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Master's Thesis

Wilkie's Theorem and the Uniform Real Schanuel Conjecture

Author:
Martijn den Besten

Supervisor:
Dr. Jaap van Oosten

Second Examiner:
Prof. Dr. Gunther Cornelissen

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Abstract

In this thesis we give a detailed proof of the model completeness of two expansions of the real ordered field, specifically the expansion by Pfaffian chains of functions and the expansion by the exponential function. The latter result is also known as Wilkie's Theorem and both of the proofs are due to Alex Wilkie. As an application of Wilkie's Theorem, we provide a modest generalization of the fact that Schanuel's conjecture over the real numbers is equivalent to a uniform version of itself, as proven by Jonathan Kirby and Boris Zilber.

Contents

1 Introduction

1.1 History and background

We will approach the set of real numbers, $\mathbb{R}$, from a model theoretic point of view. To be more precise, we shall be concerned with certain expansions of the structure $(\mathbb{R} \mid +, \cdot, -, 0, 1, <)$, the *real ordered field*. Throughout, we will refer to its language as $\mathcal{L}$. For the sake of clarity, we make this into a definition.

Definition 1.1.1. We define the language $\mathcal{L}$ as $\{+, \cdot, -, 0, 1, <\}$. We also define $\mathcal{T}$ to be the complete $\mathcal{L}$-theory $\mathrm{Th}(\mathbb{R} \mid \mathcal{L})$.

The set $\mathbb{R}$, considered as an $\mathcal{L}$-structure, was at first mainly studied by algebraists, but has also received considerable attention from model theorists. Perhaps the most famous result in this area is proven by Alfred Tarski and it can be seen as the starting point of the model theoretic study of $(\mathbb{R} \mid \mathcal{L})$. He considered the $\mathcal{L}$-theory T_{RCF} of real closed fields, consisting of

- The axioms for ordered fields.
- $\forall x \exists y (0 < x \to x = y^2)$.
- $\forall x_0, \ldots, x_{2n+1} \exists y (x_{2n+1} \neq 0 \to x_0 + x_1 y + \cdots x_{2n+1} y^{2n+1} = 0)$ for each $n \in \mathbb{N}$.

In the early 1930s he proved that this theory admits *quantifier elimination*. Recall that this means that for every $\mathcal{L}$-formula $\varphi(\vec{x})$, there exists a quantifier free formula $\psi(\vec{x})$ such that $T_{RCF} \models \forall \vec{x} (\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}))$. The subsets of $\mathbb{R}^n$ which can be defined using quantifier free $\mathcal{L}$-formulas are called semialgebraic sets. These sets are studied in real algebraic geometry and quantifier elimination implies that the projection of a semialgebraic set is semialgebraic as well. Quantifier elimination in T_{RCF} has more implications. Since every ordered field contains a copy of the rational numbers, every real closed field must contain a copy of the algebraic real numbers. As every $\mathcal{L}$-formula is equivalent to a quantifier free formula, every embedding between models of T_{RCF} is an elementary embedding. So, since the algebraic real numbers embed into every model of T_{RCF}, this implies that T_{RCF} is a complete theory. Now take any $\mathcal{L}$-sentence φ for which we want to know if $T_{RCL} \models \varphi$. Since the axioms of T_{RCL} can be effectively described (T_{RCL} is recursively enumerable) we can imagine a computer enumerating all statements provable from the axioms of T_{RCL}. Since T_{RCL} is complete, we must either encounter φ or $\neg\varphi$ after a finite amount of time. This shows that we can effectively decide whether φ is true or not. The theory T_{RCL} is said to be *decidable*. We see that $(\mathbb{R} \mid \mathcal{L})$ exhibits very good model theoretic behavior and it should come as no surprise that it has become a beloved object of study. Some time after the decidability of T_{RCL} was settled, Tarski says

> " (...) the decision problem is open (...) for the system
> obtained by introducing the operation of exponentiation."

in a discussion of related decision problems [Has12]. The question he raises is that of the decidability of the theory $\mathcal{T}_{\exp} = \mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\exp})$, where $\mathcal{L}_{\exp} = \mathcal{L} \cup \{\exp\}$ and $\exp : \mathbb{R} \to \mathbb{R}$ denotes the exponential function with base e. This problem is known as Tarski's exponential function problem. A big breakthrough in this area was achieved by Alex Wilkie in [Wil96] (a preprint was already available in 1991), in which he proves that the theory $\mathcal{T}_{\exp}$ is model complete.

Definition 1.1.2. A theory T in a language L is called *model complete* if for every L-formula $\varphi(x_1, \ldots, x_n)$, there is an existential L-formula $\psi(x_1, \ldots, x_n)$ such that

$$\Gamma \models \forall x_1, \ldots, x_n \left[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)\right].$$

Furthermore, an L-structure M is called model complete if $\mathrm{Th}(M)$ is model complete.

Recall that an existential formula consists of a string of existential quantifiers, followed by a quantifier free formula. (A universal formula is defined analogously.) There are many different equivalent ways to define what model completeness is (in fact, we shall give another one in this thesis), but the form in which it is given in Definition 1.1.2 should look very similar to the definition of quantifier elimination. Propositions containing many quantifiers are usually regarded by mathematicians as more complex than those that do not. Indeed, the complexity of a formula might be measured by counting the number of alternations of blocks of existential and universal quantifiers appearing in the prenex normal form of that formula. So in this sense, quantifier free formulas are the least complex formulas of all. Quantifier elimination is so useful, because in order to understand structures that admit elimination of quantifiers, we can restrict our study to these "easy" quantifier free formulas (and the sets they define). On the other side of the spectrum, we find structures such as $(\mathbb{N} \mid +, \cdot, 0, 1)$, which exhibits poor model-theoretic behavior. Here we encounter a hierarchy of formulas of arbitrarily high complexity (in the sense we just described), which cannot be reduced to simpler formulas. When a structure does not admit quantifier elimination, model completeness serves as the next best thing. Nonetheless, the model completeness of $\mathcal{T}_{\exp}$ does not solve Tarski's problem. Five years later, Macintyre and Wilkie essentially settle it in the following fascinating way [MW96].

Theorem 1.1.3. *If Schanuel's Conjecture is true, then $\mathcal{T}_{\exp}$ is decidable. Conversely, if $\mathcal{T}_{\exp}$ is decidable, then a weak form of Schanuel's conjecture holds.*

(For a brief introduction of Schanuel's Conjecture, we refer to Section 8.1.)

This thesis can be subdivided into three parts. In the first part we prove the model completeness of $\mathcal{T}_{Pf\restriction}$, which will be defined in the next section. This part mainly consists of proving three different Lemmas and the techniques we develop in order to prove these will also be useful in the subsequent parts. In the second part we will prove the model completeness of $\mathcal{T}_{\exp}$ (that is, Wilkie's Theorem). In both of these parts we follow the proofs given in [Wil96]. In the third part we offer slight generalization of a result by Kirby and Zilber, which states that Schanuel's Conjecture over the real numbers is equivalent to a uniform version of itself. The proof of this uses Wilkie's Theorem. This last part is mainly based on [KZ06].

1.2 Definitions and preliminary knowledge

Below we give the definition of a Pfaffian chain, which is needed to understand the First Main Theorem. The reader may also wish to glance over the appendix, in which some concepts and results regarding real analytic functions, O-minimal structures and types, which we will need along the way, are briefly summarized.

We shall be interested in certain classes of real analytic functions (in truth, truncations thereof), which we define as follows.

Definition 1.2.1. Let $m, l \in \mathbb{N}$, with $m, l \geq 1$ and let $U \subseteq \mathbb{R}^m$ be an open set, such that the closed box $[0,1]^m$ is contained in U. Now, let $G_1, \ldots, G_l : U \to \mathbb{R}$ be analytic functions and suppose that there exist polynomials $p_{i,j} \in \mathbb{R}[z_1, \ldots, z_{m+i}]$ (for $i = 1, \ldots, l$ and $j = 1, \ldots, m$) such that

$$\frac{\partial G_i}{\partial x_j}(\vec{x}) = p_{i,j}(\vec{x}, G_1(\vec{x}), \ldots, G_i(\vec{x})),$$

for all $\vec{x} \in U$. Then the sequence $G_1, \ldots, G_l$ is called a *Pfaffian chain* on U.

As we indicated above, we will actually work with truncated functions.

Definition 1.2.2. Let $m, l \in \mathbb{N}$, $U \subseteq \mathbb{R}$ and $G_1, \ldots, G_l : U \to \mathbb{R}$ be as in Definition 1.2.1 and let $F_1, \ldots, F_l$ be the corresponding truncations. That is,

$$F_i(\vec{x}) = \begin{cases} G_i(\vec{x}) & \text{if } \vec{x} \in [0,1]^m \\ 0 & \text{if } \vec{x} \in \mathbb{R}^m \setminus [0,1]^m \end{cases}$$

Now, let $C \subseteq \mathbb{R}$ by any set such that the coefficients of each $p_{i,j}$ are the value of some term in the structure $(\mathbb{R} \mid \mathcal{L}, F_1, \ldots, F_l, c)_{c \in C}$. We define the language $\mathcal{L}_{\mathrm{Pf}\upharpoonright}$ as $\mathcal{L} \cup \{F_1, \ldots, F_l\} \cup C$. Furthermore, we define the $\mathcal{L}_{\mathrm{Pf}\upharpoonright}$-theory $\mathcal{T}_{\mathrm{Pf}\upharpoonright}$ as $\mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\mathrm{Pf}\upharpoonright})$.

Remark 1.2.3. Of course, the theory $\mathcal{T}_{\mathrm{Pf}\upharpoonright}$ is dependent on the Pfaffian chain $G_1, \ldots, G_l$, even though this is not reflected in our notation. This should not cause confusion, since throughout this thesis, we will work with the fixed Pfaffian chain $G_1, \ldots, G_l$. (We will however, at some point, conveniently forget the exact details of the definitions of $G_1, \ldots, G_l$, in order to free up the variables $m, l, U, C, \ldots$)

2 Approach to the First Main Theorem

2.1 Reducing the problem

We will not keep the reader in suspense any longer and present the First Main Theorem.

Theorem 2.1.1. *The theory $\mathcal{T}_{\mathrm{Pf}\upharpoonright}$ is model complete.*

The first step in our proof of this Theorem consists of formulating a condition on structures, which is strongly related to the concept of model completeness. In our proof of the First Main Theorem, we will not verify the conditions of Definition 1.1.2 directly, but instead formulate and equivalent condition which we will verify.

Definition 2.1.2. Let L be a language and let M and N be L-structures such that $M \subseteq N$. We say that M is *existentially closed* in N if

$$N \models \varphi \text{ implies } M \models \varphi,$$

for all existential L_M-sentences φ.

In order to show how Definition 1.1.2 and Definition 2.1.2 relate to one another, we need the following Lemma (but the curious reader can already take a peek at Corollary 2.1.4).

Lemma 2.1.3. *Let L be a language and let T be a theory in the language L. Suppose that $\varphi(x_1, \ldots, x_n)$ is an L-formula such that for every inclusion $M \subseteq N$ of models of T holds that*

$$N \models \varphi(m_1, \ldots, m_n) \textit{ implies } M \models \varphi(m_1, \ldots, m_n),$$

for all $m_1, \ldots, m_n \in M$. Then there exists a universal L-formula $\psi(x_1, \ldots, x_n)$ such that

$$T \models \forall x_1, \ldots, x_n \left[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)\right].$$

Proof. Let $c_1, \ldots, c_n$ be new constants and write $\vec{c}$ and $L_{\vec{c}}$ for $(c_1, \ldots, c_n)$ and $L \cup \{c_1, \ldots, c_n\}$ respectively. We define the theory

$$\Gamma = \{\psi(\vec{c}) \mid \psi(\vec{x}) \textit{ a universal } L\textit{-formula such that } T \cup \{\varphi(\vec{c})\} \models \psi(\vec{c})\}.$$

Our goal is to prove that $T \cup \Gamma \models \varphi(\vec{c})$. So, consider an arbitrary model $M \models T \cup \Gamma$ and let $D(M)$ denote the diagram of M with respect to the language $L_{\vec{c}}$.

Claim. The theory $T \cup D(M) \cup \{\varphi(\vec{c})\}$ is consistent.

Proof. Suppose to the contrary that it is inconsistent. Then by the Compactness Theorem, it is finitely inconsistent. This means that there exist finitely many sentences $\psi_1(\vec{c}), \ldots, \psi_m(\vec{c}) \in D(M)$, such that $T \cup \{\varphi(\vec{c})\} \cup \{\psi_1(\vec{c}), \ldots, \psi_m(\vec{c})\}$ is inconsistent. We define $\Psi(\vec{c}) = \psi_1(\vec{c}) \wedge \ldots \wedge \psi_m(\vec{c}) \in D(M)$ and we note that $T \cup \{\varphi(\vec{c})\} \cup \{\Psi(\vec{c})\}$ is already inconsistent. We can write $\Psi(\vec{c}) = \Phi(\vec{c}, \vec{a})$ for some quantifier free L-formula $\Phi(\vec{x}, \vec{y})$ and constants $\vec{a}$ from M. Since the constants $\vec{a}$ do not appear in $T \cup \{\varphi(\vec{c})\}$, it follows that $T \cup \{\varphi(\vec{c})\}$ must be inconsistent with $\exists \vec{y}\, \Phi(\vec{c}, \vec{y})$. In other words, $T \cup \{\varphi(\vec{c})\} \models \forall \vec{y} \neg \Phi(\vec{c}, \vec{y})$, so $\forall \vec{y} \neg \Phi(\vec{c}, \vec{y}) \in \Gamma$, by definition of Γ. But then $M \models \forall \vec{y} \neg \Phi(\vec{c}, \vec{y})$, since $M \models \Gamma$. In particular $M \models \neg \Phi(\vec{c}, \vec{a})$, which is a contradiction with $\Phi(\vec{c}, \vec{a}) \in D(M)$. This proves our claim.

Let N be a model of $T \cup D(M) \cup \{\varphi(\vec{c})\}$. Then $M \subseteq N$ and $N \models \varphi(\vec{c})$, so we may use the special property of φ to conclude that $M \models \varphi(\vec{c})$. Since $M \models T \cup \Gamma$ was arbitrary, we conclude that $T \cup \Gamma \models \varphi(\vec{c})$. By the Compactness Theorem, there are in fact finitely many sentences

$\psi_1(\vec{c}), \ldots, \psi_m(\vec{c}) \in \Gamma$ such that $T \cup \{\psi_1(\vec{c}), \ldots, \psi_m(\vec{c})\} \models \varphi(\vec{c})$. Since the set of universal sentences is closed under conjunction (up to equivalence), we can take a universal sentence $\psi(\vec{c})$ equivalent to $\psi_1(\vec{c}) \wedge \ldots \wedge \psi_m(\vec{c})$. Then surely $T \cup \{\psi(\vec{c})\} \models \varphi(\vec{c})$, so $T \models \psi(\vec{c}) \rightarrow \varphi(\vec{c})$. On the other hand, since $\psi_1(\vec{c}), \ldots, \psi_m(\vec{c}) \in \Gamma$, it is also clear that $T \cup \{\varphi(\vec{c})\} \models \psi(\vec{c})$, so $T \models \varphi(\vec{c}) \rightarrow \psi(\vec{c})$. Since $T \models \varphi(\vec{c}) \leftrightarrow \psi(\vec{c})$ and the constants $\vec{c}$ appear nowhere in T, we must have that

$$T \models \forall x_1, \ldots, x_n \left[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)\right],$$

as required. □

As promised, we have the following corollary, linking Definition 1.1.2 and Definition 2.1.2.

Corollary 2.1.4. *Let L be a language and let T be a theory in the language L. Then the following are equivalent.*

(i) *The theory T is model complete.*

(ii) *For every pair of L-structures $M \subseteq N$, which are models of T, holds that M is existentially closed in N.*

Proof. Suppose that **(i)** holds and let $M \subseteq N$ be models of T. Let φ be an existential L_M-sentence such that $N \models \varphi$. We write $\varphi = \exists x_1, \ldots, x_r \psi(m_1, \ldots, m_s, x_1, \ldots, x_r)$, with ψ a quantifier free L-formula and $m_1, \ldots, m_s \in M$. Since T is model complete,

$$T \models \forall y_1, \ldots, y_s \left[\neg\exists x_1, \ldots, x_r\ \psi(y_1, \ldots, y_s, x_1, \ldots, x_r) \leftrightarrow \exists x_1, \ldots, x_t\ \chi(y_1, \ldots, y_s, x_1, \ldots, x_t)\right],$$

for some quantifier free L-formula χ. So in particular

$$N \models \neg\exists x_1, \ldots, x_r\ \psi(m_1, \ldots, m_s, x_1, \ldots, x_r) \leftrightarrow \exists x_1, \ldots, x_t\ \chi(m_1, \ldots, m_s, x_1, \ldots, x_t),$$

as $N \models T$ and hence

$$N \models \neg\exists x_1, \ldots, x_t\ \chi(m_1, \ldots, m_s, x_1, \ldots, x_t).$$

But if there is no tuple $x_1, \ldots, x_t$ in N such that $N \models \chi(m_1, \ldots, m_s, x_1, \ldots, x_t)$, then certainly there can be no tuple $x_1, \ldots, x_t$ in M such that $M \models \chi(m_1, \ldots, m_s, x_1, \ldots, x_t)$, since $M \subseteq N$ and χ is quantifier free. It follows that

$$M \models \neg\exists x_1, \ldots, x_t\ \chi(m_1, \ldots, m_s, x_1, \ldots, x_t),$$

and as a consequence

$$M \models \exists x_1, \ldots, x_r\ \psi(m_1, \ldots, m_s, x_1, \ldots, x_r),$$

This time because $M \models T$. So **(ii)** holds. To prove the converse, suppose that **(ii)** holds.

Claim. Every existential L-formula is equivalent to a universal L-formula, with respect to T.

Proof. To prove this claim, let $\varphi(x_1, \ldots, x_n)$ be an existential L-formula. We clearly have that for every inclusion $M \subseteq N$ of models of T holds that

$$N \models \varphi(m_1, \ldots, m_n) \textit{ implies } M \models \varphi(m_1, \ldots, m_n),$$

for all $m_1, \ldots, m_n \in M$, since T satisfies **(ii)**. Our claim now follows directly from Lemma 2.1.3.

Using the claim, we shall prove that every L-formula is equivalent, with respect to T of course, to an existential L-formula. To show this, we use induction on the number leading quantifiers of formulas in prenex normal form. Since every formula can be put in prenex normal form, this will suffice. A quantifier free formula is in particular an existential formula, so the base case is covered. Now suppose that every L-formula in prenex normal form with less than r quantifiers is equivalent to an existential formula, modulo T. Consider the following formula in prenex normal form

$$Q_1 y_1 \dots Q_r y_r \ \varphi(x_1, \dots x_n, y_1, \dots, y_r),$$

where $Q_1, \dots, Q_r$ are quantifiers and φ is a quantifier free L-formula. If Q_1 is an existential quantifier, then we are done right away, as we can apply our induction hypothesis to the formula $Q_2 y_2 \dots Q_r y_r \ \varphi$ to turn it into an existential one, at which point we can simply return $Q_1 y_1$ to the beginning of this formula. Now suppose that Q_1 is universal. In this case we also apply our induction hypothesis to the formula $Q_2 y_2 \dots Q_r y_r \ \varphi$. Our induction hypothesis tells us that this formula is equivalent, modulo T, to an existential L-formula. Then by our claim, this formula is equivalent to a universal formula, say

$$\forall y_2 \dots \forall y_s \ \psi(x_1, \dots x_n, y_1, \dots, y_s).$$

Hence

$$T \models \forall x_1, \dots, x_n \left[Q_1 y_1 \dots Q_r y_r \ \varphi \leftrightarrow \forall y_1 \dots \forall y_s \ \psi \right].$$

We apply our claim yet again, this time to the existential formula $\exists y_1 \dots \exists y_s \neg\psi$. This gives us a universal formula, say $\forall y_1 \dots \forall y_t \ \chi$, equivalent to it. But now we are done, as

$$T \models \forall x_1, \dots, x_n \left[Q_1 y_1 \dots Q_r y_r \ \varphi \leftrightarrow \exists y_1 \dots \exists y_t \neg\chi \right],$$

since $\exists y_1 \dots \exists y_s \neg\psi$ is the negation of $\forall y_1 \dots \forall y_s \ \psi$ and $\exists y_1 \dots \exists y_t \neg\chi$ is the negation of $\forall y_1 \dots \forall y_t \ \chi$. This completes the induction, so **(i)** holds. □

Thus, to prove Theorem 2.1.1, it suffices to take two arbitrary models $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$, and an arbitrary existential sentence χ in the language $\mathcal{L}_{\mathrm{Pf}\restriction,k} = \mathcal{L}_{\mathrm{Pf}\restriction} \cup k$, such that $K \models \chi$, and show that $k \models \chi$. In fact, this is more or less what we will do, but we can make our lives a litlle bit easier. It turns out that we only need to concern ourselves with existential $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-formulas χ of a special kind. In the following two Lemmas, we show exactly what we mean by this. The first of the two is formulated a bit more general than we need at this point, but we will come back and recycle this Lemma (as we will do with many other results as well).

Lemma 2.1.5. *Let L be a language of the form $\mathcal{L} \cup \mathcal{F} \cup C$, where $\mathcal{F}$ is a set of function symbols and C is a set of constants. Furthermore, let $T = \mathrm{Th}(\mathbb{R} \mid L)$. Then any existential sentence χ in the language L is equivalent, in the theory T, to a sentence of the form*

$$\exists x_1, \dots, x_r \bigwedge_{s=1}^{n} \tau_s = 0,$$

where each τ_s is a term of $\mathcal{L}_C = \mathcal{L} \cup C$ or has the form $f(x_{i_1}, \dots, x_{i_l}) - x_{i_{l+1}}$, for some $f \in \mathcal{F}$.

Proof. We begin by proving the following claim.

Claim. Each formula of the form $\sigma - y = 0$, where σ is a term of L, and with y not appearing in σ, is equivalent to a formula of the form

$$\exists x_1, \ldots, \exists x_n \left[\tau_{n+1} - y = 0 \wedge \bigwedge_{s=1}^{n} [\tau_s - x_s = 0] \right],$$

where each τ_s is a variable, a constant of L, or the form $f(x_{s_1}, \ldots, x_{s_l})$, for some function symbol f of L, and with y not appearing in $\tau_1, \ldots, \tau_{n+1}$.

Proof. We use induction over the structure of L-terms to prove this claim. The base case holds trivially, so let f be an l-ary function symbol of L and let $\sigma_1, \ldots, \sigma_l$ be terms of $\mathcal{L}_C$, for which our induction hypothesis holds true. Then the formula $f(\sigma_1, \ldots, \sigma_l) - y = 0$ is equivalent to

$$\exists x_1, \ldots, \exists x_l \left[f(x_1, \ldots, x_l) - y = 0 \wedge \bigwedge_{s=1}^{l} [\sigma_s - x_s = 0] \right].$$

Using our induction hypothesis to replace each formula $\sigma_s - x_s = 0$ now yields the result, as we are allowed to bring any existential quantifiers to the beginning of the formula, if we make sure that every new variable we introduce does not already appear in other parts of the formula.

So if we write an L-formula of the form $\sigma = 0$ as $\exists y[y = 0 \wedge \sigma - y = 0]$, with y not appearing in σ, then we can use our claim to see that it is actually equivalent to a formula of the form

$$\exists y \left[y = 0 \wedge \exists x_1, \ldots, \exists x_n \left[\tau_{n+1} - y = 0 \wedge \bigwedge_{s=1}^{n} [\tau_s - x_s = 0] \right] \right],$$

where each τ_s is a variable, a constant of $\mathcal{L}_C$, or the form $f(x_{s_1}, \ldots, x_{s_l})$, for some function symbol f of L. Moving all quantifiers to the beginning of the formula and replacing the variable y by x_{n+1} for convenience of notation, gives us a formula

$$\exists x_1, \ldots, x_{n+1} \bigwedge_{s=1}^{n+1} \tau'_s = 0, \tag{1}$$

where each τ'_s is a term of $\mathcal{L}$ or has the form $f(x_{i_1}, \ldots, x_{i_l}) - x_{i_{l+1}}$.

Now let us make the observation that every (possibly negated) atomic formula (or *literal*) of L is equivalent to a formula of the form $\exists x[\sigma = 0]$, where σ is a term in the language L. Indeed, if σ and τ are terms of L, then we have the following list of equivalences

$$\begin{aligned}
\sigma = \tau &\longleftrightarrow \exists x[\sigma - \tau = 0] \\
\sigma < \tau &\longleftrightarrow \exists x[(\tau - \sigma) \cdot x^2 - 1 = 0] \\
\neg[\sigma = \tau] &\longleftrightarrow \exists x[(\tau - \sigma) \cdot x - 1 = 0] \\
\neg[\sigma < \tau] &\longleftrightarrow \exists x[\tau - \sigma + x^2 = 0],
\end{aligned}$$

with x not appearing in σ or τ.

It follows that if we are given a set of literals $\phi_1, \ldots, \phi_n$, we can find terms $\sigma_1, \ldots, \sigma_n$, such that each ϕ_s is equivalent to $\exists x_s[\sigma_s = 0]$, and where each x_s does not appear in σ_t for $t \neq s$.

This means that we can write the disjunction of these literals in the following manner

$$\begin{aligned}\bigvee_{s=1}^{n} \phi_s &\longleftrightarrow \bigvee_{s=1}^{n} \exists x_s[\sigma_s = 0] \\ &\longleftrightarrow \exists x_1, \ldots, \exists x_n \bigvee_{s=1}^{n} \sigma_s = 0 \\ &\longleftrightarrow \exists x_1, \ldots, \exists x_n[\sigma_1 \cdots \sigma_n = 0].\end{aligned}$$

We are allowed to replace $\sigma_1 \cdots \sigma_n = 0$ by a formula of the form (1), being careful not to use the same variables twice, to arrive at

$$\bigvee_{s=1}^{n} \phi_s \longleftrightarrow \exists x_1, \ldots, x_r \bigwedge_{s=1}^{l} \tau_s = 0,$$

where each τ_s is a term of $\mathcal{L}_C$ or has the form $f(x_{i_1}, \ldots, x_{i_l}) - x_{i_{l+1}}$. Now if we take a conjunction of formulas of the kind shown on the right side of this equivalence, again with variables not appearing twice, we can move all quantifiers to the beginning of this formula, to find a formula of the exact same form, that is, of the form

$$\exists x_1, \ldots, x_r \bigwedge_{s=1}^{l} \tau_s = 0,$$

where each τ_s is a term of $\mathcal{L}_C$ or has the form $f(x_{i_1}, \ldots, x_{i_l}) - x_{i_{l+1}}$. But this means that every formula in conjunctive normal form is equivalent to a formula of this kind. Since every existential sentence is a string of existential quantifiers followed by a quantifier free formula, and every quantifier free formula is equivalent to a formula in conjunctive normal form, the lemma follows. □

Lemma 2.1.6. *Let k and K be models of $\mathcal{T}_{\mathrm{Pf}\restriction}$ such that $k \subseteq K$. Then k is existentially closed in K if and only if*

$$K \models \exists x_1, \ldots, x_r\, \chi \text{ implies } k \models \exists x_1, \ldots, x_r\, \chi,$$

for every $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-sentence $\exists x_1, \ldots, x_r\, \chi$ of the form

$$\exists x_1, \ldots, x_r \bigwedge_{s=1}^{l} \chi_s, \tag{2}$$

where each χ_s is of the form $\tau = 0$ for some term τ of $\mathcal{L}_k$ or of the form

$$F_i(y_{i_1}, \ldots, y_{i_m}) - x_{i_{m+1}} = 0 \wedge \bigwedge_{j \in S} 0 < x_{i_j} < 1,$$

(see Definition 1.2.2) for some $S \subseteq \{1, \ldots, m\}$ and where

$$y_{i_j} = \begin{cases} x_{i_j} & \text{if } j \in S \\ 0 \text{ or } 1 & \text{if } j \notin S, \end{cases}$$

for $1 \leq i_1, \ldots, i_{m+1} \leq r$.

Proof. Surely, if k is existentially closed in K and the structure K satisfies an $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-sentence ψ of the form (2), then k satisfies ψ as well, as ψ is existential. We direct our attention to the converse.

Suppose that $K \models \exists x_1, \ldots, x_r\ \chi$ implies $k \models \exists x_1, \ldots, x_r\ \chi$, for every $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-sentence χ of the form (2). Let $\exists x_1, \ldots, x_r\ \psi(x_1, \ldots, x_r)$ be an existential $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-sentence such that $K \models \exists \vec{x}\ \psi(\vec{x})$. By lemma 2.1.5 we may assume that $\exists \vec{x}\ \psi(\vec{x})$ is of the form

$$\exists x_1, \ldots, x_r \bigwedge_{s=1}^{l} \tau_s = 0,$$

where each τ_s is a term of $\mathcal{L}_k$ or has the form $F_i(x_{i_1}, \ldots, x_{i_m}) - x_{i_{m+1}}$. Since $K \models \exists \vec{x}\ \psi(\vec{x})$, there exists $\vec{a} \in K^r$ such that $K \models \psi(\vec{a})$. For all $1 \leq s \leq n$, we construct the $\mathcal{L}_{\mathrm{Pf}\restriction,k}$-formulas χ_s as follows.

- If τ_s is an $\mathcal{L}_k$-term, we let χ_s be the formula $\tau_s = 0$.
- If τ_s is of the form $F_i(x_{i_1}, \ldots, x_{i_m}) - x_{i_{m+1}}$ and it is the case that $0 \leq a_{i_1}, \ldots, a_{i_m} \leq 1$, we let χ_s be the formula

  $$F_i(y_{i_1}, \ldots, y_{i_m}) - x_{i_{m+1}} = 0 \wedge \bigwedge_{j \in S} 0 < x_{i_j} < 1,$$

  where $S = \{1 \leq j \leq m \mid 0 < p_{i_j} < 1\}$ and

  $$y_{i_j} = \begin{cases} x_{i_j} & \text{if } j \in S \\ 0 & \text{if } p_{i_j} = 0 \\ 1 & \text{if } p_{i_j} = 1. \end{cases}$$

- If τ_s is of the form $F_i(x_{i_1}, \ldots, x_{i_m}) - x_{i_{m+1}}$ and it is *not* the case that $0 \leq a_{i_1}, \ldots, a_{i_m} \leq 1$, we let χ_s be the formula

  $$x_{i_{m+1}} = 0 \wedge \exists z_1, \ldots z_{2m} \left[\left(\prod_{j=1}^{m} (x_{i_j} \cdot z_j^2 + 1) \right) \cdot \left(\prod_{j=1}^{m} ((1 - x_{i_j}) \cdot z_{m+j}^2 + 1) \right) = 0 \right]$$

  (For each τ_s of this form, we take new variables $z_1, \ldots, z_{2m}$.) Notice that χ_s is $\mathcal{T}_{\mathrm{Pf}\restriction}$-equivalent to

  $$x_{i_{m+1}} = 0 \wedge \left[\bigvee_{j=1}^{m} x_{i_j} < 0 \right] \vee \left[\bigvee_{j=1}^{m} 1 < x_{i_j} \right]$$

If we recall that each F_i vanishes outside the closed unit box, then we see that

$$\mathcal{T}_{\mathrm{Pf}\restriction} \models\ \forall x_1, \ldots, x_r [\chi_s \rightarrow \tau_s = 0],$$

for every $1 \leq s \leq l$. So if we define the formula χ by

$$\bigwedge_{s=1}^{l} \chi_s,$$

then

$$\mathcal{T}_{\mathrm{Pf}\restriction} \models \exists x_1,\ldots,x_r\ \chi(x_1,\ldots,x_r) \to \exists x_1,\ldots,x_r\ \psi(x_1,\ldots,x_r).$$

Now by construction, $K \models \chi(\vec{a})$, so $K \models \exists\vec{x}\ \chi(\vec{x})$. Furthermore, we may push any existential quantifiers present in $\chi(\vec{x})$ to the beginning of this formula, to see that $\exists\vec{x}\ \chi(\vec{x})$ is equivalent to a formula of the form (2). So by our assumption, $k \models \exists\vec{x}\ \chi(\vec{x})$ and hence $k \models \exists\vec{x}\ \psi(\vec{x})$, as desired. □

2.2 (n,r)-sequences and $\vec{\sigma}$-definable points

In broad terms, the proof of Theorem 2.1.1 will be an induction over the number of χ_s of the second form (of Lemma 2.1.6), occurring in χ. In order to systematize this induction process, we need to pad out the set of these χ_s. This is the purpose of the following definition.

Definition 2.2.1. Let $n, r \in \mathbb{N}$.

(i) A sequence $(\sigma_1,\ldots,\sigma_n)$ of terms of $\mathcal{L}_{\mathrm{Pf}\restriction}$ in the variables $x_1,\ldots,x_r$ is called an *(n,r)-sequence* if the following two conditions are satisfied.

- **(a)** For $s = 1,\ldots,n$, the component σ_s has the form $F_i(y_1,\ldots,y_m)$ for some $i = 1,\ldots,l$ and some $y_1,\ldots y_m \in \{0,1,x_1,\ldots,x_r\}$.
- **(b)** If $s = 1,\ldots,n$, $i = 2,\ldots,l$ and $\sigma_s = F_i(y_1,\ldots,y_m)$, then $s > 1$ and for some $t = 1,\ldots,s-1$ holds $\sigma_t = F_{i-1}(y_1,\ldots,y_m)$.

(ii) Those variables actually occurring in some term of an (n,r)-sequence $\vec{\sigma}$ are called *$\vec{\sigma}$-bounded.*

Remark 2.2.2. Before we continue, let us take a moment to look at a few basic properties of these sequences which will be useful to us. Firstly, any (n,r)-sequence $\vec{\sigma}$ is also an (n,r')-sequence for any $r' \geq r$ (and its $\vec{\sigma}$-bounded variables stay the same). Also notice that any initial segment of an (n,r)-sequence $\vec{\sigma}$ is also an (n',r)-sequence for $n' \leq n$. The last thing we note is that if we have a sequence satisfying **(a)** of Definition 2.2.1, but not necessarily **(b)**, then we can rearrange this sequence and pad it out in such a way that the resulting sequence will satisfy both **(a)** and **(b)** and has the same (bounded) variables.

Witnesses to formulas of the form (2) correspond to roots (on some domain) of functions generated by the components of suitable (n,r)-sequences and terms of $\mathcal{L}_k$. We will say more on this in Lemma 2.2.6. But with this in mind, it is reasonable to make the following two definitions.

Definition 2.2.3. Let K be a model of $\mathcal{T}_{\mathrm{Pf}\restriction}$ and suppose $\vec{\sigma}$ is an (n,r)-sequence. The *natural domain* of $\vec{\sigma}$ on K, denoted $D^r(\vec{\sigma},K)$, is defined to be $\prod_{i=1}^r I_i$ where

$$I_i = \begin{cases} \{x \in K \mid 0 < x < 1\} & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded,} \\ K & \text{otherwise.} \end{cases}$$

Definition 2.2.4. Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$ and let $\vec{\sigma}$ be an (n,r)-sequence. We denote by $M^r(k,K,\vec{\sigma})$ the ring of all functions $f : D^r(\vec{\sigma},K) \to K$ for which there exists a polynomial $p(x_1,\ldots,x_r,y_1,\ldots y_n) \in k[x_1,\ldots,x_r,y_1,\ldots y_n]$ such that $f(\vec{\alpha}) = p(\vec{\alpha},\vec{\sigma}(\vec{\alpha}))$ for all $\vec{\alpha} \in D^r(\vec{\sigma},K)$.

Remark 2.2.5. Let us take a look at the properties of the ring $M^r(k,K,\vec{\sigma})$, as given in Definition 2.2.4. First note that it makes sense to talk about a partial derivative, $\frac{\partial g}{\partial x_i}$, (for $i = 1,\ldots,r$) of a function $g \in M^r(k,K,\vec{\sigma})$, since the usual ε-δ definition of limits can be expressed in our language $\mathcal{L}_{\mathrm{Pf}\restriction}$. The ring $M^r(k,K,\vec{\sigma})$ is generated, over k, by the projection functions $x_1,\ldots,x_r$

and the functions $\sigma_1(\vec{x}), \ldots, \sigma_n(\vec{x})$. By Definition 2.2.1 **(i)** **(b)** and Definitions 1.2.1 and 1.2.2, the partial derivative, $\frac{\partial g}{\partial x_i}$, of such a generator $g \in M^r(k, K, \vec{\sigma})$ can be expressed as a polynomial in these generators (with coefficients in k, by Definition 1.2.2) and hence $\frac{\partial g}{\partial x_i} \in M^r(k, K, \vec{\sigma})$. By a simple application of the sum and product rule for derivatives, it follows that $M^r(k, K, \vec{\sigma})$ is closed under differentiation; it is a *differential ring.* Note that this implies in particular that the elements of $M^r(k, K, \vec{\sigma})$ are C^∞-functions.

Furthermore, by Proposition A.1.5, $M^r(\mathbb{R}, \mathbb{R}, \vec{\sigma})$ is an integral domain. By quantifying out parameters of elements $p(\vec{x}, \vec{\sigma}(\vec{x})) \in M^r(k, K, \vec{\sigma})$, this fact clearly transfers to $M^r(k, K, \vec{\sigma})$ (for an explanation of what these terms mean, please see Remark 3.1.1).

Lastly, we note that $M^r(k, K, \vec{\sigma})$ is Noetherian, as it is finitely generated over the field k.

Now let us clarify and prove the assertion we made following Remark 2.2.2.

Lemma 2.2.6. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, such that $k \subseteq K$ and suppose that for all $n, r \in \mathbb{N}$, all (n, r)-sequences $\vec{\sigma}$ and all $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ holds that if $g_1, \ldots, g_l$ have a common zero in $D^r(\vec{\sigma}, K)$, then they have a common zero in $D^r(\vec{\sigma}, k)$. Then k is existentially closed in K.*

Proof. Suppose that $K \models \exists x_1, \ldots, x_r\ \chi$, where χ is of the form (2) as described in Lemma 2.1.6. By Remark 2.2.2, we can arrange and pad out the set of functions of the form $F_i(y_{i_1}, \ldots, y_{i_m})$ appearing in the definitions of the χ_s of which χ is composed, into an (n, r)-sequence, $\vec{\sigma}$ say, for some $n, r \in \mathbb{N}$ (and in such a way that we do not introduce additional bounded variables). Then every χ_s simply states that some function $g_s \in M^r(k, K, \vec{\sigma})$ has a zero in some subset of K^r. Using this, one readily verifies that there exist $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ such that $K \models \exists x_1, \ldots, x_r\ \chi$ if and only if $g_1, \ldots, g_l$ have a common zero in $D^r(\vec{\sigma}, K)$. By the same reasoning $k \models \exists x_1, \ldots, x_r\ \chi$ if and only if $g_1, \ldots, g_l$ have a common zero in $D^r(\vec{\sigma}, k)$. The Lemma now follows by applying Lemma 2.1.6. ☐

For the next definition we make, which will play a central role in our proof, we introduce the following notation. Given $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$, an (n, r)-sequence $\vec{\sigma}$, functions $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ and indices $1 \leq i_1, \ldots, i_m \leq r$, we write

$$\frac{\partial(g_1, \ldots, g_l)}{\partial(x_{i_1}, \ldots, x_{i_m})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_{i_1}} & \cdots & \frac{\partial g_1}{\partial x_{i_m}} \\ \vdots & & \vdots \\ \frac{\partial g_l}{\partial x_{i_1}} & \cdots & \frac{\partial g_l}{\partial x_{i_m}} \end{pmatrix}$$

Definition 2.2.7. Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Also, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence. Then a point $P \in K^r$ is called $(k, \vec{\sigma})$-*definable* if there exist $g_1, \ldots, g_r \in M^r(k, K, \vec{\sigma})$ such that the following conditions are satisfied.

(i) $P \in D^r(\vec{\sigma}, K)$.

(ii) $g_1(P) = \cdots = g_r(P) = 0$.

(iii) $\det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(P) \neq 0$.

2.3 Proof of the First Main Theorem

The proof of Theorem 2.1.1 splits into proving the following three Lemmas.

Lemma 2.3.1. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence. Suppose that $g \in M^r(k, K, \vec{\sigma})$ and $g(P) = 0$ for some $P \in D^r(\vec{\sigma}, K)$. Then for some $s \in \mathbb{N}$ there exist $Q_1 \in D^r(\vec{\sigma}, K)$ and $Q_2 \in K^s$ such that $g(Q_1) = 0$ and (Q_1, Q_2) is $(k, \vec{\sigma})$-definable.*

Lemma 2.3.2. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n,r)-sequence. Suppose also that for each $s \geq r$ and each $(k,\vec{\sigma})$-definable point $(p_1,\ldots,p_s)$ of K^s, there is some $B \in k$ such that $-B < p_1,\ldots,p_s < B$. Then every $(k,\vec{\sigma})$-definable point of K^r lies in k^r.*

Lemma 2.3.3. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Let $n, r \in \mathbb{N}$ and suppose that $\vec{\sigma}' = (\sigma_1,\ldots,\sigma_{n+1})$ is an $(n+1,r)$-sequence. Let $\vec{\sigma}$ denote the (n,r)-sequence $(\sigma_1,\ldots,\sigma_n)$. Suppose that for each $s \geq r$, every $(k,\vec{\sigma})$-definable point of K^s lies in k^s. Then for each $s \geq r$ and each $(k,\vec{\sigma}')$-definable point $(p_1,\ldots,p_s)$ of K^s, there is some $B \in k$ such that $-B < p_1,\ldots,p_s < B$.*

We will present the proof of Theorem 2.1.1 using these three Lemmas momentarily, but first we prove two Lemmas, whose Corollary will provide us with the base case of an inductive argument which will combine Lemmas 2.3.2 and 2.3.3.

Lemma 2.3.4. *Suppose that k and K are models of the theory $\mathcal{T}$, with $k \subseteq K$. Let also $r \in \mathbb{N}$ and $g_1,\ldots,g_r \in k[x_1,\ldots,x_r]$. If $g_1(Q) = \cdots = g_r(Q) = 0$ and $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(Q) \neq 0$, with $Q \in K^r$, then each coordinate Q_i of $Q = (Q_1,\ldots,Q_r)$ is algebraic over k.*

Proof. Assume that $g_1,\ldots,g_r$ and Q satisfy the premise of the lemma. We work in $A = \mathrm{acl}(k)$, the algebraic closure of k. The ideal $\{f \in A[x_1,\ldots,x_r] \mid f(Q) = 0\}$ is readily seen to be a prime ideal of $A[x_1,\ldots,x_r]$, which we shall call $\mathfrak{p}$. Now if we let

$$\mathcal{V}(I) = \{P \in A^r \mid f(P) = 0 \textit{ for all } f \in I\}$$

denote the affine variety given by an ideal $I \subseteq A[x_1,\ldots,x_r]$ and we let

$$\mathcal{I}(X) = \{f \in A[x_1,\ldots,x_r] \mid f(P) = 0 \textit{ for all } P \in X\}$$

denote the vanishing ideal of a set $X \in A$, then applying Hilberts Nullstellensatz to $\mathfrak{p}$ gives $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \sqrt{\mathfrak{p}}$. Since $\mathfrak{p}$ is prime, it is equal to its own radical ideal, so $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \mathfrak{p}$. Now if $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(P) = 0$ for all $P \in \mathcal{V}(\mathfrak{p})$, then from this it would follow that $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right) \in \mathfrak{p}$, which is false by definition of $\mathfrak{p}$. Thus, we may take a point $P \in \mathcal{V}(\mathfrak{p})$, such that $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(P) \neq 0$. We define the maximal ideal $\mathfrak{m}$ by $\mathfrak{m} = (x_1 - P_1,\ldots,x_r - P_r)$. Clearly $\mathfrak{m} \subseteq \mathcal{I}(\{P\})$, so $\mathfrak{m} = \mathcal{I}(\{P\})$, since $\mathfrak{m}$ is maximal and $\mathcal{I}(\{P\})$ is proper. Hence $\mathfrak{p} \subseteq \mathfrak{m}$, as $\mathcal{I}(\mathcal{V}(\mathfrak{p})) \subseteq \mathcal{I}(\{P\})$.

We wish to prove that $\mathfrak{p} + \mathfrak{m}^2 = \mathfrak{m}$. The inclusion $\mathfrak{p} + \mathfrak{m}^2 \subseteq \mathfrak{m}$ is clear, so it suffices to show that $x_i - P_i \in \mathfrak{p} + \mathfrak{m}^2$, for each $i = 1,\ldots,r$. To this end we make a Taylor expansion of the g_j, with base point P, as follows

$$g_j(x_1,\ldots,x_r) = g_j(P) + \sum_{i=1}^{r} \frac{\partial g_j}{\partial x_i}(P)\cdot(x_i - P_i) + h_j(x_1,\ldots,x_r).$$

The polynomial h_j consists of higher order terms, which therefore all must contain a factor of the form $(x_i - P_i)(x_{i'} - P_{i'})$. In other words $h_j \equiv 0 \pmod{\mathfrak{m}^2}$. Also note that $g_j(P) = 0$ for each $1 \leq j \leq r$, as $P \in \mathcal{V}(\mathfrak{p})$, to arrive at

$$g_j(x_1,\ldots,x_r) \equiv \sum_{i=1}^{r} \frac{\partial g_j}{\partial x_i}(P)\cdot(x_i - P_i) \pmod{\mathfrak{m}^2}.$$

We can combine these r equations into the vector equation

$$\begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} \equiv \frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}(P)\cdot\begin{pmatrix} x_1 - P_1 \\ \vdots \\ x_r - P_r \end{pmatrix} \pmod{\mathfrak{m}^2}.$$

Since $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(P) \neq 0$, the matrix $\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}(P)$ has an inverse, say M, with coefficients in A. Applying this inverse gives

$$M \cdot \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} \equiv \begin{pmatrix} x_1 - P_1 \\ \vdots \\ x_r - P_r \end{pmatrix} \pmod{\mathfrak{m}^2}.$$

and hence for $i = 1, \ldots, r$, we have

$$\sum_{j=1}^{r} M_{i,j} g_j \equiv x_i - P_i \pmod{\mathfrak{m}^2}.$$

We conclude that $x_i - P_i \in \mathfrak{p} + \mathfrak{m}^2$, for each $i = 1, \ldots, r$, as $g_1, \ldots g_r \in \mathfrak{p}$.

If we localize at the maximal ideal $\mathfrak{m}$, then we find $\mathfrak{p}_\mathfrak{m} + \mathfrak{m}_\mathfrak{m}^2 = (\mathfrak{p} + \mathfrak{m}^2)_\mathfrak{m} = \mathfrak{m}_\mathfrak{m}$. It follows that $\mathfrak{m}_\mathfrak{m} \cdot (\mathfrak{m}_\mathfrak{m}/\mathfrak{p}_\mathfrak{m}) = (\mathfrak{p}_\mathfrak{m} + \mathfrak{m}_\mathfrak{m}^2)/\mathfrak{p}_\mathfrak{m} = \mathfrak{m}_\mathfrak{m}/\mathfrak{p}_\mathfrak{m}$. Since $\mathfrak{m}_\mathfrak{m}/\mathfrak{p}_\mathfrak{m}$ is finitely generated as an $A[x_1, \ldots, x_r]_\mathfrak{m}$-module and $\mathfrak{m}_\mathfrak{m}$ is the unique maximal ideal of $A[x_1, \ldots, x_r]_\mathfrak{m}$, we can apply Nakayama's Lemma to conclude that $\mathfrak{m}_\mathfrak{m}/\mathfrak{p}_\mathfrak{m} = \{0\}$ and hence $\mathfrak{m}_\mathfrak{m} = \mathfrak{p}_\mathfrak{m}$.

This implies $\mathfrak{m} = \mathfrak{p}$. For if we take some element $m \in \mathfrak{m}$, then $m \in \mathfrak{p}_\mathfrak{m}$, as $\mathfrak{m} \subseteq \mathfrak{m}_\mathfrak{m}$. So we may write $m = \frac{p}{u}$ for some $p \in \mathfrak{p}$ and $u \in A[x_1, \ldots, x_r] \setminus \mathfrak{m}$. Then $mu = p$, so $mu \in \mathfrak{p}$. But $u \notin \mathfrak{p}$, as $\mathfrak{p} \subseteq \mathfrak{m}$, so $m \in \mathfrak{p}$, since $\mathfrak{p}$ is prime. Now $x_i - P_i \in \mathfrak{m}$, so $x_i - P_i \in \mathfrak{p}$, which means that $Q_i - P_i = 0$, by definition of $\mathfrak{p}$. Since $P_i \in A = \operatorname{acl}(k)$, we conclude that each Q_i is algebraic over k. □

Lemma 2.3.5. *Let K be a model of the theory $\mathcal{T}$. Furthermore, let $n \in \mathbb{N}$ and suppose that the polynomial $f \in K[x_1, \ldots, x_n]$ vanishes on K^n. Then f is the zero polynomial.*

Proof. We use induction over n. For $n = 0$, we have $f \in K$, so the statement clearly holds. Now suppose that the Lemma holds for $n \in \mathbb{N}$. Take $f \in K[x_1, \ldots, x_{n+1}]$ and suppose that f vanishes on K^{n+1}. Then for any $p \in K$, the polynomial $f(x_1, \ldots, x_n, p)$ vanishes on K^n, so by our induction hypothesis $f(x_1, \ldots, x_n, p)$ is the zero polynomial. So if we view f as $f(x_n) \in K[x_1, \ldots, x_n][x_{n+1}]$, then f has infinitely many roots. Since $K[x_1, \ldots, x_n]$ is a domain, it follows that f must be the zero polynomial. This concludes our induction. □

Corollary 2.3.6. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$ and let $r \in \mathbb{N}$. Then every $(k, \emptyset)$-definable point of K^r lies in k^r.*

Proof. Suppose that $Q \in K^r$ is $(k, \emptyset)$-definable. By Lemma 2.3.5, the kernel of the natural ring homomorphism $k[x_1, \ldots, x_r] \to M^r(k, K, \emptyset)$ is trivial, so we may identify the ring $M^r(k, K, \emptyset)$ with $k[x_1, \ldots, x_r]$, as the homomorphism is also clearly surjective. So by definition, there exist $g_1, \ldots, g_r \in k[x_1, \ldots, x_r]$ such that $g_1(Q) = \cdots = g_r(Q) = 0$ and $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(Q) \neq 0$. Lemma 2.3.4 tells us that each coordinate Q_i of $Q = (Q_1, \ldots, Q_r)$ is algebraic over k. Fix i and let f be a nonzero polynomial with coefficients in k such that $K \models f(Q_i) = 0$. Let $n \in \mathbb{N}$ be the number of distinct roots of f in K. We define the $\mathcal{L}_k$-sentence ϕ as

$$\exists x_1, \ldots, x_n \left[\left(\bigwedge_{1 \leq s < t \leq n} x_s \neq x_t \right) \wedge \forall y \left(f(y) = 0 \to \bigvee_{s=1}^{n} x_s = y \right) \right],$$

which states that f has exactly n distinct roots. Clearly $K \models \phi$. Since $k \subseteq K$ and both k and K are models of $\mathcal{T}_{\mathrm{Pf}\restriction}$, which is complete, $k \models \phi$ must hold as well. But if $P_1, \ldots, P_n \in k$ are the distinct roots of f in k, then also $K \models f(P_j) = 0$ for all j. This means that Q_i must be among $P_1, \ldots, P_n$. We conclude that $Q_i \in k$ and hence $Q \in k^r$, as i was chosen arbitrarily. □

This Corollary provides us with the means to prove the following.

Lemma 2.3.7. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Then for all $n, r \in \mathbb{N}$ and any (n,r)-sequence $\vec{\sigma}$, every $(k,\vec{\sigma})$-definable point of K^r lies in k^r.*

Proof. The proof is by induction over n, for all values of r simultaneously. The base step, $n = 0$, is just Corollary 2.3.6.

Now suppose that $\vec{\sigma}' = (\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$ is an $(n+1, r)$-sequence, for some $r \in \mathbb{N}$ and suppose that the of the Lemma holds for $n \in \mathbb{N}$. Let $\vec{\sigma}$ be the initial segment $(\sigma_1, \ldots, \sigma_n)$ of $\vec{\sigma}'$. Then it follows from our induction hypothesis that for every $s \geq r$, every $(k, \vec{\sigma})$-definable point of K^s lies in k^s. Then Lemma 2.3.3 tells us that for each $s \geq r$ and each $(k, \vec{\sigma}')$-definable point $(p_1, \ldots, p_s)$ of K^s, there is some $B \in k$ such that $-B < p_1, \ldots, p_s < B$. But then by Lemma 2.3.2, each $(k, \vec{\sigma}')$-definable point of K^r lies in k^r, which completes our induction. □

We finish this section by giving the proof of the First Main Theorem. (But keep in mind that we still have to give the proofs of Lemmas 2.3.1, 2.3.2 and 2.3.3.)

Proof. (Of Theorem 2.1.1.) Let k and K be arbitrary models of $\models \mathcal{T}_{\mathrm{Pf}\restriction}$, such that $k \subseteq K$. We wish to apply Lemma 2.2.6. So, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n,r)-sequence and suppose that $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ have a common zero in $D^r(\vec{\sigma}, K)$. Note that a point $P \in D^r(\vec{\sigma}, K)$ is a common zero of $g_1, \ldots, g_l$ if and only if it is a zero of $g = \sum_{i=1}^{l} g_i^2$, which is also an element of $M^r(k, K, \vec{\sigma})$. We can now apply Lemma 2.3.1, which shows that some $s \in \mathbb{N}$, there exist $Q_1 \in D^r(\vec{\sigma}, K)$ and $Q_2 \in K^s$ such that $g(Q_1) = 0$ and (Q_1, Q_2) is $(k, \vec{\sigma})$-definable. By Lemma 2.3.7, (Q_1, Q_2) lies in k^r. Hence, Q_1 is a common zero of $g_1, \ldots, g_l$ in $D^r(\vec{\sigma}, k)$. So, by Lemma 2.2.6, k is existentially closed in K. Since k and K where arbitrary models of $\models \mathcal{T}_{\mathrm{Pf}\restriction}$, it follows from Corollary 2.1.4 that the theory $\models \mathcal{T}_{\mathrm{Pf}\restriction}$ is model complete. □

3 Towards Lemma 2.3.1

3.1 Germs and transfer

Before we go on to the main topic of this section, we take a brief moment to discus a technique from model theory called the *transfer principle*, as we will need to apply it in the upcoming proofs. In most of our cases, this is simply a somewhat disguised application of the fact that the theories we work in are complete. We present this discussion in the form of a Remark.

Remark 3.1.1. The principle of transfer concerns the relation between the truth of a certain statement in some structure and the truth of this same statement in another structure. It is perhaps best illustrated by means of an example. We take $\mathcal{T}_{\sin} = \mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\sin})$, where we defined $\mathcal{L}_{\sin} = \mathcal{L} \cup \{\sin\}$. Now suppose that K is another model of the theory $\mathcal{T}_{\sin}$. We take some $a \in K$ and wonder if $\sin(a \cdot x)$ is continuous in K, as a function of x, at a certain point $b \in K$. In other words, if we define the $\mathcal{L}_{\sin}$-formula $\varphi(y, z)$ by

$$\forall\varepsilon[0 < \varepsilon \to \exists\delta[0 < \delta \wedge \forall x[(z-\delta < x \wedge x < z+\delta) \to (\sin(y\cdot z)-\varepsilon < \sin(y\cdot x) \wedge \sin(y\cdot x) < \sin(y\cdot z)+\varepsilon)]]],$$

then we wonder if $K \models \varphi(a, b)$. Fortunately for us, we can "quantify out" the parameters a and b in this case. By this we mean that we can dispose of a and b by introducing two universal quantifiers, that is, we choose to show that $K \models \forall y \forall z\ \varphi(y, z)$, as this is certainly sufficient. Since $K \models \mathcal{T}_{\sin}$ and $\mathbb{R} \models \forall y \forall z\ \varphi(y, z)$, it is clearly the case that $K \models \forall y \forall z\ \varphi(y, z)$, so we are done. We have *transferred* a certain property from the structure $\mathbb{R}$ to the structure K. We can apply this principle in a more general setting, as long the property we wish to transfer can be expressed in the language we are working in. As we have seen in our example, even a simple property such as continuity leads to a relatively large formula. In our use of the transfer principle we shall therefore be a less formal and only give further details if our use is not straightforward.

From now on, we let $\mathcal{L}_{\mathcal{A}}$ be any extension of the language $\mathcal{L}$, meaning that $\mathcal{L}_{\mathcal{A}} = \mathcal{L} \cup \mathcal{A}$, for some arbitrary set of symbols $\mathcal{A}$. We also set $\mathcal{T}_{\mathcal{A}} = \mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\mathcal{A}})$. The methods we will develop in this section will first be used to give a proof of Lemma 2.3.1, but we shall be reapply them further on, which is why some of the results will be formulated in the more general setting of the theory $\mathcal{T}_{\mathcal{A}}$.

Definition 3.1.2. Let $K \models \mathcal{T}_{\mathcal{A}}$ and $n \in \mathbb{N}$, with $n \geq 1$. A *neighborhood system* $\mathcal{B}$ in K^n is a nonempty collection of nonempty definable open subsets of K^n, such that if $U_1, U_2 \in \mathcal{B}$, then also $U_1 \cap U_2 \in \mathcal{B}$.

Example 3.1.3. To give an example of a neighborhood system in K^n, let $P \in K^n$. We let $\mathcal{B}_P$ denote the set of all definable open neighborhoods of P. It is clear that $U_1, U_2 \in \mathcal{B}_P$, implies that $U_1 \cap U_2 \in \mathcal{B}$, so $\mathcal{B}_P$ is a neighborhood system in K^n.

We shall encounter $\mathcal{B}_P$ frequently, but for now let us look at a general neighborhood system $\mathcal{B}$ in K^n.

Definition 3.1.4. Consider pairs (U, f), where $U \in \mathcal{B}$ and $f : U \to K$ is an infinitely differentiable definable function. We denote the set of these pairs by $\mathcal{D}_{\mathcal{B}}^{\not\sim}$. We call two such pairs (f_1, U_1) and (f_2, U_2) equivalent if f_1 and f_2 restrict to the same function on some $U_3 \subseteq U_1 \cap U_2$, with $U_3 \in \mathcal{B}$. Let $[f, U]$ denote the equivalence class of (f, U). The equivalence classes, called germs, form a ring $\mathcal{D}_{\mathcal{B}}$, with addition and multiplication given by

$$[f_1, U_1] + [f_2, U_2] = [f_1 + f_2, U_3] \text{ and } [f_1, U_1] \cdot [f_2, U_2] = [f_1 \cdot f_2, U_3],$$

where $U_3 = U_1 \cap U_2$. Here it is implied that f_1 and f_2 are restricted to functions on U_3.

Remark 3.1.5. It is easily checked that addition and multiplication are well-defined on equivalence classes. We can add more structure to the ring $\mathcal{D}_{\mathcal{B}}$ by defining the derivatives $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$: $\mathcal{D}_{\mathcal{B}} \to \mathcal{D}_{\mathcal{B}}$, as follows

$$\frac{\partial}{\partial x_i}[f, U] = [\frac{\partial f}{\partial x_i}, U],$$

for $i = 1, \ldots, n$. Once again, one readily verifies that this operation is well-defined. This makes $\mathcal{D}_{\mathcal{B}}$ into a differential ring.

We return to Example 3.1.3, the neighborhood system $\mathcal{B}_P$, for $P \in K^n$. In this case we write $\mathcal{D}_P^{\not\partial}$ and $\mathcal{D}_P$ for $\mathcal{D}_{\mathcal{B}_P}^{\not\partial}$ and $\mathcal{D}_{\mathcal{B}_P}$ respectively. Since the point P is contained in every $U \in \mathcal{B}_P$, it is a meaningful question to ask for *the* value of a germ or its derivative at the point P. In the subsequent parts, we shall therefore write $g(P)$ when we mean $f(P)$, if $g = [f, U] \in \mathcal{B}_P$. Furthermore, we write either $d_P g$ or $d_P f$ for the vector $(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P))$ depending on convenience.

Lemma 3.1.6. *Let $K \models \mathcal{T}_{\mathcal{A}}$ and $n \in \mathbb{N}$, with $n \geq 1$, and suppose $\mathcal{B}$ is a neighborhood system in K^n. Suppose also that M is a subring of $\mathcal{D}_{\mathcal{B}}$ which is closed under differentiation and that I is a finitely generated ideal of M also closed under differentiation. Let $\{[g_1, U_1], \ldots, [g_s, U_s]\}$ be a finite set of generators for I and take*

$$Z = \{P \in \bigcap_{i=1}^{s} U_i \mid g_1(P) = \cdots = g_s(P) = 0\}.$$

Then for some $U \in \mathcal{B}$, the set $U \cap Z$ is a definable open subset of K^n.

Proof. Since I is closed under differentiation, there exist definable functions $a_{i,j}^r$, with $1 \leq i, j \leq s$ and $1 \leq r \leq n$, such that

$$\frac{\partial g_i}{\partial x_r} = \sum_{j=1}^{s} a_{i,j}^r g_j \tag{3}$$

holds for every $i = 1, \ldots, s$ and $r = 1, \ldots, n$ on some definable common domain $U \in \mathcal{B}$, which we obtain by intersecting domains if necessary. This does not pose a problem, as $\mathcal{B}$ is closed under finite intersection. Notice that this means in particular that $U \subseteq \bigcap_{i=1}^{s} U_i$. We claim that $U \cap Z$ is open in K^n. To show this, we take $P \in U \cap Z$ and $U_0 \subseteq U$ an open box containing P. It is certainly possible to take such a box, as U is open. We are done if we manage to prove that $U_0 \subseteq Z$. Since $U_0 \subseteq \bigcap_{i=1}^{s} U_i$, this means that we need to show that each $g_1, \ldots, g_s$ vanishes on U_0. Suppose that this is not the case. Then there exists $Q \in U_0$ such that $g_i(Q) \neq 0$ for at least one $i = 1, \ldots, s$. We write $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$ and consider the following sequence of vectors

$$\begin{aligned} P = Q_0 =& (p_1, p_2 \ldots, p_{n-1}, p_n) \\ Q_1 =& (q_1, p_2, \ldots, p_{n-1}, p_n) \\ Q_2 =& (q_1, q_2, \ldots, p_{n-1}, p_n) \\ &\vdots \\ Q_{n-1} =& (q_1, q_2 \ldots, q_{n-1}, p_n) \\ Q = Q_n =& (q_1, q_2, \ldots, q_{n-1}, q_n). \end{aligned}$$

Note that each of these vectors lies inside U_0, as U_0 is box shaped. Since $g_i(P) = 0$ for all $i = 1, \ldots, s$ and $g_i(Q) \neq 0$ for some $i = 1, \ldots, s$, there must be a least index m such that $g_i(Q_m) \neq 0$ for some $i = 1, \ldots, s$. Then Q_m and Q_{m-1} differ in exactly one coordinate and $g_i(Q_{m-1}) = 0$ for all $i = 1, \ldots, s$ by minimality of m. This is all we need to go to the next step in our proof, but to simplify our argument somewhat, from now on we assume that we have points $Q = (q_1, \ldots, q_n)$ and $Q' = (q_1', q_2, \ldots, q_n)$, both lying in U_0, such that $g_i(Q) \neq 0$ for at least one $i = 1, \ldots, s$ and $g_i(Q') = 0$ for all $i = 1, \ldots, s$.

We take (a, b) to be an open interval in K, containing the points q_1 and q_1', such that $(a, b) \times \{(q_2, \ldots, q_n)\} \subseteq U_0$. For any function $f : U_0 \to K$, we let $\overline{f} : (a, b) \to K$ be the result of substituting q_i for x_i in f for $i = 2, \ldots, n$. Applying this to (3) for $r = 1$, gives us the vector equation

$$\begin{pmatrix} \frac{d\overline{g}_1}{dx_1} \\ \vdots \\ \frac{d\overline{g}_s}{dx_1} \end{pmatrix} = \begin{pmatrix} \overline{a^1_{1,1}} & \cdots & \overline{a^1_{1,s}} \\ \vdots & & \vdots \\ \overline{a^1_{s,1}} & \cdots & \overline{a^1_{s,s}} \end{pmatrix} \cdot \begin{pmatrix} \overline{g}_1 \\ \vdots \\ \overline{g}_s \end{pmatrix}$$

which holds for all $x_1 \in (a, b)$.

Since we are working with definable functions, we can transfer this situation to $\mathbb{R}$, by quantifying out the parameters. By this procedure we obtain an interval (c, d) in $\mathbb{R}$, points $q, q' \in (c, d)$ and continuously differentiable functions $h_i, b_{i,j} : (c, d) \to \mathbb{R}$, with $1 \leq i, j \leq s$, such that

$$\begin{pmatrix} \frac{dh_1}{dx_1} \\ \vdots \\ \frac{dh_s}{dx_1} \end{pmatrix} = \begin{pmatrix} b_{1,1} & \cdots & b_{1,s} \\ \vdots & & \vdots \\ b_{s,1} & \cdots & b_{s,s} \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ \vdots \\ h_s \end{pmatrix}$$

for all $x \in (c, d)$. Furthermore, $h_i(q') = 0$ for all $i = 1, \ldots, s$ and $h_i(q) \neq 0$ for some $i = 1, \ldots, s$. The theory of linear differential equations teaches us that there exists an $s \times s$ matrix

$$C = \begin{pmatrix} c_{1,1} & \cdots & c_{1,s} \\ \vdots & & \vdots \\ c_{s,1} & \cdots & c_{s,s} \end{pmatrix}$$

whose entries are functions $c_{i,j} : (c, d) \to \mathbb{R}$ and is invertible for all $x \in (c, d)$, such that

$$\begin{pmatrix} h_1(x) \\ \vdots \\ h_s(x) \end{pmatrix} = C(x)^{-1} \cdot C(q') \cdot \begin{pmatrix} h_1(q') \\ \vdots \\ h_s(q') \end{pmatrix}$$

(For a proof of this fact we refer to [Mir55].) Substituting $x = q$ in this equation gives the desired contradiction, since on the one hand the linear map $C(q)^{-1} \cdot C(q')$ has a trivial kernel and on the other hand $(h_1(q), \ldots, h_s(q))^T$ is the zero vector, but $(h_1(q'), \ldots, h_s(q'))^T$ is not. □

3.2 The Implicit Function Theorem and the hat homomorphism

Recall the statement of the Implicit Function Theorem.

Theorem 3.2.1. *Let $d \in \mathbb{N} \cup \{\infty\}$ and suppose that U is open in $\mathbb{R}^{r+m}$ and $f_1, \ldots, f_m : U \to \mathbb{R}$ are C^d-functions. Assume that $(P, Q) \in U$ and $f_1(P, Q) = \ldots = f_m(P, Q) = 0$. Suppose* *furthermore that the determinant of the matrix*

$$\Delta = \begin{pmatrix} \frac{\partial f_1}{\partial x_{r+1}} & \cdots & \frac{\partial f_1}{\partial x_{r+m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{r+1}} & \cdots & \frac{\partial f_m}{\partial x_{r+m}} \end{pmatrix}$$

is nonzero at the point (P, Q). *Then there exist open neighborhoods* V_1 *of* P *and* V_2 *of* Q *with the following properties.*

(i) $V_1 \times V_2 \subseteq U$.

(ii) *For each* $\vec{x} \in V_1$ *there exists a unique point* $\vec{y} \in V_2$ *such that* $f_1(\vec{x}, \vec{y}) = \ldots f_m(\vec{x}, \vec{y}) = 0$. *This point satisfies* $\det(\Delta(\vec{x}, \vec{y})) \neq 0$.

(iii) *In this way we obtain* C^d *mappings* $\psi_1, \ldots, \psi_m : V_1 \to \mathbb{R}$ *satisfying* $\vec{\psi}(\vec{x}) = \vec{y}$. *Furthermore, for* $l = 1, \ldots, r$ *and* $\vec{x} \in V_1$ *we have*

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_l} \\ \vdots \\ \frac{\partial \psi_m}{\partial x_l} \end{pmatrix} = -\Delta^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_l} \\ \vdots \\ \frac{\partial f_m}{\partial x_l} \end{pmatrix}$$

when the left hand side is evaluated in the point $\vec{x}$ *and the right hand side is evaluated in the point* $(\vec{x}, \psi_1(\vec{x}), \ldots, \psi_m(\vec{x}))$.

Proof. A proof of this can be found in [DK04]. □

As is the case with many results from real analysis, Theorem 3.2.1 holds in arbitrary $K \models \mathcal{T}_{\mathcal{A}}$, as long as we restrict ourselves to definable sets and functions.

Theorem 3.2.2. *Suppose that* $K \models \mathcal{T}_{\mathcal{A}}$. *Let* $d \in \mathbb{N} \cup \{\infty\}$ *and suppose that* U *is a definable open in* K^{r+m} *and* $f_1, \ldots, f_m : U \to K$ *are definable* C^d*-functions. Assume that* $(P, Q) \in U$ *and* $f_1(P, Q) = \ldots = f_m(P, Q) = 0$. *Suppose furthermore that the determinant of the matrix*

$$\Delta = \begin{pmatrix} \frac{\partial f_1}{\partial x_{r+1}} & \cdots & \frac{\partial f_1}{\partial x_{r+m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{r+1}} & \cdots & \frac{\partial f_m}{\partial x_{r+m}} \end{pmatrix}$$

is nonzero at the point (P, Q). *Then there exist definable open neighborhoods* V_1 *of* P *and* V_2 *of* Q *with the following properties.*

(i) $V_1 \times V_2 \subseteq U$.

(ii) *For each* $\vec{x} \in V_1$ *there exists a unique point* $\vec{y} \in V_2$ *such that* $f_1(\vec{x}, \vec{y}) = \ldots f_m(\vec{x}, \vec{y}) = 0$. *This point satisfies* $\det(\Delta(\vec{x}, \vec{y})) \neq 0$.

(iii) *In this way we obtain definable* C^d *mappings* $\psi_1, \ldots, \psi_m : V_1 \to K$ *satisfying* $\vec{\psi}(\vec{x}) = \vec{y}$. *Furthermore for* $l = 1, \ldots, r$ *and* $\vec{x} \in V_1$ *we have*

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_l} \\ \vdots \\ \frac{\partial \psi_m}{\partial x_l} \end{pmatrix} = -\Delta^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_l} \\ \vdots \\ \frac{\partial f_m}{\partial x_l} \end{pmatrix}$$

when the left hand side is evaluated in the point $\vec{x}$ *and the right hand side is evaluated in the point* $(\vec{x}, \psi_1(\vec{x}), \ldots, \psi_m(\vec{x}))$.

Proof. Suppose that $U \subseteq \mathbb{R}^{r+m}$ is a definable open set, $(P,Q) \in U$ and $f_1, \ldots, f_m : U \to \mathbb{R}$ are definable functions which satisfy the hypothesis of Theorem 3.2.1. Let V_1, V_2 and $\vec{\psi}$ be as in the conclusion of Theorem 3.2.1. Now take some open box V_2' inside V_2 such that $Q \in V_2'$. Furthermore, take some open box V_1' in the preimage $\vec{\psi}^{-1}(V_2')$. Then the conclusion of Theorem 3.2.1 holds with V_1 and V_2 replaced by V_1' and V_2' respectively. So we may assume that V_1 and V_2 are of this shape and are therefore definable. But this means that (this stronger version of) Theorem 3.2.1 can be fully expressed in the language $\mathcal{L}_{\mathcal{A}}$. Since $\mathbb{R}$ and K are both models of the complete theory $\mathcal{T}_{\mathcal{A}}$, this means that the Theorem must also hold in K. □

Remark 3.2.3. It important to note that the functions $\psi_1(\vec{x}), \ldots, \psi_m(\vec{x})$ are definable, since they can be defined in terms of the functions $f_1, \ldots, f_m$.

The reason why we went trough the trouble of deriving Theorem 3.2.2 from Theorem 3.2.1 will become clear in the following part. For suppose that $K \models \mathcal{T}_{\mathcal{A}}$ and we are given a point $(P,Q) \in K^{r+m}$, a definable open $U \subseteq K^{r+m}$ containing this point and definable C^∞- functions $f_1, \ldots, f_m : U \to K$, satisfying the hypothesis of Theorem 3.2.2. Let V_1, V_2 and $\psi_1, \ldots, \psi_m$ be as in the conclusion of the Theorem. We write $n = r + m$ and we define the functions $\phi_1, \ldots, \phi_n : V_1 \to K$ by

$$\phi_i(\vec{x}) = \begin{cases} x_i & \text{if } i = 1, \ldots, r \\ \psi_{i-r}(\vec{x}) & \text{if } i = r+1, \ldots, n \end{cases}$$

Furthermore, we let $\phi : V_1 \to K^n$ be defined by $\phi(\vec{x}) = (\phi_1(\vec{x}), \ldots, \phi_n(\vec{x}))$. Since $\phi(P) = (P,Q)$ and each $\phi_1, \ldots, \phi_n$ is definable and infinitely differentiable, we can use this to define a mapping $\widehat{\cdot} : \mathcal{D}_{P,Q} \to \mathcal{D}_P$ which maps the germ

$$g = [f, V] \quad \text{to} \quad \hat{g} = [f \circ \phi, W],$$

where $W = V_1 \cap \phi^{-1}(V)$. In this case we shall also denote the function $f \circ \phi : W \to K$ by $\hat{f}$. One easily verifies that $W \in \mathcal{B}_P$ and that the map $\widehat{\cdot} : \mathcal{D}_{P,Q} \to \mathcal{D}_P$ is well-defined on equivalence classes. Another quick inspection reveals that this map is in fact a ring homomorphism.

Remark 3.2.4. We take a closer look at the kernel of $\widehat{\cdot}$, since we will be needing this later on. We claim that $\ker(\widehat{\cdot})$ consists of precisely those germs $g = [f, V]$ such that f vanishes on $W \cap Z$, for some $W \in \mathcal{B}_P$, with $W \subseteq V$ and

$$Z = \{(\vec{x}, \vec{y}) \in U \mid f_i(\vec{x}, \vec{y}) = 0, \text{ for } i = 1, \ldots, m\}.$$

On the one hand, if f vanishes on $W \cap Z$, then $f \circ \phi$ vanishes on $\phi^{-1}(W)$, whence

$$\hat{g} = [f \circ \phi, \phi^{-1}(W)] = 0,$$

from which we conclude that $g \in \ker(\widehat{\cdot})$. On the other hand, if $g \in \ker(\widehat{\cdot})$, then $f \circ \phi$ vanishes on some $W_1 \in \mathcal{B}_P$, with $W_1 \subseteq V_1$. Now if we take $W = W_1 \times V_2$, then f vanishes on $W \cap Z$, since every element $(\vec{x}, \vec{y}) \in W \cap Z$ must be of the form $\phi(\vec{x})$, with $\vec{x} \in W_1$. Clearly $W \in \mathcal{B}_{P,Q}$, so we are done.

Lemma 3.2.5. *Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $U \subseteq K^{r+m}$ be a definable open set, $(P,Q) \in U$ and suppose that $f_1, \ldots, f_m : U \to K$ are definable C^∞-functions which satisfy the hypothesis of Theorem 3.2.2. Then we have that for all $g \in \mathcal{D}_{P,Q}$, the vectors $d_{P,Q}f_1, \ldots, d_{P,Q}f_m, d_{P,Q}g$ (see Remark 3.1.5) are linearly independent over K if and only if $d_P\hat{g} \neq 0$.*

Proof. Let us first suppose that $d_{P,Q}f_1, \dots, d_{P,Q}f_m, d_{P,Q}g$ are linearly dependent. Since the functions $f_1, \dots, f_m$ satisfy the hypothesis of Theorem 3.2.2 we have $\det(\Delta(P,Q)) \neq 0$, so surely the vectors $d_{P,Q}f_1, \dots, d_{P,Q}f_m$ must be linearly independent. We write $g = [f_{m+1}, W]$ for notational convenience. Then we must have that

$$\sum_{i=1}^{m+1} a_i d_{P,Q} f_i = 0, \tag{4}$$

for certain $a_1, \dots, a_{m+1} \in K$, with $a_{m+1} \neq 0$. Now, by definition of ϕ, the functions $f_1 \circ \phi, \dots, f_m \circ \phi$ are identically zero on V_1, so $\frac{\partial \hat{f}_i}{\partial x_j}(P) = 0$ for $i = 1, \dots, m$ and $j = 1, \dots, r$. Therefore

$$\frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) = a_{m+1}^{-1} \sum_{i=1}^{m+1} a_i \frac{\partial \hat{f}_i}{\partial x_j}(P), \tag{5}$$

for $j = 1, \dots, r$. By the chain rule we have the following equality

$$\frac{\partial \hat{f}_i}{\partial x_j}(P) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}(P,Q) \frac{\partial \phi_l}{\partial x_j}(P), \tag{6}$$

for $j = 1, \dots, r$ and $i = 1, \dots, m+1$. We substitute this into (5) and change the order of the summation to find

$$\begin{aligned} \frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) &= a_{m+1}^{-1} \sum_{i=1}^{m+1} \left(a_i \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}(P,Q) \cdot \frac{\partial \phi_l}{\partial x_j}(P) \right) \\ &= a_{m+1}^{-1} \sum_{l=1}^{n} \left(\frac{\partial \phi_l}{\partial x_j}(P) \sum_{i=1}^{m+1} a_i \frac{\partial f_i}{\partial x_l}(P,Q) \right) \\ &= 0, \end{aligned}$$

for $j = 1, \dots, r$, by (4). Hence $d_P \hat{g} = 0$, which is what we needed to show.

Now let us suppose that the vectors $d_{P,Q}f_1, \dots, d_{P,Q}f_{m+1}$ are linearly independent. Let A be the $(m+1) \times n$ matrix with rows $d_{P,Q}f_1, \dots, d_{P,Q}f_{m+1}$. (We have set $n = r + m$, as in our construction of the hat function.) Then A determines a K-linear map from K^n to K^{m+1}, with kernel of dimension $n - (m+1) = r - 1$. For $j = 1, \dots, r$ we have

$$A \cdot \begin{pmatrix} \frac{\partial \phi_1}{\partial x_j}(P) \\ \vdots \\ \frac{\partial \phi_n}{\partial x_j}(P) \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{f}_1}{\partial x_j}(P) \\ \vdots \\ \frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) \end{pmatrix}$$

by (6). This vector is, by our earlier observation, equal to

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) \end{pmatrix}$$

Since $\frac{\partial \phi_i}{\partial x_j}(P) = \frac{\partial x_i}{\partial x_j}(P) = \delta_{i,j}$ for $1 \leq i, j \leq r$, the set of vectors

$$\{(\frac{\partial \phi_1}{\partial x_j}(P), \ldots, \frac{\partial \phi_n}{\partial x_j}(P))^T \mid j = 1, \ldots, r\}$$

is linearly independent. This means that not all of these vectors can be in the kernel of A. Therefore $\frac{\partial \hat{f}_{m+1}}{\partial x_j}(P)$ must be nonzero for at least one $j = 1, \ldots r$ and hence $d_P \hat{g} \neq 0$, as desired. □

We shall make use of this Lemma in the proof of the next Theorem. But first, let us introduce some additional notation.

Definition 3.2.6. Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $n, s \in \mathbb{N}$, with $n \geq 1$. Suppose that $g_1, \ldots, g_s$ are definable C^∞-functions with domains open in K^n. Then

$$\mathcal{V}(g_1, \ldots, g_s) = \{Q \in \bigcap_{i=1}^{s} \mathrm{dom}(g_i) \mid g_i(Q) = 0 \text{ for } i = 1, \ldots, s\}$$

and

$$\mathcal{V}_r(g_1, \ldots, g_s) = \{Q \in V(g_1, \ldots, g_s) \mid d_Q g_1, \ldots, d_Q g_s \text{ are linearly independent over } K\}$$

We are now ready to state and prove the following important technical Theorem, which we will be using repeatedly.

Theorem 3.2.7. *Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $n \in \mathbb{N}$, with $n \geq 1$. Let $P_0 \in K^n$ and suppose that M is a Noetherian subring of $\mathcal{D}_{P_0}$ which is closed under differentiation. Let $m \in \mathbb{N}$ and suppose that $[f_i, U_i] \in M$ for $i = 1, \ldots, m$. Then if $P_0 \in \mathcal{V}_r(f_1, \ldots, f_m)$, one of the following options must hold.*

(i) *$n = m$.*

(ii) *$m < n$ and for any $[h, W] \in M$, with $h(P_0) = 0$, h vanishes on $U \cap \mathcal{V}_r(f_1, \ldots, f_m)$ for some $U \in \mathcal{B}_{P_0}$, with $U \subseteq W$.*

(iii) *$m < n$ and for some $[h, W] \in M$ it holds that $P_0 \in \mathcal{V}_r(f_1, \ldots, f_m, h)$.*

Proof. Since $\mathcal{V}_r(f_1, \ldots, f_m) \neq \emptyset$, there exist m linearly independent vectors in K^n, so clearly $m \leq n$. It therefore suffices to assume that $m < n$ and to prove that **(ii)** or **(iii)** holds. We write $n = r + m$. Since $P_0 \in \mathcal{V}_r(f_1, \ldots, f_m)$, the vectors $d_{P_0} f_1, \ldots, d_{P_0} f_m$ are linearly independent over K. This means that there exists a set $S \subseteq \{1, \ldots, n\}$ of size m such that the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(P_0)\right)_{1 \leq i \leq m, j \in S}$$

has a nonzero determinant. For sake of convenience we assume that $S = \{r + 1, \ldots, n\}$. Then if we write $P_0 = (P, Q)$, with $P \in K^r$ and $Q \in K^m$, we are in the situation of Theorem 3.2.2. Let Δ be as in this Theorem. We take $\lambda = \det(\Delta)$. Then $[\lambda, U_0] \in M$, for some $U_0 \in \mathcal{B}_{P,Q}$. This is because λ is a polynomial in the derivatives of $f_1, \ldots, f_m$ and M is closed under differentiation. We write $\Lambda = [\lambda, U_0]$. Since $\lambda(P, Q) \neq 0$, the function λ is certainly nonzero on some $U_1 \in \mathcal{B}_{P,Q}$. Hence, Λ is invertible in $\mathcal{D}_{P,Q}$, since $\Lambda^{-1} = [\lambda^{-1}, U_1]$. Let $M^* = M[\Lambda^{-1}]$. The ring M^* is also closed under differentiation. It is enough to check this for a monomial $g\Lambda^{-l} \in M[\Lambda^{-1}]$, as differentiation distributes over addition. For $j = 1, \ldots, n$ we find

$$\frac{\partial}{\partial x_j}(g\Lambda^{-l}) = \frac{\partial g}{\partial x_j}\Lambda^{-l} - lg\frac{\partial \Lambda}{\partial x_j}\Lambda^{-l-1},$$

by the product rule. So $\frac{\partial}{\partial x_j}(g\Lambda^{-l})$ lies in M^*, as desired. We consider $\widehat{M}^*$, the image of M^* under the map $\widehat{\cdot} : \mathcal{D}_{P,Q} \to \mathcal{D}_P$. Since ring homomorphisms preserve subrings, $\widehat{M}^*$ is a subring of $\mathcal{D}_P$. We claim that $\widehat{M}^*$ is also closed under differentiation. For take some $\hat{g} = [\hat{f}, U] \in \widehat{M}^*$. Then by the chain rule

$$\frac{\partial}{\partial x_j}\hat{f} = \sum_{l=1}^{n} \widehat{\frac{\partial f}{\partial x_l}} \frac{\partial \phi_l}{\partial x_j} = \sum_{l=1}^{r} \widehat{\frac{\partial f}{\partial x_l}} \frac{\partial x_l}{\partial x_j} + \sum_{l=1}^{m} \widehat{\frac{\partial f}{\partial x_{r+l}}} \frac{\partial \psi_l}{\partial x_j} = \widehat{\frac{\partial f}{\partial x_j}} + \sum_{l=1}^{m} \widehat{\frac{\partial f}{\partial x_{r+l}}} \frac{\partial \psi_l}{\partial x_j}$$

for $j = 1, \ldots, r$, on some domain $V \in \mathcal{B}_P$. The equivalence classes associated with the functions $\widehat{\frac{\partial f}{\partial x_i}}$ belong to $\widehat{M}^*$, as M^* is closed under differentiation. Also recall from basic linear algebra that the entries of the matrix Δ^{-1} are polynomial expressions in the entries of the matrix Δ and the reciprocal of its determinant, λ^{-1}. So by **(iii)** of Theorem 3.2.2, the equivalence class of each $\frac{\partial \psi_l}{\partial x_j}$ is also in $\widehat{M}^*$. Hence $\frac{\partial}{\partial x_j}\hat{g} \in \widehat{M}^*$, so $\widehat{M}^*$ is closed under differentiation. We let I be the ideal $\{g \in \widehat{M}^* \mid g(P) = 0\}$.

Suppose that $I = \{0\}$. We show that **(ii)** holds in this case. Let $[h, W] \in M$, with $h(P_0) = 0$. We write $g = [h, W]$. Then $\hat{g}(P) = g(P_0) = 0$, so $\hat{g} \in I$. So, by our assumption $\hat{g} = 0$, or in other words, $g \in \ker(\widehat{\cdot})$. By our discussion of $\ker(\widehat{\cdot})$ in Remark 3.2.4, there exists $U \in \mathcal{B}_{P_0}$, with $U \subseteq W$, such that h vanishes on $U \cap \mathcal{V}(f_1, \ldots, f_m)$. So certainly h vanishes on $U \cap \mathcal{V}_r(f_1, \ldots, f_m)$.

Now suppose that $I \neq \{0\}$. We show that **(iii)** holds. Note that M^* is Noetherian, as M^* is finitely generated over M, which is Noetherian. This means that its homomorphic image, $\widehat{M}^*$ is also Noetherian. Hence, I is finitely generated. Say $I = ([g_1, U_1], \ldots, [g_s, U_s])$. Now if I where closed under differentation, we would be in the position to apply Lemma 3.1.6. However, this Lemma tells us that the functions $g_1, \ldots, g_s$ all vanish on a definable open subset of K^r, containing P. But this implies that I is the zero ideal, contrary to our assumptions. So it must be the case that I is not closed under differentiation. Hence, there exists $g \in M^*$ such that $\hat{g} \in I$, but $\frac{\partial \hat{g}}{\partial x_i} \notin I$, for some $1 \leq i \leq r$. In other words, $g(P_0) = 0$ and $\frac{\partial \hat{g}}{\partial x_i}(P) \neq 0$. Now, for some large enough $t \in \mathbb{N}$, we have $\Lambda^t g \in M$. Let us write $f = \Lambda^t g$. Then also $f(P_0) = 0$ and moreover,

$$\begin{aligned}
\frac{\partial \hat{f}}{\partial x_i}(P) &= \left(\frac{\partial}{\partial x_i}\hat{\Lambda}^t \hat{g}\right)(P) = \left(t\hat{\Lambda}^{t-1}\frac{\partial \hat{\Lambda}}{\partial x_i}\hat{g}\right)(P) + \left(\hat{\Lambda}^t \frac{\partial \hat{g}}{\partial x_i}\right)(P) \\
&= \left(\hat{\Lambda}^t \frac{\partial \hat{g}}{\partial x_i}\right)(P) \neq 0,
\end{aligned}$$

as $\lambda(P_0) \neq 0$. But this shows that $d_P \hat{f} \neq 0$, so by Lemma 3.2.5, the vectors $d_{P_0} f_1, \ldots, d_{P_0} f_m, d_{P_0} f$ are linearly independent. So if we write $[h, W]$ for f, then $P_0 \in \mathcal{V}_r(f_1, \ldots, f_m, h)$, as needed. □

We will move on to the next section, after proving a small Lemma, again using Lemma 3.2.5.

Lemma 3.2.8. *Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $n, m \in \mathbb{N}$, with $n \geq 1$ and $m < n$. Suppose that $f_1, \ldots, f_m$ are definable C^∞-functions with domains open in K^n and let $P_0 \in \mathcal{V}_r(f_1, \ldots, f_m)$. Let $[h, U] \in \mathcal{D}_{P_0}$ and assume that for some $W \in \mathcal{B}_{P_0}$, with $W \subseteq U \cap \bigcap_{i=1}^{m} \operatorname{dom}(f_i)$, holds that $h(\vec{x}) \geq h(P_0)$ for all $\vec{x} \in W \cap \mathcal{V}_r(f_1, \ldots, f_m)$. Then the vectors $d_{P_0} f_1, \ldots, d_{P_0} f_m, d_{P_0} h$ are linearly dependent.*

Proof. We write $n = r+m$. Since $P_0 \in \mathcal{V}_r(f_1,\ldots,f_m)$, the vectors $d_{P_0}f_1,\ldots,d_{P_0}f_m$ are linearly independent over K. This means that there exists a set $S \subseteq \{1,\ldots,n\}$ of size m such that the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(P_0)\right)_{1\le i\le m, j\in S}$$

has a nonzero determinant. Again, we may assume that $S = \{r+1,\ldots,n\}$ and write $P_0 = (P,Q)$, with $P \in K^r$ and $Q \in K^m$. This means that the hypothesis of Theorem 3.2.2 is satisfied. So by Lemma 3.2.5 it suffices to show that $d_P\hat{h} = 0$. Suppose to the contrary that this is not the case. Then $\frac{\partial \hat{h}}{\partial x_i}(P) \neq 0$ for some $1 \le i \le r$. For convenience we assume that $i = 1$. Let us write $P = (p_1,\ldots,p_r)$. Then by elementary calculus, there exists $p_1' \in K$ such that if we take $P' = (p_1', p_2, \ldots, p_r)$, then $P' \in \mathrm{dom}(\hat{h})$ and $\phi(P') \in W$, with $\hat{h}(P') < \hat{h}(P)$. But then $\phi(P') \in W \cap \mathcal{V}_r(f_1,\ldots,f_m)$ and $h(\phi(P')) < h(P_0)$, which we assumed to be false. We conclude that $d_P\hat{h} = 0$, so the vectors $d_{P_0}f_1,\ldots,d_{P_0}f_m, d_{P_0}h$ are linearly dependent. □

3.3 Proof of Lemma 2.3.1

In this section we give the proof of Lemma 2.3.1. We will need to prove a few auxiliary results first.

Lemma 3.3.1. *Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $n \in \mathbb{N}$, with $n \ge 1$. Suppose that the polynomial $f \in K[x_1,\ldots,x_n]$ vanishes on some nonempty open $U \subseteq K^n$. Then f vanishes on K^n.*

Proof. By applying a translation of coordinates if necessary, we may assume that $0 \in U$. Take some $P \in K^n$, not in U. Then $P \neq 0$, so if we write $P = (p_1,\ldots,p_n)$ we may take $p_1 \neq 0$, without loss of generality. Write $q_i = \frac{p_i}{p_1}$ for $i = 1,\ldots,n$. Define $g \in K[t]$ by $g(t) = f(q_1 t,\ldots,q_n t)$. Then clearly g vanishes on some open neigbourhood of 0. Since all nonzero polynomials have finitely many roots, it follows that $g = 0$. In particular $g(p_1) = 0$, so $f(P) = 0$. We conclude that f is identically zero. □

Let $n \in \mathbb{N}$, with $n \ge 1$ and suppose that $U \subseteq K^n$ is a nonempty definable open set, where $K \models \mathcal{T}_{\mathcal{A}}$. Then is is easily checked that $\{U\}$ is a neighborhood system in K^n. It is clear that we may identify $\mathcal{D}^{\not\sim}_{\{U\}}$ and $\mathcal{D}_{\{U\}}$ with the ring of definable C^∞-functions from U to K. So from now on we shall make no distinction between the three and denote all of them by $\mathcal{D}_U$. Note that by Lemma 2.3.5 and Lemma 3.3.1 we can embed $\mathbb{Z}[x_1,\ldots,x_n]$ in $\mathcal{D}_U$. We shall simply write $\mathbb{Z}[x_1,\ldots,x_n]$ for this subring. Now take $P \in U$ and consider the mapping $R_P : \mathcal{D}_U \to \mathcal{D}_P$, given by $R_P(f) = [f,U]$. One easily checks that this is a ring homomorphism and furthermore, the restriction of this homomorphism to the subring $\mathbb{Z}[x_1,\ldots,x_n]$ is injective by Lemma 3.3.1. We shall also denote this image in $\mathcal{D}_P$ by $\mathbb{Z}[x_1,\ldots,x_n]$.

Lemma 3.3.2. *Let $n \in \mathbb{N}$, with $n \ge 1$. Let $A \subseteq \mathbb{R}^n$ be a nonempty closed subset and let $\vec{a} \in \mathbb{R}^n$ be a point. Define the function $h_{\vec{a}} : A \to \mathbb{R}$ by $h_{\vec{a}}(\vec{x}) = \sum_{i=1}^n (x_i - a_i)^2$. Then $h_{\vec{a}}$ attains a minimum value on A.*

Proof. Take some $\vec{b} \in A$ and consider a closed ball $B \subseteq \mathbb{R}^n$ centered at $\vec{a}$ and containing $\vec{b}$. Since $h_{\vec{a}}$ is continuous and $A \cap B$ is compact, the function $h_{\vec{a}}$, restricted to $A \cap B$, attains a minimum value on some $\vec{c} \in A \cap B$. Clearly $h_{\vec{a}}(\vec{c})$ is also the minimum value of $h_{\vec{a}}$ on A. □

Remark 3.3.3. The analog of this Lemma will hold for definable closed sets $A \subseteq K^n$, by transfer. We shall use this fact in the following Theorem.

Theorem 3.3.4. *Suppose that $K \models \mathcal{T}_{\mathcal{A}}$. Let $n \in \mathbb{N}$, with $n \geq 1$ and let $U \subseteq K^n$ be a nonempty definable open set. Suppose that M is a Noetherian subring of $\mathcal{D}_U$ which contains $\mathbb{Z}[x_1, \ldots, x_n]$ and is closed under differentiation. Let $f \in M$ and suppose that $S \subseteq \mathcal{V}(f)$ is nonempty and definable and is furthermore open in the space $\mathcal{V}(f)$ and closed in K^n. Then there exist $f_1, \ldots, f_n \in M$ such that $S \cap \mathcal{V}_r(f_1, \ldots, f_n) \neq \emptyset$.*

Proof. For each point $Q \in S$ we define the ideal $I_Q = \{g \in M \mid g(Q) = 0\}$. Since M is Noetherian, the set $\{I_Q \mid Q \in S\}$ must have a maximal element with respect to inclusion, I_P, for some $P \in S$. Now take $m \in \mathbb{N}$ maximal such that $P \in \mathcal{V}_r(f_1, \ldots, f_m)$, for some $f_1, \ldots, f_m \in M$. Notice that we are done if $m = n$, since $S \cap \mathcal{V}_r(f_1, \ldots, f_m)$ contains P and is therefore nonempty. The rest of the proof is therefore dedicated to showing that $m < n$ leads to a contradiction.

Again, using that M is Noetherian, this ideal I_P is finitely generated, so we can write $I_P = (g_1, \ldots, g_s)$. We take

$$g = \sum_{i=1}^{s} g_i^2.$$

If $Q \in \mathcal{V}(g) \cap S$, then $g(Q) = 0$, so $g_1(Q) = \cdots = g_s(Q) = 0$. This means that I_Q contains all the generators of I_P, so $I_P \subseteq I_Q$. By maximality of I_P, we must have $I_P = I_Q$. Having made this observation, we continue by stating and proving several claims.

Claim 1. $\mathcal{V}(g) \cap S \subseteq \mathcal{V}_r(f_1, \ldots, f_m)$.

Proof. Since $P \in \mathcal{V}_r(f_1, \ldots, f_m)$, there is an $m \times m$ submatrix A of $\frac{\partial(f_1, \ldots, f_m)}{\partial(x_1, \ldots, x_n)}$ such that $\det(A) \notin I_P$. (There is such a submatrix A if and only if $d_P f_1, \ldots, d_P f_m$ are linearly independent.) For any $Q \in \mathcal{V}(g) \cap S$ holds $I_P = I_Q$, so we see that $\det(A) \notin I_Q$. However, we do have $\det(A) \in M$, as M is closed under differentiation, so this means that $\det(A)$ is nonzero at Q, which means that $d_Q f_1, \ldots, d_Q f_m$ are linearly independent.. Since $P \in \mathcal{V}_r(f_1, \ldots, f_m)$, we also have $f_1, \ldots, f_m \in I_P$ and hence $f_1, \ldots, f_m \in I_Q$, showing that $f_1(Q) = \cdots = f_m(Q) = 0$, for all $Q \in \mathcal{V}(g) \cap S$. It follows that $Q \in \mathcal{V}(g) \cap S$ implies $Q \in \mathcal{V}_r(f_1, \ldots, f_m)$, as needed.

Claim 2. Let $Q \in \mathcal{V}(g) \cap S$ and $h \in M$. Then $Q \notin \mathcal{V}_r(f_1, \ldots, f_m, h)$.

Proof. Suppose to the contrary that $Q \in \mathcal{V}_r(f_1, \ldots, f_m, h)$. Using $I_P = I_Q$, we can argue in the same way as in the proof of the previous claim to conclude that $P \in \mathcal{V}_r(f_1, \ldots, f_m, h)$. This contradicts the maximality of m, so $Q \notin \mathcal{V}_r(f_1, \ldots, f_m, h)$.

Claim 3. Let $Q \in \mathcal{V}(g) \cap S$. Then there exists $W \in \mathcal{B}_Q$, with $W \subseteq U$, such that $W \cap \mathcal{V}(g) \cap S = W \cap \mathcal{V}_r(f_1, \ldots, f_m)$.

Proof. Let M_Q be the image of M under the map $R_Q : \mathcal{D}_U \to \mathcal{D}_Q$. We wish to apply Theorem 3.2.7 to the ring M_Q, as a subring of $\mathcal{D}_Q$ and with respect to the germs $[f_1, U], \ldots, [f_m, U]$. It is clear that M_Q is Noetherian and closed under differentiation, as M is. Furthermore, $Q \in \mathcal{V}_r(f_1, \ldots, f_m)$ by our first claim, so we are indeed in the right setting to use this Theorem. By our assumption, $m < n$, so option **(i)** of the Theorem cannot hold and by our second claim, option **(iii)** cannot hold either. We have $[g, U] \in M_Q$ and since $I_P = I_Q$ we have $g(0) = 0$, so option **(ii)** of Theorem 3.2.7 tells us that g vanishes on $W_1 \cap \mathcal{V}_r(f_1, \ldots, f_m)$ for some $W_1 \in \mathcal{B}_Q$, with $W_1 \subseteq U$. Because of the way g is defined, this means that every element of I_P, and in particular f, vanishes on $W_1 \cap \mathcal{V}_r(f_1, \ldots, f_m)$. Hence $W_1 \cap \mathcal{V}_r(f_1, \ldots, f_m) \subseteq W_1 \cap \mathcal{V}(g) \cap \mathcal{V}(f)$. Since S is open in $\mathcal{V}(f)$, there exists $W_2 \in \mathcal{B}_Q$ such that $W_2 \cap S = W_2 \cap \mathcal{V}(f)$. So if we take $W = W_1 \cap W_2$, then $W \cap \mathcal{V}_r(f_1, \ldots, f_m) \subseteq W \cap \mathcal{V}(g) \cap S$. The opposite inclusion follows immediately from Claim 1, so we have proven Claim 3.

Claim 4. $S \cap \mathcal{V}(g)$ is closed in K^n.

Proof. Since g is continuous on U, the set $\mathcal{V}(g)$ is closed in U. Note that S is a subset of U, as $S \subseteq \mathcal{V}(f) \subseteq U$. It follows that $S \cap \mathcal{V}(g)$ is closed in K^n, as S is closed in K^n.

Let us finish the proof using these claims. Let $\vec{a} \in \mathbb{Z}^n$. Define the function $h_{\vec{a}} : S \cap \mathcal{V}(g) \to K$ by $h_{\vec{a}}(\vec{x}) = \sum_{i=1}^{n}(x_i - a_i)^2$. The set $S \cap \mathcal{V}(g)$ is nonempty, as it contains P and it is closed by our fourth claim. By Remark 3.3.3, it follows that the function $h_{\vec{a}}$ attains a minimum value on $S \cap \mathcal{V}(g)$. Using our third claim, there exists $W \in \mathcal{B}_Q$, with $W \subseteq U$, such that $W \cap \mathcal{V}(g) \cap S = W \cap \mathcal{V}_r(f_1, \ldots, f_m)$. This means that $h_{\vec{a}}(Q) \leq h_{\vec{a}}(\vec{x})$ for all $\vec{x} \in W \cap \mathcal{V}_r(f_1, \ldots, f_m)$. The conditions of Lemma 3.2.8 are satisfied, so by this Lemma the vectors $d_Q f_1, \ldots, d_Q f_m, d_Q h_{\vec{a}}$ are linearly dependent. This means that he vectors $d_P f_1, \ldots, d_P f_m, d_p h_{\vec{a}}$ must also be linearly dependent. For suppose that they are linearly independent. Then there is an $m \times m$ submatrix A of $\frac{\partial(f_1,\ldots,f_m,h_{\vec{a}})}{\partial(x_1,\ldots,x_n)}$ such that $\det(A) \notin I_P$. Hence $\det(A) \notin I_Q$, as $I_P = I_Q$. Now note that $h_{\vec{a}} \in M$, as $\mathbb{Z}[x_1, \ldots, x_n] \subseteq M$. So, since M is closed under differentiation, we do have $\det(A) \in M$. It follows that $\det(A)$ is nonzero at Q, which is false.

Recall that the vectors $d_P f_1, \ldots, d_P f_m$ are linearly independent, as $P \in \mathcal{V}_r(f_1, \ldots, f_m)$. Since the vectors $d_P f_1, \ldots, d_P f_m, d_p h_{\vec{a}}$ are linearly dependent, this means that the vector $d_P h_{\vec{a}}$ must be a K-linear combination of the vectors $d_P f_1, \ldots, d_P f_m$. Since this holds for any $\vec{a} \in \mathbb{Z}^n$, the vector $\frac{1}{2}(d_P h_{\vec{0}} - d_P h_{\vec{a}})$ is in the span of $d_P f_1, \ldots, d_P f_m$. One easily verifies by direct calculation that $\frac{1}{2}(d_P h_{\vec{0}} - d_P h_{\vec{a}}) = \vec{a}$, so that $\mathbb{Z}^n \subseteq \operatorname{span}(d_P f_1, \ldots, d_P f_m)$. It follows that $\operatorname{span}(d_P f_1, \ldots, d_P f_m) = K^n$, contradicting $m < n$. □

We are now ready to prove Lemma 2.3.1. For convenience, we restate the Lemma here.

Lemma 2.3.1. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n,r)-sequence. Suppose that $g \in M^r(k, K, \vec{\sigma})$ and $g(P) = 0$ for some $P \in D^r(\vec{\sigma}, K)$. Then for some $s \in \mathbb{N}$ there exist $Q_1 \in D^r(\vec{\sigma}, K)$ and $Q_2 \in K^s$ such that $g(Q_1) = 0$ and (Q_1, Q_2) is $(k, \vec{\sigma})$-definable.*

Proof. We shall first prove the Lemma under the assumption that $\mathcal{V}(g)$ is closed. After this we show that the general case essentially reduces to this special case, save for some minor details. Define $U_1 = D^r(\vec{\sigma}, K)$. Clearly U_1 is an open definable subset of K^r. We wish to apply Theorem 3.3.4 with respect to the ring $M^r(k, K, \vec{\sigma})$ as a subring of $\mathcal{D}_{U_1}$. Indeed, $M^r(k, K, \vec{\sigma})$ is a subring of $\mathcal{D}_{U_1}$ which is Noetherian and closed under differentiation (see Remark 2.2.5) and contains $\mathbb{Z}[x_1, \ldots, x_r]$. If we take $S = \mathcal{V}(g)$, then by our assumption, the hypothesis of Theorem 3.3.4 is satisfied. By this Theorem, there exist $f_1, \ldots, f_n \in M^r(k, K, \vec{\sigma})$ such that $S \cap \mathcal{V}_r(f_1, \ldots, f_n)$ is nonempty. Take some $Q_1 \in S \cap \mathcal{V}_r(f_1, \ldots, f_n)$. Then $g(Q_1) = 0$ as $Q_1 \in S$ and Q_1 is $(k, \vec{\sigma})$-definable as $Q_1 \in \mathcal{V}_r(f_1, \ldots, f_n)$, proving the Theorem, with $s = 0$.

Now, in general the set $\mathcal{V}(g)$ might not be closed. We resolve this issue by pushing possible limit points of $\mathcal{V}(g)$ that lie on the boundary of $D^r(\vec{\sigma}, K)$ out to infinity. Regard $\vec{\sigma}$ as an $(n, r+s)$-sequence, with $s = 2r$. For $i = 1, \ldots, r$ define the functions

$$g_i(x_1, \ldots, x_{r+s}) = \begin{cases} x_i \cdot x_{r+i} - 1 & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded} \\ x_i - x_{r+i} & \text{otherwise} \end{cases}$$

and

$$g_{r+i}(x_1, \ldots, x_{r+s}) = \begin{cases} (x_i - 1) \cdot x_{2r+i} - 1 & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded} \\ x_i - x_{2r+i} & \text{otherwise.} \end{cases}$$

We define $f \in M^{r+s}(k, K, \vec{\sigma})$ by $f = g^2 + \sum_{i=1}^{2r} g_i^2$. Here we restrict the functions $g_1, \ldots, g_{2r}$ to the set $D^{r+s}(\vec{\sigma}, K)$, which we will denote by U_2. Notice that $(q_1, \ldots, q_r) \in \mathcal{V}(g)$ if and only if $(q_1, \ldots, q_{r+s}) \in \mathcal{V}(f)$, where $q_{r+i} = q_{2r+i} = q_i$ if x_i is $\vec{\sigma}$-bounded and $q_{r+i} = q_i^{-1}$, $q_{2r+i} = (q_i - 1)^{-1}$ if x_i is $\vec{\sigma}$-bounded. Note that there is no danger of dividing by zero, as $0 < q_i < 1$ if x_i is $\vec{\sigma}$-bounded. Since $P \in \mathcal{V}(g)$, the set $\mathcal{V}(f)$ is also nonempty. We show that $\mathcal{V}(f)$ has no limit points on the boundary of U_2. If none of the variables $x_1, \ldots, x_r$ are $\vec{\sigma}$-bounded, then there is nothing to prove, since $U_2 = K^{r+s}$ in this case, so its boundary will be empty. So, for the sake of argument, suppose that x_1 is $\vec{\sigma}$-bounded. We only prove that $\mathcal{V}(f)$ has no limit points on the set of points in K^{r+s} satisfying the equation $x_1 = 0$, as any boundary points of U_2 not in this set can be dealt with in a similar fashion. Regard the function g_1 as defined on the entire space K^{r+s}. Then $\mathcal{V}(g_1)$ is closed in K^{r+s}, as g_1 is continuous. Clearly none of the points in $\{\vec{x} \in K^{r+s} \mid x_1 = 0\}$ lie in $\mathcal{V}(g_1)$, so $\mathcal{V}(g_1)$ has no limit points satisfying $x_1 = 0$. Now note that $\mathcal{V}(f) \subseteq \mathcal{V}(g_1)$, because of the way f is defined. Hence, f has no limit points satisfying $x_1 = 0$. Since $\mathcal{V}(f)$ has no limit points on the boundary of U_2, we find

$$\mathrm{Cl}_{K^{r+s}}(U_2) \cap \mathrm{Cl}_{K^{r+s}}(\mathcal{V}(f)) = U_2 \cap \mathrm{Cl}_{K^{r+s}}(\mathcal{V}(f)) = \mathrm{Cl}_{U_2}(\mathcal{V}(f)).$$

Since $\mathcal{V}(f)$ is closed in U_2 by continuity of f, the set $\mathrm{Cl}_{U_2}(\mathcal{V}(f))$ is just $\mathcal{V}(f)$. It follows that $\mathcal{V}(f)$ is closed in K^{r+s}, as it can be written as the intersection of two closed sets. But now we can argue just as in the special case at the beginning of this proof, only now with $M^{r+s}(k, K, \vec{\sigma})$ as a subring of $\mathcal{D}_{U_2}$, taking $S = \mathcal{V}(f)$ and $s = 2r$. $\square$

4 Towards Lemma 2.3.2

4.1 Results by Khovanskii and Van den Dries

We use this section to present some miscellaneous results from Askold Khovanskii and Lou van den Dries and we will derive several consequences that will be needed in the proofs in the upcoming sections.

The following Proposition is by Khovanskii.

Proposition 4.1.1. *Suppose that $h_1, \ldots, h_l$ is any Pfaffian chain of functions on $\mathbb{R}^{n+m}$ and let $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$. Then there is a natural number N such that for any $Q \in \mathbb{R}^n$, the set*

$$\{P \in \mathbb{R}^m \mid g_1(P,Q) = \cdots g_m(P,Q) = 0 \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)}\right)(P,Q) \neq 0\}$$

contains at most N elements.

Proof. A proof of this can be found in [Kho80]. □

For our purposes we need the following more general form of this result.

Lemma 4.1.2. *For each $i = 1, \ldots, m+n$, let J_i be either $\mathbb{R}$ or the interval $(0,1)$. Suppose that $h_1, \ldots, h_l$ is any Pfaffian chain of functions on $\prod_{i=1}^{m+n} J_i$. Suppose that $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$, as a ring of functions defined on $\prod_{i=1}^{m+n} J_i$. Then there is a natural number N such that for any $Q \in \mathbb{R}^n$, the set*

$$\{P \in \prod_{i=1}^{m} J_i \mid g_1(P,Q) = \cdots g_m(P,Q) = 0 \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)}\right)(P,Q) \neq 0\}$$

contains at most N elements.

Proof. For $i = 1, \ldots, m+n$, we define the functions $\alpha_i, \beta_i : \mathbb{R}^{m+n} \to \mathbb{R}$ by

$$\alpha_i(\vec{x}) = \begin{cases} 1 & \text{if } J_i = \mathbb{R} \\ \frac{1}{\pi(1+x_i^2)} & \text{if } J_i = (0,1) \end{cases}$$

and

$$\beta_i(\vec{x}) = \begin{cases} x_i & \text{if } J_i = \mathbb{R} \\ \frac{1}{2} + \frac{1}{\pi}\arctan(x_i) & \text{if } J_i = (0,1). \end{cases}$$

Then the map $\vec{\beta} = (\beta_1, \ldots, \beta_{m+n}) : \mathbb{R}^{m+n} \to \prod_{i=1}^{m+n} J_i$ is an analytic bijection and the functions $h_i \circ \vec{\beta} : \mathbb{R}^{m+n} \to \mathbb{R}$ are analytic, for $i = 1, \ldots, l$. Using that

$$\frac{\partial h_i \circ \vec{\beta}}{\partial x_j}(\vec{x}) = \sum_{l=1}^{m+n} \frac{\partial \beta_l}{\partial x_j}(\vec{x}) \cdot \frac{\partial h_i}{\partial x_l}(\vec{\beta}(\vec{x})),$$

for $i = 1, \ldots, l$ and $j = 1, \ldots, m+n$, it is easily checked that the sequence $\alpha_1, \beta_1, \ldots, \alpha_{m+n}, \beta_{m+n}, h_1 \circ \vec{\beta}, \ldots, h_l \circ \vec{\beta}$ is a Pfaffian chain on $\mathbb{R}^{m+n}$. Furthermore

$$g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta} \in \mathbb{R}[x_1, \ldots, x_{m+n}, \alpha_1, \beta_1, \ldots, \alpha_{m+n}, \beta_{m+n}, h_1 \circ \vec{\beta}, \ldots, h_l \circ \vec{\beta}].$$

Fix $Q \in \prod_{i=m+1}^{m+n}$ and suppose that $P \in \prod_{i=1}^{m} J_i$ is a point such that $g_1(P,Q) = \cdots g_m(P,Q) = 0$ and $\det\left(\frac{\partial(g_1,\ldots,g_m)}{\partial(x_1,\ldots,x_m)}\right)(P,Q) \neq 0$. If we define $(P',Q') = \vec{\beta}^{-1}(P,Q)$, then surely $g_1 \circ \vec{\beta}(P',Q') = \cdots = g_m \circ \vec{\beta}(P',Q') = 0$. Furthermore, one readily verifies that

$$\frac{\partial(g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta})}{\partial(x_1,\ldots,x_m)}(P',Q') = \frac{\partial(g_1,\ldots,g_m)}{\partial(x_1,\ldots,x_m)}(P,Q) \cdot \frac{\partial(\beta_1,\ldots,\beta_m)}{\partial(x_1,\ldots,x_m)}(P',Q'),$$

using that

$$\frac{\partial g_i \circ \vec{\beta}}{\partial x_j}(\vec{x}) = \sum_{l=1}^{m+n} \frac{\partial \beta_l}{\partial x_j}(\vec{x}) \cdot \frac{\partial g_i}{\partial x_l}(\vec{\beta}(\vec{x})),$$

by the chain rule. The matrix $\frac{\partial(\beta_1,\ldots,\beta_m)}{\partial(x_1,\ldots,x_m)}(P',Q')$ is diagonal, so it is easy to see that

$$\det\left(\frac{\partial(\beta_1,\ldots,\beta_m)}{\partial(x_1,\ldots,x_m)}\right)(P',Q') = \prod_{i=1}^{m} \alpha_i(P',Q') \neq 0,$$

from which it follows that

$$\det\left(\frac{\partial(g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta})}{\partial(x_1,\ldots,x_m)}\right)(P',Q') \neq 0.$$

We now use that the inverse of $\vec{\beta}$ is calculated pointwise, that is, $\vec{\beta}^{-1} = (\beta_1^{-1}, \ldots, \beta_{m+n}^{-1})$. Combined with our calculations above, this implies that $(\beta_1^{-1}, \ldots, \beta_m^{-1})$ is a injection from

$$\{P \in \prod_{i=1}^{m} J_i \mid g_1(P,Q) = \cdots g_m(P,Q) = 0 \text{ and } J(g_1,\ldots,g_m)(P,Q) \neq 0\} \tag{7}$$

to

$$\{P' \in \mathbb{R}^m \mid g_1 \circ \vec{\beta}(P',Q') = \cdots g_m \circ \vec{\beta}(P',Q') = 0 \text{ and } J(g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta})(P',Q') \neq 0\}. \tag{8}$$

By Proposition 4.1.1, the set (8) contains at most N elements, for some $N \in \mathbb{N}$, independent of Q', hence the set (7) also contains at most N elements, independent of Q, as needed. □

The fact that the bound N is uniform in Q allows us to transfer this result to a situation we are interested in.

Corollary 4.1.3. *Suppose that $n, r_1, r_2 \in \mathbb{N}$ and that $\vec{\sigma}$ is an $(n, r_1 + r_2)$-sequence. Suppose further that $k, K \models \mathcal{T}_{\mathrm{Pf}\upharpoonright}$, $k \subseteq K$, and that $g_1, \ldots, g_{r_1} \in M^{r_1+r_2}(k, K, \vec{\sigma})$. Then there exists $N \in \mathbb{N}$ such that for each $Q \in K^{r_2}$ the set*

$$\{P \in K^{r_1} \mid (P,Q) \in D^{r_1+r_2}(\vec{\sigma}, K), g_1(P,Q) = \cdots = g_{r_1}(P,Q) = 0$$
$$\text{and } \det\left(\frac{\partial(g_1,\ldots,g_{r_1})}{\partial(x_1,\ldots,x_{r_1})}\right)(P,Q) \neq 0\}$$

contains at most N elements.

Proof. Let $a_1, \ldots, a_m \in k$ be the parameters from k appearing in $g_1, \ldots, g_{r_1}$. Now take

$$h_1, \ldots, h_{r_1} \in \mathbb{Z}[x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}, z_1, \ldots, z_n, w_1, \ldots, w_m]$$

such that $g_i(\vec{x}, \vec{y}) = h_i(\vec{x}, \vec{y}, \vec{\sigma}(\vec{x}, \vec{y}), \vec{a})$ for all $i = 1, \ldots, r_1$ and $(\vec{x}, \vec{y}) \in D^{r_1+r_2}(\vec{\sigma}, K)$. We write $f_i(\vec{x}, \vec{y}, \vec{z}) = h_i(\vec{x}, \vec{y}, \vec{\sigma}(\vec{x}, \vec{y}), \vec{z})$, for $i = 1, \ldots, r_1$. Now note that the functions $f_1, \ldots, f_{r_1}$ are definable without parameters. This means that we can transfer Lemma 4.1.2 (applied to the Pfaffian chain $\sigma_1, \ldots, \sigma_n$) to K and find that there exists $N \in \mathbb{N}$, such that for each $(Q_1, Q_2) \in K^{r_2+m}$ the set

$$\{P \in K^{r_1} \mid (P, Q_1, Q_2) \in D^{r_1+r_2}(\vec{\sigma}, K) \times K^m, f_1(P, Q_1, Q_2) = \cdots = f_{r_1}(P, Q_1, Q_2) = 0 \text{ and } \det\left(\frac{\partial(f_1, \ldots, f_{r_1})}{\partial(x_1, \ldots, x_{r_1})}\right)(P, Q_1, Q_2) \neq 0\}$$

contains at most N elements. But if we take $Q_1 = Q$ and $Q_2 = \vec{a}$, then this is exactly the set

$$\{P \in K^{r_1} \mid (P, Q) \in D^{r_1+r_2}(\vec{\sigma}, K), g_1(P, Q) = \cdots = g_{r_1}(P, Q) = 0 \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_{r_1})}{\partial(x_1, \ldots, x_{r_1})}\right)(P, Q) \neq 0\},$$

so we are done. □

The following result is also due to Khovanskii. Using Theorem 3.3.4 and some model theoretic arguments, we can deduce it from Proposition 4.1.1.

Theorem 4.1.4. *Suppose that $h_1, \ldots, h_l$ is a Pfaffian chain of functions on $\mathbb{R}^{m+n}$ and let $g \in \mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$. Then there is a natural number N such that for any $Q \in \mathbb{R}^n$ the set*

$$\{P \in \mathbb{R}^m \mid g(P, Q) = 0\}$$

has at most N components. By a component of a set $S \subseteq \mathbb{R}^m$ we mean a set $X \subseteq S$, such that X is both open and closed in the subspace S.

Proof. We argue by contradiction, so assume that the theorem is false. Then for each $i \in \mathbb{N}$ we can find a point $Q^i \in \mathbb{R}^n$ such that the set $\{P \in \mathbb{R}^m \mid g(P, Q^i) = 0\}$ has pairwise disjoint and nonempty components $C_0^i, \ldots, C_i^i$. (We can take these components disjoint, as the set of components forms a Boolean algebra.)

Now expand the language $\mathcal{L}$ to the language $\mathcal{L}'$, by adding symbols for:

- The functions $h_1, \ldots, h_l$.
- A unary relation for the set of natural numbers.
- A map $i \mapsto Q^i$, for natural numbers i.
- An $(m+2)$-ary relation expressing that $(x_1, \ldots, x_m) \in C_j^i$.

We will leave these symbols unspecified.

We let the $\mathcal{L}'$-structure K be a $(2^{\aleph_0})^+$-saturated elementary extension of $(\mathbb{R} \mid \mathcal{L}')$. Let us prove a few facts about the natural numbers in K. First of all, K contains nonstandard natural numbers, as K is $(2^{\aleph_0})^+$-saturated and the finitely satisfiable partial type

$$p(x) = \{x \in \mathbb{N}\} \cup \{y < x \mid y \in \mathbb{N}\}$$

has only $\aleph_0$ many parameters. Denote the set of all natural numbers in K, both standard and nonstandard, by $\mathcal{N}$. The set $\mathcal{N}$ is a definable subset of K, because of the relation symbol we have added.

We claim that if $a \in K$ is a nonstandard natural number in K, then the set $\{x \in \mathcal{N} \mid x \leq a\}$ has at least size $(2^{\aleph_0})^+$. For suppose that this in not the case. Then it is clear that the set

$$\mathcal{N}_{\ll a} = \{x \in \mathcal{N} \mid \forall y \in \mathbb{Z}\, [x < y + a]\}$$

has cardinality less than $(2^{\aleph_0})^+$. But now the type

$$q(x) = \{y < x \mid y \in \mathcal{N}_{\ll a}\} \cup \{x < y + a \mid y \in \mathbb{Z}\}$$

has less than $(2^{\aleph_0})^+$ parameters and is finitely satisfiable in K, yet there is no element in K satisfying the type. This contradicts the fact that K is $(2^{\aleph_0})^+$-saturated, proving the claim.

We take $a \in K$ to be some fixed nonstandard natural number. Now define

$$M = \mathbb{R}[x_1, \ldots, x_m, Q^a, h_1(x_1, \ldots, x_m, Q^a), \ldots, h_l(x_1, \ldots, x_m, Q^a)].$$

Then M is a Noetherian ring, as it is finitely generated over $\mathbb{R}$. The ring M consists of functions definable in K and it is closed under differentiation, as $h_1, \ldots, h_l$ is a Pfaffian chain. Furthermore, M contains $\mathbb{Z}[x_1, \ldots, x_m]$. Note that $g(x_1, \ldots, x_m, Q^a) \in M$ and that

$$\mathcal{V}(g(x_1, \ldots, x_m, Q^a)) = \{P \in K^m \mid g(P, Q^a) = 0\}$$

is closed in K^m. The sets C_i^a, with $i \leq a$ and $i \in \mathcal{N}$, are both open and closed in $\mathcal{V}(g(x_1, \ldots, x_m, Q^a))$ by definition, hence also closed in K^m, as $V(g(x_1, \ldots, x_m, Q^a))$ is closed in K^m. This means that we can apply Theorem 3.3.4 for each $i \leq a$ with $i \in \mathcal{N}$. So for each such i, there exist $f_1^i, \ldots f_m^i \in M$, such that $C_i^a \cap \mathcal{V}_r(f_1^i, \ldots, f_m^i) \neq \emptyset$. This implies that there exists a map

$$F : \{i \in \mathcal{N} \mid i \leq a\} \to \bigcup_{i \leq a} \mathcal{V}_r(f_1^i, \ldots, f_m^i),$$

such that each $F(i)$ lies in $C_i^a \cap \mathcal{V}_r(f_1^i, \ldots, f_m^i)$. Such a function F is an injection, as $C_i^a \cap C_j^a = \emptyset$ for $i \neq j$, so the codomain of F must be of at least size $(2^{\aleph_0})^+$.

On the other hand, by Proposition 4.1.1, there is a natural number N such that for any $Q \in \mathbb{R}^n$ the set

$$\{P \in \mathbb{R}^m \mid f_1^i(P, Q) = \cdots = f_m^i(P, Q) = 0 \text{ and } \det\left(\frac{\partial(f_1^i, \ldots, f_m^i)}{\partial(x_1, \ldots, x_m)}\right) \neq 0\}$$

contains at most N elements. Since K is an elementary extension of $\mathbb{R}$, as an $\mathcal{L}'$-structure, the same must hold when we replace $\mathbb{R}$ by K, so in particular the set

$$\{P \in K^m \mid f_1^i(P, Q^a) = \cdots = f_m^i(P, Q^a) = 0 \text{ and } \det\left(\frac{\partial(f_1^i, \ldots, f_m^i)}{\partial(x_1, \ldots, x_m)}\right) \neq 0\}$$

contains no more than N elements. So each $\mathcal{V}_r(f_1^i, \ldots, f_m^i)$ is finite, which implies that the cardinality of the set

$$\bigcup_{i \leq a} \mathcal{V}_r(f_1^i, \ldots, f_m^i)$$

is limited by the number of distinct functions in M. But $|M| = 2^{\aleph_0}$, as M is finitely generated over $\mathbb{R}$. We have arrived at a contradiction, so we conclude that the theorem holds. □

Definition 4.1.5. For each m and each analytic function $f : U \to \mathbb{R}$, where U is some open neighborhood of the closed box $[0,1]^m$ in $\mathbb{R}^m$, let $\widetilde{f} : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$\widetilde{f}(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in [0,1]^m \\ 0 & \text{if } \vec{x} \in \mathbb{R}^m \setminus [0,1]^m \end{cases}$$

Let $\mathcal{F}$ be a collection of function symbols for each such function $\widetilde{f}$. We let $\mathcal{L}_{\text{an}\restriction} = \mathcal{L} \cup \mathcal{F}$ and $\mathcal{T}_{\text{an}\restriction} = \text{Th}(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$.

The result below is due to Van den Dries.

Proposition 4.1.6. *The following two statements hold for* $(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$.

(i) *The structure* $(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$ *is O-minimal.*

(ii) *If* $e \in \mathbb{R}$ *and* $f : (e, \infty) \to \mathbb{R}$ *is a function, definable in* $(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$ *with parameters in* $\mathbb{R}$*, then there exists* $d \geq e$ *such that on* (d, ∞)*, the function* f *can be represented by a convergent Puiseux series*

$$f(x) = \sum_{i=p}^{\infty} a_i \cdot x^{-i/q},$$

with $q \in \mathbb{Z}_{\geq 1}$*,* $p \in \mathbb{Z}$*,* $a_i \in \mathbb{R}$*, for* $i \in \mathbb{Z}_{\geq p}$*. Furthermore* $a_p \neq 0$*, if* f *is not eventually identically zero.*

Proof. A proof of this can be found in [vdD86]. □

We have two corollaries to this Proposition.

Corollary 4.1.7. *Every model* K *of the theory* $\mathcal{T}_{\text{Pf}\restriction}$ *is O-minimal.*

Proof. Since $\mathcal{L}_{\text{Pf}\restriction} \subseteq \mathcal{L}_{\text{an}\restriction}$, every set definable (with parameters from $\mathbb{R}$) in the structure $(\mathbb{R} \mid \mathcal{L}_{\text{Pf}\restriction})$ is also definable (with parameters from $\mathbb{R}$) in the structure $(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$. So from Proposition 4.1.6 **(i)** we may conclude that $(\mathbb{R} \mid \mathcal{L}_{\text{Pf}\restriction})$ is O-minimal.

The Corollary now follows directly from Proposition A.2.5. □

Corollary 4.1.8. *Suppose that* $K \models \mathcal{T}_{\text{Pf}\restriction}$*,* $e \in K$ *and* $g : (e, \infty) \to K$ *is a* K*-definable function, which is not identically zero. Then there exists* $s \in \mathbb{Q}$ *and a nonzero* $a \in K$*, such that* $K \models \lim_{x\to\infty} g(x)x^s = a$*.*

Proof. Let $\phi(\vec{z}, x, y)$ be an $\mathcal{L}_{\text{Pf}\restriction}$-formula, such that $\phi(\vec{b}, x, y)$ defines the graph of g in K, for some set of parameters $\vec{b}$ from K. We define the $\mathcal{L}_{\text{Pf}\restriction}$-formula $\psi(\vec{z})$ by

$$\exists u[(\forall x > u \exists! y\ \varphi(\vec{z}, x, y)) \wedge (\forall x > u \exists w > x\ \neg\varphi(\vec{z}, x, 0))].$$

Then $K \models \psi(\vec{b})$.

Now suppose that $\vec{\alpha}$ is a set of parameters from $\mathbb{R}$ such that $\mathbb{R} \models \psi(\vec{\alpha})$ and let $f_{\vec{\alpha}} : (e, \infty) \to \mathbb{R}$, for some $e \in \mathbb{R}$, be the the function whose graph is defined by $\phi(\vec{\alpha}, x, y)$ in $\mathbb{R}$. Note that every function definable in $(\mathbb{R} \mid \mathcal{L}_{\text{Pf}\restriction})$ is in particular definable in $(\mathbb{R} \mid \mathcal{L}_{\text{an}\restriction})$. This means that we may apply that Proposition 4.1.6 **(ii)**. Hence, there is some $d \geq e$, such that if $x \geq d$, then we have

$$f_{\vec{\alpha}}(x) = \sum_{i=p}^{\infty} a_i \cdot x^{-i/q},$$

with $q \in \mathbb{Z}_{\geq 1}$, $p \in \mathbb{Z}$, $a_i \in \mathbb{R}$, for $i \in \mathbb{Z}_{\geq p}$ and $a_p \neq 0$. Then clearly

$$\lim_{x \to \infty} f_{\vec{\alpha}}(x) \cdot x^{p/q} = a_p.$$

Furthermore, we may differentiate this series termwise to arrive at

$$f'_{\vec{\alpha}}(x) = \sum_{i=p}^{\infty} -\frac{ia_i}{q} \cdot x^{-(i/q)-1}.$$

We see that

$$\lim_{x \to \infty} f'_{\vec{\alpha}}(x) \cdot x^{(p/q)+1} = -\frac{pa_p}{q}.$$

Combining the two limits gives

$$\lim_{x \to \infty} -f'_{\vec{\alpha}}(x) \cdot x / f_{\vec{\alpha}}(x) = \frac{p}{q}.$$

Let $\chi(\vec{z}, y)$ be an $\mathcal{L}_{\mathrm{Pf}\restriction}$-formula formalizing the statement

$$\lim_{x \to \infty} -f'_{\vec{\alpha}}(x) \cdot x / f_{\vec{\alpha}}(x) = y.$$

Then, as we have shown, the $\mathcal{L}_{\mathrm{Pf}\restriction}$-formula $\exists \vec{z}[\psi(\vec{z}) \wedge \chi(\vec{z}, y)]$ defines a set of rational numbers $S \subseteq \mathbb{Q}$. Since $(\mathbb{R} \mid \mathcal{L}_{\mathrm{Pf}\restriction})$ is O-minimal by Corollary 4.1.7, this set must be finite, say $S = \{s_1, \ldots, s_n\}$.

From what we have seen so far follows that

$$\mathbb{R} \models \forall \vec{z} \left[\psi(\vec{z}) \to \exists y \left(y \neq 0 \wedge \bigvee_{i=1}^{n} \lim_{x \to \infty} f_{\vec{z}}(x) \cdot x^{s_i} = y \right) \right].$$

Since this statement can be formalized in the language $\mathcal{L}_{\mathrm{Pf}\restriction}$, it must also be true in K. Since $K \models \psi(\vec{b})$ and $f_{\vec{b}}(x) = g(x)$ for sufficiently large x, the result follows. □

4.2 Pfaffian chains of unrestricted functions

The reader may have noticed already that in not many of our proofs we have used the fact that the functions in our Pfaffian chain are truncated. We will not let this greater generality go to waste. First we make a few definitions which will look familiar.

Definition 4.2.1. Let $m, l \in \mathbb{N}$, and let $H_1, \ldots, H_l : \mathbb{R}^m \to \mathbb{R}$ be a Pfaffian chain. Recall that this means that there exist polynomials $p_{i,j} \in \mathbb{R}[z_1, \ldots, z_{m+i}]$ (for $i = 1, \ldots, l$ and $j = 1, \ldots, m$) such that

$$\frac{\partial H_i}{\partial x_j}(\vec{x}) = p_{i,j}(\vec{x}, H_1(\vec{x}), \ldots, H_i(\vec{x})),$$

for all $\vec{x} \in \mathbb{R}^m$. Now, let $C \subseteq \mathbb{R}$ by any set such that the coefficients of each $p_{i,j}$ are the value of some term in the structure $(\mathbb{R} \mid \mathcal{L}, H_1, \ldots, H_l, c)_{c \in C}$. We define the language $\mathcal{L}_{\mathrm{Pf}}$ as $\mathcal{L} \cup \{H_1, \ldots, H_l\} \cup C$. Furthermore, we define the $\mathcal{L}_{\mathrm{Pf}}$-theory $\mathcal{T}_{\mathrm{Pf}}$ as $\mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\mathrm{Pf}})$.

Definition 4.2.2. Let $n, r \in \mathbb{N}$.

(i) A sequence $(\sigma_1, \ldots, \sigma_n)$ of terms of $\mathcal{L}_{\mathrm{Pf}}$ in the variables $x_1, \ldots, x_r$ is called an (n, r)-sequence if the following two conditions are satisfied.

(a) For $s = 1, \ldots, n$, the component σ_s has the form $H_i(y_1, \ldots, y_m)$ for some $i = 1, \ldots, l$ and some $y_1, \ldots y_m \in \{x_1, \ldots, x_r\}$.

(b) If $s = 1, \ldots, n$, $i = 2, \ldots, l$ and $\sigma_s = H_i(y_1, \ldots, y_m)$, then $s > 1$ and for some $t = 1, \ldots, s-1$ holds $\sigma_t = H_{i-1}(y_1, \ldots, y_m)$.

(ii) Those variables actually occurring in some term of an (n, r)-sequence $\vec{\sigma}$ are called $\vec{\sigma}$-bounded.

Of course, we have now provided two conflicting definitions of what an (n, r)-sequence is: one for the language $\mathcal{L}_{\mathrm{Pf}\restriction}$ and one for the language $\mathcal{L}_{\mathrm{Pf}}$. This should now lead to confusion however, as it will always be clear from the context which of the two is meant in a given situation. We give two more "shadow definitions".

Definition 4.2.3. Let K be a model of $\mathcal{T}_{\mathrm{Pf}}$ and suppose $\vec{\sigma}$ is an (n, r)-sequence. We put $D^r(\vec{\sigma}, K) = K^r$.

Definition 4.2.4. Let $k, K \models \mathcal{T}_{\mathrm{Pf}}$, with $k \subseteq K$ and let $\vec{\sigma}$ be an (n, r)-sequence. We denote by $M^r(k, K, \vec{\sigma})$ the ring of all functions $f : K^r \to K$ for which there exists a polynomial $p(x_1, \ldots, x_r, y_1, \ldots y_n) \in k[x_1, \ldots, x_r, y_1, \ldots y_n]$ such that $f(\vec{\alpha}) = p(\vec{\alpha}, \vec{\sigma}(\vec{\alpha}))$ for all $\vec{\alpha} \in K^r$.

The reason behind introducing these definitions now is that in the upcoming sections we will develop techniques for the theories $\mathcal{T}_{\mathrm{Pf}\restriction}$ and $\mathcal{T}_{\mathrm{Pf}}$ simultaneously. We will use these techniques in the $\mathcal{T}_{\mathrm{Pf}\restriction}$ case in our proof of the First Main Theorem. The techniques in the $\mathcal{T}_{\mathrm{Pf}}$ case will be used in the proof of the Second Main Theorem.

Remark 4.2.5. Since we will be needing this later on, we ask the reader to verify that Corollary 4.1.3 also holds with $\mathcal{T}_{\mathrm{Pf}\restriction}$ replaced by $\mathcal{T}_{\mathrm{Pf}}$, using the same proof. (In fact, we do not even need Lemma 4.1.2 in this proof, since we can invoke Proposition 4.1.1 directly.)

Lemma 4.2.6. *Every $\mathcal{L}_{\mathrm{Pf}}$-term is part of a Pfaffian chain of $\mathcal{L}_{\mathrm{Pf}}$-terms.*

Proof. We use induction on terms. Clearly every constant and every variable of $\mathcal{L}_{\mathrm{Pf}}$ is part of a Pfaffian chain, namely the chain consisting of just that constant or variable. Now suppose that for each $i = 1, \ldots, m$ we are given a Pfaffian chain $g_1^i, \ldots, g_{n_i}^i$, of terms of $\mathcal{L}_{\mathrm{Pf}}$. Take some $1 \leq t \leq l$. We show that the term $H_t(g_{n_1}^1, \ldots, g_{n_m}^m)$ is part of a Pfaffian chain. We claim that the following chain of functions is a Pfaffian chain

$$g_1^1, \ldots, g_{n_1}^1, g_1^2, \ldots, g_{n_2}^2, \ldots, g_1^m, \ldots, g_{n_m}^m, H_1(g_{n_1}^1, \ldots, g_{n_m}^m), \ldots, H_t(g_{n_1}^1, \ldots, g_{n_m}^m).$$

For $j = 1, \ldots, t$, we check that the derivatives of the function $H_j(g_{n_1}^1, \ldots, g_{n_m}^m)$ satisfy the conditions of Definition 1.2.1. This is trivial for the other functions in the chain. Recall that by the chain rule

$$\frac{\partial}{\partial x_s} H_j(g_{n_1}^1, \ldots, g_{n_m}^m) = \sum_{i=1}^{m} \frac{\partial g_{n_i}^i}{\partial x_s} \frac{\partial H_j}{\partial x_i}(g_{n_1}^1, \ldots, g_{n_m}^m). \tag{9}$$

Since $H_1, \ldots, H_l$ is a Pfaffian chain, there exist polynomials $p_1, \ldots p_m$ such that

$$\frac{\partial H_j}{\partial x_i}(g_{n_1}^1, \ldots, g_{n_m}^m) = p_i(\vec{x}, H_1(g_{n_1}^1, \ldots, g_{n_m}^m), \ldots, H_j(g_{n_1}^1, \ldots, g_{n_m}^m))$$

and by our induction hypothesis, there exist polynomials, $q_1, \ldots, q_m$, such that

$$\frac{\partial g_{n_i}^i}{\partial x_s} = q_i(\vec{x}, g_1^i, \ldots, g_{n_i}^i),$$

for each $i = 1 \ldots, m$. If we substitute these expressions into (9), then we see that $\frac{\partial}{\partial x_s} H_j(g^1_{n_1}, \ldots, g^m_{n_m})$ indeed is of the right form. A similar argument can be made regarding the function symbols $\cdot, +, -$. This shows that our chain of functions is indeed a Pfaffian chain, so this concludes our induction. □

As we already divulged, we shall be developing Theorems for $\mathcal{T}_{\mathrm{Pf}\restriction}$ and $\mathcal{T}_{\mathrm{Pf}}$ simultaneously. In the $\mathcal{T}_{\mathrm{Pf}\restriction}$ situation, we can use the quite powerful result of Corollary 4.1.7, which we do not have in the $\mathcal{T}_{\mathrm{Pf}}$ case. Using Lemma 4.2.6, we can prove the following Corollary to Theorem 4.1.4, which will serve as a substitute for this.

Corollary 4.2.7. *Suppose that $\phi(x_1, \ldots, x_p)$ is an existential formula in the language $\mathcal{L}_{\mathrm{Pf}}$. Then there exists $N \in \mathbb{N}$ such that for all $r_2, \ldots, r_p \in \mathbb{R}$, the set*

$$\{r_1 \in \mathbb{R} \mid \mathbb{R} \models \phi(r_1, \ldots, r_p)\}$$

is a union of at most N open intervals and N points.

Proof. By Lemma 2.1.5, we may suppose that ϕ has the form

$$\exists y_1, \ldots, y_n \bigwedge_{i=1}^{m} \tau_i = 0,$$

where each $\tau_i(\vec{x}, \vec{y})$ is an $\mathcal{L}_{\mathrm{Pf}}$-term. Then clearly $\phi(\vec{x})$ is equivalent to $\exists \vec{y}(f(\vec{x}, \vec{y}) = 0)$, where $f = \tau_1^2 + \cdots + \tau_m^2$. Since f is a term of $\mathcal{L}_{\mathrm{Pf}}$, Lemma 4.2.6 tells us that f is part of some Pfaffian chain of functions, $h_1, \ldots, h_t : \mathbb{R}^{p+n} \to \mathbb{R}$, say. So surely, $f \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_t]$. Then by Theorem 4.1.4, there exists $N_0 \in \mathbb{N}$ such that for all $r_2, \ldots, r_p \in \mathbb{R}$, the set

$$\{(p, q_1, \ldots, q_n) \in \mathbb{R}^{1+n} \mid f(p, r_1, \ldots, r_p, q_1, \ldots, q_n) = 0\}$$

has at most N_0 components. Let us call this set $Z(r_2, \ldots, r_p)$ for convenience. Now note that

$$\begin{aligned} &\{r_1 \in \mathbb{R} \mid \phi(r_1, \ldots, r_p)\} \\ =&\{r_1 \in \mathbb{R} \mid \exists q_1, \ldots, q_n(f(r_1, \ldots, r_p, q_1, \ldots, q_n) = 0)\} \\ =&\pi[Z(r_2, \ldots, r_p)], \end{aligned}$$

where $\pi : \mathbb{R}^{1+n} \to \mathbb{R}$ is the projection onto the first coordinate. Since π is continuous, $\pi[Z(r_2, \ldots, r_p)]$ can have at most the same number of components as $Z(r_2, \ldots, r_p)$ has.

Hence, the Boolean algebra B, formed by the components of $\{r_1 \in \mathbb{R} \mid \mathbb{R} \models \phi(r_1, \ldots, r_p)\}$, has size at most N_0. Since B is finite, it must be atomic and its set of atoms is certainly not larger than N_0 as well. Now note that every atom $a \in B$ is a connected subset of $\mathbb{R}$, for otherwise it would split up into two components. Hence, every atom $a \in B$ is either a point or an interval. This shows that $\{r_1 \in \mathbb{R} \mid \mathbb{R} \models \phi(r_1, \ldots, r_p)\}$ can we written as a union of N_0 intervals and $2N_0$ points. So setting $N = 2N_0$ suffices. □

4.3 Parametrization Theorems

From this point on, we let $\mathcal{T}_{\mathrm{Pf}(\restriction)}$ be either $\mathcal{T}_{\mathrm{Pf}\restriction}$ or $\mathcal{T}_{\mathrm{Pf}}$ and similarly, we let $\mathcal{L}_{\mathrm{Pf}(\restriction)}$ be either $\mathcal{L}_{\mathrm{Pf}\restriction}$ or $\mathcal{L}_{\mathrm{Pf}}$. In this section we show that under certain conditions, curves that are implicitly defined in models of $\mathcal{T}_{\mathrm{Pf}(\restriction)}$, can be explicitly parametrized by finitely many definable C^∞-functions, defined on open intervals. First, we need two Lemmas, one analytic and one combinatorial in nature.

(In [Wil96] the author remarks in passing that the following result requires f to be continuous. Perhaps he had a proof in mind that is only valid for continuous functions.)

Lemma 4.3.1. *Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$. Let $f : (a,b) \to \mathbb{R}^n$, for some $n \in \mathbb{N}$. Then either $\lim_{x \uparrow b} \|f(x)\| = \infty$ or $(b, \vec{c})$ is a limit point of* graph(f) *for some $\vec{c} \in \mathbb{R}^n$.*

Proof. Suppose that $\lim_{x \uparrow b} \|f(x)\| \neq \infty$. Then

$$\exists R \, \forall \delta > 0 \, \exists x \in (b - \delta, b) \, (\|f(x)\| \leq R).$$

Fix such an R and take for every $\delta_m = \frac{1}{m}$, with $m \in \mathbb{N}$ and $m \geq 1$, an element $x_m \in (b - \delta_m, b)$ such that $\|f(x)\| \leq R$. Then $(f(x_m))_m$ is a bounded sequence in $\mathbb{R}^n$. By the Bolzano-Weierstrass Theorem, this sequence has a convergent subsequence. Let $\vec{c}$ be the limit of this subsequence. Then clearly $(b, \vec{c})$ is a limit point of graph(f). □

Lemma 4.3.2. *Let $n, N \in \mathbb{N}$, with $n, N \geq 1$. Then there exist $Q_1, \ldots, Q_s \in \mathbb{Z}^n$, where $s = n \cdot N^2 + 1$, with the property that for any field K of characteristic 0 and any distinct $P_1, \ldots, P_m \in K^n$, with $m \leq N$, there exists $1 \leq i \leq s$, such that the dot products $Q_i \cdot P_1, \ldots, Q_i \cdot P_m$ are distinct elements of K.*

Proof. Let us prove two claims.

Claim 1. A vector space V over an infinite field F can never be written as a finite union of proper subspaces.

Proof. Suppose to the contrary that

$$V = \bigcup_{i=1}^{l} V_i,$$

where the $V_i \subseteq V$ are proper subspaces of V. Without loss of generality we may assume that

$$V_1 \nsubseteq \bigcup_{i=2}^{l} V_i,$$

for otherwise we might as well remove V_1 from this union. Pick $v \in V_1$ and let $u \in V \setminus V_1$. Then u is nonzero, so the set $A = \{v + x \cdot u \mid x \in F \setminus \{0\}\}$ is infinite, as F is infinite. Also note that $A \cap V_1 = \emptyset$, since otherwise u would be in V_1. This means that one of the sets $V_2, \ldots, V_l$, let us say V_2, must contain at least two (in fact infinitely many) elements from A. But this implies that $u \in V_2$ and hence also $v \in V_2$. Since v was arbitrary, we find

$$V_1 \subseteq \bigcup_{i=2}^{l} V_i,$$

which is false, proving the claim.

Claim 2. For any $t \in \mathbb{N}$ there exists a t-element set, $\{Q_1, \ldots, Q_t\} \subseteq \mathbb{Z}^n$, such that any subset of size less than or equal to n is linearly independent over $\mathbb{Q}$.

Proof. We construct such a set recursively. Certainly $\emptyset$ satisfies these conditions for $t = 0$. Now suppose that the set $A = \{Q_1, \ldots, Q_t\}$ meets our criteria. Set $\mathcal{A} = \{X \subseteq A \mid |X| < n\}$ and consider

$$B = \bigcup_{X \in \mathcal{A}} \operatorname{span}(X),$$

where span(X) denotes the linear span of X in the vector space $\mathbb{Q}^n$. Then B is a proper subset of $\mathbb{Q}^n$ by our first claim, so there exists a point $Q \in \mathbb{Q}^n \setminus B$. We take some nonzero $q \in \mathbb{Q}$ such that $q \cdot Q \in \mathbb{Z}^n$. Now if we let $Q_{t+1} = q \cdot Q$, then any subset of $\{Q_1, \ldots, Q_t, Q_{t+1}\}$, of size less than or equal to n, is linearly independent over $\mathbb{Q}$ by choice of Q, so we are done.

Take $Q_1, \ldots, Q_s \in \mathbb{Z}^n$ such that any n of them are linearly independent over $\mathbb{Q}$. This is equivalent to the statement that all $n \times n$ submatrices of $\left(Q_1^T, \ldots, Q_s^T\right)$ have nonzero determinant. If K is a field of characteristic 0, then these determinants are also nonzero in K, so any n vectors among $Q_1, \ldots, Q_s$ are also linearly independent over K.

Suppose that the lemma is false. Then there exists a field K of characteristic 0 and distinct $P_1, \ldots, P_m \in K^n$, with $m \leq N$, such that for each $1 \leq i \leq s$ we have $Q_i \cdot P_{\alpha_i} = Q_i \cdot P_{\beta_i}$, for some $1 \leq \alpha_i < \beta_i \leq m$. Let $f : \{1, \ldots, s\} \to \{1, \ldots, m\} \times \{1, \ldots, m\}$ be the function defined by $f(i) = (\alpha_i, \beta_i)$. Since the domain of f has size $n \cdot N^2 + 1$ and the codomain of f has size $m^2 \leq N^2$, there must exist $1 \leq \alpha < \beta \leq m$ and $1 \leq i_1 < \ldots < i_n \leq s$ such that $f(i_j) = (\alpha, \beta)$ for all i_j, by the pigeonhole principle. By definition of f, this means that $Q_{i_j} \cdot (P_\alpha - P_\beta) = 0$, for all i_j, hence $(P_\alpha - P_\beta) \cdot \left(Q_{i_1}^T, \ldots, Q_{i_n}^T\right) = (0, \ldots, 0)$. Since $P_\alpha - P_\beta \neq (0, \ldots, 0)$, this contradicts the fact that the matrix $\left(Q_{i_1}^T, \ldots, Q_{i_n}^T\right)$ is invertible. ☐

Theorem 4.3.3. *Let k and K be models of $\mathcal{T}_{\mathrm{Pf}(\restriction)}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$, with $r \geq 2$, and let $\vec{\sigma}$ be an (n, r)-sequence. Take $g_1, \ldots, g_{r-1} \in M^r(k, K, \vec{\sigma})$ and suppose that $\mathcal{V}(g_1, \ldots, g_{r-1})$ is closed in K^r and moreover, for all $P \in \mathcal{V}(g_1, \ldots, g_{r-1})$,*

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0.$$

Then there exists a finite set, $\mathcal{S}$, of pairs (I, ϕ), satisfying the following conditions.

(i) *For each $(I, \phi) \in \mathcal{S}$, I is a nonempty open interval in K and $\phi : I \to K^{r-1}$ is a definable C^∞-function.*

(ii) *For each $(I, \phi) \in \mathcal{S}$ holds that if $\sup(I) \in K$ (that is, $\sup(I) \neq \infty$), then*

$$\lim_{x \uparrow \sup(I)} \|\phi(x)\| = \infty,$$

and similarly, if $\inf(I) \in K$ (meaning $\inf(I) \neq -\infty$), then

$$\lim_{x \downarrow \inf(I)} \|\phi(x)\| = \infty.$$

(iii) *The set $\mathcal{V}(g_1, \ldots, g_{r-1})$ is equal to the union*

$$\bigcup \{\mathrm{graph}(\phi) \mid (I, \phi) \in \mathcal{S}\}$$

and this union is disjoint.

Proof. For an element $p_1 \in K$, we write

$$\mathcal{V}_{p_1} = \{(p_2, \ldots, p_r) \in K^{r-1} \mid (p_1, \ldots, p_r) \in \mathcal{V}(g_1, \ldots, g_{r-1})\}.$$

By Corollary 4.1.3 (also see Remark 4.2.5), there is some $N \in \mathbb{N}$ such that for each $p_1, \in K$, the set V_{p_1} contains at most N elements. Let $s = (r-1) \cdot N^2 + 1$ and take $Q_1, \ldots, Q_s \in \mathbb{Z}^n$ as in Lemma 4.3.2. For $i = 1, \ldots, s$, we write

$$Q_i \cdot \mathcal{V}_{p_1} = \{Q_i \cdot (p_2, \ldots, p_r) \mid (p_2, \ldots, p_r) \in \mathcal{V}_{p_1}\} \subseteq K.$$

Now for each $m = 1, \dots, N$ and $i = 1, \dots, s$ we define the set

$$A_{m,i} = \{p_1 \in K \mid m = |\mathcal{V}_{p_1}| = |Q_i \cdot \mathcal{V}_{p_1}|\}.$$

Note that the sets $A_{m,i}$ are definable in K using parameters, so if $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}\restriction}$, then each $A_{m,i}$ is a finite union of intervals and points by Corollary 4.1.7. To get this same result for $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}}$, we need to argue a little bit further.

Claim 1. Each $A_{m,i}$ can be defined by a Boolean combination of existential $\mathcal{L}_{\mathrm{Pf}(\restriction)}$-formulas with parameters from K.

Proof. Since $|\mathcal{V}_{p_1}| \geq |Q_i \cdot \mathcal{V}_{p_1}|$ always holds, it suffices to find formulas $\chi_1(x)$ and $\chi_2(x)$ expressing $m \geq |\mathcal{V}_x|$ and $|Q_i \cdot \mathcal{V}_x| \geq m$ respectively, as their conjunction will then define $A_{m,i}$. We define $\chi_1(x)$ by

$$\forall \vec{y}_1, \dots, \vec{y}_{m+1} \left[\left(\bigwedge_{p=1}^{m+1} \bigwedge_{q=1}^{r-1} g_q(x, \vec{y}_p) = 0 \right) \rightarrow \left(\bigvee_{1 \leq p < q \leq m+1} \vec{y}_p = \vec{y}_q \right) \right]$$

and we define $\chi_2(x)$ by

$$\exists x_1, \dots, x_m \exists \vec{y}_1, \dots, \vec{y}_m \left[\left(\bigwedge_{1 \leq p < q \leq m} x_p \neq x_q \right) \wedge \left(\bigwedge_{p=1}^{m} \bigwedge_{q=1}^{r-1} g_q(x, \vec{y}_p) = 0 \right) \wedge \left(\bigwedge_{j=1}^{m} x_j = Q_1 \cdot \vec{y}_j \right) \right].$$

Then $\chi_1(x)$ and $\chi_2(x)$ express the desired properties. Furthermore, $\chi_1(x)$ is a negated existential formula and $\chi_2(x)$ is an existential formula, so this proves the claim.

Now note that the collection of subsets of K which can be written as a finite union of points and intervals forms a Boolean algebra. In the case $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}}$, Corollary 4.2.7 also holds in K, by transfer. So, using our claim, each $A_{m,i}$ is a finite union of intervals and points, just like we saw earlier in the case $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}\restriction}$.

It follows that there exists $t \in \mathbb{N}$ and $a_1, \dots, a_t \in K$, such that

$$a_0 < a_1 < \cdots < a_t < a_{t+1},$$

where $a_0 = -\infty$ and $a_{t+1} = \infty$, with the property that for each $j = 0, \dots, t$, each $m = 1, \dots, N$ each $i = 1, \dots, s$ and each pair of points $p, q \in (a_j, a_{j+1})$ holds that $p \in A_{m,i}$ if and only if $q \in A_{m,i}$. For $p \in K$, we let $m(p) = |\mathcal{V}_p|$. Furthermore, we let $i(p)$ be the least i such that $|Q_i \cdot \mathcal{V}_p| = m(p)$. Since $|\mathcal{V}_p| \leq N$, such an i exists by virtue of Lemma 4.3.2. By definition of the $a_0, \dots, a_{t+1}$, the values of $m(p)$ and $i(p)$ do not depend on the choice of $p \in (a_j, a_{j+1})$, within each interval. We may therefore denote these numbers by m_j and i_j respectively. For each $j = 0, \dots, t$ such that $m_j \geq 1$, we can define functions $\phi_{j,l} : (a_j, a_{j+1}) \to K^{r-1}$, for every $l = 1, \dots, m_j$, such that for $x \in (a_j, a_{j+1})$,

$$\phi_{j,l}(x) = \vec{y}$$

if and only if

$$\exists \vec{y}_1, \dots, \vec{y}_{m_j} \left[\left(\bigwedge_{i=1}^{m_j} (x, \vec{y}_i) \in \mathcal{V}(g_1, \dots, g_{r-1}) \right) \wedge \left(\bigwedge_{i=1}^{m_j - 1} Q_{i_j} \cdot \vec{y}_i < Q_{i_j} \cdot \vec{y}_{i+1} \right) \wedge \vec{y} = \vec{y}_l \right].$$

Clearly

$$(a_j, a_{j+1}) \times K^{r-1} \cap \mathcal{V}(g_1, \dots, g_{r-1}) = \bigcup \{\mathrm{graph}(\phi_{j,l}) \mid 1 \leq l \leq m_j\},$$

where the union is disjoint. We shall now argue that each $\phi_{j,l}$ is infinitely differentiable. Take a point $x \in (a_j, a_{j+1})$. Since each point $(x, \phi_{j,l}(x))$ lies in $\mathcal{V}(g_1, \dots, g_{r-1})$, by Theorem 3.2.2 there exist C^∞-functions $\theta_1, \dots, \theta_{m_j}$ defined on a neighborhood of x such that $\theta_l(x) = \phi_{j,l}(x)$, for each $l = 1, \dots, m_j$. Note that this implies

$$Q_{i_j} \cdot \theta_1(x) < \cdots < Q_{i_j} \cdot \theta_{m_j}(x).$$

Since the functions $Q_{i_j} \cdot \theta_1, \dots, Q_{i_j} \cdot \theta_{m_j}$ are continuous, the inequalities

$$Q_{i_j} \cdot \theta_1(z) < \cdots < Q_{i_j} \cdot \theta_{m_j}(z)$$

hold for all z in some small neighborhood of x. Furthermore, the points $(z, \theta_l(z))$ lie in $\mathcal{V}(g_1, \dots, g_{r-1})$, for each $l = 1, \dots, m_j$. This means that the functions θ_l and $\phi_{j,l}$ must coincide in some neighborhood of x, for each $l = 1, \dots, m_j$. This clearly implies that the functions $\phi_{j,1}, \dots, \phi_{j,m_j}$ are of class C^∞.

Now take $j < t$ and also fix $1 \leq l \leq m_j$. Then $\sup((a_j, a_{j+1}) \in K$. By transferring Lemma 4.3.1 to K, we have that either $\lim_{x \uparrow a_{j+1}} \|\phi_{j,l}(x)\| = \infty$ or $(a_{j+1}, p_2, \dots, p_r)$ is a limit point of $\mathrm{graph}(\phi_{j,l})$ for some $(p_2, \dots, p_r) \in K^{r-1}$. Suppose that the latter is true. Since $\mathrm{graph}(\phi_{j,l}) \subseteq \mathcal{V}(g_1, \dots, g_{r-1})$ it is also a limit point of $\mathcal{V}(g_1, \dots, g_{r-1})$, and since this set is closed by hypothesis, we have $(a_{j+1}, p_2, \dots, p_r) \in \mathcal{V}(g_1, \dots, g_{r-1})$. By Theorem 3.2.2, there exists an open neighborhood $U \subseteq K^{r-1}$ of $(p_2, \dots, p_r)$ and positive $\varepsilon \in K$, with

$$a_j < a_{j+1} - \varepsilon < a_{j+1} < a_{j+1} + \varepsilon < a_{j+2}$$

and a definable C^∞-function $\theta : (a_{j+1} - \varepsilon, a_{j+1} + \varepsilon) \to U$, such that $\theta(a_{j+1}) = (p_2, \dots, p_r)$ and

$$(a_{j+1} - \varepsilon, a_{j+1} + \varepsilon) \times U \cap \mathcal{V}(g_1, \dots, g_{r-1}) = \mathrm{graph}(\theta).$$

Claim 2. The functions θ and $\phi_{j,l}$ coincide on the interval $(a_{j+1} - \varepsilon, a_{j+1})$.

Proof. Since intervals in $\mathbb{R}$ are connected, intervals in K are *definably* connected, meaning that they can not be written as the disjoint union of two definable open sets in a nontrivial way. So to prove our claim, it suffices to prove that the definable set

$$A = \{p \in (a_{j+1} - \varepsilon, a_{j+1}) \mid \theta(p) = \phi_{j,l}(p)\}$$

is open, closed and nonempty. Clearly the set A is closed, as θ and ϕ are both continuous. Furthermore, since $(a_{j+1}, p_2, \dots, p_r)$ is a limit point of $\mathrm{graph}(\phi_{j,l})$, the set $(a_{j+1} - \varepsilon, a_{j+1}) \times U$ must contain points of $\mathrm{graph}(\phi_{j,l})$, which are then automatically also points of $\mathrm{graph}(\theta)$, so A is nonempty. Lastly, to show that A is open, pick a point $p \in A$. Then $\theta(p) = \phi_{j,l}(p)$, so

$$Q_{i_{j+1}} \cdot \phi_{j,1}(p) < \ldots < Q_{i_{j+1}} \cdot \phi_{j,l-1}(p) < Q_{i_{j+1}} \cdot \theta(p) < Q_{i_{j+1}} \cdot \phi_{j,l+1}(p) < \ldots < Q_{i_{j+1}} \cdot \phi_{j,m_j}(p).$$

Again, by continuity, these inequalities hold for all points in some neighborhood of p, and hence $\theta(q) = \phi_{j,l}(q)$ for all points q in this neigbourhood. It follows that A is open, proving the claim.

By a similar argument, there exists $1 \leq l' \leq m_{j+1}$ such that θ coincides with the function $\phi_{j+1,l'}$ on the interval $(a_{j+1}, a_{j+1} + \varepsilon)$. This shows that $\phi_{j,l}$, $\phi_{j+1,l'}$ and $\{(a_{j+1}, p_2, \dots, p_r)\}$ can be glued together to form a definable C^∞-function from (a_j, a_{j+2}) to K^{r-1}. The Theorem follows by performing these gluings exhaustively. As a final detail, we should point out that every point P on the line $\{a_{j+1}\} \times K^{r-1}$ lying in $\mathcal{V}(g_1, \dots, g_{r-1})$ will be part of some gluing in the end. By Theorem 3.2.2, such a point is part of the graph of some definable C^∞-function $\theta : (a_{j+1} - \varepsilon, a_{j+1} + \varepsilon) \to K^{r-1}$. Subsequently, one can show that θ must coincide with some $\phi_{j,l}$ on the interval $(a_{j+1} - \varepsilon, a_{j+1})$, using the ideas above, showing that P is part of the same gluing as $\phi_{j,l}$. □

We will refer to the set $\mathcal{S}$, as given in Theorem 4.3.3 as a *parametrization* of $\mathcal{V}(g_1, \ldots, g_{r-1})$. Such a parametrization gives us a firm grasp on the set $\mathcal{V}(g_1, \ldots, g_{r-1})$, and in fact, it lies at the heart of the proof of Lemma 2.3.2. The idea is that if $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ is closed in the model $(k \mid \mathcal{L}_{\mathrm{Pf}(\restriction)})$, then we can apply Theorem 4.3.3 with $K = k$, to obtain a parametrization $\mathcal{S}'$ of $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ in $(k \mid \mathcal{L}_{\mathrm{Pf}(\restriction)})$. Our goal then is to derive some connection between $\mathcal{S}$ and $\mathcal{S}'$. The following Lemma serves as a first step in that direction.

Lemma 4.3.4. *Let k and K be models of $\mathcal{T}_{\mathrm{Pf}(\restriction)}$, with $k \subseteq K$. Let $n, r \in \mathbb{N}$, with $r \geq 2$, and let $\vec{\sigma}$ be an (n, r)-sequence. Take $g_1, \ldots, g_{r-1} \in M^r(k, K, \vec{\sigma})$ and suppose that $\mathcal{V}(g_1, \ldots, g_{r-1})$ is closed in K^r and furthermore, for all $P \in \mathcal{V}(g_1, \ldots, g_{r-1})$,*

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0.$$

Suppose also that every $(k, \vec{\sigma})$-definable point of $\mathcal{V}(g_1, \ldots, g_{r-1})$ lies in k^r. We write

$$K^- = \{\alpha \in K \mid -\beta < \alpha < \beta \text{ for some } \beta \in k\}.$$

Now take $\alpha \in K^-$ and $P \in K^{r-1}$ such that $\|P\| \in K^-$ and $(\alpha, P) \in \mathcal{V}(g_1, \ldots, g_{r-1})$. Then there exist $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2 \in k$, with $\gamma_2 < \gamma_1 < \alpha < \beta_1 < \beta_2$ and $\|P\| < B_1 < B_2$, $m \in \mathbb{N}$, with $m \geq 1$, and K-definable C^∞-functions $\phi_i : (\gamma_2, \beta_2) \to K^{r-1}$, such that

(i) *$\|\phi_i(p)\| < B_1$, for $i = 1, \ldots, m$ and $p \in (\gamma_2, \beta_2)$.*

(ii) *The set*

$$\mathcal{V}(g_1, \ldots, g_{r-1}) \cap \big((\gamma_2, \beta_2) \times \{Q \in K^{r-1} \mid \|Q\| < B_2\}\big)$$

is equal to

$$\bigcup_{i=1}^{m} \mathrm{graph}(\phi_i)$$

and this union is disjoint.

(iii) *If $p \in (\gamma_2, \beta_2)$, with $p \in k$, then $\phi_i(p) \in k^{r-1}$, for $i = 1, \ldots, m$.*

Furthermore, if $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ is closed in k^r, there exist k-definable C^∞-functions ψ_i : $(\gamma_2, \beta_2) \to K^{r-1}$, for $i = 1, \ldots, m$, such that **(i)** *and* **(ii)** *hold with ψ_i in place of ϕ_i, where all notions are interpreted in k.*

Proof. As in the proof of Theorem 4.3.3, we write

$$\mathcal{V}_\alpha = \{(p_2, \ldots, p_r) \in K^{r-1} \mid (\alpha, p_2, \ldots, p_r) \in \mathcal{V}(g_1, \ldots, g_{r-1})\}.$$

Let $m \in \mathbb{N}$ be the number of points Q satisfying $Q \in \mathcal{V}_\alpha$ and $\|Q\| \in K^-$. Recall that the number of these points is indeed finite by Corollary 4.1.3 (and Remark 4.2.5) and note that $m \geq 1$, as P is such a point. We denote these points by $P_1, \ldots, P_m$. Take $B \in k$ such that $\|P_i\| < B$ for each $i = 1, \ldots, m$ and choose $B' \in k$ with $B < B'$. Let $\mathcal{S}$ be as in Theorem 4.3.3 and for each $i = 1, \ldots, m$, let (I_i, ϕ_i) be the element of $\mathcal{S}$ such that $\alpha \in I_i$ and $\phi_i(\alpha) = P_i$. We write

$$I = \bigcap_{i=1}^{m} I_i.$$

Consider the set $A^+ \subseteq K$, consisting of those elements $p \in I$, with $\alpha \leq p$, such that for all $q \in [\alpha, p]$ and $i = 1, \ldots, m$ holds that $\|\phi_i(q)\| < B$ and $\phi_1(q), \ldots, \phi_m(q)$ are the only points $Q \in \mathcal{V}_q$ satisfying $\|Q\| \leq B'$. Keep in mind that the set A^+ depends on B and B', even though our notation does not reflect this. We shall write $A^+_{B,B'}$ whenever we need to emphasize this fact. Note that for $i = 1, \ldots, m$, the set

$$\{q \in I_i \mid \|\phi_i(q)\| < B\}$$

is open in K by continuity of ϕ_i. Furthermore, if $(J, \phi) \in \mathcal{S} \setminus \{(I_1, \phi_1), \ldots, (I_m, \phi_m)\}$, then the set

$$\{q \in J \mid \|\phi(q)\| \geq B'\}$$

is not only closed in J, by continuity of ϕ, but also closed in K, as it has no limit points on the boundary of J, by part **(ii)** of Theorem 4.3.3. Combing these two facts with part **(iii)** of Theorem 4.3.3 shows that A^+ is an interval in K of the form $[\alpha, \beta)$, with $\beta \in K \cap \{\infty\}$. Note that certainly $\alpha \in A^+$, by choice of B and B', so $\alpha < \beta$. If $\beta = \infty$, we simply take $\beta_1, \beta_2 \in k$ such that $\alpha < \beta_1 < \beta_2$. This is possible, as $\alpha \in K^-$.

Suppose on the other hand that $\beta \in K$. Then we claim that $\beta \in k$. First we need that $\beta \in I$. If I is unbounded on the right, then this is certainly true. If I is bounded on the right, then by part **(ii)** of Theorem 4.3.3, there is $1 \leq i \leq m$ and $q \in I$ such that $\|\phi_i(q)\| \geq B$. Since $\beta \leq q$ by definition of A^+, it follows that in this case we also have $\beta \in I$. This implies that there is some $Q \in \mathcal{V}_\beta$ such that either $\|Q\| = B$ or $\|Q\| = B'$. This follows from the fact that β is the least element (greater that α) such that $\beta \notin A^+$ and the fact that the set $\mathcal{V}(g_1, \ldots, g_{r-1})$ is parameterized by the finitely many continuous functions from $\mathcal{S}$. We define the function $h : D^r(\vec{\sigma}, K) \to K$ by

$$h(x_1, \ldots, x_r) = \left(\sum_{i=2}^{r} x_i^2 \right) - B^2,$$

in the case $\|Q\| = B$ or

$$h(x_1, \ldots, x_r) = \left(\sum_{i=2}^{r} x_i^2 \right) - (B')^2,$$

in the case $\|Q\| = B'$. Then $h \in M^r(k, K)$ and h vanishes at the point (β, Q). However, for no point $q \in [\alpha, \beta)$ does there exist $P \in \mathcal{V}_q$ such that $B \leq \|P\| \leq B'$, by definition of A^+. Hence, h does not vanish on $\mathcal{V}(g_1, \ldots, g_r) \cap W$ for any open neighborhood W of (β, Q). Define the subring

$$M = \{[f, D^r(\vec{\sigma}, K)] \mid f \in M^r(k, K, \vec{\sigma})\}$$

of $\mathcal{D}_{(\beta,Q)}$. Note that M Noetherian and closed under differentiation, as $M^r(k, K, \vec{\sigma})$ is. We wish to apply Theorem 3.2.7 with respect to the point $(\beta, Q) \in K^r$ and the functions $[g_i, D^r(\vec{\sigma}, K)] \in M$, for $i = 1, \ldots, r-1$. Since $(\beta, Q) \in \mathcal{V}(g_1, \ldots, g_{r-1})$ and

$$\det \left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)} \right) (\beta, Q) \neq 0,$$

by assumption, we have $(\beta, Q) \in \mathcal{V}_r(g_1, \ldots, g_{r-1})$, so we may indeed apply the Theorem. Because $r - 1 < r$, either **(ii)** or **(iii)** of Theorem 3.2.7 must hold. Option **(ii)**, however, is not possible by what we have just proven. This means that **(iii)** must hold, so (β, Q) is $(k, \vec{\sigma})$-definable as a direct consequence. By our hypothesis, this implies that $(\beta, Q) \in k^r$, proving our claim.

We take $\beta_1 = \beta$ and choose $B_1, B_2 \in k$ such that $B < B_1 < B_2 < B'$. Then $A^+_{B_1,B_2} = [\alpha, \beta')$ for some $\beta' \in k \cup \{\infty\}$. Using the continuity of the functions ϕ, for $(J, \phi) \in \mathcal{S}$, it is not difficult to verify that $\beta_1 < \beta'$. If $\beta' \in k$, we take $\beta_2 = \beta'$. In case $\beta' = \infty$, we take $\beta_2 = \beta_1 + 1$.

Analogously, by defining the set A^- in the obvious way, using the same B, B', B_1 and B_2 as before, we find γ_1 and γ_2 as asserted in the statement of the Lemma.

We move on to proving the last statement of the Lemma, so suppose that $\mathcal{V}(g_1, \dots, g_{r-1}) \cap k^r$ is closed in k^r. As a preliminary result, we will show that for a point $\gamma_2 < p < \beta_2$ and $i = 1, \dots, m$, holds that $\phi_i(p) \in k^{r-1}$. Take such a point $p \in k$ and suppose that $(p, Q) \in \mathcal{V}(g_1, \dots, g_{r-1})$. We define the function $h : D^r(\vec{\sigma}, K) \to K$ by

$$h(x_1, \dots, x_r) = x_1 - p.$$

Then $h \in M^r(k, K)$ and h vanishes at the point (p, Q). Also, h does not vanish on $\mathcal{V}(g_1, \dots, g_r) \cap W$ for any open neighborhood W of (p, Q). We can therefore apply Theorem 3.2.7 and our assumption on $(k, \vec{\sigma})$-definable points in the same way as before to conclude that $Q \in k^{r-1}$. Since each $\phi_i(p)$ is such a point, we find that $\phi_i(p) \in k^{r-1}$ for $i = 1, \dots, m$. Since $\mathcal{V}(g_1, \dots, g_{r-1})$ has a quantifier free definition, we find, using **(ii)**, that for every point $\gamma_2 < p < \beta_2$, there are exactly m points $Q \in k^{r-1}$ satisfying

$$k \models (p, Q) \in \mathcal{V}(g_1, \dots, g_{r-1}) \wedge \|Q\| < B_2.$$

Furthermore, by **(i)**, these points satisfy $\|Q\| < B_1$. Let $Q_1, \dots, Q_m$ be these points for $p = \frac{\gamma_2+\beta_2}{2}$. Let $\mathcal{S}'$ be a parametrization of $\mathcal{V}(g_1, \dots, g_{r-1})$ in k, using Theorem 4.3.3. This means that we apply the Theorem, setting $K = k$. It is not difficult to verify that the hypotheses of Theorem 4.3.3 are satisfied. In particular k models that

$$\det \left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)} \right) (P) \neq 0,$$

for each $P \in \mathcal{V}(g_1, \dots, g_{r-1}) \cap k^r$, since this can be expressed without using quantifiers, as $M^r(k, K, \vec{\sigma})$ is closed under differentiation. For each $i = 1, \dots, m$, let (I'_i, ψ_i) be the element of $\mathcal{S}'$ such that $p \in I'_i$ and $\psi_i(p) = Q_i$. We are done if we manage to show that $(\gamma_2, \beta_2) \subseteq I'_i$ for each $i = 1, \dots, m$. Suppose that this is not the case. Then $\sup(I'_i) \in (\gamma_2, \beta_2)$ or $\inf(I'_i) \in (\gamma_2, \beta_2)$, for some $i = 1, \dots, m$. In either case, there is a point $q \in (\gamma_2, \beta_2) \cap I'_i$ such that $\|\psi_i(p)\| \geq B_1$, by **(ii)** of Theorem 4.3.3. Now by transfer from $\mathbb{R}$, Intermediate Value Theorem holds in k. The Intermediate Value Theorem, when applied to the points $p, q \in (\gamma_2, \beta_2) \cap I'_i$, tells us that there exists a point $x \in (\gamma_2, \beta_2) \cap I'_i$ such that $\|\psi_i(x)\| = B_1$. But this is clearly in violation of **(i)** and **(ii)** of this Lemma. □

Remark 4.3.5. At first sight, it might seem "obvious" that given $\gamma_2 < \alpha < \beta_2$ as in Lemma 4.3.4, there exist $\gamma_1, \beta_1 \in k$ such that $\gamma_2 < \gamma_1 < \alpha < \beta_1 < \beta_2$. In general however, there is no reason to assume that this is true.

4.4 Proof of Lemma 2.3.2.

We are almost ready to present the proof of Lemma 2.3.2. We shall in fact be proving the Lemma not only for $\mathcal{T}_{\mathrm{Pf}\restriction}$, but also for $\mathcal{T}_{\mathrm{Pf}}$, right after we prove the following simple result.

Lemma 4.4.1. *Suppose that (a, b) is an interval in $\mathbb{R}$ and let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Suppose that for each $x \in (a, b)$ such that $f(x) = 0$, we have $f'(x) > 0$. Then f has at most one zero on (a, b).*

Proof. Suppose to the contrary that f has at least two distinct zeros $x_1, x_2 \in (a,b)$. We may assume that $x_1 < x_2$. Since $f(x_1) = 0$, we have by hypothesis that $f'(x_1) > 0$, so there exists $\varepsilon > 0$ such that $f(x) > 0$ for all $x \in (x_1, x_1 + \varepsilon)$. Consider the set $A = \{x \in (a,b) \mid x_1 + \varepsilon \leq x \text{ and } f(x) = 0\}$. The set A is closed in (a,b) by continuity of f, and it is nonempty, as it contains x_2. Furthermore, A is bounded from below by x_1. This means that the infimum of A, let us call it x_3, is an element of A. Note that x_3 is the smallest point strictly greater than x_1, such that $f(x_3) = 0$. Now, $f(x_3) = 0$, so there exists $\eta > 0$ such that $f(x) < 0$ for all $x \in (x_3 - \eta, x_3)$. Since $f(x_1 + \frac{\varepsilon}{2}) > 0$ and $f(x_3 - \frac{\eta}{2}) < 0$, there must exists some $x_1 + \frac{\varepsilon}{2} < x_4 < x_3 - \frac{\eta}{2}$ such that $f(x_4) = 0$, by the Intermediate Value Theorem. This contradicts the minimality of x_3. ☐

Lemma 4.4.2. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}(\restriction)}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n,r)-sequence. Suppose also that for each $s \geq r$ and each $(k,\vec{\sigma})$-definable point $(p_1, \dots, p_s)$ of K^s holds that $p_1, \dots, p_s \in K^-$ (using the notation from Lemma 4.3.4). Then every $(k,\vec{\sigma})$-definable point of K^r lies in k^r.*

Proof. Before we get to the main part of this proof, we handle the cases $r = 0, 1$ separately. If $\vec{\sigma}$ is an $(n,1)$-sequence, then a point $Q \in K$ is $(k,\vec{\sigma})$-definable if there exists $g \in M^s(k,K,\vec{\sigma})$ with $Q \in D^s(\vec{\sigma}, K)$, $g(Q) = 0$ and $g'(Q) \neq 0$. It is clear that the points $Q \in K$ satisfying these equations for a fixed g are isolated. This means that in the case $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}\restriction}$, the set of these points is finite by Corollary 4.1.7. In the case $\mathcal{T}_{\mathrm{Pf}(\restriction)} = \mathcal{T}_{\mathrm{Pf}}$, we note that the properties $g(Q) = 0$ and $g'(Q) \neq 0$ can be expressed without using quantifiers, as $M^s(k,K,\vec{\sigma})$ is closed under differentiation. In this case, Corollary 4.2.7 (after transfer to K) tells us that the set of these points is finite. So, in both cases we can reason as in Corollary 2.3.6, to conclude that k and K must have exactly the same $(k,\vec{\sigma})$-definable points. The case $r = 0$ is trivial.

From now on we assume that $r \geq 2$. We use induction on n. The case $n = 0$ is proven in Corollary 2.3.6 (this result also holds for $\mathcal{T}_{\mathrm{Pf}}$, with the same proof). Let $(\vec{\sigma}, \sigma_{n+1})$ be an $(n+1, r)$-sequence such that for all $s \geq r$, every $(k, (\vec{\sigma}, \sigma_{n+1}))$-definable point of K^s lies in $(K^-)^s$. Let $s \geq r$ and suppose that the point $P \in D^s((\vec{\sigma}, \sigma_{n+1}), K)$ is $(k,\vec{\sigma})$-definable. We need to make an observation about such a point P. Since every $\vec{\sigma}$-bounded variable is in particular $(\vec{\sigma}, \sigma_{n+1})$-bounded, we have $D^s((\vec{\sigma}, \sigma_{n+1}), K) \subseteq D^s(\vec{\sigma}, K)$. Furthermore, if $g \in M^s(k,K,\vec{\sigma})$, then its restriction to $D^s((\vec{\sigma}, \sigma_{n+1}), K)$ lies in $M^s(k, K, (\vec{\sigma}, \sigma_{n+1}))$. This shows that P is also $(k, (\vec{\sigma}, \sigma_{n+1}))$-definable and hence $P \in (K^-)^s$. By induction hypothesis on $\vec{\sigma}$, it follows that $P \in k^s$.

Now let $Q \in K^r$ be $(k, (\vec{\sigma}, \sigma_{n+1}))$-definable. We need to show that $Q \in k^r$. By definition

$$Q \in D^r((\vec{\sigma}, \sigma_{n+1}), K) \tag{10}$$

and there exist $g_1, \dots, g_r \in M^r(k, K, (\vec{\sigma}, \sigma_{n+1}))$, such that

$$g_1(Q) = \cdots = g_r(Q) = 0 \tag{11}$$

$$\det\left(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)}\right)(Q) \neq 0 \tag{12}$$

We shall prove that $Q \in k^r$ under some extra assumptions, which we will justify later. These extra assumptions are

$$g_1, \dots, g_{r-1} \in M^r(k, K, \vec{\sigma}) \tag{13}$$

$$\mathcal{V}(g_1, \dots, g_{r-1}) \text{ is closed in } K^r \text{ and } \mathcal{V}(g_1, \dots, g_{r-1}) \cap k^r \text{ is closed in } k^r \tag{14}$$

$$\mathcal{V}(g_1, \ldots, g_{r-1}) \subseteq D^r((\vec{\sigma}, \sigma_{n+1}), K) \tag{15}$$

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0 \text{ for all } P \in \mathcal{V}(g_1, \ldots, g_{r-1}) \tag{16}$$

$$\text{For all } P \in \mathcal{V}(g_1, \ldots, g_{r-1}), \text{ if } g_r(P) = 0, \text{ then } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(P) < 0. \tag{17}$$

By our observation and (15), every $(k, \vec{\sigma})$-definable point of $\mathcal{V}(g_1, \ldots, g_{r-1})$ lies in k^r. Using our extra assumptions, one easily verifies that the other hypotheses of Lemma 4.3.4 are also satisfied. Recall that $Q \in (K^-)^r$ by our assumptions on $(\vec{\sigma}, \sigma_{n+1})$, so we may apply Lemma 4.3.4 with $(\alpha, P) = Q$. Let $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2$ and ϕ_i, ψ_i (with $i = 1, \ldots, m$) be as in the Lemma. Now let the function ϕ be one of the ϕ_i. For $t \in (\gamma_2, \beta_2)$, we have $(t, \phi(t)) \in \mathcal{V}(g_1, \ldots, g_{r-1})$. So by (15), $(t, \phi(t)) \in D^r((\vec{\sigma}, \sigma_{n+1}), K)$. We may therefore define, for any $g \in M^s(k, K, (\vec{\sigma}, \sigma_{n+1}))$, the function $\overline{g} : (\gamma_2, \beta_2) \to K$ by $\overline{g}(t) = g(t, \phi(t))$. Note that $\overline{g}$ a is definable C^∞-function. The derivative of $\overline{g}$ is given by

$$\frac{d\overline{g}}{dt}(t) = \overline{\frac{\partial g}{\partial x_1}}(t) + \sum_{i=2}^{r} \overline{\frac{\partial g}{\partial x_i}}(t) \cdot \frac{d\phi^i}{dt}(t), \tag{18}$$

where $\phi = (\phi^2, \ldots, \phi^r)$. Now write

$$J(x_1, \ldots, x_r) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1}, g)}{\partial(x_1, \ldots, x_r)}\right)$$

and

$$J_1(x_1, \ldots, x_r) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right).$$

Claim 1. $\frac{d\overline{g}}{dt}(t) = (-1)^{r+1}\overline{J}(t)\overline{J_1}(t)^{-1}$.

Proof. Note that $\overline{J_1}(t) \neq 0$, by (16). Define

$$A = \begin{pmatrix} \overline{\frac{\partial g_1}{\partial x_1}} \\ \vdots \\ \overline{\frac{\partial g_{r-1}}{\partial x_1}} \end{pmatrix} \qquad B = \begin{pmatrix} \overline{\frac{\partial g_1}{\partial x_2}} & \cdots & \overline{\frac{\partial g_1}{\partial x_r}} \\ \vdots & & \vdots \\ \overline{\frac{\partial g_{r-1}}{\partial x_2}} & \cdots & \overline{\frac{\partial g_{r-1}}{\partial x_r}} \end{pmatrix}$$

$$C = \begin{pmatrix} \overline{\frac{\partial g}{\partial x_1}} \end{pmatrix} \qquad D = \begin{pmatrix} \overline{\frac{\partial g}{\partial x_2}} & \cdots & \overline{\frac{\partial g}{\partial x_r}} \end{pmatrix}$$

Then

$$(-1)^{r+1}\overline{J}(t)\overline{J_1}(t)^{-1} = (-1)^{r+1} \det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \det\left[B^{-1}\right]$$

This is equal to

$$
\begin{aligned}
&\det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] \det\left[\begin{pmatrix} 0 & 1 \\ B^{-1} & 0 \end{pmatrix}\right] \\
=&\det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ B^{-1} & 0 \end{pmatrix}\right] \\
=&\det\left[\begin{pmatrix} I & A \\ D \cdot B^{-1} & C \end{pmatrix}\right] \\
=&\det\left[\begin{pmatrix} B & 0 \\ D & 1 \end{pmatrix} \cdot \begin{pmatrix} B^{-1} & B^{-1} \cdot A \\ 0 & C - D \cdot B^{-1} \cdot A \end{pmatrix}\right] \\
=&\det\left[\begin{pmatrix} B & 0 \\ D & 1 \end{pmatrix}\right] \det\left[\begin{pmatrix} B^{-1} & B^{-1} \cdot A \\ 0 & C - D \cdot B^{-1} \cdot A \end{pmatrix}\right] \\
=&\det[B] \cdot (C - D \cdot B^{-1} \cdot A) \cdot \det\left[B^{-1}\right] \\
=&C - D \cdot B^{-1} \cdot A
\end{aligned}
$$

Now, if we take $g = g_j$, with $j = 1, \ldots, r-1$, in (18), then the left hand side is equal to zero, as $(t, \phi(t)) \in \mathcal{V}(g_1, \ldots, g_{r-1})$. This shows that $A + B \cdot \phi^T = 0$. We find

$$
\begin{aligned}
&C - D \cdot B^{-1} \cdot A \\
=&C + D \cdot B^{-1} \cdot B \cdot \phi^T \\
=&C + D \cdot \phi^T \\
=&\frac{d\overline{g}}{dt}(t),
\end{aligned}
$$

proving our claim.

From now on we assume that r is even. The argument is easily modified in the case that r is odd.

Claim 2. If $p \in (\gamma_2, \beta_2)$ and $\overline{g}_r(p) = 0$, then $\frac{d\overline{g}_r}{dt}(p)$ has the same sign as $\overline{J}_1(p)$.

Proof. Take $g = g_r$ in Claim 1. By (17), we have $\overline{J}(p) < 0$. Claim 2 now follows immediately from Claim 1, as r is even.

Claim 3. The function $\overline{g}_r$ has at most one zero.

Proof. Notice that by (16), $\overline{J}_1$ is nonzero on its entire domain. Since $\overline{J}_1$ is continuous and definable, it has constant sign on (γ_2, β_2), by transfer of the Intermediate Value Theorem to K. Without loss of generality we take $\overline{J}_1$ positive. Then for each $p \in (\gamma_2, \beta_2)$ such that $\overline{g}_r(p) = 0$, we have $\frac{d\overline{g}_r}{dt}(p) > 0$ by Claim 3. The claim follows from transferring Lemma 4.4.1 to K.

Now notice that (13) - (17) all hold with k in place of K and $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ in place of $\mathcal{V}(g_1, \ldots, g_{r-1})$. This is because each statement implies the corresponding statement for k and $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$. This means that our three claims also hold if we take ϕ to be one of the ψ_i. For any $g \in M^r(k, K, (\vec{\sigma}, \sigma_{n+1}))$, let $\overline{g}(\phi_i; \cdot)$ be the function from $\{t \in K \mid \gamma_2 < t < \beta_2\}$ to K obtained as above, with $\phi = \phi_i$ and let $\overline{g}(\psi_i; \cdot)$ be the function from $\{t \in k \mid \gamma_2 < t < \beta_2\}$ to k obtained by taking $\phi = \psi_i$.

We write $Q = (q_1, \ldots, q_r)$. Let i_0 be the number such that $\phi_{i_0}(q_1) = (q_2, \ldots, q_r)$. Let us assume that $\overline{J}_1(\phi_{i_0}; q_1) > 0$, as the case $\overline{J}_1(\phi_{i_0}; q_1) < 0$ is similar. We define

$$\mathcal{S} = \{1 \leq i \leq m \mid \overline{J}_1(\phi_i; q_1) > 0\}.$$

By (16) and the Intermediate Value Theorem in K, we have for each $i \in \mathcal{S}$ and each $t \in (\gamma_2, \beta_2)$ that $\overline{J}_1(\phi_i; t) > 0$. Similarly, for each $i \in \{1, \dots, m\} \setminus \mathcal{S}$ and for each $t \in (\gamma_2, \beta_2)$, we have $\overline{J}_1(\phi_i; t) < 0$. This holds in particular for $t = \gamma_1$. By part **(iii)** of Lemma 4.3.4, $\phi_i(\gamma_1) \in k^{r-1}$ for $i = 1, \dots, m$. This means that there is a subset $\mathcal{S}'$ of $\{1, \dots, m\}$ such that

$$\{\psi_i(\gamma_1) \mid i \in \mathcal{S}'\} = \{\phi_i(\gamma_1) \mid i \in \mathcal{S}\}.$$

Then $\overline{J}_1(\psi_i; \gamma_1) > 0$ for $i \in \mathcal{S}'$ and $\overline{J}_1(\psi_i; \gamma_1) < 0$ for $i \in \{1, \dots, m\} \setminus \mathcal{S}'$. So by the Intermediate Value Theorem in k we have for each $i \in \mathcal{S}'$ and each $t \in (\gamma_2, \beta_2) \cap k$ that $\overline{J}_1(\psi_i; t) > 0$ and for each $i \in \{1, \dots, m\} \setminus \mathcal{S}'$ and each $t \in (\gamma_2, \beta_2) \cap k$, we have $\overline{J}_1(\psi_i; t) < 0$. Using part **(iii)** of Lemma 4.3.4 again, it follows that for each $t \in (\gamma_2, \beta_2) \cap k$,

$$\{\psi_i(t) \mid i \in \mathcal{S}'\} = \{\phi_i(t) \mid i \in \mathcal{S}\}.$$

Now take $\gamma_3, \beta_3 \in k$, with $\gamma_2 < \gamma_3 < \gamma_1$ and $\beta_1 < \beta_3 < \beta_2$, such that for all $i = 1, \dots, m$, the functions $\overline{g}_r(\phi_i; \cdot)$ and $\overline{g}_r(\psi_i; \cdot)$ are nonzero at γ_3 and β_3. It is possible to do this, as there are only finitely many points that need to be avoided, by claim 3. Take $i \in \mathcal{S}$. If $\overline{g}_r(\phi_i; \gamma_3) < 0$ and $\overline{g}_r(\phi_i; \beta_3) > 0$, then $\overline{g}_r(\phi_i; \cdot)$ clearly has a zero between γ_3 and β_3, by the Intermediate Value Theorem in K. Conversely, if $\overline{g}_r(\phi_i; \cdot)$ has a zero at some point $p \in (\gamma_3, \beta_3)$, then it must be the case that $\overline{g}_r(\phi_i; \gamma_3) < 0$ and $\overline{g}_r(\phi_i; \beta_3) > 0$, as $\frac{d\overline{g}_r}{dt}(p) > 0$, by claim 2, and p is the only zero of $\overline{g}_r(\phi_i; \cdot)$ in this interval, by claim 3. Also note that if $\overline{g}_r(\phi_i; \cdot)$ does not have a zero in (γ_3, β_3), then $\overline{g}_r(\phi_i; \gamma_3)$ and $\overline{g}_r(\phi_i; \beta_3)$ have the same sign. The same argument can be made regarding $\overline{g}_r(\psi_i; \cdot)$, with respect to $(\gamma_3, \beta_3) \cap k$, for $i \in \mathcal{S}'$. It follows that

$$\begin{aligned}&|\{i \in \mathcal{S} \mid \exists t \in (\gamma_3, \beta_3)\ \overline{g}_r(\phi_i; t) = 0\}| \\ =&|\{i \in \mathcal{S} \mid \overline{g}_r(\phi_i; \gamma_3) < 0\}| - |\{i \in \mathcal{S} \mid \overline{g}_r(\phi_i; \beta_3) < 0\}|\end{aligned}$$

and

$$\begin{aligned}&|\{i \in \mathcal{S}' \mid \exists t \in (\gamma_3, \beta_3) \cap k\ \overline{g}_r(\psi_i; t) = 0\}| \\ =&|\{i \in \mathcal{S}' \mid \overline{g}_r(\psi_i; \gamma_3) < 0\}| - |\{i \in \mathcal{S}' \mid \overline{g}_r(\psi_i; \beta_3) < 0\}|.\end{aligned}$$

But by part **(ii)** and **(iii)** of Lemma 4.3.4 the two "right" hand sides are equal. It follows that every point $P = (p_1, \dots, p_r)$ of K^r satisfying $P \in \mathcal{V}(g_1, \dots, g_{r-1})$, $g_r(P) = 0$, $J_1(P) > 0$, $\gamma_3 < p_1 < \beta_3$ and $\|(p_2, \dots, p_r)\| < B_1$ lies in k^r. But this means that $Q \in k^r$, as Q is such a point. We have therefore proven the Lemma, once we can show that we may assume (13) - (17). We shall do so now.

Our aim is to modify $(\vec{\sigma}, \sigma_{n+1})$ to $(\vec{\sigma}', \sigma'_{n+1})$, construct $h_1, \dots, h_s \in M^s(k, K, (\vec{\sigma}', \sigma'_{n+1}))$, for some $s \geq r$, and find a point $Q' \in K^s$ such that (10) - (17) are satisfied for $(\vec{\sigma}', \sigma'_{n+1})$, $h_1, \dots, h_s$ and Q' in place of $(\vec{\sigma}, \sigma_{n+1})$, $g_1, \dots, g_r$ and Q. Furthermore, the coordinates of Q will occur among the coordinates of Q'. This will clearly be sufficient. We will develop our modifications in four stages. Each of these stages will preserve (10) - (12), as well as all the previous stages. To avoid bulky notation, we revert to the original notation at the end of each stage.

Stage 1. We may assume that for each $(\vec{\sigma}, \sigma_{n+1})$-bounded variable x, there are variables y, z such that both $xy^2 - 1$ and $(1 - x)z^2 - 1$ occur among $g_1, \dots, g_r$.

Proof. Suppose that x_i is $(\vec{\sigma}, \sigma_{n+1})$-bounded. Define $g_{r+1}, g_{r+2} \in M^{r+2}(k, K, (\vec{\sigma}, \sigma_{n+1}))$ by $g_{r+1}(x_1, \dots, x_{r+2}) = x_i x_{r+1}^2 - 1$ and $g_{r+2}(x_1, \dots, x_{r+2}) = (1 - x_i)x_{r+2}^2 - 1$. By (10), $0 < q_i < 1$, so we can can take $q_{r+1} = \frac{1}{\sqrt{q_i}}$ and $q_{r+2} = \frac{1}{\sqrt{1-q_i}}$. Then (10) and (11) are clearly satisfied for

$g_1, \ldots, g_{r+2}$ and (Q, q_{r+1}, q_{r+2}). Furthermore,

$$\frac{\partial(g_1, \ldots, g_{r+2})}{\partial(x_1, \ldots, x_{r+2})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_r} & 0 & 0 \\ \frac{\partial g_{r+1}}{\partial x_1} & \cdots & \frac{\partial g_{r+1}}{\partial x_r} & \frac{\partial g_{r+1}}{\partial x_{r+1}} & 0 \\ \frac{\partial g_{r+2}}{\partial x_1} & \cdots & \frac{\partial g_{r+2}}{\partial x_r} & 0 & \frac{\partial g_{r+2}}{\partial x_{r+2}} \end{pmatrix}$$

so

$$\det\left(\frac{\partial(g_1, \ldots, g_{r+2})}{\partial(x_1, \ldots, x_{r+2})}\right) = \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right) \cdot \frac{\partial g_{r+1}}{\partial x_{r+1}} \cdot \frac{\partial g_{r+2}}{\partial x_{r+2}}$$

and hence

$$\det\left(\frac{\partial(g_1, \ldots, g_{r+2})}{\partial(x_1, \ldots, x_{r+2})}\right)(Q, q_{r+1}, q_{r+2}) = \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(Q) \cdot 4\sqrt{q_i}\sqrt{1-q_i}),$$

which is nonzero by (12). It follows that (12) also holds for the new system. We can now apply this process until we have treated each $(\vec{\sigma}, \sigma_{n+1})$-bounded variable x_i.

Stage 2. We may assume that $g_1, \ldots, g_{r-1} \in M^r(k, K, \vec{\sigma})$ and that g_r has the form $\sigma_{n+1}(x_1, \ldots, x_r) - x_e$, where x_e is not $(\vec{\sigma}, \sigma_{n+1})$-bounded.

Proof. By definition of $M^r(k, K, (\vec{\sigma}, \sigma_{n+1}))$, there exist $h_1, \ldots, h_r \in M^r(k, K, \vec{\sigma})[x_{r+1}]$ such that

$$g_i(x_1, \ldots, x_r) = h_i(x_1, \ldots, x_r, \sigma_{n+1}(x_1, \ldots, x_r)),$$

for $i = 1, \ldots, r$. Take $Q' = (Q, \sigma_{n+1}(Q))$ and $h_{r+1} = \sigma_{n+1}(x_1, \ldots, x_r) - x_{r+1}$. Certainly (10) and (11) hold for $h_1, \ldots, h_{r+1}$ and Q'. Note that Stage 1 and Stage 2 are also satisfied. We only need to check that (12) holds for our new system. Consider

$$\frac{\partial(h_1, \ldots, h_{r+1})}{\partial(x_1, \ldots, x_{r+1})} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_r} & \frac{\partial h_1}{\partial x_{r+1}} \\ \vdots & & \vdots & \vdots \\ \frac{\partial h_r}{\partial x_1} & \cdots & \frac{\partial h_r}{\partial x_r} & \frac{\partial h_{r+1}}{\partial x_{r+1}} \\ \frac{\partial \sigma_{n+1}}{\partial x_1} & \cdots & \frac{\partial \sigma_{n+1}}{\partial x_r} & -1 \end{pmatrix}$$

Now for each $i = 1, \ldots, r$, multiply row $r+1$ by $\frac{\partial h_i}{\partial x_{r+1}}$ and add the result to row i (recall that the resulting matrix will have the same determinant as the original one). Since $\frac{\partial g_i}{\partial x_j} = \frac{\partial h_i}{\partial x_j} + \frac{\partial \sigma_{n+1}}{\partial x_j}\frac{\partial h_i}{\partial x_{r+1}}$ for $i, j = 1, \ldots r$, by the chain rule, the resulting matrix is equal to

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_r} & 0 \\ \frac{\partial \sigma_{n+1}}{\partial x_1} & \cdots & \frac{\partial \sigma_{n+1}}{\partial x_r} & -1 \end{pmatrix}$$

It follows that

$$\det\left(\frac{\partial(h_1, \ldots, h_{r+1})}{\partial(x_1, \ldots, x_{r+1})}\right)(Q') = -\det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(Q),$$

which is nonzero by the original (12).

Stage 3. We may assume that for all $P \in D^r((\vec{\sigma}, \sigma_{n+1}), K)$, if $g_i(P) = 0$ for $i = 1, \dots, r-1$, then $\det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_2,\dots,x_r)}\right)(P) \neq 0$.

Proof. Since

$$\det\left(\frac{\partial(g_1,\dots,g_r)}{\partial(x_1,\dots,x_r)}\right) = \sum_{i=1}^{r}(-1)^{r+i} \cdot \det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_1,\dots,x_{i-1},x_{i+1},\dots,x_r)}\right),$$

there must be some $1 \leq i \leq r$ such that

$$\det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_1,\dots,x_{i-1},x_{i+1},\dots,x_r)}\right)(Q) \neq 0,$$

by (12). We now relabel the variables in such a way that we may assume $i = 1$. It is important to note that an (n,r)-sequence for which the variables are permuted is still an (n,r)-sequence. Furthermore, the definable points of the permuted sequence are simply coordinate transformations of the original sequence. It is also clear that (10) - (12) still hold, as well as Stages 1 and 2. We define $h \in M^{r+1}(k, K, (\vec{\sigma}, \sigma_{n+1}))$ by

$$h(x_1,\dots,x_{r+1}) = x_{r+1} \cdot \det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_2,\dots,x_r)}\right) - 1.$$

Furthermore, we take $q_{r+1} = \det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_2,\dots,x_{r-1})}\right)(Q)^{-1}$ and define $Q' = (Q, q_{r+1})$. Then $g_1, \dots, g_{r-1}, h, g_r$ and Q' satisfy Stages 1 and 2, along with (10) and (11). For (12), note that

$$\frac{\partial(g_1,\dots,g_{r-1},h,g_r)}{\partial(x_1,\dots,x_{r+1})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_{r-1}}{\partial x_1} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} & 0 \\ \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_r} & \frac{\partial h}{\partial x_{r+1}} \\ \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_r} & 0 \end{pmatrix}$$

so that

$$\det\left(\frac{\partial(g_1,\dots,g_{r-1},h,g_r)}{\partial(x_1,\dots,x_{r+1})}\right) = -\det\left(\frac{\partial(g_1,\dots,g_r)}{\partial(x_1,\dots,x_r)}\right) \cdot \frac{\partial h}{\partial x_{r+1}}$$

and hence

$$\det\left(\frac{\partial(g_1,\dots,g_{r-1},h,g_r)}{\partial(x_1,\dots,x_{r+1})}\right)(Q') = -\det\left(\frac{\partial(g_1,\dots,g_r)}{\partial(x_1,\dots,x_r)}\right)(Q) \cdot q_{r+1}^{-1},$$

which is nonzero by the original (12). Lastly, we check that Stage 3 is satisfied. Suppose that $P \in D^{r+1}((\vec{\sigma}, \sigma_{n+1}), K)$ and $g_1(P) = \cdots = g_{r-1}(P) = h(P) = 0$, with $P = (p_1, \dots, p_{r+1})$. Since $h(P) = 0$, it follows that

$$p_{r+1}^{-1} = \det\left(\frac{\partial(g_1,\dots,g_{r-1})}{\partial(x_2,\dots,x_r)}\right)(p_1,\dots,p_r),$$

which is nonzero. Since

$$\frac{\partial(g_1,\ldots,g_{r-1},h)}{\partial(x_2,\ldots,x_{r+1})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_{r-1}}{\partial x_2} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} & 0 \\ \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_r} & \frac{\partial h}{\partial x_{r+1}} \end{pmatrix}$$

we have

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1},h)}{\partial(x_2,\ldots,x_{r+1})}\right) = \det\left(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_2,\ldots,x_r)}\right)\cdot\frac{\partial h}{\partial x_{r+1}},$$

so that

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1},h)}{\partial(x_2,\ldots,x_{r+1})}\right)(P) = p_{r+1}^{-2},$$

which is nonzero, as desired.

Stage 4. We may assume that for all $P \in D^r((\vec{\sigma},\sigma_{n+1}),K)$, if $g_i(P)=0$ for $i=1,\ldots,r$, then $\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(P) < 0$.

Proof. As in the proof of Stage 2, there exists $h \in M^r(k,K,\vec{\sigma})[z]$ such that

$$\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right) = h(x_1,\ldots,x_r,\sigma_{n+1}(x_1,\ldots,x_r)).$$

We define $H \in M^{r+1}(k,K,\vec{\sigma})$ by

$$H(x_1,\ldots,x_{r+1}) = x_{r+1}\cdot h(x_1,\ldots,x_r,x_e) - 1,$$

where x_e is the same variable as given in Stage 2. Now $g_r(Q)=0$, so $\sigma_{n+1}(Q)=q_e$, by Stage 2. This shows that $h(Q,q_e) = \det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(Q)$, which is nonzero by (12). We can therefore take $q_{r+1} = h(Q,q_e)^{-1}$ and define $Q' = (Q,q_{r+1})$. One easily verifies that (10) and (11), as well as Stages 1 and 2 are satisfied for $g_1,\ldots,g_{r-1},H,g_r$ and Q'. We check that Stage 4 is satisfied. Note that (12) will then also immediately hold. Take $P \in D^{r+1}((\vec{\sigma},\sigma_{n+1}),K)$ and suppose that

$$g_1(P) = \cdots = g_{r-1}(P) = H(P) = g_r(P) = 0.$$

First of all, we have

$$\frac{\partial(g_1,\ldots,g_{r-1},H,g_r)}{\partial(x_1,\ldots,x_{r+1})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_{r-1}}{\partial x_1} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} & 0 \\ \frac{\partial H}{\partial x_1} & \cdots & \frac{\partial H}{\partial x_r} & \frac{\partial H}{\partial x_{r+1}} \\ \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_r} & 0 \end{pmatrix}$$

so

$$\det\left(\frac{\partial(g_1,\ldots,g_{r-1},H,g_r)}{\partial(x_1,\ldots,x_{r+1})}\right) = -\det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)\cdot\frac{\partial H}{\partial x_{r+1}}.$$

Since $g_r(P) = 0$, we have $\sigma_{n+1}(P) = p_e$, by Stage 2, so $h(P, p_e) = \det\left(\frac{\partial(g_1,\ldots,g_r)}{\partial(x_1,\ldots,x_r)}\right)(P)$. It follows that

$$\frac{\partial(g_1, \ldots, g_{r-1}, H, g_r)}{\partial(x_1, \ldots, x_{r+1})}(P) = -h(P, p_e)^2,$$

as needed. This final thing we need to verify is that Stage 3 is still satisfied by our new system. So suppose that $P \in D^{r+1}((\vec{\sigma}, \sigma_{n+1}), K)$ is a point such that $g_1(P) = \cdots = g_{r-1}(P) = H(P) = 0$. Now

$$\frac{\partial(g_1, \ldots, g_{r-1}, H)}{\partial(x_2, \ldots, x_{r+1})} = \begin{pmatrix} \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_{r-1}}{\partial x_2} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} & 0 \\ \frac{\partial H}{\partial x_2} & \cdots & \frac{\partial H}{\partial x_r} & \frac{\partial H}{\partial x_{r+1}} \end{pmatrix}$$

so

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, H)}{\partial(x_2, \ldots, x_{r+1})}\right) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right) \cdot \frac{\partial H}{\partial x_{r+1}}$$

and hence

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, H)}{\partial(x_2, \ldots, x_{r+1})}\right)(P) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(p_1, \ldots, p_r) \cdot h(P, p_e).$$

But this last expression is nonzero, by Stage 3 and the fact that $H(P) = 0$.

Now that we have applied our four stages, let us check that they indeed give us (13) - (17). Property (13) is satisfied by Stage 2. Furthermore, (14) follows from Stage 1, as possible limit points of $\mathcal{V}(g_1, \ldots, g_{r-1})$ that lie on the boundary of $D^r((\vec{\sigma}, \sigma_{n+1}), K)$ are pushed out towards infinity, in a similar way as in the proof of Lemma 2.3.1. We shall therefore not go through the details again. Additionally, Stage 1 forces that each coordinate of $P \in K^r$, associated to a $(\vec{\sigma}, \sigma_{n+1})$-bounded variable, lies between 0 and 1, if $g_1(P) = \cdots = g_r(P) = 0$. Now note that the value of $g_r(P)$ is irrelevant for this argument, by Stage 2. So it is already the case that each $(\vec{\sigma}, \sigma_{n+1})$-bounded coordinate of P lies between 0 and 1, if $g_1(P) = \cdots = g_{r-1}(P) = 0$, which implies (15). Lastly, (16) and (17) satisfied because of Stages 3 and 4 respectively. $\square$

5 Proof of Lemma 2.3.3

In this section, we give a proof of Lemma 2.3.3, which will finish the proof of the First Main Theorem. We need one small other Lemma first.

Lemma 5.0.3. *Let $m \in \mathbb{N}$ and suppose that $U \subseteq \mathbb{R}^m$ is an open set containing $[0,1]^m$. Then there exists a positive rational number ε such that $B_\varepsilon(P) \subseteq U$ for all $P \in [0,1]^m$. Here $B_\varepsilon(P)$ denotes the open ball in $\mathbb{R}^m$ with center P and radius ε.*

Proof. Consider the set $V = [-1,2]^m \cap (\mathbb{R}^m \setminus U)$. If V is empty, then we can take $\varepsilon = \frac{1}{2}$. Otherwise, we define the function $f : [0,1]^m \times V \to \mathbb{R}$ by $h(x,y) = \|x-y\|$. Since h is continuous and $[0,1]^m \times V$ is compact, h takes on a minimum value, δ say, by the Extreme Value Theorem. Note that $\delta > 0$ as $U \cap V = \emptyset$. Now any rational number $0 < \varepsilon < \delta$ suffices. □

Recall the statement of Lemma 2.3.3.

Lemma 2.3.3. *Let $k, K \models \mathcal{T}_{\mathrm{Pf}\restriction}$, with $k \subseteq K$. Let $n, r \in \mathbb{N}$ and suppose that $\vec{\sigma}' = (\sigma_1, \ldots, \sigma_{n+1})$ is an $(n+1, r)$-sequence. Let $\vec{\sigma}$ denote the (n,r)-sequence $(\sigma_1, \ldots, \sigma_n)$. Suppose that for each $s \geq r$, every $(k, \vec{\sigma})$-definable point of K^s lies in k^s. Then for each $s \geq r$ and each $(k, \vec{\sigma}')$-definable point $(p_1, \ldots, p_s)$ of K^s, there is some $B \in k$ such that $-B < p_1, \ldots, p_s < B$.*

Our proof strategy will be to find two conflicting estimates for the quantity $\sigma_{n+1}(\overline{p}_1, \ldots, \overline{p}_m) - \overline{p}_e$, for certain $\overline{p}_1, \ldots, \overline{p}_m \in k$ (which we will properly introduce), assuming that the Lemma is false. One of the estimates we obtain by polynomial approximations using Taylor's Theorem. The other estimate relies on Corollary 4.1.8, which is the reason that this proof only works for $\mathcal{T}_{\mathrm{Pf}\restriction}$, but not for $\mathcal{T}_{\mathrm{Pf}}$. (Indeed, one easily checks that the result of Corollary 4.1.8 is in general not true for T_{Pf}, by considering the Pfaffian chain "exp".)

Proof. (Of Lemma 2.3.3) Take $\vec{\sigma}$ and $\vec{\sigma}'$ as in the hypothesis of the Lemma. Let $Q = (q_1, \ldots, q_r)$ be a $(k, \vec{\sigma}')$-definable point of K^r. With the same reasoning as in the proof of Lemma 2.3.2, we may assume that $r \geq 2$. We may also apply Stages 1 up to 4, as in the proof of Lemma 2.3.2, as one easily verifies that we are justified in doing so in this situation. This gives us $g_1, \ldots, g_r \in M^r(k, K, (\vec{\sigma}, \sigma_{n+1}))$ with the following properties.

$$g_1, \ldots, g_{r-1} \in M^s(k, K, \vec{\sigma}) \tag{19}$$

g_r has the form $\sigma_{n+1}(x_1, \ldots, x_r) - x_e$, where x_e is not $(\vec{\sigma}, \sigma_{n+1})$-bounded. (20)

$$g_i(Q) = 0, \text{ for } i = 1, \ldots, r \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(Q) \neq 0. \tag{21}$$

$$\mathcal{V}(g_1, \ldots, g_{r-1}) \subseteq D^r((\vec{\sigma}, \sigma_{n+1}), K) \tag{22}$$

$\mathcal{V}(g_1, \ldots, g_{r-1})$ is closed in K^r and $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ is closed in k^r. (23)

$$\det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0 \text{ for all } P \in \mathcal{V}(g_1, \ldots, g_{r-1}.) \tag{24}$$

$$\text{For all } P \in \mathcal{V}(g_1, \ldots, g_{r-1}), \text{ if } g_r(P) = 0, \text{ then } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(P) \neq 0. \tag{25}$$

This allows us to prove the following Claim.

Claim 1. Suppose that $\chi(x_1, \ldots, x_r)$ is an $\mathcal{L}$-formula, with parameters from k. Suppose furthermore that there exists $P \in \mathcal{V}(g_1, \ldots, g_{r-1})$ such that $K \models \chi(P)$. Then there exists $P' \in \mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ such that $k \models \chi(P')$.

Proof. Since K and k are models of $\mathcal{T}_{\mathrm{Pf}\restriction}$, they are in particular models of $\mathcal{T}$, which admits elimination of quantifiers. Since χ is a formula in the language $\mathcal{L}$, we may therefore assume that χ is quantifier free. Then by Lemma 2.1.5, we may take χ to be of the form

$$\exists x_{r+1}, \ldots, x_{r+t} \bigwedge_{i=1}^{l} \tau_i = 0,$$

where each τ_i is a term of $\mathcal{L}_k$. Let ρ be the sum $\tau_1^2 + \cdots + \tau_l^2$. Then we may assume that χ is of the form $\exists x_{r+1}, \ldots, x_{r+t}\ \rho(x_1, \ldots, x_{r+t}) = 0$. We define $g = \rho + \sum_{i=1}^{r-1} g_i^2$. Note that $g \in M^{r+t}(k, K, \vec{\sigma})$, by (19). Furthermore, using (22),

$$\mathcal{V}(g_1, \ldots, g_{r-1}) \times K^t \subseteq D^r((\vec{\sigma}, \sigma_{n+1}), K) \times K^t = D^{r+t}((\vec{\sigma}, \sigma_{n+1}), K) \subseteq D^{r+t}(\vec{\sigma}, K),$$

so by our assumption regarding χ, there exists a point $P \in D^{r+t}(\vec{\sigma}, K)$, such that $g(P) = 0$. Lemma 2.3.1 now gives us a point $(Q_1, Q_2) \in D^{(r+t)+s}(\vec{\sigma}, K)$, for some $s \in \mathbb{N}$, which is $(k, \vec{\sigma})$-definable, such that $g(Q_1) = 0$. By hypothesis on $\vec{\sigma}$, this means that $(Q_1, Q_2) \in k^{(r+t)+s}$. Take $P' \in k^r$ to be the the first r coordinates of Q_1. Since ρ is always non-negative, $g_1(P') = \cdots = g_{r-1}(P') = 0$ and

$$k \models \exists x_{r+1}, \ldots, x_{r+t}\ \rho(P', x_{r+1}, \ldots, x_{r+t}) = 0,$$

as $\rho(Q_1) = 0$. But this is exactly what we needed to show.

From this point on, we suppose that $Q \notin (K^-)^r$ and work towards a contradiction.

Claim 2. $q_1 \notin k$.

Proof. Suppose to the contrary that $q_1 \in k$. We define $h \in M^r(k, K, \vec{\sigma})$ by $h(x_1, \ldots, x_r) = x_1 - q_1$. Then $h(Q) = g_1(Q) = \cdots = g_{r-1}(Q) = 0$ and

$$\frac{\partial(h, g_1, \ldots, g_{r-1})}{\partial(x_1, \ldots, x_r)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_r} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_{r-1}}{\partial x_1} & \frac{\partial g_{r-1}}{\partial x_2} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} \end{pmatrix}$$

So

$$\det\left(\frac{\partial(h, g_1, \ldots, g_{r-1})}{\partial(x_1, \ldots, x_r)}\right)(Q) = \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(Q) \neq 0,$$

by (24). Hence, Q is a $(k, \vec{\sigma})$-definable point, so $Q \in k^r$, by assumption on $\vec{\sigma}$. In particular $Q \in (K^-)^r$, which is false.

Now, by (19), (23) and (24), the conditions of Theorem 4.3.3 are satisfied, if we set $K = k$ in the Theorem. This means that there exists a parametrization of $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ in k. We write $\{(I_j, \psi_j) \mid 1 \leq j \leq N\}$ for this parametrization, for some $N \in \mathbb{N}$. Furthermore, let $I_j = (a_j, b_j)$, with $a_j \in k \cup \{-\infty\}$ and $b_j \in k \cup \{\infty\}$, for $j = 1, \ldots, N$. Note that $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r \neq \emptyset$ by Claim 1.

Claim 3. If $q_1 \in K^-$, then there is some $j = 1, \ldots, N$, such that either $0 < q_1 - a_j < \alpha$ for all positive $\alpha \in k$ or $0 < b_j - q_1 < \alpha$ for all positive $\alpha \in k$.

Proof. Suppose that $q_1 \in K^-$. There must be at least one $j = 1, \ldots, N$, such that $a_j < q_1 < b_j$, for otherwise there exist $a, b \in k$, with $a < q_1 < b$, such that there is no point of $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ which satisfies the formula $a < x_1 < b$. Since $Q \in \mathcal{V}(g_1, \ldots, g_{r-1})$ does satisfy this formula, this contradicts Claim 1. This guarantees the existence of

$$a = \max\{a_j \mid 1 \leq j \leq N \text{ and } a_j < q_1 < b_j\}$$

and

$$b = \min\{b_j \mid 1 \leq j \leq N \text{ and } a_j < q_1 < b_j\}.$$

To find a contradiction, we suppose that there is some $\alpha \in k$, with $\alpha > 0$, such that $q_1 - a > \alpha$ and $b - q_1 > \alpha$. Clearly, if $a \neq -\infty$ and $b \neq \infty$, then $a < a + \alpha < q_1 < b - \alpha < b$. We can now define $\gamma = a + \alpha$ and $\beta = b - \alpha$, which have the property that $[\gamma, \beta] \subseteq I_j$ for each j such that $a_j < q_1 < b_j$, by maximality of a and minimality of b. If either $a = -\infty$ or $b = \infty$, then we can certainly also find $\gamma, \beta \in k$ with this property and such that $\gamma < q_1 < \beta$, as $q_1 \in K^-$. By transfer of the Extreme Value Theorem to k and continuity of $\psi_1, \ldots, \psi_N$, there exists $B \in k$ such that $\|\psi_j(t)\| < B$ for all j such that $a_j < q_1 < b_j$ and all $t \in [\gamma, \beta]$. Now take

$$c = \max(\{\gamma\} \cup \{b_j \mid 1 \leq j \leq N \text{ and } b_j < q_1\})$$

and

$$d = \min(\{\beta\} \cup \{a_j \mid 1 \leq j \leq N \text{ and } a_j > q_1\}).$$

Consider a point $P \in \mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ such that $c < p_1 < d$. By construction of c and d, the point P must be equal to $(p_1, \psi_j(p_1))$, for some j such that $a_j < c < q_1 < d < b_j$, since the a_i and b_i are all unequal to q_1 by Claim 2. This means that $\|(p_2, \ldots, p_r)\| < B$. What we gather from this is that there is no point in $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ satisfying the formulas $c < x_1 < d$ and $\|(x_2, \ldots, x_r)\| \geq B$. However, since $q_1 \in K^-$ and $Q \notin (K^-)^r$, it must be the case that $\|(q_2, \ldots, q_r)\| \geq B$, so $Q \in \mathcal{V}(g_1, \ldots, g_{r-1})$ does satisfy these formulas. But this contradicts Claim 1.

Claim 4. We may assume that $q_1 > \alpha$ for all $\alpha \in k$.

Proof. Suppose that this is not already the case. Then the following three possibilities are left.

(a) $q_1 < \alpha$ for all $\alpha \in k$, or

(b) $0 < q_1 - a < \alpha$ for all positive $\alpha \in k$, or

(c) $0 < b - q_1 < \alpha$ for all positive $\alpha \in k$, for some $a, b \in k$. If $q_1 \notin K^-$, then **(a)** holds and if $q_1 \in K^-$, then **(b)** or **(c)** holds by Claim 3. We define $h \in M^{r+1}(k, K, \vec{\sigma})$ by

$$h(x_1, \dots, x_{r+1}) = \begin{cases} x_1 + x_{r+1} & \text{in case } \textbf{(a)} \\ x_{r+1}(x_1 - a) - 1 & \text{in case } \textbf{(b)} \\ x_{r+1}(b - x_1) - 1 & \text{in case } \textbf{(c)} \end{cases}$$

Furthermore, we define

$$q_{r+1} = \begin{cases} -q_1 & \text{in case } \textbf{(a)} \\ \frac{1}{q_1 - a} & \text{in case } \textbf{(b)} \\ \frac{1}{b - q_1} & \text{in case } \textbf{(c)} \end{cases}$$

It is clear that $q_{r+1} > \alpha$ for all $\alpha \in k$, if we define q_1 in this way. In each case, if we let $Q' = (Q, q_{r+1})$, then

$$h(Q') = g_1(Q') = \cdots = g_r(Q') = 0.$$

It is easy to check that (19) and (20)hold for the new system $h, g_1, \dots, g_r, Q'$. To see that (21) also holds for this system is not too difficult as well, as

$$\frac{\partial(h, g_1, \dots, g_r)}{\partial(x_1, \dots, x_{r+1})} = \begin{pmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_r} & \frac{\partial h}{\partial x_{r+1}} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_r} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial g_r}{\partial x_1} & \cdots & \frac{\partial g_r}{\partial x_r} & 0 \end{pmatrix}$$

so

$$\frac{\partial(h, g_1, \dots, g_r)}{\partial(x_1, \dots, x_{r+1})}(Q') = (-1)^{r+1} \cdot \frac{\partial h}{\partial x_{r+1}}(Q') \cdot \frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)}(Q)$$

which is nonzero by the old (21) and by the fact that

$$\frac{\partial h}{\partial x_{r+1}}(Q') = \begin{cases} 1 & \text{in case } \textbf{(a)} \\ q_1 - a & \text{in case } \textbf{(b)} \\ b - q_1 & \text{in case } \textbf{(c)} \end{cases}$$

is nonzero. The fact that (22) hold for this new system follows directly from the old (22), as

$$\mathcal{V}(h, g_1, \dots, g_{r-1}) \subseteq \mathcal{V}(g_1, \dots, g_{r-1}) \times K \subseteq D^r((\vec{\sigma}, \sigma_{n+1}), K) \times K = D^{r+1}((\vec{\sigma}, \sigma_{n+1}), K).$$

For (23), regard h as being defined on the entire space K^{r+1} and note that

$$\mathcal{V}(h, g_1, \dots, g_{r-1}) = (\mathcal{V}(h, g_1, \dots, g_{r-1}) \times K) \cap h^{-1}(\{0\})$$

is closed in K^{r+1}, by continuity of h and by the old (23). In the same way

$$\mathcal{V}(h, g_1, \dots, g_{r-1}) \cap k^{r+1} = ((\mathcal{V}(g_1, \dots, g_{r-1}) \cap k^r) \times k) \cap h^{-1}(\{0\})$$

is closed in k^{r+1}. In fact, (24) holds as well, but we will not be needing this. However,

$$\det\left(\frac{\partial(h, g_1, \dots, g_{r-1})}{\partial(x_1, \dots, x_r)}\right) = \begin{pmatrix} \frac{\partial h}{\partial x_1} & 0 & \cdots & 0 \\ \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_r} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_{r-1}}{\partial x_1} & \frac{\partial g_{r-1}}{\partial x_2} & \cdots & \frac{\partial g_{r-1}}{\partial x_r} \end{pmatrix}$$

So if we take $P \in K^{r+1}$ such that $h(P) = g_1(P) = \cdots = g_{r-1}(P) = 0$, then

$$\det\left(\frac{\partial(h, g_1, \ldots, g_{r-1})}{\partial(x_1, \ldots, x_r)}\right)(P) = \frac{\partial h}{\partial x_1}(P) \cdot \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0,$$

by the old (24) and since $\frac{\partial h}{\partial x_1}(P) \neq 0$ whenever $h(P) = 0$. We now relabel the variables, as in Stage 2 of Lemma 2.3.2, such that x_{r+1} becomes x_1. This does not alter the status of 19)-(23) or (25), so our new system satisfies 19)-(25), as well as the statement in our Claim.

Claim 5. There exists a finite set $S \subseteq k$, an element $B \in k$ and a positive rational number θ such that

(i) $0 \leq a \leq 1$ for all $a \in S$.

(ii) For any $P \in K^r$, with $p_1 > B$ and $P \in \mathcal{V}(g_1, \ldots, g_{r-1})$ and any i such that the variable x_i is $(\vec{\sigma}, \sigma_{n+1})$-bounded, there exists $a \in S$ such that $|p_i - a| < p_1^{-\theta}$.

Proof. Note that x_1 is not $(\vec{\sigma}, \sigma_{n+1})$-bounded, as $Q \in D^r((\vec{\sigma}, \sigma_{n+1}), K)$ by 22 and $q_1 > 1$ by Claim 4. By Claim 1, it suffices to prove Claim 5 with K replaced by k and $\mathcal{V}(g_1, \ldots, g_{r-1})$ replaced by $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$. We shall therefore work in k. Let $\mathcal{S}$ be a parametrization of $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ in k, as in Theorem 4.3.3. Suppose that $(I, \psi) \in \mathcal{S}$, such that I is unbounded to the right. We write $\psi = (\psi_2, \ldots, \psi_r)$. Let x_i be a $(\vec{\sigma}, \sigma_{n+1})$-bounded variable and recall that we must therefore have $2 \leq i \leq r$. By (22), we have $0 < \psi_i(t) < 1$ for all $t \in I$. By Corollary 4.1.8, there is a rational number s and a nonzero element $a_i \in k$, such that $\lim_{t\to\infty} \psi_i(t)t^s = a_i$. Since $0 < \psi_i(t) < 1$ for all $t \in I$, this can only happen if $s \geq 0$. If $s = 0$, we put $b_i = a_i$ and if $s > 0$, we put $b_i = 0$. Then in either case, $\lim_{t\to\infty} \psi_i(t) = b_i$ and $0 \leq b_i \leq 1$. Now consider the function $\psi_i - b_i$ and assume that it is not eventually identically zero. Then we can apply Corollary 4.1.8 once more to find $\lim_{t\to\infty}(\psi_i(t) - b_i)t^{s_i} = c$ for some rational number s_i and a nonzero element $c \in k$. Since $\lim_{t\to\infty} \psi_i(t) - b_i = 0$ and $c \neq 0$, it must be the case that $s_i > 0$. Let $\theta_i = \frac{s_i}{2}$. Then

$$\lim_{t\to\infty} (\psi_i(t) - b_i)t^{\theta_i} = \left(\lim_{t\to\infty} (\psi_i(t) - b_i)t^{s_i}\right) \cdot \left(\lim_{t\to\infty} t^{-\theta_i}\right) = 0,$$

so $|\psi_i(t) - b_i| < t^{-\theta_i}$ for all $t \in k$, larger than some $B_i \in k$. If $\psi_i - b_i$ is eventually identically zero, then there is clearly also a positive rational number θ_i and some $B_i \in k$ such that $|\psi_i(t) - b_i| < t^{-\theta_i}$ for all $t \in k$, larger than B_i. We take S to be the set of the b_i, over all $(\psi, I) \in \mathcal{S}$, with I unbounded on the right. We let θ be the minimum of the θ_i over all $(\psi, I) \in \mathcal{S}$, with I unbounded on the right. Furthermore, we take C to be the maximum of the B_i, taken over all $(\psi, I) \in \mathcal{S}$, with I unbounded on the right. Then we let B be the maximum of C and the right endpoints of the intervals I, with $(\psi, I) \in \mathcal{S}$, which are bounded on the right. Then S, B and θ satisfy the statement of the Claim.

Claim 6. There exists a positive integer μ and an element $B' \in k$ such that for any $P \in \mathcal{V}(g_1, \ldots, g_r) \cap k^r$ with $p_1 > B'$ holds that $|g_r(P)| > p_1^{\mu}$.

Proof. By (25) and Corollary 4.1.3 (with $r_1 = r$ and $r_2 = 0$) the function g_r has only finitely many zeros on $\mathcal{V}(g_1, \ldots, g_r) \cap k^r$. Let $\mathcal{S}$ be a parametrization of $\mathcal{V}(g_1, \ldots, g_{r-1}) \cap k^r$ in k and suppose that $(I, \psi) \in \mathcal{S}$, such that I is unbounded to the right. The function $g_r(t, \psi(t))$ has only finitely many zeros, so we we can apply Corollary 4.1.8. According to Corollary 4.1.8, $\lim_{t\to\infty} g_r(t, \psi(t))t^s = a$, for some rational number s and some nonzero element $a \in k$. Now let

η be a positive integer, strictly larger than s. Then

$$\begin{aligned}&\lim_{t\to\infty}|g_r(t,\psi(t))|\cdot t^{\eta}=\left(\lim_{t\to\infty}|g_r(t,\psi(t))|\cdot t^{s}\right)\cdot\left(\lim_{t\to\infty}t^{\eta-s}\right)\\=&|a|\cdot\lim_{t\to\infty}t^{\eta-s}=\infty,\end{aligned}$$

so $|g_r(t,\psi(t))|>t^{-\eta}$ for all $t\in k$, larger than some $B\in k$. Now, like in the proof of Claim 5, we take μ to be the maximum of all the η, over all $(\psi,I)\in\mathcal{S}$, with I unbounded on the right. We let C to be the maximum of all the B, taken over all $(\psi,I)\in\mathcal{S}$, with I unbounded on the right. Then we let B' be the maximum of C and the right endpoints of the intervals I, with $(\psi,I)\in\mathcal{S}$, which are bounded on the right. Then μ and B' satisfy the claim.

We shall now find another estimate for g_r using polynomials in order to find a contradiction with Claim 6. By (20), $g_r(x_1,\dots,x_r)$ has the form $\sigma_{n+1}(x_1,\dots,x_r)-x_e$, and by Definition 2.2.1, $\sigma_{n+1}(x_1,\dots,x_r)$ has the form $F_i(y_1,\dots,y_m)$ for some $i=1,\dots,l$ and $y_1,\dots,y_m\in\{0,1,x_1,\dots,x_m\}$. Working in $\mathbb{R}$, consider the function $G_i:U\to\mathbb{R}$, with U and G_i as given in Definition 1.2.1. We shall write F and G for F_i and G_i respectively. Since U is open and contains $[0,1]^m$, we can apply Lemma 5.0.3 to find a positive rational number ε_0, such that $B_{\varepsilon_0}(P)\subseteq U$ for each $P\in[0,1]^m$. We set $\varepsilon=\frac{\varepsilon_0}{2}$. Since G is a C^∞-function, we may apply Taylor's Theorem to G, which states that

$$G(p_1+t_1,\dots,p_m+t_m)=\sum_{i=0}^{\lambda}\left[\frac{1}{i!}\left(\sum_{j=1}^{m}t_j\frac{\partial}{\partial x_j}\right)^{i}G\right](P)+R_\lambda,\tag{26}$$

for $P=(p_1,\dots,p_m)\in[0,1]^m$, $(t_1,\dots,t_m)\in B_\varepsilon(0)$ and $\lambda\in\mathbb{N}$, where

$$R_\lambda=\left[\frac{1}{(\lambda+1)!}\left(\sum_{j=1}^{m}t_j\frac{\partial}{\partial x_j}\right)^{\lambda+1}G\right](P'),\tag{27}$$

for some $P'\in B_\varepsilon(P)$. Since G is a C^∞-function, G and all of its derivatives are bounded (not necessarily uniformly) on the set

$$\mathrm{Cl}\left(\bigcup_{P\in[0,1]}B_\varepsilon(P)\right)\subseteq\bigcup_{P\in[0,1]}B_{\varepsilon_0}(P)\subseteq U,$$

as it is compact, so in particular G and all of its derivatives are bounded on $\bigcup_{P\in[0,1]}B_\varepsilon(P)$. This means that for each $\lambda\in\mathbb{N}$, there exists $C_\lambda\in\mathbb{N}$ such that for all $(t_1,\dots,t_m)\in B_\varepsilon(0)$ we can make the estimate

$$|R_\lambda|<C_\lambda\cdot(\max\{|t_i|\mid 1\le i\le m\})^{\lambda+1}.\tag{28}$$

Since G is part of a Pfaffian chain, the polynomials given in Definition 1.2.1 allow us to write

$$\sum_{i=0}^{\lambda}\left[\frac{\lambda!}{i!}\left(\sum_{j=1}^{m}t_j\frac{\partial}{\partial x_j}\right)^{i}G\right](P)=\sum_{\deg(\pi)\le\lambda}\tau_\pi^\lambda(P)\cdot\pi(t_1,\dots,t_m),$$

where each $\tau_\pi^\lambda(x_1,\dots,x_m)$ is some term of $\mathcal{L}_{\mathrm{Pf}\restriction}$ and we sum over all monomials π with $\deg(\pi)\le\lambda$. Since F is the restriction of G to $[0,1]^m$, we have for $P\in[0,1]^m$ and $(t_1,\dots,t_m)\in B_\varepsilon(0)$, with $(p_1+t_1,\dots,p_m+t_m)\in B_\varepsilon(P)\cap[0,1]^m$, that

$$\begin{aligned}&|\lambda!\cdot F(p_1+t_1,\dots,p_m+t_m)-\sum_{\deg(\pi)\le\lambda}\tau_\pi^\lambda(P)\cdot\pi(t_1,\dots,t_m)|\\&<\lambda!\cdot C_\lambda\cdot(\max\{|t_i|\mid 1\le i\le m\})^{\lambda+1},\end{aligned}\tag{29}$$

using (26) and (28). We wish to apply (29) in K. As we have stated before, $\sigma_{n+1}(x_1,\dots,x_r)$ has the form $F(y_1,\dots,y_m)$ for some $y_1,\dots,y_m\in\{0,1,x_1,\dots,x_r\}$. We define for each point $(p_1,\dots,p_r)\in K^r$ and $i=1,\dots,m$,

$$p_i'=\begin{cases}0 & \textit{if } y_i=0\\ 1 & \textit{if } y_i=1\\ p_j & \textit{if } y_i=x_j\end{cases}$$

As a result of the above definition, we have $0\le p_i\le 1$, for $i=1,\dots,m$, whenever $(p_1,\dots,p_r)\in D^r((\vec{\sigma},\sigma_{n+1}),K)$. By (22) this in particular the case for $(p_1,\dots,p_r)\in\mathcal{V}(g_1,\dots,g_r)$. We also note the fact that $\sigma_{n+1}(p_1,\dots,p_r)=F(p_1',\dots,p_r')$ for these points.

Now take S,θ and B as in Claim 5 and take μ and B' as in Claim 6. Furthermore, let λ_0 be an integer greater than $\frac{\mu+1}{\theta}$. Recall that the point $Q\in K^r$ we have in consideration lies in $\mathcal{V}(g_1,\dots,g_r)$ and that $q_1>B$, by Claim 4. By Claim 5, we can therefore take, for each $i=1,\dots,m$, an element $a_i\in S\cup\{0,1\}$ such that $|q_i-a_i|<q_1^{-\theta}$. Notice that $(q_1'-a_1,\dots,q_r'-a_r)\in B_\varepsilon(0)$, as $0\le q_1^{-\theta}<\varepsilon$, using Claim 4 and the fact that θ and ε are both positive rational numbers. It follows that $(q_1',\dots,q_r')\in B_\varepsilon(a_1,\dots,a_r)\cap[0,1]^m$. Since $g_r(Q)=0$, we have $F(q_1',\dots,q_r')=q_e$ by (20), so by applying (29) in K, we find

$$\begin{aligned}&|\lambda_0!\cdot q_e-\sum_{\deg(\pi)\le\lambda_0}\tau_\pi^{\lambda_0}(a_1,\dots,a_m)\cdot\pi(q_1'-a_1,\dots,q_r'-a_r)|\\&<\lambda_0!\cdot C_{\lambda_0}\cdot q_1^{-\theta(\lambda_0+1)}.\end{aligned}\tag{30}$$

Here we used that $\max\{|q_i'-a_i|\mid 1\le i\le m\}<q_1^{-\theta}$. Furthermore we have

$$q_1>\max\{B',2C_{\lambda_0},(\frac{\varepsilon}{m})^{-\theta^{-1}}\},\tag{31}$$

by Claim 4. As already stated above, we also have

$$|q_i'-a_i|<q_1^{-\theta}\text{ for }i=1,\dots,m.\tag{32}$$

Now, each $\tau_\pi^{\lambda_0}(a_1,\dots,a_m)$ is simply an element of k. It is not difficult to see that we can express the conjunction of (30), (31) and (32) as $\chi(q_1,\dots,q_r)$, where $\chi(x_1,\dots,x_r)$ is a formula in the language $\mathcal{L}$ with parameters from k. By Claim 1, this means that there exists $(\overline{p}_1,\dots,\overline{p}_r)\in\mathcal{V}(g_1,\dots,g_r)\cap k^r$ such that (30), (31) and (32) hold in k, with $(\overline{p}_1,\dots,\overline{p}_r)$ in place of $(q_1,\dots,q_r)$. We claim that we may apply (29) in k, with $p_i=a_i$ and $t_i=\overline{p}_i-a_i$ to give us

$$\begin{aligned}&|\lambda_0!\cdot F(\overline{p}_1,\dots,\overline{p}_m)-\sum_{\deg(\pi)\le\lambda_0}\tau_\pi^{\lambda_0}(a_1,\dots,a_m)\cdot\pi(\overline{p}_1-a_1,\dots,\overline{p}_m-a_m)|\\&<\lambda_0!\cdot C_{\lambda_0}\cdot(\max\{|\overline{p}_i-a_i|\mid 1\le i\le m\})^{\lambda_0+1}.\end{aligned}$$

Indeed, by the new (31) and (32), $|\overline{p}'_i - a_i| < \overline{p}_1^{-\theta} < \frac{\varepsilon}{m}$, for $i = 1, \ldots, m$, so that $(\overline{p}'_1 - a_1, \ldots, \overline{p}'_m - a_m) \in B_\varepsilon(0)$. Secondly, since $(\overline{p}_1, \ldots, \overline{p}_m) \in \mathcal{V}(g_1, \ldots, g_r) \cap k^r$, we have $(\overline{p}'_1, \ldots, \overline{p}'_m) \in [0,1]^m$, so

$$(\overline{p}'_1, \ldots, \overline{p}'_m) \in B_\varepsilon(a_1, \ldots, a_m) \cap [0,1]^m.$$

This shows that our use of (29) is justified. We apply the new (32) to get

$$\begin{aligned} &|\lambda_0! \cdot F(\overline{p}_1, \ldots, \overline{p}_m) - \sum_{\deg(\pi) \leq \lambda_0} \tau_\pi^{\lambda_0}(a_1, \ldots, a_m) \cdot \pi(\overline{p}_1 - a_1, \ldots, \overline{p}_m - a_m)| \\ <&\lambda_0! \cdot C_{\lambda_0} \cdot \overline{p}_1^{\theta(\lambda_0+1)}. \end{aligned}$$

Using the triangle inequality, we can now combine this with the new (30), which says that

$$\begin{aligned} &|\lambda_0! \cdot \overline{p}_e - \sum_{\deg(\pi) \leq \lambda_0} \tau_\pi^{\lambda_0}(a_1, \ldots, a_m) \cdot \pi(\overline{p}'_1 - a_1, \ldots, \overline{p}'_r - a_r)| \\ <&\lambda_0! \cdot C_{\lambda_0} \cdot \overline{p}_1^{-\theta(\lambda_0+1)}. \end{aligned}$$

to arrive at

$$|\lambda_0! \cdot F(\overline{p}'_1, \ldots, \overline{p}'_m) - \lambda_0! \cdot p_e| < 2\lambda_0! \cdot C_{\lambda_0} \cdot q_1^{-\theta(\lambda_0+1)}.$$

This shows that

$$\begin{aligned} &|g_r(\overline{p}_1, \ldots, \overline{p}_m)| = |F(\overline{p}'_1, \ldots, \overline{p}'_m) - p_e| \\ <&2C_{\lambda_0} \cdot \overline{p}_1^{-\theta(\lambda_0+1)} < 2C_{\lambda_0} \cdot \overline{p}_1^{-\mu-1} < \overline{p}_1^{-\mu}, \end{aligned}$$

using the fact that $\mu + 1 < \theta(\lambda_0 + 1)$ by choice of λ_0 and using that $\overline{p}_1 > 2C_{\lambda_0}$ by the new (31). But this contradicts Claim 6. $\square$

6 Approach to the Second Main Theorem

6.1 Reducing the problem

Recall that the Second Main Theorem concerns the following language and theory.

Definition 6.1.1. Define $\mathcal{L}_{\text{exp}} = \mathcal{L} \cup \{\text{exp}\}$ and $\mathcal{T}_{\text{exp}} = \text{Th}(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$, where exp is the unrestricted exponential function $x \mapsto e^x : \mathbb{R} \to \mathbb{R}$.

Our goal in this section is to give a proof of the following Theorem.

Theorem 6.1.2. *The theory $\mathcal{T}_{\text{exp}}$ is model complete.*

A large part of what is needed for this proof has already been set up in the previous sections. We will modify and combine some of the results used in the proof of the First Main Theorem below in such a way that they are suitable for our current application.

Lemma 6.1.3. *Let $k, K \models \mathcal{T}_{\text{Pf}}$, with $k \subseteq K$. Furthermore, let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence. Suppose that $g \in M^r(k, K, \vec{\sigma})$ and $g(P) = 0$ for some $P \in K^r$. Then there exist $Q \in K^r$ such that $g(Q) = 0$ and Q is $(k, \vec{\sigma})$-definable.*

Note that this Lemma is just a slightly stronger version of Lemma 2.3.1, but for $\mathcal{T}_{\text{Pf}}$ instead of $\mathcal{T}_{\text{Pf}\upharpoonright}$. The proof is not very exciting; it is just a trimmed version of the proof of Lemma 2.3.1, as we can drop some of the extra steps we needed when dealing with truncated functions.

Proof. (Of Lemma 6.1.3.) Let $U = K^r$. Since Remark 2.2.5 also applies to Definition 4.2.4, $M^r(k, K, \vec{\sigma})$ is a subring of $\mathcal{D}_U$ which is Noetherian and closed under differentiation. Note also that $M^r(k, K, \vec{\sigma})$ contains $\mathbb{Z}[x_1, \ldots, x_r]$. If we take $S = \mathcal{V}(g)$, then the hypothesis of Theorem 3.3.4 is satisfied, with respect to the ring $M^r(k, K, \vec{\sigma})$ as a subring of $\mathcal{D}_U$. By this Theorem, there exist $f_1, \ldots, f_n \in M^r(k, K, \vec{\sigma})$ such that $S \cap \mathcal{V}_r(f_1, \ldots, f_n)$ is nonempty. Take some $Q \in S \cap \mathcal{V}_r(f_1, \ldots, f_n)$. Then $g(Q) = 0$ as $Q \in S$ and Q is $(k, \vec{\sigma})$-definable as $Q \in \mathcal{V}_r(f_1, \ldots, f_n)$, proving the Theorem. ☐

Lemma 6.1.4. *Let $k, K \models \mathcal{T}_{\text{Pf}}$, such that $k \subseteq K$ and suppose that for all $n, r \in \mathbb{N}$, all (n, r)-sequences $\vec{\sigma}$ and all $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ holds that if $g_1, \ldots, g_l$ have a common zero in K^r, then they have a common zero in k^r. Then k is existentially closed in K.*

Proof. Suppose that $K \models \chi$, where χ is an existential $\mathcal{L}_{\text{Pf},k}$-formula. By Lemma 2.1.5 we may suppose that χ is of the form

$$\exists x_1, \ldots, x_s \bigwedge_{i=1}^{l} \tau_i = 0,$$

where each τ_i is a term of $\mathcal{L}_{\text{Pf},k}$ or has the form $H(x_{i_1}, \ldots, x_{i_m}) - x_{i_{m+1}}$ (see Definition 4.2.1). By Remark 2.2.2 (which also applies to Definition 4.2.2), we can arrange and pad out the set of functions of the form $H_i(x_{i_1}, \ldots, x_{i_m})$ appearing among the τ_i into an (n, r)-sequence, $\vec{\sigma}$ say, for some $n, r \in \mathbb{N}$ (and in such a way that we do not introduce additional bounded variables). Then $K \models \chi$ simply means that some functions $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ have a common zero in K^r. By the same reasoning, $k \models \chi$ if and only if $g_1, \ldots, g_l$ have a common zero in k^r. So, by the hypothesis of the Lemma,

$$K \models \chi \text{ implies } k \models \chi,$$

which is what we needed to show. ☐

Theorem 6.1.5. *Suppose that for each pair of models $k, K \models \mathcal{T}_{\text{Pf}}$, with $k \subseteq K$ holds that for all $n, r \in \mathbb{N}$ and all (n, r)-sequences $\vec{\sigma}$, every $(k, \vec{\sigma})$-definable point $P \in K^r$ lies in $(K^-)^r$. Then $\mathcal{T}_{\text{Pf}}$ is model complete.*

Proof. Let k and K be arbitrary models of $\models \mathcal{T}_{\text{Pf}}$, such that $k \subseteq K$. We will apply Lemma 6.1.4. Let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence and suppose that $g_1, \ldots, g_l \in M^r(k, K, \vec{\sigma})$ have a common zero P in K^r. Note that P is a common zero of $g_1, \ldots, g_l$ if and only if it is a zero of $g = \sum_{i=1}^{l} g_i^2$, which is also an element of $M^r(k, K, \vec{\sigma})$. Lemma 6.1.3 then tells us that there exist $Q \in K^r$ such that $g(Q) = 0$ and Q is $(k, \vec{\sigma})$-definable. Now, by the hypothesis of the current Lemma, the hypothesis of Lemma 4.4.2 is satisfied (for $\mathcal{T}_{\text{Pf}(\restriction)} = \mathcal{T}_{\text{Pf}}$). Hence, $Q \in k^r$, as Q is $(k, \vec{\sigma})$-definable, so k is existentially closed in K by Lemma 6.1.4. Since k and K where arbitrary, it follows that $\mathcal{T}_{\text{Pf}}$ is model complete by Corollary 2.1.4. □

6.2 Proof of the Second Main Theorem

Let us fix two models $K, k \models \mathcal{T}_{\exp}$, with $k \subseteq K$, for the remainder of this section.

Remark 6.2.1. In order to prove Theorem 6.1.2, it suffices to show that for all (n, r)-sequences $\vec{\sigma}$, every $(k, \vec{\sigma})$-definable point $\vec{\alpha} \in K^r$ lies in $(K^-)^r$ (by Theorem 6.1.5, with $\mathcal{T}_{\text{Pf}} = \mathcal{T}_{\exp}$). In our specific case, $\vec{\sigma}$ is of the form $(\exp(y_1), \ldots, \exp(y_n))$, with each $y_i \in \{x_1, \ldots, x_r\}$. So certainly $\vec{\alpha}$ is $(k, \vec{\sigma}')$-definable, where $\vec{\sigma}'$ is the (r, r)-sequence $(\exp(x_1), \ldots, \exp(x_r))$. Hence, simply by writing out what it means to be $(k, \vec{\sigma}')$-definable, it is enough to prove that each $r \in \mathbb{N}$ and each $\vec{\alpha} \in K^r$ for which there are $f_1, \ldots, f_r \in k[x_1, \ldots, x_r, \exp(x_1), \ldots, \exp(x_n)]$, such that

$$f_1(\vec{\alpha}) = \cdots = f_r(\vec{\alpha}) = 0$$

and

$$\det\left(\frac{\partial(f_1, \ldots, f_r)}{\partial(x_1, \ldots, x_r)}\right)(\vec{\alpha}) \neq 0,$$

holds that $\vec{\alpha} \in (K^-)^r$. Our method of proof is to use induction on the number of distinct $\exp(x_i)$ actually occurring in $f_1, \ldots, f_r$. The idea behind the proof is that we can eliminate exponentials by introducing new variables and their exponentials, but in such a way that only values of the new variables lying between 0 and 1 will be relevant. At the base case we can then apply the model completeness of the structure $(\mathbb{R} \mid \exp \restriction [0, 1])$, which follows from the First Main Theorem.

It turns out to be more convenient to work with functions that are not truncated, so to work around this, we introduce the following function.

Definition 6.2.2. In any model $K_0 \models \mathcal{T}_{\exp}$, we define the function $e : K_0 \to K_0$ by $e(x) = \exp((1 + x^2)^{-1})$. Furthermore, we let $\mathcal{L}_e = \mathcal{L} \cup \{e\}$ and $\mathcal{T}_e = \text{Th}(\mathbb{R} \mid \mathcal{L}_e)$.

Notice that, since the function $x \mapsto (1 + x^2)^{-1}$ is a definable bijection between $[0, \infty)$ and $(0, 1]$, the functions e and $\exp \restriction_{[0,1]}$ contain essentially the same information. In fact, we have the following Lemma.

Lemma 6.2.3. *Let $K_0 \models \mathcal{T}_e$ and define the language $\mathcal{L}_{\exp\restriction} = \mathcal{L} \cup \{\exp \restriction_{[0,1]}\}$. Then the structures $(K_0 \mid \mathcal{L}_e)$ and $(K_0 \mid \mathcal{L}_{\exp\restriction})$, where the function (symbol) $\exp \restriction_{[0,1]}$ is interpreted in the obvious way, have the same definable sets. Moreover, they have the same existentially definable sets.*

Proof. We prove that for every formula of the form $t = x$, where t is an $\mathcal{L}_e$-term, there is an existential $\mathcal{L}_{\exp\restriction}$-formula $\phi_t(x)$, such that $t = x$ and $\phi_t(x)$ define the same sets. It is worth pointing out that $\phi_t(x)$ may implicitly depend on variables other than x. Our proof uses induction over the term t. The base case is satisfied, because if t is a variable or a constant, then we can just take $\phi_t(x) \equiv t = x$. Now suppose that f is a function symbol other than e. (So f is $+, \cdot$ or $-$.) Suppose furthermore that we have $\mathcal{L}_{\exp\restriction}$-formulas $\phi_t(x)$ and $\phi_s(x)$ corresponding to the $\mathcal{L}_e$ formulas $t = x$ and $s = x$ respectively. Then

$$\phi(x) \equiv \exists y_1 y_2 [f(y_1, y_2) = x \wedge \phi_t(y_1) \wedge \phi_t(y_2)]$$

corresponds to the formula $f(t, s) = x$ and is (equivalent to) an existential formula. Lastly, suppose that the $\mathcal{L}_{\exp\restriction}$-formula $\phi_t(x)$ corresponds to the $\mathcal{L}_e$ formula $t = x$. Then

$$\phi(x) \equiv \exists y_1 y_2 [\exp\restriction_{[0,1]}(y_1) = x \wedge 1 = (1 + y_2^2) \cdot y_1 \wedge \phi_t(y_2)]$$

corresponds to the formula $e(t) = x$ and is existential, up to equivalence. This completes our induction.

It is easily verified that for an atomic or negated atomic $\mathcal{L}_e$-formula, χ say, there is an existential $\mathcal{L}_{\exp\restriction}$-formula, ϕ_χ, defining the same set. For if t and s are $\mathcal{L}_e$-terms, then

$$\begin{array}{lll} \chi \equiv t = s & \text{corresponds to} & \phi_\chi \equiv \exists y [\phi_t(y) \wedge \phi_s(y)], \\ \chi \equiv \neg(t = s) & \text{corresponds to} & \phi_\chi \equiv \exists y_1 y_2 [\neg(y_1 = y_2) \wedge \phi_t(y_1) \wedge \phi_s(y_2)], \\ \chi \equiv t < s & \text{corresponds to} & \phi_\chi \equiv \exists y_1 y_2 [y_1 < y_2 \wedge \phi_t(y_1) \wedge \phi_s(y_2)] \text{ and} \\ \chi \equiv \neg(t < s) & \text{corresponds to} & \phi_\chi \equiv \exists y_1 y_2 [\neg(y_1 < y_2) \wedge \phi_t(y_1) \wedge \phi_s(y_2)]. \end{array}$$

Recall that every formula can be written as a string of quantifiers followed by a formula in conjunctive normal form. So every $\mathcal{L}_e$-formula is equivalent to a formula of the form

$$Q_1 x_1 \ldots Q_n x_n \bigwedge_{i=1}^{m} \bigvee_{j=1}^{l_i} \chi_i^j, \tag{33}$$

where the $Q_1 \ldots Q_n$ are quantifiers and each χ_i^j is an atomic $\mathcal{L}_e$-formula or a negated atomic $\mathcal{L}_e$-formula. But then the $\mathcal{L}_{\exp\restriction}$-formula

$$Q_1 x_1 \ldots Q_n x_n \bigwedge_{i=1}^{m} \bigvee_{j=1}^{l_i} \phi_{\chi_i^j} \tag{34}$$

defines the same set. Furthermore, since each formula $\phi_{\chi_i^j}$ is existential, (34) is equivalent to an existential formula if (33) is existential. We have now shown that every (existentially) definable set of $(K_0 \mid \mathcal{L}_e)$ is also an (existentially) definable set of $(K_0 \mid \mathcal{L}_{\exp\restriction})$. We omit the proof of the converse, as it is similar. □

Corollary 6.2.4. *The theory $\mathcal{T}_e$ is model complete.*

Proof. This is an immediate consequence of Lemma 6.2.3 and the fact that the theory $\mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\exp\restriction})$ is model complete by Theorem 2.1.1. □

It is also convenient to introduce the following family of rings.

Definition 6.2.5. Let $n \in \mathbb{N}$ and $s \subseteq \{1, \ldots, n\}$. By M_n^s we denote the ring of functions $K^n \to K$ generated (as a ring) over k (considered as a field of constant functions) by

- x_i, for $i = 1, \ldots, n$.

- $(1+x_i^2)^{-1}$, for $i = 1, \dots, n$.
- $e(x_i)$, for $i = 1, \dots, n$.
- $\exp(x_i)$ for $i \in s$.

Remark 6.2.6. Since the derivatives of each of the generators of M_n^s lie in M_n^s, the ring M_n^s is closed under differentiation, by the sum and product rule. In particular we have $\det\left(\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\right) \in M_n^s$, for $f_1, \dots, f_n \in M_n^s$. Furthermore, the functions in M_n^s are K-definable and C^∞. Note also $\mathbb{Z}[x_1, \dots, x_n]$ is a subring of M_n^s and that M_n^s is Noetherian, as it is finitely generated over k.

The properties of the rings M_n^s mentioned in Remark 6.2.6 allow us to use many of the results we have already proven. In the following Proposition, we give these results in a form that is suited to our needs.

Proposition 6.2.7. *Let $n \in \mathbb{N}$ and let $s \subseteq \{1, \dots, n\}$.*

(i) *Suppose that $f \in M_n^s$, $\vec{\alpha} \in K^n$ and $f(\vec{\alpha}) = 0$. Then there exist $f_1, \dots, f_n \in M_n^s$ and $\vec{\beta} \in K^n$ such that $f(\vec{\beta}) = f_1(\vec{\beta}) = \cdots f_n(\vec{\beta}) = 0$ and $\det\left(\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\right)(\vec{\beta}) \neq 0$.*

(ii) *If, in* **(i)**, *$\vec{\alpha}$ is an isolated zero of f, then we may take $\vec{\beta} = \vec{\alpha}$.*

(iii) *Let $f_1, \dots, f_n \in M_n^s$. Then there are only finitely many $\vec{\gamma} \in K^n$ such that $f_1(\vec{\gamma}) = \cdots = f_n(\vec{\gamma})$ and $\det\left(\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)}\right)(\vec{\gamma}) \neq 0$.*

Proof. For **(i)**, we apply Theorem 3.3.4, with $\mathcal{T}_\mathcal{A} = \mathcal{T}_{\exp}$, $M = M_n^s$, $U = K$ and $S = V(f)$. The conditions of Theorem 3.3.4 are satisfied by Remark 6.2.6 and by the fact that $S \neq \emptyset$, as $\vec{\alpha} \in S$. This gives us the desired result immediately.

For **(ii)**, we apply Theorem 3.2.7. To be precise, we set $\mathcal{T}_\mathcal{A} = \mathcal{T}_{\exp}$, $P_0 = \vec{\alpha}$ and

$$M = \{[g \restriction_U, U] \mid \ g \in M_n^s \text{ and } U \subseteq K^n \text{ open, with } \vec{\alpha} \in U\ \}$$

and we apply Theorem 3.2.7 repeatedly for $m = 0, \dots, n-1$. At each stage m, we acquire a function $f_{m+1} \in M_n^s$ by using **(iii)** of Theorem 3.2.7, satisfying $\vec{\alpha} \in \mathcal{V}_r(f_1, \dots, f_{m+1})$. Once we reach $\vec{\alpha} \in \mathcal{V}_r(f_1, \dots, f_n)$ we have our desired result. In order for this to work, we need to show that option **(ii)** of Theorem 3.2.7 cannot hold at any stage. (It is clear that option **(i)** never holds.) Suppose to the contrary that this is the case for some $m < n$ and set $r = n - m$. Then by taking $[h, W] = [f, K^n]$, we find that f vanishes on $U \cap \mathcal{V}_r(f_1, \dots, f_m)$, for some open neighborhood U of $\vec{\alpha}$. Since $\vec{\alpha} \in \mathcal{V}_r(f_1, \dots, f_m)$, the vectors $d_{\vec{\alpha}} f_1, \dots, d_{\vec{\alpha}} f_m$ are linearly independent over K. This means that there exists a set $S \subseteq \{1, \dots, n\}$ of size m such that the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(\vec{\alpha})\right)_{1 \leq i \leq m, j \in S}$$

has a nonzero determinant. By relabeling our variables we assume that $S = \{r+1, \dots, n\}$, which means that we can apply Theorem 3.2.2 at the point $\vec{\alpha}$. But then by **(ii)** of Theorem 3.2.2, α is clearly not an isolated point of $U \cap \mathcal{V}_r(f_1, \dots, f_m)$, contrary to our assumption. So indeed **(ii)** of Theorem 3.2.7 does not hold.

For **(iii)**, we write $s = \{i_1, \dots, i_m\}$ and note that the sequence

$$H_1(x) = (1+x^2)^{-1}, H_2(x) = e(x), H_3(x) = \exp(x)$$

is a Pfaffian chain on $\mathbb{R}$. Now take $\mathcal{L}_{\text{Pf}}$ and $\mathcal{T}_{\text{Pf}}$ as in Definition 4.2.1, for H_1, H_2, H_3. Then the sequence

$$\vec{\sigma} = ((1+x_1^2)^{-1}, \ldots, (1+x_n^2)^{-1}, e(x_1), \ldots, e(x_n), \exp(x_{i_1}), \ldots, \exp(x_{i_m}))$$

is a $(2n+m, n)$-sequence with respect to $\mathcal{L}_{\text{Pf}}$. Note that $M^n(k, K, \vec{\sigma})$ (as in Definition 4.2.4) is the same as M_n^s (as in Definition 6.2.5). By Remark 4.2.5, we can apply Corollary 4.1.3, with $r_1 = n$ and $r_2 = 0$ to conclude that **(iii)** holds. □

We will now give a proof of the Second Main Theorem, assuming that for certain elements of K, we can find a linear combination which is "small" in some sense. (This condition is formulated in (36).)

Proof. (Of Theorem 6.1.2.) Let us assume that the Theorem is false. Then by Remark 6.2.1, it follows that there exists $m \in \mathbb{N}$ such that the following statement is true.

> For some $n \in \mathbb{N}$, with $n \geq m$, there exists $\vec{\alpha} \in K^n$, $l \in \{1, \ldots, n\}$ and $s \subseteq \{1, \ldots, n\}$, with $|s| = m$, such that for some $f_1, \ldots, f_n \in M_n^s$ holds that $f_1(\vec{\alpha}) = \cdots f_n(\vec{\alpha}) = 0$ and $\det\left(\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}\right)(\vec{\alpha}) \neq 0$. Furthermore, $|\alpha_l| > b$ for all $b \in k$, and if $m > 0$, then $l \in s$. (35)

At first sight, this statement might look a bit more complicated than necessary, as we could take $n = m$ and $s = \{1, \ldots, m\}$. However, we should keep in mind that our strategy is to reduce m at the cost of increasing n. So, let us choose m minimal such that (35) holds. We claim that $m > 0$.

To prove this claim, suppose that $m = 0$. Since $K \models \mathcal{T}_{\exp}$, it has an obvious interpretation as an $\mathcal{L}_e$-structure. Similarly, we can consider k as an $\mathcal{L}_e$-structure. Clearly $K, k \models \mathcal{T}_e$ and k is an $\mathcal{L}_e$-substructure of K. By (35), there exists $\vec{\alpha} \in K^n$ and $f_1, \ldots, f_n \in M_n^{\emptyset}$, such that $f_1(\vec{\alpha}) = \cdots f_n(\vec{\alpha}) = 0$ and $\det\left(\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}\right)(\vec{\beta}) \neq 0$. By Proposition 6.2.7, there exist only finitely many such $\vec{\alpha} \in K^n$, say N. But we can express the fact that there are at least N solutions to these equations using an existential $\mathcal{L}_e$-sentence with parameters from k. Since $\mathcal{T}_e$ is model complete by Corollary 6.2.4, this means that these solutions must all lie in k^n. But this contradicts that $|\alpha_l| > b$ for all $b \in k$, by (35), proving the claim.

Now, for our minimal m, which is nonzero as we have just seen, take $n, \vec{\alpha}, l, s$ and $f_1, \ldots, f_n$ as in (35). Eventually, we we will be able to show the following fact.

> There exist $n_i \in \mathbb{Z}$, for $i \in s$, not all zero, and $c \in k$ such that $0 < c + \sum_{i \in s} n_i \alpha_i < 1$. (36)

Let us assume this for now and continue with the rest of the proof. Note that since $|\alpha_l| > b$ for all $b \in k$, it cannot be the case that $n_i = 0$ for all $i \in s \setminus \{l\}$. So, for convenience we suppose that $1 \in s$, $n_1 \neq 0$, and $l \neq 1$. We may furthermore assume that $n_1 > 0$, for if this is not the case, we simply replace each n_i by $-n_i$ and c by $1 - c$ in (36). We now set $\alpha_{n+1} = \exp(\alpha_1)$ and we take $\alpha_{n+2} \in K$ such that $\alpha_{n+2} > 0$ and

$$(1 + \alpha_{n+2}^2)^{-1} = c + \sum_{i \in s} n_i \alpha_i.$$

this is possible, as K is a real closed field. For each $i = 1, \dots, n$, we let $g_i(x_1, \dots, x_{n+1})$ be the result of replacing $\exp(x_1)$ by x_{n+1} in $f_i(x_1, \dots, x_n)$. Then each g_i is an element of $M_{n+1}^{s\setminus\{l\}}$ and it is not difficult to verify that $(\alpha_1, \dots, \alpha_{n+2})$ is a solution to the following system of equations.

$$\begin{aligned} g_1(x_1, \dots, x_{n+1}) &= 0 \\ &\vdots \\ g_n(x_1, \dots, x_{n+1}) &= 0 \end{aligned} \tag{37}$$

$$(1 + x_{n+2}^2)^{-1} - c - \sum_{i \in s} n_i x_i = 0 \tag{38}$$

$$\left[x_{n+1}^{n_1} \cdot \exp(c) \cdot \prod_{j \in s^+} \exp(x_j)^{n_j} \right] - \left[e(x_n + 2) \cdot \prod_{j \in s^-} \exp(x_j)^{-n_j} \right] = 0, \tag{39}$$

where $s^{\pm} = \{j \in s \mid j > 1, \pm n_j > 0\}$. The last equation is obtained by rewriting (38) as

$$n_1 x_1 + c + \sum_{j \in s^+} n_j x_j = (1 + x_{n+2}^2)^{-1} + \sum_{j \in s^-} -n_j x_i,$$

exponentiating both sides and subsequently replacing $\exp(x_1)$ by x_{n+1}. After this, it is simply rearranged and we have written $e(x_{n+2})$ for $\exp((1 + x_{n+2}^2)^{-1})$.

Recall that $f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0$ and $\det\left(\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)}\right)(\vec{\alpha}) \neq 0$ and that $f_1, \dots, f_n$ are C^∞-functions. If it where the case that $K = \mathbb{R}$, then the Inverse Function Theorem would tell us that the function $(f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible on some open neighborhood U of $\vec{\alpha} \in \mathbb{R}^n$. Then in particular, $\vec{\alpha}$ is the unique solution to $f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0$ on some open neighborhood $U \subseteq \mathbb{R}^n$ of $\vec{\alpha}$, which we may take to be definable. Fortunately, $K \models \mathcal{T}_{\exp}$, so even if $K \neq \mathbb{R}$, we may suppose that $\vec{\alpha}$ is the only solution of $f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0$ on some definable open neighborhood $U \subseteq K^n$ of $\vec{\alpha}$, by transfer.

We claim that $(\alpha_1, \dots, \alpha_{n+2})$ is the only solution of the system (37) - (39) contained in the open subset $U \times K_{>0} \times K_{>0} \subseteq K^{n+2}$. For suppose that $(\beta_1, \dots, \beta_{n+2})$ is such a solution. Then in particular, $(\beta_1, \dots, \beta_{n+2})$ must satisfy (38) and if we just remember how we obtained (39) from (38), we see that $(\beta_1, \dots, \beta_n, \exp(\beta_1), \beta_{n+2})$ satisfies (39). Since $(\beta_1, \dots, \beta_{n+2})$ also satisfies (39), we get

$$\beta_{n+1}^{n_1} \cdot \exp(c) \cdot \prod_{j \in s^+} \exp(\beta_j)^{n_j} = e(\beta_n + 2) \cdot \prod_{j \in s^-} \exp(\beta_j)^{-n_j} = \exp(\beta_1)^{n_1} \cdot \exp(c) \cdot \prod_{j \in s^+} \exp(\beta_j)^{n_j}.$$

It follows that $\beta_{n+1}^{n_1} = \exp(\beta_1)^{n_1}$, so since n_1 is nonzero and since β_{n+1} and $\exp(\beta_1)$ are both positive, we may conclude that $\beta_{n+1} = \exp(\beta_1)$. This means that $g_i(\beta_1, \dots, \beta_n, \exp(\beta_1)) = 0$ for $i = 1, \dots, n$, so each $f_i(\beta_1, \dots, \beta_n) = 0$, by definition of the g_i. By uniqueness of the solution for $f_1(\vec{x}) = \cdots = f_n(\vec{x}) = 0$ in U, this shows that $\beta_i = \alpha_i$ for $i = 1, \dots, n$. This automatically gives us $\beta_{n+1} = \exp(\beta_1) = \exp(\alpha_1) = \alpha_{n+1}$. And lastly, by (38),

$$(1 + \beta_{n+2}^2)^{-1} = c + \sum_{i \in s} n_i \beta_i = c + \sum_{i \in s} n_i \alpha_i = (1 + \alpha_{n+2}^2)^{-1},$$

which tells us that $\beta_{n+2} = \alpha_{n+2}$, as β_{n+2} and α_{n+2} are both positive, proving our claim.

Now let f be the sum of the squares of the $n+2$ functions appearing in (37) - (39). By our claim, $(\alpha_1, \ldots, \alpha_{n+2})$ is an isolated zero of f. Note furthermore that $f \in M_{n+2}^{s \setminus \{1\}}$ (using that c and $\exp(c)$ lie in k). By parts **(i)** and **(ii)** of Proposition 6.2.7, there exist $h_1, \ldots, h_{n+2} \in M_{n+2}^{s \setminus \{1\}}$ such that $h_1(\alpha_1, \ldots, \alpha_{n+2}) = \cdots = h_{n+2}(\alpha_1, \ldots, \alpha_{n+2}) = 0$ and $\det\left(\frac{\partial(h_1,\ldots,h_{n+2})}{\partial(x_1,\ldots,x_{n+2})}\right)(\alpha_1, \ldots, \alpha_{n+2}) \neq 0$. But this shows that (35) holds for $m-1$, contradicting the minimality of m. $\square$

We have now proven the Second Main Theorem. However, we still have a debt to pay. This debt is the proof of (36). In the upcoming sections, we show that we were justified in assuming (36).

7 Towards condition 36

7.1 Dimensions for O-minimal expansions

In the subsequent parts, we let $\mathcal{L}_{\mathcal{O}}$ be any extension of the language $\mathcal{L}$, such that $(\mathbb{R} \mid \mathcal{L}_{\mathcal{O}})$ is an O-minimal structure. Furthermore, we set $\mathcal{T}_{\mathcal{O}} = \mathrm{Th}(\mathbb{R} \mid \mathcal{L}_{\mathcal{O}})$. Recall that this means that every model $K \models \mathcal{T}_{\mathcal{O}}$ is also O-minimal. In this section, we give two notions of dimension for such a structure K and we discuss some of their properties.

Definition 7.1.1. Given a language L and an L-structure M, we say that M has *definable Skolem functions*, if for every L-formula $\phi(\vec{x}, y)$, there exists a function $f(\vec{x})$, definable in the language L, such that whenever $\vec{a} \in M$, with $M \models \exists y \phi(\vec{a}, y)$, then $M \models \phi(\vec{a}, f(\vec{a}))$.

Furthermore, we say a theory T in a language L has definable Skolem functions, if for every L-formula $\phi(\vec{x}, y)$, there exists a function $f(\vec{x})$, definable in the language L, such that whenever $M \models T$ and $\vec{a} \in M$, with $M \models \exists y \phi(\vec{a}, y)$, then $M \models \phi(\vec{a}, f(\vec{a}))$.

Remark 7.1.2. It is known that for O-minimal structures endowed with an additive group structure, definable Skolem functions exist. (This is a direct consequence of Proposition A.2.6.) Since $\mathcal{T}_{\mathcal{O}}$ is the complete $\mathcal{L}_{\mathcal{O}}$-theory of the additive group $\mathbb{R}$, it follows that $\mathcal{T}_{\mathcal{O}}$ admits definable Skolem functions. We are indifferent to the exact inner workings of these functions, so let us just agree upon some unspecified, but fixed set of definable Skolem functions.

Definition 7.1.3. Let $K \models \mathcal{T}_{\mathcal{O}}$. For any subset $A \subseteq K$, we denote by $\mathrm{Dcl}(A)$ the closure of A under the definable functions of $\mathcal{T}_{\mathcal{O}}$ in K. That is

$$\mathrm{Dcl}(A) = \{f(a_1, \dots, a_n) \mid n \in \mathbb{N}, a_1, \dots, a_n \in A \text{ and } f \text{ a definable (partial) function }\}.$$

Remark 7.1.4. Using the fact that $\mathcal{T}_{\mathcal{O}}$ has definable Skolem functions, it is not difficult to verify that $\mathrm{Dcl}(A)$ is (the domain of) a substructure of K. In fact $\mathrm{Dcl}(A) \preceq K$, by the Tarski-Vaught Test.

Remark 7.1.5. By convention, a 0-place definable function is a definable element of K, so $\mathrm{Dcl}(\emptyset)$ is the same as $\mathrm{Dcl}(\{0\})$ for example, as 0 is part of our language. Note that if we take $k = \mathrm{Dcl}(\{0\})$, then there exists an embedding of $\mathcal{L}_{\mathcal{O}}$-structures $k \to \mathbb{R}$, which sends an element $f^k(0) \in k$ to $f^{\mathbb{R}}(0) \in \mathbb{R}$. Recall that an ordered field F is called *Arichimedean* if for every positive $x, y \in F$, there exists $n \in \mathbb{N}$ such that $y < nx$. Since $\mathbb{R}$ is Archimedean, so is k by this embedding.

Lemma 7.1.6. *A structure $K \models \mathcal{T}_{\mathcal{O}}$ together with this closure operation satisfy the requirements for being a so-called pregeometry, which means that*

(i) Dcl *is monotone increasing and dominates* id, *so* $A \subseteq \mathrm{Dcl}(A) \subseteq \mathrm{Dcl}(B)$ *whenever* $A \subseteq B$.

(ii) Dcl *is idempotent, meaning that* $\mathrm{Dcl}(A) = \mathrm{Dcl}(\mathrm{Dcl}(A))$.

(iii) Dcl *is of finite character, which means that for every* $a \in \mathrm{Dcl}(A)$, *there is some finite subset* $B \subseteq A$ *such that* $a \in \mathrm{Dcl}(B)$.

(iv) Dcl *has the exchange property, so if* $a \in \mathrm{Dcl}(A \cup \{b\}) \setminus \mathrm{Dcl}(A)$, *then* $b \in \mathrm{Dcl}(A \cup \{a\})$.

Proof. Let $A \subseteq B \subseteq K$. To prove **(i)**, take $a \in A$. Then the 0-place definable function $\varphi(x) \equiv x = a$ shows that $a \in \mathrm{Dcl}(A)$. Hence $A \subseteq \mathrm{Dcl}(A)$. The fact that $\mathrm{Dcl}(A) \subseteq \mathrm{Dcl}(B)$ is clear.

For **(ii)**, note that $\mathrm{Dcl}(A) \subseteq \mathrm{Dcl}(\mathrm{Dcl}(A))$ by **(i)**. Now let $c \in \mathrm{Dcl}(\mathrm{Dcl}(A))$. Then by definition, $c = f(b_1, \ldots, b_n)$, with $b_1, \ldots, b_n \in \mathrm{Dcl}(A)$ and f a definable function. For each b_i, we have a definable function g_i, such that $g_i(a_1, \ldots, a_m) = b_i$, for some $a_1, \ldots, a_m \in A$. But then $f(g_1(\vec{x}), \ldots, g_n(\vec{x}))$ is a definable function and $c = f(g_1(\vec{a}), \ldots, g_n(\vec{a}))$, so $c \in \mathrm{Dcl}(A)$. Hence $\mathrm{Dcl}(A) \supseteq \mathrm{Dcl}(\mathrm{Dcl}(A))$.

Property **(iii)** is clear.

For **(iv)**, let $a \in \mathrm{Dcl}(A \cup \{b\})$. We show that either $b \in \mathrm{Dcl}(A \cup \{a\})$ or $a \in \mathrm{Dcl}(A)$. By definition, there exists a definable function f, with parameters from A, such that $f(b) = a$. We define the set $B = \{x \in K \mid f(x) = a\}$. By O-minimality of K, B is a finite union of points and intervals. Now, if b is a boundary point of B, then there exists a formula $\varphi(x)$, with parameters from $A \cup \{a\}$, such that only b satisfies $\varphi(x)$. (We can express that b is the left or right endpoint of the i-th interval of B, and we can express that b is the j-th isolated point of B.) Hence, $\varphi(x)$ is a 0-place definable function witnessing that $b \in \mathrm{Dcl}(A \cup \{a\})$.

On the other hand, suppose that b is not a boundary point of B. Then there exist an interval $(c_1, c_2) \subseteq B$ such that $b \in (c_1, c_2)$. Note that we can define the set C_l of left endpoints (lying in K) of the intervals on which f is constant by

$$\begin{aligned} C_l = \{x \in K \mid \exists y > x[\\ &\forall z_1, z_2((x < z_1 < y \wedge x < z_2 < y) \to f(z_1) = f(z_2)) \\ &\wedge \neg \exists w < x(\forall z_1, z_2((w < z_1 < y \wedge w < z_2 < y) \to f(z_1) = f(z_2))]\}. \end{aligned}$$

In the same way we can define C_r, the set of right endpoints of the intervals on which f is constant. Take $d_1 \in C_l \cup \{-\infty\}$ and $d_2 \in C_r \cup \{\infty\}$ such that $f(x) = a$ for all $x \in (d_1, d_2)$. Since both C_l and C_r clearly do not contain any intervals, they must be finite. This means that each of the points of C_l and C_r are definable using parameters from A. But then there exists an $\mathcal{L}_{\mathcal{O}}$-formula $\varphi(x)$, with parameters from A, asserting *"x is the value f takes on the interval (d_1, d_2)"*. This shows that $a \in \mathrm{Dcl}(A)$. □

Definition 7.1.7. Let $K \models \mathcal{T}_{\mathcal{O}}$. We call a set $A \subseteq K$ *independent* if $a \notin \mathrm{Dcl}(A \setminus \{a\})$ for all $a \in A$. A set $A \subseteq K$ is said to be a *basis* for K if A is independent and generates K, meaning that $K = \mathrm{Dcl}(A)$.

Lemma 7.1.8. *Let $K \models \mathcal{T}_{\mathcal{O}}$. Then any basis for K has the same cardinality.*

Proof. Let B be a basis for K with minimal cardinality. Suppose first that $|B|$ is finite, say $|B| = n$. Now let $m \in \mathbb{N}$ be the largest number such that for some basis B' of K, $|B'| \neq n$ and $|B' \cap B| = m$. Suppose that $m = n$. Then $B \subseteq B'$ and there exists at least one $a \in B' \setminus B$. But then $a \in \mathrm{Dcl}(B' \setminus \{a\})$, as $B \subseteq B' \setminus \{a\}$, contradicting the fact that B' is independent. So, since $|B' \cap B| = m < n$ and $|B'| \neq m$, by minimality of n, there exists $b' \in B' \setminus B$. By independence of B', $B' \setminus \{b'\}$ does not generate K. This means that there must be some $b \in B$ such that $b \notin \mathrm{Dcl}(B' \setminus \{b'\})$, for otherwise

$$K = \mathrm{Dcl}(B) \subseteq \mathrm{Dcl}(\mathrm{Dcl}(B' \setminus \{b'\})) = \mathrm{Dcl}(B' \setminus \{b'\}).$$

Consider $B'' = (B' \setminus \{b'\}) \cup \{b\}$. We note that $|B'' \cap B| = m + 1$ and $|B''| = |B'| \neq n$. We show that B'' is a basis for K, contradicting the maximality of m. First of all, $b \in \mathrm{Dcl}((B' \setminus \{b'\}) \cup \{b'\}) \setminus \mathrm{Dcl}(B' \setminus \{b'\})$, so by the exchange property, $b' \in \mathrm{Dcl}((B'')$. It follows that $B' \subseteq \mathrm{Dcl}((B'')$ and hence $\mathrm{Dcl}((B'') = K$, so B'' generates K. To prove that B'' is independent, let $a \in B''$ and suppose to the contrary that $a \in \mathrm{Dcl}(B'' \setminus \{a\})$. If $a = b$, then we immediately find that $b \in \mathrm{Dcl}(B' \setminus \{b'\})$, which is false, so we may suppose that $a \neq b$. Since B' is independent, $a \notin \mathrm{Dcl}(B' \setminus \{a\})$, so certainly $a \notin \mathrm{Dcl}(B' \setminus \{b', a\})$ and hence $a \in \mathrm{Dcl}((B' \setminus \{b', a\}) \cup \{b\}) \setminus$

$\mathrm{Dcl}(B' \setminus \{b', a\})$, as $(B' \setminus \{b', a\}) \cup \{b\} = B'' \setminus \{a\}$. Then by the exchange property, $b \in \mathrm{Dcl}((B' \setminus \{b', a\}) \cup \{a\}) = \mathrm{Dcl}((B' \setminus \{b'\})$, which is false.

Now suppose that B is infinite. Let B' be any other basis for K. Then $|B| \leq |B'|$, by choice of B. We show that $|B'| \leq |B|$. For every $b \in B$, there is a finite set $B_b \subseteq B'$ such that $b \in \mathrm{Dcl}(B_b)$, since Dcl is of finite character. Hence $K = \mathrm{Dcl}(B) = \mathrm{Dcl}(\bigcup_{b \in B} B_b)$, so the subset $\bigcup_{b \in B} B_b \subseteq B'$ must be equal to B', by independence of B'. But since B' is infinite and each B_b is finite, $\bigcup_{b \in B} B_b = B'$ can only hold if $|B'| \leq |B|$. □

By Lemma 7.1.8, we can now unambiguously define the dimension of K.

Definition 7.1.9. Given $K \models \mathcal{T}_{\mathcal{O}}$, we define the dimension of K, denoted $\dim(K)$, to be the cardinality of any basis for K.

Lemma 7.1.10. *Let $K \models \mathcal{T}_{\mathcal{O}}$. Then any independent subset $A \subseteq K$ can be extended to a basis for K.*

Proof. Let $\mathcal{S} = \{B \subseteq K \mid A \subseteq B \text{ and } B \text{ independent }\}$. Then $\mathcal{S}$ is a poset, ordered by $\subseteq$. We apply Zorn's Lemma to $\mathcal{S}$. Note that $A \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$. Now let $\{B_i \mid i \in I\}$ be a nonempty chain in $\mathcal{S}$ and take $\mathcal{B} = \bigcup_{i \in I} B_i$. Then $\mathcal{B}$ is independent, for if $a \in \mathcal{B}$ and $a \in \mathrm{Dcl}(\mathcal{B} \setminus \{a\})$, but then also $a \in \mathrm{Dcl}(\mathcal{B}' \setminus \{a\})$, for some finite subset $\mathcal{B}' \subseteq \mathcal{B}$. For some sufficiently large index $i \in I$, we have $a \in B_i$ and $\mathcal{B}' \subseteq B_i$, so $a \in \mathrm{Dcl}(B_i \setminus \{a\})$, contradicting that B_i is independent. We conclude that $\mathcal{B}$ is an upper bound for $\{B_i \mid i \in I\}$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element. But such a maximal independent set is clearly a basis for K, containing A, so we have proven the Lemma. □

We will also work with closures relative to substructures.

Definition 7.1.11. If $k, K \models \mathcal{T}_{\mathcal{O}}$ and $k \subseteq K$, then we can define the closure of A under the k-definable functions of $\mathcal{T}_{\mathcal{O}}$ in K by

$$\mathrm{Dcl}_k(A) = \{f(a_1, \ldots, a_n) \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A \text{ and } f \text{ a } k\text{-definable (partial) function }\}.$$

Remark 7.1.12. We call $\mathrm{Dcl}_k(A)$ the definable closure of A *over* k. Lemma 4.4.2, Remark 7.1.4 as well as Lemma 7.1.10 still hold true in this new situation (but now over k) and we denote the cardinality of a basis of K over k by $\dim_k(K)$.

Lemma 7.1.13. *Let $k_0, k_1, K \models \mathcal{T}_{\mathcal{O}}$, with $k_0 \subseteq k_1 \subseteq K$. Then* $\dim_{k_0}(K) = \dim_{k_0}(k_1) + \dim_{k_1}(K)$.

Proof. Let A be a basis for k_1 over k_0. Then A is and independent set (with respect to Dcl_{k_0}) so by Lemma 7.1.10 and Remark 7.1.12, A can be extended to a basis B (over k_0) for K. We write B as a disjoint union $B = A \cup C$. Then C generates K over k_1, since

$$K = \mathrm{Dcl}_{k_0}(A \cup C) = \mathrm{Dcl}_{k_0}(k_1 \cup C) = \mathrm{Dcl}_{k_1}(C).$$

Furthermore, C is independent (with respect to Dcl_{k_1}), for if $a \in C$ and $a \in \mathrm{Dcl}_{k_1}(C \setminus \{a\})$, then also $a \in \mathrm{Dcl}_{k_0}(A \cup C \setminus \{a\})$, as

$$\mathrm{Dcl}_{k_0}(A \cup C \setminus \{a\}) = \mathrm{Dcl}_{k_0}(k_1 \cup C \setminus \{a\}) = \mathrm{Dcl}_{k_1}(C \setminus \{a\}).$$

But this is false, as $A \cup C$ is an independent set, with respect to Dcl_{k_0}. We conclude that C is a basis for K over k_1. Hence, $\dim_{k_0}(K) = |B| = |A| + |C| = \dim_{k_0}(k_1) + \dim_{k_1}(K)$. □

We introduce another notion of dimension for models $K \models \mathcal{T}_{\mathcal{O}}$. An element $a \in K$ is called *finite* if $|a| < n$ for some $n \in \mathbb{N}$ and *infinitesimal* if $|a| < \frac{1}{n}$ for all $n \in \mathbb{N} \setminus \{0\}$. The set of finite elements of K is denoted by $\mathrm{Fin}(K)$ and forms a convex subring of K, with as unique maximal ideal $\mu(K)$, the set of infinitesimals in K. Note that the set $\mathrm{Fin}(K) \setminus \mu(K)$ forms a subgroup of $K \setminus \{0\}$ under multiplication.

Definition 7.1.14. Given $K \models \mathcal{T}_{\mathcal{O}}$, we define the quotient group

$$V(K) = (K \setminus \{0\})/(\mathrm{Fin}(K) \setminus \mu(K)),$$

which we shall call the *value group* of K.

The value group of K basically allows us to ignore the "standard part" of K and studying this group gives us information about the nature of the infinite elements contained in K. Although it might seems natural to write "$\cdot$" for the group operation of $V(K)$ at this point, we shall actually use "+" for reasons that will become clear momentarily. Since n-th roots exist for all positive elements of K and all $n \in \mathbb{N}$, this makes $V(K)$ into a divisible group. This allows us view $V(K)$ as a vector space over $\mathbb{Q}$, explaining our preference for using "+".

Definition 7.1.15. Given $K \models \mathcal{T}_{\mathcal{O}}$, we denote the dimension of $V(K)$ as a $\mathbb{Q}$-vector space by $\mathrm{valdim}(K)$.

We can generate an order on the group $V(K)$ by setting $a/(\mathrm{Fin}(K) \setminus \mu(K)) > 0$, if and only if $a \in \mu(K)$. (This order is well-defined on equivalence classes.)

Definition 7.1.16. Let $K \models \mathcal{T}_{\mathcal{O}}$. The map $\nu_K : K \to V(K) \cup \{\infty\}$, extending the quotient map $K \setminus \{0\} \to V(K)$ by setting $\nu_K(0) = \infty$, is called the *valuation map* of K. We extend the order of $V(K)$ to $V(K) \cup \{\infty\}$ by setting $\infty > \alpha$ for all $\alpha \in V(K)$. Furthermore, we extend the addition operation on $V(K)$ to $V(K) \cup \{\infty\}$ by setting $\alpha + \infty = \infty + \alpha = \infty$ for all $\alpha \in V(K)$.

Remark 7.1.17. The map $\nu_K : K \to V(K) \cup \{\infty\}$ map satisfies the following properties, which are not difficult to verify.

(i) $\nu_K(x \cdot y) = \nu_K(x) + \nu_K(y)$ for all $x, y \in K$.

(ii) $\nu_K(x + y) \geq \min(\nu_K(x), \nu_K(y))$ for all $x, y \in K$, with equality when $\nu_K(x) \neq \nu_K(y)$.

As in Remark 7.1.12, we have a notion of dimension relative to a substructure.

Definition 7.1.18. Let k and K be models of $\mathcal{T}_{\mathcal{O}}$, with $k \subseteq K$. Then $\nu_K[k \setminus \{0\}]$ is a $\mathbb{Q}$-vector subspace of $V(K)$, as k is a real closed subfield of K. We denote the dimension of $V(K)$ over $\nu_K[k \setminus \{0\}]$ by $\mathrm{valdim}_k(K)$.

Lemma 7.1.19. *Let $k_0, k_1, K \models \mathcal{T}_{\mathcal{O}}$, with $k_0 \subseteq k_1 \subseteq K$. Then* $\mathrm{valdim}_{k_0}(K) = \mathrm{valdim}_{k_0}(k_1) + \mathrm{valdim}_{k_1}(K)$.

Proof. Recall that given three ($\mathbb{Q}$-)vector spaces $V_0 \subseteq V_1 \subseteq V_2$, we have $d_0 = d_1 + d_2$, where d_0 is the dimension of V_2 over V_0, d_1 is the dimension of V_1 over V_0 and d_2 is the dimension of V_2 over V_1. This means that $\mathrm{valdim}_{k_0}(K) = d + \mathrm{valdim}_{k_1}(K)$, where d is the dimension of the $\mathbb{Q}$-vector space $\nu_K[k_1 \setminus \{0\}]$ over its subspace $\nu_K[k_0 \setminus \{0\}]$. It is not difficult to verify that the map $\nu_K[k_1 \setminus \{0\}] \to V(k_1)$ given by $x/(\mathrm{Fin}(K) \setminus \mu(K)) \mapsto x/(\mathrm{Fin}(k_1) \setminus \mu(k_1))$ is an isomorphism of $\mathbb{Q}$-vecor spaces and that the subspace $\nu_K[k_0 \setminus \{0\}] \subseteq \nu_K[k_1 \setminus \{0\}]$ corresponds to the subspace $\nu_{k_1}[k_0 \setminus \{0\}] \subseteq V(k_1)$ under this isomorphism. It follows that the dimension of $\nu_K[k_1 \setminus \{0\}]$ over $\nu_K[k_0 \setminus \{0\}]$ is the same as the dimension of $V(k_1)$ over $\nu_{k_1}[k_0 \setminus \{0\}]$. Hence $\mathrm{valdim}_{k_0}(K) = \mathrm{valdim}_{k_0}(k_1) + \mathrm{valdim}_{k_1}(K)$. $\square$

Now that we have defined these two different notions of dimensions, we can explain how these relate to one another and how we intend to use them. We will show that if $k, K \models \mathcal{T}_{\mathcal{O}}$, with $\dim_k(K)$ finite and $\mathcal{T}_{\mathcal{O}}$ is *smooth* (see Definition 7.1.20), then $\mathrm{valdim}_k(K) \leq \dim_k(K)$. We will also prove that the theory $\mathcal{T}_e$ is smooth. The proof of condition (36) relies heavily on these two facts.

Definition 7.1.20.

(i) We say that the theory $\mathcal{T}_{\mathcal{O}}$ is satisfies condition S_1 if for any $K \models \mathcal{T}_{\mathcal{O}}$ and any K-definable function $f : K \to K$, there exists $N \in \mathbb{N}$ such that $|f(x)| \leq x^N$ for all sufficiently large $x \in K$.

(ii) The theory $\mathcal{T}_{\mathcal{O}}$ satisfies condition S_2 if for any $\mathcal{L}_{\mathcal{O}}$-formula $\phi(x_1, \dots, x_n)$ there are $m, p \in \mathbb{N}$ and C^∞-functions $F_i : \mathbb{R}^{n+m} \to \mathbb{R}$, for $i = 1, \dots, p$, which are definable without parameters and are such that

$$\mathbb{R} \models \forall \vec{x} \left(\phi(\vec{x}) \leftrightarrow \exists \vec{y} \left(\|\vec{y}\| \leq 1 \wedge \bigvee_{i=1}^{p} (N_i(\vec{y}) \wedge F_i(\vec{x}, \vec{y}) = 0) \right) \right),$$

where, if $\vec{y} = y_1, \dots, y_m$, $\|\vec{y}\| = \max\{|y_i| \mid i = 1, \dots, m\}$ and $N_i(\vec{y})$ is a formula of the form $\bigwedge_{j \in s_i} y_j \neq 0$ for some $s_i \subseteq \{1, \dots, m\}$.

(iii) If the theory $\mathcal{T}_{\mathcal{O}}$ satisfies both S_1 and S_2, then $\mathcal{T}_{\mathcal{O}}$ is said to be smooth.

Theorem 7.1.21. *Suppose $\mathcal{T}_{\mathcal{O}}$ satisfies S_1. Let $K \models \mathcal{T}_{\mathcal{O}}$ and suppose that R is a convex subring of K. Let I be the ideal of R consisting of those elements of R which are not invertible in R. (I is the unique maximal ideal of R.) Then there exists $k_0 \preceq K$ such that $k_0 \subseteq R$ and such that for each $a \in R$, $k_0 \cap (a + I)$ contains exactly one element. We say that k_0 splits R.*

Proof. Let $\mathcal{S} = \{k \preceq K \mid k \subseteq R\}$. Then $\mathcal{S}$ is a poset, ordered by $\preceq$. We wish to apply Zorn's Lemma to $\mathcal{S}$. To prove that $\mathcal{S} \neq \emptyset$, we show that $\mathcal{S}$ contains $\mathrm{Dcl}(\{0\})$. As we have seen in Remark 7.1.4, $\mathrm{Dcl}(\{0\}) \preceq K$. Now take some positive $x \in \mathrm{Dcl}(\{0\})$. By the Archimedean property (see Remark 7.1.5), $x < n$, for some $n \in \mathbb{N}$. Since R is a subring of K, it contains $\mathbb{Z}$. By convexity of R, x must be an element of R, as needed. Now let $\mathcal{C} = \{k_j \mid j \in J\}$ be a nonempty chain in $\mathcal{S}$. By Tarski's Elementary Chain Theorem, we have $k_j \preceq \bigcup_{j \in J} k_j$ for all $j \in J$. (A proof of this Theorem can be found in [Ges] for example.) It is clear that this is an upper bound for $\mathcal{C}$, so the requirements of Zorn's Lemma are met. We let k_0 be a maximal element of $\mathcal{S}$. Then $k_0 \preceq K$ and $k_0 \subseteq R$. Moreover, for each $a \in R$, the set $k_0 \cap (a + I)$ contains at most one element, for if $b, c \in k_0 \cap (a + I)$ are unequal, then $b - c \in I$ so that $(b - c)^{-1} \notin R$, contradicting $k_0 \subseteq R$.

We claim that for all $a \in R$, there exists $\alpha \in k_0$ such that $\alpha > a$. Suppose to the contrary that this claim is false for some a. Consider $\mathrm{Dcl}(k_0 \cup \{a\})$. We have $k_0 \preceq \mathrm{Dcl}(k_0 \cup \{a\}) \preceq K$. Since $a \in \mathrm{Dcl}(k_0 \cup \{a\})$, but $a \notin k_0$, there must be some element of $\mathrm{Dcl}(k_0 \cup \{a\})$ which is not in R, by maximality of k_0. We can write this element as $f(a)$, where f is a k_0-definable function. Since $k_0 \models \mathcal{T}_{\mathcal{O}}$, there exists $b \in k_0$ and $N \in \mathbb{N}$ such that $k_0 \models \forall x > b\ (|f(x)| \leq x^N)$, as $\mathcal{T}_{\mathcal{O}}$ satisfies S_1. Since $k_0 \preceq K$ and $a > b$, we have $K \models |f(a)| \leq a^N$. But this contradicts the fact that R is a convex subring of K.

Now suppose that $a \in R$ and that $k_0 \cap (a + I) = \emptyset$. Then certainly $a \notin k_0$, so once more $\mathrm{Dcl}(k_0 \cup \{a\})$ contains an element which is not in R. So if we can to show that $f(a) \in R$, for any k_0-definable function $K \to K$, then we will have found a contradiction, as every element of $\mathrm{Dcl}(k_0 \cup \{a\})$ is of this form. So let f be such a function. By O-minimality of k_0, there exist elements $a_1 < \cdots < a_n$, such that if we set $a_0 = -\infty$ and $a_{n+1} = \infty$, then f is monotone, in k_0, on the interval (a_i, a_{i+1}) for each $i = 0, \dots, n$. By our claim, a must lie in such an interval in K, say (b, c), with $b, c \in k_0$. Since k_0 is an elementary substructure of K, f must also be monotone in K on the interval (b, c). Since $k_0 \cap (a + I) = \emptyset$, we must have $c - a, a - b > \beta$ for all $\beta \in I$, which implies that $(c - a)^{-1}, (a - b)^{-1} \in R$. Using our claim a second time gives us an element $d \in k_0$ such that $d > (c - a)^{-1}, (a - b)^{-1}$. Since $d^{-1} \in k_0$ and $b < b + d^{-1} < a < c - d^{-1} < c$, it follows that either $f(b + d^{-1}) \leq f(a) \leq f(c - d^{-1})$ or $f(b + d^{-1}) \geq f(a) \geq f(c - d^{-1})$, by monotonicity of f. But this means that $f(a) \in R$, as R is convex. $\square$

Theorem 7.1.22. *Suppose that $\mathcal{T}_{\mathcal{O}}$ is smooth and $K \models \mathcal{T}_{\mathcal{O}}$. If* $\dim(K)$ *is finite, then* $\operatorname{valdim}(K) \leq \dim(K)$.

Proof. We use induction over $\dim(K)$. If K is Archimedean, which is equivalent to $\mu(K) = \{0\}$ and to $K = \operatorname{Fin}(K)$, then clearly $\operatorname{valdim}(K) = 0$, so we are done in this case. By this same observation, we are also done if $\dim(K) = 0$, for then $K = \operatorname{Dcl}(\emptyset)$, which is Archimedean, as we have seen. So suppose that $\dim(K) = n > 0$ and $\mu(K) \neq \{0\}$.

Claim 1. There exists $a \in K$ with $a > 0$ such that for all $b \in K$ with $b > 0$ we have $a^m < b$ for some $m \in \mathbb{N}$.

Proof. Since $\dim(K) = n$, we may write $K = \operatorname{Dcl}(\{c_1, \ldots, c_n\})$, where $c_1, \cdots, c_n \in K$ forms a basis for K. Let $K_i = \operatorname{Dcl}(\{c_1, \ldots, c_i\})$ for $i = 0, \ldots, n$. We use induction over i, up to and including n, to show that our claim holds for each K_i. It is an easy consequence of the fact that $K_0 = \operatorname{Dcl}(\emptyset)$ is Archimedean that there exists an element $a_0 \in K_0$ with $a_0 > 0$ such that for all $b \in K_0$ with $b > 0$ we have $a_0^m < b$ for some $m \in \mathbb{N}$. (Just take $a_0 = \frac{1}{2}$ for example.) Now suppose that the claim holds for some $i = 0, \ldots, n-1$, that is, there exists $a_i \in K_i$ with $a_i > 0$ such that for all $b \in K_i$ with $b > 0$ we have $a_i^m < b$ for some $m \in \mathbb{N}$. Then if for all $b \in K_{i+1}$ with $b > 0$ we have $a_i^m < b$ for some $m \in \mathbb{N}$, then we are done, as we can take $a_{i+1} = a_i$. If this is not the case, then there exists some positive $\beta \in K_{i+1}$ such that $\beta < a_i^m$ for all $m \in \mathbb{N}$. Clearly β is not an element of K_i, so $\{c_1, \ldots, c_i, \beta^{-1}\}$ is an independent subset of K_{i+1} and

$$K_{i+1} = \operatorname{Dcl}(\{c_1, \ldots, c_{i+1}\}) = \operatorname{Dcl}(\{c_1, \ldots, c_i, \beta^{-1}\}) = \operatorname{Dcl}_{K_i}(\{\beta^{-1}\}).$$

This means that every element of K_{i+1} is equal to $f(\beta^{-1})$ for some K_i-definable function f. Since $K_i \models \mathcal{T}_{\mathcal{O}}$, there exists $c \in K_i$ and $m \in \mathbb{N}$ such that $K_i \models \forall x > c\ (|f(x)| \leq x^m)$, by property S_1. Since $K_i \preceq K_{i+1}$ and certainly $\beta^{-1} > c$, we have $K_{i+1} \models |f(\beta^{-1})| \leq \beta^{-m}$. This shows that $a_{i+1} = \beta$ behaves as needed, which concludes the induction.

Take $a \in K$ as in Claim 1. We define $R = \{b \in K \mid |b| < a^{-\frac{1}{m}}$ *for all* $m \in \mathbb{N}\}$. Then R is a convex subring of K and its unique maximal ideal is Archimedean in the sense that for all $x, y \in I \setminus \{0\}$, there exists $m \in \mathbb{N}$ such that $|x|^m < |y|$. By Theorem 7.1.21, there is $k \preceq K$ such that k splits R. Note that $k \neq K$, as $a^{-1} \notin k$, so $\dim(k) < n$. Say $\dim(k) = n - r$, with $r \in \mathbb{N} \setminus \{0\}$. Take $c_1, \ldots, c_r \in K$ such that $\{c_1, \ldots, c_r\}$ forms a basis for K over k. We may suppose that $c_1, \ldots, c_r \in I$, for if $c_i \notin R$, then we can replace c_i by $c_i^{-1} \in I$ and if $c_i \in R$, then we can replace c_i by the unique element $\eta \in I$ such that $c_i + \eta \in k$, using the fact that k splits R. We take k^* to be the algebraic closure of the field $k(c_1, \ldots, c_r)$ in K. It is easy to check that $\nu_K[k^* \setminus \{0\}]$ and $\nu_K[k \setminus \{0\}]$ form linear subspaces of $V(K)$.

Claim 2. We have $\dim_{\mathbb{Q}}(\nu_K[k^* \setminus \{0\}]) \leq \dim_{\mathbb{Q}}(\nu_K[k \setminus \{0\}]) + r$, where $\dim_{\mathbb{Q}}$ means the dimension as a $\mathbb{Q}$-vector space.

Proof. Suppose to the contrary that $\dim_{\mathbb{Q}}(\nu_K[k^* \setminus \{0\}]) > \dim_{\mathbb{Q}}(\nu_K[k \setminus \{0\}]) + r$. Since $\nu_K[k \setminus \{0\}] \subseteq \nu_K[k^* \setminus \{0\}]$ as a $\mathbb{Q}$-vector subspace, this means that we can find elements $a_1, \ldots, a_{r+1} \in k^* \setminus \{0\}$, such that the vectors $\nu_K(a_1), \ldots, \nu_K(a_{r+1}) \in \nu_K[k^* \setminus \{0\}]$ are $\mathbb{Q}$-linearly independent over $\nu_K[k \setminus \{0\}]$. We claim that elements $a_1, \ldots, a_{r+1}$ are algebraically independent over k.

For suppose that they are algebraically dependent over k. Then $p(a_1, \ldots, a_{r+1}) = 0$, where p is some nontrivial polynomial with coefficients in k. We write

$$p(a_1, \ldots, a_{r+1}) = \sum_{\eta \in S} b_\eta a^\eta = 0,$$

with each $b_\eta \in k$ nonzero and where η is a multi-index ranging over some finite subset $S \subseteq \mathbb{N}^{r+1}$. We wish to show that $\nu_K(b_\eta a^\eta) = \nu_K(b_{\eta'} a^{\eta'})$, for two distinct $\eta, \eta' \in S$. Take $\tau \in S$ such that $\nu_K(b_\tau a^\tau)$ is minimal. Suppose to the contrary that $\nu_K(b_\tau a^\tau) \neq \nu_K(b_\eta a^\eta)$ for all other $\eta \in S$. Let $S' \subseteq S$ be a subset containing τ, such that

$$\nu_K \left(\sum_{\eta \in S'} b_\eta a^\eta \right) = \nu_K(b_\tau a^\tau)$$

and let $\eta' \in S \setminus S'$. Then

$$\nu_K \left(b_{\eta'} a^{\eta'} + \sum_{\eta \in S'} b_\eta a^\eta \right) = \min \left(\nu_K(b_{\eta'} a^{\eta'}), \nu_K \left(\sum_{\eta \in S'} b_\eta a^\eta \right) \right)$$
$$= \min(\nu_K(b_{\eta'} a^{\eta'}), \nu_K(b_\tau a^\tau)) = \nu_K(b_\tau a^\tau).$$

by **(ii)** of Remark 7.1.17. Starting at $S' = \{\tau\}$, we can keep adding terms inductively until $S' = S$, to arrive at

$$\nu_K \left(\sum_{\eta \in S} b_\eta a^\eta \right) = \nu_K(b_\tau a^\tau).$$

But this is false, as

$$\nu_K \left(\sum_{\eta \in S} b_\eta a^\eta \right) = \nu_K(0) = \infty.$$

We conclude that there do exist distinct $\eta, \eta' \in S$, such that $\nu_K(b_\eta a^\eta) = \nu_K(b_{\eta'} a^{\eta'})$. Explicitly writing out components and rearranging gives

$$\nu_K(b_\eta b_{\eta'}^{-1}) + \sum_{i=1}^{r+1} (\eta_i - \eta_i') \nu_K(a_i) = 0.$$

But this shows that the vectors $\nu_K(a_1), \ldots, \nu_K(a_{r+1})$ are $\mathbb{Q}$-linearly dependent over $\nu_K[k \setminus \{0\}]$, which is false. We conclude that the elements $a_1, \ldots, a_{r+1}$ are algebraically independent over k.

Recall that the map $\nu_K[k \setminus \{0\}] \to V(k)$ given by $x/(\mathrm{Fin}(K) \setminus \mu(K)) \mapsto x/(\mathrm{Fin}(k) \setminus \mu(k))$ is an isomorphism of $\mathbb{Q}$-vector spaces. Combined with our second claim, this gives $\dim_{\mathbb{Q}}(\nu_K[k^* \setminus \{0\}]) \leq \mathrm{valdim}(k) + r$, from which it follows that $\dim_{\mathbb{Q}}(\nu_K[k^* \setminus \{0\}]) \leq \dim(k) + r = n$, by our induction hypothesis. This means that it would suffice to show that the map $\nu_K : k^* \to V(K)$ is surjective, as this implies $\dim_{\mathbb{Q}}(\nu_K[k^* \setminus \{0\}]) = \mathrm{valdim}(K)$. So let $d \in K \setminus \{0\}$. We must find some $\alpha \in k^*$ such that $\nu_K(\alpha) = \nu_K(d)$. Note that $\nu_K(-x) = \nu_K(x)$ and $\nu_K(x^{-1}) = -\nu_K(x)$ for all $x \in K \setminus \{0\}$ and also note that $\nu_K(x) \in \nu_K[k \setminus \{0\}]$ for all $x \in R \setminus I$, as a consequence of the fact that k splits

R. We may therefore assume that $d \in I$ and $d > 0$. Let $f : K^r \to K$ be a k-definable function such that $f(c_1, \dots, c_r) = d$. Let the graph of f be defined by the formula $\phi(\vec{\gamma}, x_1, \dots, x_r, x)$, where $\vec{\gamma}$ are parameters from k and $\phi(\vec{z}, x_1, \dots, x_r, x)$ is a formula in the language $\mathcal{L}_{\mathcal{O}}$. We can now apply property S_2, by transferring it to K, to find

$$K \models \phi(\vec{\gamma}, c_1, \dots, c_r, d) \leftrightarrow \exists \vec{y} \left(\|\vec{y}\| \leq 1 \wedge \bigvee_{i=1}^{p} (N_i(\vec{y}) \wedge F_i(\vec{\gamma}, c_1, \dots, c_r, d, \vec{y}) = 0) \right),$$

with N_i and F_i as in **(ii)** of Definition 7.1.20 and $\vec{y} = y_1, \dots, y_m$ for some $m \in \mathbb{N}$. Now take F to be one of the F_i such that the conjunction holds and take $s = s_i$ (the same index as F_i), with s_i also as in **(ii)** of Definition 7.1.20. This means that for all $x \in K$, $f(c_1, \dots, c_r) = x$ if and only if there exist $b_1, \dots, b_m \in K$, with $b_i \neq 0$ for $i \in s$ and $|b_i| \leq 1$ for $i = 1, \dots, m$, such that $F(\vec{\gamma}, c_1, \dots, c_r, x, b_1, \dots, b_m) = 0$. From now on, we suppress the parameters $\vec{\gamma}$ and write just $F(x_1, \dots, x_r, x, y_1, \dots, y_m)$.

Now take $\beta_1, \dots, \beta_m \in K$ such that $\beta_i \neq 0$ for $i \in s$, $|\beta_i| \leq 1$ for $i = 1, \dots, m$ and $F(c_1, \dots, c_r, d, \beta_1 \dots, \beta_m) = 0$. Since $\beta_1, \dots, \beta_m \in R$, there exist $\beta_1^0, \dots, \beta_m^0 \in k$ such that $\beta_i - \beta_i^0 \in I$ for each $i = 1, \dots, m$, as k splits R. Since $c_1 \in I$ is nonzero, as it is part of a basis for K over k, we can take $N \in \mathbb{N}$ large enough that $|\beta_i| > |c_1|^N$, using the Archimedean property of I.

Define the set

$$A = \{(x_1, \dots, x_m) \in K^m \mid \; |c_1|^N \leq |x_i| \text{ for } i \in s \text{ and } |x_i| \leq 1 \text{ for } i = 1, \dots, m\}$$

and consider the function $h : K^{1+m} \to K$, which we define by

$$h(x, x_1, \dots, x_m) = |F(c_1, \dots, c_r, x, x_1, \dots, x_m)|.$$

Since F is a C^∞-function, h is certainly continuous. By transfer of the Extreme Value Theorem to K, h must attain a minimum on the set $([0,1] \setminus (\frac{d}{2}, \frac{2d}{3})) \times A$, as this set is closed, bounded and definable. Let γ be this minimum and note that $\gamma > 0$, as $\gamma = 0$ would imply that $f(c_1, \dots, c_r) = d'$, for some $d' \neq d$, by choice of F. So again, by the Archimedean property of I, we may take $N' \in \mathbb{N}$ large enough that $\gamma > |c_1|^{N'}$. Since γ is the minimum of h on the set $([0,1] \setminus (\frac{d}{2}, \frac{2d}{3})) \times A$, and $\gamma > |c_1|^{N'}$, it follows that if we where to find a point $(\alpha, \beta_1', \dots, \beta_m') \in [0,1] \times A$ such that $|F(c_1, \dots, c_r, \alpha, \beta_1', \dots, \beta_m')| \leq |c_1|^{N'}$, then $\frac{d}{2} < \alpha < \frac{2d}{3}$. But then $\nu_K(\alpha) = \nu_K(d)$, so to finish our proof, it would certainly be sufficient to find such points $\alpha, \beta_1', \dots, \beta_m'$ in k^*.

Let $\lambda \in \mathbb{N}$ and consider the Taylor expansion of degree λ of the function $F : K^{r+1+m} \to K$, at the point $\vec{\omega} = (0, \dots, 0, \beta_1^0, \dots, \beta_m^0) \in k^{r+1+m}$. The justification, of course, is that we can transfer Taylors Theorem from $\mathbb{R}$ to K. We write $\rho_\lambda(y_1, \dots, y_r, x, x_1, \dots, x_m)$ for this expansion, which is a polynomial with coefficients in k, as F is k-definable and $\vec{\omega} \in k^{r+1+m}$. Recall that for $\vec{z} \in B_t(\vec{\omega})$ we have

$$F(\vec{z}) = \rho_\lambda(\vec{z}) + R_\lambda,$$

where

$$R_\lambda = \left[\frac{1}{(\lambda+1)!} \left(\sum_{j=1}^{r+1+m} z_j \frac{\partial}{\partial x_j} \right)^{\lambda+1} F \right] (\vec{v}),$$

for some $\vec{v} \in B_t(\vec{\omega})$. Since all he derivatives of F are continuous, they are bounded on the set $B_1(\vec{\omega})$ (as they are certainly bounded on its closure). We can calculate these bounds in k, and these are certain to also hold in K, as $k \preceq K$. Hence, there exists a positive element B_λ of k such that

$$\text{for all } t \in K, \text{ with } 1 > t > 0 \text{ and all} \\ \vec{z} \in K^{r+1+m} \text{ with } \|\vec{z} - \vec{\omega}\| < t \text{ holds } |F(\vec{z}) - \rho_\lambda(\vec{z})| < B_\lambda \cdot t^{\lambda+1}. \tag{40}$$

Let

$$t_0 = 2(r+1+m) \cdot \max\{|c_1|, \ldots, |c_r|, d, |\beta_1 - \beta_1^0|, \ldots, |\beta_m - \beta_m^0|\}.$$

Then $t_0 \in I$ and $t_0 > 0$, so by the Archimedean property of I, we may take $\lambda_0 \in \mathbb{N}$ large enough that

$$t_0^{\lambda_0+1} < (2B_{\lambda_0})^{-1} \cdot |c_1|^{N'}. \tag{41}$$

We set $\lambda = \lambda_0$, $t = t_0$ and $\vec{z} = (c_1, \ldots, c_r, d, \beta_1, \ldots, \beta_m)$ in (40), which gives us

$$|\rho_{\lambda_0}(c_1, \ldots, c_r, d, \beta_1, \ldots, \beta_m)| < \frac{1}{2} \cdot |c_1|^{N'}, \tag{42}$$

using (41) and the fact that $F(c_1, \ldots, c_r, d, \beta_1, \ldots, \beta_m) = 0$. Because of the way A is defined, we also clearly have

$$(d, \beta_1, \ldots, \beta_m) \in [0,1] \times A. \tag{43}$$

Furthermore,

$$\|(c_1, \ldots, c_r, d, \beta_1, \ldots, \beta_m) - \vec{\omega}\| < ((2B_{\lambda_0})^{-1} \cdot |c_1|^{N'})^{(\lambda_0+1)^{-1}}, \tag{44}$$

by (41) and choice of t_0. Now, we can express the conjunction of (42), (43) and (44) as $\psi(d, \beta_1, \ldots, \beta_m)$, where $\psi(x, x_1, \ldots, x_m)$ is an $\mathcal{L}$-formula with parameters in k^*. Since both K and k^* are real closed fields, k^* is an elementary substructure of K, when regarded as $\mathcal{L}$-structures, since the theory of real closed fields admits quantifier elimination. This means that there must be elements $\alpha, \beta_1', \ldots, \beta_m' \in k^*$ such that $\psi(\alpha, \beta_1', \ldots, \beta_m')$ holds, or in other words

$$|\rho_{\lambda_0}(c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m')| < \frac{1}{2} \cdot |c_1|^{N'}, \tag{45}$$

$$(\alpha, \beta_1', \ldots, \beta_m') \in [0,1] \times A \tag{46}$$

and

$$\|(c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m') - \vec{\omega}\| < ((2B_{\lambda_0})^{-1} \cdot |c_1|^{N'})^{(\lambda_0+1)^{-1}}. \tag{47}$$

By (47), we are allowed to apply (40), with $\lambda = \lambda_0$, $t = ((2B_{\lambda_0})^{-1} \cdot |c_1|^{N'})^{(\lambda_0+1)^{-1}}$ and $\vec{z} = (c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m')$, which gives us

$$|F(c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m') - \rho_{\lambda_0}(c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m')| < \frac{1}{2} \cdot |c_1|^{N'}.$$

We combine this with (45), using the triangle inequality to arrive at

$$|F(c_1, \ldots, c_r, \alpha, \beta_1', \ldots, \beta_m')| < |c_1|^{N'}.$$

But this is exactly what we needed to achieve. □

Theorem 7.1.23. *Suppose that $\mathcal{T}_{\mathcal{O}}$ is smooth, $k, K \models \mathcal{T}_{\mathcal{O}}$, with $k \subseteq K$ and that* $\dim_k(K)$ *is finite. Then* $\text{valdim}_k(K) \leq \dim_k(K)$.

Proof. Since $\dim_k(K)$ is finite, there exist $k = k_0 \preceq k_1 \preceq \cdots \preceq k_n = K$, such that $\dim_{k_i}(k_{i+1}) = 1$ for each $i = 0, \ldots, n-1$. So since

$$\dim_k(K) = \sum_{i=0}^{n-1} \dim_{k_i}(k_{i+1}) \quad \text{and} \quad \text{valdim}_k(K) = \sum_{i=0}^{n-1} \text{valdim}_{k_i}(k_{i+1}),$$

by Lemma 7.1.13 and Lemma 7.1.19, it is enough to prove the inequality asserted in the Theorem just for the case $\dim_k(K) = 1$. So from now on we assume that we are in this situation. Since $\dim_k(K) = 1$, K is generated (over k) by a single element, say $a \in K$. Now suppose to the contrary that $\text{valdim}_k(K) \geq 2$. Then there exist k-definable functions $f, g : K \to K$ such that $\nu_K(f(a))$ and $\nu_K(g(a))$ are $\mathbb{Q}$-linearly independent over $\nu_K[k \setminus \{0\}]$.

Consider K as an $\mathcal{L}_{\mathcal{O}} \cup \{P\}$-structure, where P is a unary relation symbol, which we interpret as the domain of k. Now let K^* be an $\aleph_0$-saturated elementary extension of K, as an $\mathcal{L}_{\mathcal{O}} \cup \{P\}$-structure. Then K^* has an elementary $\mathcal{L}_{\mathcal{O}}$-substructure, k', consisting of those elements of K^* satisfying P. It follows directly from the fact that K^* is $\aleph_0$-saturated as an $\mathcal{L}_{\mathcal{O}} \cup \{P\}$-structure, that k' is $\aleph_0$-saturated as an $\mathcal{L}_{\mathcal{O}}$-structure. Now let $K' = \text{Dcl}_{k'}(a)$.

Claim. $\nu_{K'}(f(a))$ and $\nu_{K'}(g(a))$ are $\mathbb{Q}$-linearly independent over $\nu_{K'}[k' \setminus \{0\}]$.

Proof. Suppose that this is not the case. Then there exist $b \in k' \setminus \{0\}$ and $p, q \in \mathbb{Q}$, not both zero, such that $p\nu_{K'}(f(a)) + q\nu_{K'}(g(a)) + \nu_{K'}(b) = 0$. In other words,

$$n^{-1} < |f(a)|^p \cdot |g(a)|^q \cdot |b| < n,$$

for some $n \in \mathbb{N}$. Since in particular $b \in K^*$ and $K \preceq K^*$ as $\mathcal{L}_{\mathcal{O}} \cup \{P\}$-structures, there must exist some $b_0 \in k$, such that

$$n^{-1} < |f(a)|^p \cdot |g(a)|^q \cdot |b_0| < n.$$

But this contradicts the fact that $\nu_K(f(a))$ and $\nu_K(g(a))$ are linearly independent over $\nu_K[k \setminus \{0\}]$.

Note that $a \notin k'$, as $P(a)$ is false in K, so $\dim_{k'}(K') = 1$. Furthermore, by our claim, $\text{valdim}_{k'}(K') \geq 2$, which means that we are back where we started, but now with k' as an $\aleph_0$-saturated structure. We may therefore continue our proof with the strengthened hypothesis that k is $\aleph_0$-saturated.

Let k_0 be some elementary substructure of k, with $\dim(k_0)$ finite and such that f and g are k_0-definable. (We could take $\text{Dcl}(A)$, where A is the set of parameters occurring in f and g for example.) Consider the partial type

$$\Theta(x) = \{|f(x)|^p \cdot |g(x)|^q \cdot |b| \leq n^{-1} \ \vee \ |f(x)|^p \cdot |g(x)|^q \cdot |b| \geq n \mid n \in \mathbb{N} \setminus \{0\},\ b \in k_0,\ p, q \in \mathbb{Q} \text{ not both zero}\}.$$

Clearly a realizes $\Theta(x)$ in K, which means that $\Theta(x)$ is finitely satisfiable in k. We may write $\Theta(x)$ in such a way that the only parameters occurring in it are from the basis of k_0, which is finite. So, since k is $\aleph_0$-saturated, $\Theta(x)$ is realized in k by some element, a_1, say. Now take $k_1 = \text{Dcl}_{k_0}(a_1)$. Note that a_1 cannot possibly be an element of k_0, for then we could take $b = f(a_1)$, so that $\frac{1}{2} < |f(a_1)|^1 \cdot |g(a_1)|^0 \cdot |b| < 2$, contradicting the fact that a_1 realizes $\Theta(x)$. This shows that $\dim(k_1) = \dim(k_0) + 1$. Furthermore, $\nu_{k_1}(f(a_1))$ and $\nu_{k_1}(g(a_1))$ are $\mathbb{Q}$-linearly independent over $\nu_{k_1}[k_0 \setminus \{0\}]$, by definition of $\Theta(x)$, which shows that $\text{valdim}(k_1) \geq \text{valdim}(k_0) + 2$. But now we can repeat this argument, with k_1 in place of k_0 to find $k_1 \preceq k_2 \preceq k$ such that $\dim(k_2) = \dim(k_0) + 2$ and $\text{valdim}(k_2) \geq \text{valdim}(k_0) + 4$. In fact, we can continue this process to find, for every $l \in \mathbb{N}$, an elementary substructure of k_l of k such that $\dim(k_l) = \dim(k_0) + l$ and $\text{valdim}(k_l) \geq \text{valdim}(k_0) + 2l$. Setting $l = \dim(k_0) + 1$ gives us the inequality

$$\text{valdim}(k_l) \geq \text{valdim}(k_0) + \dim(k_l) + 1,$$

contradicting Theorem 7.1.22. □

7.2 Proof of condition 36

In this section we show that we where allowed to use condition(36) in our proof of the Second Main Theorem. First we prove that the results from the previous section are applicable to the theory $\mathcal{T}_e$.

Theorem 7.2.1. *The theory $\mathcal{T}_e$ is smooth.*

Proof. The theory $\mathcal{T}_{\exp\restriction}$ is O-minimal by Corollary 4.1.7. Now, by Lemma 6.2.3, the models of $\mathcal{T}_e$ and $\mathcal{T}_{\exp\restriction}$ have the same definable sets. Hence $\mathcal{T}_e$ is O-minimal as an immediate consequence.

To show that $\mathcal{T}_e$ satisfies S_1, let $K \models \mathcal{T}_e$ and let $f : K \to K$ be a definable function. By Lemma 6.2.3, the function $f : K \to K$ is also definable in $(K \mid \mathcal{L}_{\exp\restriction})$. Now if $\lim_{x\to\infty} f(x) = 0$, then S_1 is certainly satisfied. If not, then by Corollary 4.1.8, there is $s \in \mathbb{Q}$ and a nonzero $a \in K$ such that $\lim_{x\to\infty} f(x)x^s = a$. So clearly if we take $N \in \mathbb{N}$ larger than $-s$, then $|f(x)| \leq x^N$ for all sufficiently large $x \in K$, as needed.

We show that $\mathcal{T}_e$ satisfies S_2. Consider the function $e^* : \mathbb{R} \to \mathbb{R}$ defined by $e^*(x) = \exp(x^2 \cdot (1 + x^2)^{-1})$. Note that $e^*(x) = e(x^{-1})$ for all $x \in \mathbb{R} \setminus \{0\}$. Since $e^*(0) = 1$, it follows that e^* is definable in $(\mathbb{R} \mid \mathcal{L}_e)$ without parameters. Notice furthermore that both e and e^* are C^∞-functions. Now, let $\phi(x_1, \ldots, x_n)$ be any $\mathcal{L}_e$-formula. Since $\mathcal{T}_e$ is model complete by Corollary 6.2.4, $\phi(x_1, \ldots, x_n)$ is equivalent to some existential formula $\psi(x_1, \ldots, x_n)$. By Lemma 2.1.5, $\psi(x_1, \ldots, x_n)$ is equivalent to a formula of the form

$$\exists y_1, \ldots, y_m \bigwedge_{i=1}^{l} \tau_i = 0,$$

where each τ_i is a term of $\mathcal{L}$ or of the form $e(z_1) - z_2 = 0$, with $z_1, z_2 \in \{y_1, \ldots y_m, x_1, \ldots, x_n\}$. It is clear that this formula is in turn equivalent to the formula $\exists y_1, \ldots y_m(\tau_1 \cdots \tau_l = 0)$. This shows that there is a polynomial $\rho \in \mathbb{Z}[z_1, \ldots, z_{2m+2n}]$, such that

$$\begin{aligned} \mathbb{R} \models & \forall x_1, \ldots, x_n[\phi(x_1, \ldots, x_n) \\ & \leftrightarrow \exists y_1, \ldots, y_m\ \rho(y_1, \ldots, y_m, e(y_1), \ldots, e(y_m), x_1, \ldots, x_n, e(x_1), \ldots, e(x_n)) = 0]. \end{aligned} \tag{48}$$

For a subset $s \subseteq \{1, \ldots, m\}$, let $G_s(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be the result of replacing y_j by y_j^{-1} and $e(y_j)$ by $e^*(y_j)$ in

$$\rho(y_1, \ldots, y_m, e(y_1), \ldots, e(y_m), x_1, \ldots, x_n, e(x_1), \ldots, e(x_n)).$$

For a sufficiently large $r \in \mathbb{N}$, the function

$$\left(\prod_{j\in s} y_j\right)^r G_s(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

is a C^∞-function on $\mathbb{R}$, which we shall denote by F_s. We now call $p = 2^m$ and let $\{s_i \mid i = 1, \dots, p\}$ be an enumeration of the subsets of $\{1, \dots, m\}$. For $i = 1, \dots, p$, we write $N_i(\vec{y})$ to denote $\bigwedge_{j \in s_i} y_j \neq 0$. Lastly, we write F_i for F_{s_i}. We claim that

$$\mathbb{R} \models \forall \vec{x} \left(\phi(\vec{x}) \leftrightarrow \exists \vec{y} \left(\|\vec{y}\| \leq 1 \wedge \bigvee_{i=1}^{p} (N_i(\vec{y}) \wedge F_i(\vec{x}, \vec{y}) = 0) \right) \right),$$

where $\|\vec{y}\| = \max\{|y_i| \mid i = 1, \dots, m\}$. Once we manage to prove this claim, then we are done, as this is precisely the definition of S_2. To show that our claim is true, we use (48) and suppose that

$$\mathbb{R} \models \exists y_1, \dots, y_m \; \rho(y_1, \dots, y_m, e(y_1), \dots, e(y_m), a_1, \dots, a_n, e(a_1), \dots, e(a_n)) = 0,$$

for certain $a_1, \dots, a_n \in \mathbb{R}$. This means that

$$\mathbb{R} \models \rho(b_1, \dots, b_m, e(b_1), \dots, e(b_m), a_1, \dots, a_n, e(a_1), \dots, e(a_n)) = 0,$$

for some $b_1, \dots, b_m \in \mathbb{R}$. Let s_{i_0} be the subset of $\{1, \dots, m\}$ such that $j \in s_{i_0}$ exactly when $|b_j| > 1$. Now let $\beta_j = b_j^{-1}$ for $j \in s_{i_0}$ and $\beta_j = b_j$ for $j \in \{1, \dots, m\} \setminus s_{i_0}$. Then $\max\{|\beta_j| \mid j = 1, \dots, m\} \leq 1$ and $N_{i_0}(\vec{\beta})$ are satisfied. Moreover, $G_{s_{i_0}}(a_1, \dots, a_n, \beta_1, \dots, \beta_m) = 0$ by definition of $G_{s_{i_0}}$, so certainly $F_{i_0}(a_1, \dots, a_n, \beta_1, \dots, \beta_m) = 0$. It follows that

$$\mathbb{R} \models \exists \vec{y} \left(\|\vec{y}\| \leq 1 \wedge \bigvee_{i=1}^{p} (N_i(\vec{y}) \wedge F_i(\vec{a}, \vec{y}) = 0) \right).$$

That the converse implication also holds, should be clear from the definitions of the N_i and F_i, so we have proven our claim. □

Before we can apply Theorem 7.1.23 (to the theory $\mathcal{T}_e$), we require the following result on ordered vector spaces.

Lemma 7.2.2. *Let V be an ordered $\mathbb{Q}$-vector space and let U be a subspace of V with dimension $n \in \mathbb{N}$ over U. Then there exists a basis $0 < v_1 < \cdots < v_n$ for V over U, with the following property. If v is an element of V, which we write as*

$$v = u_0 + \sum_{i=1}^{n} q_i v_i,$$

with $u_0 \in U$ and $q_1, \dots, q_n \in \mathbb{Q}$, which has the property that $v > u$ for all $u \in U$, then $|v| > q v_j$ for some positive $q \in \mathbb{Q}$, where $j = \max\{i \mid q_i \neq 0\}$.

Proof. The first thing we will show is that the convex subspaces of V are linearly ordered by inclusion. To demonstrate this, let W_1, W_2 be distinct convex subspaces of V. Then without loss of generality we may suppose that $W_1 \setminus W_2 \neq \emptyset$. We can therefore take some $w_1 \in W_1 \setminus W_2$, which we may assume is positive. Now let $w_2 \in W_2$ be arbitrary. Then by convexity of W_2, the inequality $|w_2| < w_1$ must hold, as $w_1 \notin W_2$. But this means that $w_2 \in W_1$, by convexity of W_1, and hence $W_2 \subsetneq W_1$, as needed.

We can therefore create a chain

$$U = W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_l = V,$$

where each W_{i+1} is the smallest convex subspace of V, strictly containing W_i. Note that this chain must be finite, as the dimension of V over U is finite. For each $i = 1, \ldots, l-1$, we let $0 < w_1^i < \cdots < w_{m_i}^i$ be a basis for W_{i+1} over W_i. Then

$$0 < w_1^1 < \cdots < w_{m_1}^1 < \cdots < w_1^{l-1} < \cdots < w_{m_{l-1}}^{l-1}$$

is a basis for V over U and we shall write this as $0 < v_1 < \cdots < v_n$. Suppose that we are given $v \in V$, written as

$$v = u_0 + \sum_{i=1}^{n} q_i v_i,$$

with $u_0 \in U$ and $q_1, \ldots, q_n \in \mathbb{Q}$, which has the property that $v > u$ for all $u \in U$. Then $\{i \mid q_i \neq 0\} \neq \emptyset$, so we have some $j = \max\{i \mid q_i \neq 0\}$. By definition, $v_j = w_{j_0}^{i_0}$ for some i_0, j_0. Verify that we can therefore write $v = x + y$, with $x \in W_{i_0}$ and $y \in W_{i_0+1} \setminus W_{i_0}$, with y nonzero. Recall that our goal is to find some positive $q \in \mathbb{Q}$ such that $|v| > qv_j$. Suppose to the contrary that $|x + y| \leq qv_j$ for all positive $q \in \mathbb{Q}$. We note that the inequality $|x| \leq \frac{1}{2}|y|$ must hold, for otherwise $y \in W_{i_0}$, by convexity of W_{i_0}. But then

$$\frac{1}{2}|y| \leq |y| - |x| \leq |x + y| < qv_j$$

for all positive $q \in \mathbb{Q}$ and hence $|y| < qv_j$ for all positive $q \in \mathbb{Q}$. It follows that the convex closure of the subspace of V generated by y lies strictly between W_{i_0} and W_{i_0+1}. Since the existence of such a subspace is impossible by definition of W_{i_0} and W_{i_0+1}, this proves the Lemma. □

Suppose that k and K are models of $\mathcal{T}_{\exp}$, with $k \subseteq K$. Then these two structures also determine models of $\mathcal{T}_e$ (see Definition6.2.2). We shall denote these models of $\mathcal{T}_e$ by k' and K' respectively. (So K and K' have the same underlying ordered field, but $K \models \mathcal{T}_{\exp}$ and $K' \models \mathcal{T}_e$ and the same holds for k and k'.)

Since $\mathcal{T}_e$ is model complete by Corollary 6.2.4, every $\mathcal{L}_e$-formula φ is equivalent to an existential $\mathcal{L}_e$-formula ψ. Similarly, $\neg\varphi$ is equivalent to some existential $\mathcal{L}_e$-formula χ, so φ is equivalent to $\neg\chi$, which is universal. Since universal formulas are preserved downward and existential formulas are preserved upward, $k' \subseteq K'$ implies $k' \preceq K'$.

Now let k^* be a model of the theory $\mathcal{T}_e$, such that $k' \subseteq k^* \subseteq K'$. Then for each $a \in k^*$, $\exp(a)$ is an element of K, but it need not be an element of k^*, so it is worthwhile to define $E(k^*) = \{a \in k^* \mid \exp(a) \in k^*\}$. Because k^* is a model of $\mathcal{T}_e$, it is in particular a real closed field, so it is closed under taking rational powers of positive elements. Using this, it is not hard to verify that $E(k^*)$ is a $\mathbb{Q}$-vector subspace of k^*, as an additive group. In turn, $E(k^*)$ contains $\mathrm{Fin}(k^*)$ as a $\mathbb{Q}$-vector subspace. To see this, consider an element $a \in \mathrm{Fin}(k^*)$. Since $a \in \mathrm{Fin}(k^*)$, we can take an element $m \in \mathbb{Z}$ of the same sign as a and such that $|a| \leq |m|$. Then the equation $\frac{m}{1+b^2} = a$ holds for some $b \in k^*$, as k^* is a real closed field. But then $\exp(a) = e(b)^m$, which lies in k^*, as needed. For the sake of completeness, we also point out that k is a $\mathbb{Q}$-vector subspace of $E(k^*)$.

Lemma 7.2.3. *Let $k, K \models \mathcal{T}_{\exp}$ and $k^* \models \mathcal{T}_e$, such that $k' \subseteq k^* \subseteq K'$, as introduced above. Suppose that $\dim_{k'}(k^*) = n$, with $n \in \mathbb{N}$, as models of $\mathcal{T}_e$. Suppose also that $E(k^*)$ is at least n-dimensional over its $\mathbb{Q}$-vector subspace $k + \mathrm{Fin}(k^*) = \{x + y \mid x \in k,\ y \in \mathrm{Fin}(k^*)\}$. Then for each $a \in E(k^*)$, there exists $b \in k$ such that $|a| < b$.*

Proof. Suppose that the Lemma is false. We write U for the subspace $k + \mathrm{Fin}(k^*)$. Let α be an element of $E(k^*)$ such that such that $\alpha > b$ for all $b \in k$ and choose a subspace V of $E(k^*)$, with

$U \subseteq V$ and containing α, such that V is exactly n-dimensional over U. Let $0 < v_1 < \cdots < v_n$ be a basis for V over U as given in Lemma 7.2.2. Since $\alpha > b$, for every $b \in k$, we must surely also have that $\alpha > b$ for every $b \in U$. It follows that there is some v_j such that $v_j > b$ for every $b \in U$ and we take j minimal such that this is the case.

Claim. The elements $\nu_K(\exp(v_1)), \ldots, \nu_K(\exp(v_n))$ of the value group $V(K)$ are linearly independent over $\nu_K[k \setminus \{0\}]$.

Proof. Suppose not. Then there exist $q_1, \ldots, q_n \in \mathbb{Q}$, not all zero, and $c \in k$ such that

$$\nu_K(c) + \sum_{i=1}^{n} q_i \nu_K(\exp(v_i)) = 0.$$

We may certainly suppose that $c > 0$, so that $c = \exp(d)$ for some $d \in k$. The above equation is then equivalent to

$$\exp(d + \sum_{i=1}^{n} q_i v_i) \in \mathrm{Fin}(K) \setminus \mu(K),$$

using the basic properties om the maps ν_K and exp. Since $1 + x \leq \exp(x)$ (and hence $x - 1 \geq -\exp(-x)$) for all $x \in K$, one readily verifies that this implies $d + \sum_{i=1}^{n} q_i v_i \in \mathrm{Fin}(K)$ and consequently $d + \sum_{i=1}^{n} q_i v_i \in \mathrm{Fin}(k^*)$. But this contradicts the fact that $v_1, \ldots, v_n$ are linearly independent over U.

Now, by Theorems 7.1.23 and 7.2.1 and our assumption that $\dim_{k'}(k^*) = n$, we have $\mathrm{valdim}_{k'}(k^*) \leq n$, meaning that the dimension of $V(k^*)$ over its subspace $\nu_{k^*}(k') = \nu_{k^*}(k)$ is less than or equal to n. Recall that we have an isomorphism of $\mathbb{Q}$-vector spaces $\nu_K[k^* \setminus \{0\}] \to V(k^*)$, given by $x/(\mathrm{Fin}(K) \setminus \mu(K)) \mapsto x/(\mathrm{Fin}(k^*) \setminus \mu(k^*))$ and that the subspace $\nu_K[k \setminus \{0\}] \subseteq \nu_K[k^* \setminus \{0\}]$ corresponds to the subspace $\nu_{k^*}[k \setminus \{0\}] \subseteq V(k^*)$ under this isomorphism. This means that the dimension of $\nu_K[k^* \setminus \{0\}]$ over $\nu_K[k \setminus \{0\}]$ is less than or equal to n. But $\nu_K(\exp(v_1)), \ldots, \nu_K(\exp(v_n) \in \nu_K[k^* \setminus \{0\}]$, as $v_1, \ldots, v_n \in E(k^*)$, so by our Claim, they must span the space $\nu_K[k^* \setminus \{0\}]$ over $\nu_K[k \setminus \{0\}]$. In particular

$$\nu_K(v_j) = \nu_K(c) + \sum_{i=1}^{n} p_i \nu_K(\exp(v_i)),$$

for a certain $c \in k \setminus \{0\}$ and $p_1, \ldots, p_n \in \mathbb{Q}$. Again, we may write $c = \exp(d)$ for some $d \in k$ to get

$$\nu_K(v_j) = \nu_K(\exp(d + \sum_{i=1}^{n} p_i v_i)),$$

which is the same as saying that

$$\frac{v_j}{N} < \exp(d + \sum_{i=1}^{n} p_i v_i) < N v_j, \tag{49}$$

for some $N \in \mathbb{N} \setminus \{0\}$. Now since $1 < \frac{v_j}{N}$, the left inequality of (49) tells us that $0 < d + \sum_{i=1}^{n} p_i v_i$. Furthermore, we cannot have $p_j = p_{j+1} = \cdots = p_n = 0$, as this implies $0 < d + \sum_{i=1}^{n} p_i v_i < b$, for some $b \in k$, by choice of v_j. This leads to $\frac{v_j}{N} < \exp(b)$, which contradicts our choice of v_j, as $N \cdot \exp(b) \in k$. Thus $p_j \geq p_{j'}$, where $j' = \max\{i \mid p_i \neq 0\}$, from which it follows that there exists

$q \in \mathbb{Q}$, positive, such that $d + \sum_{i=1}^{n} p_i v_i > q p_j$, by choice of $v_1, \ldots, v_n$, using Lemma 7.2.2. By the right inequality of (49), we must therefore have $\exp(q v_j) < N v_j$. But by simply reasoning in $\mathbb{R}$, there exists $r \in \mathbb{N}$ such that $\exp(qx) \geq Nx$, for all $x > r$, because $\lim_{r \to \infty} \frac{Nr}{\exp(qr)} = 0$. We have derived a contradiction, since surely $v_j > r$ for all $r \in \mathbb{N}$. $\square$

We return to the context in which we formulated (36).

Lemma 7.2.4. *Let $n, m \in \mathbb{N}$, with $n \geq m > 0$ and let $\vec{\alpha} \in K^n$, $l \in \{1, \ldots, n\}$, $s \subseteq \{1, \ldots, n\}$, with $|s| = m$ and $l \in s$. Let also $f_1, \ldots, f_n \in M_n^s$ be such that $f_1(\vec{\alpha}) = \ldots = f_n(\vec{\alpha}) = 0$ and $\det\left(\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right)(\vec{\alpha}) \neq 0$. Lastly, suppose that $|\alpha_l| > b$ for all $b \in k$. Then the set $\{\alpha_i \mid i \in s\}$ is $\mathbb{Q}$-linearly dependent over $k + \mathrm{Fin}(K)$.*

Proof. Define the submodel $k^* \subseteq K'$ by

$$k^* = \mathrm{Dcl}_{k'}(\{\alpha_i \mid 1 \leq i \leq n\} \cup \{\exp(\alpha_i) \mid i \in s\}),$$

where the closure is taken with respect to the definable functions of $\mathcal{T}_e$. Then $k^* \models \mathcal{T}_e$ and $k' \subseteq k^* \subseteq K'$.

Claim. $\dim_{k'}(k^*) \leq m$.

Proof. Suppose for convenience that $s = \{1, \ldots, m\}$ and set $\alpha_{n+i} = \exp(\alpha_i)$ for $i = 1, \ldots, m$. We will show that $\{\alpha_i \mid 1 \leq i \leq n+m\}$ contains an m-element subset which generates k^* over k'. To this end, we take $g_i \in M_n^{\emptyset}[x_{n+1}, \ldots, x_{n+m}]$ such that $g_i(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_{n+m})) = f_i(x_1, \ldots, x_n)$ for each $i = 1, \ldots, n$, and we let $g_{n+i}(x_1, \ldots, x_{n+m}) = x_{n+i} - \exp(x_i)$ for each $i = 1, \ldots, m$. Clearly then, $g_1(\alpha_1, \ldots, \alpha_{n+m}) = \cdots = g_{n+m}(\alpha_1, \ldots, \alpha_{n+m}) = 0$. We shall now demonstrate that $\det\left(\frac{\partial(g_1,\ldots,g_{n+m})}{\partial(x_1,\ldots,x_{n+m})}\right)(\alpha_1, \ldots, \alpha_{n+m}) \neq 0$. We split up the matrix $\frac{\partial(g_1,\ldots,g_{n+m})}{\partial(x_1,\ldots,x_{n+m})}$ into four blocks

$$A = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{\partial g_1}{\partial x_{n+1}} & \cdots & \frac{\partial g_1}{\partial x_{n+m}} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_{n+1}} & \cdots & \frac{\partial g_n}{\partial x_{n+m}} \end{pmatrix}$$

$$C = \begin{pmatrix} \frac{\partial g_{n+1}}{\partial x_1} & \cdots & \frac{\partial g_{n+1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_{n+m}}{\partial x_1} & \cdots & \frac{\partial g_{n+m}}{\partial x_n} \end{pmatrix} \qquad D = \begin{pmatrix} \frac{\partial g_{n+1}}{\partial x_{n+1}} & \cdots & \frac{\partial g_{n+1}}{\partial x_{n+m}} \\ \vdots & & \vdots \\ \frac{\partial g_{n+m}}{\partial x_{n+1}} & \cdots & \frac{\partial g_{n+m}}{\partial x_{n+m}} \end{pmatrix}$$

and we note that D is simply I_m, the $m \times m$ identity matrix. Now obtain the matrix B' from B by adding $n-m$ columns of zeros on the right. Similarly, obtain C' from C by adding $n-m$ rows of zeros on the bottom. Lastly, we let $D' = I_n$. This gives us four $n \times n$ matrices A, B', C', D' and it is not difficult to verify that

$$\det\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \det\left[\begin{pmatrix} A & B' \\ C' & D' \end{pmatrix}\right]$$

Furthermore,

$$\det\left[\begin{pmatrix} A & B' \\ C' & D' \end{pmatrix}\right] = \det\left[\begin{pmatrix} A & B' \\ C' & D' \end{pmatrix}\right] \det\left[\begin{pmatrix} D' & 0 \\ -C' & I_n \end{pmatrix}\right]$$
$$= \det\left[\begin{pmatrix} A & B' \\ C' & D' \end{pmatrix} \cdot \begin{pmatrix} D' & 0 \\ -C' & I_n \end{pmatrix}\right] = \det\left[\begin{pmatrix} AD' - B'C' & B' \\ C'D' - D'C' & D' \end{pmatrix}\right]$$
$$= \det\left[\begin{pmatrix} AD' - B'C' & B' \\ 0 & D' \end{pmatrix}\right] = \det\left[AD' - B'C'\right] = \det\left[A - B'C'\right].$$

Now, for $i = 1, \dots, n$ and $j = 1, \dots, m$, we have

$$\frac{\partial f_i}{\partial x_j}(\alpha_1, \dots, \alpha_n) = \frac{\partial g_i}{\partial x_j}(\alpha_1, \dots, \alpha_{n+m}) - \exp(\alpha_j) \cdot \frac{\partial g_i}{\partial x_{n+j}}(\alpha_1, \dots, \alpha_{n+m}),$$

by the chain rule and for $i = 1, \dots, n$ and $j = m+1, \dots, n$ we have

$$\frac{\partial f_i}{\partial x_j}(\alpha_1, \dots, \alpha_n) = \frac{\partial g_i}{\partial x_j}(\alpha_1, \dots, \alpha_{n+m}).$$

Since C' is a diagonal matrix with entries $\exp(x_1), \dots, \exp(x_m), 0, \dots, 0$ on its diagonal, this shows that

$$\det [A - B'C'] (\alpha_1, \dots, \alpha_{n+m}) = \det \left(\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \right) (\alpha_1, \dots, \alpha_n) \neq 0,$$

as desired. It follows that the row vectors $(\frac{\partial(g_1)}{\partial(x_1)}, \dots, \frac{\partial(g_1)}{\partial(x_{n+m})}), \dots, (\frac{\partial(g_n)}{\partial(x_1)}, \dots, \frac{\partial(g_n)}{\partial(x_{n+m})})$ evaluated at $(\alpha_1, \dots, \alpha_{n+m})$ are linearly independent over K. Hence, there exists a subset $u \subseteq \{1, \dots, n+m\}$ of size n such that the matrix

$$\left(\frac{\partial(g_i)}{\partial(x_j)} \right)_{1 \leq i \leq n, j \in u}$$

evaluated at $(\alpha_1, \dots, \alpha_{n+m})$ is invertible. We relabel $(x_1, \dots, x_{n+m})$ in such a way that $u = \{1, \dots, n\}$ and we relabel $(\alpha_1, \dots, \alpha_{n+m})$ accordingly. Then

$$\det \left(\frac{\partial(g_1, \dots, g_n)}{\partial(x_1, \dots, x_n)} \right) (\alpha_1, \dots, \alpha_{n+m}) \neq 0$$

and clearly still $g_1(\alpha_1, \dots, \alpha_{n+m}) = \cdots = g_n(\alpha_1, \dots, \alpha_{n+m}) = 0$. Furthermore, $g_1, \dots, g_n \in M_{n+m}^{\emptyset}$. Now consider the functions $h_i(x_1, \dots, x_n) = g_i(x_1, \dots, x_n, \alpha_{n+1}, \dots, \alpha_{n+m})$ for $i = 1, \dots, n$. Then

$$\det \left(\frac{\partial(h_1, \dots, h_n)}{\partial(x_1, \dots, x_n)} \right) (\alpha_1, \dots, \alpha_n) \neq 0$$

and $h_1(\alpha_1, \dots, \alpha_n) = \cdots = h_n(\alpha_1, \dots, \alpha_n) = 0$. So, by Proposition 6.2.7 **(iii)** (using $k = K$ in the definition of $M_n^{\emptyset}$, to ensure that $h_1, \dots, h_n \in M_n^{\emptyset}$), there are only finitely many such points. Since the h_i are k'-definable over $\alpha_{n+1}, \dots, \alpha_{n+m}$, this implies that

$$\alpha_1, \dots, \alpha_n \in \mathrm{Dcl}_{k'}(\{\alpha_i \mid n+1 \leq i \leq n+m\})$$

and hence

$$k^* = \mathrm{Dcl}_{k'}(\{\alpha_i \mid 1 \leq i \leq n+m\}) = \mathrm{Dcl}_{k'}(\{\alpha_i \mid n+1 \leq i \leq n+m\}),$$

proving our claim.

By our claim and by the fact that $\alpha_l \in E(k^*)$ (since $l \in s$), Lemma 7.2.3 tells us that $E(k^*)$ can have at most dimension $m-1$ over $k + \mathrm{Fin}(K)$. But $\{\alpha_i \mid i \in s\} \subseteq E(k^*)$, so since $|s| = m$, the set $\{\alpha_i \mid i \in s\}$ must be $\mathbb{Q}$-linearly dependent over $k + \mathrm{Fin}(K)$. □

We are now ready to justify (36). Since $\{\alpha_i \mid i \in s\}$ is $\mathbb{Q}$-linearly dependent over $k + \mathrm{Fin}(K)$, there exist $a \in k$, $b \in \mathrm{Fin}(K)$ and $n_i \in \mathbb{Z}$, for $i \in s$, not all zero, such that $a + b + \sum_{i \in s} n_i \alpha_i = 0$. Since $b \in \mathrm{Fin}(K)$, there exists $q \in \mathbb{Q}$ such that $0 < q - b < 1$. We can then take $c = q + a \in k$ to get $0 < c + \sum_{i \in s} n_i \alpha_i < 1$, as needed. This finishes the proof of the Second Main Theorem.

8 An application of Wilkie's Theorem

8.1 Schanuel's Conjecture

Schanuel's Conjecture is a conjecture made by Stephen Schanuel in the 1960s about the transcendence degree of certain field extensions of $\mathbb{Q}$. The conjecture can be formulated as follows.

Conjecture 8.1.1. *Suppose that* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$*, such that*

$$\text{trdeg}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n, \exp(\alpha_1), \ldots, \exp(\alpha_n))) < n,$$

where $\text{trdeg}_{\mathbb{Q}}(\mathbb{Q}(\alpha_1, \ldots, \alpha_n))$ *stands for the transcendence degree of* $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ *over* $\mathbb{Q}$*. Then there are* $m_1, \ldots, m_n \in \mathbb{Z}$*, not all zero, such that* $\sum_{i=1}^n m_i\alpha_i = 0$*.*

The conjecture neatly summarizes many known results from transcendental number theory. The special case where $\alpha_1, \ldots, \alpha_n$ are all algebraic is the Lindemann-Weierstrass Theorem for example. But the truth of Conjecture 8.1.1 would also settle a large number currently unanswered questions. For instance, setting $\alpha_1 = 1$ and $\alpha_2 = \pi i$ would prove that π and e are algebraically independent. Unfortunately, a proof of Schanuel's Conjecture is generally considered to be out of reach at the present day.

In the upcoming part, we will prove a modest generalization of the result found in [KZ06]. This paper is centered around the real form of Schanuel's Conjecture, which is the following statement.

Conjecture 8.1.2. *Suppose that* $a_1, \ldots, a_n \in \mathbb{R}$*, such that*

$$\text{trdeg}_{\mathbb{Q}}(\mathbb{Q}(a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_n))) < n.$$

Then there are $m_1, \ldots, m_n \in \mathbb{Z}$*, not all zero, such that* $\sum_{i=1}^n m_i a_i = 0$*.*

In [KZ06], the authors manage to put a uniform bound on the coefficients $m_1, \ldots, m_n$ by proving that Conjecture 8.1.2 is equivalent to the statement below, which is suitably called the uniform real version of Schanuel's conjecture.

Conjecture 8.1.3. *Let* $V \subseteq \mathbb{R}^{2n}$ *be an algebraic variety, with* $\dim(V) < n$*. Then there exists* $N \in \mathbb{N}$*, such that if*

$$(a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_1)) \in V,$$

there are $m_1, \ldots, m_n \in \mathbb{Z}$*, not all zero, with* $|m_i| \leq N$ *for each* $i = 1, \ldots, n$*, such that* $\sum_{i=1}^n m_i a_i = 0$*.*

We shall formulate yet another form of Schanuel's conjecture, as well as an accompanying uniform version and we shall prove that these two are equivalent. The result of [KZ06] will easily follow as a special case of this equivalence.

8.2 Schanuel's Conjecture for matrices

Let $d \in \mathbb{N}$, with $d \geq 1$. We let $G \subseteq M_{d\times d}$ be a definable collection of real $d \times d$ matrices, with real entries and real eigenvalues. We will identify $M_{d\times d}$ with $\mathbb{R}^{d^2}$ and when we say definable, we will from now on always mean definable in $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$. For G we could for instance simply take the set of all those matrices in $M_{d\times d}$ with real eigenvalues. Other interesting examples include the (noncommutative) ring of all upper (or lower) triangular matrices in $M_{d\times d}$ and the ring of all diagonal matrices in $M_{d\times d}$.

Our goal is to formulate forms of (the uniform) Schanuel's Conjecture for G in such a way that they reduce to Conjectures 8.1.2 and 8.1.3 for $G = \mathbb{R}$. Since we are working with G, which might not be commutative, it is dangerous to assume that theorems and definitions from commutative algebra still hold in this situation. It is for example no longer obvious what we mean "algebraic variety" or "dimension". In order to make this clear, we will have to make a few definitions.

Definition 8.2.1. By $\mathbb{Q}\langle x_1, \ldots, x_n \rangle$, we denote the monoid ring of M over $\mathbb{Q}$, where M is the free monoid generated by $x_1, \ldots, x_n$. (This is essentially the same as the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$, only the variables $x_1, \ldots, x_n$ do not commute among each other.)

Definition 8.2.2. We call a subset $V \subseteq G^n$ an *algebraic set* if

$$V = \{(A_1, \ldots, A_n) \in G^n \mid \ f(A_1, \ldots, A_n) = 0 \text{ for all } f \in S\},$$

for some finite $S \subseteq \mathbb{Q}\langle x_1, \ldots, x_n \rangle$. An *algebraic variety* is a nonempty algebraic set which cannot be written as a union of two proper algebraic subsets.

Definition 8.2.3. Let $V \subseteq G^n$ be an algebraic variety. A *chain* in V of length $m \in \mathbb{N}$ is a sequence of proper inclusions $V_0 \subsetneq \cdots \subsetneq V_m$, where each $V_i \subseteq V$ is an algebraic variety. We define the dimension of V by

$$\dim(V) = \sup\{\text{length}(C) \mid \ C \text{ is a chain in } V\} \in \mathbb{N} \cup \{\infty\}.$$

Remark 8.2.4. Note that for $G = \mathbb{R}$, our definition of an algebraic variety $V \subseteq \mathbb{R}^n$ coincides with the conventional definition of an algebraic variety. The same is true for the dimension of V.

We will also have to define an analogue of the exponential function on matrices.

Definition 8.2.5. We define $\exp : M_{d\times d} \to M_{d \times d}$ by

$$\exp(X) = \sum_{n=1}^{\infty} X^n.$$

(It is known that this sum converges for all $X \in M_{d\times d}$.)

We are now ready to define our version of Schanuel's Conjecture.

Conjecture 8.2.6. *Let $V \subseteq G^{2n}$ be an algebraic variety with $\dim(V) < n$. Then if*

$$(A_1, \ldots, A_n, \exp(A_1), \ldots, \exp(A_1)) \in V,$$

there are $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, such that $\sum_{i=1}^n m_i A_i = 0$.

We define the uniform version as follows.

Conjecture 8.2.7. *Let $V \subseteq G^{2n}$ be an algebraic variety, with $\dim(V) < n$. Then there exists $N \in \mathbb{N}$, such that if*

$$(A_1, \ldots, A_n, \exp(A_1), \ldots, \exp(A_1)) \in V,$$

there are $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero, with $|m_i| \leq N$ for each $i = 1, \ldots, n$, such that $\sum_{i=1}^n m_i A_i = 0$.

Remark 8.2.8. Recall that if $V \subseteq \mathbb{R}^n$ is an algebraic variety, then $\dim(V) = \text{trdeg}_{\mathbb{Q}}(\mathbb{Q}(a_1, \ldots, a_n))$, for $(a_1, \ldots, a_n) \in V$. Combining this fact with Remark 8.2.4 shows that for $G = \mathbb{R}$, Conjectures 8.2.6 and 8.2.7 reduce to Conjectures 8.1.2 and 8.1.3 respectively.

8.3 Buchheim's formula and Analytic cell decomposition

Our strategy is to show that the function $\exp : G \to M_{d\times d}$ is definable in the structure $(\mathbb{R} \mid \mathcal{L}_{\exp})$, by which we mean that every component of exp is definable, when G and $M_{d\times d}$ are viewed as subsets of $\mathbb{R}^{d^2}$. If we prove this, then we can use the argument from [KZ06], with some minor alterations. In our proof we will make use of Buchheim's formula (50).

Remark 8.3.1. Recall that the minimal polynomial of a matrix $A \in M_{d\times d}$ is the monic polynomial with coefficients in $\mathbb{R}$, of minimal degree that annihilates A.

Proposition 8.3.2. *Let $A \in M_{n\times n}(\mathbb{R})$ be a matrix with minimal polynomial $q(t) = (t - \eta_1)^{r_1}\cdots(t-\eta_\nu)^{r_\nu}$, where $\eta_1,\ldots,\eta_\nu$ are distinct and all $r_i \geq 1$. Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be an analytic function. Suppose that each η_i is in its domain D and each η_i with $r_i > 1$ is in the interior of D. Suppose furthermore that $g(t) = (t-\lambda_1)^{s_1}\cdots(t-\lambda_\mu)^{s_\mu}$ is a monic polynomial that annihilates A, where $\lambda_1,\ldots,\lambda_\mu$ are distinct and all $s_i \geq 1$. Then*

$$f(A) = \sum_{i=1}^{\mu}\left[\left(\sum_{l=0}^{s_i-1}\frac{1}{l!}\varphi_i^{(l)}(\lambda_i)(A-\lambda_i I)^l\right)\prod_{j=1, j\neq i}^{\mu}(A-\lambda_j I)^{s_j}\right] \tag{50}$$

where $\varphi_i(t) = f(t)\frac{(t-\lambda_i)^{s_i}}{g(t)}$ and $\varphi_i^{(l)}$ is the l-th derivative of φ_i.

Proof. A proof of this can be found in [HJ91]. □

Remark 8.3.3. We will sometimes write $\mathbb{R} \models A = B$ for $A = (a_{i,j})_{1\leq i,j\leq d}$ and $B = (b_{i,j})_{1\leq i,j\leq d}$ elements of $M_{d\times d}$. This is of course shorthand for

$$\mathbb{R} \models \bigwedge_{1\leq i,j\leq d} a_{i,j} = b_{i,j}.$$

Lemma 8.3.4. *The function $\exp : G \to M_{d\times d}(\mathbb{R})$ is definable in the structure $(\mathbb{R} \mid \mathcal{L}_{\exp})$.*

Proof. We may safely assume that $d > 1$, as the Lemma is certainly true for $d = 1$. Let $X = (x_{i,j})_{1\leq i,j\leq d}$ and $\vec{y} = (y_1,\ldots,y_d)$. We define the $\mathcal{L}_{\exp}$-formula

$$\psi(X,\vec{y}) \equiv y_1 \leq \cdots \leq y_d \ \wedge\ \forall t(\det(tI - X) = (t-y_1)\cdots(t-y_d)).$$

Then given a matrix $A \in G$, $\mathbb{R} \models \psi(A,\lambda_1,\ldots,\lambda_d)$ if and only if $\lambda_1,\ldots,\lambda_d$ are the eigenvalues of A, in ascending order, counting their multiplicities. (Recall that all the eigenvalues of A are real.) To define $\exp : G \to M_{d\times d}(\mathbb{R})$, we want to make use of Buchheim's formula. In order to do this, we will need to make 2^{n-1} case distinctions, accounting for the all possible different multiplicities of the zero's of the characteristic polynomials. To this end we let $S = \{0,1\}^{\{1,\ldots,d-1\}}$ be the set of binary strings of length $d-1$. We let $\sigma(0)$ and $\sigma(1)$ stand for the symbols "=" and "<" respectively and for each $\tau \in S$, we define the $\mathcal{L}_{\exp}$-formula

$$\theta_\tau(y_1,\ldots,y_d) \equiv y_1\sigma(\tau_1)y_2\sigma(\tau_2)\cdots\sigma(\tau_{d-1})y_d.$$

Also for $\tau \in S$, set $\mu_\tau = 1 + \sum_{i=1}^{d-1}\tau_i$ and for $i = 1,\ldots,\mu_\tau - 1$, let $\rho_\tau(i)$ denote the position of the i-th 1 in the sequence τ. Furthermore, we define $\rho_\tau(0) = 0$ and $\rho_\tau(\mu_\tau) = d$ and for $i = 1,\ldots,\mu_\tau$ we set $s_\tau(i) = \rho_\tau(i) - \rho_\tau(i-1)$. Verify that if $\mathbb{R} \models \psi(A,\lambda_1,\ldots,\lambda_d)$, then there exists a unique $\tau \in S$ such that $\mathbb{R} \models \theta_\tau(\lambda_1,\ldots,\lambda_d)$ and that for this τ holds that

$$\det(tI - A) = (t-\lambda_{\rho_\tau(1)})^{s_\tau(1)}\cdots(t-\lambda_{\rho_\tau(\mu_\tau)})^{s_\tau(\mu_\tau)},$$

with $\lambda_{\rho_\tau(1)}, \ldots, \lambda_{\rho_\tau(\mu_\tau)}$ distinct and $s_\tau(1), \ldots, s_\tau(\mu_\tau)$ positive. For $\tau \in S$ and $i = 1, \ldots, \mu_\tau$ let

$$\varphi_{\tau,i}(t, u_1, \ldots, u_{\mu_\tau}) = \frac{\exp(t)}{(t-u_1)^{s_\tau(1)} \cdots (t-u_{i-1})^{s_\tau(i-1)}(t-u_{i+1})^{s_\tau(i+1)} \cdots (t-u_{\mu_\tau})^{s_\tau(\mu_\tau)}}$$

and note that it is a definable function. Now, let $Z = (z_{i,j})_{1 \leq i,j \leq d}$ and define the $\mathcal{L}_{\exp}$-formula

$$\chi_\tau(X, y_1, \ldots, y_d, Z) \equiv$$
$$Z = \sum_{i=1}^{\mu_\tau} \left[\left(\sum_{l=0}^{s_\tau(i)-1} \frac{1}{l!} \frac{\partial^l \varphi_{\tau,i}}{\partial t^l}(y_{\rho_\tau(i)}, y_{\rho_\tau(1)}, \ldots, y_{\rho_\tau(\mu_\tau)}) \cdot (X - y_{\rho_\tau(i)} I)^l \right) \prod_{j=1, j\neq i}^{\mu_\tau} (X - y_{\rho_\tau(j)} I)^{s_\tau(j)} \right]$$

(Compare this with (50).) Then if $\mathbb{R} \models \psi(A, \lambda_1, \ldots, \lambda_d)$ and $\tau \in S$ is the unique element such that $\mathbb{R} \models \theta_\tau(\lambda_1, \ldots, \lambda_d)$, it follows that $\exp(A)$ is the unique Z such that $\mathbb{R} \models \chi_\tau(A, \lambda_1, \ldots, \lambda_d, Z)$. This is because, by the Cayley-Hamilton Theorem, every $d \times d$ matrix satisfies its own characteristic equation, which means that the conditions of Proposition 8.3.2 are satisfied. The function $\exp : G \to M_{d\times d}(\mathbb{R})$ can therefore be defined by the $\mathcal{L}_{\exp}$-formula

$$\exists \vec{y}[\psi(X, \vec{y}) \wedge \bigvee_{\tau \in S} (\theta_\tau(\vec{y}) \wedge \chi_\tau(X, \vec{y}, Z))].$$

□

The heart of the proof used in [KZ06] is based on the analytic analog of Proposition A.2.4, which we shall eventually formulate and prove. For this proof we will be needing Corollary 8.3.5 (to Theorem 6.1.2), Lemma 8.3.6 and Theorem 8.3.7 as ingredients.

Corollary 8.3.5. *The structure* $(\mathbb{R} \mid \mathcal{L}_{\exp})$ *is O-minimal.*

Proof. Let $\phi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}_{\exp}$-formula and let $a_1, \ldots, a_n$ be parameters from $\mathbb{R}$. By Theorem 6.1.2 we may suppose that $\phi(x, y_1, \ldots, y_n)$ is an existential formula. By Corollary 4.2.7, the set

$$\{x \in \mathbb{R} \mid \mathbb{R} \models \phi(x, a_1, \ldots, a_n)\}$$

is a finite union of points and open intervals. It follows that $(\mathbb{R} \mid \mathcal{L}_{\exp})$ is O-minimal. □

Lemma 8.3.6. *Let $h_1(\vec{x}, \vec{y}), \ldots, h_l(\vec{x}, \vec{y})$ be a Pfaffian chain of $\mathcal{L}_{\exp}$-terms, with $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_m)$, and let $g(\vec{x}, \vec{y}) \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_l]$. Then there are finitely many m-tuples $f_1 = (f_{1,1}, \ldots, f_{1,m}), \ldots, f_m = (f_{s,1}, \ldots, f_{s,m})$, with $f_{i,j} \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_l]$ such that*

$$\mathcal{T}_{\exp,\mathbb{R}} \models \forall \vec{x}[\exists \vec{y}(g(\vec{x}, \vec{y}) = 0) \leftrightarrow$$
$$\exists \vec{y}(g(\vec{x}, \vec{y}) = 0 \wedge \bigvee_{1 \leq i \leq s} (f_i(\vec{x}, \vec{y}) = 0 \wedge \det \left(\frac{\partial(f_{i,1}, \ldots, f_{i,m})}{\partial(y_1, \ldots, y_m)} \right) (\vec{x}, \vec{y}) \neq 0)].$$

Proof. Let $K \models \mathcal{T}_{\exp,\mathbb{R}}$ and let $a_1, \ldots, a_n \in K$. For every $f \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_l]$, we let $h_{\vec{a}} : K^m \to K$ be given by $h_{\vec{a}}(\vec{y}) = h(\vec{a}, \vec{y})$. Now define $M_{\vec{a}}$ be the ring of all these functions $f_{\vec{a}}$. We note that $M_{\vec{a}}$ is closed under differentiation, as $h_1, \ldots, h_l$ is a Pfaffian chain, $M_{\vec{a}}$ contains $\mathbb{Z}[y_1, \ldots, y_m]$ and $M_{\vec{a}}$ is Noetherian, as it is finitely generated over $\mathbb{R}$. This means that we are in a position to apply Theorem 3.3.4, with $\mathcal{T}_{\mathcal{A}} = \mathcal{T}_{\exp,\mathbb{R}}$, $M = M_{\vec{a}}$, $U = K^m$ and $S = \mathcal{V}(g_{\vec{a}})$. This Theorem tells us that if we assume $K \models \exists \vec{y}(g(\vec{a}, \vec{y}) = 0)$, then there exist $f_1, \ldots, f_m \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_l]$ such that

$$K \models \exists \vec{y}(g(\vec{a}, \vec{y}) = 0 \wedge f_1(\vec{a}, \vec{y}) = \cdots = f_m(\vec{a}, \vec{y}) = 0 \wedge \det\left(\frac{\partial(f_1, \ldots, f_m)}{\partial(y_1, \ldots, y_m)}\right)(\vec{a}, \vec{y}) \neq 0).$$

For every $K \models \mathcal{T}_{\exp,\mathbb{R}}$ and every $a_1, \ldots, a_n \in K$, we define the $\mathcal{L}_{\exp,\mathbb{R}}$-formula

$$\phi_{K,\vec{a}}(\vec{x}) \equiv \exists \vec{y}(g(\vec{x}, \vec{y}) = 0 \wedge f_1(\vec{x}, \vec{y}) = \cdots = f_m(\vec{x}, \vec{y}) = 0 \wedge \det\left(\frac{\partial(f_1, \ldots, f_m)}{\partial(y_1, \ldots, y_m)}\right)(\vec{x}, \vec{y}) \neq 0),$$

where the $f_1, \ldots, f_m$ implicitly depend on K and $a_1, \ldots, a_n$ of course. Consider the theory

$$T = \mathcal{T}_{\exp,\mathbb{R}} \cup \{\neg\phi_{K,\vec{a}}(\vec{c}) \mid K \models \mathcal{T}_{\exp,\mathbb{R}} \text{ and } a_1, \ldots, a_n \in K\},$$

where $\vec{c}$ are new constants. For every $K \models T$ and $a_1, \ldots, a_n$, the statement $K \models \phi_{K,\vec{a}}(\vec{a})$ is a consequence of $K \models \exists \vec{y}(g(\vec{a}, \vec{y}) = 0)$, as $K \models \mathcal{T}_{\exp,\mathbb{R}}$. So, since $K \models \neg\phi_{K,\vec{c}}(\vec{c})$, we must have $K \models \neg\exists \vec{y}(g(\vec{c}, \vec{y}) = 0)$ for every $K \models T$ and hence $T \models \neg\exists \vec{y}(g(\vec{c}, \vec{y}) = 0)$. By the Compactness Theorem, there are finitely many

$$\neg\phi_1(\vec{c}), \ldots, \neg\phi_s(\vec{c}) \in \{\neg\phi_{K,\vec{a}}(\vec{c}) \mid K \models \mathcal{T}_{\exp,\mathbb{R}} \text{ and } a_1, \ldots, a_n \in K\}$$

such that

$$\mathcal{T}_{\exp,\mathbb{R}} \cup \{\neg\phi_1(\vec{c}), \ldots, \neg\phi_s(\vec{c})\} \models \neg\exists \vec{y}(g(\vec{c}, \vec{y}) = 0),$$

so

$$\mathcal{T}_{\exp,\mathbb{R}} \models \exists \vec{y}(g(\vec{c}, \vec{y}) = 0) \to \bigvee_{1 \le i \le s} \phi_i(\vec{c})$$

and therefore

$$\mathcal{T}_{\exp,\mathbb{R}} \models \forall \vec{x}[\exists \vec{y}(g(\vec{x}, \vec{y}) = 0) \to \bigvee_{1 \le i \le s} \phi_i(\vec{x})],$$

as the constants $\vec{c}$ do not appear in $\mathcal{T}_{\exp,\mathbb{R}}$. But this is easily rearranged to a statement of the form

$$\begin{aligned}\mathcal{T}_{\exp,\mathbb{R}} \models &\forall \vec{x}[\exists \vec{y}(g(\vec{x}, \vec{y}) = 0) \to \\ &\exists \vec{y}(g(\vec{x}, \vec{y}) = 0 \wedge \bigvee_{1 \le i \le s} (f_i(\vec{x}, \vec{y}) = 0 \wedge \det\left(\frac{\partial(f_{i,1}, \ldots, f_{i,m})}{\partial(y_1, \ldots, y_m)}\right)(\vec{x}, \vec{y}) \neq 0)],\end{aligned}$$

proving the Lemma, as the implication the other way around is trivial. $\square$

The Theorem below is known as the *Analytic Implicit Function Theorem.*

Theorem 8.3.7. *Suppose that U is open in $\mathbb{R}^{r+m}$ and $f_1, \ldots, f_m : U \to \mathbb{R}$ are analytic functions. Assume that $(P, Q) \in U$ and $f_1(P, Q) = \ldots = f_m(P, Q) = 0$. Suppose furthermore that the determinant of the matrix*

$$\Delta = \begin{pmatrix} \frac{\partial f_1}{\partial x_{r+1}} & \cdots & \frac{\partial f_1}{\partial x_{r+m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{r+1}} & \cdots & \frac{\partial f_m}{\partial x_{r+m}} \end{pmatrix}$$

is nonzero at the point (P, Q). Then there exist open neighborhoods V_1 of P and V_2 of Q with the following properties.

(i) $V_1 \times V_2 \subseteq U$.

(ii) *For each* $\vec{x} \in V_1$ *there exists a unique point* $\vec{y} \in V_2$ *such that* $f_1(\vec{x}, \vec{y}) = \ldots f_m(\vec{x}, \vec{y}) = 0$. *This point satisfies* $\det(\Delta(\vec{x}, \vec{y})) \neq 0$.

(iii) *In this way we obtain analytic mappings* $\psi_1, \ldots, \psi_m : V_1 \to \mathbb{R}$ *satisfying* $\vec{\psi}(\vec{x}) = \vec{y}$. *Furthermore, for* $l = 1, \ldots, r$ *and* $\vec{x} \in V_1$ *we have*

$$\begin{pmatrix} \frac{\partial \psi_1}{\partial x_l} \\ \vdots \\ \frac{\partial \psi_m}{\partial x_l} \end{pmatrix} = -\Delta^{-1} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial x_l} \\ \vdots \\ \frac{\partial f_m}{\partial x_l} \end{pmatrix}$$

when the left hand side is evaluated in the point $\vec{x}$ *and the right hand side is evaluated in the point* $(\vec{x}, \psi_1(\vec{x}), \ldots, \psi_m(\vec{x}))$.

Proof. A proof of this can be found in [FG02]. □

The following definitions are basically the same as Definitions A.2.2 and A.2.3, only with "continuous" replaced by "analytic".

Definition 8.3.8. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones. An analytic $(i_1, \ldots, i_n)$-*cell* is a definable subset of $\mathbb{R}$, defined by induction as follows. (When we say definable, we mean definable in the language $\mathcal{L}_{\text{exp}}$, with constants from $\mathbb{R}$.)

(i) An analytic (0)-cell is a one-element set $\{r\} \subseteq \mathbb{R}$ and an analytic (1)-cell is an interval $(a, b) \subseteq \mathbb{R}$, with $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$.

(ii) If C is an analytic $(i_1, \ldots, i_n)$-cell and $f : C \to \mathbb{R}$ is a definable continuous analytic function, then its graph $\{(\vec{x}, y) \in C \times \mathbb{R} \mid f(\vec{x}) = y\}$ is an analytic $(i_1, \ldots, i_n, 0)$-cell.

(iii) If A is an analytic $(i_1, \ldots, i_n)$-cell and $f, g : C \to \mathbb{R}$ are definable continuous analytic functions or the constant functions $\pm\infty$ and $f(\vec{x}) < g(\vec{x})$ for all $\vec{x} \in C$, then $\{(\vec{x}, y) \in C \times \mathbb{R} \mid f(\vec{x}) < y < g(\vec{x})\}$ is an analytic $(i_1, \ldots, i_n, 1)$-cell.

Definition 8.3.9. Let $n \in \mathbb{N}$, with $n \geq 1$. An analytic *decomposition* of $\mathbb{R}^n$ is a special kind of of partition of $\mathbb{R}^n$ into finitely many analytic cells. The definition is by induction on n.

(i) An analytic decomposition of $\mathbb{R}$ is a finite collection of intervals and points of the form

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_m, \infty), \{a_1\}, \ldots, \{a_m\}\},$$

with $a_1 < \cdots < a_m$ real numbers.

(ii) An analytic decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into analytic cells C, such that the set of projections $\pi[C]$ is an analytic decomposition of $\mathbb{R}$. (Here, $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^m$ is the projection on the first n coordinates.)

As promised, we give a proof of the Analytic Cell Decomposition Theorem for $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$. The proof is based on that given in [vdDM94].

Theorem 8.3.10. *For every* $n \in \mathbb{N}$ *with* $n \geq 1$, *the following holds.*

(I_n) *Given any definable sets* $A_1, \ldots, A_l \subseteq \mathbb{R}^n$, *there is an analytic decomposition of* $\mathbb{R}^n$, *partitioning each of* $A_1, \ldots, A_l$.

(II$_n$) *For each definable function $f : A \to \mathbb{R}$, with $A \subseteq \mathbb{R}^n$, there is an analytic decomposition of $\mathbb{R}^n$, partitioning A, such that each restriction $f \restriction C : C \to \mathbb{R}$ is analytic, for each cell $C \subseteq A$ in the decomposition.*

Proof. We use induction on n, in the following manner. First we show that (I$_1$) holds. Then we prove (I$_n$) $\Rightarrow$ (II$_n$) and (I$_n$) + (II$_n$) $\Rightarrow$ (I$_{n+1}$) for all positive $n \in \mathbb{N}$.

Verify that (I$_1$) is simply given by (I$_1$) of Proposition A.2.4. Now suppose that (I$_n$) holds and let $f : A \to \mathbb{R}$ be a definable function with $A \subseteq \mathbb{R}^n$. Then by Theorem 6.1.2, there exists an existential $\mathcal{L}_{\text{exp}}$-formula ϕ, such that $\mathbb{R} \models \forall \vec{x}, y[(\vec{x}, y) \in \text{graph}(f) \leftrightarrow \phi(x, y)]$. By Lemma 2.1.5, we may assume that ϕ is of the form

$$\exists z_1, \ldots, z_m \bigwedge_{i=1}^{r} \tau_i = 0,$$

where each τ_i is an $\mathcal{L}_{\text{exp}}$-term. This gives us an $\mathcal{L}_{\text{exp}}$-term, $F = \tau_1^2 + \cdots \tau_r^2$, such that

$$\mathbb{R} \models \forall \vec{x}, y[(\vec{x}, y) \in \text{graph}(f) \leftrightarrow \exists \vec{z}(F(\vec{x}, y, \vec{z}) = 0)].$$

Lemma 4.2.6 tells us that F is part of a Pfaffian chain of $\mathcal{L}_{\text{exp}}$-terms, say $h_1, \ldots, h_l$. Since $F(\vec{x}, y, \vec{z}) \in \mathbb{R}[\vec{x}, y, \vec{z}, h_1, \ldots, h_l]$, we can use Lemma 8.3.6 to find finitely many $(1+m)$-tuples $f_1 = (f_{1,1}, \ldots, f_{1,1+m}), \ldots, f_{1+m} = (f_{s,1}, \ldots, f_{s,1+m})$, with $f_{i,j} \in \mathbb{R}[\vec{x}, y, \vec{z}, h_1, \ldots, h_l]$ such that

$$\begin{aligned} \mathbb{R} \models &\forall \vec{x}[\exists y, \vec{z}(F(\vec{x}, y, \vec{z}) = 0) \leftrightarrow \\ &\quad \exists y, \vec{z}(F(\vec{x}, y, \vec{z}) = 0 \wedge \bigvee_{1 \le i \le s} (f_i(\vec{x}, y, \vec{z}) = 0 \wedge \det\left(\frac{\partial(f_{i,1}, \ldots, f_{i,1+m})}{\partial(y, z_1, \ldots, z_m)}\right)(\vec{x}, y, \vec{z}) \neq 0)]. \end{aligned}$$

This means that $A = \bigcup_{1 \le i \le s} A_i$, where

$$A_i = \{\vec{x} \in A \mid \mathbb{R} \models \exists \vec{z}(f_i(\vec{x}, f(\vec{x}), \vec{z}) = 0 \wedge \det\left(\frac{\partial(f_{i,1}, \ldots, f_{i,1+m})}{\partial(y, z_1, \ldots, z_m)}\right)(\vec{x}, f(\vec{x}), \vec{z}) \neq 0)\},$$

for $i = 1, \ldots, s$.

Next, we fix some A_i and use ordinary cell decomposition ((II$_n$) of Proposition A.2.4) to find a decomposition $\mathcal{D}_i$ of $\mathbb{R}^n$, partitioning A_i, such that the restriction $f \restriction C$ is continuous for each cell $C \subseteq A_i$ in $\mathcal{D}_i$.

(In [vdDM94] it is claimed that at this point it follows from Theorem 8.3.7 that f is analytic when restricted to C. I was unable to verify this claim, however. I shall therefore use an alternative approach.)

For a cell $C \subseteq A_i$ in $\mathcal{D}_i$, consider the set

$$B = \{(\vec{x}, \vec{z}) \in C \times \mathbb{R}^m \mid \mathbb{R} \models f_i(\vec{x}, f(\vec{x}), \vec{z}) = 0 \wedge \det\left(\frac{\partial(f_{i,1}, \ldots, f_{i,1+m})}{\partial(y, z_1, \ldots, z_m)}\right)(\vec{x}, f(\vec{x}), \vec{z}) \neq 0\}.$$

Then $\pi[B] = C$, as $C \subseteq A_i$, where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the projection on the first n coordinates. Then by Proposition A.2.6, there exists a definable function $g : C \to \mathbb{R}^m$ such that $\text{graph}(g) \subseteq B$. In other words, for all $\vec{a} \in C$,

$$\mathbb{R} \models f_i(\vec{a}, f(\vec{a}), g(\vec{a})) = 0 \wedge \det\left(\frac{\partial(f_{i,1}, \ldots, f_{i,1+m})}{\partial(y, z_1, \ldots, z_m)}\right)(f_i(\vec{a}, f(\vec{a}), g(\vec{a})) \neq 0.$$

Write $g = (g_1, \ldots, g_m)$.

We now apply (II_n) of Proposition A.2.4 in m stages. Starting out with $\mathcal{P}_i^0 = \{C \in \mathcal{D}_i \mid C \subseteq A_i\}$, we obtain $\mathcal{P}_i^j$ from $\mathcal{P}_i^{j-1}$, for $j = 1, \ldots, m$, in the following manner. If $\mathcal{P}_i^j$ contains a cell C such that $g_j \restriction C$ is not continuous, we use (II_n) of Proposition A.2.4 to obtain a partition P of C, such that the restriction of g_j is continuous when restricted to each individual cell in this partition. We now replace C in $\mathcal{P}_i^{j-1}$ by the cells in P. Applying this process exhaustively gives us $\mathcal{P}_i^j$. Verify that $\mathcal{P}_i^j$ is finite partition of A_i, such that for each cell $C \in \mathcal{P}_i^j$, the functions $f, g_1, \ldots, g_j$ are continuous when restricted to C.

We claim that for each cell $C \in \mathcal{P}_i^m$, the restriction $f \restriction C$ is analytic. For take some $\vec{a} \in C$. Then $f_i(\vec{a}, f(\vec{a}), g(\vec{a})) = 0$ and $\det\left(\frac{\partial(f_{i,1},\ldots,f_{i,1+m})}{\partial(y,z_1,\ldots,z_m)}\right)(\vec{a}, f(\vec{a}), g(\vec{a})) \neq 0$. Since (the interpretations of) the function symbols present in $\mathcal{L}_{\exp}$ are analytic and analyticity is preserved under composition, the functions $f_{i,1}, \ldots, f_{i,1+m} \in \mathbb{R}[\vec{x}, y, \vec{z}, h_1, \ldots, h_l]$ are analytic. We can therefore apply Theorem 8.3.7, to obtain open neighborhoods V_1 of $\vec{a}$ and V_2 of $(f(\vec{a}), g(\vec{a}))$ and analytic functions $\psi_1, \ldots, \psi_{1+m} : V_1 \to \mathbb{R}$ as described in the Theorem. By reducing the size of V_1 if necessary, we may assume that $(f(\vec{x}), g(\vec{x})) \in V_2$ for each $\vec{x} \in C \cap V_1$, by continuity of f and g on C. Since $f_i(f(\vec{x}), g(\vec{x})) = 0$ for $\vec{x} \in C \cap V_1$, the functions (f, g) and $(\psi_1, \ldots, \psi_{1+m})$ must coincide on $C \cap V_1$, by uniqueness of $(\psi_1, \ldots, \psi_{1+m})$. In particular, $f(\vec{x}) = \psi_1(\vec{x})$ for $\vec{x} \in C \cap V_1$. Hence, $f \restriction C$ is analytic at the point $\vec{a}$ and since $\vec{a}$ was arbitrary, $f \restriction C$ is analytic. Finally, using our induction hypothesis, we apply (I_n) to the collection $\bigcup_{1 \le i \le s} \mathcal{P}_i^m$. This gives us an analytic decomposition of $\mathbb{R}^n$, partitioning A, such that $f \restriction C$ is analytic for each cell in the decomposition.

To derive (I_{n+1}) from $(\mathrm{I}_n) + (\mathrm{II}_n)$, let $A_1, \ldots, A_l \subseteq \mathbb{R}^{n+1}$. Then by (I_{n+1}) of Proposition A.2.4, there exists a decomposition $\mathcal{D}$ of $\mathbb{R}^{n+1}$, partitioning each of $A_1, \ldots, A_l$. Let C be a $(i_1, \ldots, i_n, 0)$-cell in this decomposition. Then by definition there is a definable continuous function $f : \pi[C] \to \mathbb{R}$, such that $C = \mathrm{graph}(f)$. By (II_n), there is an analytic decomposition $\mathcal{D}_C$ of $\mathbb{R}^n$, partitioning $\pi[C]$, such that each restriction $f \restriction C'$ is analytic, for each cell $C' \subseteq \pi[C]$ in the decomposition. Now if on the other hand $C = \{(\vec{x}, y) \in \pi[C] \times \mathbb{R} \mid f(\vec{x}) < y < g(\vec{x})\}$ is a $(i_1, \ldots, i_n, 1)$-cell, in the decomposition $\mathcal{D}$, then we can proceed similarly, only now we get two analytic decompositions $\mathcal{D}_f, \mathcal{D}_g$ of $\mathbb{R}^n$, such that each restriction $f \restriction C'$ is analytic for $C' \in \mathcal{D}_f$ and each restriction $g \restriction C''$ is analytic for $C'' \in \mathcal{D}_g$. We write $\mathcal{D}_C = \mathcal{D}_f \cup \mathcal{D}_g$ in this case. Next, we apply (I_n) on the finite collection $\bigcup_{C \in \mathcal{D}} \mathcal{D}_C$ of subsets of $\mathbb{R}^n$, to find an analytic decomposition $\mathcal{D}'$ of $\mathbb{R}^n$, partitioning each cell $C' \in \mathcal{D}_C$, for each cell $C \in \mathcal{D}$. Now suppose that $C \in \mathcal{D}$ is an $(i_1, \ldots, i_n, 0)$-cell, say $C = \mathrm{graph}(f)$. Then by construction of $\mathcal{D}'$, the projection $\pi[C]$ is partitioned by analytic cells $C_1, \ldots, C_m \in \mathcal{D}'$, such that the restrictions $f \restriction C_i$ are analytic. Thus, $C = \bigcup_{1 \le i \le m} \mathrm{graph}(f \restriction C_i)$ can be partitioned into finitely many analytic cells. A similar treatment can be given to the $(i_1, \ldots, i_n, 1)$-cells in $\mathcal{D}$. Applying this to each individual cell in the decomposition $\mathcal{D}$, gives us an analytic decomposition of $\mathbb{R}^{n+1}$, partitioning each of $A_1, \ldots, A_l$, as desired. $\square$

8.4 Uniformity comes for free

In this section we finish the proof of the fact that Conjecture 8.2.6 implies Conjecture 8.2.7. For this, we need one last Lemma.

Lemma 8.4.1. *Let $C \subseteq \mathbb{R}^n$ be an analytic $(i_1, \ldots, i_n)$-cell and write $m = i_1 + \cdots + i_n$. Then there exists a definable analytic diffeomorphism $\theta : B \to C$, where $B \subseteq \mathbb{R}^m$ is an open box. (For $m = 0$, B is a point.)*

Proof. We use induction on n. For $n = 1$, we can take θ to be the identity, as C is a point or an open interval in this case.

Suppose that C is an analytic $(i_1, \ldots, i_n, 0)$-cell. Then $C = \text{graph}(f)$, where $f : \pi[C] \to \mathbb{R}$ is an analytic function. By the induction hypothesis, there exists a definable analytic diffeomorpism $\varphi : B \to \pi[C]$, where B is the product of $i_1 + \cdots + i_n$ open intervals. Then if we define $\theta : B \to C$ by $\theta(\vec{x}) = (\varphi(\vec{x}), f(\varphi(\vec{x})))$, the map θ is a definable analytic diffeomorphism between B and C, as needed.

Next, suppose that C is an analytic $(i_1, \ldots, i_n, 1)$-cell, say $C = \{(\vec{x}, y) \in \pi[C] \times \mathbb{R} \mid f(\vec{x}) < y < g(\vec{x})\}$. Again, by our induction hypothesis, there exists an analytic diffeomorpism $\varphi : B \to \pi[C]$, where B is the product of $i_1 + \cdots + i_n$ open intervals.

- If $f \neq -\infty$ and $g \neq \infty$, we define $\theta : B \times (0, 1) \to C$ by

$$\theta(\vec{x}, y) = (\varphi(\vec{x}), (1 - y) \cdot f(\vec{x}) + y \cdot g(\vec{x})).$$

- If $f \neq -\infty$ and $g = \infty$, we define $\theta : B \times (0, \infty) \to C$ by

$$\theta(\vec{x}, y) = (\varphi(\vec{x}), f(\vec{x}) + y).$$

- If $f = -\infty$ and $g \neq \infty$, we define $\theta : B \times (-\infty, 0) \to C$ by

$$\theta(\vec{x}, y) = (\varphi(\vec{x}), g(\vec{x}) + y).$$

- If $f = -\infty$ and $g = \infty$, we define $\theta : B \times \mathbb{R} \to C$ by

$$\theta(\vec{x}, y) = (\varphi(\vec{x}), y).$$

In each case, θ is a definable analytic diffeomorphism between an open box in $\mathbb{R}^m$ and C, with $m = i_1 + \cdots + i_n + 1$, as required. □

Theorem 8.4.2. *Conjecture 8.2.6 implies Conjecture 8.2.7.*

Proof. Assume Conjecture 8.2.6. Let $V \subseteq G^{2n}$ be an algebraic variety, with $\dim(V) < n$ and let

$$W = \{(X_1, \ldots, X_n) \in G^n \mid (X_1, \ldots, X_n, \exp(X_1), \ldots, \exp(X_n)) \in V\}$$

Then by Lemma 8.3.4, the set W is definable in the structure $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$. Theorem 8.3.10 then allows us to partition W into finitely many analytic cells. Let C be an $(i_1, \ldots, i_n)$-cell in this partition and let $\theta : B \to C$ be a definable analytic diffeomorphism from an open box $B \subseteq \mathbb{R}^m$ to C, with $m = i_1 + \cdots i_n$, as given in Lemma 8.4.1. Let $\vec{X}, \vec{Y} \in C$ and let $\sigma : [0, 1] \to B$ be the path of uniform speed along the line segment from $\theta^{-1}(\vec{X})$ to $\theta^{-1}(\vec{Y})$. Then $\gamma = \theta \circ \sigma$ is a definable analytic path from $\vec{X}$ to $\vec{Y}$ in C.

By Conjecture 8.2.6, every point in $\vec{Z} \in W$ satisfies an equation of the form $\sum_{i=1}^{n} m_i Z_i = 0$, with $m_1, \ldots, m_n \in \mathbb{Z}$, not all zero. This is in particular true for the points in the image of γ. Since only countably many such equations exist, at least one of these, say $h(\vec{Z}) = \sum_{i=1}^{n} m_i Z_i = 0$, must be satisfied by infinitely points in the image of γ. Then $\{t \in [0, 1] \mid h(\gamma(t)) = 0\}$ is an infinite definable subset of $[0, 1]$, so by O-minimality of $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$ it must contain an open interval. Since $h \circ \gamma : [0, 1] \to M_{n \times n}(\mathbb{R})$ is an analytic function which is zero on a subinterval of $[0, 1]$, it must be identically zero on $[0, 1]$. Hence, $\vec{X}$ and $\vec{Y}$ satisfy the same equation h. Since these point where arbitrary points of C, each point of C satisfies this equation h. Because we partitioned W into finitely many cells, it is clear that there exists a uniform bound N on the coefficients of these equations, as described in Conjecture 8.2.7. □

9 Concluding remarks

9.1 Possible generalization

Let us address a question that one might have about Theorem 8.4.2. Is it necessary for the eigenvalues of the matrices in G to be real? Or can we also find a proof for Theorem 8.4.2 with $G = M_{d\times d}$ for example? The answer appears to be no, at least not with the methods we have at our disposal. This is because we cannot hope to improve on the result of Lemma 8.3.4 to show that the function $\exp : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is definable in the structure $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$. For suppose that it where definable. Then setting $n = 2$ shows that the function

$$x \mapsto \exp\left[\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}\right]$$

is definable. Since

$$\exp\left[\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}$$

this means that in particular the function $x \mapsto \sin(x)$ is definable in $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$. But then $\{x \in \mathbb{R} \mid \sin(x) = 0\}$ would be a definable set, which is clearly false by O-minimality of $(\mathbb{R} \mid \mathcal{L}_{\text{exp}})$.

9.2 Acknowledgment

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A Appendix

A.1 Real analytic functions of several variables

Let us fix $n \in \mathbb{N}$, with $n \geq 1$. For elements $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ and variables $x = (x_1, \dots, x_n)$, we will sometimes use the notation $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$. In this context, the element μ is called a *multi-index*. A formal expression of the form

$$\sum_{\mu \in \mathbb{N}^n} a_\mu (x - \alpha)^\mu,$$

with $\alpha \in \mathbb{R}^n$ and $a_\mu \in \mathbb{R}$, for each μ, is called a power series in n variables. Recall that if such a series converges *absolutely* at a point $x \in \mathbb{R}^n$, then the series converges to a value in $\mathbb{R}$, independent of the order of summation.

Definition A.1.1. Let A be a subset of $\mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is called *real analytic* if for each $\alpha \in A$, there exists a neighborhood of α such that the function f may be represented by an absolutely convergent power series on the intersection of this neighborhood with A. A vector valued function $f = (f_1, \dots, f_m) : A \to \mathbb{R}^m$ is called real analytic if all of its components $f_i : A \to \mathbb{R}$ are real analytic.

As the reader is surely aware, analytic functions enjoy many useful properties. We will make ample use of some of their basic properties and for the sake of completeness, we shall list these (without proof) after the following definition.

Definition A.1.2. Let $U \subseteq \mathbb{R}^n$ be an open set. A function $f : U \to \mathbb{R}$ is of class C^l, or a C^l-function, if the partial derivatives $\frac{\partial^l f}{\partial x^\mu} : U \to \mathbb{R}$ exist and are continuous for all $\mu \in \mathbb{N}^n$ such that $\mu_1 + \cdots + \mu_n = l$. The class C^∞ is defined as the intersection of the classes C^l, over all $l \in \mathbb{N}$.

Proposition A.1.3. *Let $U \subseteq \mathbb{R}^n$ be an open set and suppose that $f : U \to \mathbb{R}$ is a real analytic function. Then for each $i = 1, \dots, n$, the derivative $\frac{\partial f}{\partial x_i} : U \to \mathbb{R}$ exists and is analytic. Hence all higher order derivatives of f are analytic and in particular f is a C^∞-function.*

Proposition A.1.4. *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets and suppose that $f : U \to V$ and $g : U \to \mathbb{R}$ are real analytic functions. Then their composition $g \circ f : U \to \mathbb{R}$ is analytic.*

Proposition A.1.5. *Let $U \subseteq \mathbb{R}^n$ be an open set. Then the set of real analytic functions $U \to \mathbb{R}$ forms a ring. Moreover, if U is connected, then this ring is an integral domain.*

A.2 O-minimal structures

Given a language L, an L-structure M is called *minimal* if every subset of M which is definable with parameters from M is is quantifier-free definable just using equality. This means that these definable sets are either finite or cofinite. By analogy, if every definable subset of M is quantifier-free definable using equality and *in*equality, then we say that this structure is *order minimal* or *O-minimal*.

Definition A.2.1. Let L be a language containing "<" and let M be an infinite L-structure which is linearly ordered (by "<"). Then M is called an *O-minimal* structure if every subset of M, definable in L with parameters from M, is a finite union of intervals and points.

Many nice properties of definable subsets of M^n for all $n \in \mathbb{N}$ follow from this condition on just the definable subsets of M. One of these properties (and perhaps the most significant one) is the Cell Decomposition Theorem. This Theorem characterizes the definable sets and shows that all definable functions are piecewise continuous. The Cell Decomposition Theorem is stated below the following two definitions which we shall need first. For this, we temporarily fix the O-minimal structure M, in the language L.

Definition A.2.2. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones. An $(i_1, \ldots, i_n)$-*cell* is a definable subset of M, defined by induction as follows. (When we say definable, we mean definable in the language L, with constants from M.)

(i) A (0)-cell is a one-element set $\{m\} \subseteq M$ and a (1)-cell is an interval $(a, b) \subseteq M$, with $a \in M \cup \{-\infty\}$ and $b \in M \cup \{\infty\}$.

(ii) If C is an $(i_1, \ldots, i_n)$-cell and $f : C \to M$ is a definable continuous function, then its graph $\{(\vec{x}, y) \in C \times M \mid f(\vec{x}) = y\}$ is an $(i_1, \ldots, i_n, 0)$-cell.

(iii) If A is an $(i_1, \ldots, i_n)$-cell and $f, g : C \to M$ are definable continuous functions or the constant functions $\pm\infty$ and $f(\vec{x}) < g(\vec{x})$ for all $\vec{x} \in C$, then $\{(\vec{x}, y) \in C \times M \mid f(\vec{x}) < y < g(\vec{x})\}$ is an $(i_1, \ldots, i_n, 1)$-cell.

Definition A.2.3. Let $n \in \mathbb{N}$, with $n \geq 1$. A *decomposition* of M^n is a special kind of of partition of M^n into finitely many cells. The definition is by induction on n.

(i) A decomposition of M is a finite collection of intervals and points of the form

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_m, \infty), \{a_1\}, \ldots, \{a_m\}\},$$

with $a_1 < \cdots < a_m$ elements of M.

(ii) A decomposition of M^{n+1} is a finite partition of M^{n+1} into cells C, such that the set of projections $\pi[C]$ is a decomposition of M. (Here, $\pi : M^{n+1} \to M^m$ is the projection on the first n coordinates.)

Proposition A.2.4. *For every $n \in \mathbb{N}$ with $n \geq 1$, the following holds.*

(I_n) *Given any definable sets $A_1, \ldots, A_l \subseteq M^n$, there is a decomposition of M^n, partitioning each of $A_1, \ldots, A_l$.*

(II_n) *For each definable function $f : A \to M$, with $A \subseteq M^n$, there is a decomposition of M^n, partitioning A, such that each restriction $f \restriction C : C \to M$ is continuous, for each cell $C \subseteq A$ in the decomposition.*

Proof. A proof of this can be found in [vdD98]. □

It turns out that models of the complete theory of an O-minimal structure are themselves again O-minimal. This result is not trivial and the analogous statement for minimal structures does not hold.

Proposition A.2.5. *If M is an O-minimal L-structure and $N \models \mathrm{Th}(M \mid L)$, then N is O-minimal as well.*

Proof. A proof of this can be found in [KPS86]. □

If additionally M is an ordered Abelian group, then there is even more we can say. The following Proposition is known as *definable choice* for O-minimal structures.

Proposition A.2.6. *Suppose that* $\{0, -, +\} \subseteq L$ *and* M *is an ordered group with respect to addition. If* $A \subseteq M^{m+n}$ *is a definable set and* $\pi : M^{m+n} \to M^m$ *is the projection on the first* m *coordinates, then there exists a definable map* $f : \pi[A] \to \mathbb{R}^n$*, such that* $\mathrm{graph}(f) \subseteq A$.

Proof. A proof of this can be found in [vdD98]. □

A.3 Types and saturated models

One of the most important notions in model theory is that of a *type*. Loosely speaking, a type is a (possibly infinite) list of properties describing how an element might behave.

Definition A.3.1. A *partial* n*-type* in L is a set of L-formulas of (the same) n variables.

It is also possible to define *complete* n-types, but we will not be needing this concept. Since we shall only be concerned with partial types, there will be no harm in sometimes just referring to them as "types". A partial n-type in the variables $x_1, \ldots, x_n$ is usually written as $p(x_1, \ldots, x_n)$. If M is an L-structure with $a_1, \ldots, a_n \in M$ and $M \models \varphi(a_1, \ldots, a_n)$ for every $\varphi(x_1, \ldots, x_n) \in p(x_1, \ldots, x_n)$, then it is said that $(a_1, \ldots, a_n)$ *realizes* p in M. If M is an L-structure which contains some n-tuple that realizes p, then we say that p is realized in M.

Definition A.3.2. If M is an L-structure and $p(x_1, \ldots, x_n)$ is a partial n-type in L, then p is *finitely satisfiable* in M if all finite subsets of p are realized in M.

Next, we introduce the concept of a *saturated* model. Such a structure realizes as many types as can be reasonable expected. Such a model is "rich" in some sense. Saturation is defined relative to some cardinal number, as we will allow the use of parameters from some fixed set smaller than this cardinal number.

Definition A.3.3. Let M be an L-structure and let κ be cardinal number. We say that M is κ*-saturated* if for any subset $A \subseteq M$, with $|A| < \kappa$, and any partial 1-type $p(x)$ in L_A which is finitely satisfiable in M, the type $p(x)$ is in fact realized in M.

The following Proposition shows that we can always extend a given model in such a way that the resulting structure is saturated. This will be our main tool when working with types.

Proposition A.3.4. *Let* M *be an* L*-structure and let* κ *be a cardinal number. Then there exists an elementary extension* $M \preceq N$ *which is* κ *saturated.*

Proof. A proof of this can be found in [Poi00]. □

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