follows:we will prove that $|\log(\varepsilon(s)) - \prod_\rho(1 - \frac{s}{\rho})| \leq |s - \frac{1}{2}|^{1+\epsilon}$ holds for $\epsilon > 0$ and $|s - \frac{1}{2}|$ big enough. Then we will prove that this implies that both expressions differ by a constant. This proves both expressions are identical, and we will determine this constant.
Let's start at the begin and prove:

Lemma 2.3 *Let $\epsilon > 0$, then $Re[\log(\frac{\varepsilon(s)}{\prod_\rho(1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})})] \leq |s - \frac{1}{2}|^{1+\epsilon}$ for all sufficiently large $|s - \frac{1}{2}|$.*

Proof: Define the following two functions:

$$\begin{aligned} f_R(s) &= Re[\log \frac{\varepsilon(s)}{\prod\limits_{|\rho-\frac{1}{2}|\leq 2R} (1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})}] \\ g_R(s) &= Re[\log \frac{1}{\prod\limits_{|\rho-\frac{1}{2}|> 2R} (1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})}] \end{aligned}$$

First we'll find an estimate for $f_R(s)$. Look at the circle $|s - \frac{1}{2}| = 4R$. Using the inverse triangle inequality, we can deduce the following:

$$|1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}| \geq |1 - |\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}|| = |1 - \frac{4R}{|\rho - \frac{1}{2}|}|$$

Since $|\rho - \frac{1}{2}| \leq 2R$, the modulus of every term in the product is greater than 1. Therefore, the following holds:

$$f_R(s) \leq Re\log(\varepsilon(s)) = \log(|\varepsilon(s)|)$$

Using our estimate for $|\varepsilon(s)|$ and the fact that $\lim_{x\to\infty} \frac{\log(x)}{x^a} = 0$ for all $a > 0$, we can do the following:

$$f_R(s) \leq \log|\varepsilon(s)| \leq \log((4R)^{4R}) \leq 4R \cdot \log 4R \leq R^{1+\epsilon}$$

The last part holds for big enough values of R. Now take a look at the disc $|s - \frac{1}{2}| \leq 4R$. Since $\varepsilon(\rho) = 0$, we would expect trouble since $\varepsilon(s)$ is inside a logarithm. This is almost true, for $\rho \leq 2R$, there is a term in the denominator that will ensure that $f_R(\rho)$ is well defined. For $2R < \rho \leq 4R$, the situation is more troublesome. In this case, simply eliminate a small open neighborhood of ρ to ensure that our function is analytic. Notice that $f_R(s)$ is the real part of an analytic function, which makes it a harmonic function. For harmonic functions, we know they obtain a maximum on the boundary of their domain. The only boundary on our domain is the circle $|s - \frac{1}{2}| = 4R$, and the boundaries of the open sets around $2R < \rho \leq 4R$. Because there are no poles to counter $\varepsilon(s)$ to go to zero in these points, the limit actually goes to zero. Since $\log(x) \xrightarrow{x\downarrow 0} -\infty$, the boundary around these ρ can never generate a maximum because our opens around ρ can be chosen arbitrarily small. This implies that the maximum is attained at $|s - \frac{1}{2}| = 4R$ at which $f_R(s) \leq R^{1+\epsilon}$ holds for R big enough, proving that this inequality holds for the entire disc $|s - \frac{1}{2}| \leq 4R$.

We will need the same estimate for $g_R(s)$. To find it, we need an easily derived inequality. Let $|t| \leq \frac{1}{2}$.

$$Re\log\frac{1}{1-t} = -Re\log(1-t) = Re\int_0^t \frac{dy}{1-y} \leq \left|\int_0^t \frac{dy}{1-y}\right| \leq |t| \cdot \mathrm{Max}_{y\in[0,t]}\frac{1}{|1-y|} = 2|t|$$

We can use this inequality directly in $g_R(s)$. Set $|s - \frac{1}{2}| = R$, this ensures $|\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}| \leq \frac{1}{2}$, which is needed for the inequality.

$$g_R(s) \quad = \quad Re\left[\log\frac{1}{\prod\limits_{|\rho-\frac{1}{2}|>2R}(1-\dfrac{s-\frac{1}{2}}{\rho-\frac{1}{2}})}\right] \leq \sum_{|\rho-\frac{1}{2}|\geq 2R} 2\cdot\left|\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right|^2 = 2\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\left|\frac{R^2}{(\rho-\frac{1}{2})^2}\right|$$

The trick is to force an ϵ into the inequality. We can do that by rewriting the square:

$$2\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\left|\frac{R^2}{(\rho-\frac{1}{2})^2}\right| = 2\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\left|(\frac{R}{(\rho-\frac{1}{2})})^{1-\epsilon}(\frac{R}{(\rho-\frac{1}{2})})^{1+\epsilon}\right|$$

Because $|\rho - \frac{1}{2}| \geq 2R$, the fraction $|\frac{R}{(\rho-\frac{1}{2})}|$ is smaller than $\frac{1}{2}$:

$$\begin{aligned} 2\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\left|(\frac{R}{(\rho-\frac{1}{2})})^{1-\epsilon}(\frac{R}{(\rho-\frac{1}{2})})^{1+\epsilon}\right| &\leq 2\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\left|(\frac{1}{2})^{1-\epsilon}(\frac{R^{1+\epsilon}}{(\rho-\frac{1}{2})^{1+\epsilon}})\right| \\ &= 2^{\epsilon}R^{1+\epsilon}\cdot\sum_{|\rho-\frac{1}{2}|\geq 2R}\frac{1}{|\rho-\frac{1}{2}|^{1+\epsilon}} \end{aligned}$$

The convergence of this sum was proved in lemma 2.2. This means that if $R \to \infty$, the sum goes to zero; it is bounded. Since R is large, it is at least larger than 2 and we can do the following:

$$2^{\epsilon}R^{1+\epsilon} = R\cdot(2R)^{\epsilon} \leq R\cdot R^{2\epsilon} = R^{1+2\cdot\epsilon}$$

Since we can pick ϵ arbitrarily small, we conclude that for small ϵ and large enough R, $g_R(s) \leq R^{1+\epsilon}$.

Now we want to prove $f_R(s) + g_R(s) \leq R^{1+\epsilon}$ for R large enough. First, let ϵ decrease to ϵ', and pick R large enough for $f_R(s) \leq R^{1+\epsilon'}$ and $g_R(s) \leq R^{1+\epsilon'}$. It follows that $f_R(s) + g_R(s) \leq 2R^{1+\epsilon'}$. Simply pick ϵ' such that $2 \leq R^{\epsilon-\epsilon'}$ holds. We conclude

$$f_R(s) + g_R(s) \leq 2R^{1+\epsilon'} \leq R^{\epsilon-\epsilon'}R^{1+\epsilon'} \leq R^{1+\epsilon}$$

holds, proving our assertion.

2.5 Expressing $\varepsilon(s)$ as an infinite product

The strategy of our proof is pretty similar to Liouville's proof that entire functions that are bounded, are constant. But in his proof, the entire function has a bounded modulus. In our case, the only bounded part is the real part of our function. We will first find an estimate of the modulus, using the maximal value of the real part on $B(0, R)$, then we will pretty much just prove Liouville's theorem. The estimate follows from the following lemma:

Lemma 2.4 *Let $f(s)$ be analytic on $B(0, R)$ and $f(0) = 0$. Define $M = Max_{|s|=R}Re[f(s)]$. Let $r < R$. Then $|f(s)| \leq 2r\frac{M}{R-r}$ holds for all $|s| \leq r$.*

Proof: Define $\phi(s) = \frac{f(s)}{s(2M-f(s))}$, and let $u(s)$ and $v(s)$ denote the real, respectively imaginary part of $f(s)$. We know that the real part of an analytic function is a harmonic function. Harmonic functions take on a maximum on the boundary of their domain. This means that $Re[f(s)] = u(s) \leq M$ for all $s \in B(0, R)$. Indirectly, this also implies $|2M - u(s)| \geq u(s)$ on the entire disc. Using this last fact, we can find an estimate for the modulus of $\phi(s)$ on $|s| \leq R$:

$$|\phi(s)| = \frac{(u^2+v^2)^{\frac{1}{2}}}{|s|((2M-u)^2+v^2)^{\frac{1}{2}}} \leq \frac{(u^2+v^2)^{\frac{1}{2}}}{R(u^2+v^2)^{\frac{1}{2}}} = \frac{1}{R}$$

The logical thing to do, is to express $f(s)$ in terms of $\phi(s)$ and see if this gives us an estimate for $|f(s)|$.

$$f(s) = \frac{2M\phi(s)}{1+s\phi(s)}$$

Notice that $|1 + s\phi(s)| \geq |1 - |s\phi(s)||$, which is just the inverse triangle inequality. Now, set $|s| = r < R$, and notice that $|s\phi(s)| \leq r\frac{1}{R} < 1$. Plugging this into the equation:

$$|f(s)| = |\frac{2Ms\phi(s)}{1+s\phi(s)}| \leq \frac{2M|s\phi(s)|}{|1-|s\phi(s)||} \leq \frac{2Mr|\phi(s)|}{1-\frac{r}{R}} \leq \frac{2Mr\frac{1}{R}}{1-\frac{r}{R}} = \frac{2Mr}{R-r}$$

The maximum modulus principle ensures that this inequality holds for all $|s| \leq r$, which proves our lemma.

We'll use the lemma to prove the following theorem, which more or less tells us that entire functions are not easily bounded:

Theorem: *Let $f(s)$ be an entire function satisfying $f(-s) = f(s)$. If for every $\epsilon > 0$ there exists an R such that $Re[f(s)] < \epsilon|s|^2$ for all s satisfying $|s| \geq R$, then $f(s)$ is constant.*

Proof: If $f(s)$ satisfies the condition of the theorem, then so does $f(s)+A$, were $A \in \mathbb{C}$ is some constant. Therefore, without loss of generality, assume $f(0)=0$, since we can pick A to be $-f(0)$. Because $f(s)$ is analytic, it admits a power series representation. The coefficients of this representation satisfy the following formula:

$$a_n = \frac{1}{2\pi i}\int_{\partial B(\frac{R}{2})} \frac{f(s)}{s^{n+1}} ds$$

Here, $\partial B(\frac{R}{2})$ denotes the boundary of $B(0, \frac{R}{2})$. Now, let ϵ and R be as in the theorem and take the absolute value of the above integral. Applying the Cauchy-Schwarz inequality yields the following inequality:

$$|a_n| = |\frac{1}{2\pi i}\int_{\partial B(\frac{R}{2})} \frac{f(s)}{s^{n+1}} ds| \leq \frac{1}{2\pi}\int_0^{2\pi} |\frac{f(\frac{1}{2}Re^{it})}{(\frac{1}{2}Re^{it})^{n+1}} \frac{1}{2} iRe^{it}| = \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\frac{1}{2}Re^{it})|}{R^n} 2^n dt$$

The condition the theorem has set on $f(s)$, makes it possible for us to apply the lemma on $|f(\frac{1}{2}Re^{it})|$. Notice that the maximum value M, is smaller than $\epsilon|s|^2 = \epsilon R^2$. Therefor, the lemma proves:

$$\frac{2^n}{R^n}|f(\frac{1}{2}Re^{it})| \leq \frac{2^n}{R^n}\frac{2\epsilon R^2 \frac{1}{2}}{R - \frac{1}{2}R} = \frac{2^{n+1}}{R^{n-2}}\epsilon$$

For $n \geq 2$, this means that the integral is smaller than $\epsilon 2^{n+1}$ since the R is a very large value. It follows that $|a_n| \leq \epsilon 2^{n+1}$. But this should hold for all $\epsilon > 0$, which proves $a_n = 0$ for $n \geq 2$. We assumed $a_0 = 0$, which means $f(s) = a_1 s$. But $f(-s) = f(s)$, which is only possible if $a_1 = 0$, which proves $f(s) = 0$ for all $s \in \mathbb{C}$. We conclude that f is constant, which proves the theorem.

Now we can finally prove the asserted product representation of $\varepsilon(s)$. Define:

$$F(s) = \frac{\varepsilon(s)}{\prod_\rho (1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})}$$

As stated before, all singularities caused by $\prod_\rho(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})^{-1}$, are canceled by zeroes of ε. The value in these points is therefore well defined. On other parts of $\mathbb{C}$, F is just a composition of analytic functions and as a result, $F(s)$ is an entire function.
The way we where able to make sense of the infinite product throughout this chapter, by first looking at the zeroes inside a disc of finite radius and letting the radius go to infinity. Therefore, we can write the product in the following way: $\prod_\rho(1-\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}) = \prod_{Im[\rho]>0}(1-\frac{s(1-s)}{\rho(1-\rho)})$. Translating s to $s+\frac{1}{2}$ gives term of the form $1-\frac{s^2-\frac{1}{4}}{\rho(1-\rho)}$. This proves that in the product, $\frac{1}{2}+s$ and $\frac{1}{2}-s$ are equivalent. Remembering $\varepsilon(s) = \varepsilon(1-s)$ and applying the same translation gives: $\varepsilon(\frac{1}{2}+s) = \varepsilon(\frac{1}{2}-s)$. We conclude that $F(\frac{1}{2}+s) = F(\frac{1}{2}-s)$, it is an even function in $s-\frac{1}{2}$.
Since $F(s)$ is non-zero, we can apply the logarithm which returns a value that is unique up to $2\pi k i$, $k \in \mathbb{Z}$. Notice that, once again, we defined the logarithm as a path integral: $\log(F(s)) = \int_0^s \frac{F'(t)}{F(t)}dt + \log(F(0))$. In the preceding section, we proved $Re[\log(F(s))] \leq R^{1+\epsilon}$, for $|s-\frac{1}{2}| = R$ large enough. In this paragraph we have proved that this, along with $\log(F(s))$ being an even function in $s-\frac{1}{2}$, implies that this function is constant. This proves:

$$\varepsilon(s) = C \cdot \prod_\rho (1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}})$$

C is just some constant. We can eliminate this constant, by dividing by $\varepsilon(0)$, which is nonzero. We get:

$$\frac{\varepsilon(s)}{\varepsilon(0)} = \prod_\rho (1 - \frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}) \cdot (1 - \frac{0-\frac{1}{2}}{\rho-\frac{1}{2}})^{-1}$$

Now notice that the terms inside the product, are just linear factors of s. We can find the usual $a + bs$ expression for these factors by simply filling in values of s. $s = 0$ implies that the term is one, $s = \rho$ implies that the term is zero. This means that it is equal to $1 - \frac{s}{\rho}$. We conclude

$$\varepsilon(s) = \varepsilon(0) \cdot \prod_{\rho}(1 - \frac{s}{\rho})$$

3 $Li(x)$ en $J(x)$

3.1 Plugging $\varepsilon(s)$ in $J(x)$

In chapter 1, we defined the function $\varepsilon(s)$. With some simple algebra, we can deduce an expression for $\zeta(s)$:

$$\zeta(s) = \varepsilon(s)\frac{2}{s(s-1)}\pi^{\frac{s}{2}}\frac{1}{\Gamma(\frac{s}{2})} \tag{7}$$

We want to plug this expression into (5), which means we want to take the logarithm of the above expression. We can eliminate a term in the logarithm by using the fact that $\frac{s}{2}\Gamma(\frac{s}{2}) = \Gamma(\frac{s}{2}+1)$. For the sake of notation, define $\Pi(x) = \Gamma(x+1)$. $\log(\zeta(s))$ will now get the following form:

$$\log(\zeta(s)) = \log\left(\varepsilon(s)\frac{1}{s-1}\pi^{\frac{s}{2}}\frac{1}{\Pi(\frac{s}{2})}\right) = \log(\varepsilon(s)) - \log(s-1) + \frac{s}{2}\log(\pi) - \log(\Pi(\frac{s}{2}))$$

Directly plugging this into $J(x)$ is possible, but we do not want to evaluate the entire integral at once. We want to split it up in smaller parts, dealing with one logarithm at the time, but thit is not possible because not every integral will be convergent. Take for example the $\frac{s}{2}\log(\pi)$ part. The resulting integral would be:

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{s}{2}\log(\pi)\frac{x^s}{s}ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{1}{2}\log(\pi)x^s ds = \frac{\log(\pi)x^a}{4\pi}\int_{-\infty}^{\infty}x^{iy}dy$$

The last integral expression is definitely non-convergent. In order to make our lives a little easier, we once again turn to a trick from Fourier analysis: integration by parts. Through this operation we can obtain a new expression for $J(x)$:

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log(\zeta(s))}{s}x^s ds = \frac{1}{2\pi i}\frac{\log(\zeta(s))}{s}\frac{1}{\log(x)}x^s\Bigg|_{s=a-i\infty}^{a+i\infty} - \frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(\zeta(s))}{s}\right]x^s$$

To prove that the term in the middle is zero, it is enough to show that for $y \to \infty$, $\log(\zeta(a \pm iy))$ is finite since $x^{a\pm iy}$ will just oscillate and $\frac{1}{a\pm iy}$ will go to zero. This can be done by writing $\zeta(s)$ as an Euler product, which is allowed because $a > 1$, and taking the Taylor expansion of $\log(1-x)$ in $x = 0$ (the logarithm equals its own Taylor expansion).

$$\begin{aligned}
|\log(\zeta(s))| = |\log(\prod_{p\in P}(1-\frac{1}{p^s})^{-1})| &= |-\sum_{p\in P}\log(1-p^s)| = |\sum_{p\in P}\sum_{n=1}^{\infty}\frac{p^{sn}}{n}| \\
&\leq \sum_{p\in P}\sum_{n=1}^{\infty}|\frac{p^{-sn}}{n}| = \sum_{p\in P}\sum_{n=1}^{\infty}\frac{p^{-an}}{n} = \log(\zeta(a))
\end{aligned}$$

Since $\zeta(a)$ is the sum of positive elements, it is non-zero and since we already proved $\zeta(a)$ to be finite, $\log(\zeta(a))$ is finite. This proves $|\log(\zeta(s))|$ to be finite which, as if it where collateral damage, also proves $\zeta(s)$ to be non-zero for $Re(s) > 1$. We conclude:

$$J(x) = -\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}[\frac{\log(\zeta(s))}{s}]x^s$$

All that's left to do, is to evaluate each of the four integrals. As it turns out, the hardest integral we need to evaluate, is closely related to the logarithmic integral function $Li(x)$. We will explore this function in greater detail in the next paragraphs.

3.2 $F(\beta)$

Define:

$$Li(x) = \lim_{\epsilon \downarrow 0} \left(\int_0^{1-\epsilon} \frac{ds}{\log(s)} + \int_{1+\epsilon}^{x} \frac{ds}{\log(s)} \right)$$

It will turn out that some of our integrals are equal, or at least for some part, to $Li(x)$. Therefore, it is no coincidence that our helper functions are of a similar form. The tricky part is that these helper functions are not globally defined but locally. This is also exactly why we need them: we need useful expressions for $Li(x)$ on certain parts of $\mathbb{C}$. We'll start off by defining the function $F(\beta)$.

$$F(\beta) = \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds} \left[\frac{\log(\frac{s}{\beta} - 1)}{s} \right] ds$$

Notice that this F is not in any way related to the function F from the end of chapter 2. It will turn out that $F(1) = Li(x)$, which is the reason that we want to take a closer look at F. Since changing the definition of our logarithm in $J(x)$ might change the value of $J(x)$, we do not allow $\frac{s}{\beta} - 1$ to be a non-positive real number. We can extend the domain of the logarithmic expression, by defining $\log(\frac{s}{\beta} - 1) = \log(s - \beta) - \log(\beta)$. This is well defined as long as β is not a non-positive real number, notation: $\beta > 0$. Since $s = a + iy$ and we are integrating over y from $-\infty$ to ∞, $\log(s - \beta)$ will be well defined for all real values of y as long as $Re(\beta) < a$.
We'll prove $F(\beta)$ is absolutely convergent. We can ignore the oscillating x^s part of the integral since $|x^s| = x^a |x^{iy}| = x^a$ is just a constant. Calculating the derivative yields:

$$\frac{d}{ds} \frac{\log(\frac{s}{\beta} - 1)}{s} = -\frac{\log(\frac{s}{\beta} - 1)}{s^2} + \frac{1}{s(\frac{s}{\beta} - 1)} \frac{1}{\beta}$$

This leads to the inequality:

$$\left| \frac{\log(\frac{s}{\beta} - 1)}{s^2} + \frac{1}{s(s-\beta)} \right| \leq \left| \frac{\log(\frac{s}{\beta} - 1)}{s^2} \right| + \left| \frac{1}{s(s-\beta)} \right|$$

When we plug $s = a + iy$ into the right term, we get the following situation: a 2^{nd} order polynomial $f(t)$ and we need to calculate the integral $\frac{1}{|f|}$ over the real line. Notice that $\frac{1}{|f|}$ is a meromorphic function on $\mathbb{C}$ except for negative reals. Since none of the roots of f lie on the real line, Cauchy's residue theorem is applicable. The contour will be a semicircle whose straight part is exactly the line $a + iy$, and whose half circle lies on the right part of this line. Since $\frac{1}{|f|}(r)$ is $O(\frac{1}{r^2})$ using big O notation, the semi circle will not contribute to the integral and we can conclude that $\int_{a-i\infty}^{a+i\infty} \frac{1}{|f|}(s) ds$ is finite. The same reasoning will hold for the left term. We only need to remark that $|\log(a + iy)| \approx \log(a + |y|)$ for large values of y, and that $\frac{\log(a+|y|)}{|y|} \to 0$ for $|y| \to \infty$. The same reasoning for the left term, will prove it is absolutely convergent as well and we conclude that F is absolutely convergent. Notice that this reasoning also holds when $f(t)$ is a polynomial of a higher degree than 2. We will need this in chapter 4.

Notice that $\frac{d}{d\beta} \log(\frac{s}{\beta} - 1) \frac{1}{s} = \frac{1}{\frac{s}{\beta} - 1} \frac{-s}{\beta^2} \frac{1}{s} = \frac{1}{(\beta - s)\beta}$. This derivative is continuous is both variables on our domain. Combining this with the absolute convergence from above we are allowed to take the derivative inside the integral and swap the order of differentiation:

$$\frac{d}{d\beta} F(\beta) = F'(\beta) = \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{d\beta} \frac{d}{ds} \left[\frac{\log(\frac{s}{\beta} - 1)}{s} \right] ds = \frac{1}{2\pi i} \frac{1}{\beta \log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds} \left[\frac{1}{\beta - s} \right] ds$$

Using integration by parts we can make this integral a little bit nicer:

$$\begin{aligned}
F'(\beta) &= \frac{1}{2\pi i}\frac{1}{\beta\log(x)}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}\left[\frac{1}{\beta-s}\right] ds \\
&= \frac{1}{2\pi i}\frac{1}{\beta\log(x)}\, x^s \cdot \frac{1}{\beta-s}\Big|_{a-i\infty}^{a+i\infty} - \frac{1}{2\pi i}\frac{1}{\beta}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{1}{\beta-s} ds \\
&= -\frac{1}{2\pi i\beta}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{1}{\beta-s} ds
\end{aligned}$$

Although it may not be the prettiest, it is an integral we will see more often. That's why, for its calculation, we'll prove a lemma.

Lemma 3.1 *Let $Re(\beta) < a$, then*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{s-\beta} x^s ds = \begin{cases} x^\beta & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}$$

Proof We can write $\frac{1}{s-\beta}$ in integral form: $\int_1^\infty x^{-s+\beta+1} dx = \int_1^\infty x^{-a-iy+\beta+1} dx$. This is allowed since $Re(\beta) < a$. Substituting $x = e^\lambda$ results in $\frac{1}{a+iy-\beta} = \int_0^\infty e^{(-a+\beta)\lambda} e^{-iy\lambda} d\lambda$. Using Fourier-inversion, we get:

$$\int_{-\infty}^{\infty} \frac{1}{a+iy-\beta} e^{iy\beta} dy = \begin{cases} 2\pi e^{\lambda(\beta-a)} & \text{for } \lambda > 0 \\ 0 & \text{for } \lambda < 0 \end{cases}$$

Substituting $x = e^\lambda$ and multiplying both sides with $x^a \frac{1}{2\pi}$:

$$\frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty} \frac{1}{s-\beta} x^s (-i) ds = \begin{cases} x^\beta & \text{for } x > 1 \\ 0 & \text{for } x < 1 \end{cases}$$

This closes our proof.

Since x will be our variable for which we can count the number of primes below x, the case $x < 1$ is not that interesting. For that reason we will, using **Lemma 3.1**, immediately write $F'(\beta) = \frac{1}{\beta} x^\beta$ which is the result we need for $F(x)$ right now.

3.3 $G(\beta)$ and $H(\beta)$

Let the contour C^+ be the path from zero to $1-\epsilon$, followed by a semicircle around 1 through the upper half plane which in turn is followed by the path from $1+\epsilon$ to x. Define the following function:

$$G(\beta) = \int_{C^+} \frac{t^{\beta-1}}{\log(t)} dt$$

The path C^+ depends on the variable ϵ, but for small values of ϵ, all the curves $C^+(\epsilon)$ are homotopic and the paths do not cross a pole of $G(\beta)$. This can be seen by noticing that $\frac{1}{\log(t)}$ has poles at $2\pi ki$ for all $k \in \mathbb{Z}$ and $t^{\beta-1}$ is analytic for $Re(\beta) > 0$. Because a finite path is compact, one can take the maximum of $\frac{t^{\beta-a}}{\log(t)}$ since it is a continuous function on this path. This maximum will depent on x, but this is not important since this is value is fixed. Multiplying this maximum with the length of the path will give us an upper estimate $M(x)$ for $G(\beta)$. Now we can use the Cauchy-Schwarz:

$$|\int_{C^+} \frac{t^{\beta-1}}{\log(t)} dt| \leq \int_{C^+} |\frac{t^{\beta-1}}{\log(t)}| \leq M$$

We conclude that the integral is absolutely convergent, which proves $G(\beta)$ to be well defined. This fact, and the fact that $\frac{d}{d\beta}\frac{t^{\beta-1}}{\log(t)}$ is continuous in t and β, will also allow us the take the derivative over β inside the integral:

$$\frac{d}{d\beta}G(\beta) = G'(\beta) = \int_{C^+} \frac{d}{d\beta}\frac{t^{\beta-1}}{\log(t)}dt = \int_{C^+} t^{\beta-1}dt = \left.\frac{t^\beta}{\beta}\right|_0^x = x^\beta = F'(\beta)$$

This means that $F(\beta)$ and $G(\beta)$ only differ by a constant, which is a remarkable result since they do not look very similar. Notice that $G(\beta)$ is far more easy to determine than $F(\beta)$, which is the reason why we want to know by what constant F and G differ. There is more. Since F is defined on $Re(\beta) < a$ and G is defined on $Re(\beta) > 0$, this result only holds on $0 < Re(\beta) < a$. But both F and G are analytic since both are complex differentiable. Therefore, their equivalence holds up to a constant for $Re(\beta) > 0$ thanks to the principle of analytic continuation. To find this constant, we need to define yet another function:

$$H(\beta) = \frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log(1-\frac{s}{\beta})}{s}\right] x^s ds$$

H is not related to the function H from chapter 1. This function looks a lot like $F(\beta)$ and is conflicted with similar problems for its logarithmical component. This time, we define $\log(1-\frac{s}{\beta})$ to be $\log(s-\beta) - \log(-\beta)$. As a result, β may not be a non-negative real number: $\beta < 0$. Both F and H are defined on the upper half plane, which means that $H(\beta) - F(\beta)$ is defined there as well, and it turns out that we can simply calculate this value:

$$\begin{aligned}
H(\beta) - F(\beta) &= \frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}\left[\frac{\log(s-\beta) - \log(-\beta) - \log(s-\beta) + \log(\beta)}{s}\right] ds \\
&= \frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}\left[\frac{\log(\beta) - \log(-\beta)}{s}\right] ds \\
&= \frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}\left[\frac{\pi i}{s}\right] ds
\end{aligned}$$

This last step is motivated by the fact that we can write $-1 = \lim_{t\downarrow -1} e^{\pi i t}$, plugging this in:

$$\log(\beta) - \log(-\beta) = \log(\beta) - \log(\lim_{t\downarrow -1} e^{\pi i t}\beta) = \log(\lim_{t\downarrow -1} e^{-\pi i t}) = \lim_{t\downarrow -1} -\pi i t = \pi i$$

Using integration by parts, we can calculate the integral:

$$\begin{aligned}
\frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}\left[\frac{\pi i}{s}\right] ds &= \frac{1}{2\pi i \log(x)} \left. x^s \cdot \frac{\pi i}{s}\right|_{a-i\infty}^{a+i\infty} - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{\pi i}{s} ds \\
&= -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{\pi i}{s} ds
\end{aligned}$$

We can evaluate this last integral using **Lemma 3.1**, and conclude that $H(\beta) - F(\beta) = -i\pi$ on the upper half plane. We could have taken minus one to be $\lim_{t\uparrow 1} e^{\pi i t + 2\pi k}$ with $k \in \mathbb{Z}$, which would have changed the value of our final integral. This is not that important. Since $e^{2\pi i k} = 1$ for all $k \in \mathbb{Z}$, $\log(z)$ is uniquely defined up to a factor $2\pi i k$. Therefore, all complex numbers can be seen to belong to a specific equivalence class. At this point we are choosing as our representative, the number $-i\pi$. At the end of this derivation, the next paragraph, it will turn out that our result is independent of the choice we make over here.

At this point we conclude $F(\beta) = H(\beta) + \pi i$ on the upper half plane. We already computed that F and G differ by a constant, which means that G and H also differ by a constant. We can determine these constants, by computing $\lim_{T\to\infty} G(\sigma + Ti)$ and $\lim_{T\to\infty} H(\sigma + Ti)$ where σ is a fixed real number satisfying $0 < \sigma < a$. This last requirement makes sure that we are taking the limit inside a domain where both G and H are well defined. We'll compute these constants in the next paragraph.

3.4 $Li(x)$

Now we will finally arrive at the wanted result: $Li(x) = F(1)$. As promised, we will first calculate

$$\lim_{T\to\infty} G(\sigma + Ti) = \lim_{T\to\infty} \int_{C^+} \frac{t^{\beta-1}}{\log(t)} dt$$

Substitute $t = e^u$. Now we can make our path of integration easier. Instead of C^+, let the path run from $i\delta - \infty$ to $i\delta + \log(x)$, and from $i\delta + \log(x)$ to $\log(x)$. Of course, δ is assumed to be small enough to make sure that the path does not enclose another pole. Since the two paths are homotopic, the value of the evaluated integrals are the same.

$$G(\beta) = \int_{i\delta-\infty}^{i\delta+\log(x)} \frac{e^{u(\beta-1)}}{u} e^u du + \int_{i\delta+\log(x)}^{\log(x)} \frac{e^{u(\beta-1)}}{u} e^u du$$

For the left integral, substitute $u = z + i\delta$ and plug in $\beta = \sigma + Ti$:

$$\int_{-\infty}^{\log(x)} \frac{e^{(z+i\delta)\beta}}{z + i\delta} dz = \int_{-\infty}^{\log(x)} \frac{e^{(z+i\delta)(\sigma+Ti)}}{z + i\delta} dz = e^{-\delta T} e^{i\sigma\delta} \int_{-\infty}^{\log(x)} \frac{e^{z(\sigma+Ti)}}{z + i\delta} dz$$

When T goes to infinity, the term in front of the integral will go to zero, while the part inside the integral will oscillate. Since the integral is absolutely convergent, the limit $T \to \infty$ over the left integral will go to zero. For the right integral, substitute $u = \log(x) + iw$:

$$\int_{i\delta+\log(x)}^{\log(x)} \frac{e^{u(\beta)}}{u} du = \int_0^{\delta} \frac{e^{\log(x)\beta} e^{iw\beta}}{\log(x) + iw} (-i) dw = -ix^{\sigma+iT} \int_0^{\delta} \frac{e^{iw\sigma} e^{-wT}}{\log(x) + iw} dw$$

This time, when T goes to infinity, the term in front of the integral will oscillate. Because the integral is absolutely convergent and since the function inside the integral is continuous in w and T, we are allowed to pull the limit inside the integral. As a result, we are integrating over zero and the right integral becomes zero as well, when T goes to infinity. This entails that

$$\lim_{T\to\infty} G(\sigma + iT) = 0$$

Notice that the above reasoning fails when we let T tend to $-\infty$. This is an important fact and it will return in chapter 4.

We'll take the same limit for $H(\beta)$ and we will see that it becomes zero as well. First, we'll remove the differential operator from $H(\beta)$ by carrying out the differentiation:

$$\frac{d}{ds} \frac{\log(1 - \frac{s}{\beta})}{s} = -\frac{\log(1 - \frac{s}{\beta})}{s^2} + \frac{1}{s(1 - \frac{s}{\beta})} \frac{-1}{\beta} = -\frac{\log(1 - \frac{s}{\beta})}{s^2} + \frac{1}{\beta(s - \beta)} - \frac{1}{\beta s}$$

Integrating the first part, gives the following situation:

$$\lim_{Im(\beta)\to\infty} \frac{1}{2\pi i \log(x)} \int_{a-i\infty}^{a+i\infty} -\frac{\log(1 - \frac{s}{\beta})}{s^2} x^s ds$$

We already established that $F(\beta)$ is absolutely convergent by splitting the integral up in multiple parts which are absolutely convergent. The above integral was one of them. Because of this property, we can use Lebesgue dominated convergence theorem to pull the limit inside the integral. Since $\frac{s}{\beta}$ will become zero, the logarithm will go to zero, which proves that the entire expression will go to zero. Notice that this trick would not have worked on $F(\beta)$ itself, since for that function the term inside the logarithm goes to zero. This is the main reason to define $H(\beta)$. We are left with:

$$\begin{aligned}\lim_{Im(\beta)\to\infty} \int_{a-i\infty}^{a+i\infty} \left(\frac{1}{\beta(s - \beta)} - \frac{1}{\beta s}\right) x^s ds &= \lim_{Im(\beta)\to\infty} \left[\frac{1}{\beta} \int_{a-i\infty}^{a+i\infty} \frac{1}{s - \beta} x^s ds - \frac{1}{\beta} \int_{a-i\infty}^{a+i\infty} \frac{1}{s} x^s ds\right] \\ &= \lim_{Im(\beta)\to\infty} \frac{x^\beta}{\beta} - \frac{x^0}{\beta}\end{aligned}$$

We evaluated the integrals using **Lemma 3.1**. The absolute value of the denominator will go to infinity when $Im(\beta) \to \infty$, while the other terms have a constant norm: the limit is zero. We summarize: H and G differ by a constant and $\lim_{Im(\beta)\to\infty} G(\beta) = 0 = \lim_{Im(\beta)\to\infty} H(\beta)$. This means $G = H$. Since $F = H + \pi i$ on the upper half plane, $\lim_{Im(\beta)\to\infty} F(\beta) = \pi i$. But F and G also differ by a constant: $F = G + \pi i$, which holds for $0 < Re(\beta) < a$. Since $a > 1$, we can use this equality to calculate $F(1)$:

$$F(1) = G(1) + \pi i = \int_0^{1-\epsilon} \frac{dt}{\log(t)} + \int_B \frac{dt}{\log(t)} + \int_{1+\epsilon}^{x} \frac{dt}{\log(t)} + \pi i$$

Here, B denotes the semicircle in the upper half plane with radius ϵ. This integral can be evaluated by proving that $\frac{1}{\log(t)}$ has a simple pole at $t = 1$ because in that case, the value of the integral is half of $2\pi i$ times the residue at that point. This can be done by noticing the following: $\lim_{t\to1} \frac{t-1}{\log(t)|} = 1$, this is a consequence of L'Hospital's rule. It implies that the residue at $t = 1$ equals 1. Still, we will conclude that the integral becomes $-\pi i$ since it goes around $t = 1$ clockwise. The Cauchy residue formula on the other hand, works with paths that go around residues counterclockwise, which is why we include the minus sign.
As we discussed before, $C^+(\epsilon)$ denotes a collection of paths that are homotopic to each other: we might just as well let ϵ tend to zero. We conclude:

$$F(1) = \int_0^{1-\epsilon} \frac{dt}{\log(t)} + \int_B \frac{dt}{\log(t)} + \int_{1+\epsilon}^{x} \frac{dt}{\log(t)} + \pi i = \lim_{\epsilon\downarrow0} \int_0^{1-\epsilon} \frac{dt}{\log(t)} + \int_{1+\epsilon}^{x} \frac{dt}{\log(t)} = Li(x)$$

It may not feel like a result, but it is. On one hand, F shows up in our expressions of $J(x)$, while on the other hand, $Li(x)$ is a reasonably well known function. Even though its definition is a little abstract, the values of $Li(x)$ can be numerically calculated and this allows us to draw its graph:

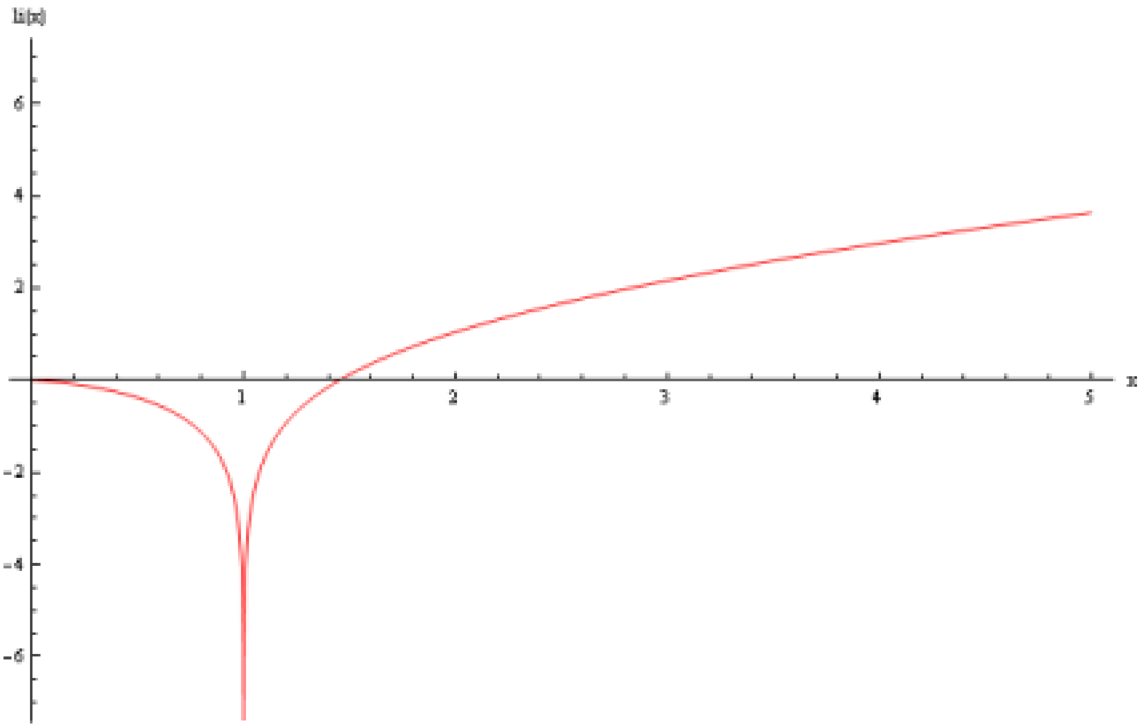

Figure 3: The graph of $Li(x)$.

In the next chapter, we'll show just how important $Li(x)$ is for the prime counting function which we want to construct.

4 Constructing $\pi(x)$

4.1 Plugging $\varepsilon(s)$ into the new expression of $J(x)$

Just as we did before, we want to plug $\varepsilon(s)$ into $\log(\zeta(s))$ which is inside the definition of $J(x)$. Remember that $\Pi(x) = \Gamma(x+1)$. We already established:

$$\log(\zeta(s)) = -\log(s-1) + \log(\varepsilon(s)) + \frac{s}{2}\log(\pi) - \log(\Pi(\frac{s}{2})) \tag{8}$$

and we want to plug this into:

$$J(x) = -\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}[\frac{\log(\zeta(s))}{s}]x^s ds$$

Let's write out explicitly what we need to calculate:

$$\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty} x^s \cdot \left(\frac{d}{ds}[\frac{\log(s-1)}{s}] - \frac{d}{ds}[\frac{\log(\varepsilon(s))}{s}] - \frac{d}{ds}[\frac{\frac{s}{2}\log(\pi)}{s}] + \frac{d}{ds}\frac{\log(\Pi(\frac{s}{2}))}{s}\right) ds \tag{9}$$

Cutting this integral in four parts is legal as long as the integral over each part converges, we will show explicitly that this is true. In multiple occasions we will assume that the order of integration and summation is allowed to be switched. We will look at this in more detail in the next paragraph.
The first term, counting from left to right, should be very familiar. It is equal to $F(1)$, which we proved to equal $Li(x)$. Since we assume $x > 1$, this value is finite.

The second term is more difficult. In chapter 2, we established the following product representation : $\varepsilon(s) = \varepsilon(0) \cdot \prod_\rho (1 - \frac{s}{\rho})$. Putting this in the integral yields :

$$\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}[\frac{\log(\varepsilon(0)) + \sum_\rho \log(1-\frac{s}{\rho}))}{s}]ds$$

We'll start by using integration by parts for the $\varepsilon(0)$ term:

$$\begin{aligned}\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}[\frac{\log(\varepsilon(0))}{s}]ds &= \frac{1}{2\pi i}\frac{1}{\log(x)}\, x^s \cdot \frac{\log(\varepsilon(0))}{s}\Big|_{a-i\infty}^{a+i\infty} - \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{\log(\varepsilon(0))}{s}ds \\ &= -\log(\varepsilon(0))x^0 = -\log(\frac{1}{2}) = \log(2)\end{aligned}$$

To compute the integral, we used the results of paragraph 1.7 and **Lemma 3.1**. Assume that we are allowed to swap the order of summation and integration. Then the remaining terms are equal to $H(\rho)$. For $Im(\rho) > 0$ we know $H(\rho) = G(\rho) = Li(x) - i\pi$. The requirement $Re(\rho) > 0$ is satisfied since there are no non-trivial zeroes with $Re(\rho) \le 0$. But there are zeroes with $Im(\rho) < 0$, where $H = G$ does not hold. Indeed, using the same trick once again and letting $Im(\beta)$ tend to $-\infty$, we will get 0 for H ($|H|$ is symmetric in the real line), while G will diverge. There is a way to avoid this. Instead of integrating over C^+, integrate over C^- which equals C^+ except for the semicircle around 1 which lies in the lower half plane. Call this function K. Exactly the same reasoning as we showed for G, will now prove $H = K$ on the lower half plane and $K(1) = Li(x) + i\pi$. The factor $i\pi$ does not have a minus sign this time since the path around $t = 1$ has become counterclockwise. We conclude that we can put the integral in the following form:

$$\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds}[\frac{\sum_\rho \log(1-\frac{s}{\rho}))}{s}]ds = \sum_\rho H(\rho) = \sum_{Im(\rho)>0}\left[\int_{C^+}\frac{t^{\rho-1}}{\log(t)}dt + \int_{C^-}\frac{t^{-\rho}}{\log(t)}dt\right] \tag{10}$$

Substituting $t = u^{\frac{1}{\rho}}$ transforms the integral over C^+ into:

$$\int_{C^+} \frac{t^{\rho-1}}{\log(t)} = \int_0^{x^\rho} \frac{u^{\frac{\rho-1}{\rho}} \cdot u^{\frac{1}{\rho}-1}}{\log(u^{\frac{1}{\rho}})} \frac{1}{\rho} du = \int_0^{x^\rho} \frac{du}{\log(u)} = Li(x^\rho) - i\pi$$

The path of integration between 0 and x^ρ is of course, taken to be in the upper half plane, avoiding $u = 1$. Now we do the same for the C^- integral with the substitution $u^{\frac{1}{1-\rho}}$:

$$\int_{C^-} \frac{t^{-\rho}}{\log(t)} dt = \int_0^{x^{1-\rho}} \frac{u^{\frac{-\rho}{\rho-1}} \cdot u^{\frac{1}{1-\rho}-1}}{\log(u^{\frac{1}{\rho-1}})} \frac{1}{1-\rho} du = Li(x^{1-\rho}) + i\pi$$

Thanks to the pairwise summation, we can write:

$$\frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds} [\frac{\sum_\rho \log(1 - \frac{s}{\rho}))}{s}] ds = \sum_{Im(\rho)>0} [Li(x^\rho) + Li(x^{1-\rho})] \tag{11}$$

This will be our final representation for this integral.

The third integral drops out, since $\frac{d}{ds}\frac{1}{2}\log(\pi) = 0$. For the last integral we will again use an infinite product expression:

$$\Pi(\frac{s}{2}) = \prod_{n=1}^{\infty} (1 + \frac{s}{2n})^{-1} \cdot (1 + \frac{1}{n})^{\frac{s}{2}} \tag{12}$$

which will result in the following integral:

$$\frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds} \left[\frac{\sum_{n=1}^{\infty} - \log(1 + \frac{s}{2n}) + \frac{s}{2} \log(1 + \frac{1}{n})}{s} \right] ds \tag{13}$$

Assume for now that we can pull $\frac{d}{ds}$ into the sum. Then it is clear that the right term in the summation becomes zero. Assuming that we can bring the integral into the summation gives:

$$-\sum_{n=1}^{\infty} \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-i\infty}^{a+i\infty} x^s \cdot \frac{d}{ds} \frac{\log(1 + \frac{s}{2n})}{s} ds = -\sum_{n=1}^{\infty} H(-2n) \tag{14}$$

Notice that this is well defined since H is well defined on the negative real axis. But all the previous relations with F and G are no longer valid, since they do not life on the negative real line. This is why, just as we did for G, we define a new but closely related alternative:

$$E(\beta) = -\int_x^{\infty} \frac{t^{\beta-1}}{\log(t)} dt$$

E is absolutely convergent for $Re(\beta) < 0$ since $\int_x^\infty t^{\beta-1} dt$ is finite in this situation, and $\frac{1}{\log(t)}$ is smaller than $\frac{1}{\log(x)}$, which is finite. Again, we are allowed to pull $\frac{d}{d\beta}$ inside the integral to obtain the derivative of E:

$$\frac{d}{d\beta} E(\beta) = E'(\beta) = -\int_x^{\infty} \frac{d}{d\beta} \frac{t^{\beta-1}}{\log(t)} dt = \frac{t^\beta}{\beta}\bigg|_x^\infty = \frac{x^\beta}{\beta} = H'(\beta)$$

We conclude that E and H differ by a constant, and taking the limit $\beta \to -\infty$ shows that both E and H become zero. The constant is zero which means that $E = H$ on $Re(\beta) < 0$. Replace H with E in the summation. All the terms in the summation are positive which means that the summation is absolutely convergent if it is convergent. Therefore, if we assume that we are allowed to swap the order of summation and integration and find a finite expression, our swap was legal. This will be the case:

$$\begin{aligned}
-\sum_{n=1}^{\infty} H(-2n) &= -\sum_{n=1}^{\infty} E(-2n) = \sum_{n=1}^{\infty} \int_{x}^{\infty} \frac{t^{-2n-1}}{\log(t)} dt \\
&= \int_{x}^{\infty} \sum_{n=1}^{\infty} \frac{t^{-2n-1}}{\log(t)} dt = \int_{x}^{\infty} \frac{1}{t\log(t)} [\sum_{n=0}^{\infty} t^{-2n} - 1] dt = \int_{x}^{\infty} \frac{1}{t(t^2-1)\log(t)} dt
\end{aligned}$$

Notice that the integral we found is convergent and decreasing with respect to x. This was the last integral to be evaluated, which means that we have found a new expression for $J(x)$. Notice that all our integrals turned out to be finite which in retrospect allowed us to chop the original integral in four pieces. We arrive at the main result of Riemann's paper and this will allow us to give an analytic expression for $\pi(x)$. We conclude:

$$J(x) = Li(x) - \sum_{Im(\rho)>0} [Li(x^{\rho}) + Li(x^{1-\rho})] - \log(2) + \int_{x}^{\infty} \frac{1}{t(t^2-1)\log(t)} dt \tag{15}$$

4.2 The loose ends, switching summation and integration

First we will tackle the problems for the last integral. We'll first prove that (13) and (14) are equal;

$$\frac{d}{ds} \sum_{n=1}^{\infty} \frac{\log(1+\frac{s}{2n}) + \frac{s}{2}\log(\pi)}{s} = \sum_{n=1}^{\infty} \frac{d}{ds} \frac{\log(1+\frac{s}{2n})}{s}$$

Notice that for a finite value of s, there is a finite value N such that for all $n > N$, $|\frac{s}{2n}| < 1$. In this region, $\log(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ and the summation is absolutely convergent. This proves that it is uniformly convergent as well. Since for $|a| < |s|$, $|\log(1+a)| \leq |\log(1+s)|$, this proves that the summation is uniformly convergent on every closed disc around zero. This proves the following:

$$\sum_{n=1}^{\infty} \frac{d}{ds} \frac{\log(1+\frac{s}{2n})}{s} = \sum_{n=1}^{\infty} \frac{d}{ds} \frac{\log(1+\frac{s}{2n}) + \frac{s}{2}\log(\pi)}{s} = \frac{d}{ds} \sum_{n=1}^{\infty} \frac{\log(1+\frac{s}{2n}) + \frac{s}{2}\log(\pi)}{s}$$

This proves the first assumption we made and it also proves that we can integrate termwise over a finite domain:

$$\frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-iT}^{a+iT} x^s \cdot \frac{d}{ds} \frac{\log(\Pi(\frac{s}{2}))}{s} ds = -\sum_{n=1}^{\infty} \frac{1}{2\pi i} \frac{1}{\log(x)} \int_{a-iT}^{a+iT} x^s \cdot \frac{d}{ds} \frac{\log(1+\frac{s}{2n})}{s} ds$$

It will take some doing, but we will be able to prove that the integral will be smaller than a constant times n^{-2}, which proves that the summation on the right is absolutely convergent. This will allow us to pull the limit $T \to \infty$ inside the summation, proving that summation and integration are allowed to be swapped. First we will substitute $s = a + iv$:

$$\int_{a-iT}^{a+iT} x^s \cdot \frac{d}{ds} \frac{\log(1+\frac{s}{2n})}{s} ds = \int_{-T}^{T} x^{a+iv} \cdot \frac{d}{idv} \frac{\log(1+\frac{a+iv}{2n})}{a+vi} idv$$

Our strategy is to pull a factor n^{-2} out of the integral. For this reason, substitute $v = y \cdot 2n$:

$$\begin{aligned}
\int_{-T}^{T} x^{a+iv} \cdot \frac{d}{idv} \frac{\log(1+\frac{a+iv}{2n})}{a+vi} idv &= \int_{-\frac{T}{2n}}^{\frac{T}{2n}} x^{a+iy2n} \cdot \frac{d}{2ndy} \frac{\log(1+\frac{a}{2n}+y)}{a+i2ny} 2ndy \\
&= \frac{x^a}{2n} \int_{-\frac{T}{2n}}^{\frac{T}{2n}} x^{iy2n} \cdot \frac{d}{dy} \frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy} dy
\end{aligned}$$

Since a sum over n^{-1} will diverge, we will need to retrieve another factor n^-1 out of the integral. This can be achieved using integration by parts:

$$\begin{aligned}
&\frac{x^a}{2n}\int_{-\frac{T}{2n}}^{\frac{T}{2n}} x^{iy2n}\cdot\frac{d}{dy}\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}dy \quad = \\
&\frac{x^a}{2n}\frac{x^{iy2n}}{i2n\log(x)}\cdot\frac{d}{dy}\left.\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}\right|_{\frac{-T}{2n}}^{\frac{T}{2n}} \quad - \quad \frac{x^a}{2n}\int_{-\frac{T}{2n}}^{\frac{T}{2n}}\frac{x^{iy2n}}{i2n\log(x)}\cdot\frac{d^2}{dy^2}\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}dy \\
= &\frac{x^a}{2n}\frac{1}{i2n\log(x)}\left(x^{iy2n}\cdot\frac{d}{dy}\left.\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}\right|_{\frac{-T}{2n}}^{\frac{T}{2n}} \quad - \quad \int_{-\frac{T}{2n}}^{\frac{T}{2n}} x^{iy2n}\cdot\frac{d^2}{dy^2}\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}dy\right)
\end{aligned}$$

The factor on the left is of the form n^{-2} multiplied with a constant. If we can prove the part on the right to have a finite maximum, the modulus of the integral will be smaller than Cn^{-2} for some constant C, independent of n and T. Since the sum over Cn^{-2} is finite, the entire sum is absolutely convergent which in retrospect proves that we can switch summation and integration. This can all be done by computing the derivative in the integral:

$$\frac{d}{dy}\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy} \quad = \quad -i\frac{\log(a+\frac{a}{2n}+y)}{(\frac{a}{2n}+iy)^2}+\frac{1}{(\frac{a}{2n}+iy)(1+\frac{a}{2n}+iy)}$$

Notice that this function is bounded for real values of y. So, when we let $|T|\to\infty$ to compute the constant term after our partial integration, this will return finite values. These values will be independent of n which means that the constant terms of the partial integration, will be smaller than a certain constant A for all n. Therefore, our case reduces to proving the convergence of the following (B is a certain constant, independent of n):

$$n^{-2}\left(A-B\cdot\int_{-\frac{T}{2n}}^{\frac{T}{2n}}\frac{d^2}{dy^2}\frac{\log(1+\frac{a}{2n}+y)}{\frac{a}{2n}+iy}\frac{x^{iy2n}}{i2n\log(x)}dy\right)$$

In chapter 3, we proved the absolute convergence of F by using Cauchy's residue theorem. The same line of reasoning, tells us that

$$\frac{d}{dy}\left[-i\frac{\log(a+\frac{a}{2n}+y)}{(\frac{a}{2n}+iy)^2}+\frac{1}{(\frac{a}{2n}+iy)(1+\frac{a}{2n}+iy)}\right]$$

is absolutely integrable over the real line. Since this holds for all n, we conclude that there is a constant C such that:

$$|\int_{a-iT}^{a+iT} x^s\cdot\frac{d}{ds}\frac{\log(1+\frac{s}{2n})}{s}ds| \le \int_{a-iT}^{a+iT}|x^s\cdot\frac{d}{ds}\frac{\log(1+\frac{s}{2n})}{s}|ds \le C\cdot n^{-2}$$

This proves the absolute convergence of the summation and proves the legality of the switch of integration and summation.

The only thing thats left, is to prove that (10) is correct:

$$\frac{1}{2\pi i}\frac{1}{\log(x)}\int_{a-i\infty}^{a+i\infty} x^s\cdot\frac{d}{ds}[\frac{\sum_\rho(1-\frac{s}{\rho}))}{s}]ds = \sum_\rho\frac{1}{2\pi i\log(x)}\int_{a-i\infty}^{a+i\infty} x^s\cdot\frac{d}{ds}\left[\log(1-\frac{s}{\beta})\right]ds \quad (16)$$

Riemann, in is original paper, did not prove the legality of most of his changes in the order of summation and integration, but just indicated how it could be proved, except for the term above. He stated that this operation is legal, but that it would take another, closer evaluation to prove its correctness. This gap in the proof, was closed by the work of Von Mangoldt. At this point, we will not go into the details but just assume its correctness. A proof can be found in ([E],48 - 66).

4.3 Constructing $\pi(x)$ analytically

Lets finally start counting primes analytically. In the chapter 1 we defined

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

. Remember we used a trick from Fourier analysis to make our notation nicer. This means that we should think of $\pi(x)$ as $\frac{\pi(x+0)+\pi(x-0)}{2}$. Define $b_n = \frac{1}{n}J(x^{\frac{1}{n}})$ As stated before, we can find $\pi(x)$ by using the generalized Möbius inversion formula. The idea behind this inversion formula is as follows:
We know $b_1 = J(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$
This means that:

$$\begin{aligned} b_1 - b_2 &= \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) - \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \frac{1}{4}\pi(x^{\frac{1}{4}}) - \frac{1}{2\cdot 2}\pi(x^{\frac{1}{2\cdot 2}}) \cdots \\ &= \pi(x) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \frac{1}{5}\pi(x^{\frac{1}{5}}) + \cdots \end{aligned}$$

Notice that the indices of the terms on the right are all non-zero modulo 2. We can do the same for other prime numbers to ensure they do no longer live on the right side of this equation, for example 3:

$$\begin{aligned} b_1 - b_2 - b_3 &= \pi(x) + \frac{1}{3}J(x^{\frac{1}{3}}) - \frac{1}{3}J(x^{\frac{1}{3}}) + \frac{1}{5}\pi(x^{\frac{1}{5}}) - \frac{1}{6}\pi(x^{\frac{1}{6}}) + \frac{1}{7}J(x^{\frac{1}{7}}) + \cdots \\ &= \pi(x) + \frac{1}{5}\pi(x^{\frac{1}{5}}) - \frac{1}{6}\pi(x^{\frac{1}{6}}) + \frac{1}{7}J(x^{\frac{1}{7}}) + \cdots \end{aligned}$$

Notice that the introduction of $-b_3$ has caused the reprisal of π-terms with indices which are zero modulo 2. But there are still no pure powers of 2 or 3 on the right side of the equation. Since this holds for all primes, the only indices left in the following equation are non-pure powers of primes (and 1):

$$b_1 - \sum_{p\in P} b_p = \pi(x) - \frac{1}{6}\pi(x^{\frac{1}{6}}) - \frac{1}{10}\pi(x^{\frac{1}{10}}) - \frac{1}{14}\pi(x^{\frac{1}{14}}) - \cdots$$

Notice that for composite numbers, for example 6, there still remains a $-\frac{1}{6}\pi(x^{\frac{1}{6}})$ term. Indeed, for p and q distinct primes, both b_q and b_p will give a factor $\frac{-2}{q\cdot p}\pi(x^{\frac{1}{q\cdot p}})$, which in combination with $\pi(x)$ results in $\frac{-1}{q\cdot p}\pi(x^{\frac{1}{q\cdot p}})$. Therefore, adding the terms $b_{q\cdot p}$ will neutralize these terms. Let $P \times P$ denote all natural numbers that are the product of two distinct primes.

$$b_1 - \sum_{p\in P} b_p + \sum_{k\in P\times P} b_k = \pi(x) + \frac{1}{30}\pi(x^{\frac{1}{30}}) + \cdots$$

Notice that the 30-term survives. This is a result of the fact that in this procedure, we counted 30 multiple times. It arises in b_j with $j = 1,2,3,6,5,10,15$ for which there are three cases in which there is a minus sign in front. To fix this, all numbers which are the product of three different primes should be subtracted again. But in this case, we count will count products of four primes double. The procedure to fix this goes on ad infinitum and in the end we get:

$$b_1 - \sum_{p\in P} b_p + \sum_{k\in P\times P} b_k - \sum_{t\in P\times P\times P} b_t + \cdots = \pi(x)$$

The left side is well defined since for large values of values of n, $x^{\frac{1}{n}}$ goes to 1 and since $\pi(1) = 0$, $\pi(x^{\frac{1}{n}}) \stackrel{n\to\infty}{\longrightarrow} 0$.
But why wouldn't there be any terms on the right side which are divisible by a square? Well,

lets take a look at 2^n where $n > 1$. Define $a_n = \frac{1}{n}\pi(x^{\frac{1}{n}})$. The term $a(2^n)$ only arises in b_1 and b_2. Since there is a minus sign in front of b_2, it is clear that there will not be any terms of the form a_{2^n} on the right side. Now look at $a_{2^n \cdot 3}$. This one arises in b_1, b_2, b_3 and b_6. Notice that the amount of terms has doubled, it are the terms of $a(2^n)$ and two new ones which cancel: it doesn't appear on the right side. Notice that $a_{2^n \cdot 3^2}$ will have the same terms which means it will also not appear on the right side of the equation. Proving this holds can be seen using induction. We'll prove it a bit more formally below.

Remember the definition of the Möbius-function μ. Let $t \in \mathbb{N}$.

$$\mu(t) = \begin{cases} 1 & \text{when t = 1} \\ (-1)^n & \text{when t is the product of n distinct primes} \\ 0 & \text{when t is divisible by a square} \end{cases}$$

Our goal is to prove the following:

$$\pi(x) = \sum_{k \in \mathbb{N}} \frac{\mu(k)}{k} J(x^{\frac{1}{k}})$$

As mentioned before, since the function $x^{1/x}$ is smaller than 2 for positive real numbers and there are no primes below two, $J(x)$ can be expressed in a finite sum:

$$J(x) = \sum_{n=1}^{\infty} \frac{1}{n}\pi(x^{\frac{1}{n}}) = \sum_{1 \leq n \leq x} \frac{1}{n}\pi(x^{\frac{1}{n}})$$

Fix a value x and denote $F_x(\frac{1}{n}) = \frac{1}{n}\pi(x^{\frac{1}{n}})$ and $G_x(\frac{1}{n}) = \frac{1}{n}J(x^{\frac{1}{n}})$. In this notation, we can follow the usual proof of the inversion formula and this proof is therefore equivalent:

Generalized Möbius Inversion Formula *Let $x > 1$ and let $G_x(m) = \sum_{1 \leq n \leq x \cdot m} F_x(\frac{m}{n})$. The following holds: $\sum_{1 \leq n \leq x} \mu(n) G_x(\frac{1}{n}) = F_x(1)$*

Proof:

$$\sum_{1 \leq n \leq x} \mu(n) G_x(\frac{1}{n}) = \sum_{1 \leq n \leq x} \mu(n) \sum_{1 \leq m \leq \frac{x}{n}} F_x(\frac{1}{n \cdot m}) = \sum_{1 \leq n \leq x} \mu(n) \sum_{1 \leq m \leq \frac{x}{n}} \sum_{1 \leq r \leq x} [r = m \cdot n] F_x(\frac{1}{n \cdot m})$$

Here, the notation [.] stands for a boolean operator that is zero when $[r \neq n \cdot m]$ and 1 otherwise. Now we switch the order of the summations:

$$\sum_{1 \leq n \leq x} \mu(n) \sum_{1 \leq m \leq \frac{x}{n}} \sum_{1 \leq r \leq x} [r = m \cdot n] F_x(\frac{1}{n \cdot m}) = \sum_{1 \leq r \leq x} F_x(\frac{1}{r}) \sum_{1 \leq n \leq x} \mu(n) \sum_{1 \leq m \leq \frac{x}{n}} [m = \frac{r}{n}]$$

If we take a closer look at the last two summations, it turns out that the only surviving terms n, are those that divide r:

$$\sum_{1 \leq r \leq x} F_x(\frac{1}{r}) \sum_{1 \leq n \leq x} \mu(n) \sum_{1 \leq m \leq \frac{x}{n}} [m = \frac{r}{n}] = \sum_{1 \leq r \leq x} F_x(\frac{1}{r}) \sum_{n|r} \mu(n)$$

We know that $\mu(t)$ is a arithmetic function and filling in $t = p^k$ with p a prime, we see that $\sum_{n|r} \mu(n)$ is zero when $r \neq 1$ and 1 otherwise. Therefore, the last sum can be simplified:

$$\sum_{1 \leq r \leq x} F_x(\frac{1}{r}) \sum_{n|r} \mu(n) = \sum_{1 \leq r \leq x} F_x(\frac{1}{r})[r = 1] = F_x(1)$$

This finishes our proof. Notice that $F_x(1) = \pi(x)$ and that we have found an expressions for $\pi(x)$ is terms of G_x. We conclude:

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} J(x^{\frac{1}{k}})$$

It may not look like much, but it is a pretty big deal. Instead of counting primes, we are now able to calculate the number of primes. And even though $J(x)$ is not the nicest expression imaginable, it allows its self to be well approximated. As it turn out, $Li(x)$ is the most significant term in our result for $J(x)$. For this reason, Riemann suggests the approximation:

$$R(x) = \sum_{k \in \mathbb{N}} \frac{\mu(k)}{k} Li(x^{\frac{1}{k}}) \tag{17}$$

It turns out that this is a very good approximation. In fact, this formula more or less proves why Gauss his empirically validated approximation, $Li(x) - Li(2)$, works. It is just the first term in the summation.

4.4 Heuristics for Riemann's approximation

Riemann's approximation is based on the fact that the main term in $J(x)$ is $Li(x)$. Let's examine this for a moment. $Li(x)$ is a monotonically increasing function which takes pretty large values, for example $Li(10^{10}) \approx 4.5 \times 10^8$. This is a much larger value than $\log(2) \approx 0.6$, therefore it is negligible in equation (15) for large values of x. For the integral term, we can find a estimate, which proves this value to be small as well. We assume $x \geq e$:

$$\begin{aligned}\int_x^{\infty} \frac{1}{t(t^2-1)\log(t)} dt \leq \int_e^{\infty} \frac{1}{t(t^2-1)\log(t)} dt &\leq \frac{1}{\log(e)} \int_x^{\infty} \frac{1}{t(t^2-1)} dt \\ &= \frac{1}{1}\left[-\log(t) + \frac{1}{2}\log(1-t^2)\right]_e^{\infty} \approx 0.07\end{aligned}$$

Since this is very rough approximation, the actual value in $x = 10^10$ of this term is seen to be even smaller. Therefore, it can be omitted to get a good approximation for $J(x)$.

What remains is the sum over the non-trivial roots of $\zeta(s)$. It is very hard to find a proper estimate of this term. Riemann himself struggled with it and was not able to give a solid argument why this term is small with respect to $Li(x)$. He did however notice that the approximation you get when you remove the term to get $R(x)$, works better than $Li(x)$ for $x \leq 3 \cdot 10^6$ ([R], 9). This was the best known approximation at that time. Using the work we did in this paper, one can also see why this was a good approximation, it is simply the biggest expression in the first term of our exact formula for $\pi(x)$. Using the computer, we can easily give a stonger heuristic argument for Riemann's approximation, which can be found in the following table:

x	$\pi(x)$	Error $Li(x)$	Error $R(x)$
10^6	78,498	130	29
10^7	664,579	339	88
10^8	5,761,455	754	97
10^9	50,847,534	1,701	79
10^{10}	455,052,511	3,104	1828

In the table, we rounded off the value for the errors. This suggests that Riemann's intuition seems to be right. But it also suggests that the sum over the non-trivial $\zeta(s)$ roots, is small in comparison to $Li(x)$. It seems strange that a term which is only conditionally convergent, takes a small value but it seems to be true.

Notice that at this point, we have not mentioned the Riemann hypothesis at all. This is not a coincidence. In his original paper Riemann only mentions 'his' hypothesis once. His main goal was to prove equation (15) and for this he does not need the hypothesis. For completeness, lets state the hypothesis:

The Riemann Hypothesis *Let ρ be a root of $\varepsilon(s)$. Then $Re(\rho) = \frac{1}{2}$*

Notice that we did not need this result for any of the theorems and lemma's we proved. But, as stated before, the most mysterious term in equation (15) is the sum over the roots of $\varepsilon(s)$. The hypothesis gives a general property of roots of ε and it allows us to make estimates for this term. For example, Schoenfeld showed that the Riemann hypothesis is equivalent to
$|\pi(x) - Li(x)| < \frac{1}{8 \cdot \pi}\sqrt{x}\log(x)$ for x large enough. For this reason, the correct or incorrectness of the hypothesis would give us more information about the distribution of primes. This explains why the hypothesis is so interesting to us, we use primes all the time but we do not now much about there distribution on the real line. The Riemann zeta function seems to be our best approach in this situation and therefore the Riemann hypothesis is so interesting.

4.5 The Riemann Hypothesis

The proof we used for our expression of $J(x)$ is mainly due to Riemann's original paper. In this paper, Riemann left out a lot of important details and did not fully prove all of his assumptions about the order of integration and summation. The product formula of $\varepsilon(s)$ for example, was first rigorously proved in 1893 by Hadamard, more that 30 years after Riemann's paper ([E],38). Though he did not have the means to prove all his assumptions, he was however convinced about his formula for $J(x)$. It turns out that he was correct. The first proof of this fact, was published by Von Mangoldt who used an entirely different approach than Riemann ([E],38). After proving the functional equation, Riemann started with inverting the following formula:

$$\frac{1}{s}\log(\zeta(s)) = \int_0^\infty x^{-s-1}J(x)dx$$

As a result he gets an expression for $J(x)$ in terms of $\log(\zeta(s))$. But the logarithm of the zeta function is not the prettiest thing in the world and it is hard to express it into elementary functions. Von Mangoldt realized this and first differentiated over s before inverting, to get the equation:

$$\psi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} -\frac{\zeta'(s)}{\zeta(s)}x^s\frac{ds}{s}$$

Here, $\psi(x)$ is the sum over all prime powers smaller than x, where each prime power p^n is weighted with a factor $\log(p)$. $\frac{\zeta'(s)}{\zeta(s)}$ is a meromorphic function with poles at the roots of $\zeta(s)$. It turns out that $\psi(x)$ is an easier function to work. This is the reason that Riemanns original result, equation (15), is not the standard way to represent $\pi(x)$ or an equivalent function. Take for example the prime number theorem which states that $\pi(x) \sim Li(x)$. This is equivalent to proving $\psi(x) \sim x$. In 1896, Hadamard and la Vallée Poussin independently proved the second statement, thereby proving the prime number theorem.([E],38). The proof can be found in ([E], 68 - 77).

The previous paragraph's indicate the importance and usefulness of the Riemann zeta function and specifically the roots of $\varepsilon(s)$. This is more or less summerized in the amount of attempts to prove the Riemann hypothesis which vary greatly. On startegy that was suggested by Hilbert and Pólya is to use a self-adjoint operator. This works in the following way: define an operator T working on $\frac{1}{2} + iy$ such that the roots of $\varepsilon(s)$ are its eigenvalues. Since eigenvalues of self-adjoint operators are real, this would prove that all roots of $\varepsilon(s)$ are of the form $\frac{1}{2} + iy$.

An interesting indication that this might work comes from random matrix theory. In 1972 Montgomery tryed to find an expression for the amount of zeroes between $\frac{1}{2} + iA$ and $\frac{1}{2} + iB$. He conjectured an expression for this number, which was supposed to hold for large values of $A < B$ ([C], 8). The expression he found turned out to be the pair correlation function for the eigenvalues of large hermitian matrices. This connection implies that there might be a connection between hermitian operators (matrices) and the zeroes. It turns out that for zeroes with a large modulus, the conjecture seems to be true. Therefore, it may be interesting to investigate the correlation function more closely and see where the correspondence with the zeroes of $\varepsilon(s)$ actually comes from.

5 References

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