TOPOLOGICAL TWISTING AND ELLIPTICALLY FIBERED CALABI-YAU SPACES

ERIK M.W. LUMENS

Utrecht University, The Netherlands

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Supervisors:

Professor S. Vandoren	Assistant Professor M. Kool
Institute For Theoretical Physics	Mathematical Institute
Utrecht University	Utrecht University

Abstract

In this thesis we look at topological twistings. These are certain operations that counter holonomy effects when defining a global symmetry on a curved space. The settings that we will encounter also demand some mathematical work to fully comprehend all the details. This work also focuses on the description and construction of elliptically fibered Calabi-Yau spaces.

In section 1 we repeat the basics of bosonic string theory. After this we briefly mention the different superstring theories. Furthermore, we discuss the dualities that relate the superstring theories together. Finally, we discuss F-theory and the importance of it in what follows.

In section 2 we discuss Calabi-Yau manifolds. We start with some definitions from differential geometry and continue with some calculations on the Calabi-Yau conditions as subspaces of certain projective spaces. We also discuss Hodge numbers and calculate the general Hodge diamond.

In section 3 we start by defining topological field theories. After we discussed some examples, we continue with $\mathcal{N} = (2, 2)$ SUSY and twist it to obtain two $\mathcal{N} = 2$ SUSY's.

In section 4 we discuss toroidal dimensional reduction of $\mathcal{N} = 4$ SYM theory in four dimensions. We start however with describing the geometry of the torus. This will help us understand the symmetries that will be apparent in our theory later on. Next, we perform the dimensional reduction and determine the dynamical phases.

In section 5 we give an overview of the concepts needed in toric geometry to understand Batyrev's construction. We do this using multiple examples.

In section 6 we perform Batyrev's construction to obtain Calabi-Yau manifolds as subspaces of some bigger projective space. We will detail the construction and explain all the necessary components. Again, working with an example the construction will be made apparent. Furthermore, we discuss the software library *PolyTori* that was written specifically by the author for this thesis. It aids in the calculation of Hodge numbers for Calabi-Yau threefolds. Furthermore, it can perform toric data manipulations.

In section 7 we discuss elliptic threefolds based on paper [12] on this subject. Given a elliptic threefold we determine its discrimant locus, and determine the structure of the elliptic fiber. Again this will be made explicit by working out some examples.

In section 8 we can finally apply all this knowledge to our last example of this thesis, namely type IIB superstring theory on a D3-brane. We will perform a topological duality twist here, using the $SL(2, \mathbb{Z})$-duality apparent in this theory. Using intersection theory, we ultimately derive the central charges of this theory.

This work is by no means self-contained. Although I try to introduce concepts when I use them, some background in physics and mathematics is assumed.

Dedicated to my parents

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Contents

Chapter 1

The Superstring

In this chapter we will study superstring theory. After a short review of bosonic string theory, we will introduce fermion fields to the action and discuss type I, type IIA and type IIB superstring theory. We will not go into much detail about heterotic superstring theories ($SO(32)$ and $E_8 \times E_8$). Furthermore, we will write down the different dualities between these 10-dimensional theories and show that they can be seen as different limits of an 11-dimensional supergravity theory, namely M-theory. Furthermore, we will discuss F-theory, a 12-dimensional theory and see how we can arrive at type IIB superstring theory in a certain limit. Since this is all standard literature, we will mostly quote results. Details can be found in [14] and [22].

1.1 A short overview of the bosonic string

A one-dimensional object sweeps out a world-sheet Σ, that can be described by two parameters, σ and τ. Here τ can be seen as the 'timelike' coordinate and σ as the 'spacelike' coordinate. This two-dimensional world-sheet can be embedded into an ambient space, of dimension D. We denote the D embedding functions by $X^\mu(\tau, \sigma)$, where $\mu = 0, ..., D-1$. Let us change notation: denote $\sigma^1 = \tau$ and $\sigma^2 = \sigma$. The simplest Poincaré and reparametrization invariant action is the Nambu-Goto action, defined by

$$S_{NG}[X] = \frac{-1}{2\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-\det h_{ab}}. \tag{1.1}$$

Here $h_{ab} = \partial_a X^\mu \partial_b X_\mu$, where a, b (and in more generality lower-cased alphabet letters, unless otherwise stated) run over the world-sheet coordinates (σ^1, σ^2). h_{ab} is the induced metric of the ambient space on the world-sheet. The constant α' is called the Regge slope. Because of the square root in the above expression, this action is not very practical to work with. We can improve this by introducing an independent, flat world-sheet metric γ_{ab}. Let us take it to have Lorentzian signature $(-, +)$. We obtain the following action, a functional of both X and γ, called the Polyakov action:

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-\gamma}\gamma^{ab} h_{ab}. \tag{1.2}$$

Here $\gamma = \det \gamma_{ab}$. If we fill in the equation of motion for γ, we retrieve the Nambu-Goto action. The Polyakov action is invariant under the following symmetries.

Poincaré invariance:

$$X'^{\mu}(\sigma^1,\sigma^2) = \Lambda^{\mu}_{\nu} X^{\nu}(\sigma^1,\sigma^2) + a^{\mu}$$
$$\gamma'_{ab}(\sigma^1,\sigma^2) = \gamma_{ab}(\sigma^1,\sigma^2)$$

Diffeomorphism invariance:

$$X'^{\mu}(\sigma'^1,\sigma'^2) = X^{\mu}(\sigma^1,\sigma^2)$$
$$\partial_a \sigma'^c \partial_b \sigma'^d \gamma'_{cd}(\sigma'^1,\sigma'^2) = \gamma_{ab}(\sigma^1,\sigma^2)$$

Weyl invariance:

$$X'^{\mu}(\sigma^1,\sigma^2) = X^{\mu}(\sigma^1,\sigma^2)$$
$$\gamma'_{ab}(\sigma^1,\sigma^2) = e^{2\omega(\sigma^1,\sigma^2)} \gamma_{ab}(\sigma^1,\sigma^2)$$

We can now use the Polyakov action (equation 1.2) to derive the equation of motion for X, by varying the action with respect to this field. In doing so, we obtain the equation of motion

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^{\mu}) = 0. \tag{1.3}$$

We have to be careful of surface terms. For world-sheets with boundary (i.e. open strings), we can take the parameter space to be

$$-\infty < \sigma^1 < +\infty \qquad \text{and} \qquad 0 \leq \sigma^2 \leq \pi.$$

Then the surface term

$$-\frac{1}{2\pi\alpha'} \int_{-\infty}^{+\infty} d\sigma^1 \sqrt{-\gamma}\partial^2 X_{\mu} \delta X^{\mu} \Big|_{\sigma^2=0}^{\sigma^2=\pi} \tag{1.4}$$

will vanish if we consider one of the following two boundary conditions:

Neumann boundary condition:

$$\partial_2 X^{\mu}(\sigma^1, 0) = 0 = \partial_2 X^{\mu}(\sigma^1, \pi),$$

Dirichlet boundary condition:

$$\delta X^{\mu}(\sigma^1, 0) = 0 = \delta X^{\mu}(\sigma^1, \pi).$$

In the case that there is no boundary, imposing periodic conditions will get rid of any surface term. In this case, the parameter space is

$$-\infty < \sigma^1 < +\infty \qquad \text{and} \qquad 0 \leq \sigma^2 \leq 2\pi.$$

The periodic conditions are as follows:

$$X^{\mu}(\sigma^1, 0) = X^{\mu}(\sigma^1, 2\pi),$$
$$\partial^2 X^{\mu}(\sigma^1, 0) = \partial^2 X^{\mu}(\sigma^1, 2\pi),$$
$$\gamma_{ab}(\sigma^1, 0) = \gamma_{ab}(\sigma^1, 2\pi).$$

We will now look at compactification of one or more dimensions of our theory. While it is not the most realistic compactification of string theory, it is quite interesting to compactify our theory at this stage, since it results in T-duality and D-branes in a natural way. In general, we have a theory in D dimensions, and we could compactify one or multiple dimensions. So if τ runs over a particular subset of $\{0, ..., D-1\}$, then $X^\tau \cong X^\tau + 2\pi R_\tau$. This is called toroidal compactification. We will study the case where we compactify only one dimension. So let us consider the closed bosonic string in $D = 26$ dimensions (so that the theory is Weyl invariant), with X^{25} periodic, i.e. $X^{25} \cong X^{25} + 2\pi R$ for $R > 0$. A consequence of this is that the center-of-mass momentum is quantized:

$$k = \frac{n}{R}, \qquad n \in \mathbb{Z}. \tag{1.5}$$

Furthermore, the closed string may wind around the compact dimension with winding number w:

$$X^{25}(\sigma^2 + 2\pi) = X^{25}(\sigma^2) + 2\pi R w, \qquad w \in \mathbb{Z}. \tag{1.6}$$

The left- and right-moving momenta in the compact dimension are given by the equations

$$p_L = \frac{n}{R} + \frac{wR}{\alpha'}, \tag{1.7a}$$

$$p_R = \frac{n}{R} - \frac{wR}{\alpha'}. \tag{1.7b}$$

The mass-shell condition becomes

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{1.8}$$

where $N, \tilde{N}$ are the number operators for the left- and right-moving spectrum.

Notice that the spectrum 1.8 is invariant under the transformation

$$R \longrightarrow \frac{\alpha'}{R}, \qquad n \longleftrightarrow w. \tag{1.9}$$

The exchange of winding number and momentum is equivalent to performing the transformation

$$p_L^{25} \longrightarrow p_L^{25}, \qquad p_R^{25} \longrightarrow -p_R^{25}. \tag{1.10}$$

Let us define a new field $Y^\mu(z, \bar{z})$ in the following way:

$$Y^\mu(z, \bar{z}) = X^\mu(z, \bar{z}), \qquad \mu = 0, ..., 24 \tag{1.11a}$$

$$Y^{25}(z, \bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z}) \tag{1.11b}$$

with $z, \bar{z}$ the complex coordinates $z = \sigma^1 + i\sigma^2, \bar{z} = \sigma^1 - i\sigma^2$. The only change in the conformal field theory of using Y^{25} instead of X^{25} is the sign change of equation 1.10. So, a theory with X^μ with X^{25} compactified on radius R is equivalent to a theory with Y^μ and Y^{25} compactified on a radius $\frac{\alpha'}{R}$. This equivalence is called T-duality. The space of inequivalent theories under this duality is $R \geq \sqrt{\alpha'}$.

For the open string with Neumann boundary conditions, there is no quantum number comparable to the winding number w. Since $m^2 \sim \frac{1}{R^2}$ for the open string, when

$R \to 0$ the states with momentum not equal to zero go to infinite mass. There is no new continuum of states as was the case for the closed string. Thus, the resulting states move in 25 spacetime dimensions. In more detail, the interior of the open string can still vibrate in 26 dimensions, but the endpoints have to be restricted to move in a 25-dimensional hyperplane. We can see this in the following way. From equation 1.11, we can check that

$$\partial_a X^{25} = \epsilon_{ab} \partial^b Y^{25}. \tag{1.12}$$

Thus, if X has Neumann boundary conditions in X^{25} then Y^{25} has Dirichlet boundary conditions ($\partial_1 Y^{25} = 0$ at $\sigma^2 = 0, \pi$) and vice-versa.

At first people rarely considered Dirichlet boundary conditions, because it is odd to consider fixed end points $X^\mu(\sigma^1, \bar{\sigma}^2) = c^\mu$, with $\bar{\sigma}^2 = 0, \pi$, since it breaks Lorentz invariance. However, as we have seen they follow in a natural way from T-duality. If we consider the following set up:

$$\begin{aligned} \partial_2 X^a &= 0, & a &= 0, ..., p \\ X^I &= c^I, & I &= p+1, ..., D-1 \end{aligned}$$

then we restrict the end-points of the string to lie in a $(p+1)$-dimensional hypersurface. The hypersurface is called a Dp-brane, with p the number of spatial dimensions. The Lorentz group is broken, so that it splits as

$$SO(1, D-1) \longrightarrow SO(1, p) \times SO(D-p-1).$$

If we apply T-duality to a circle transverse to a Dp-brane it will turn into a $D(p+1)$-brane. If we apply T-duality to a circle orthogonal to the Dp-brane, we recover a $D(p-1)$-brane. We will revisit T-duality when we talk about the different dualities that exist in superstring theory.

So far, we have discussed bosonic string theory. The fields on the world-sheet are γ_{ab} and the X^μ, which are bosonic. The spectrum contains a tachyon and it involves bosonic excitations only. Also, the theory is only Weyl invariant when requiring $D = 26$. We would like to get rid of the tachyon and we would like to have space-time fermions in our theory. We can do this by introducing fermionic excitations and requiring supersymmetry. This theory is only consistent in $D = 10$ dimensions. The resulting theory is called superstring theory. There are five different superstring theories, originating from a choice we make when adding fermions to the theory. These are:

Type I: closed superstrings with left- and right-moving fermion fields and $\mathcal{N} = 2$ supersymmetry + open superstrings with Neumann boundary conditions.

Type IIA and type IIB: closed superstrings with left- and right-moving fermion fields and $\mathcal{N} = 2$ supersymmetry + open superstrings with Neumann and Dirichlet boundary conditions.

Heterotic $SO(32)$ and $E_8 \times E_8$: closed superstrings with only right-moving fermion fields and $\mathcal{N} = 1$ supersymmetry.

Define the cylindrical coordinate $w = \sigma^1 + i\sigma^2$, with $w \cong w + 2\pi$. Let us introduce the matter fermion action:

$$\frac{1}{4\pi} \int d^2 w \left(\psi^\mu \partial_{\bar{w}} \psi_\mu + \tilde{\psi}^\mu \partial_w \tilde{\psi}_\mu \right). \tag{1.13}$$

If we require Lorentz invariance and periodicity under the cylindrical coordinates for the closed string, we are left with the following two possible conditions on the periodicity of the right-movers ψ^μ:

Ramond (R): $\psi^\mu(w + 2\pi) = +\psi^\mu(w)$

Neveu-Schwarz (NS): $\psi^\mu(w + 2\pi) = -\psi^\mu(w)$

Similarly for the left-movers $\tilde{\psi}^\mu$. The fact that ψ^μ and $\tilde{\psi}^\mu$ are spinors is apparent in these conditions, the spin group forms a double cover of the special orthogonal group. We can write

$$\psi^\mu(w + 2\pi) = \exp(2\pi i v)\psi^\mu(w) \tag{1.14a}$$

$$\tilde{\psi}^\mu(\bar{w} + 2\pi) = \exp(-2\pi i \tilde{v})\tilde{\psi}^\mu(\bar{w}) \tag{1.14b}$$

with $v, \tilde{v}$ taking values 0 or 1/2. Taking X^μ periodic, there are four different kinds of the closed superstring: NS-NS, NS-R, R-NS and R-R. We can denote these as $(v, \tilde{v})$.

In the open string, we need to satisfy the following boundary condition in order for the surface term in the equation of motion to vanish:

$$\psi^\mu(\sigma^1, 0) = \exp(2\pi i v)\tilde{\psi}^\mu(\sigma^1, 0), \qquad \psi^\mu(\sigma^1, \pi) = \exp(2\pi i v')\tilde{\psi}^\mu(\sigma^1, \pi). \tag{1.15}$$

We can combine ψ^μ and $\tilde{\psi}^\mu$ into a single field for the extended range $0 \leq \sigma^2 \leq 2\pi$:

$$\psi^\mu(\sigma^1, \sigma^2) = \begin{cases} \psi^\mu(\sigma^1, \sigma^2) & 0 \leq \sigma^2 \leq \pi \\ \tilde{\psi}^\mu(\sigma^1, 2\pi - \sigma^2) & \pi \leq \sigma^2 \leq 2\pi \end{cases}$$

The boundary condition 1.15 thus becomes a periodicity condition on the extended field, giving one field with the R or NS periodicity condition.

We can define an operator F, called the fermion number. This operator counts the number of fermionic excitations. The operator $exp(\pi i F) = (-1)^F$ is then invariant under the Lorentz generators. This means that states in the massless spectrum with even and odd fermionic number do not mix.

For the closed superstring, we thus get 16 different sectors, labeled by $(\alpha, F, \tilde{\alpha}, \tilde{F})$, where $\alpha = 1 - 2v$. By the level matching condition, $L_0 = \tilde{L}_0$, we cannot pair NS$-$ with NS$+$, R$-$ and R$+$. After imposing some consistency conditions, which can be found in [22, p. 26], we find the following remaining interesting sectors:

Type IIA: (NS+,NS+) (R+,NS+) (NS+,R$-$) (R+,R$-$),

Type IIB: (NS+,NS+) (R+,NS+) (NS+,R+) (R+,R+).

In order to obtain type IIA and IIB we perform a projection from the full spectrum down to the eigenspace of $\exp(\pi i F)$ and $\exp(\pi i \tilde{F})$ called the GSO projection. In type IIA we project onto all sectors with

$$\exp(\pi i F) = +1, \qquad \exp(\pi i \tilde{F}) = (-1)^{\tilde{\alpha}}, \tag{1.16}$$

while to obtain the type IIB theory we project onto all sectors with

$$\exp(\pi i F) = +1, \qquad \exp(\pi i \tilde{F}) = +1. \tag{1.17}$$

We will explicitly give the IIB action in section 1.3.

1.2 M-theory and the web of dualities

In this section we take a closer look at S- and T- duality and in what way they connect the superstring theories we discussed.

S-duality links weak- and strong string coupling regimes. It comes forth as an invariance of the action under a $SL(2,\mathbb{Z})$ transformation, which takes certain 2-form potentials into each other. Type IIB is self-dual under S-duality. In section 1.3 we not only give the action of type IIB superstring theory, but we also show how the fields in the action transform under this $SL(2,\mathbb{Z})$ invariance. The way in which the complexified coupling constant of the type IIB theory transforms under S-duality is actually one of the main reasons Cumrun Vafa explored the idea of F-theory. M-theory plays an important part in this story, since it allows for a completely geometric interpretation of F-theory. More on this later in the section on F-theory.

Starting with type IIA superstring theory, from the GSO projection 1.16 we know that the Ramond sectors do not have the same chirality on left and right movers. Let us compactify the $\mu = 9$ direction, we know that then under T-duality: $Y^9_R(\bar{z}) = -X^9_R(\bar{z})$. By superconformal invariance we must also reflect the fermionic right moving fields:

$$T : \tilde{\psi}^9(\bar{z}) \longrightarrow -\tilde{\psi}^9(\bar{z}).$$

Thus, the chirality of the right-moving Ramond sector is reversed. So, in this example we see that T-duality connects IIA and IIB, since they only differ in the Ramond sectors having different or the same chirality on left and right movers respectively. The same happens if we T-dualize an odd number of dimensions:

$$\text{IIA} \longrightarrow \text{IIB}, \qquad\qquad \text{IIB} \longrightarrow \text{IIA}.$$

If we T-dualize an even number of dimensions:

$$\text{IIA} \longrightarrow \text{IIA}, \qquad\qquad \text{IIB} \longrightarrow \text{IIB},$$

since the chiralities of Ramond sectors will stay the same or different on left and right movers in this case.

For the type I theory, T-duality will result in D-branes and other structures just as we have seen in section 1.1.

It can be shown that there exists a single superstring theory, all the different theories we mentioned are different limits of the parameter space on which this superstring theory lives. M-theory, an 11 dimensional theory, is also a limit of this single theory. By compactifying one its dimensions, we can link M-theory with type IIA and $E_8 \times E_8$ heterotic. Edward Witten conjectured M-theory, but left the meaning of the letter M open until a better formulation of the theory is known. The relations between the different superstring theories is shown in figure 1.1.

1.3 F-theory

F-theory is a 12-dimensional theory in string theory first proposed by Cumrun Vafa in [25]. It provides an elegant framework to analyse questions in IIB superstring theory. Let us analyse how F-theory arises naturally out of considerations within this framework.

$$\begin{tikzcd}
\text{Type I} \arrow[r, leftrightarrow] & SO(32) \arrow[r, leftrightarrow, dashed] & E_8 \times E_8 \arrow[dr, leftrightarrow] & \text{Type IIA} \arrow[r, leftrightarrow, dashed] \arrow[d, leftrightarrow] & \text{Type IIB} \arrow[loop right, leftrightarrow] \\
& & & \text{M-theory} &
\end{tikzcd}$$

Figure 1.1: A diagram depicting the various string theory dualities. The dashed lines depict T-duality, the full lines depict S-duality.

Type IIB supergravity in ten dimensions has $\mathcal{N} = 2$ supersymmetry and 32 supercharges. Let us write down an action for IIB which gives the correct equations of motion for the fields in question:

$$\begin{aligned} S_{\text{IIB}} = \frac{2\pi}{l_s^8} \Bigg[&\int d^{10}x \sqrt{-g} R \\ &-\frac{1}{2} \int \left(\frac{1}{(\text{Im } \tau)^2} d\tau \wedge *d\bar{\tau} + \frac{1}{\text{Im } \tau} G_3 \wedge *\bar{G}_3 + \frac{1}{2} \tilde{F}_5 \wedge *\tilde{F}_5 + C_4 \wedge H_3 \wedge F_3 \right) \Bigg], \end{aligned}$$

where we define

$$\begin{aligned} \tau &= C_0 + ie^{-\phi}, \\ G_3 &= F_3 - \tau H_3, \\ \tilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3, \\ F_p &= dC_{p-1} \qquad \text{for } p = 1, 3, 5, \\ H_3 &= dB_2, \end{aligned}$$

and use the ten dimensional Einstein frame metric, l_s is the string length. To get all the equations of motion, one also needs the selfduality constraint $\tilde{F}_5 = *\tilde{F}_5$. The action S_{IIB} of type IIB is invariant under the $SL(2, \mathbb{Z})$-duality or S-duality as was mentioned before. If we take an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \tag{1.18}$$

the fields in the action transform as follows:

$$\begin{aligned} \tau &\mapsto \frac{a\tau + b}{c\tau + d}, \\ \begin{pmatrix} H \\ F \end{pmatrix} &\mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} H \\ F \end{pmatrix}, \\ \tilde{F}_5 &\mapsto F_5, \\ g_{MN} &\mapsto g_{MN}. \end{aligned}$$

By varying each term of the action separately, we can quite easily show its invariance under the above transformation. We can make a few remarks now. First of all, notice that τ transforms under S-duality as the modulus of a two-torus under the geometrical $SL(2,\mathbb{Z})$ reparametrization gauge symmetry of the torus. For more information about the geometry of a torus, see chapter 4.1. So, if we interpret τ as being the modulus of a two-torus, changing the modulus by a $SL(2,\mathbb{Z})$-transformation will amount to doing a S-duality transformation on the axion-dilaton field. We might thus obtain this action by compactifying a 12-dimensional theory along this torus with modulus τ, where F_3 and H_3 perform the role as components of some twelve dimensional form $\hat{F}_4$ reduced along one of the two 1-cycles of the two-torus respectively.

These observations do not lead very straightforwardly to a 12-dimensional theory. There are some problems with this approach, the first one being that there exists no theory of 12-dimensional supergravity with metric signature $(1,11)$. Also, reducing the proposed $\hat{F}_4$ along T^2 would result in a 2-form fields strength in the 10-dimensional IIB action. We do not see this field strength. We can circumvent these problems by using a different approach altogether; making use of the web of dualities we mentioned briefly above. We start with M-theory on a small two-torus with modulus τ, with this M-theory living on a 9-dimensional manifold M_9 with a T^2 fibration over this manifold. Since we have two 1-cycles on T^2, we can take one of the two circles to be the M-theory circle, which gives a weakly coupled IIA on the remaining small circle. We now use T-duality on this small circle, to obtain IIB on a large circle. When we take the limit of vanishing T^2 in M-theory, this becomes an uncompactified IIB-theory. If we perform this procedure fiberwise, we obtain T^2 fibrations. This thus gives us IIB compactifications with varying τ interpreted as either the modulus of T^2 or as a varying axion-dilaton field. We thus obtain the F-theory idea through this series of performed dualities.

Next, let us study F-theory from this M-theory perspective. We start by giving the 11-dimensional low energy action of M-theory:

$$S_M = \frac{2\pi}{l_M^9}\left[\int d^{11}x\sqrt{-g}R - \frac{1}{2}\int\left(G_4\wedge *G_4 - \frac{1}{6}C_3\wedge G_4\wedge G_4\right) + ...\right], \tag{1.19}$$

with $G_4 = dC_3$ and l_M the Planck length in 11 dimensions. We can compactify this theory on $M_9\times T^2$, with the metric given by

$$ds_M^2 = \frac{v}{\tau_2}\left((dx+\tau_1 dy)^2 + \tau_2^2 dy^2\right) + ds_9^2. \tag{1.20}$$

Here, x, y are coordinates on the torus with periodicity one. Furthermore, the torus is parametrized by area v and modulus $\tau = \tau_1 + i\tau_2$. We can let v and τ depend on the coordinates of the manifold M_9, we then get a T^2-fibration. We have two 1-cycles: one around the x-direction and one around the y-direction. We will call the first the A-cycle and the second the B-cycle, since we will reduce from M to IIA along the A-cycle and subsequently T-dualize along the B-cycle to obtain IIB. We can write down a relation between the metric of M-theory compactified on a circle and type IIA metrics:

$$ds_M^2 = L^2 e^{\frac{4\chi}{3}}(dx + C_1)^2 + e^{-\frac{2\chi}{3}} ds_{\mathrm{IIA}}^2. \tag{1.21}$$

Here L is a length scale that gives the size of the M-theory circle, furthermore:

$$\begin{aligned} C_1 &= \tau_1 dy, \\ e^{\frac{4\chi}{3}} &= \frac{v}{L^2\tau_2}, \\ ds^2_{\mathrm{IIA}} &= \frac{\sqrt{v}}{L\sqrt{\tau_2}}(v\tau_2 dy^2 + ds_9^2). \end{aligned}$$

To go from IIA to IIB, we now want to T-dualize the geometry described above. As we have discussed T-duality maps IIA to IIB, while sending the circle length L_A to $L_B = \frac{l_s}{L_A}$, the string coupling changes to $g_{\mathrm{IIB}} = \frac{l_s}{L_A} g_{\mathrm{IIA}}$ and $C_0 = (C_1)_y$. We find as a final result:

$$\begin{aligned} C_0 + \frac{i}{g_{\mathrm{IIB}}} &= \tau, \\ ds^2_{\mathrm{IIB},E} &= \frac{\sqrt{v}}{L}\left(\frac{L^2 l_s^4}{v^2} dy^2 + ds_9^2\right), \end{aligned}$$

in the Einstein frame. We now make a further assumption that $M_9 = \mathbb{R}^{1,2} \times B_6$, with B_6 a Kähler manifold of complex dimension three. We furthermore ask that T^2 depends on the coordinates of B_6. As we noted earlier, this gives us an elliptic fibration. To obtain a supersymmetryic solution the resulting space must be Calabi-Yau. In elleptic fibrations τ will depend holomorphically on the base coordinates, while v remains constant. We can thus set $L = \sqrt{v}$. So that in the Einstein frame the metric gets a nice form:

$$ds^2_{\mathrm{IIB}} = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + \frac{l_s^4}{v} dy^2 + ds^2_{B_6}. \tag{1.22}$$

If we now send $v \to 0$, all the while keeping l_s at a finite value, we find that the theory decompactifies to $\mathbb{R}^{1,3} \times B_6$. We have a non-trivial dilaton $\tau(b)$, with $b \in B_6$. The IIB configuration we obtain will be supersymmetric since we started from a supersymmetric configuration in the M-theory case and obtained our new configuration by applying dualities. Note that what was part of the Calabi-Yau in M-theory becomes part of a noncompact and visible space in IIB, where we also get Lorentz invariance in the $v \to 0$ limit.

In conclusion, F-theory compactified on an elliptic fibration means the type IIB geometry obtained by compactifying M-theory on the same elliptic fibration and following the procedure above in the limit of vanishing elliptic fiber size v. We will use this viewpoint and some ideas in this section in later chapters. They will form an important part in the topological duality twisting we will perform later.

Chapter 2

Calabi-Yau manifolds

In this chapter we will discuss Calabi-Yau spaces. We start with an introduction that uses concepts from differential geometry to help define what a Calabi-Yau manifold is. After giving multiple definitions, we will discuss the Hodge structure of these spaces. It will become apparent that the Hodge diamond for Calabi-Yau threefolds (the main interest of this paper) can be written in a simplified form. We will perform a quick calculation to determine the Hodge diamond of the quintic.

In the mathematical part of this thesis we will construct Calabi-Yau manifolds using Batyrev's construction. This technique constructs Calabi-Yau spaces as subspaces of certain projective spaces. We will analyse the Calabi-Yau condition that these subspaces need to abide by in order for them to be able to be Calabi-Yau spaces.

For an overview of the techniques and definitions used in this chapter, the reader is advised to consult [10], [13] and [6].

2.1 Calabi-Yau manifolds from differential geometry

Let us begin this chapter by recalling some definitions.

Definition 1. If $\mathcal{M}$ is a real manifold of dimension $2m$, with a given open cover $\{U_i\}_{i\in I}$ and for each open subset U_i a coordinate chart $\psi_i : U_i \to \mathbb{C}^m$ with ψ_i a homeomorphism, then $(\mathcal{M}, \{U_i, \psi_i\})$ determines a complex manifold if on the intersections the transition maps are holomorphic maps from $\mathbb{C}^m$ to $\mathbb{C}^m$.

Another definition for a complex manifold can be given in terms of an (almost) complex structure and the corresponding Nijenhuis tensor:

Definition 2. Take $\mathcal{M}$ a real manifold of dimension $2m$. Take $J \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ to be the almost complex structure on $\mathcal{M}$. Define the Nijenhuis tensor to be:

$$\mathcal{N}_J(v, w) = [v, w] + J[v, Jw] + J[Jv, w] - [Jv, Jw]$$

with $v, w \in \Gamma(T\mathcal{M})$. If $\mathcal{N}_J \equiv 0$ then J is a complex structure on $\mathcal{M}$ and $(\mathcal{M}, J)$ is a complex manifold.

These definitions are equivalent. Take $(\mathcal{M}, J)$ a complex manifold. We can always define a Riemannian metric on a manifold, so let us define g to be the Riemannian metric on $\mathcal{M}$.

Definition 3. A Riemannian metric g on a complex manifold $(\mathcal{M}, J)$ is called a hermitian metric, if the following condition holds:

$$g(v, w) = g(Jv, Jw)$$

for all $v, w \in \Gamma(T\mathcal{M})$. In other words,

$$g : T^{(1,0)}\mathcal{M} \otimes T^{(0,1)}\mathcal{M} \to \mathbb{C}$$

is positive definite.

The above condition is called the hermitian condition. We can now define a hermitian form:

Definition 4. If $(\mathcal{M}, J)$ a complex manifold with hermitian metric g, then a hermitian form on $\mathcal{M}$ is defined by:

$$\omega(v, w) = g(Jv, w),$$

with $v, w \in \Gamma(T\mathcal{M})$.

We can now define what is called a Kähler manifold.

Definition 5. Take $(\mathcal{M}, J)$ to be a complex manifold, with hermitian metric g and ω the corresponding hermitian form. If $d\omega = 0$, g is called a Kähler metric, ω the Kähler form and $\mathcal{M}$ a Kähler manifold.

An example of a Kähler manifold is $\mathbb{CP}^m$. Note that all submanifolds of Kähler manifolds are Kähler manifolds in their own right, since restriction of the Kähler form to a complex submanifold is also a closed, positive, $(1, 1)$-form. In particular this means that all submanifolds of complex projective space are Kähler. We can now define a Calabi-Yau manifold.

Definition 6. A Calabi-Yau manifold of real dimension $2m$ is a compact Kähler manifold $(\mathcal{M}, J, g)$ with one of the following properties:

1. zero Ricci form;
2. vanishing first Chern class;
3. $Hol(g) = SU(m)$;
4. with trivial canonical bundle;
5. admits a globally defined and nowhere vanishing holomorphic m-form.

2.2 Hodge structure

In this section we discuss the Hodge structure of a Calabi-Yau manifold. We start with some general definitions in de Rham and Dolbeault cohomology, after that we determine the Hodge structure for Calabi-Yau manifolds. Finally, we discuss the example of the quintic threefold as a hypersurface in $\mathbb{CP}^4$.

2.2.1 Cohomology

A complex manifold $\mathcal{M}$ has an exterior derivative $d : \Lambda^k T^*\mathcal{M} \to \Lambda^{k+1} T^*\mathcal{M}$, which is closed: $d^2 = 0$. Using this, we can define the de Rham complex as the cochain complex of exterior differential forms on $\mathcal{M}$. The de Rham complex is given by:

$$0 \to \Lambda^0 T^*\mathcal{M} \to \Lambda^1 T^*\mathcal{M} \to \Lambda^2 T^*\mathcal{M} \to \cdots \to \Lambda^{\dim \mathcal{M}} T^*\mathcal{M} \to 0$$

with the arrows the exterior derivative. We say that two exterior forms are equivalent if they differ by an exact form. We now define the de Rham cohomology group $H^k_{\mathrm{dR}}(\mathcal{M})$ to be the set of equivalence classes under this equivalence, that is the set of closed forms in $\Lambda^k T^*\mathcal{M}$ modulo the exact ones.

We can now define the Betti numbers to be

$$b^k = \dim_{\mathbb{R}} H^k_{\mathrm{dR}}(\mathcal{M}, \mathbb{R}), \tag{2.1}$$

and the Euler class χ to be equal to

$$\chi = \sum_{k=0}^{\dim(\mathcal{M})} (-1)^k b^k. \tag{2.2}$$

The above story holds true for real manifolds. However, since we are working on complex manifolds, we can split up the exterior derivative and exterior forms into holomorphic and anti-holomorphic parts. We write

$$\Lambda^k T^*_{\mathbb{C}}\mathcal{M} = \bigoplus_{j=0}^{k} \Lambda^{j,k-j}\mathcal{M},$$

with

$$\Lambda^{p,q}\mathcal{M} = \Lambda^p T^{*(1,0)}\mathcal{M} \otimes \Lambda^q T^{*(0,1)}\mathcal{M},$$

and $d = \partial + \bar{\partial}$. Here,

$$\partial : \Lambda^{p,q}\mathcal{M} \to \Lambda^{p+1,q}\mathcal{M}, \qquad \text{and} \qquad \bar{\partial} : \Lambda^{p,q}\mathcal{M} \to \Lambda^{p,q+1}\mathcal{M}.$$

One can easily show that $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. We can now use this last property to define the Dolbeault cohomology group in the same fashion as we did for the de Rham cohomology group, now however the differential of the cochain will be $\bar{\partial}$. We obtain:

$$0 \to \Lambda^{p,0}\mathcal{M} \to \Lambda^{p,1}\mathcal{M} \to \Lambda^{p,2}\mathcal{M} \to \cdots \to \Lambda^{p,\dim_{\mathbb{C}} \mathcal{M}}\mathcal{M} \to 0$$

and we define the (p,q)-th Dolbeault cohomology group to be

$$H^{p,q}_{\bar\partial}(\mathcal{M}) = \frac{\ker(\bar\partial : \Lambda^{p,q}\mathcal{M} \to \Lambda^{p,q+1}\mathcal{M})}{\operatorname{im}(\bar\partial : \Lambda^{p,q-1}\mathcal{M} \to \Lambda^{p,q}\mathcal{M})}. \tag{2.3}$$

We define the Hodge numbers to be

$$h^{p,q} = \dim H^{p,q}_{\bar\partial}(\mathcal{M}).$$

2.2.2 Hodge diamond

In this subsection, we make some general remarks on the Hodge structure of a Calabi-Yau threefold. Denote the threefold by $\mathcal{M}$. Then, the relevant Hodge diamond is given by:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}. \tag{2.4}$$

Our goal is to determine all the Hodge numbers and thus complete this diamond.

For a Calabi-Yau threefold, the Hodge diamond immediately simplifies. We know that the canonical bundle is trivial, i.e. $\Lambda^{3,0}\mathcal{M} = \mathcal{M} \times \mathbb{C}$. So, $h^{3,0} = 1$. Thus, there exists a unique holomorphic volume form $\Omega \in \Lambda^{3,0}M$. For every cohomology class $[\alpha]$ given by $\alpha \in \Lambda^{0,p}M$ there exists a unique cohomology class $[\beta]$ with $\beta \in \Lambda^{0,3-p}M$, with $p = 0, 1, 2, 3$, such that

$$\int_{\mathcal{M}} \alpha \wedge \beta \wedge \Omega = 1.$$

So, $h^{o,p} = h^{0,3-p}$. We already had the equalities $h^{p,q} = h^{q,p}$ (by applying complex conjugation) and $h^{p,q} = h^{3-p,3-q}$ (by applying Hodge star). So, since $h^{3,0} = 1$, we have:

$$h^{3,0} = h^{0,3} = h^{0,0} = h^{3,3} = 1.$$

Furthermore, assuming $\mathcal{M}$ is simply connected, we find $b_1 = 0$. This is because if $\mathcal{M}$ is simply connected, any closed path is path homotopic to a constant loop. So, the integral over a closed path is equal to the constant integral, so zero. So, the integral of a one-form around any closed loop is zero. So every one-form on $\mathcal{M}$ is conservative. We know that closed one-forms are exact if and only if they are conservative, thus every closed one-form is exact. Hence $H^1(M) = 0 = b_1$. Thus $h^{0,1} = h^{1,0} = 0$ since $b_1 = h^{1,0} + h^{0,1}$ and they are non-negative. We find

$$h^{1,0} = h^{0,1} = h^{0,2} = h^{2,0} = h^{2,3} = h^{3,2} = h^{3,1} = h^{1,3} = 0.$$

Thus, our Hodge diamond in the case of a simply connected Calabi-Yau manifold $\mathcal{M}$. becomes:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{2.5}$$

We will now apply this story the example of the quintic. What is thus left is determining $h^{1,1}$ and $h^{1,2}$.

2.2.3 Quintic threefold

In this section we will determine the Hodge diamond for a quintic threefold in a four dimensional complex projective space $\mathbb{CP}^4$. We have:

$$\mathbb{CP}^4 = \{[x_0 : x_1 : x_2 : x_3 : x_4] | x_0, ..., x_4 \in \mathbb{C} \setminus \{0\}\},$$

with the equivalence classes given by

$$[x_0 : x_1 : x_2 : x_3 : x_4] = \{(y_0, y_1, y_2, y_3, y_4) \in \mathbb{C}^5 \setminus \{0\}\}/\sim,$$

and the equivalence relation given by

$$\lambda(x_0, x_1, x_2, x_3, x_4) \sim (y_0, y_1, y_2, y_3, y_4), \qquad ,$$

for $(x_0, x_1, x_2, x_3, x_4), (y_0, y_1, y_2, y_3, y_4) \in \mathbb{C}^5 \setminus \{0\}$. A quintic threefold is a degree five, dimension three hypersurface in $\mathbb{CP}^4$. An example is Fermat's quintic threefold, given by the equation $f_F([v : w : x : y : z]) = v^5 + w^5 + x^5 + y^5 + z^5 = 0$. Denote the quintic threefold by X.

Next, let us state and use the Lefschetz hyperplane theorem to determine $h^{1,1} = h^{2,2}$.

Theorem 1. *Let V be an n-dimensional complex projective algebraic variety in $\mathbb{CP}^N$, and let W be a hyperplane section of V such that $V \setminus W$ is smooth. Then the natural map*

$$H_k(W, \mathbb{Z}) \to H_k(V, \mathbb{Z})$$

in singular homology is an isomorphism for $k < n - 1$ and is surjective for $k = n - 1$.

In our case, $V = \mathbb{CP}^4$, so $n = 4$. So, for $k = 0, 1, 2$ the map in the theorem is an isomorphism. Now,

$$H_p(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p \text{ even}, 0 \leq p \leq 2n \\ 0, & \text{otherwise} \end{cases} .$$

So, $H_0(X) = H_0(\mathbb{CP}^4) = \mathbb{Z}$, $H_1(X) = H_1(\mathbb{CP}^4) = 0$ and $H_2(X) = H_2(\mathbb{CP}^4) = \mathbb{Z}$. Thus, $b_0 = b_2 = 1$ and $b_1 = 0$. Let us use $b_2 = 1$:

$$b_2 = 1 = h^{2,0} + h^{1,1} + h^{0,2} = 2h^{2,0} + h^{1,1}$$

This has as the only solution $h^{2,0} = 0$ (which we found earlier through other means) and $h^{1,1} = 1$. Through symmetry we thus find $h^{1,1} = 1 = h^{2,2}$. We now still need to find $h^{2,1} = h^{1,2}$.

We will need some results for this. First off, we have the following relationship between the Euler characteristic and the Betti numbers of our quintic:

$$\chi = \sum_{k=0}^{2n} (-1)^k b_k, \tag{2.6}$$

and between the Euler characteristic and the top Chern class of our threefold:

$$\chi = \int_X c_3(X). \tag{2.7}$$

Equation 2.6 implies that

$$\begin{aligned}\chi &= b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6 = 2b_0 - 2b_1 + 2b_2 - b_3 \\ &= 2 - 0 + 2(h^{1,1} + h^{2,0} + h^{0,2}) - (h^{3,0} + h^{0,3} + h^{1,2} + h^{2,1}) \\ &= 2 + 2h^{1,1} - 2 - 2h^{2,1} = 2(h^{1,1} - h^{2,1}) \\ &= 2(1 - h^{2,1}),\end{aligned}$$

where in the second step we used Poincaré duality. So, we have found that

$$\int_X c_3(X) = 2(1 - h^{2,1}). \tag{2.8}$$

We now want to find an expression for $c_3(X)$.

We have the following exact sequence, called an Euler sequence, given by

$$0 \longrightarrow \Omega_{\mathbb{P}^n_A/\mathrm{Spec}\ A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0, \tag{2.9}$$

for A a ring. Taking the dual of this exact sequence, we obtain:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(1)^{n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n_A} \longrightarrow 0, \tag{2.10}$$

which is also exact.

Remark 1. Note that for an exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

the dual sequence is given by

$$0 \longrightarrow Z^* \longrightarrow Y^* \longrightarrow X^* \longrightarrow 0.$$

A sequence is exact if and only if its dual sequence is exact.

We know from the properties of Chern classes that if we have an exact sequence of locally free sheaves on any nonsingular quasi-projective variety given by $0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$, then $c_t(\mathcal{E}) = c_t(\mathcal{E}') \cdot c_t(\mathcal{E}'')$. Thus, in our case

$$c(\mathcal{O}_{\mathbb{P}^n_A}(1)^{n+1}) = c(\mathcal{O}_{\mathbb{P}^n_A}) \cdot c(\mathcal{T}_{\mathbb{P}^n_A})$$

Now $c(\mathcal{O}_{\mathbb{CP}^n}) = 1 + 0H = 1$, with $H \in A^1(X)$ the class of a hyperplane section of X. Furthermore,

$$\mathcal{O}_{\mathbb{P}^n_A}(1)^{n+1} = \underbrace{\mathcal{O}_{\mathbb{P}^n_A}(1) \bigoplus \cdots \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(1)}_{n+1 \text{ times}},$$

so $\mathcal{O}_{\mathbb{P}^n_A}(1)^{n+1}$ splits trivially into $n+1$ copies of the invertible sheave $\mathcal{O}_{\mathbb{P}^n_A}(1)$, so $c(\mathcal{O}_{\mathbb{P}^n_A}(1)^{n+1}) = c(\mathcal{O}_{\mathbb{P}^n_A}(1))^{n+1}$ by the properties of Chern classes. Thus, we find

$$c(\mathcal{T}_{\mathbb{CP}^n}) = c(\mathcal{O}_{\mathbb{CP}^n}(1))^{n+1}. \tag{2.11}$$

Now, $\mathcal{O}_{\mathbb{CP}^n}(1)$ is a line bundle, thus $c(\mathcal{O}_{\mathbb{CP}^n}(1)) = 1 + c_1(\mathcal{O}_{\mathbb{CP}^n}(1))$. Writing

$$c_1(\mathcal{O}_{\mathbb{CP}^n}(1)) = x,$$

we thus find

$$c(\mathcal{T}_{\mathbb{CP}^n}) = (1+x)^{n+1}. \tag{2.12}$$

If X is a smooth hypersurface in $\mathbb{CP}^n$ defined as the zero locus of a degree d polynomial, we have an exact sequence given by

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \underbrace{\mathcal{T}_{\mathbb{CP}^n} \otimes \mathcal{O}_X}_{=\mathcal{T}_{\mathbb{CP}^n}|_X} \longrightarrow \underbrace{\mathcal{N}_{X/\mathbb{CP}^n}}_{=\mathcal{O}_{\mathbb{CP}^n}(d)|_X} \longrightarrow 0. \tag{2.13}$$

So, $c(\mathcal{T}_{\mathbb{CP}^n}) = c(X) \cdot c(\mathcal{O}_{\mathbb{CP}^n}(d))$. Now, $c(\mathcal{O}_{\mathbb{CP}^n}(d)) = 1 + dH = 1 + dx$. So,

$$c(X) = \frac{(1+x)^{n+1}}{1+dx}$$

Let us Taylor this expression:

$$c(X) = (1 + (n+1)x + \ldots)(1 - dx + \ldots) = 1 - dx + (n+1)x + \ldots$$

We obtain the Calabi-Yau condition for X, namely that $n - d + 1 = 0$. Since in our case $n = 4$ and $d = 5$, we find that the first Chern class of the quintic X vanishes and that indeed X is a Calabi-Yau threefold in $\mathbb{CP}^4$! Thus, indeed our initial simplification of the general Hodge diamond for threefolds to a diamond for a Calabi-Yau threefold was justified.

Let us calculate the remaining Hodge numbers $h^{2,1} = h^{1,2}$. First we derived the total Chern class for X:

$$\begin{aligned} c(X) &= \frac{(1+x)^5}{1+5x} \\ &= \left(1 + 5x + \frac{5(5-1)}{2}x^2 + \frac{5(5-1)(5-2)}{3!}x^3\right)\left(1 - 5x + 25x^2 - 125x^3\right) \\ &= \left(1 + 5x + 10x^2 + 10x^3\right)\left(1 - 5x + 25x^2 - 125x^3\right) \\ &= 1 + 10x^2 - 40x^3 \end{aligned}$$

So, $c_3(X) = -40x^3$. Thus, looking back at equation 2.7, we now compute

$$\chi = \int_X c_3(X) = \int_X (-40x^3) = \int_{\mathbb{CP}^n} (-40x^3) \wedge \eta_X$$

where in the last equality we used Poincaré duality. What is the form η_X? If $\mathcal{N}_X = E|_X$ of E bundle over $\mathbb{CP}^n$, then it follows that $\eta_X = c_k(E)$ with k the rank of the bundle E. In our case, we have $E = \mathcal{O}_{\mathbb{CP}^4}(5)$, so $\eta_X = c_1(\mathcal{O}_{\mathbb{CP}^4}(5)) = 5x$. Thus,

$$\chi = \int_{\mathbb{CP}^4} (-40x^3) \wedge \eta_X = -200 \int_{\mathbb{CP}^4} x^4.$$

Since x is Poincaré dual to a hyperplane in $\mathbb{CP}^4$ and four hyperplanes intersect at a point, we find that $\int_{\mathbb{CP}^4} x^4 = 1$. Thus, using equation 2.8:

$$2(h^{1,1} - h^{2,1}) = -200 \Leftrightarrow h^{1,1} + 100 = h^{2,1}.$$

Thus, we conclude that $h^{2,1} = h^{1,2} = 101$. The Hodge diamond for the quintic X becomes:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}.$$

Note that it took a long calculation to determine the missing Hodge numbers in this diamond. When we discuss Batyrev's construction, we will see that we can produce Calabi-Yau manifolds inside a toric ambient space. We will get the missing Hodge numbers as part of this construction. They are derived directly from the toric data used to describe the ambient space.

2.3 Calabi-Yau condition

In the previous example of the quintic, we needed a calculation to determine $c_1(X)$. The first Chern class vanishing is one the Calabi-Yau conditions and we thus now spend some time on calculating the Calabi-Yau conditions for hypersurfaces in various (products of) (weighted) projective spaces.

2.3.1 CY condition for surfaces in $\mathbb{CP}^n$

Suppose that we are looking at a surface embedded in $\mathbb{CP}^n$, defined polynomials $\{p_i\}_{i \in I}$ with $I = \{1, ..., l\}$ and $\deg p_i = d_i$. Again, we have an exact sequence:

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \underbrace{\mathcal{T}_{\mathbb{CP}^n} \otimes \mathcal{O}_X}_{=\mathcal{T}_{\mathbb{CP}^n}|_X} \longrightarrow \mathcal{N}_{X/\mathbb{CP}^n} \longrightarrow 0.$$

We would like to show, using the above exact sequence, that

$$c(X) = \frac{(1+x)^{n+1}}{\prod_{i=1}^{l}(1 + d_i x)}, \tag{2.14}$$

and from here derive the Calabi-Yau condition for such embedded surfaces in $\mathbb{CP}^n$. We again have $c(\mathcal{T}_{\mathbb{CP}^n}) = (1+x)^{n+1}$ by the Euler sequence. Furthermore, by the defining exact sequence for the normal bundle:

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathbb{CP}^n} \bigotimes \mathcal{O}_X \longrightarrow \mathcal{N}_{X/\mathbb{CP}^n} \longrightarrow 0,$$

we find that $c(\mathcal{T}_{\mathbb{CP}^n}) = c(X) \cdot c(\mathcal{N}_{X/\mathbb{CP}^n})$. Now, if $\mathcal{N}_{X/\mathbb{CP}^n}$ admits a filtration of the form:

$$\mathcal{N}_{X/\mathbb{CP}^n} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq ... \supseteq \mathcal{E}_l = 0$$

such that $\mathcal{E}_{i-1}/\mathcal{E}_i = \mathcal{O}_{\mathbb{CP}^n}(d_i)$ for $i \in I$. Then, by properties of Chern classes, we find that

$$c(\mathcal{N}_{X/\mathbb{CP}^n}) = \prod_{i=1}^{l} c(\mathcal{O}_{\mathbb{CP}^n}(d_i)) = \prod_{i=1}^{l} (1 + d_i x).$$

Let us find this filtration. Consider the following sequence of subvarieties:

$$\mathbb{CP}^n = X_0 \supseteq X_1 \supseteq ... \supseteq X_{l-1} \supseteq X_l = X,$$

where X_i is the hypersurface in $\mathbb{CP}^n$ generated by $f_1, f_2, ..., f_i$. This induces the following sequence of subsheaves:

$$\mathcal{T}_{\mathbb{CP}^n} = \mathcal{T}_{X_0} \supseteq \mathcal{T}_{X_1} \supseteq ... \supseteq \mathcal{T}_{X_{l-1}} \supseteq \mathcal{T}_{X_l} = \mathcal{T}_X.$$

From this, we get the following sequence of subsheaves

$$0 = \mathcal{N}_{\mathbb{CP}^n/\mathbb{CP}^n} = \mathcal{N}_{X_0/\mathbb{CP}^n} \subseteq \mathcal{N}_{X_1/\mathbb{CP}^n} \subseteq ... \subseteq \mathcal{N}_{X_{l-1}/\mathbb{CP}^n} \subseteq \mathcal{N}_{X_l} = \mathcal{N}_{X/\mathbb{CP}^n}.$$

It follows that $\mathcal{N}_{X_i/\mathbb{CP}^n}/\mathcal{N}_{X_{i-1}/\mathbb{CP}^n} = \mathcal{N}_{X_i/X_{i-1}}$. We see that defining

$$\mathcal{E}_i = \mathcal{N}_{X_{l-i}/\mathbb{CP}^n}, \tag{2.15}$$

gives us

$$\mathcal{E}_{i-1}/\mathcal{E}_i = \mathcal{N}_{X_{l-i+1}/X_{l-i}}. \tag{2.16}$$

Now, by definition of the X_i with $i \in I$, we have that $\mathcal{N}_{X_{l-i+1}/X_{l-i}} = \mathcal{O}_{\mathbb{CP}^n}(d_{l-i+1})$. Thus, the successive quotients of the $\mathcal{E}_i$ are all invertible sheaves. By a property of Chern classes, we thus find for the total Chern class of the normal sheaf:

$$c(\mathcal{N}_X/\mathbb{CP}^n) = \prod_{i=1}^{l} c(\mathcal{O}_{\mathbb{CP}^n}(d_{l-i+1})) = \prod_{i=1}^{l} c(\mathcal{O}_{\mathbb{CP}^n}(d_i)) = \prod_{i=1}^{l} (1 + d_i x).$$

Thus,

$$\begin{aligned} c(X) &= \frac{(1+x)^{n+1}}{\prod_{i=1}^{l}(1 + d_i x)} \\ &= (1 + (n+1)x + ...) \prod_{i=1}^{l} (1 - d_i x + ...) \\ &= 1 + (n+1)x - \sum_{i=1}^{l} d_i x + ... \end{aligned}$$

So, in order for $c_1(X) = 0$, we need

$$\sum_{i=1}^{l} d_i = n + 1. \tag{2.17}$$

For $l = 1$ this reduces to the Calabi-Yau condition derived in section 2.2.3.

2.3.2 CY condition for surfaces in $\mathbb{CP}^{n_1} \times ... \times \mathbb{CP}^{n_r}$

In this section, we try to generalize what we have seen so far to the case where X is a hypersurface embedded in $M = \mathbb{CP}^{n_1} \times ... \times \mathbb{CP}^{n_r}$, defined by equations $f_1, ..., f_l$ with $\deg f_i = d_i = (d_{i1}, ..., d_{ir}) \in \mathbb{N}^r$. This means that if $p \in M$ with $p = (x_1, ..., x_r)$ with $x_i \in \mathbb{CP}^{n_i}$, and $x_i = [y_{i0} : y_{i1} : ... : y_{in_i}]$ with $y_{ij} \in \mathbb{C}$ for $i \in \{1, ..., r\}$ and $j \in \{1, ..., n_i\}$, then:

$$\begin{aligned} &f_j(y_{10}, y_{11}, ..., y_{1n_1}, ..., y_{(i-1)n_{i-1}}, \lambda y_{i0}, ..., \lambda y_{in_i}, y_{(i+1)0}, ..., y_{rn_r}) \\ &= \lambda^{d_{ji}} f_j(y_{10}, ..., y_{i0}, ..., y_{in_i}, ..., y_{rn_r}), \end{aligned}$$

for $\lambda \in \mathbb{R} \setminus \{0\}$.

Now, $X \subseteq M$ can be written as $X = X_1 \times ... \times X_r$ with $X_i \subseteq \mathbb{CP}^{n_i}$ for $i \in \{1, ..., r\}$. So, we have the following two exact sequences for all $i \in \{1, ..., r\}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{CP}^{n_i}} \longrightarrow \mathcal{O}(1)^{n_i+1} \longrightarrow \mathcal{T}_{\mathbb{CP}^{n_i}} \longrightarrow 0, \tag{2.18a}$$

$$0 \longrightarrow \mathcal{T}_{X_i} \longrightarrow \mathcal{T}_{\mathbb{CP}^{n_i}} \otimes \mathcal{O}_{X_i} \longrightarrow \mathcal{N}_{X_i/\mathbb{CP}^{n_i}} \longrightarrow 0. \tag{2.18b}$$

If we take a direct sum over $i \in \{1, ..., r\}$ for the upper and lower exact sequence separately, we find:

$$0 \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{CP}^{n_i}} \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}(1)^{n_i+1} \longrightarrow \mathcal{T}_M \longrightarrow 0, \tag{2.19a}$$

$$0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_M \otimes \mathcal{O}_X \longrightarrow \mathcal{N}_{X/M} \longrightarrow 0. \tag{2.19b}$$

Now, using properties of Chern classes:

$$c(\bigoplus_{i=1}^{r} \mathcal{O}(1)^{n_i+1}) = c(\mathcal{T}_M) \cdot c(\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{CP}^{n_i}}) = c(\mathcal{T}_M) \cdot \prod_{i=1}^{r} c(\mathcal{O}_{\mathbb{CP}^{n_i}}) = c(\mathcal{T}_M),$$

and

$$c(\bigoplus_{i=1}^{r} \mathcal{O}(1)^{n_i+1}) = \prod_{i=1}^{r} c(\mathcal{O}(1)^{n_i+1}) = \prod_{i=1}^{r} c(\mathcal{O}(1))^{n_i+1} = \prod_{i=1}^{r} (1+x)^{n_i+1},$$

so we find:

$$c(\mathcal{T}_M) = (1+x)^{r+\sum_{i=1}^{r} n_i}. \tag{2.20}$$

Furthermore, using what we derived in section 2.3.1, we find:

$$c(\mathcal{N}_{X/M}) = c(\bigoplus_{i=1}^{r} \mathcal{N}_{X_i/\mathbb{CP}^{n_i}}) = \prod_{i=1}^{r} c(\mathcal{N}_{X_i/\mathbb{CP}^{n_i}}) = \prod_{i=1}^{r} \prod_{j=1}^{l} (1 + d_{ji}x). \tag{2.21}$$

Thus, from $c(\mathcal{T}_M) = c(X) \cdot c(\mathcal{N}_{X/M})$ we derive:

$$c(X) = \frac{(1+x)^{r+\sum_{i=1}^{r} n_i}}{\prod_{i=1}^{r}\prod_{j=1}^{l}(1+d_{ji}x)}. \tag{2.22}$$

Let us now Taylor expand this to arrive at the desired Calabi-Yau condition:

$$\begin{aligned} c(X) &= \frac{(1+x)^{r+\sum_{i=1}^{r} n_i}}{\prod_{i=1}^{r}\prod_{j=1}^{l}(1+d_{ji}x)} \\ &= \left(1 + \left(r + \sum_{i=1}^{r} n_i\right) x + \ldots\right) \prod_{i=1}^{r}\prod_{j=1}^{l}(1 - d_{ji}x + \ldots) \\ &= \prod_{i=1}^{r}\prod_{j=1}^{l}(1 - d_{ji}x + \ldots) + (r + \sum_{i=1}^{r} n_i)x + \ldots \\ &= 1 - \sum_{i=1}^{r}\sum_{j=1}^{l} d_{ji}x + rx + \sum_{i=1}^{r} n_i x + \ldots \end{aligned}$$

So, $c_1(X) = -\sum_{i=1}^{r}\sum_{j=1}^{l} d_{ji}x + rx + \sum_{i=1}^{r} n_i x$ and we find the Calabi-Yau condition in this generalized case to be

$$\sum_{i=1}^{r}\sum_{j=1}^{l} d_{ji} = r + \sum_{i=1}^{r} n_i. \tag{2.23}$$

For $r = 1$, this simplifies to equation 2.17.

2.3.3 CY condition for surfaces in $\mathbb{CP}^n(a_0, \ldots, a_n)$

In this subsection, we try to generalize what we have seen in section 2.3.1 to the general case of a weighted projective space. A weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is the projective variety $\mathrm{Proj}(k[x_0, \ldots, x_n])$ associated to the graded ring $k[x_0, \ldots, x_n]$ where the variable x_i has degree a_i. Take f to be the polynomial that defines our surface X. Denote $\mathbb{P}(a_0, \ldots, a_n) = \mathbb{P}$. Then,

$$\mathcal{T}_{\mathbb{P}} = \mathcal{T}_X \oplus \mathcal{N}_{X/\mathbb{P}}.$$

Furthermore,

$$\mathcal{T}_{\mathbb{P}} = (\mathcal{O}(a_0) \oplus \ldots \oplus \mathcal{O}(a_n))/\mathcal{O}.$$

Here, $\mathcal{O}(a_i)$ is a line bundle with $c(\mathcal{O}(a_i)) = 1 + a_i x$. It follows that

$$c(\mathcal{T}_{\mathbb{P}}) = \prod_{i=0}^{n}(1 + a_i x).$$

Since f is a fibre coordinate on $\mathcal{N}_{X/\mathbb{P}}$, $\mathcal{N}_{X/\mathbb{P}} = \mathcal{O}(d)$ and $c(\mathcal{N}_{X/\mathbb{P}}) = 1 + dx$. We thus find

$$c(\mathcal{T}_X) = \frac{\prod_{i=0}^{n}(1 + a_i x)}{1 + dx}.$$

Using a Taylor expansion, we can rewrite this fraction as

$$\begin{aligned} c(\mathcal{T}_X) &= \frac{\prod_{i=0}^{n}(1+a_i x)}{1+dx} \\ &= (1+a_0 x)(1+a_1 x)...(1+a_n x)(1-dx+...) \\ &= 1+(a_0+...a_n-d)x+... \end{aligned}$$

So, in conclusion, we find that

$$c_1(\mathcal{T}_X) = \sum_{i=0}^{n} a_i - d \tag{2.24}$$

and the Calabi-Yau condition becomes $d = \sum_{i=0}^{n} a_i$ which, when we set $a_0 = ... = a_n = 1$, simplifies to the answer we found before.

Chapter 3

Topological String Theory

In this chapter we will study the basics of topological string theory. Some mathematical background is assumed. For a description of the mathematical definitions and techniques used, the reader can for example turn to [27] or [16]. For an introduction to topological string theory, the reader should turn to [19]. Another helpful reference is [26]. With this study of topological string theory we will find some nice examples of the main objects of study in this thesis. Namely, we will encounter our first example of a topological twist, as it was first performed by Edward Witten. Also, we will find that these theories live in a ten dimensional space with therein a Calabi-Yau space. Their importance will become apparent later on.

3.1 Topological field theories

We are ultimately interested in quantum field theories because they give us information about physical observables. What is meant by physical depends on the theory in question. Observables can be written as correlation functions of products of operators $\mathcal{O}(x)$. These correlators are generally calculated in a particular background. That means, for example, a particular choice of a manifold over which we integrate, a choice of metric on this manifold or the choice of a particular coupling constant. A topological field theory is a theory in which the observables of the theory do not depend on the choice of metric, i.e.

$$\frac{\delta}{\delta h^{\alpha\beta}}\langle\mathcal{O}\rangle = \frac{\delta}{\delta h^{\alpha\beta}}\int D[\phi]\mathcal{O}e^{iS[\phi]} = 0, \tag{3.1}$$

for $h^{\alpha\beta}$ a metric. After treating an important example in section 3.1.1, we will see in section 3.1.2 that cohomological field theories are in particular topological.

3.1.1 An example: Chern-Simons theory on a 3-manifold

Chern-Simons theory is an example of topological field theory of the "Schwarz-type". These are theories where both the action and the fields do not depend on the metric.

Take $\pi : E \to M$ a vector bundle over a 3-manifold M, with structure group G and connection A. The structure group G will play the part of a gauge group and the connection A has to be interpreted as the gauge field. The Chern-Simons Lagrangian is then given by:

$$L_{\mathrm{CS}} = \mathrm{tr}\left(A \wedge dA - \frac{2}{3} A \wedge A \wedge A\right). \tag{3.2}$$

If we choose the corresponding action to be

$$S = \frac{k}{4\pi} \int_M L_{\mathrm{CS}}, \qquad k \in \mathbb{Z}, \tag{3.3}$$

then under a gauge transformation $A \longrightarrow gAg^{-1} - gdg^{-1}$ with $g \in G$, $\delta S = 2\pi km$ with $m \in Z$ if M does not have a boundary. It was shown that Chern-Simons theory on a 3-manifold is anomaly-free. That is, we can define a path integral measure $\mathcal{D}A$ in a way such that it is gauge-invariant. We thus find that the partition function of the Chern-Simons theory for fixed E is a topological invariant of M.

3.1.2 Cohomological field theories

Chern-Simons theory is a topological field theory in a trivial way: at no point did we introduce a metric in the above story, so surely the observables of the theory will not depend on the metric. We will now discuss theories where we do introduce a metric, but where the correlation functions and partition function are metric independent. These theories are called cohomological field theories, or field theories of "Witten-type".

Start with an action S, that has a global symmetry $\delta_\epsilon S = 0$. Assume that the path integral measure is also invariant under this symmetry. The symmetry is generated by a corresponding operator Q, which acts on the Hilbert space of states and on other operators of the theory. Operators change under this symmetry in the following way:

$$\delta_\epsilon \mathcal{O} = \begin{cases} i\epsilon[Q, \mathcal{O}] & \mathcal{O} \text{ or } Q \text{ is bosonic,} \\ i\epsilon\{Q, \mathcal{O}\} & \mathcal{O} \text{ and } Q \text{ are fermionic.} \end{cases} \tag{3.4}$$

From now on in this section, we will denote both the commutator as the anti-commutator with $\{\cdots\}$. States of the Hilbert space which are invariant under the global symmetry are called symmetric. A symmetric state $|\psi\rangle$ satisfies $Q|\psi\rangle = 0$. So, if the symmetry is not spontaneously broken, the vacuum state of the theory will be symmetric: $Q|0\rangle = 0$. We call all operators annihilated by Q closed operators, all operators $\mathcal{O}$ that can be written as $\mathcal{O} = \{Q, \mathcal{O}'\}$ in terms of some other operator $\mathcal{O}'$ are called exact operators.

We need four requirements to have a cohomological field theory:

1. The symmetry is nilpotent: $Q^2 = 0$.

2. The vacuum state is symmetric: $Q|0\rangle = 0$.

3. Physical operators $\mathcal{O}$ are closed: $\{Q, \mathcal{O}\} = 0$.

4. The energy-momentum tensor is Q-exact: $T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}$, for some operator $G_{\alpha\beta}$.

A consequence of the requirement that the symmetry is not spontaneously broken together with the requirement that physical operators are closed is that all physical operators are described by the cohomology ring of Q. We can see this in the following way. Any correlator over a Q-exact operator is zero, since:

$$\begin{aligned}
&\langle 0\,|\,\mathcal{O}_1\cdots\mathcal{O}_{i-1}\{Q,\mathcal{O}'_i\}\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle = \langle 0\,|\,\mathcal{O}_1\cdots\mathcal{O}_{i-1}(Q\mathcal{O}'_i \pm \mathcal{O}'_i Q)\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle \\
&= \langle 0\,|\,\mathcal{O}_1\cdots\mathcal{O}_{i-1}Q\mathcal{O}'_i\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle \pm \langle 0\,|\,\mathcal{O}_1\cdots\mathcal{O}_{i-1}\mathcal{O}'_i Q\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle \\
&= \pm\,\langle 0\,|\,Q\mathcal{O}_1\cdots\mathcal{O}_{i-1}\mathcal{O}'_i\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle \pm \langle 0\,|\,Q\mathcal{O}_1\cdots\mathcal{O}_{i-1}\mathcal{O}'_i\mathcal{O}_{i+1}\cdots\mathcal{O}_n\,|\,0\rangle \\
&= 0.
\end{aligned}$$

Therefore, we have the equivalence $\mathcal{O}_i \sim \mathcal{O}_i + \{Q,\mathcal{O}'\}$. So all physical operators of the theory are the operators which are closed, modulo this equivalence. This is where the name "cohomological" comes from.

We advertised above that cohomological field theories are in particular topological field theories. Let us show that the fourth requirement above implies the independence of the observables on the choice of the metric:

$$\begin{aligned}
\frac{\delta}{\delta h^{\alpha\beta}}\langle\mathcal{O}_1\cdots\mathcal{O}_n\rangle &= \frac{\delta}{\delta h^{\alpha\beta}}\left(\int D\phi\mathcal{O}_1\cdots\mathcal{O}_n e^{iS[\phi]}\right) = \int D\phi\mathcal{O}_1\cdots\mathcal{O}_n\frac{\delta e^{iS[\phi]}}{\delta h^{\alpha\beta}} \\
&= i\int D\phi\mathcal{O}_1\cdots\mathcal{O}_n\frac{\delta S}{\delta h^{\alpha\beta}}e^{iS[\phi]} = i\int D\phi\mathcal{O}_1\cdots\mathcal{O}_n\{Q,G_{\alpha\beta}\}e^{iS[\phi]} \\
&= i\langle\mathcal{O}_1\cdots\mathcal{O}_n\{Q,G_{\alpha\beta}\}\rangle = 0,
\end{aligned}$$

where we assumed that the operators $\mathcal{O}_i$ do not depend on the metric $h^{\alpha\beta}$, and we used that $\frac{\delta S}{\delta h^{\alpha\beta}} = T_{\alpha\beta}$. So, we see that all cohomological field theories are topological in nature.

An interesting choice for a Lagrangian such that the fourth requirement is satisfied is taking a Lagrangian which itself is Q-exact: $L = \{Q,V\}$. Plugging Planck's constant back explicitly into our description, we obtain:

$$e^{\frac{i}{\hbar}S} = e^{\frac{i}{\hbar}\left\{Q,\int_M V\right\}}.$$

We now find that the derivative of all correlators with respect to $\hbar$ is zero, since taking such a derivative will bring a Q-exact operator into the correlator. So, all physical correlators are independent of $\hbar$. We can therefore take the limit $\hbar \longrightarrow 0$ and compute everything semiclassically. This is called localization. So, an important property of cohomological theories is that the semiclassical approximation is exact.

3.2 Twisting supersymmetric field theories

In this section we derive our first example of a cohomological field theory, namely twisted $\mathcal{N}=2$ supersymmetric theory on a string-worldsheet Σ with arbitrary metric. We will arrive at this theory by a so-called twisting procedure of a $\mathcal{N}=(2,2)$ supersymmetric theory. In section 3.2.1 we will discuss the properties of this theory and see that it is almost topological. In section 3.2.2 we discuss the twisting procedure, which results in two different models: the A- and B-model.

3.2.1 $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions

Superspace

We assume for the moment that $\Sigma = \mathbb{C}$, we generalize to non-flat worldsheets later on. Define complex coordinates $z, \bar{z}$ on the worldsheet. The notation $\mathcal{N} = (p, q)$ means that there are p irreducible spinor supercharges with positive $U(1)$-charge and q irreducible spinor supercharges with negative $U(1)$-charge. We will describe a $\mathcal{N} = (2, 2)$ supersymmetric theory in the so-called superspace. Add four additional coordinates to the existing $z, \bar{z}$: $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$. These four are fermionic coordinates. The $\pm$-superscript denotes the handedness of the fermionic coordinates, i.e. the way these coordinates transform under rotations of the worldsheet $z \longrightarrow e^{i\alpha} z$ with $\alpha \in [0, 2\pi)$ (Lorentz rotations):

$$\theta^\pm \longrightarrow e^{\pm i\frac{\alpha}{2}} \theta^\pm, \qquad \bar{\theta}^\pm \longrightarrow e^{\pm i\frac{\alpha}{2}} \bar{\theta}^\pm. \tag{3.5}$$

The factor $\frac{1}{2}$ comes from the spin 1/2 representation being a double cover of $SO(2) \simeq U(1)$. Under complex conjugation $z \longleftrightarrow \bar{z}$, we have

$$\theta^\pm \longleftrightarrow \bar{\theta}^\mp. \tag{3.6}$$

A superfield Ψ is a function depending on these coordinates: $\Psi = \Psi(z, \bar{z}, \theta^\pm, \bar{\theta}^\pm)$. Since $\theta^\pm, \bar{\theta}^\pm$ are fermionic coordinates they anti-commute with each other. Thus, every superfield can be written down in a finite Taylor expression of in total 16 terms:

$$\Psi(z, \bar{z}, \theta^\pm, \bar{\theta}^\pm) = \phi(z, \bar{z}) + \psi_+(z, \bar{z})\theta^+ + \psi_-(z, \bar{z})\theta^- + ... \,. \tag{3.7}$$

The functions $\phi, \psi_+, \psi_-, ...$ can be ordinary valued or Grassmann-valued.

Symmetries of superspace

Denote the measure of superspace by

$$dz d\bar{z} d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- = d^2 z d^4 \theta.$$

Symmetries of superspace are coordinate transformations that leave this measure invariant. We are interested in the linear coordinate transformations which result in a symmetry.

The Poincaré group is a symmetry group of superspace. The generators of the group, the Hamiltonian and momentum operators, can be written as

$$H = -i(\partial_+ - \partial_-), \tag{3.8a}$$
$$P = -i(\partial_+ + \partial_-), \tag{3.8b}$$

where $\partial_+ = \partial_z$ and $\partial_- = \partial_{\bar{z}}$. The $U(1)$ Lorentz generator can in turn be written as:

$$M = 2z\partial_+ - 2\bar{z}\partial_- + \theta^+ \frac{d}{d\theta^+} - \theta^- \frac{d}{d\theta^-} + \bar{\theta}^+ \frac{d}{d\bar{\theta}^+} - \bar{\theta}^- \frac{d}{d\bar{\theta}^-}. \tag{3.9}$$

The operators satisfy the Poincaré algebra

$$[M, H] = -2P, \qquad [M, P] = -2H. \tag{3.10}$$

The second set of symmetries are bosonic- and fermionic shifts in the fermionic- and bosonic coordinates respectively. The former schematically looks like $\theta \longrightarrow \theta + c$ and is generated by $\frac{\partial}{\partial\theta}$, while the latter schematically looks like $z \longrightarrow z + c\theta$, with c a bosonic valued field and is generated by $\theta\partial_\pm$. We are only interested in the fermionic shifts in the bosonic coordinate with $U(1)$ charge equal to $\pm\frac{1}{2}$, so we are left with four generators from the fermionic shifts and four generators from the bosonic shifts. Let us now group these 8 generators in complex combinations:

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm, \qquad \bar{Q}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\partial_\pm, \tag{3.11a}$$

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm, \qquad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm. \tag{3.11b}$$

The non-zero anti-commutators between these operators are

$$\{Q_\pm, \bar{Q}_\pm\} = +P \pm H, \tag{3.12a}$$

$$\{D_\pm, \bar{D}_\pm\} = -P \mp H, \tag{3.12b}$$

while the non-zero commutators with rotation generator M are

$$[M, Q_\pm] = \mp Q_\pm, \qquad [M, \bar{Q}_\pm] = \mp\bar{Q}_\pm, \tag{3.13a}$$

$$[M, D_\pm] = \mp D_\pm, \qquad [M, \bar{D}_\pm] = \mp\bar{D}_\pm. \tag{3.13b}$$

A superfield where $\bar{D}_\pm\Psi = 0$ is called a chiral superfield, while a superfield which satisfies $D_\pm\Psi = 0$ is called an anti-chiral field.

The final set of symmetries we will discuss is the so-called R-symmetry. We already discussed the action of the $U(1)$ group on all coordinates in superspace, we can however also look at the case when $U(1)$ acts on a subset of these coordinates while leaving the measure invariant. A final condition is that rotated chiral superfields have to remain chiral. We are left with two independent rotations of the fermionic coordinates, R_V ("vector") and R_A ("axial"), which are given for $\alpha \in [0, 2\pi)$ by:

$$R_V(\alpha) : \theta^\pm \longrightarrow e^{-i\alpha}\theta^\pm, \qquad \bar{\theta}^\pm \longrightarrow e^{i\alpha}\bar{\theta}^\pm, \tag{3.14a}$$

$$R_A(\alpha) : \theta^\pm \longrightarrow e^{\mp i\alpha}\theta^\pm, \qquad \bar{\theta}^\pm \longrightarrow e^{\pm i\alpha}\bar{\theta}^\pm. \tag{3.14b}$$

These rotations form $U(1)$ rotation groups, and are called the vector- and axial R-symmetry groups. The corresponding symmetry generators are the operators:

$$F_V = -\theta^+\frac{d}{d\theta^+} - \theta^-\frac{d}{d\theta^-} + \bar{\theta}^+\frac{d}{d\bar{\theta}^+} + \bar{\theta}^-\frac{d}{d\bar{\theta}^-} \tag{3.15a}$$

$$F_A = -\theta^+\frac{d}{d\theta^+} + \theta^-\frac{d}{d\theta^-} + \bar{\theta}^+\frac{d}{d\bar{\theta}^+} - \bar{\theta}^-\frac{d}{d\bar{\theta}^-} \tag{3.15b}$$

which have non-zero commutators with the Q-operators:

$$[F_V, Q_\pm] = +Q_\pm, \qquad [F_V, \bar{Q}_\pm] = -\bar{Q}_\pm \tag{3.16a}$$

$$[F_A, Q_\pm] = \pm Q_\pm, \qquad [F_A, \bar{Q}_\pm] = \mp\bar{Q}_\pm. \tag{3.16b}$$

R-symmetry as a symmetry of the quantum theory

We can now wonder whether the R_V and R_A symmetries are also symmetries of the quantum theory. In order to check this, we'll need to show that the measure

$$\prod_i D\phi^i D\psi_+^i D\psi_-^i DF^i D\bar{\phi}^i D\bar{\psi}_+^i D\bar{\psi}_-^i D\bar{F}^i \tag{3.17}$$

is invariant under the R-symmetry rotations. This is a non-trivial calculation, we will limit ourselves here by saying that it can be done. See for example [26]. We can now make the following conclusion:

- R_V symmetry is present in the underlying quantum theory for any Kähler target space if the R-charges of the fields vanish.
- The R_A symmetry is present in the underlying quantum theory if the target space is a Calabi-Yau space.

$\mathcal{N} = (2,2)$ supersymmetric Lagrangian

The Lagrangian density for the $\mathcal{N} = (2,2)$ supersymmetric theory becomes:

$$\mathcal{L} = -g_{i\bar{j}}\partial^\alpha \phi^i \partial_\alpha \bar{\phi}^j - 2ig_{i\bar{j}}\bar{\psi}_-^j \Delta_+ \psi_-^i - 2ig_{i\bar{j}}\bar{\psi}_+^j \Delta_- \psi_+^i - R_{i\bar{j}k\bar{l}}\psi_+^i \psi_-^k \bar{\psi}_+^j \bar{\psi}_-^l. \tag{3.18}$$

Here the α summation is done over the worldsheet coordinates, where the worldsheet metric is given by:

$$\eta^{+-} = \eta^{-+} = 2, \qquad \eta^{--} = \eta^{++} = 0.$$

The space-time metric is given by

$$g_{i\bar{j}}(\phi, \bar{\phi}) = \frac{d^2 K}{d\phi^i d\bar{\phi}^j},$$

with K the Kähler potential of the target space. The covariant derivatives $\Delta_\pm$ are given by

$$\Delta_\pm \psi^i = \partial_\pm \psi^i + \Gamma^i_{jk} \partial_\pm \phi^j \psi^k.$$

3.2.2 Twisted $\mathcal{N} = 2$ supersymmetry

Following from equations 3.12, we see that

$$\{\bar{Q}_+ + Q_-, Q_+ - \bar{Q}_-\} = 2H, \tag{3.19a}$$
$$\{\bar{Q}_+ + Q_-, Q_+ + \bar{Q}_-\} = 2P, \tag{3.19b}$$
$$\{\bar{Q}_+ + \bar{Q}_-, Q_+ - Q_-\} = 2H, \tag{3.19c}$$
$$\{\bar{Q}_+ + \bar{Q}_-, Q_+ + Q_-\} = 2P. \tag{3.19d}$$

Defining

$$Q_A = \bar{Q}_+ + Q_-, \qquad\qquad Q_B = \bar{Q}_+ + \bar{Q}_-, \tag{3.20}$$

we see that the Hamiltonian and momentum operator are both Q_A and Q_B exact. That is, they can be written as $\{Q_{A/B}, \cdots\}$. Furthermore,

$$\begin{aligned} Q_A^2 &= \bar{Q}_+^2 + Q_-^2 + \bar{Q}_+ Q_- + Q_- \bar{Q}_+ = \{\bar{Q}_+, Q_-\} = 0 \\ Q_B^2 &= \bar{Q}_+^2 + \bar{Q}_-^2 + \bar{Q}_+ \bar{Q}_- + \bar{Q}_- \bar{Q}_+ = \{\bar{Q}_+, \bar{Q}_-\} = 0, \end{aligned}$$

using that the only non-zero commutators were given by equations 3.12. We have thus constructed a theory which is very similar to a cohomological field theory. They are not actual cohomological theories yet, because we should be able to define them using arbitrary worldsheet metrics, so that the theory is metric independent as a whole. Up until this point the $\mathcal{N} = (2, 2)$ theory we have constructed lives on flat worldsheets.

In terms of the action, we thus replace worldsheet derivatives with covariant derivatives; we obtain nontrivial connections on non-flat worldsheets. But how will we define supersymmetries corresponding to the $Q_\pm$ operators? We would like to write

$$\delta \Phi^i = \epsilon^+ Q_+ \Phi^i, \tag{3.21}$$

but in order to define a global symmetry, ϵ^+ needs to be a covariantly constant spinor so that we can pull it outside covariant derivatives in the action to its invariance under these symmetries. Notice however that for a general worldsheet metric, there is no covariantly constant spinor field. The reason has to do with the fact that the value at two arbitrary points of a covariant constant field should be related by parallel transport. If we would parallel transport the spinor ϵ^+ around a closed curve, it would give the same answer up to rotation. Only very special metrics make sure that around such a closed curve this rotation is the identity. A way around this problem is the twisting procedure, which we will describe in the next section. After twisting we obtain two different $\mathcal{N} = 2$ theories: the A- and B-model. We will describe their construction and highlight their most important properties.

Twisting procedure

The twisting procedure relies on the fact that for a trivial bundle we can always choose a covariantly constant field. We can thus solve the above problem by making sure the supercharge that we are interested in lives in a trivial bundle. In our case, we want that the supercharge transforms as a scalar under the Lorentz group. So, in order to construct a Q-symmetric theory, we need to construct a different Lorentz group for which some of the Q-operators will transform with spin equal to zero. We define two new Lorentz symmetry generators:

$$M_A = M - F_V, \tag{3.22a}$$

$$M_B = M - F_A. \tag{3.22b}$$

We declare these to be the new generators for the Lorentz symmetry. When we compute the commutators with the symmetry generators we see that

$$\begin{aligned}
[M_A, Q_+] &= -2Q_+ \\
[M_A, Q_-] &= 0 \\
[M_B, Q_+] &= -2Q_+ \\
[M_B, Q_-] &= 2Q_- \\
[M_A, \bar{Q}_+] &= 0 \\
[M_A, \bar{Q}_-] &= 2\bar{Q}_- \\
[M_B, \bar{Q}_+] &= 0 \\
[M_B, \bar{Q}_-] &= 0
\end{aligned}$$

and thus for $Q_A = \bar{Q}_+ + Q_-$ and $Q_B = \bar{Q}_+ + \bar{Q}_-$ we find:

$$[M_A, Q_A] = 0, \qquad\qquad [M_B, Q_B] = 0.$$

Thus, for for M_A, the Q_A operator has become a scalar, similar for B. So, we have generalized the theory to an arbitrary worldsheet. These twisted theories are now true cohomological field theories.

Note one important thing. We know that we have two types of R-symmetry, vector and axial. We know that vector R-symmetry is a quantum symmetry for any Kähler target space, while axial R-symmetry is a quantum symmetry only when the first Chern class of the target space vanishes. So, the A-twisting can be done for any Kähler target space, but the B-twisting only for a target space that is also Calabi-Yau. What topological data do these theories compute for the target space?

A-model

Define $\phi : \Sigma \to M$ to be the embedding map of our worldsheet. After the twist, the fermion fields will be sections of the following bundles:

$$\begin{aligned}
\psi_+^i &\equiv \psi_z^i \in \Gamma(\Omega^{1,0} \otimes \phi^*(T^{(1,0)}M)), \\
\psi_-^i &\equiv \chi^i \in \Gamma(\phi^*(T^{(1,0)}M)), \\
\bar{\psi}_+^i &\equiv \chi^{\bar{i}} \in \Gamma(\phi^*(T^{(0,1)}M)), \\
\bar{\psi}_-^i &\equiv \psi_{\bar{z}}^{\bar{i}} \in \Gamma(\Omega^{0,1} \otimes \phi^*(T^{(0,1)}M)).
\end{aligned}$$

Let us now rewrite the Lagrangian density in terms of these fields. Let us start with equation 3.18:

$$\begin{aligned}
\mathcal{L} &= -g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j - 2ig_{i\bar{j}}\bar{\psi}_-^j\Delta_+\psi_-^i - 2ig_{i\bar{j}}\bar{\psi}_+^j\Delta_-\psi_+^i - R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\bar{\psi}_+^j\bar{\psi}_-^l \\
&= -2g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^j - 2g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^j - 2ig_{i\bar{j}}\psi_{\bar{z}}^{\bar{j}}\Delta_z\chi^i - 2ig_{i\bar{j}}\psi_z^i\Delta_{\bar{z}}\chi^{\bar{j}} - R_{i\bar{j}k\bar{l}}\psi_z^i\psi_{\bar{z}}^{\bar{j}}\chi^k\chi^{\bar{l}}
\end{aligned}$$

where we used that fermionic fields anticommute, as well as the symmetries of the Riemann tensor. We also dropped terms with a total derivative, since these do not contribute to the physics. Multiplying with a coupling constant t, we get the expression:

$$\mathcal{L} = -2t(g_{i\bar{j}}\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^j + g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^j + ig_{i\bar{j}}\psi_{\bar{z}}^{\bar{j}}\Delta_z\chi^i + ig_{i\bar{j}}\psi_z^i\Delta_{\bar{z}}\chi^{\bar{j}} + \frac{1}{2}R_{i\bar{j}k\bar{l}}\psi_z^i\psi_{\bar{z}}^{\bar{j}}\chi^k\chi^{\bar{l}}) \quad (3.23)$$

Note, that we can write

$$\mathcal{L}_{\alpha\beta} = -t(g_{i\bar{j}}\partial_\alpha\phi^i\partial_\beta\bar{\phi}^{\bar{j}} + i\psi^i_\alpha\Delta_\beta\chi_i + \frac{1}{4}R_{i\bar{j}k\bar{l}}\psi^i_\alpha\psi^{\bar{j}}_\beta\chi^k\chi^{\bar{l}})$$

So that $\mathcal{L} = \eta^{\alpha\beta}\mathcal{L}_{\alpha\beta}$. Thus, we can immediately generalize the Lagrangian to curved worldsheets by changing $\eta^{\alpha\beta}$ to a general $h^{\alpha\beta}$ metric.

Next, let us show that the Lagrangian is almost Q_A-exact, up to some term. One finds that it is possible to write

$$\underbrace{\mathcal{L} - 2tg_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}})}_{\equiv\mathcal{L}'} = -it\{Q_A, g_{i\bar{j}}(\psi^i_z\partial_{\bar{z}}\bar{\phi}^{\bar{j}} + \partial_z\phi^i\psi^{\bar{j}}_{\bar{z}})\}.$$

Let us now work out the difference in the action:

$$\begin{aligned} S - S' &= 2t\int_\Sigma d^2z g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\bar{\phi}^{\bar{j}} - \partial_{\bar{z}}\phi^i\partial_z\bar{\phi}^{\bar{j}}) \\ &= 2t\int_\Sigma g_{i\bar{j}}d\phi^i\wedge d\bar{\phi}^{\bar{j}} \\ &= t\int_\Sigma \phi^*(2g_{i\bar{j}}dz^i\wedge d\bar{z}^{\bar{j}}) \\ &= t\int_\Sigma \phi^*(\omega) \\ &= t\int_{\phi(\Sigma)} \omega = t\omega\cdot\beta \end{aligned}$$

with ω the Kähler form of the target space and $\beta\in H_2(\mathcal{M})$ the homology class of $\phi(\Sigma)$ in $\mathcal{M}$. Writing out the contributions, we find that

$$e^{-S} = e^{-S+S'}e^{-S'} = e^{-t\omega\cdot\beta}e^{-S'},$$

where the first term does not depend on the metric on the worldsheet and where the second term is Q_A-exact. So, using the arguments in section 3.1.2 we can conclude that our theory is topological with respect to the metric present on the worldsheet. The model is half topological in the sense that it only depends on the Kähler class of the target space, but not on the complex structure of the target space, it thus depends on half of the moduli space of the worldsheet and target space metrics. Furthermore, since S' is Q_A-exact, dS'/dt will be Q_A-exact as well. So,

$$\frac{d}{dt}\int e^{-S} = \int \underbrace{(-\omega\cdot\beta - \frac{dS'}{dt})}_{\text{not dependent on } t}e^{-S}$$

For $t\to\infty$, the classical limit, the theory can thus be calculated exactly.

Finally, let us look at the local physical operators and see what they tell us. A general local operator has the following form:

$$\mathcal{O}_F = F_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}.$$

We cannot have any z-indices in this expression, because these can only be removed covariantly by introducing the metric of the worldsheet, making the expression non-topological. Using the transformation rules of the fields, we obtain the following equation:

$$\begin{aligned}\{Q_A, \mathcal{O}_F\} &= \frac{\partial F_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)}{\partial\bar{\phi}^k}\chi^{\bar{k}}\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}^q} + \frac{\partial F_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)}{\partial\phi^k}\chi^{k}\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}^q} \\ &= dF_{i_1\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}^q}.\end{aligned}$$

So, when we view $F(\phi)$ as a complex differential form on the target manifold, $\{Q_A, \mathcal{O}_F\} = \mathcal{O}_{dF}$. We thus find that there is a group isomorphism between the local physical operators in the A-model and the de Rham cohomology elements on $\mathcal{M}$.

B-model

Let us start our discussion of the B-model by again listing which bundles the fields are sections of after the twist:

$$\begin{aligned}\psi_+^i &\in \Gamma(\Omega^{1,0}\otimes\phi^*(T^{(1,0)}M)),\\ \psi_-^i &\in \Gamma(\Omega^{0,1}\otimes\phi^*(T^{(1,0)}M)),\\ \bar{\psi}_+^i &\in \Gamma(\phi^*(T^{(0,1)}M)),\\ \bar{\psi}_-^i &\in \Gamma(\phi^*(T^{(0,1)}M)).\end{aligned}$$

We again relabel to make transformation properties more manifest in the notation:

$$\begin{aligned}\eta^{\bar{i}} &= \bar{\psi}_+^i + \bar{\psi}_-^i,\\ \theta^i &= g_{i\bar{j}}(\bar{\psi}_+^j - \bar{\psi}_-^j),\\ \rho_z^i &= \psi_+^i,\\ \rho_{\bar{z}}^i &= \psi_-^i.\end{aligned}$$

Let us rewrite the Lagrangian of equation 3.18.

$$\begin{aligned}\mathcal{L} &= -g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j - 2ig_{i\bar{j}}\bar{\psi}_-^j\Delta_+\psi_-^i - 2ig_{i\bar{j}}\bar{\psi}_+^j\Delta_-\psi_+^i - R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\bar{\psi}_+^j\bar{\psi}_-^l\\ &= -g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j + ig_{i\bar{j}}\eta^{\bar{j}}(\Delta_{\bar{z}}\rho_z^i + \Delta_z\rho_{\bar{z}}^i) + i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) - R_{i\bar{j}k\bar{l}}\rho_z^i\rho_-^k\bar{\psi}_+^j\bar{\psi}_-^l\\ &= -g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j + ig_{i\bar{j}}\eta^{\bar{j}}(\Delta_{\bar{z}}\rho_z^i + \Delta_z\rho_{\bar{z}}^i) + i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) - \frac{1}{2}R_{i\bar{j}kl}\rho_z^i\rho_-^k\eta^{\bar{j}}\theta^l\end{aligned}$$

In the same way as before we can write this as $\mathcal{L} = \eta^{\alpha\beta}\mathcal{L}_{\alpha\beta}$. So, again a generalization to arbitrary worldsheet metric is straightforward. Furthermore, in the same way as before we introduce a coupling constant t, so that the Lagrangian becomes:

$$\mathcal{L} = -t(g_{i\bar{j}}\partial^\alpha\phi^i\partial_\alpha\bar{\phi}^j - ig_{i\bar{j}}\eta^{\bar{j}}(\Delta_{\bar{z}}\rho_z^i + \Delta_z\rho_{\bar{z}}^i) - i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) + \frac{1}{2}R_{i\bar{j}kl}\rho_z^i\rho_-^k\eta^{\bar{j}}\theta^l) \quad (3.24)$$

Again, we can express the B-model action in an almost exact fashion:

$$S = -it\int_\Sigma\{Q_B, V\} - t\int_\Sigma(i\theta_i(\Delta_{\bar{z}}\rho_z^i - \Delta_z\rho_{\bar{z}}^i) + \frac{1}{2}R_{i\bar{j}kl}\rho_z^i\rho_{\bar{z}}^k\eta^{\bar{j}}\theta^l), \quad (3.25)$$

with $V = g_{i\bar{j}}(\rho^i_z \partial_{\bar{z}} \bar{\phi}^j + \rho^i_{\bar{z}} \partial_z \bar{\phi}^j)$. The latter term of the right-hand side of this equation is antisymmetric in $z, \bar{z}$-indices. Thus, we can write it as a $(1,1)$-form, whose integral over a two-dimensional manifold is in general independent of the metric. So, the only dependence on the metric is in the $\{Q_B, V\}$ term. Thus, we find again that theory is topological with respect to the worldsheet metric. Note also that V does not depend on θ and that the rest term only has linear dependence in θ. So, we can rescale these fields with t, removing the t-dependence. We can take t large again and calculate exact results. Furthermore, we state that the theory does depend on the choice of complex structure on the target space $\mathcal{M}$ and not on the Kähler structure of the target space. Note that this is exactly the opposite of what we found in the A-model case.

Next, let us analyse the local operators. A general local, metric-independent operator takes the form:

$$\mathcal{O}_F = F^{j_1 \cdots j_q}_{\bar{i}_1 \cdots \bar{i}_p}(\phi, \bar{\phi}) \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_p}. \tag{3.26}$$

Studying the variation of such an operator, one finds the following equation:

$$\{Q_B, \mathcal{O}_F\} = -\frac{\partial F^{j_1 \cdots j_q}_{\bar{i}_1 \cdots \bar{i}_p}(\phi, \bar{\phi})}{\partial \bar{\phi}^k} \eta^{\bar{k}} \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_p}.$$

Where in the A-model we found a isomorphism with the de Rham cohomology group, here we find a relation with Dolbeault cohomology:

$$\{Q_B, \mathcal{O}_F\} = \mathcal{O}_{\bar{\partial} F}. \tag{3.27}$$

Thus, we have group isomorphism between the cohomology group of Q_B of physical operators and the Dolbeault cohomology of $\mathcal{M}$:

$$\mathcal{H}(Q_B) = H_{\text{Dolb}}(\mathcal{M}, T^{1,0}\mathcal{M}).$$

3.3 Topological strings

The theories we have constructed up until this point have had a fixed background metric $h^{\alpha\beta}$ on the worldsheet. We will introduce an integral over all possible metrics of the worldsheet in this section, and go from topological field theories to topological string theories in this manner. If we want to couple a theory to gravity, we need to perform three crucial steps:

1. Write the Lagrangian in a covariant way.

2. Introduce an Einstein-Hilbert term in such a way that symmetries are preserved.

3. Integrate the resulting theory over the space of all metrics on the worldsheet.

We will focus on the last step. The first two steps are analysed in [3].

Our method of approach will be to first do the integral over all conformally equivalent metrics. After that we are left with a finite-dimensional integral over a collection of worldsheet moduli, which we then discuss. In doing the integral over all conformally equivalent metrics, we will need to worry about conformal anomalies. A conformal anomaly happens when the central charge is non-zero, making it hard to define the quantum theory. The energy-momentum tensor $T_{\alpha\beta}$ is a conserved current, so $\partial_\alpha T^\alpha_\beta = 0$. It follows that $T_{zz} = T(z)$ and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$. Furthermore, it is known that $T_{z\bar{z}} = T_{\bar{z}z} = 0$. The central charge of the theory arises out of the Virasoro algebra that the Laurent modes of the $T(z)$ expansion follow:

$$T(z) = \sum L_m z^{-m-2}. \tag{3.28}$$

The commutator between two Laurent modes can be calculated in conformal field theory and shown to be

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}. \tag{3.29}$$

The same story goes for the anti-holomorphic case. In conformal field theory, one now requires for physical states to satisfy the constraint $L_m |\psi\rangle = 0$ for all $m \in \mathbb{Z}$. One thus surely needs $c = 0$. In string theory, this happens when the dimension of the target space $\mathcal{M}$ is ten. We will show that things turn out to be different in topological string theory.

We will denote holomorphic by left-movers and anti-holomorphic by right-movers. When working on open strings, the left- and right-moving parts are complex conjugates of each other, so that we are left with one single algebra. In the closed case (which we are studying), left- and right-movers are independent so we have two sets of Virasoro algebras: one for L_m and one for $\bar{L}_m$. Any global $U(1)$-symmetry of our theory will have a conserved current J_α. Since it is conserved, it will have to parts: $J_z = J(z)$ and $J_{\bar{z}} = \bar{J}(\bar{z})$. Again in the closed case, left- and right-movers will be independent, so we really have two symmetry generators F_L and F_R for the two left- and right-moving $U(1)$ symmetries with corresponding currents J and $\bar{J}$. Since R-symmetries are $U(1)$-symmetries, this discussion holds for R-symmetries and noticing that $F_V + F_A$ only acts on left-moving quantities, while $F_V - F_A$ only acts on right-moving quantities, we discern that we can split these two generators into

$$F_V = \frac{1}{2}(F_L + F_R), \tag{3.30a}$$

$$F_A = \frac{1}{2}(F_L - F_R). \tag{3.30b}$$

Indeed, now $F_V + F_A = F_L$ and $F_V - F_A = F_R$. We can again do a Laurent expansion, now for the left-moving current:

$$J(z) = \sum J_m z^{-m-1}. \tag{3.31}$$

The commutation relations with J itself and with L are given by

$$[L_m, J_n] = -nJ_{m+n}, \tag{3.32a}$$

$$[J_m, J_n] = \frac{c}{3}m\delta_{m+n}. \tag{3.32b}$$

The corresponding conserved charges are

$$F_L = \oint_{z=0} J(z)dz = 2\pi i J_0,$$

$$F_R = \oint_{\bar{z}=0} \bar{J}(\bar{z})d\bar{z} = 2\pi i \bar{J}_0.$$

These are found by integrating the currents over a space-like slice, which amounts to the radial direction in the $z, \bar{z}$-plane. Note that the generator of Lorentz rotations can be written as $M = 2\pi i(L_0 - \bar{L}_0)$. So, when we twist the theory, we obtain:

$$\begin{aligned}
M_A &= M - F_V = M - \frac{1}{2}(F_L + F_R) = 2\pi i(L_0 - \bar{L}_0) - \frac{1}{2}2\pi i(J_0 + \bar{J}_0) \\
&= 2\pi i\left[(L_0 - \frac{1}{2}J_0) - (\bar{L}_0 + \frac{1}{2}\bar{J}_0)\right], \\
M_B &= M - F_A = M - \frac{1}{2}(F_L - F_R) = 2\pi i(L_0 - \bar{L}_0) - \frac{1}{2}2\pi i(J_0 - \bar{J}_0) \\
&= 2\pi i\left[(L_0 - \frac{1}{2}J_0) - (\bar{L}_0 - \frac{1}{2}\bar{J}_0)\right]
\end{aligned}$$

So after the twisting procedure, we find that there is a difference between the A and B model:

$$\begin{aligned}
L_{0,\text{ A}} &= L_0 - \frac{1}{2}J_0, \\
\bar{L}_{0,\text{ A}} &= \bar{L}_0 + \frac{1}{2}\bar{J}_0, \\
L_{0,\text{ B}} &= L_0 - \frac{1}{2}J_0, \\
\bar{L}_{0,\text{ B}} &= \bar{L}_0 - \frac{1}{2}\bar{J}_0.
\end{aligned}$$

For both twistings the left-moving sector has the same Lorentz generator. This generator should again correspond to a current, which in [17] is shown to be

$$\tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z).$$

Looking at the Laurent expansion of this new current, we obtain:

$$\begin{aligned}
\tilde{T}(z) &= T(z) + \frac{1}{2}\partial J(z) \\
&= \sum_m L_m z^{-m-2} + \frac{1}{2}\partial \sum J_m z^{-m-1} \\
&= \sum_m L_m z^{-m-2} + \frac{1}{2}\sum(-m-1)J_m z^{-m-2} \\
&= \sum_m \left(L_m - \frac{1}{2}(m+1)J_m\right) z^{-m-2}
\end{aligned}$$

So, defining

$$\tilde{L}_m = L_m - \frac{1}{2}(m+1)J_m,$$

we get new Laurent modes. These will satisfy a different algebra than the Virasoro algebra. Let us calculate the commutator, using the commutation relations given above.

$$\begin{aligned}
[\tilde{L}_m, \tilde{L}_n] &= [L_m, L_n] - \frac{1}{2}(n+1)[L_m, J_n] - \frac{1}{2}(m+1)[J_m, L_n] + \frac{1}{4}(m+1)(n+1)[J_m, J_n] \\
&= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} + \frac{1}{2}(n+1)nJ_{m+n} - \frac{1}{2}(m+1)mJ_{m+n} \\
&+ \frac{c}{12}(m+1)(-m+1)m\delta_{m+n} \\
&= (m-n)\tilde{L}_{m+n}
\end{aligned}$$

Notice that after the twisting, the central charge is gone! This has a very deep implication: topological strings will be well-defined on a target space of arbitrary dimension. We can now integrate over conformal equivalent metrics. After this integral we have a finite dimensional integral left. This is because locally we know we can always use a conformal transformation to transform an arbitrary worldsheet metric into the flat case. However, globally one cannot always do this. For example, the complex modular parameter τ of a torus worldsheet cannot be gauged away in this manner. So, for the $g = 1$ case we still have an integral left over 1 complex modular parameter. For $g = 0$, there will be no complex modular parameter over which we need to integrate. For $g > 1$, the dimension of the moduli space is equal to $3g - 3$.

Let us focus on the $g > 1$ case. We need to find a way to integrate over this moduli space of dimension $3g - 3$. We will first need a volume element over this space. The moduli space we have left after having done the integration over conformal equivalent metrics labels the complex structures possible on the worldsheet. So, a tangent vector is an infinitesimal change in complex structure on the worldsheet. We can parametrize these changes of complex structure in the following way:

$$\begin{aligned}
dz &\mapsto dz + \epsilon\mu_{\bar{z}}^{z}(z)d\bar{z} \\
d\bar{z} &\mapsto d\bar{z} + \epsilon\mu_{z}^{\bar{z}}(\bar{z})dz
\end{aligned}$$

So, the tangent space of the moduli space is spanned by these $\mu_i(z, \bar{z}), \bar{\mu}_i(z, \bar{z})$ of which three are $3g - 3$ of each kind. We can now define the integration over moduli space as

$$\int_{M_g} \prod_{i=1}^{3g-3} \left(dm^i d\bar{m}^i \int_\Sigma G_{zz}(\mu_i)_{\bar{z}}^{z} \int_\Sigma G_{\bar{z}\bar{z}}(\bar{m}u_i)_{z}^{\bar{z}} \right), \tag{3.33}$$

with G_{zz} the supersymmetric partner under Q of T_{zz}. We have now performed the full integration over the space of all metrics on the worldsheet, the theory is now a topological string theory. We can however, say one more important thing about these twisted theories.

The worldsheet metric is invariant under R-symmetry rotations, but the measure is not since G-insertions will transform under R-symmetry. From the $N = 2$ superconformal algebra, we know that the G and $\bar{G}$ insertions have vector charge of zero and an axial charge equal to 2. So, looking at the complete measure, the vector charge will remain the same (namely zero), but we get an axial charge contribution of $6g - 6$ from G and $\bar{G}$. If we add the contribution from fermion zero-modes, equal to $-2m(g - 1)$, we find that the total axial R-symmetry charge of the measure is equal to

$$c_A = 6(g-1) - 2m(g-1),$$

so if the dimension of the target space is three, the axial charge of the measure will vanish for any genus. Hence, the partition function is non-zero at any genus. So, a Calabi-Yau threefold will be the most interesting target space for these topological string theories.

Next, let us give some closing remarks, which will tie this chapter together with the other chapters. First of all, from chapter 2 we know that the Calabi-Yau manifolds have the following Hodge diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{3.34}$$

Since there is a symmetry in the above diamond in the horizontal- and vertical direction, the Hodge numbers that classify the Calabi-Yau manifold are $h^{1,1}$ and $h^{1,2}$. In chapter 6 when we talk about the Batyrev construction, we will construct for every Calabi-Yau manifold X a so-called mirror manifold Y. This "mirror" is also a Calabi-Yau manifold. We call (X, Y) a mirror pair. There exists a nice relation between the Hodge numbers of the mirror pair:

$$h^{1,1}(X) = h^{1,2}(Y) \qquad \text{and} \qquad h^{1,2}(X) = h^{1,1}(Y). \tag{3.35}$$

There is a fundamental relation between physics on the A/B-model and a particular mirror pair (X, Y). Namely, the physical interesting quantities in the A-model on the Calabi-Yau space X are equivalent to the physical interesting quantities in the B-model on the Calabi-Yau space Y and vice versa. In practice, it relates symplectic- and complex geometric calculations on these mirror pairs. This can be used to relate difficult calculations to equivalent, but far easier, calculations on the mirror manifold. A mirror manifold to a particular Calabi-Yau manifold can thus be seen as the Calabi-Yau manifold that describes the same underlying physics as the original Calabi-Yau space.

Chapter 4

Toroidal Dimensional Reduction

In this chapter we will look at the techniques necessary to reduce the number of dimensions of space-time on which our theory works, by applying what is known as toroidal dimensional reduction. We start with some theory on tori in general. Next, we look at the Kaluza-Klein reduction, since this will give us some intuition for the more general cases handled later. After that we start with the torus case for gauge group $G = U(1)$. Next, we will generalize to an arbitrary gauge group G. For our purposes we will be interested in $\mathcal{N} = 4$ Super Yang-Mills (SYM) in dimension $D = 4$. We will show in this chapter how to reduce this theory on a two-torus T^2 to dimension $D = 2$.

4.1 Shapes of a torus and the modular group

Let us define coordinates x_1, x_2 on T^2, the torus. The coordinates are periodic: $x_1 = x_1 + 1$ and $x_2 = x_2 + 1$. Let us define a metric $g_{ij}(x_1, x_2)$, with $i, j \in \{1, 2\}$. This metric then has three independent components: g_{11}, g_{22} and $g_{12} = g_{21}$. We want to discuss the shape of a torus and so we are not interested in its overall size. This means we have Weyl symmetry, or invariance under local rescaling:

$$g_{ij}(x_1, x_2) = e^{2\phi(x_1,x_2)} g_{ij}(x_1, x_2).$$

This means that in defining the shape of a torus, we are dealing with three geometric parameters, namely g_{11}, g_{12}, g_{22}, and three functions of the coordinates x_1, x_2: x_1', x_2' (redefinition of local coordinates) and ϕ (local rescaling). All are local parameters, and depend on the coordinates x_1, x_2. Now we studied the local parameters, are there any global parameters that cannot be changed by coordinate- or Weyl transformations?

Let us consider a manifold that is a torus topologically, with a metric g_{ij} satisfying:

$$g_{ij}(x_1, x_2) = g_{ij}(x_1 + 1, x_2) = g_{ij}(x_1, x_2 + 1). \tag{4.1}$$

For a torus, the Euler characteristic $\chi = 0$, so:

$$\chi = \int d^2x \sqrt{g} R = 0,$$

with R the Ricci curvature. For every T^2, there now exists a Weyl transformation such that $R(x_1, x_2) = 0$ for all $(x_1, x_2) \in T^2$ and such that the periodicity of equation 4.1

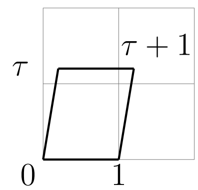


Figure 4.1: A depiction of the periodic identifications on T^2 for modular parameter $\tau \in \mathbb{H}$.

is not violated. Now because we are working in two dimensions, the only surviving component of the Riemann curvature tensor is R_{1212}, and now because $R = 0$ we find that $R_{ijkl} = 0$. We see that for any two-torus we can choose a coordinates such that the metric is periodic and constant due to Ricci-flatness. So the shape of a torus is defined by the three parameters g_{11}, g_{22} and g_{12}, which are all periodic real functions of x_1, x_2. Note that we can use the last remaining Weyl rescaling to identify $(g_{11}, g_{12}, g_{22}) = e^{2\phi}(g_{11}, g_{12}, g_{22})$ so that we are left with only two real parameters. We will organize these two real conformal parameters into one single complex parameter τ, the modular parameter of the torus.

Note that by a linear transformation of x_1, x_2 we can bring the flat torus to a nice form: $g_{ij}(x_1, x_2) = \delta_{ij}$, but now with different periodic identifications for the coordinates on T^2:

$$(x_1, x_2) \sim (x_1 + 1, x_2) \sim (x_1 + \mathrm{Re}(\tau), x_2 + \mathrm{Im}(\tau)).$$

We show this identification in figure 4.1. The metric can be rewritten in terms of the complex coordinate $z = x_1 + ix_2$: $ds^2 = (dx_1)^2 + (dx_2)^2 = dzd\bar{z}$. The "normal" torus is obtained by taking $\tau = i$, however we see that $\tau \in \mathbb{C}$ describes infinitely many shapes of T^2. The real part of τ tilts the torus in a particular direction, while the imaginary part determines the thickness of the torus. An important question is whether all values of $\mathbb{C}$ for τ produce a different shape of the torus. Let us answer this question.

Our torus has a complex coordinate z on its surface, with identification

$$z \sim z + 1 \sim z + \tau.$$

First, notice that we can identify $\tau \sim -\tau$ since these differ by a multiple of τ. Notice that $\mathrm{Im}(\tau) = 0$ does not give us a non-singular finite two-dimensional manifold that we want. We can thus use these two comments to require that $\mathrm{Im}(\tau) > 0$. So, $\tau \in \mathbb{H}$. Next, notice that we can define a lattice

$$\{m + n\tau | m, n \in \mathbb{Z}\}$$

of $\mathbb{C}$ which is spanned by the basis vectors 1 and τ. We can perform a change of basis, so that we obtain basis vectors $a + b\tau$ and $c + d\tau$ with $a, b, c, d \in \mathbb{Z}$. These new basis vectors span a new lattice, where elements are of the form

$$m(a + b\tau) + n(c + d\tau)$$

with $m, n \in \mathbb{Z}$. These are all integer combinations of 1 and τ and thus lie in the previous lattice. If the converse holds as well, the two basis define the same identification. So we want

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

to have integer values. So, we have to demand that $ad - bc = \pm 1$. Due to $\tau \in \mathbb{H}$, we get a stronger identification. So for our change of basis to define the same lattice,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/(A \sim -A);$$

we find the modular group. Let us rescale the new basis: $(a + b\tau, c + d\tau) \sim (1, \frac{c+d\tau}{a+b\tau}$. Now, after a relabelling $a \to d, b \to c, c \to b, d \to a$ we find

$$\tau \sim \tau' = \frac{a\tau + b}{c\tau + d}$$

with $ad - bc = 1$. So τ and τ', found by a modular transformation, define the same shape of the torus. The next question is we could answer is how many inequivalent values of τ there are.

Every matrix of $SL(2,\mathbb{Z})$ can be written as a product of the following matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T^{\pm 1} = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$

Notice that

$$S : \tau \mapsto \frac{-1}{\tau}, \qquad T^{\pm} : \tau \mapsto \tau \pm 1.$$

The fundamental domain is now given by

$$-\frac{1}{2} < \text{ Re}(\tau) < \frac{1}{2}; \qquad \text{Im}(\tau) > 0, \qquad |\tau| > 1$$

and is plotted in figure 4.2. If you are in this region, any transformation

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

which is not the identity will get you out of the fundamental domain, while you can always move into this region by applying S, T and T^{-1} consecutively.

4.2 Kaluza-Klein reduction on a circle

To get a good grip on the different techniques and concepts that we will using in the more general cases, we start investigating the machinery of dimensional reduction on a simple case. Namely, in this section we will look at the Kaluza-Klein reduction for $\mathcal{N} = 4$ super Yang-Mills in four dimensions, with a $U(1)$ gauge group on $S^1(R)$, a circle with radius R. We start with the following action:

$$S[\phi, A, \Psi] = \int d^4x^{\hat{\mu}} \left(\partial_{\hat{\mu}}\phi^i \partial^{\hat{\mu}}\phi^i - \frac{1}{4}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} - \frac{1}{2}\Psi^I_A(\sigma^{\hat{\mu}})_{A\dot{A}}\partial_{\hat{\mu}}\bar{\Psi}_{I,\dot{A}} \right). \tag{4.2}$$

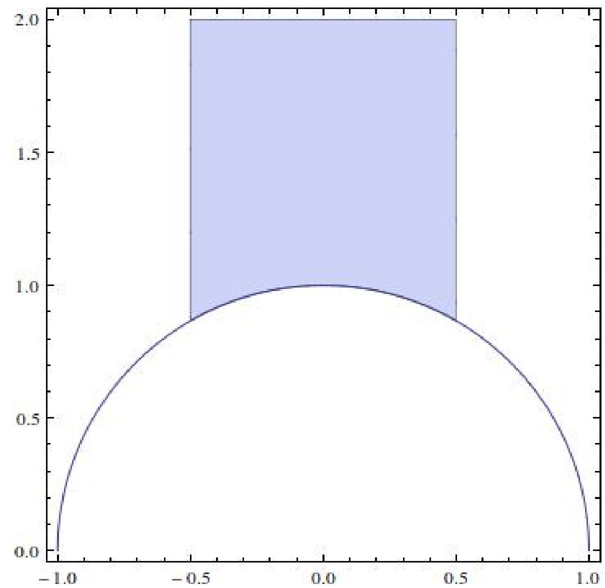


Figure 4.2: The fundamental domain of the modular parameter τ of T^2.

Note that this is not the full $\mathcal{N} = 4$ SYM action, there are some terms missing. However, these terms will be all we need in order to explain the concepts and machinery of the Kaluza-Klein reduction. We have scalar fields ϕ^i (with $i \in \{1, ...6\}$) and vector field $A^{\hat{\mu}}$ which both correspond to bosons. The last term has the fermion fields Ψ^I, with $I = 1, ..., 4 = \mathcal{N}$. We are working on the manifold $\mathcal{M} = \mathbb{R}^{1,2} \times S^1(R)$. The coordinates $x^{\hat{\mu}} = (x^0, x^1, x^2, x^3)$, with $x^{\hat{\mu}} = (x^\mu, x^3)$ with $x^3 \sim x^3 + 2\pi R$. We can thus write down a Fourier decomposition for each field in the action:

$$\phi^i(x^{\hat{\mu}}) = \sum_n \phi^i_n(x^\mu) e^{i\frac{n}{R}x^3}, \tag{4.3a}$$

$$A^{\hat{\nu}}(x^{\hat{\mu}}) = \sum_n A^{\hat{\nu}}_n(x^\mu) e^{i\frac{n}{R}x^3}, \tag{4.3b}$$

$$\Psi^I_A(x^{\hat{\mu}}) = \sum_n \Psi^I_{A,n}(x^\mu) e^{i\frac{n}{R}x^3}. \tag{4.3c}$$

Let us start with the scalar fields. We will try to rewrite the term in which these fields appear:

$$\begin{aligned} S[\phi] &= \int d^4x^{\hat{\mu}} \partial_{\hat{\nu}}\phi^i \partial^{\hat{\nu}}\phi^i \\ &= \int d^3x^\mu \int_0^{2\pi R} dx^3 \sum_{n,m} \left[(\partial_\nu \phi^i_n(x^\mu))(\partial^\nu \phi^i_m(x^\mu)) + \phi^i_n(x^\mu)\phi^i_m(x^\mu)\frac{in}{R}\frac{im}{R} \right] e^{i\frac{m+n}{R}x^3}. \end{aligned}$$

Notice that

$$\frac{1}{2\pi} \int_0^{2\pi} dx e^{i(n-m)x} \equiv \delta_{n,m},$$

so in our case

$$\frac{1}{2\pi} \int_0^{2\pi R} dx e^{i\frac{(n-m)}{R}x} = \delta_{n,m}.$$

Thus,

$$S[\phi] = \int d^3x^\mu \int_0^{2\pi R} dx^3 \sum_{n,m} \left[(\partial_\nu \phi_n^i(x^\mu))(\partial^\nu \phi_m^i(x^\mu)) + \phi_n^i(x^\mu)\phi_m^i(x^\mu)\frac{in}{R}\frac{im}{R} \right] e^{i\frac{m+n}{R}x^3}$$
$$= 2\pi \int d^3x^\mu \sum_n \left[(\partial_\nu \phi_n^i(x^\mu))(\partial^\nu \phi_{-n}^i(x^\mu)) + \phi_n^i(x^\mu)\phi_{-n}^i(x^\mu)\frac{n^2}{R^2} \right]$$

We thus get a Kaluza-Klein tower of fields in dimension three with mass $m = \frac{n^2}{R^2}$. Thus, when we look at the limit $R \to 0$, all the massive fields decouple from the theory since these fields cannot propagate any more:

$$\frac{1}{k^2 + m} \to 0.$$

Interaction terms have this as well. So, only remaining term in the theory in the small radius limit is when $n = 0$:

$$S[\phi] = 2\pi \int d^3x^\mu \partial_\nu \phi_0^i \partial^\nu \phi_0^i \tag{4.4}$$

which is the same term as in the four dimensional case, but now with $\nu = 0, 1, 2$. So, the six scalar fields ϕ^i after dimensional reduction on a circle are changed into ϕ_0^i, which are again six scalar fields. If we would do the same process again, but now from $\mathcal{M} = \mathbb{R}^{1,1} \times S^1(R')$ we would find using the same methods as before:

$$S[\phi] = 2\pi \int d^3x^\mu \partial_\nu \phi_{0,0}^i \partial^\nu \phi_{0,0}^i. \tag{4.5}$$

So,

$$\phi^i \quad \xrightarrow{R \to 0} \quad \phi_0^i \quad \xrightarrow{R' \to 0} \quad \phi_{0,0}^i,$$

going from four to three dimensions by compactifying the x^3-direction and then taking its radius R to zero, and subsequently going from three to two dimensions by compactifying the x^2-direction and then taking its radius R' to zero. In each step we have six scalar fields.

Next, let us look at the term housing the gauge vector field $A_{\hat\mu}$. Here $F_{\hat\mu\hat\nu} = \partial_{\hat\mu} A_{\hat\nu} - \partial_{\hat\nu} A_{\hat\mu}$. Let us do the same calculation:

$$S[A] = -\frac{1}{4}\int d^4x^{\hat\mu} F_{\hat\mu\hat\nu}F^{\hat\mu\hat\nu} = -\frac{1}{4}\int d^3x^\mu \int_0^{2\pi R} dx^3 \left(F_{\mu\nu}F^{\mu\nu} + 2F_{\mu 3}F^{\mu 3} \right)$$
$$= -\frac{\pi}{2}\int d^3x^\mu \sum_n [\partial_\mu A_{\nu,n}\partial^\mu A_{-n}^\nu - \partial_\mu A_{\nu,n}\partial^\nu A_{-n}^\mu - \partial_\nu A_{\mu,n}\partial^\mu A_{-n}^\nu + \partial_\nu A_{\mu,n}\partial^\nu A_{-n}^\mu$$
$$+ 2\left(\partial_\mu A_{3,n}\partial^\mu A_{-n}^3 + \partial_\mu A_{3,n}\frac{in}{R}A_{-n}^\mu - A_{\mu,n}\frac{in}{R}\partial^\mu A_{-n}^3 + \frac{n^2}{R^2}A_{\mu,n}A_{-n}^\mu \right)]$$

Again, taking the $R \to 0$ limit, we find that we are left with:

$$S[A] = -\frac{\pi}{2}\int d^3x^\mu [\partial_\mu A_{\nu,0}\partial^\mu A_0^\nu - \partial_\mu A_{\nu,0}\partial^\nu A_0^\mu - \partial_\nu A_{\mu,0}\partial^\mu A_0^\nu + \partial_\nu A_{\mu,0}\partial^\nu A_0^\mu + 2\partial_\mu A_{3,0}\partial^\mu A_0^3]$$
$$= -\frac{\pi}{2}\int d^3x^\mu [F_{\mu\nu,0}F^{\mu\nu,0} + 2\partial_\mu A_{3,0}\partial^\mu A_0^3]$$

We thus do not find the same form of the action as what we started with. This is different from the scalar field case. We go from a four dimensional vector to a three dimensional vector and a scalar after a dimensional reduction. Let us denote $A_0^3 = \phi$. So our theory went from $A^{\hat{\mu}} \to (A^\mu, \phi)$. We want to take it one step further and go from three to two dimensions. We can do this by dualizing the theory first and afterwards performing the dimensional reduction on the dual theory.

In three dimensions we look at the following action:

$$S[A] = -\frac{1}{4}\int d^3x F_{\mu\nu}(A)F^{\mu\nu}(A). \tag{4.6}$$

Now define,

$$\tilde{S}[F,\lambda] = \int d^3x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \epsilon^{\mu\nu\rho}\lambda\partial_\mu F_{\nu\rho}\right). \tag{4.7}$$

Let us calculate the equation of motion for λ:

$$\tilde{S}[F,\lambda+\delta\lambda] - S[F,\lambda] = \int d^3x \left(\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}\right)\delta\lambda.$$

So, the equation of motion for lambda is

$$\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0. \tag{4.8}$$

Next, we need the equation of motion for F:

$$\begin{aligned}
\delta_F\tilde{S} &= \tilde{S}[F+\delta F,\lambda] - S[F,\lambda] \\
&= \int d^3x \left[-\frac{1}{4}\left(g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}\delta F_{\rho\sigma} + F^{\mu\nu}\delta F_{\mu\nu}\right) + \epsilon^{\mu\nu\rho}\lambda\partial_\mu\delta F_{\nu\rho}\right] \\
&= \int d^3x \left(-\frac{1}{2}g^{\mu\kappa}g^{\nu\tau}F_{\mu\nu} - \epsilon^{\mu\kappa\tau}\partial_\mu\lambda\right)\delta F_{\kappa\tau} \\
&= \int d^3x \left(-\frac{1}{2}(F^{\kappa\tau} - \epsilon^{\mu\kappa\tau}\partial_\mu\lambda\right)\delta F_{\kappa\tau}
\end{aligned}$$

So, the equation of motion for F is:

$$F_{\mu\nu} = -2\epsilon_{\mu\nu\rho}\partial^\rho\lambda. \tag{4.9}$$

If we fill in equation 4.8 into the action $\tilde{S}$, we get:

$$\tilde{S}[F,\lambda] = \int d^3x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda\underbrace{\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}}_{=0}\right) = -\frac{1}{4}\int d^3x F_{\mu\nu}F^{\mu\nu}.$$

Note that satisfying the equation of motion for λ, means that F is closed two-form and can be written as $F = dA$. So, it follows that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. So, with λ on-shell, $\tilde{S}[F,\lambda] = S[A]$. Now, let us look at the case when the equation of motion for

F, equation 4.9, is satisfied:

$$\begin{aligned}
\tilde{S}[F,\lambda] &= \int d^3x \left(-\frac{1}{4}(-2\epsilon_{\mu\nu\rho}\partial^\rho\lambda)(-2\epsilon^{\mu\nu\tau}\partial_\tau\lambda) + \epsilon^{\mu\nu\rho}\lambda\partial_\mu(-2\epsilon_{\nu\rho\tau}\partial^\tau\lambda) \right) \\
&= \int d^3x \left(-\underbrace{\epsilon_{\mu\nu\rho}\epsilon^{\mu\nu\tau}}_{=\delta^\tau_\rho}\partial^\rho\lambda\partial_\tau\lambda - 2\underbrace{\epsilon^{\mu\nu\rho}\epsilon_{\nu\rho\tau}}_{=\delta^\mu_\tau}\lambda\partial_\mu\partial^\tau\lambda \right) \\
&= \int d^3x\partial^\tau\lambda\partial_\tau\lambda.
\end{aligned}$$

So, we find the action for a scalar field. This is thus the dual picture of the $S[A]$ action we defined above in three dimensions. So, in going from four to three to two dimensions, the procedure looks as follows:

$$A_{\hat{\mu}} \to (A_\mu, \phi) \to (\lambda, \phi) \to (\lambda_0, \phi_0)$$

where we first perform a Kaluza-Klein reduction, then a dualization to a theory consisting of two scalars and finally another Kaluza-Klein reduction.

Finally, we want to repeat this procedure for the fermion fields. Starting from the action term housing the fermion fields and using the Fourier decomposition, we perform the Kaluza-Klein reduction:

$$\begin{aligned}
S[\Psi] &= -\frac{1}{2}\int d^4x^{\hat{\mu}}\Psi^I_A(\sigma^{\hat{\mu}})_{A\dot{A}}\partial_{\hat{\mu}}\bar{\Psi}_{I,\dot{A}} \\
&= -\frac{1}{2}\int d^3x \int_0^{2\pi R} dx^3 \sum_{n,m}\left(\Psi^I_{A,n}(\sigma^\mu)_{A\dot{A}}\partial_\mu\bar{\Psi}_{I\dot{A},m} - \frac{im}{R}\Psi^I_{A,n}(\sigma^3)_{A\dot{A}}\bar{\Psi}_{I\dot{A},m}\right)e^{i\frac{n-m}{R}x^3} \\
&= -\pi\int d^3x\sum_n\left(\Psi^I_{A,n}(\sigma^\mu)_{A\dot{A}}\partial_\mu\bar{\Psi}_{I\dot{A},n} - \frac{in}{R}\Psi^I_{A,n}(\sigma^3)_{A\dot{A}}\bar{\Psi}_{I\dot{A},n}\right)
\end{aligned}$$

The mass will be $m = \frac{n}{R}$. If we take $R \to 0$, we again that massive terms decouple from the theory. Leaving only one term within the sum: $n = 0$. So, after the dimensional reduction we are left with:

$$S[\Psi] = -\pi\int d^3x\Psi^I_{A,0}(\sigma^\mu)_{A\dot{A}}\partial_\mu\bar{\Psi}_{I\dot{A},0}, \tag{4.10}$$

which is the same form for the action that we started with. We thus go from eight fermions in dimension four to eight fermions in dimension three. The same trick will give us eight fermions in dimension two. Now let us do a consistency check and verify that the number of fermions is equal to the number of bosons after dimensional reduction, so that we have not broken supersymmetry. In four dimensions we had eight fermions, two degrees of freedom from the gauge field and six scalar fields. So in four dimensions the degrees of freedom on both the boson as the fermion the side is equal to eight. In two dimensions we still have eight fermions and six scalar fields and we obtained two scalar fields from the gauge field reduction. Thus, also in dimension two we find that the number of bosons and fermions is equal.

4.3 Reduction of $\mathcal{N} = 4$ SYM in $D = 4$ on T^2 for $G = U(1)$

In the previous section we have looked at the Kaluza-Klein reduction on S^1 and have actually done a dimensional reduction on $S^1 \times S^1$ with two different radii. Now we come to the main interest of this section, namely the reduction on a torus T^2. The space we will be looking at in this section is $\mathcal{M} = \mathbb{R}^{1,1} \times T^2$. Again we will use $\hat{\mu}$ indices on the manifold $\mathcal{M}$, while we use μ for the $\mathbb{R}^{1,1}$ part and i for the T^2 part. We follow the techniques in [7], however applied to a different situation.

Let us start by defining a normalization of the gauge fields in the compact directions, done by demanding invariance under large gauge transformations:

$$\oint_{S^1_i} 2A_i \mapsto \oint_{S^1_i} 2A_i + 2\pi.$$

We can define Wilson line variables

$$\varphi_1(x^0, x^1) \equiv 2 \oint S^1_3 A_3 = 2\pi R A_{3,0}$$
$$\varphi_2(x^0, x^1) \equiv 2 \oint S^1_4 A_4 = 2\pi R A_{4,0}$$

These variables thus obey

$$\varphi_1 \sim \varphi_1 + 2\pi, \qquad \varphi_2 \sim \varphi_2 + 2\pi.$$

They thus parametrize a torus $(\varphi_1, \varphi_2) \in \Gamma \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi\mathbb{Z})$, with $\Gamma \simeq \mathbb{Z}^2$. We saw in section 4.1 that the symmetry group of T^2 is given by the modular group $PSL(2, \mathbb{Z})$. So, under a $SL(2, \mathbb{Z})$ transformation of the torus symmetry group, the Wilson line variables transform as:

$$\begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varphi_2 \\ \varphi_1 \end{pmatrix}$$

for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Next, we want to introduce a metric on $\mathcal{M}$. Denote this metric as $\tilde{g}$. Then

$$\tilde{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

with g_{ij} the metric on the torus. Take coordinates on the torus $z = x^2 + \tau x^3$, then the metric is given by

$$ds^2 = |dz|^2 = (dx^2 + \tau dx^3)(dx^2 + \bar{\tau} dx^3) = (dx^2)^2 + 2\mathrm{Re}(\tau) dx^2 dx^3 + |\tau|^2 (dx^3)^2. \quad (4.11)$$

So, we expect something like

$$\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}.$$

We now multiply each row of this matrix by $1/\tau_2$ and we add an overall normalization $\mathcal{V}$ equal to the volume of the torus. We thus obtain as metric:

$$g_{ij} = \frac{\mathcal{V}}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{4.12}$$

We will also need the inverse metric, a short calculation gives us:

$$g^{ij} = \frac{1}{\mathcal{V}\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}. \tag{4.13}$$

Furthermore, the determinant is

$$g = \left(\frac{\mathcal{V}}{\tau_2}\right)^2 (|\tau|^2 - \tau_1^2) = \mathcal{V}^2.$$

Starting again from the action in equation 4.2 (only now multiplied by $\sqrt{g}$), we take things one term at a time. Let us start with the bosonic scalar fields. Reducing over T^2, we obtain:

$$\begin{aligned} S[\phi] &= \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} \partial_{\hat{\nu}}\phi^i \partial^{\hat{\nu}}\phi^i \\ &= \mathcal{V}\int d^2x \int_{T^2} dx^2 dx^3 \left(\partial_\nu \phi^i \partial_\rho \phi^i \eta^{\rho\nu} + \partial_j \phi^i \partial_k \phi^i g^{jk}\right) \\ &= \mathcal{V}\int d^2x \partial_\nu \phi^i \partial^\nu \phi^i. \end{aligned}$$

We obtain the same form of the action as the one we started with. The number of scalar fields stays the same. Notice that any τ-dependence drops out of the equation here, since in the second line only the second term in the integral carries g^{ij}, but is nullified by there being only derivatives parallel to the torus direction.

Next, let us take a look at the gauge vector field. We again start from the appropriate term in the action. Let us begin:

$$\begin{aligned} S[A] &= -\frac{1}{4}\int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} = -\frac{\mathcal{V}}{4}\int d^2x \int_{T^2} dx^2 dx^3 (F_{\mu\nu}F^{\mu\nu} + 2F_{i\nu}F^{i\nu} + 2F_{ij}F^{ij}) \\ &= -\frac{\mathcal{V}}{4}\int d^2x (F_{\mu\nu}F^{\mu\nu} + \frac{2\mathcal{V}}{\tau_2}(\partial_\nu\varphi_1\partial^\nu\varphi_1|\tau|^2 - 2\partial_\nu\varphi_1\partial^\nu\varphi_2\tau_1 + \partial_\nu\varphi_2\partial^\nu\varphi_2)) \end{aligned}$$

Now, defining

$$z = \varphi_2 - \tau\varphi_1 \tag{4.14a}$$
$$\bar{z} = \varphi_2 - \bar{\tau}\varphi_1 \tag{4.14b}$$

then we can write

$$\begin{aligned} \partial_\mu z \partial^\mu \bar{z} &= \partial_\mu(\varphi_2 - \tau\varphi_1)\partial^\mu(\varphi_2 - \bar{\tau}\varphi_1) \\ &= \partial_\mu\varphi_1\partial^\mu\varphi_1|\tau|^2 - 2\partial_\mu\varphi_1\partial^\mu\varphi_2\tau_1 + \partial_\mu\varphi_2\partial^\mu\varphi_2). \end{aligned}$$

So,

$$S[A] = -\frac{\mathcal{V}}{4}\int d^2x (F_{\mu\nu}F^{\mu\nu} + \frac{2\mathcal{V}}{\tau_2}\partial_\mu z \partial^\mu \bar{z}) = \frac{\mathcal{V}^2}{2\tau_2}\int d^2x \partial_\mu z \partial^\mu \bar{z} \tag{4.15}$$

where the last equality was obtained by noticing that F harbours no degrees of freedom in dimension two. We find a complex scalar term in the action obtained by toroidal compactification. It is interesting to look at what happens when we act with a symmetry of the torus on this action. The question is whether such a symmetry is also a symmetry of the Lagrangian. Let us check this. Under $SL(2,\mathbb{Z})$, we have the following:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix}\varphi_2\\ \varphi_1\end{pmatrix} \mapsto \begin{pmatrix}a\varphi_2+b\varphi_1\\ c\varphi_2+d\varphi_1\end{pmatrix}.$$

Thus the complex scalar field z changes as:

$$\begin{aligned}
z' &= \varphi_2' - \tau'\varphi_1' \\
&= a\varphi_2 + b\varphi_1 - \frac{a\tau+b}{c\tau+d}(c\varphi_2+d\varphi_1) \\
&= \frac{(ad-bc)\varphi_2 + (bc-ad)\tau\varphi_1}{c\tau+d} \\
&= \frac{\varphi_2 - \tau\varphi_1}{c\tau+d} \\
&= \frac{z}{c\tau+d}.
\end{aligned}$$

Similarly,

$$\bar{z}' = \frac{\bar{z}}{c\bar{\tau}+d}.$$

How does τ_2 changer under this element of $SL(2,\mathbb{Z})$? Let us calculate this:

$$\begin{aligned}
\frac{a\tau+b}{c\tau+d} &= \frac{ac(\tau_1^2+\tau_2^2) + (ad+bc)\tau_1 + bd + i(ad-bc)\tau_2}{(c\tau+d)(c\bar{\tau}+d)} \\
&= \frac{ac|\tau|^2 + (ad+bc)\tau_1 + bd}{|c\tau+d|^2} + i\frac{\tau_2}{|c\tau+d|^2}.
\end{aligned}$$

So, indeed an $SL(2,\mathbb{Z})$ transformation leaves this Lagrangian invariant. Finally, let us examine the fermion term. We have:

$$\begin{aligned}
S[\Psi] &= -\frac{1}{2}\int d^4x^{\hat{\mu}}\sqrt{\tilde{g}}\Psi^I_A(\sigma^{\hat{\mu}})_{A\dot{A}}\partial_{\hat{\mu}}\bar{\Psi}_{I,\dot{A}} \\
&= -\frac{\mathcal{V}}{2}\int d^2x \int_{T^2} dx^2dx^3(\Psi^I_A(\sigma^{\mu})_{A\dot{A}}\partial_\mu\bar{\Psi}_{I,\dot{A}} + \Psi^I_A(\sigma^i)_{A\dot{A}}\partial_i\bar{\Psi}_{I,\dot{A}}) \\
&= -\frac{\mathcal{V}}{2}\int d^2x\Psi^I_A(\sigma^{\mu})_{A\dot{A}}\partial_\mu\bar{\Psi}_{I,\dot{A}}.
\end{aligned}$$

We find the same form for the action and the same number of fermions after reduction. Notice that also after toroidal compactification the number of fermions and bosons remains equal to each other.

4.4 Reduction of $\mathcal{N}=4$ SYM in $D=4$ on T^2 for G arbitrary

Now we have treated the case $G=U(1)$ extensively to get a feeling for the procedures, we can now look at what happens for arbitrary gauge group G. For an introduction into non-Abelian gauge theories, the reader can for example turn to [21].

The fields take values in Lie algebra $\mathfrak{g}$ of the Lie group G. This means that if the generators of $\mathfrak{g}$ are $\tau_{\tilde{a}}$, we can write the bosonic and fermionic fields as

$$\begin{aligned} A_\mu &= A_\mu^{\tilde{a}} \tau_{\tilde{a}}, \\ \phi_i &= \phi_i^{\tilde{a}} \tau_{\tilde{a}}, \\ \Psi^I_\mu &= \Psi^{I,\tilde{a}} \tau_{\tilde{a}}. \end{aligned}$$

The gauge multiplet transforms under the adjoint representation of $\mathfrak{g}$. Since G in general will not be an Abelian group, some of our expressions have to be altered:

$$\begin{aligned} F_{\mu\nu} \to F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \\ \partial_\mu \to D_\mu &= \partial_\mu + i[A_\mu, \cdot]. \end{aligned}$$

This time we will look at the full $\mathcal{N} = 4$ SYM Lagrangian in dimension four. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} = \mathrm{tr} \Bigg(&-\frac{1}{2g^2} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} + \frac{\theta_I}{8\pi^2} F_{\hat{\mu}\hat{\nu}} \tilde{F}^{\hat{\mu}\hat{\nu}} - i \bar{\Psi}^I \bar{\sigma}^{\hat{\mu}} D_{\hat{\mu}} \Psi_I - D_{\hat{\mu}} \phi^i D^{\hat{\mu}} \phi^i + g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] \\ &+ g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \Bigg), \end{aligned}$$

where we changed notation somewhat from the previous sections. Also, define

$$\tilde{F}_{\hat{\mu}\hat{\nu}} = \frac{1}{2} \epsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} F^{\hat{\rho}\hat{\sigma}}. \tag{4.16}$$

We will now perform the reduction one term at a time. Let us start with the first term:

$$S_1 = -\frac{1}{2g^2} \mathrm{tr} \int d^4 x^{\hat{\mu}} \sqrt{\tilde{g}} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}}. \tag{4.17}$$

We find:

$$\begin{aligned} S_1 &= -\frac{1}{2g^2} \mathrm{tr} \int d^4 x^{\hat{\mu}} \sqrt{\tilde{g}} F_{\hat{\mu}\hat{\nu}} F^{\hat{\mu}\hat{\nu}} \\ &= -\frac{1}{2g^2} \mathrm{tr} \int d^4 x^{\hat{\mu}} \sqrt{g} \left(F_{\mu\nu} F^{\mu\nu} + 2 F_{\mu i} F_{\nu j} \eta^{\mu\nu} g^{ij} + F_{ij} F_{kl} g^{ik} g^{jl} \right) \\ &= -\frac{\mathcal{V}}{2g^2} \mathrm{tr} \int d^2 x \left(F_{\mu\nu} F^{\mu\nu} + 2 (\partial_\mu \varphi_i + i[A_\mu, \varphi_i]) (\partial_\nu \varphi_j + i[A_\nu, \varphi_j]) \eta^{\mu\nu} g^{ij} \right. \\ &\left. - [\varphi_i, \varphi_j][\varphi_k, \varphi_l] g^{ik} g^{jl} \right). \end{aligned}$$

Next, let us analyze the second term. This term is given by

$$S_2 = \frac{\theta_I}{8\pi^2} \mathrm{tr} \int d^4 x^{\hat{\mu}} \sqrt{\tilde{g}} F_{\hat{\mu}\hat{\nu}} \tilde{F}^{\hat{\mu}\hat{\nu}}. \tag{4.18}$$

After starting the dimensional reduction procedure, we find:

$$
\begin{aligned}
S_2 &= \frac{\theta_I}{8\pi^2} \text{ tr} \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} F_{\hat{\mu}\hat{\nu}} \tilde{F}^{\hat{\mu}\hat{\nu}} \\
&= \frac{\theta_I \mathcal{V}}{16\pi^2} \text{ tr} \int d^2x \int_{T^2} dx^2 dx^3 \left(F^{\mu\nu} \epsilon_{\mu\nu ij} g^{ik} g^{jl} F_{kl} + F_{ij} g^{ik} g^{jt} \epsilon_{kt\mu\nu} F^{\mu\nu} \right. \\
&\left. +4F_{\mu i} \eta^{\mu\kappa} g^{ij} \epsilon_{\kappa j\nu k} \eta^{\nu\tau} g^{kl} F_{\tau l} \right) \\
&= \frac{\theta_I \mathcal{V}}{16\pi^2} \text{ tr} \int d^2x \left(F^{\mu\nu} \epsilon_{\mu\nu ij} g^{ik} g^{jl} i[\varphi_k, \varphi_l] + i[\varphi_i, \varphi_j] g^{ik} g^{jt} \epsilon_{kt\mu\nu} F^{\mu\nu} \right. \\
&\left. +4(\partial_\mu \varphi_i + i[A_\mu, \varphi_i]) \eta^{\mu\kappa} g^{ij} \epsilon_{\kappa j\nu k} \eta^{\nu\tau} g^{kl} (\partial_\tau \varphi_l + i[A_\tau, \varphi_l]) \right) .
\end{aligned}
$$

Let us now focus on the third term:

$$
S_3 = - \text{ tr} \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} i \bar{\Psi}^I \bar{\sigma}^{\hat{\mu}} D_{\hat{\mu}} \Psi_I, \tag{4.19}
$$

containing the fermionic fields. Let us start:

$$
\begin{aligned}
S_3 &= - \text{ tr} \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} i \bar{\Psi}^I \bar{\sigma}^{\hat{\mu}} D_{\hat{\mu}} \Psi_I \\
&= -\mathcal{V} \text{ tr} \int d^2x \int_{T^2} dx^2 dx^3 i \bar{\Psi}^I (\bar{\sigma}^\mu D_\mu \Psi_I + \bar{\sigma}_i g^{ij} D_j \Psi_I) \\
&= -\mathcal{V} \text{ tr} \int d^2x i \bar{\Psi}^I (\bar{\sigma}^\mu D_\mu \Psi_I + \bar{\sigma}_i g^{ij} [\varphi_j, \Psi_I]) \\
&= -\mathcal{V} \text{ tr} \int d^2x i (\bar{\Psi}^I \bar{\sigma}^\mu D_\mu \Psi_I + \bar{\Psi}^I \bar{\sigma}_i g^{ij} [\varphi_j, \Psi_I]).
\end{aligned}
$$

We find the original term in two dimensions lower plus an extra coupling term. Next, let us do the fourth term:

$$
S_4 = - \text{ tr} \int d^4x^{\hat{\mu}} D_{\hat{\mu}} \phi^i D^{\hat{\mu}} \phi^i. \tag{4.20}
$$

We again split the sum in term containing toroidal indices and terms that do not and perform the toroidal reduction. We find:

$$
\begin{aligned}
S_4 &= - \text{ tr} \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}} D_{\hat{\mu}} \phi^i D^{\hat{\mu}} \phi^i \\
&= -\mathcal{V} \text{ tr} \int d^2x \int_{T^2} dx^2 dx^3 \left(D_\mu \phi^i D^\mu \phi^i + D_j \phi^i D^j \phi^i \right) \\
&= -\mathcal{V} \text{ tr} \int d^2x \left(D_\mu \phi^i D^\mu \phi^i - g^{jk} [\varphi_j, \phi^i][\varphi_k, \phi^i] \right) .
\end{aligned}
$$

We obtain an extra coupling term. Finally, we group the remaining terms of the full action together:

$$
S_5 = \text{tr} \int d^4x^{\hat{\mu}} \sqrt{\tilde{g}^{-1}} \left(g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] + g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \right) , \tag{4.21}
$$

since these do not contain any $\hat{\mu}$-indices. We thus can, in a straightforward manner, integrate over the toroidal dimensions. Let us do this:

$$\begin{aligned}
S_5 &= \mathrm{tr} \int d^4 x^{\hat{\mu}} \sqrt{\tilde{g}} \left(g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] + g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \right) \\
&= \mathcal{V}\ \mathrm{tr} \int d^2 x \int_{T^2} dx^2 dx^3 \left(g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] + g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \right) \\
&= \mathcal{V}\ \mathrm{tr} \int d^2 x \left(g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] + g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \right).
\end{aligned}$$

We obtain the same terms that we started with. Let us now group all terms together again, to get the full $N = 4$ SYM action after dimensional reduction on a torus. We find:

$$\begin{aligned}
S = \mathcal{V}\mathrm{tr} \int d^2 x \Bigg[&-\frac{1}{2g^2} \big(F_{\mu\nu} F^{\mu\nu} + 2(\partial_\mu \varphi_i + i[A_\mu, \varphi_i])(\partial_\nu \varphi_j + i[A_\nu, \varphi_j]) \eta^{\mu\nu} g^{ij} \\
&- [\varphi_i, \varphi_j][\varphi_k, \varphi_l] g^{ik} g^{jl} \big) + \frac{\theta_I}{16\pi^2} \big(F^{\mu\nu} \epsilon_{\mu\nu ij} g^{ik} g^{jl} i[\varphi_k, \varphi_l] + i[\varphi_i, \varphi_j] g^{ik} g^{jt} \epsilon_{kt\mu\nu} F^{\mu\nu} \\
&+ 4(\partial_\mu \varphi_i + i[A_\mu, \varphi_i]) \eta^{\mu\kappa} g^{ij} \epsilon_{\kappa j \nu k} \eta^{\nu\tau} g^{kl} (\partial_\tau \varphi_l + i[A_\tau, \varphi_l]) \big) \\
&- i(\bar{\Psi}^I \bar{\sigma}^\mu D_\mu \Psi_I + \bar{\Psi}^I \bar{\sigma}_i g^{ij} [\varphi_j, \Psi_I]) - \big(D_\mu \phi^i D^\mu \phi^i - g^{jk} [\varphi_j, \phi^i][\varphi_k, \phi^i] \big) \\
&+ \left(g C_i^{IJ} \Psi_I [\phi^i, \Psi_J] + g \bar{C}_{iIJ} \bar{\Psi}^I [\phi^i, \bar{\Psi}^J] + \frac{g^2}{2} [\phi^i, \phi^j]^2 \right) \Bigg].
\end{aligned}$$

4.5 Determining the dynamical phases

Next, let us determine the dynamical phases for this theory. We look at a specific group $G = SU(2)$. To analyse the dynamics of the $\mathcal{N} = 4$ SYM theory, we look at the potential energy term in the action, given by

$$V = \mathrm{tr} \frac{g^2}{2} [\phi^i, \phi^j]^2. \tag{4.22}$$

If $G = SU(2)$, we have for the Lie algebra $\mathfrak{g}$ of G:

$$\mathfrak{g} = \{ A \in M_{2\times 2}(\mathbb{C}) |\ \mathrm{tr}(A) = 0, A^\dagger = -A \}. \tag{4.23}$$

Thus, we can write $\mathfrak{g}$ as the span:

$$\begin{aligned}
\mathfrak{g} &= \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\rangle \\
&= \langle i\sigma_1, i\sigma_2, i\sigma_3 \rangle,
\end{aligned}$$

with the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.24}$$

These satisfy the usual commutation relation

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c. \tag{4.25}$$

Let us now write the potential term in equation 4.22 in terms of the Lie algebra generators. Because we have an expression for these terms we can apply the commutators and trace and find a meaningful expression. Let us do this:

$$\begin{aligned}
V &= \operatorname{tr}\frac{g^2}{2}[\phi^i,\phi^j]^2 \\
&= \frac{g^2}{2}\operatorname{tr}\sum_{i,j=1}^{6}[\phi^{ia}i\sigma_a,\phi^{jb}i\sigma_b]^2 \\
&= \frac{g^2}{2}\operatorname{tr}\sum_{i,j=1}^{6}(-\phi^{ia}\phi^{jb}[\sigma_a,\sigma_b])^2 \\
&= -4g^2\operatorname{tr}\sum_{i,j=1}^{6}(\phi^{i1}\phi^{j2}\sigma_3-\phi^{i2}\phi^{j1}\sigma_3+\phi^{i2}\phi^{j3}\sigma_1-\phi^{i3}\phi^{j2}\sigma_1+\phi^{i3}\phi^{j1}\sigma_2-\phi^{i1}\phi^{j3}\sigma_2)^2.
\end{aligned}$$

Now, all cross-products in the $(\cdots)^2$-term will have trace zero and thus drop out of the expression because of the properties of Pauli matrices. We find:

$$\begin{aligned}
V &= -4g^2\operatorname{tr}\sum_{i,j=1}^{6}(\phi^{i1}\phi^{j2}\sigma_3-\phi^{i2}\phi^{j1}\sigma_3+\phi^{i2}\phi^{j3}\sigma_1-\phi^{i3}\phi^{j2}\sigma_1+\phi^{i3}\phi^{j1}\sigma_2-\phi^{i1}\phi^{j3}\sigma_2)^2 \\
&= -8g^2\operatorname{tr}\sum_{i,j=1}^{6}\left((\phi^{i1}\phi^{j2}-\phi^{i2}\phi^{j1})^2+(\phi^{i2}\phi^{j3}-\phi^{i3}\phi^{j2})^2+(\phi^{i3}\phi^{j1}-\phi^{i1}\phi^{j3})^2\right).
\end{aligned}$$

If we now demand that this expression is equal to zero, we find the ground state. The above expression is zero if

$$\phi^{i1}=0,\qquad \phi^{i2}=0\qquad\text{and}\qquad \phi^{i3}\text{ arbitrary.} \tag{4.26}$$

This means that

$$\phi^i=\begin{pmatrix} x^i & 0\\ 0 & -x^i\end{pmatrix}. \tag{4.27}$$

Next, let us look at the kinetic term when $V=0$. The kinetic term is given by:

$$K=-\operatorname{tr}D_\mu\phi^iD^\mu\phi^i. \tag{4.28}$$

We do the same thing as before, only this time using the expression for ϕ^i given in equation 4.27. We find:

$$\begin{aligned}
K &= -\operatorname{tr}D_\mu\phi^iD^\mu\phi^i \\
&= -\operatorname{tr}\left((\partial_\mu\phi^i+i[A_\mu,\phi^i])(\partial^\mu\phi^i+i[A^\mu,\phi^i])\right) \\
&= -\operatorname{tr}\left(\partial_\mu\phi^i\partial^\mu\phi^i+i[A_\mu,\phi^i]\partial^\mu\phi^i+\partial_\mu\phi^i i[A^\mu,\phi^i])-[A_\mu,\phi^i][A^\mu,\phi^i]\right).
\end{aligned}$$

Let us analyse the last term of this expression, since this will tell us something about which components of the gauge field get a mass contribution. We obtain:

$$\begin{aligned}
T &= \operatorname{tr}[A_\mu,\phi^i][A^\mu,\phi^i] \\
&= \operatorname{tr}A^a_\mu x^i[\sigma_a,\sigma_3]A^{\mu b}x^i[\sigma_b,\sigma_3] \\
&= |x|^2\operatorname{tr}A^a_\mu\epsilon_{a3c}\sigma_cA^{\mu b}\epsilon_{b3d}\sigma_d \\
&= |x|^2\operatorname{tr}\left(A^2_\mu\sigma_1-A^1_\mu\sigma_2\right)\left(A^{\mu2}\sigma_1-A^{\mu1}\sigma_2\right) \\
&= 2|x|^2(A^2_\mu A^{\mu2}+A^1_\mu A^{\mu1}).
\end{aligned}$$

We thus have two cases:

- $|x| = 0$: the superconformal phase. No components of the gauge field get a mass term.
- $|x| \neq 0$: the Coulomb branch. The components $A^{\mu 1}, A^{\mu 2}$ of the gauge field get a mass term, $A^{\mu 3}$ does not get a mass term.

In chapter 8, we will discuss the relation between $\mathcal{N} = 4$ SYM and type IIB superstring theory on a D3-brane. In order to relate the central charges we obtain there to a central charge calculation we could do here, we will need one extra assumption, which we will introduce. This is due to the topological duality twist we perform in chapter 8, for which we need some extra information. However, this chapter will help us in understanding how the theory changes after dimensional reduction. Namely, in chapter 8 we perform a reduction over a Riemann curve C of genus g. Specifying $g = 1$ then thus brings us back to the case studied in this chapter.

Chapter 5

Toric Geometry

In this chapter, we will work out the preliminaries involved with constructing elliptically fibered Calabi-Yau spaces. In our construction, these Calabi-Yau spaces will emerge as toric hypersurfaces in some toric ambient space. We follow Batyrev's construction in this regard and discuss this in chapter 6. Chapter seven of [11] gives a good overview of why we are interested in these spaces.

In order to understand Batyrev's construction, we need to take a look at some of the basics in toric geometry. We start out with asking the question what a toric variety is. After that, we discuss the fan- and polyhedron description of toric varieties. We apply this to the example of $\mathbb{P}^2$. Furthermore, we discuss singularities and how to resolve them.

For an introduction to toric varieties, the reader should turn to [5].

5.1 Overview of Toric geometry

5.1.1 The set-up

To see what a toric variety is, we first need to deal with some definitions. Toric varieties will be constructed from a lattice N, of dimension $\dim N = n$. We can set $N \simeq \mathbb{Z}^n$. The lattice N has a dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. We denote the dual pairing by $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Denote their respective real completions by $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Definition 7. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set

$$\sigma = \{a_1 v_1 + ... + a_s v_s \in N_{\mathbb{R}} | a_1, ..., a_s \geq 0\},$$

for fixed $v_1, ..., v_s \in N_{\mathbb{R}}$

Definition 8. A convex rational polyhedral cone in $N_{\mathbb{R}}$ is a set

$$\sigma = \{a_1 v_1 + ... + a_s v_s \in N_{\mathbb{R}} | a_1, ..., a_s \geq 0\},$$

for fixed $v_1, ..., v_s \in N$.

Definition 9. A strongly convex polyhedral cone in $N_{\mathbb{R}}$ is a set

$$\sigma = \{a_1 v_1 + ... + a_s v_s \in N_{\mathbb{R}} | a_1, ..., a_s \geq 0\},$$

with $v_1, ..., v_s \in N_{\mathbb{R}}$, such that σ contains no line through the origin. This means that we cannot have $v_i = -v_j$ for any $i \neq j$.

We will denote a strongly convex rational polyhedral cone by the word cone from now on. The dual of a cone $\sigma \subset N_{\mathbb{R}}$ is denoted by $\sigma^\vee \subset M_{\mathbb{R}}$ and is given by

$$\sigma^\vee = \{u \in M_{\mathbb{R}} | \langle u, v \rangle \geq 0 \forall v \in \sigma\}.$$

The next definition we need is that of a fan.

Definition 10. A fan Σ in N is a set of cones σ in $N_{\mathbb{R}}$ such that each face of a cone in Σ is also a cone in Σ and the intersection of two cones in Σ is a face of each.

We will be able to construct toric varieties starting from a given cone or (more generally) a given fan. This construction is explained in section 5.1.2. There is another way to construct toric varieties, namely when we start from a convex polyhedra. Let us define what this is first.

Definition 11. A convex polyhedra Δ in $N_{\mathbb{R}}$ is the convex hull of a finite set of points in $N_{\mathbb{R}}$. A rational convex polyhedra is a convex polyhedra which is the convex hull of a finite set of points in N.

The construction using a polyhedra instead of a fan is explained in section 5.1.3. The two ways of constructing toric varieties are not entirely equivalent to each other. We will show how and when we can go from one description to the other. Let us start with the fan description.

5.1.2 Fan description

We start with a given fan Σ in $N_{\mathbb{R}}$. For each cone $\sigma \in \Sigma$, determine the dual cone $\sigma^\vee$ in $M_{\mathbb{R}}$. This determines a commutative semigroup

$$S_\sigma = \sigma^\vee \cap M. \tag{5.1}$$

The corresponding group algebra to this commutative semigroup is then $\mathbb{C}[S_\sigma]$, a finitely generated commutative $\mathbb{C}$-algebra. The corresponding affine variety to σ is given by

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]). \tag{5.2}$$

We define $\mathbb{P}_\Sigma$, the toric variety corresponding to the fan Σ, as the disjoint union

$$\mathbb{P}_\Sigma = \bigsqcup_{\sigma \in \Sigma} U_\sigma \tag{5.3}$$

with U_σ glued to U_τ on $U_{\sigma \cap \tau}$ for $\sigma, \tau \in \Sigma$.

We can thus define a toric variety as being a variety obtained in this way. With this construction we can also immediately see where the name 'toric variety' comes from. Take $\Sigma = \sigma = (0)$ the origin, which is a cone. We have $\sigma^\vee = M_{\mathbb{R}}$ and thus $S_\sigma = M$. If $e_1^*, ..., e_n^*$ form a basis for M then $\pm e_1^*, ..., \pm e_n^*$ are generators for the semigroup S_σ. Thus,

$$\mathbb{C}[S_\sigma] = \mathbb{C}[M] = \mathbb{C}[X_1^\pm, ..., X_n^\pm],$$

and $U_\sigma = \mathrm{Spec}(\mathbb{C}[M]) = (\mathbb{C}^*)^n$. Now, $(\mathbb{C}^*)^n = T$ the torus of algebraic groups. Since (0) is a cone in every fan Σ, we see that each toric variety $\mathbb{P}_\Sigma$ contains a torus as a dense open subset.

5.1.3 Polyhedra description

Another definition for a toric variety is given in terms of the polyhedra construction. Take Δ a rational convex n-dimensional polyhedra in $M_{\mathbb{R}}$. Define the $n+1$-convex cone supporting Δ by

$$C_\Delta = 0 \cup \{(x_0, ..., x_n) \in \mathbb{R} \oplus M_{\mathbb{R}} | \left(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right) \in \Delta, x_0 > 0\}. \tag{5.4}$$

Furthermore, define S_Δ to be the graded subring of $\mathbb{C}[X_0, X_1^{\pm}, ..., X_n^{\pm}]$ with complex basis consisting of monomials $X_0^{m_0}...X_n^{m_n}$ such that $(m_0, ..., m_n) \in C_\Delta$. Then the n-dimensional projective toric variety corresponding to the graded ring S_Δ is given by

$$\mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta). \tag{5.5}$$

We also define $\mathcal{O}_\Delta(1)$ to be the ample invertible sheaf on $\mathbb{P}_\Delta$ corresponding to the graded S_Δ-module $S_\Delta(-1)$.

5.1.4 Comparing covariant and contravariant definition

The definition of toric varieties in terms of the polyhedra construction is also called the contravariant definition. It is less general than the construction in terms of fans, which is called the covariant definition. First of all, the contravariant definition makes a choice for the ample invertible sheaf $\mathcal{O}_\Delta(1)$. It also gives us an embedding of $\mathbb{P}_\Delta$ in projective space. In general there are infinitely many ample sheaves on $\mathbb{P}_\Delta$. We thus get infinitely many rational polyhedra which correspond to isomorphic toric varieties. If we want a one-to-one correspondence, we need to use the covariant definition. This also gives us the freedom to construct affine and quasi-projective toric varieties (instead of just projective varieties) as well as complete toric varieties which are not quasi-projective.

Let us now show how to go from one piece of toric data to the other. In other words, let us show how to construct a fan out of a polyhedra and how to construct a polyhedra out of a fan.

Constructing $\Sigma(\Delta)$

For every l-dimensional face $\Theta \subset \Delta \subset M_{\mathbb{R}}$, we define the convex n-dimensional cone $\sigma^\vee(\Theta) \subset M_{\mathbb{R}}$ consisting of all vectors $\lambda(p - p')$ for which $\lambda \in \mathbb{R}_{\geq 0}$, $p \in \Delta$ and $p' \in \Theta$. Take $\sigma(\Theta) \subset N_{\mathbb{R}}$ to be the dual cone relative to $\sigma^\vee(\Theta)$. The set $\Sigma(\Delta)$ is given by all cones $\sigma(\Theta)$. We then have $\mathbb{P}_\Sigma \simeq \mathbb{P}_\Delta$.

Constructing $\Delta(\Sigma)$

To go from a fan to a polyhedra description, we need a support function.

Definition 12. Take Σ a fixed rational polyhedral fan. $h : N_{\mathbb{R}} \to \mathbb{R}$ is called a support function for Σ if h is linear on all cones $\sigma \subset \Sigma$. A support function h is called integral if $h(N) \subset \mathbb{Z}$. h is called upper convex if $h(x + x') \leq h(x) + h(x')$ for all $x, x' \in N_{\mathbb{R}}$. h is called strictly upper convex if h is upper convex and for all two distinct n-dimensional cones σ, σ' in Σ, $h_\sigma = h|_\sigma$ and $h_{\sigma'} = h|_{\sigma'}$ are two distinct linear equations.

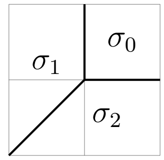


Figure 5.1: The fan Σ_B of $\mathbb{P}^2$, drawn in N.

For h integral strictly upper convex, h corresponds one-to-one to a T-linearised ample invertible sheave $\mathcal{O}(D_h)$ over $\mathbb{P}_\Sigma$, with D_h a T-invariant $\mathbb{R}$-Cartier divisor D_h on P_Σ. We now define the convex polyhedron $\Delta(\Sigma)$ as follows:

$$\Delta(\Sigma) = \bigcap_{\sigma \in \Sigma^{(n)}} (-h_\sigma + \sigma^\vee) \tag{5.6}$$

with $(h_\sigma : N \to \mathbb{Z}) \in M$ and $\Sigma^{(n)}$ the set of all n-dimensional cones in Σ. Then one has $\mathbb{P}_\Delta \simeq \mathbb{P}_\Sigma$ and $\mathcal{O}_\Delta(1) \simeq \mathcal{O}(D_h)$.

Note that the construction of Δ from a fan Σ such that their respective toric varieties are isomorphic is not unique and depends on the choice of the support function. There is an important example where we can make a natural choice.

Definition 13. A toric variety $\mathbb{P}_\Sigma$ is a toric $\mathbb{R}$-Fano variety if the anti-canonical support function h_K, corresponding to the anti-canonical divisor $-K_\Sigma = \mathbb{P}_\Sigma \backslash T$, is strictly upper convex. A toric $\mathbb{R}$-Fano variety is called a Gorenstein toric Fano variety if h_K is integral.

For every $\mathbb{R}$-Gorenstein toric variety $\mathbb{P}_\Sigma$ we have the unique integral strictly upper convex support function h_K.

For every toric $\mathbb{R}$-Fano variety, we can associate two convex polyhedra to a given fan Σ:

$$\Delta(\Sigma, h_K) = \bigcap_{\sigma \subset \Sigma^{(n)}} (-\left. h_K \right|_\sigma + \sigma^\vee) \subset M_\mathbb{R},$$
$$\Delta^*(\Sigma, h_K) = \{v \in N_\mathbb{R} | h_K(v) \leq 1\} \subset N_\mathbb{R}.$$

Let us now look at some examples, to better understand how both constructions work and how we can relate them.

5.2 $\mathbb{P}^2$

5.2.1 Fan description of $\mathbb{P}^2$

Let us start by constructing $\mathbb{P}^2$ in the language of fans. We have $\dim N = 2$, with fixed basis e_1, e_2. Consider the cone σ_0 generated by e_1 and e_2, the cone σ_1 generated by e_2 and $-e_1 - e_2$ and finally the cone σ_2 generated by e_1 and $-e_1 - e_2$. Denote the corresponding fan by Σ_B, which we draw in figure 5.1. Denote $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual space of N. We have that $\dim M = 2$ and M is generated by e_1^* and e_2^*, subjected to the relations $\langle e_i^*, e_i \rangle = 1$, $\langle e_i^*, e_j \rangle = 0$ for $i, j = 1, 2$ and $i \neq j$. Let us determine the dual cones for this fan. We have that $\sigma_i^\vee$ for $i = 0, 1, 2$ is given by:

$$\sigma_i^\vee = \{x \in M_\mathbb{R} | \langle x, y \rangle \ \geq 0 \forall y \in \sigma_i\}. \tag{5.7}$$

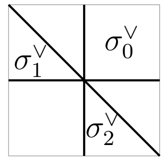


Figure 5.2: The dual fan Σ_B^* of $\mathbb{P}^2$, drawn in M.

Thus,

$$\begin{aligned}
\sigma_0^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_1 + be_2\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{ae_1^* + be_2^* | a, b \in \mathbb{R}_{\geq 0}\}, \\
\sigma_1^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_2 + b(-e_1 - e_2)\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{-ae_1^* + b(-e_1^* + e_2^*) | a, b \in \mathbb{R}_{\geq 0}\}, \\
\sigma_2^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_1 + b(-e_1 - e_2)\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{-ae_2^* + b(e_1^* - e_2^*) | a, b \in \mathbb{R}_{\geq 0}\}.
\end{aligned}$$

Together, these form the dual fan Σ_B^* of $\mathbb{P}^2$, drawn in figure 5.2. The associated semi-groups $S_{\sigma_i} = \sigma_i^\vee \cap M$ are given by:

$$\begin{aligned}
S_{\sigma_0} &= \{ae_1^* + be_2^* | a, b \in \mathbb{N}\}, \\
S_{\sigma_1} &= \{-ae_1^* + b(-e_1^* + e_2^*) | a, b \in \mathbb{N}\}, \\
S_{\sigma_2} &= \{-ae_2^* + b(e_1^* - e_2^*) | a, b \in \mathbb{N}\}.
\end{aligned}$$

The corresponding group algebras are given by:

$$\begin{aligned}
\mathbb{C}[S_{\sigma_0}] &= \mathbb{C}[X, Y], \\
\mathbb{C}[S_{\sigma_1}] &= \mathbb{C}[X^{-1}, X^{-1}Y], \\
\mathbb{C}[S_{\sigma_2}] &= \mathbb{C}[Y^{-1}, XY^{-1}],
\end{aligned}$$

with X, Y the elements in $\mathbb{C}[M]$ corresponding to the dual basis. We obtain a variety $U_{\sigma_i} = \mathrm{Spec}(\mathbb{C}[S_{\sigma_i}])$ for $i = 0, 1, 2$. Thus we obtain the following coordinate patches: $U_{\sigma_0} \simeq \mathbb{C}^2$ with coordinates (X, Y), $U_{\sigma_1} \simeq \mathbb{C}^2$ with coordinates $(X^{-1}, X^{-1}Y)$ and $U_{\sigma_2} \simeq \mathbb{C}^2$ with coordinates (Y^{-1}, XY^{-1}). What remains is to glue these varieties together to obtain the toric variety $\mathbb{P}_{\Sigma_B}$ corresponding to Σ_B. We define $\mathbb{P}_{\Sigma_B}$ as

$$\mathbb{P}_{\Sigma_B} = \bigsqcup_{\sigma \in \Sigma_B} U_\sigma \tag{5.8}$$

with U_σ glued to U_τ on $U_{\sigma\cap\tau}$ for $\sigma, \tau \in \Sigma_B$. Take $[T_0 : T_1 : T_2]$ homogeneous coordinates on $\mathbb{P}^2$. Then we have

$$\begin{aligned}
X &= \frac{T_1}{T_0}, Y = \frac{T_2}{T_0} && \text{on } U_{\sigma_0} \text{ with } T_0 \neq 0, \\
X^{-1} &= \frac{T_0}{T_1}, X^{-1}Y = \frac{T_2}{T_1} && \text{on } U_{\sigma_1} \text{ with } T_1 \neq 0, \\
Y^{-1} &= \frac{T_0}{T_2}, XY^{-1} = \frac{T_1}{T_2} && \text{on } U_{\sigma_2} \text{ with } T_2 \neq 0.
\end{aligned}$$

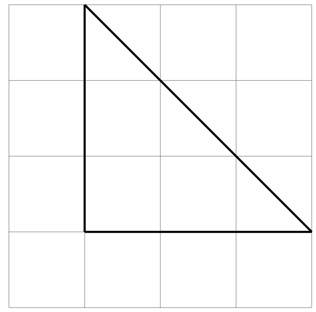


Figure 5.3: The polyhedra corresponding to $\mathbb{P}^2$.

The patching maps are given by:

$$\begin{aligned}
[T_0 : T_1 : T_2] &\longleftrightarrow [T_1 : T_0 : T_2] \qquad && \text{on } U_{\sigma_0 \cap \sigma_1}, \\
[T_0 : T_1 : T_2] &\longleftrightarrow [T_2 : T_0 : T_1] \qquad && \text{on } U_{\sigma_0 \cap \sigma_2}, \\
[T_1 : T_0 : T_2] &\longleftrightarrow [T_2 : T_0 : T_1] \qquad && \text{on } U_{\sigma_1 \cap \sigma_2}.
\end{aligned}$$

So, indeed $\mathbb{P}_{\Sigma_B} = \mathbb{P}^2$.

5.2.2 Polyhedra description of $\mathbb{P}^2$

Let us study the polyhedra description of $\mathbb{P}^2$. Take Δ_B to be the 2-dimensional polyhedra in $M_{\mathbb{R}}$, given by:

$$\Delta_B = \operatorname{conv}\left(\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}\right\}\right).$$

The polyhedra Δ_B is depicted in figure 5.3. The corresponding convex cone supporting Δ_B is then given by:

$$C_{\Delta_B} = 0 \cup \{(x_0, x_1, x_2) \in \mathbb{R} \oplus M_{\mathbb{R}} | \left(\frac{x_1}{x_0}, \frac{x_2}{x_0} \in \Delta_B, x_0 > 0\right)\}.$$

So, at height $x_0 = 1$ we have as level section the original Δ_B, at height $x_0 = k \in \mathbb{Z}$ we have as level section the original shape of Δ_B, but all distances between vertices have been multiplied by k. So,

$$S_{\Delta_B} = \mathbb{C}[X_0 X_1^{-1} X_2^{-1}, X_0 X_1^2 X_2^{-1}, X_0 X_1^{-1} X_2^2] \subset \mathbb{C}[X_0, X_1^{\pm 1}, X_2^{\pm 1}].$$

Define

$$X = X_0 X_1^{-1} X_2^{-1}, \qquad Y = X_0 X_1^2 X_2^{-1}, \qquad Z = X_0 X_1^{-1} X_2^2,$$

then for $X \neq 0$:

$$\frac{Y}{X} = X_1^3; \qquad \frac{Z}{X} = X_2^3,$$

for $Y \neq 0$:

$$\frac{X}{Y} = X_1^{-3}; \qquad \frac{Z}{Y} = X_1^{-3} X_2^3,$$

and for $Z \neq 0$:

$$\frac{X}{Z} = X_2^{-3}; \qquad \frac{Y}{Z} = X_1^3 X_2^{-3}.$$

The 2-dimensional toric variety corresponding to the graded ring S_{Δ_B} is

$$\mathbb{P}_{\Delta_B} = \mathrm{Proj}(S_{\Delta_B}) = \mathbb{P}^2,$$

where the choice for ample invertible sheaf is $\mathcal{O}_{\Delta_B}(1) = \mathcal{O}(3)$.

5.2.3 Duality between toric data for $\mathbb{P}^2$

From polyhedra to fan Given $\Delta_B = \mathrm{conv}(\{(2,-1),(-1,2),(-1,-1)\})$, the fan Σ_B is constructed as follows. The zero-dimensional faces of Δ_B are $\theta_0 = (2,-1)$, $\theta_1 = (-1,2)$ and $\theta_2 = (-1,-1)$. Then

$$\sigma^\vee(\theta_i) = \sigma_i^\vee = \{\lambda(p-\theta_i)|\lambda \geq 0, p \in \Delta\},$$

for $i = 0,1,2$. So, we find that

$$\begin{aligned}
\sigma_0^\vee &= \left\langle \begin{pmatrix}-1\\2\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}} = \left\langle \begin{pmatrix}-3\\3\end{pmatrix}, \begin{pmatrix}-3\\0\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_1^\vee &= \left\langle \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}} = \left\langle \begin{pmatrix}3\\-3\end{pmatrix}, \begin{pmatrix}0\\-3\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_2^\vee &= \left\langle \begin{pmatrix}2\\-1\end{pmatrix} - \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} - \begin{pmatrix}-1\\-1\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}} = \left\langle \begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}.
\end{aligned}$$

In other words,

$$\begin{aligned}
\sigma_0^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_2 + b(-e_1 - e_2)\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{-ae_1^* + b(-e_1^* + e_2^*) | a, b \in \mathbb{R}_{\geq 0}\}, \\
\sigma_1^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_1 + b(-e_1 - e_2)\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{-ae_2^* + b(e_1^* - e_2^*) | a, b \in \mathbb{R}_{\geq 0}\}, \\
\sigma_2^\vee &= \{x \in M_\mathbb{R} | \langle x, ae_1 + be_2\rangle \ \geq 0 \forall a, b \in \mathbb{R}_{\geq 0}\} \\
&= \{ae_1^* + be_2^* | a, b \in \mathbb{R}_{\geq 0}\}.
\end{aligned}$$

This is what we found for the dual cones in the fan description after we relabel the indexes. Thus taking the dual fan of the fan generated by $\sigma_0^\vee, \sigma_1^\vee$ and $\sigma_2^\vee$, we get back our original Σ_B (since $(\sigma_i^\vee)^\vee = \sigma_i$ for all $i = 0,1,2$).

From fan to polyhedra We have

$$\begin{aligned}
\Delta_B(\Sigma_B, h_K) &= \bigcap_{\sigma \in \Sigma^{(n)}} (-h_\sigma + \sigma^\vee) \\
&= (-h_{\sigma_0} + \sigma_0^\vee) \bigcap (-h_{\sigma_1} + \sigma_1^\vee) \bigcap (-h_{\sigma_2} + \sigma_2^\vee)
\end{aligned}$$

Define $D = -K = D_1 + D_2 + D_3$, where D_i are divisors corresponding to the v_i^* coordinate $\{x_i = 0\}$, with v_i spanning Σ. We have

$$v_0 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_1 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad v_2 \equiv \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Now the coefficients a_i in front of the D_i in the definition of D can be written in terms of a function $\psi_D(v_i) = a_i$, with

$$\psi_D(v) = \langle u(\sigma), v \rangle .$$

Then $u(\sigma_i) = h_{\sigma_i}$. We thus need to find what the $u(\sigma_i)$ are:

$$\begin{aligned}
\langle u(\sigma_0), v_0 \rangle &= \left\langle u(\sigma_0), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = a_0 = 1, \langle u(\sigma_0), v_1 \rangle = \left\langle u(\sigma_0), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = a_1 = 1; \\
\langle u(\sigma_1), v_1 \rangle &= \left\langle u(\sigma_1), \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = a_1 = 1, \langle u(\sigma_1), v_2 \rangle = \left\langle u(\sigma_1), \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\rangle = a_2 = 1; \\
\langle u(\sigma_2), v_0 \rangle &= \left\langle u(\sigma_2), \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle = a_0 = 1, \langle u(\sigma_2), v_2 \rangle = \left\langle u(\sigma_2), \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\rangle = a_2 = 1.
\end{aligned}$$

We find:

$$u(\sigma_0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad u(\sigma_1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \qquad u(\sigma_2) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\Delta_B(\Sigma_B, h_K) &= (-h_{\sigma_0} + \sigma_0^\vee) \bigcap (-h_{\sigma_1} + \sigma_1^\vee) \bigcap (-h_{\sigma_2} + \sigma_2^\vee) \\
&= \left(- \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sigma_0^\vee \right) \bigcap \left(- \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \sigma_1^\vee \right) \bigcap \left(- \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \sigma_2^\vee \right).
\end{aligned}$$

When we take the intersections of these spaces, we indeed find Δ_B. Another interesting thing we can do is calculate Δ_B^*. Let us do this. We have

$$\Delta_B^*(\Sigma_B, h_K) = \{y \in N_{\mathbb{R}} | h_K(y) \leq 1\}.$$

We know that for $y \in \sigma_i$ it holds that

$$h_K(y) = \psi_D(y) = \langle u(\sigma_i), y \rangle .$$

So, on σ_0:

$$y_1 + y_2 \leq 1, y_1 \geq 0, y_2 \geq 0;$$

on σ_1:

$$y_2 \leq 1 + 2y_1, y_1 \leq 0, y_2 \geq y_1$$

and on σ_2:

$$y_2 \geq -\frac{1}{2} + \frac{1}{2} y_1, y_2 \leq 0, y_1 \geq y_2.$$

When we take the union of the spaces carved out by these conditions, we find Δ_B^* depicted in figure 5.4. Note that the vertices of Δ_B^* correspond to vectors that generate the cones of Σ_B. This is a general remark. As a final note, we present all the toric data together in figure 5.5.

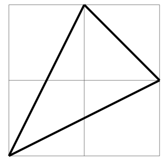


Figure 5.4: The dual polyhedra corresponding to $\mathbb{P}^2$.

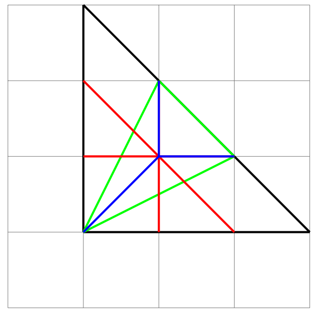


Figure 5.5: Toric data corresponding to $\mathbb{P}^2$: Δ_B (black), Δ_B^* (green), Σ_B (blue) and Σ_B^* (red).

5.3 Singularities

In this section we will mainly give some definitions on singularities in toric varieties. We will also show how to resolve some of these singularities in terms of the toric description. We also need the general blow-up constructions on projective varieties in a later stage. These can be found in [10] or [9].

5.3.1 Types of singularities

Let us start with some definitions. Using the symbols defined above, denote by $\mathbb{T}$ the algebraic torus over $\mathbb{C}$. Denote by $\mathbb{T}_\sigma$ the $\mathbb{T}$-orbit of σ in $\mathbb{P}_\Delta$, with σ an s-dimensional cone in $N_\mathbb{Q}$. Denote by $\mathbb{A}_\sigma$ the n-dimensional affine toric variety associated with σ: $\mathrm{Spec}(\sigma^\vee \cap M)$. It now suffices to look at one point $p_\sigma \in \mathbb{A}_\sigma$ if we want to investigate toric singularities in our space. Take $n_1, ..., n_r \in N$ with $r \geq s$ primitive N-integral generators of all 1-dimensional faces of the s-dimensional cone σ.

Proposition 1. *The point $p_\sigma \in \mathbb{A}_\sigma$ is $\mathbb{Q}$-factorial if and only if the cone σ is simplicial, i.e. $r = s$.*

Proposition 2. *The point $p_\sigma \in \mathbb{A}_\sigma$ is $\mathbb{Q}$-Gorenstein if and only if the elements $n_1, ..., n_r$ are contained in an affine hyperplane $H_\sigma = \{y \in N_\mathbb{Q} | \langle k_\sigma, y\rangle = 1\}$, for some $k_\sigma \in M_\mathbb{Q}$. The point $p_\sigma \in \mathbb{A}_\sigma$ is Gorenstein if and only if $k_\sigma \in M$.*

Furthermore, we have the following if we assume that $\mathbb{A}_\sigma$ is $\mathbb{Q}$-Gorenstein:

Proposition 3. *$\mathbb{A}_\sigma$ has at p_σ at most terminal singularity if and only if*

$$N \cap \sigma \cap \{y \in N_\mathbb{Q} | \langle k_\sigma, y\rangle \leq 1\} = \{0, n_1, ..., n_r\}.$$

Proposition 4. *$\mathbb{A}_\sigma$ has at p_σ at most canonical singularity if and only if*

$$N \cap \sigma \cap \{y \in N_\mathbb{Q} | \langle k_\sigma, y\rangle < 1\} = \{0\}.$$

All Gorenstein toric singularities will thus be canonical. These four propositions can be found in [23]. In chapter 6 we will discuss how to get rid of certain singularities in our toric varieties. This will an important procedure in the construction of smooth Calabi-Yau spaces. In [5] the reader can find how to spot non-singular surfaces by means of the corresponding fan and how to resolve these singularities that occur.

5.3.2 Cyclic quotient singularities

Weighted projective spaces have singularities intrinsic to the space itself. For example, look at $\mathbb{P}(1,2,3)$. In the neighbourhood of $[0:1:0]$ we have the identification

$$(x_0, 1, x_2) \sim (-x_0, 1, -x_2),$$

where we took $\lambda = -1$ as the common factor. We find a $\mathbb{Z}_2$ identification on the space. Similarly, we find a $\mathbb{Z}_3$ identification for $a_2 = 3$ in the neighbourhood of $[0:0:1]$:

$$(x_0, x_1, 1) \sim (e^{\frac{2\pi i}{3}} x_0, e^{\frac{4\pi i}{3}} x_1, 1) \sim (e^{\frac{4\pi i}{3}} x_0, e^{\frac{2\pi i}{3}} x_1, 1).$$

These singularities are called cyclic quotient singularities and are the only singularities that a weighted projective space has.

Definition 14. Take $r > 0$; $a_0, ..., a_n \in \mathbb{Z}$; $x_0, ..., x_n$ coordinates in $\mathbb{A}^n$. Define a $\mathbb{Z}_r$ action on $\mathbb{A}^n$ by

$$x_i \mapsto \epsilon^{a_i} x_i,$$

for all $i \in \{0, ..., n\}$ and ϵ a primitive rth root of unity. A singularity $q \in \mathbb{P}$, with $\mathbb{P}$ a weighted projective space with weights a_i, is a quotient singularity of type $\frac{1}{r}(a_0, ..., a_n)$ if $(\mathbb{P}, q)$ is isomorphic to analytic neighbourhood of $(\mathbb{A}^n, 0)/\mathbb{Z}_r$.

Let us define some notation. Consider $P_i = [0 : ... : 0 : 1 : 0 : ... : 0] \in \mathbb{P}^n(a_0, ..., a_n)$, where the 1 is in the ith spot. We denote by P_iP_j than the corresponding toric stratum, and by Δ the fundamental simplex which is the union of coordinate hyperplanes $P_0...\hat{P}_i...P_n$ (where the hatted P_i is left out). Denote with $h_{i,j,...} = \text{hcf}(a_i, a_j, ...)$.

Definition 15. The singular locus $\mathbb{P}^n_{\text{sing}}(a_0, ..., a_n)$ of $\mathbb{P}^n(a_0, ..., a_n)$ is defined by:

- P_i is a singularity of type
$$\frac{1}{a_i}(a_0, ..., \hat{a}_i, ..., a_n).$$

- Each generic point p of the edge P_iP_j has an analytic neighbourhood U_p which is analytically isomorphic to a neighbourhood with singularity of type
$$\frac{1}{h_{i,j}}(a_0, ..., \hat{a}_i, ..., \hat{a}_j, ..., a_n).$$

- A similar situation occurs for higher dimensional toric strata. The singularities only occur on the fundamental simplex Δ.

Chapter 6

Batyrev's construction

The main reference in this chapter is the original paper by Batyrev: [1]. The first three sections will be a summary of the information we need from that paper. Having dealt with most definitions from toric geometry necessary in chapter 5, we start this chapter by discussing Laurent polynomials and MPCP-desingularization. After discussing reflexive pairs, we explain how all these concepts come together in constructing a smooth Calabi-Yau manifold. Furthermore, we will state Batyrev's result for the Hodge structure of these Calabi-Yau subspaces and the Euler characteristic when the Calabi-Yau in question is a threefold. Next, we want to discuss mirror symmetry and how it manifests itself in this language. Furthermore, we discuss code specifically written by the author to calculate the Hodge structure of Calabi-Yau spaces given certain toric data. The software can also handle some language from toric geometry. We will present some of the code and methods used in this software package named *PolyTori*. Finally, we discuss the quintic as an example again.

6.1 Hypersurfaces in toric varieties

Take M an abelian group of rank n.

6.1.1 Δ/Σ-regularity

We begin with some definitions:

Definition 16. A Laurent polynomial f is a finite linear combination of elements of M:

$$f(x) = \sum c_m X^m$$

with c_m complex coefficients and $m \in M$. The Newton polyhedra $\Delta(f)$ is the convex hull in $M_{\mathbb{Q}}$ of all elements m such that $c_m \neq 0$.

The Laurent polynomial f together with its Newton polyhedra Δ define an affine hypersurface:

$$Z_{f,\Delta} = \{X \in \mathbb{T} | f(X) = 0\} \tag{6.1}$$

with $\mathbb{T}$ the n-dimensional algebraic torus. Denote by $\bar{Z}_{f,\Delta}$ the closure in $\mathbb{P}_\Delta$.

Definition 17. For any l-dimensional face $\Theta \subset \Delta$, define

$$Z_{f,\Theta} = \bar{Z}_{f,\Delta} \bigcap \mathbb{T}_{\Theta},$$

with $\mathbb{T}_{\Theta}$ the l-dimensional $\mathbb{T}$-orbit in $\mathbb{P}_{\Delta}$ of Θ.

Denote by $L(\Delta)$ the space of all Laurent polynomials with a fixed Newton polyhedron Δ.

Definition 18. A $f \in L(\Delta)$ and the corresponding $Z_{f,\Delta}$, $\bar{Z}_{f,\Delta}$ are said to be Δ-regular if for all $\Theta \subset \Delta$, $Z_{f,\Theta}$ is empty or a smooth subvariety of codimension one in $\mathbb{T}_{\Theta}$.

Of course, like we saw in chapter 5, we can describe a toric variety in terms of its polyhedra data or its fan data. Let us extend this notion to Σ-regular hypersurfaces in $\mathbb{P}_{\Sigma}$.

Definition 19. Take $\bar{Z}_{f,\Sigma}$ to be the closure in $\mathbb{P}_{\Sigma}$ of the affine hypersurface Z_f defined by the Laurent polynomial f. Define

$$Z_{f,\sigma} = \bar{Z}_{f,\Sigma} \bigcap \mathbb{T}_{\sigma}.$$

f, Z_f and $\bar{Z}_{f,\Sigma}$ are Σ-regular if for every s-dimensional cone $\sigma \in \Sigma$ the corresponding variety $Z_{f,\sigma}$ is empty or a smooth subvariety of codimension one in $\mathbb{T}_{\sigma}$.

6.1.2 MPCP-desingularization

We now discuss how to desingularize toric singularities. Let us start with some definitions:

Definition 20. Take $\phi : W' \to W$ a proper birational morphism of normal $\mathbb{Q}$-Gorenstein algebraic varieties. ϕ is called crepant if $\phi^* K_W = K_{W'}$.

Definition 21. Take $\phi : W' \to W$ a projective birational morphism of normal $\mathbb{Q}$-Gorenstein algebraic varieties. ϕ is called a maximal projective crepant partial desingularization (MPCP-desingularization) of W if ϕ is crepant and W' has only $\mathbb{Q}$-factorial terminal singularities.

We can define such an MPCP-desingularization of a toric variety by means of a triangulation of the corresponding toric polyhedra. Let us discuss what the properties of such a triangulation have to be.

Definition 22. Take A a finite subset in $\Delta \cap \mathbb{Z}^n$. Call A admissible if all vertices of the integral polyhedron Δ are contained within A.

Definition 23. Take A admissable in $\Delta \cap \mathbb{Z}^n$. Define an A-triangulation of Δ to be a finite collection of simplices $\mathcal{T} = \{\theta\}$ with vertices in A, having the following properties:

- If θ' is the face of $\theta \in \mathcal{T}$ then $\theta' \in \mathcal{T}$.
- The vertices of all $\theta \in \mathcal{T}$ lie in $\Delta \cap \mathbb{Z}^n$.
- The intersection of two simplices is empty or a common face of both.

- Δ is given as the union of all simplices in $\mathcal{T}$.
- Every element of A is a vertex of some simplex $\theta \in \mathcal{T}$.

The A-triangulation of an integral convex polyhedron Δ is called maximal if $A = \Delta \cap \mathbb{Z}^n$. We say that the A-triangulation is projective if and only if there exists a strictly upper convex function $\alpha(\mathcal{T}) : A \to \mathbb{Q}$. The following theorem in [1] then says:

Theorem 2. *Take $\mathbb{P}_\Sigma$ a toric Fano variety with only Gorenstein singularities. Then $\mathbb{P}_\Sigma$ admits at least one MPCP-desingularization*

$$\tilde{\phi} : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma.$$

Moreover, MPCP-desingularizations of $\mathbb{P}_\Sigma$ are defined by maximal projective triangulations of the polyhedron $\Delta^*(\Sigma, h_K)$, with h_K the integral strictly upper convex support function associated with the anticanonical divisor $\mathbb{P}_\Sigma \setminus \mathbb{T}$ on $\mathbb{P}_\Sigma$.

Furthermore, we need the following proposition, also from [1]:

Proposition 5. *Take $\mathbb{P}_\Sigma$ a projective toric variety with only Gorenstein singularities. Assume that*

$$\tilde{\phi} : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Sigma,$$

is a MPCP-desingularization of $\mathbb{P}_\Sigma$. Then $\bar{Z}_{\tilde{\phi}^* f, \Sigma'}$ is a MPCP-desingularization of $\tilde{Z}_{f,\Sigma}$.

We will use this last proposition to desingularize our Calabi-Yau variety once we have constructed it in the next section.

6.2 Calabi-Yau hypersurfaces in toric varieties

The idea now is that Δ (with some properties) generates an ambient space $\mathbb{P}_\Delta$ with therein a family of hypersurfaces $\bar{Z}_f$ which are Calabi-Yau varieties. We then perform a MPCP-desingularization to smoothen the surface and to obtain a Calabi-Yau manifold.

We again start with a definition.

Definition 24. Take Δ compact, convex set in $M_\mathbb{Q}$ containing the zero vector in its interior, then Δ^* defined by

$$\Delta^* = \{y \in N_\mathbb{Q} | \langle x, y \rangle \geq -1 \forall x \in \Delta\},$$

is called the dual set relative to Δ.

Take $p \in M_\mathbb{Q}$, H an affine hyperplane in $M_\mathbb{Q}$ defined as the set

$$H = \{x \in M_\mathbb{Q} | \langle x, l \rangle = c\}$$

for some integer c and $l \in N_\mathbb{Q}$. Then $|c - \langle p, l \rangle|$ is called the integral distance between H and the point p. We then arrive at the following definition for a reflexive pair:

Definition 25. The pair (Δ, M), for Δ convex integral polyhedron in $M_{\mathbb{Q}}$ (of dimension n) containing the zero in its interior, is called reflexive if the integral distance between 0 and all affine hyperplanes generated by $(n-1)$-dimensional faces of Δ equals one.

If (Δ, M) is a reflexive pair then Δ is a reflexive polyhedron. Furthermore, if (Δ, M) is a reflexive pair then (Δ^*, N) is a reflexive pair as well ([1]). Furthermore, there is a one-to-one correspondence between l-dimensional faces of Δ and $n-l-1$-dimensional faces of Δ^*.

Let us now define what a Calabi-Yau variety is.

Definition 26. A complex normal irreducible n-dimensional projective algebraic variety W with only Gorenstein canonical singularities we will call a Calabi-Yau variety if W has trivial canonical bundle and $H^i(W, \mathcal{O}_W) = 0$ for all $0 < i < n$.

The following theorem then gives the relation between reflexive polytopes and Calabi-Yau varieties:

Theorem 3. *Take Δ a n-dimensional integral polyhedron in $M_{\mathbb{Q}}$, $\mathbb{P}_\Delta$ the corresponding n-dimensional projective toric variety, $\mathcal{F}(\Delta)$ the family of projective Δ-regular hypersurfaces $\bar{Z}_f$ in $\mathbb{P}_\Delta$. Then the following are equivalent:*

- *the family $\mathcal{F}(\Delta)$ consists of Calabi-Yau varieties with canonical singularities;*
- *$\mathcal{O}_\Delta(1)$ on $\mathbb{P}_\Delta$ is anticanonical; i.e. $\mathbb{P}_\Delta$ is toric Fano variety with Gorenstein singularities;*
- *Δ contains only one integral point m_0 in its interior, and $(\Delta - m_0, M)$ is a reflexive pair.*

Using this theorem, and everything we have seen this chapter, we can thus make the following summary of the Batyrev construction.

Take Δ a reflexive polyhedron, denote by Δ^* its dual reflexive polyhedron. There is a maximal projective triangulation $\mathcal{T}$ of Δ^*. $\mathcal{T}$ defines a MPCP-desingularization

$$\phi_{\mathcal{T}} : \hat{\mathbb{P}}_\Delta \to \mathbb{P}_\Delta, \tag{6.2}$$

of the Gorenstein toric variety $\mathbb{P}_\Delta$. Take $\bar{Z}_f$ a Δ-regular Calabi-Yau hypersurface in $\mathbb{P}_\Delta$ (by the previous theorem). Denote

$$\hat{Z}_f = \phi_{\mathcal{T}}^{-1}(\bar{Z}_f). \tag{6.3}$$

Then we have that

$$\phi_{\mathcal{T}} : \hat{Z}_f \to \bar{Z}_f, \tag{6.4}$$

is a MPCP-desingularization of $\bar{Z}_f$. We will call it the toroidal MPCP-desingularization of $\bar{Z}_f$ corresponding to a maximal projective triangulation $\mathcal{T}$ of Δ^*. Note that there exists at least one such desingularization $\hat{Z}_f$ for any Δ-regular hypersurface in $\mathbb{P}_\Delta$. The codimension of singularities of $\hat{Z}_f$ is always at least four. So when the dimension of Δ is smaller or equal to four, we obtain a smooth Calabi-Yau manifold. $\dim(\Delta) = 4$ is our focus in this work, so we obtain Calabi-Yau threefolds!

In this thesis we are interested in compact Calabi-Yau manifolds. For them, the Laurent polynomial corresponding to a reflexive polyhedron Δ takes the following form:

$$W_\Delta = \sum_{\nu^{(i)}\in\Delta} a_i \prod_{\nu^{*(k)}\in\Delta^*} Y_k^{\langle \nu^{(i)},\nu^{*(k)}\rangle+1}, \tag{6.5}$$

with every ray in the fan Σ corresponding to a toric divisor D_i in the Chow group of $\mathbb{P}_\Delta$ and the coordinates Y_i corresponding to $\{Y_i = 0\} = D_i$. The coefficients a_i parametrize the complex structure of the Calabi-Yau $X = \{W_\Delta = 0\}$. The mirror is given by:

$$W_{\Delta^*} = \sum_{\nu^{*(i)}\in\Delta^*} a_i^* \prod_{\nu^{(k)}\in\Delta} X_k^{\langle \nu^{*(i)},\nu^{(k)}\rangle+1}, \tag{6.6}$$

in the coordinate ring of $\mathbb{P}_{\Delta^*}$.

Batyrev gives in [1] two formulae for the Hodge numbers of the Calabi-Yau threefolds $\hat{Z}_f$ defined by Δ-regular Laurent polynomials f with Newton polyhedron a reflexive four-dimensional polyhedron Δ. They are given by:

$$h^{1,1}(\hat{Z}_f) = l(\Delta^*) - 5 - \sum_{\text{codim } \Theta^*=1} l^*(\Theta^*) + \sum_{\text{codim } \Theta^*=2} l^*(\Theta^*)\cdot l^*(\Theta); \tag{6.7a}$$

$$h^{2,1}(\hat{Z}_f) = l(\Delta) - 5 - \sum_{\text{codim } \Theta=1} l^*(\Theta) + \sum_{\text{codim } \Theta=2} l^*(\Theta)\cdot l^*(\Theta^*), \tag{6.7b}$$

with $l(P)$ counting the number of integral points in $P \cap M$, $l^*(P)$ counting the number of integral points in the interior of P for P a compact convex subset in $M_{\mathbb{Q}}$, and $\Theta^* \leftrightarrow \Theta$ dual faces in Δ^*, Δ respectively. Note the difference in dimension of these dual faces as was described above. Furthermore, it is clear the both formulae are interchangeable under $\Delta \leftrightarrow \Delta^*$. This is a manifestation of mirror symmetry. It is quite remarkable that topological invariants of our space like Hodge numbers are described in terms of pure geometrical data described by counting integral points in toric data. Calculating these numbers by hand can still be quite a difficult task. By writing software we can speed up these calculations drastically. Calculating the Hodge numbers is one of the main goals of the *PolyTori* software package. The Euler number can then be easily calculated by

$$\frac{1}{2}e(\hat{Z}_f) = h^{1,1}(\hat{Z}_f) - h^{2,1}(\hat{Z}_f). \tag{6.8}$$

6.3 Mirror symmetry

In this section we treat the theory of minimal and maximal pairs, and its consequences for mirror symmetry. We will work in the category $\mathcal{C}_n$ of reflexive pairs (Δ, M) of dimension n. We need to define the arrows in this category. Let us do this.

Definition 27. Take (Δ, M) and (Δ', M') two reflexive pairs of equal dimension. A finite morphism of reflexive pair

$$\phi : (\Delta, M) \to (\Delta', M'), \tag{6.9}$$

is a homomorphism of lattices $\phi : M \to M'$ with $\phi(\Delta) = \Delta'$.

So the morphisms in $\mathcal{C}_n$ are the finite morphisms of reflexive pairs. We can do the same for the dual reflexive pairs (Δ^*, N), where the morphisms are denoted by ϕ^*. The dual of

$$\phi : (\Delta, M) \to (\Delta', M'),$$

would be

$$\phi^* : ((\Delta')^*, N') \to (\Delta^*, N).$$

We thus have a dual category $\mathcal{C}_n^*$ and we can define an involutive functor

$$\mathrm{Mir} : \mathcal{C}_n \to \mathcal{C}_n^* : (\Delta, M) \to (\Delta^*, N), \tag{6.10}$$

which is an isomorphism between $\mathcal{C}_n$ and $\mathcal{C}_n^*$. Let us now define minimal- and maximal morphisms/pairs.

Definition 28. Take $\phi_0 : (\Delta_0, M_0) \to (\Delta, M)$ a finite morphism of reflexive pair. ϕ_0 is said to be minimal if for all finite morphisms

$$\psi : (\Delta', M') \to (\Delta, M),$$

there exists a unique morphism

$$\phi : (\Delta_0, M_0) \to (\Delta', M'),$$

such that $\phi_0 = \psi \circ \phi$. If ϕ_0 is the identity morphism, then (Δ, M) is a minimal reflexive pair.

Definition 29. Take $\phi^0 : (\Delta, M) \to (\Delta^0, M^0)$ a finite morphism of reflexive pair. ϕ^0 is said to be maximal if for all finite morphisms

$$\psi : (\Delta, M) \to (\Delta', M'),$$

there exists a unique morphism

$$\phi : (\Delta', M') \to (\Delta^0, M^0),$$

such that $\phi^0 = \phi \circ \psi$. If ϕ^0 is the identity morphism, then (Δ, M) is a maximal reflexive pair.

Denote by M_Δ the sublattice in M generated by vertices of Δ, by N_{Δ^*} the sublattice in N generated by vertices of Δ^*. Then (Δ, M_Δ) and (Δ^*, N_{Δ^*}) are minimal reflexive pairs. Denote by (Δ, M^Δ) the dual of (Δ^*, N_{Δ^*}) and by (Δ^*, N^{Δ^*}) the dual of (Δ, M_Δ). Then (Δ, M^Δ) and (Δ^*, N^{Δ^*}) are maximal reflexive pairs.

Definition 30. The quotients

$$\pi_1(\Delta) = N^{\Delta^*}/N_{\Delta^*}; \tag{6.11a}$$

$$\pi_1(\Delta^*) = M^\Delta/M_\Delta, \tag{6.11b}$$

are called the fundamental groups of Δ and Δ^* respectively. They are isomorphic dual finite abelian groups. The finite abelian group $\pi_1(\Delta, M)$ is defined as the quotient N/N_{Δ^*} and is called the fundamental group of the pair (Δ, M).

We need one more definition before we can apply this to our Calabi-Yau hypersurfaces.

Definition 31. Take (Δ, M) a reflexive pair, and assume that there exists a isomorphism between maximal reflexive pairs

$$\phi : (\Delta, M^{\Delta}) \to (\Delta^*, N^{\Delta^*}),$$

then we call Δ a selfdual reflexive polyhedron.

In [1] the following theorem is proved, for

$$\phi : (\Delta_1, M_1) \to (\Delta_2, M_2),$$

a finite morphism of reflexive pairs.

Theorem 4. *The Calabi-Yau hypersurfaces in $\mathbb{P}_{\Delta_1,M_1}$ are quotients of some Calabi-Yau hypersurfaces in $\mathbb{P}_{\Delta_2,M_2}$ by the action of the dual to $M_2/\phi(M_1)$ finite abelian group.*

We then obtain:

Proposition 6. *Take (Δ, M) a reflexive pair with Δ a selfdual reflexive polyhedron. Then $\mathcal{F}(\Delta), \mathcal{F}(\Delta^*)$ are quotients respectively by $\pi_1(\Delta, M)$ and $\pi_1(\Delta^*, N)$ of subfamilies of the family of Calabi-Yau hypersurfaces corresponding to two isormophic maximal reflexive pairs (Δ, M^{Δ}) and (Δ^*, N^{Δ^*}). Moreover, the order of $\pi_1(\Delta)$ is the product of the orders of $\pi_1(\Delta, M)$ and $\pi_1(\Delta^*, N)$.*

6.4 Software package *PolyTori*

In this section we would like to highlight some of the functionalities available in the *PolyTori* software package. We will not present a line-for-line analysis of how the code works. Instead, we offer in this section some example code that should allow people to use the package. Note that as of writing, work is still being done on the library.

6.4.1 Installation

The most recent version of the *Polytori* library can be found at `https://github.com/eriklumens/PolyTori`. The reader can download the source from there. The build has been tested on a linux system. One needs the following libraries:

- OpenGL
- GLEW
- GLFW
- CMake

To get the appropriate version of GLEW, just run

```
$ sudo apt-get install libglew-dev
```

To properly install GLFW, obtain GLFW source files from their official website, then run

```
$ sudo apt-get install xorg-dev libglu1-mesa-dev
```

After which you go to downloaded source files directory and run

```
$ cmake .
$ make
$ sudo make install
```

After we have installed all the necessary libraries, navigate to the root directory of the repository and run

```
$ cd build
$ cmake .
$ make
```

Then open

```
$ ./bin/PolyToriApp
```

to run the program. Most of the functionality is at the moment in

```
application/src/main.cpp
```

If you want to change how the program works you'll have to do it there.

6.4.2 Functionality

Now we have installed the software package, we want to use it! Let us discuss which problems the software can describe and/or solve.

Defining a lattice

We start by defining a lattice. In the next example we define a two-dimensional lattice, by specifying its basis vectors.

```
//Set up basisvectors
std::vector<std::vector<double> > basis2(2,std::vector<double>(2));
basis2[0][0] = 1;
basis2[0][1] = 0;
basis2[1][0] = 0;
basis2[1][1] = 1;
```

```
    //Define lattice
    Lattice myLattice(2,basis2);
```

Fan description

We can define a fan in any dimension by specifying the different cones it is made out of. The cones are given by the vectors that specify the rays.

```
    //Set up cones
    Cone Sigma0 = Cone({{1,0},{0,1}},myLattice);
    Cone Sigma11 = Cone({{0,1},{-1,0}},myLattice);
    Cone Sigma12 = Cone({{-1,0},{-1,-1}},myLattice);
    Cone Sigma13 = Cone({{-1,-1},{-2,-3}},myLattice);
    Cone Sigma21 = Cone({{-2,-3},{-1,-2}},myLattice);
    Cone Sigma22 = Cone({{-1,-2},{0,-1}},myLattice);
    Cone Sigma23 = Cone({{0,-1},{1,0}},myLattice);

    //Define fan
    std::vector<Cone> myCones = {Sigma0, Sigma11, Sigma12, Sigma13,
    Sigma21, Sigma22, Sigma23};
    Fan myFan(myCones,myLattice);
```

Polyhedron description

We describe a polyhedron by specifying its vertices of which it is the convex hull. In the next example, we define Δ^* of $\mathbb{P}^2(1,2,3)$.

```
    //Set up polytope vertices
    std::vector<std::vector<double> > projTwo(3,std::vector<double>(2));
    projTwo[0][0] = 1;
    projTwo[0][1] = 0;
    projTwo[1][0] = 0;
    projTwo[1][1] = 1;
    projTwo[2][0] = -2;
    projTwo[2][1] = -3;

    //Define polytope
    Polytope myPolytope(projTwo,myLattice);
```

2D description

The code allows for the calculation of dual fans and dual polyhedra but only in two dimensions. One can also draw the toric data using OpenGL.

```
    //Calculating dual polyhedra and fans
    Polytope myDualPolytope = myPolytope.getCorrespondingDualPolytope();
    Fan myDualFan = myFan.getCorrespondingDualFan();
```

```
    //Drawing toric data
    myPolytope.drawPolytope();
    myDualPolytope.drawPolytope();
    myFan.drawFan();
    myDualFan.drawFan();
```

Calabi-Yau threefolds

To specify a Calabi-Yau threefold, we start by defining a four-dimensional lattice. We can get a list of the integral points of a polytope of dimension smaller than four. We can also get a list of the two- and three-dimensional faces of the polyhedron, given by the vertices of these faces. Furthermore, the Hodge numbers can be calculated in general if the number of vertices of a polytope is equal to five. Some successes have been made in generalizing the code to more then five vertices, but only in the case of a very specific construction for obtaining an elliptic fibration (studied in the next chapter). Here we will give an example of how that piece of the code works.

```
    //Set up basis vectors
    std::vector<std::vector<double> > basis4(4,std::vector<double>(4));
    basis4[0][0] = 1;
    basis4[0][1] = 0;
    basis4[0][2] = 0;
    basis4[0][3] = 0;
    basis4[1][0] = 0;
    basis4[1][1] = 1;
    basis4[1][2] = 0;
    basis4[1][3] = 0;
    basis4[2][0] = 0;
    basis4[2][1] = 0;
    basis4[2][2] = 1;
    basis4[2][3] = 0;
    basis4[3][0] = 0;
    basis4[3][1] = 0;
    basis4[3][2] = 0;
    basis4[3][3] = 1;

    //Define lattice
    Lattice myLattice4(4,basis4);

    //Define four-dimensional polytopes
    Polytope polP
   ({{12,-6,-1,-1},{-6,12,-1,-1},{-6,-6,-1,-1},{0,0,2,-1},{0,0,-1,1}},
   myLattice4);
    Polytope polPDual
   ({{1,0,-2,-3},{0,1,-2,-3},{-1,-1,-2,-3},{0,0,1,0},{0,0,0,1}},
   myLattice4);

    //Obtain integer points
    std::vector<std::vector<double> > pointsP = polP.
   getIntegerpoints4DPolytope();
    std::vector<std::vector<double> > pointsPDual = polPDual.
   getIntegerpoints4DPolytope();
```

```
    //Get two- and three-dimensional faces polytope
    std::vector<std::vector<std::vector<double> > > faces2D = polP.
   get2DFacesOf4DPolytope();
    std::vector<std::vector<std::vector<double> > > faces3D = polP.
   get3DFacesOf4DPolytope();

    //Obtain Hodge numbers
    int hOneOne = polP.hodgeOneOne(polPDual);
    int hTwoOne = polP.hodgeTwoOne(polPDual);
```

Note that the library is specifically written to cater to the needs of someone working with the construction highlighted in chapter 7 to obtain elliptically fibered Calabi-Yau spaces following [12]. We will discuss why in the next chapter.

6.5 Quintic revisited

In this section we revisit the quintic threefold that we first analysed in section 2.2.3. Using Batyrev's construction, we will generate this threefold as a subspace of $\mathbb{P}^4$ and calculate its mirror.

Define,

$$\Delta = \mathrm{conv}\left(\left\{\begin{pmatrix}4\\-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\4\\-1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\4\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\\4\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}\right\}\right).$$

We see that there exists a linear relation among the vertices:

$$\begin{pmatrix}4\\-1\\-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\4\\-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\\4\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\\-1\\4\end{pmatrix}+\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}=0,$$

so if we denote the vertices by p_i, then the weights w_i in the linear relation

$$\sum_{i=0}^{4} w_i p_i = 0 \tag{6.12}$$

are all equal to one. We can calculate the dual reflexive simplex to the reflexive simplex Δ, and find:

$$\Delta^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}\right\}\right).$$

Denote the vertices of Δ^* by l_i. Let us now calculate the associated matrix $B(\Delta)$ whose matrix indices are given by

$$b_{ij} = \langle p_i, l_j \rangle . \tag{6.13}$$

We find:

$$B(\Delta) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} \tag{6.14}$$

This matrix has some important properties, but the main one is probably that it entirely depends only on the weights w_i of Δ. We could have given any reflexive simplex $\Delta = \mathrm{conv}(\{p_i\})$ with

$$\sum_{i=0}^{4} p_i = 0,$$

and the matrix $B(\Delta)$ would have been the same. We can now define the homomorphism ι_Δ in the following way:

$$\begin{aligned} \iota_\Delta : & M \to \mathbb{Z}^5 \\ & m \mapsto (\langle m, l_0 \rangle, ..., \langle m, l_4 \rangle). \end{aligned}$$

We now define $\Delta(w)$ as the image of Δ under ι_Δ:

$$\Delta(w) = \mathrm{conv}\left(\left\{ \begin{pmatrix} 4 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 4 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 4 \end{pmatrix} \right\}\right).$$

The image of ι_Δ is the 4-dimensional sublattice $M(w)$ of $\mathbb{Z}^5$, defined by the equation $\sum_{i=0}^4 w_i x_i = \sum_{i=0}^4 x_i = 0$. We then have a maximal reflexive pair $(\Delta(w), M(w))$, whose corresponding toric Fano variety is the weighted projective space

$$\mathbb{P}^4(w_0, ..., w_4) = \mathbb{P}^4(1, ..., 1) = \mathbb{P}^4.$$

Define $d_i = b_{ii} + 1$ and $d = \mathrm{l.c.m.}(\{d_i\})$, then for all i, $d_i = 5$ and $d = 5$. It follows that

$$v_0^{d_0} + v_1^{d_1} + v_2^{d_2} + v_3^{d_3} + v_4^{d^4} = v_0^5 + v_1^5 + v_2^5 + v_3^5 + v_4^5 = 0, \tag{6.15}$$

defines a $\Delta(w)$-regular Calabi-Yau hypersurface of Fermat-type in $\mathbb{P}_{\Delta(w)} = \mathbb{P}^4$. This is precisely an equation for a quintic threefold in $\mathbb{P}^4$!

Furthermore, we can calculate the fundamental group $\pi_1(\Delta)$ of the reflexive simplex Δ. Namely, it is isomorphic to the kernel of the surjective homomorphism

$$(\mu_{d_0} \times .... \times \mu_{d_4})/\mu_d \to \mu_d, \tag{6.16}$$

with the homomorphism to μ_d the product of the complex numbers in $\mu_{d_0}, ..., \mu_{d_4}$ and the embedding of μ_d in $\mu_{d_0} \times .... \times \mu_{d_4}$ defined by

$$g \mapsto (g^{w_0}, ..., g^{w_4}) = (g, ..., g). \tag{6.17}$$

We find $\pi_1(\Delta) \cong (\mathbb{Z}/5\mathbb{Z})^3$. The family $\mathcal{F}(\Delta)$ consists of quotients by $\pi_1(\Delta)$ of Calabi-Yau hypersurfaces in $\mathbb{P}^4$ whose equations are invariant under the canonical diagonal action of $\pi_1(\Delta)$ on $\mathbb{P}^4$. We can use *PolyTori* to calculate the Hodge diamond of the quintic, we obtain:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{6.18}$$

For its mirror we find:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{6.19}$$

This is the same result that we obtained through a lengthy calculation in section 2.2.3.

In this chapter we have constructed Calabi-Yau hypersurfaces in toric ambient spaces. In the next chapter we will revisit this example and others, this time in the framework of elliptic fibrations.

Chapter 7

Elliptic threefolds

In this chapter we construct Calabi-Yau manifolds with an elliptic fibration. While the main technique used is still the Batyrev construction, we also use reference [12] to obtain the elliptic structure. The main idea is the following. We start with a particular two-dimensional base polyhedron, either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or a blow-up of $\mathbb{P}^2$ in up to eight points. These surfaces are the so called (almost) Del Pezzo surfaces. This surface will play the part of the base over which we have an elliptic fibration. The elliptic curve will then be a curve inside a toric space which is given by another (almost) Del Pezzo surface. We combine the toric data of these two spaces in a specific way such that we obtain a four-dimensional polyhedron. The Laurent polynomial that corresponds to this polyhedron gives us the Calabi-Yau threefold inside the toric ambient space corresponding to the four-dimensional polyhedron. In this chapter we start by discussing the (almost) Del Pezzo surfaces, after which we specify how this construction is done. Afterwards, we treat some important examples.

7.1 (Almost) Del Pezzo surfaces

This section deals with the theory of (almost) Del Pezzo surfaces necessary for the construction in this chapter. For an overview the reader can consult [4]. Let us start with some definitions.

Definition 32. A Del Pezzo surface is a smooth projective geometrically irreducible surface whose anti-canonical bundle is ample.

The anti-canonical bundle of a Del Pezzo surface S is denoted by $-K_S$. We can classify Del Pezzo surfaces by their degree, defined in the following way.

Definition 33. The degree of a Del Pezzo surface is a positive integer, defined by

$$\deg S = K_S \cdot K_S,$$

i.e. the self-intersection of the canonical class.

Every Del Pezzo surface is birationally equivalent to $\mathbb{P}^2$, since they are geometrically rational. The degree of a Del Pezzo surface S lies between $1 \leq \deg S \leq 9$. When $\deg S \geq 2$, $-K_S$ is very ample. The degree of $\mathbb{P}^2$ is nine. If we blow up $\mathbb{P}^2$ in $r \leq 8$

points, the degree of the resulting space which is a Del Pezzo surface is equal to $9 - r$. The classification of these spaces can be found in [4]. We give the corresponding toric data in figure 7.1. These are all polyhedrons up to a modular transformation, who give rise to a Del Pezzo surface. The modular transformation does not change the corresponding space, nor the Lauren polynomial. Next, let us introduce almost Del Pezzo surfaces.

Definition 34. An almost Del Pezzo surface S has a non-trivial anti-canonical bundle $-K_S$, which has at least one non-zero section at any point of S.

Again, these surfaces have a finite classification. Their toric data is given in figure 7.2, note that the polyhedra are self-dual. Note that Del Pezzo surfaces are a specific instance of a much larger set: Fano varieties.

Definition 35. A Fano variety is a complete variety whose anti-canonical bundle is ample.

A Del Pezzo surface is thus a Fano variety of dimension two. All Fano varieties in dimension d can be classified by d-dimensional reflexive polyhedra. Note that indeed all polyhedra in figures 7.1 and 7.2 are reflexive. Indeed, we have given all 16 two-dimensional reflexive polyhedra. We will now use these polyhedra as building blocks to construct four-dimensional reflexive polyhedra.

7.2 Construction

Denote the two-dimensional base polyhedron by Δ_B, its dual by Δ_B^*. Furthermore, denote the two-dimensional fiber polyhedron by Δ_F and its dual by Δ_F^*. Our goal is to construct a four-dimensional polyhedron Δ and its dual Δ^* with some nice properties. The anti-canonical hypersurface in $\mathbb{P}_{\Delta^*}$ should give rise to an elliptically fibered Calabi-Yau threefold over the toric base $\mathbb{P}_{\Delta_B^*}$.

The construction goes as follows. Let us say that Δ_B has n vertices, Δ_F has m vertices. Take $\{v_i\}_{i=1,\dots,n}$ the vertices of Δ_B, $\{v_i^*\}_{i=1,\dots,n}$ the vertices of Δ_B^*, $\{\nu_i\}_{i=1,\dots,m}$ the vertices of Δ_F and $\{\nu_i^*\}_{i=1,\dots,m}$ the vertices of Δ_F^*. Δ^* is then defined as the convex hull of the following points:

$$\Delta^* = \text{conv}\left(\left\{\begin{pmatrix} v_1^* \\ \nu_k^* \end{pmatrix}, \dots, \begin{pmatrix} v_n^* \\ \nu_k^* \end{pmatrix}, \begin{pmatrix} \vec{0} \\ \nu_1^* \end{pmatrix}, \dots, \begin{pmatrix} \vec{0} \\ \nu_m^* \end{pmatrix}\right\}\right), \tag{7.1}$$

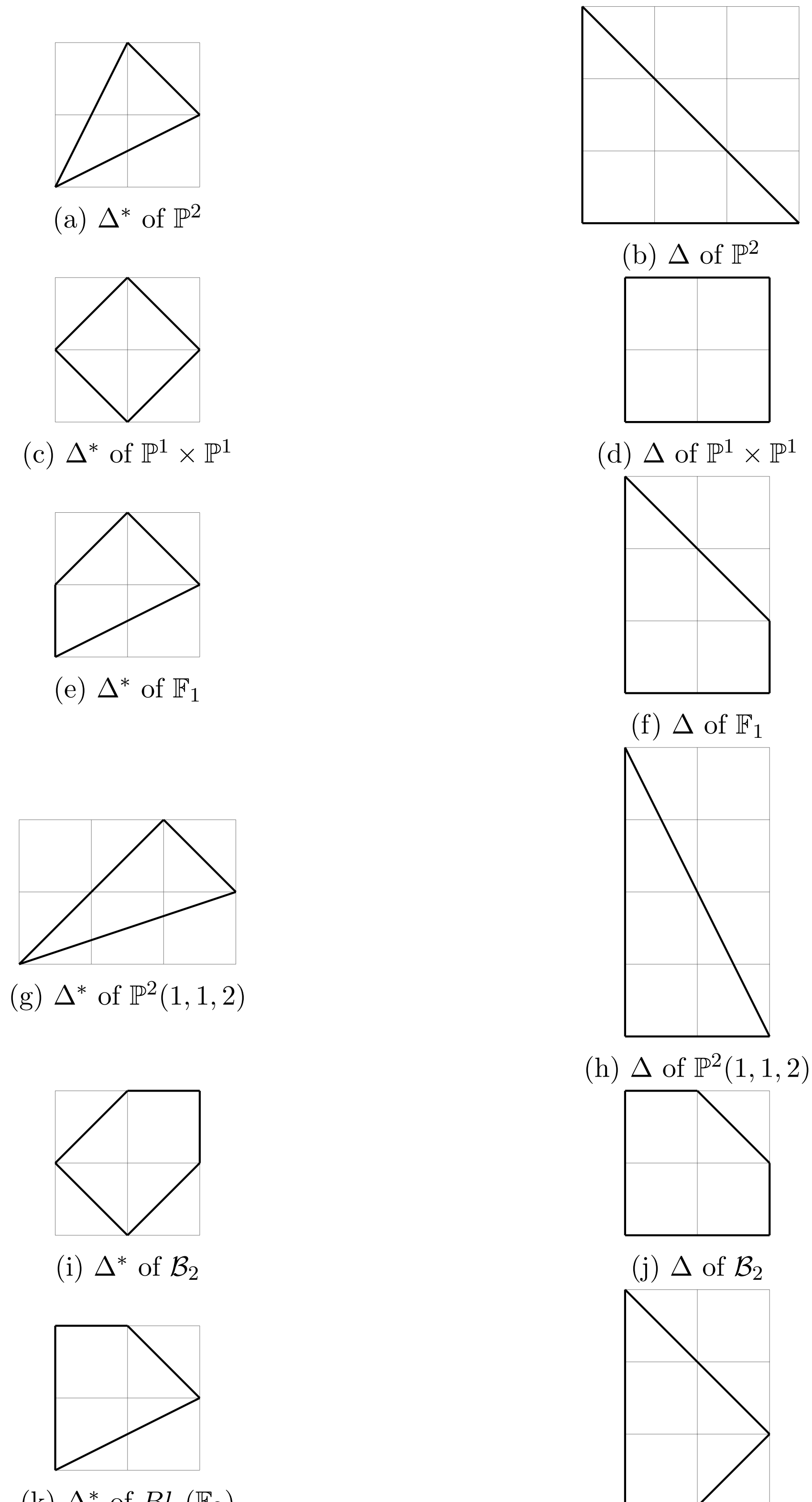
and Δ as

$$\Delta = \text{conv}\left(\left\{\begin{pmatrix} s_{kl}v_1 \\ \nu_l \end{pmatrix}, \dots, \begin{pmatrix} s_{kl}v_n \\ \nu_l \end{pmatrix}, \begin{pmatrix} \vec{0} \\ \nu_1 \end{pmatrix}, \dots, \begin{pmatrix} \vec{0} \\ \nu_m \end{pmatrix}\right\}\right), \tag{7.2}$$

where $\vec{0}$ is the zero-vector in two dimensions, s_{kl} is defined as

$$s_{kl} = \langle \nu_k^*, \nu_l \rangle + 1 \tag{7.3}$$

and we made a choice for a vector ν_k^* in Δ_F^* and ν_l in Δ_F such that s_{kl} is not zero. Different choices give rise to different toric spaces. Note that s_{ij} in general will be zero or positive. Since we scaled Δ_B with a positive number there will be more points inside the embedded space than we had before.



(a) Δ^* of $\mathbb{P}^2$

(b) Δ of $\mathbb{P}^2$

(c) Δ^* of $\mathbb{P}^1 \times \mathbb{P}^1$

(d) Δ of $\mathbb{P}^1 \times \mathbb{P}^1$

(e) Δ^* of $\mathbb{F}_1$

(f) Δ of $\mathbb{F}_1$

(g) Δ^* of $\mathbb{P}^2(1,1,2)$

(h) Δ of $\mathbb{P}^2(1,1,2)$

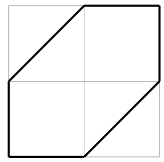
(i) Δ^* of $\mathcal{B}_2$

(j) Δ of $\mathcal{B}_2$

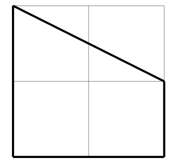
(k) Δ^* of $Bl_p(\mathbb{F}_2)$

(l) Δ of $Bl_p(\mathbb{F}_2)$

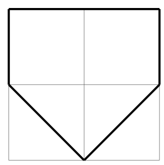
Figure 7.1: All toric Del Pezzo surfaces, up to modular transformations.



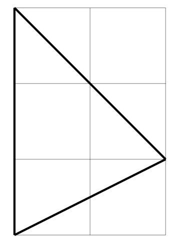
(a) Δ and Δ^* of $\mathcal{B}_3$



(b) Δ and Δ^* of $D5$



(c) Δ and Δ^* of $Bl_{p_1,p_2}(\mathbb{F}_2)$



(d) Δ and Δ^* of $\mathbb{P}^2(1,2,3)$

Figure 7.2: All almost toric Del Pezzo surfaces, up to modular transformations.

For Δ and Δ^* the following features hold. Δ_B is a face of Δ and the image along the projection under the fiber polyhedron, Δ_F is a subpolyhedron and it shares the unique inner point, the origin, with Δ. Furthermore, denote by Γ the lattice generated by Δ, by Γ_F the lattice generated by Δ_F and by Γ_B the lattice generated by Δ_B. Then we have the exact sequence of lattices

$$0 \longrightarrow \Gamma_F \longrightarrow \Gamma \longrightarrow \Gamma_B \longrightarrow 0. \tag{7.4}$$

From [5] we now know that all conditions are satisfied for there to exist a fibration

$$\mathbb{P}_\Delta \to \mathbb{P}_{\Delta_B}, \tag{7.5}$$

over $\mathbb{P}_{\Delta_B}$ with $\mathbb{P}_{\Delta_F}$ as the generic fiber. Furthermore, if we want this fibration to be a smooth and flat one, we demand the existence of a fan $\Sigma(\Delta)$, whose cones have lattice volume one and which is defined by a triangulation of Δ that lifts from a fan Σ_{Δ_B}. The hypersurface $W_{\Delta^*} = 0$ then becomes an elliptic fibration whose generic fiber is defined as the section of the anti-canonical bundle $\mathbb{P}_{\Delta_F}$. For $\mathbb{P}_{\Delta^*}$ this condition is satisfied, so that $X = \{W_\Delta = 0\}$ is a smooth and flat elliptic fibration. However, due to the scaling, $X^* = \{W_{\Delta^*} = 0\}$ is in general only a non-flat elliptic fibration. Note that because Δ and Δ^* are reflexive X and X^* are Calabi-Yau manifolds and mirrors of each other. We can calculate their Hodge numbers using the techniques from chapter 6.

In *PolyTori*, this construction can be done by giving the program the initial two-dimensional toric data and a choice for the fibre vertices with which you want to calculate the scale factor. The user also needs to specify whether he wants to calculate the polyhedron or its dual. Then using the following constructor:

```
Polytope(Polytope polytopeBase, Polytope polytopeFiber, unsigned int
choiceFiber, unsigned int choiceFiberDual, bool isDual);
```

we can perform the construction highlighted above. The constructor then defines a new four-dimensional polytope, with which the user can work as normal. Note that *PolyTori* can only calculate Hodge numbers for polyhedra with more than 5 vertices if they were obtained by the above construction. This is because due to geometrical restraints, it is possible to obtain faces in a much easier way when the vertices are distributed over the four-dimensional space as they are. Normally it would be hard to tell which vertices link with which. However, since the vertices can be grouped into two sets, depending on in which of two two-dimensional planes the vertices lie, it is possible to obtain the Hodge numbers using this method. This library makes it possible for the user to obtain the Hodge numbers given any form of a polyhedron describing a toric space. Since there are usually multiple polyhedra specifying the same toric space, this is an advantage.

Let us discuss some examples of these techniques and discuss their elliptic structure.

7.3 Examples

In this section, we treat some examples. If the ambient space is not known, we specify the base- and fiber space.

7.3.1 $\mathbb{P}^4(1,1,1,6,9)$

Write in this section $\mathbb{P}^4(1,1,1,6,9) = \mathbb{P}$.

Construction

A weighted projective space in four dimensions is given by a polyhedron with five vertices, for which these five vertices $\{p_i\}_{i=0,\ldots,4}$ have the following relation between them:

$$\sum_{i=0}^{4} a_i p_i = 0, \tag{7.6}$$

with $\{a_i\}_{i=0,\ldots,4}$ the weights of the weighted projective space, just as we have seen in chapter 6. Following the construction procedure explained in the previous section, we start with specifying a base and fiber polyhedron:

$$\Delta_B = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.7a}$$

$$\Delta_B^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.7b}$$

$$\Delta_F = \mathrm{conv}\left(\left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.7c}$$

$$\Delta_F^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-2\\-3\end{pmatrix}\right\}\right) \tag{7.7d}$$

Our base manifold is thus $\mathbb{P}^2$, our fiber manifold is $\mathbb{P}^2(1,2,3) \simeq \mathbb{P}^2(1,6,9)$. Note that for the fiber manifold we took a modular transformation of the polyhedron depicted in figure 7.2d, making the resulting polyhedron no longer self-dual. We now form the four-dimensional polyhedron from this data, choosing

$$\nu_k^* = \begin{pmatrix}-2\\-3\end{pmatrix} \text{ and } \nu_l = \begin{pmatrix}-1\\-1\end{pmatrix},$$

such that $s_{kl} = 6$. We find:

$$\Delta = \operatorname{conv}\left(\left\{\begin{pmatrix}12\\-6\\-1\\-1\end{pmatrix}, \begin{pmatrix}-6\\12\\-1\\-1\end{pmatrix}, \begin{pmatrix}-6\\-6\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}\right) \tag{7.8a}$$

$$\Delta^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\\-2\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\\-3\end{pmatrix}, \begin{pmatrix}-1\\-1\\-2\\-3\end{pmatrix}, \begin{pmatrix}0\\0\\-2\\-3\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}\right) \tag{7.8b}$$

Note that the fourth vertex in both polyhedra are contained in the subpolyhedron spanned by the first three vertices. The polyhedra are thus given as the convex hull of the five other vertices. Using our software, we find the following Hodge diamond for the Calabi-Yau threefold $X = \{W_\Delta = 0\}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 272 & & 272 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}. \tag{7.9}$$

For its mirror $X^* = \{W_{\Delta^*} = 0\}$ we find:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 272 & & 0 & \\ 1 & & 2 & & 2 & & 1 \\ & 0 & & 272 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}. \tag{7.10}$$

We do not give the equations for these Calabi-Yau subspaces just yet, we will do that in a later stage. The fundamental group $\pi_1(\Delta)$ of the reflexive simplex Δ is found to be $\mathbb{Z}_{18} \times \mathbb{Z}_6$.

Singular locus

Since our base manifold is $\mathbb{P}^2$, we are interested in the rational map

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2, \tag{7.11}$$

which is not defined on the locus $S = \{x_0 = x_1 = x_2 = 0\}$. We have,

$$\begin{aligned} S &= \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\} \\ &= \mathbb{P}^1(6, 9), \end{aligned}$$

so S is a line in $\mathbb{P}$.

Next, let us find the singular locus of $\mathbb{P}$. We follow the notation introduced in section 5.3.2. We find that of all the toric strata in the fundamental simplex Δ of $\mathbb{P}$, the following harbour cyclic quotient singularities:

- $P_3 \longleftrightarrow \frac{1}{6}(1,1,1,9)$.
- $P_4 \longleftrightarrow \frac{1}{9}(1,1,1,6)$.
- $P_3P_4 \longleftrightarrow \frac{1}{3}(1,1,1)$.

We thus find that the coordinate hyperplane P_3P_4 given by

$$P_3P_4 = \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\}, \tag{7.12}$$

harbours all singularities. Looking at what we found above, we find that:

$$S = \mathbb{P}_{\text{sing}}. \tag{7.13}$$

We will see that this identification has some important consequences.

Define $X \subset \mathbb{P}$ to be a generic smooth hypersurface. If $X = \{f = 0\}$ then X is a threefold, since it is generated by one equation. From section 2.3.3, we know that demanding

$$\deg f = d = \sum_{i=0}^{4} a_i = 1 + 1 + 1 + 6 + 9 = 18, \tag{7.14}$$

will satisfy the Calabi-Yau condition. Note that $\deg x_0 = 1$, $\deg x_1 = 1$, $\deg x_2 = 1$, $\deg x_3 = 6$ and $\deg x_4 = 9$. So, we can write f as:

$$f(x_0, ..., x_4) = \sum_i b_i g_i(x_0, x_1, x_2) h_i(x_3, x_4) + c_0 x_3^3 + c_1 x_4^2, \tag{7.15}$$

with b_i, c_0, c_1 constants and for all i: $\deg(g_i(x_0, x_1, x_2) h_i(x_3, x_4)) = 18$ and g_i not depending trivially on all of x_0, x_1, x_2. We can now calculate how many singular points X has. Starting from a point $s \in S$, we know that $x_0(s) = x_1(s) = x_2(s) = 0$. So,

$$f(s) = c_0 x_3^3(s) + c_1 x_4^2(s) = 0 \Longleftrightarrow x_4^2 = c x_3^3,$$

with c a constant. If we choose a value for x_3: $x_3 = c^{-3}$, then $x_4 = \pm 1$. However, $(0, 0, 0, c^{-3}, 1) \sim (0, 0, 0, c^{-3}, -1)$ define the same point, since

$$(0, 0, 0, c^{-3}, 1) \sim (0, 0, 0, \lambda^6 c^{-3}, \lambda^9),$$

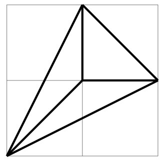
choosing $\lambda = -1$ then shows that there is only one singular point in X. So, $X \cap S = \{[0 : 0 : 0 : c^{-3} : 1]\}$.

Next, we would like to blow-up the ambient space $\mathbb{P}$ in its singular locus, so that the pre-image of X under the blow-up map will have no singularities. We are then left with a Calabi-Yau manifold as we shall see. Define ψ to be the blow-up map

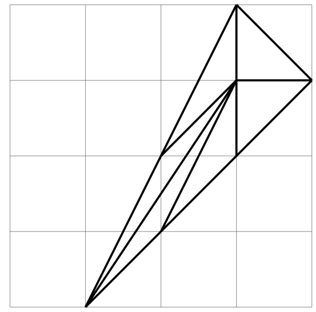
$$\psi : \tilde{\mathbb{P}} \to \mathbb{P}, \tag{7.16}$$

where $\tilde{\mathbb{P}} = \bar{\Gamma}_\phi$ with Γ_ϕ the closure of the graph of the rational map ϕ defined above. So, if

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2 : [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [\xi_0 : \xi_1 : \xi_2], \tag{7.17}$$

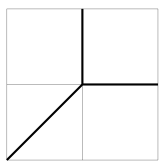


(a) Δ_B^* with maximal projective triangulation

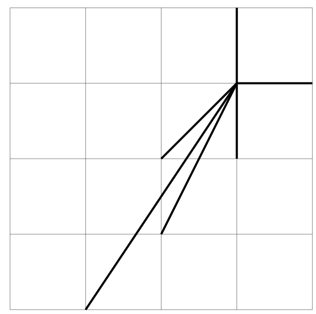


(b) Δ_F^* with maximal projective triangulation

Figure 7.3: The base- and fiber polyhedra for $\mathbb{P}^4(1,1,1,6,9)$ and a maximal projective triangulation on them.



(a) Σ'_B of maximal projective triangulation



(b) Σ'_F of maximal projective triangulation

Figure 7.4: The base- and fiber fans for $\mathbb{P}^4(1,1,1,6,9)$ and a maximal projective triangulation on them.

then for $x = [x_0 : x_1 : x_2 : x_3 : x_4], \xi = [\xi_0 : \xi_1 : \xi_2]$ we have

$$\begin{aligned}
\Gamma_\phi &= \{(x, \xi(x)) \in \mathbb{P} \times \mathbb{P}^2 | x \in \mathbb{P}\}, \\
\bar{\Gamma}_\phi &= \{(x, \xi(x)) \in \Gamma_\phi | x_i \xi_j = x_j \xi_i \text{ for } i, j \in \{0,1,2\}\}.
\end{aligned}$$

Then $\tilde{X} = \psi^{-1}(X \setminus S)$ (where we take the closure) is the proper transform of X.

This space corresponds to the smooth Calabi-Yau manifold obtained from Batyrev's construction starting from Δ and using the MPCP-desingularization. Let us show why. First of, Δ_B^* and Δ_F^* have the maximal projective triangulations depicted in figure 7.3. The corresponding fans are depicted in figure 7.4. A cone is non-singular if its base vectors span the entirety of the lattice. We see that the cones in figure 7.4a are all non-singular, that is because we know that $\mathbb{P}^2$ is non-singular. Next, we look at what would be the original fan Σ_F (constructed from the vertices of Δ_F^*) and the new fan Σ'_F depicted in figure 7.4b. The following cones are singular:

$$\begin{aligned}
\sigma_1 &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_2 &= \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}.
\end{aligned}$$

The cone

$$\sigma_1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}},$$

is non-singular. Following [5], we desingularize Σ_F and we see that we obtain Σ'_F. Furthermore, using *PolyTori*, we find that the only integral points in Δ^* are:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

We see that a fan Σ' corresponding to a maximal projective triangulation of Δ will have gotten rid of the same singularity as we just did for $\mathbb{P}^2(1,6,9)$ with the desingularization process we performed by constructing Σ'_F. The singular locus this corresponded to was $\mathbb{P}^1(6,9) \subset \mathbb{P}^2(1,6,9)$. We see from the above integral points that S is the embedding of this $\mathbb{P}^1(6,9)$ into $\mathbb{P}$. And so, by blowing up in S, we have actually constructed $\mathbb{P}_{\Sigma'}$, which is thus equal to $\tilde{\mathbb{P}}$.

Elliptic structure

We now turn our attention to the elliptic structure on $\tilde{X}$. We have a map

$$\pi : \tilde{\mathbb{P}} \to \mathbb{P}^2 : \tilde{X} \to \mathbb{P}^2, \tag{7.18}$$

defined by projection on the $\mathbb{P}^2$-part in $\mathbb{P} \times \mathbb{P}^2$. Let us look at a generic element $p \in \mathbb{P}^2$. Take $\xi_0(p) \neq 0$ and define $X_1(p) = \frac{\xi_1(p)}{\xi_0(p)}$, $X_2(p) = \frac{\xi_2(p)}{\xi_0(p)}$. Then for $\tilde{\mathbb{P}} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{\mathbb{P}}_p$: $x_1 = x_0 X_1(p)$, $x_2 = x_0 X_2(p)$ and $x_1 \xi_2(p) = x_2 \xi_1(p)$. Write $x_0 = z$, $x_3 = x$, $x_4 = y$. f was a function on $\mathbb{P}$, whose locus defined X. When $q \in X \setminus S$, $\phi(q)$ is well-defined and we defined the proper transform as

$$\tilde{X} = \{(q, \phi(q)) | q \in X \setminus S, x_i(q)\xi_j(\phi(q)) = x_j(q)\xi_i(\phi(q))\}.$$

Since $q \in X \setminus S \subset X$, $f(q) = 0$. We can rewrite f as:

$$f = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^3 + c_1 y^2. \tag{7.19}$$

For $f = 0$, we thus find $\tilde{X} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{X}_p$. Acting with π on this set will give p. X_1, X_2 are functions on $\mathbb{P}^2$, they are fixed for a given choice of $p \in \mathbb{P}^2$. So, we are left with a function f that depends on the coordinates x, y and z. The coordinate z has degree one, the coordinate x has degree six and the coordinate y has degree 9. The function f can thus be seen as a homogeneous equation of degree 18 in $\mathbb{P}^2(1,6,9)$. Furthermore, $\tilde{\mathbb{P}}_p \simeq \mathbb{P}^2(1,6,9)$ and $\tilde{X}_p \simeq \mathbb{P}^2(1,6,9) \cap \{f = 0\}$. We would like to show that the fibers are elliptic curves in $\mathbb{P}^2(1,6,9)$. Taking $f = 0$, we find

$$0 = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^3 + c_1 y^2,$$

which corresponds to $\tilde{X} \cup E$, with E the exceptional divisor. Now, taking $z = 0$ corresponds to the exceptional divisor E. We are then left with:

$$c_0 x^3 + c_1 y^2 = 0, \tag{7.20}$$

but since the solution of this is only one point, the closure of $\tilde{X} \cup E \setminus E$ will be the smooth manifold $\tilde{X}$.

Finally, let us prove that the section we found is indeed the only section. We have

$$\tilde{\sigma} : \mathbb{P}^2 \to \tilde{\mathbb{P}} : (t_0, t_1, t_2) \mapsto (f_0, f_1, f_2, x, y, t_0, t_1, t_2), \tag{7.21}$$

with the equations $f_i t_j = f_j t_i$ implying $f_i = t_i$ what is left to determine is $x \in |\mathcal{O}(6)|$ and $y \in |\mathcal{O}(9)|$. So, these are respectively degree six and degree nine functions in three variables, with 21 and 45 degrees of freedom left. However, f was a degree 18 polynomial. So, there are 171 equations determining these degrees of freedom. The system is thus overdetermined, and no such solution exists in general. There are thus no other sections and the section we found is unique.

7.3.2 $\mathbb{P}^4(1,1,1,3,3)$

Write $\mathbb{P}^4(1,1,1,3,3) = \mathbb{P}$.

Construction

In this example we take as base $\mathbb{P}^2$ and as fiber $\mathbb{P}^2$. We thus have as our starting data:

$$\Delta_B = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.22a}$$

$$\Delta_B^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.22b}$$

$$\Delta_F = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.22c}$$

$$\Delta_F^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.22d}$$

We obtain for the four-dimensional polyhedra:

$$\Delta = \mathrm{conv}\left(\left\{\begin{pmatrix}-3\\6\\-1\\2\end{pmatrix}, \begin{pmatrix}-3\\-3\\-1\\2\end{pmatrix}, \begin{pmatrix}6\\-3\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\2\\-1\end{pmatrix}\right\}\right) \tag{7.23a}$$

$$\Delta^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}\right) \tag{7.23b}$$

We find the following Hodge diamond for the Calabi-Yau threefold $X = \{W_\Delta = 0\}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 4 & & 0 & \\ 1 & & 112 & & 112 & & 1 \\ & 0 & & 4 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{7.24}$$

For its mirror $X^* = \{W_{\Delta^*} = 0\}$ we find:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 112 & & 0 & \\ 1 & & 4 & & 4 & & 1 \\ & 0 & & 112 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{7.25}$$

The fundamental group $\pi_1(\Delta)$ is given by $\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Singular locus

We are studying another example where the base manifold will be $\mathbb{P}^2$. Thus, the rational map of interest looks the same:

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2, \tag{7.26}$$

which is not defined on the locus $S = \{x_0 = x_1 = x_2 = 0\}$. We have,

$$\begin{aligned} S &= \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\} \\ &= \mathbb{P}^1(3,3) \\ &= \mathbb{P}^1, \end{aligned}$$

where we used a property of weighted projective spaces. So, S is a line in $\mathbb{P}$.

Next, let us again find the singular locus of $\mathbb{P}$. Of all the toric strata in the fundamental simplex Δ of $\mathbb{P}$, the following harbour cyclic quotient singularities:

- $P_3 \longleftrightarrow \frac{1}{3}(1,1,1,3)$.
- $P_4 \longleftrightarrow \frac{1}{3}(1,1,1,3)$.
- $P_3P_4 \longleftrightarrow \frac{1}{3}(1,1,1)$.

We thus find that the coordinate hyperplane P_3P_4 given by

$$P_3P_4 = \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\}, \tag{7.27}$$

harbours all singularities. Looking at what we found above, we find again that:

$$S = \mathbb{P}_{\text{sing}}. \tag{7.28}$$

Define $X \subset \mathbb{P}$ to be a generic smooth hypersurface. If $X = \{f = 0\}$ then X is a threefold, since it is generated by one equation. From section 2.3.3, we know that demanding

$$\deg f = d = \sum_{i=0}^{4} a_i = 1+1+1+3+3 = 9, \tag{7.29}$$

will satisfy the Calabi-Yau condition. Note that in this example $\deg x_0 = 1$, $\deg x_1 = 1$, $\deg x_2 = 1$, $\deg x_3 = 3$ and $\deg x_4 = 3$. So, we can write f as:

$$f(x_0,...,x_4) = \sum_i b_i g_i(x_0,x_1,x_2)h_i(x_3,x_4) + c_0x_3^3 + c_1x_4^3 + c_2x_3^2x_4 + c_3x_3x_4^2, \tag{7.30}$$

with b_i, c_0, c_1, c_2, c_3 constants and for all i: $\deg(g_i(x_0,x_1,x_2)h_i(x_3,x_4)) = 9$ and g_i again not depending trivially on all of x_0, x_1, x_2. Let us now calculate how many singular points X has. Starting from a point $s \in S$, we know that $x_0(s) = x_1(s) = x_2(s) = 0$. So,

$$f(s) = c_0x_3^3(s) + c_1x_4^3(s) = 0 \Longleftrightarrow x_4^3 = cx_3^3,$$

with c a constant and where we set $c_2, c_3 = 0$. If we choose a value for x_3: $x_3 = c^{-3}$, then $x_4 \in \{e^{1,\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$. However, $(0,0,0,c^{-3},1)$, $(0,0,0,c^{-3},e^{\frac{2\pi i}{3}})$ and $(0,0,0,c^{-3},e^{\frac{4\pi i}{3}})$ this time do not define the same point, since

$$\begin{aligned}
&(0,0,0,c^{-3},1) \sim (0,0,0,\lambda^3c^{-3},\lambda^3),\\
&(0,0,0,c^{-3},e^{\frac{2\pi i}{3}}) \sim (0,0,0,\lambda^3c^{-3},\lambda^3e^{\frac{2\pi i}{3}}),\\
&(0,0,0,c^{-3},e^{\frac{4\pi i}{3}}) \sim (0,0,0,\lambda^3c^{-3},\lambda^3e^{\frac{4\pi i}{3}}),
\end{aligned}$$

but there is no solution for λ that reaches the other two solutions for this particular distribution of the constants. So,

$$X \cap S = \{[0:0:0:c^{-3}:1], [0:0:0:c^{-3}:e^{\frac{2\pi i}{3}}], [0:0:0:c^{-3}:e^{\frac{4\pi i}{3}}]\}.$$

Next, we would like to blow-up the ambient space $\mathbb{P}$ in its singular locus, so that the pre-image of X under the blow-up map will have no singularities. We are then left with a Calabi-Yau manifold as we shall see. Define ψ to be the blow-up map

$$\psi : \tilde{\mathbb{P}} \to \mathbb{P}, \tag{7.31}$$

where $\tilde{\mathbb{P}} = \bar{\Gamma}_\phi$ with Γ_ϕ the closure of the graph of the rational map ϕ defined above. So, if

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2 : [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [\xi_0 : \xi_1 : \xi_2], \tag{7.32}$$

then for $x = [x_0 : x_1 : x_2 : x_3 : x_4], \xi = [\xi_0 : \xi_1 : \xi_2]$ we have

$$\begin{aligned}
\Gamma_\phi &= \{(x,\xi(x)) \in \mathbb{P} \times \mathbb{P}^2 | x \in \mathbb{P}\},\\
\bar{\Gamma}_\phi &= \{(x,\xi(x)) \in \Gamma_\phi | x_i\xi_j = x_j\xi_i \text{ for } i,j \in \{0,1,2\}\}.
\end{aligned}$$

Then $\tilde{X} = \psi^{-1}(X \setminus S)$ (where we take the closure) is the proper transform of X.

Elliptic structure

We now turn our attention to the elliptic structure on $\tilde{X}$. We have a map

$$\pi : \tilde{\mathbb{P}} \to \mathbb{P}^2 : \tilde{X} \to \mathbb{P}^2, \tag{7.33}$$

defined by projection on the $\mathbb{P}^2$-part in $\mathbb{P} \times \mathbb{P}^2$. Let us look at a generic element $p \in \mathbb{P}^2$. Take $\xi_0(p) \neq 0$ and define $X_1(p) = \frac{\xi_1(p)}{\xi_0(p)}$, $X_2(p) = \frac{\xi_2(p)}{\xi_0(p)}$. Then for $\tilde{\mathbb{P}} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{\mathbb{P}}_p$: $x_1 = x_0 X_1(p)$, $x_2 = x_0 X_2(p)$ and $x_1 \xi_2(p) = x_2 \xi_1(p)$. Write $x_0 = z$, $x_3 = x$, $x_4 = y$. f was a function on $\mathbb{P}$, whose locus defined X. When $q \in X \setminus S$, $\phi(q)$ is well-defined and we defined the proper transform as

$$\tilde{X} = \{(q, \phi(q)) | q \in X \setminus S, x_i(q)\xi_j(\phi(q)) = x_j(q)\xi_i(\phi(q))\}.$$

Since $q \in X \setminus S \subset X$, $f(q) = 0$. We can rewrite f as:

$$f = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^3 + c_1 y^3 + c_2 x^2 y + c_3 x y^2. \tag{7.34}$$

For $f = 0$, we thus find $\tilde{X} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{X}_p$. Acting with π on this set will give p. X_1, X_2 are functions on $\mathbb{P}^2$, they are fixed for a given choice of $p \in \mathbb{P}^2$. So, we are left with a function f that depends on the coordinates x, y and z. The coordinate z has degree one, the coordinate x has degree three and the coordinate y has degree three. The function f can thus be seen as a homogeneous equation of degree 9 in $\mathbb{P}^2(1,3,3) \simeq \mathbb{P}^2$. Furthermore, $\tilde{\mathbb{P}}_p \simeq \mathbb{P}^2$ and $\tilde{X}_p \simeq \mathbb{P}^2 \cap \{f = 0\}$. We would like to show that the fibers are elliptic curves in $\mathbb{P}^2$. Taking $f = 0$, we find

$$0 = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^3 + c_1 y^3 + c_2 x^2 y + c_3 x y^2,$$

which corresponds to $\tilde{X} \cup E$. Now, taking $z = 0$ corresponds to the exceptional divisor E, we are left with:

$$c_0 x^3 + c_1 y^3 + c_2 x^2 y + c_3 x y^2 = 0, \tag{7.35}$$

and if we subtract its solutions from our hypersurface and then take the closure we obtain a smooth manifold: $(\tilde{X} \cup E) \setminus E$ gives $\tilde{X}$.

Next, let us proof in the same way as before that the sections we found are the only ones. We are dealing with a defining function f of degree 9, so there are 45 different equations determining the system. We found that $x \in |\mathcal{O}(3)|$ and $y \in |\mathcal{O}(3)|$. So, there are respectively six and six variables to determine, which leads to a total of 12. The system again is overdetermined and we have proven that the sections we have found are indeed the only ones.

7.3.3 $\mathbb{P}^4(1,1,1,3,6)$

Write $\mathbb{P}^4(1,1,1,3,6) = \mathbb{P}$.

Construction

This will be the third and final example where the base manifold will be $\mathbb{P}^2$, the fiber is $\mathbb{P}^2(1,1,2)$. We thus have:

$$\Delta_B = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.36a}$$

$$\Delta_B^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.36b}$$

$$\Delta_F = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\3\end{pmatrix}\right\}\right) \tag{7.36c}$$

$$\Delta_F^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-2\\-1\end{pmatrix}\right\}\right) \tag{7.36d}$$

We obtain for the four-dimensional polyhedra:

$$\Delta = \mathrm{conv}\left(\left\{\begin{pmatrix}-4\\8\\-1\\3\end{pmatrix},\begin{pmatrix}-4\\-4\\-1\\3\end{pmatrix},\begin{pmatrix}8\\-4\\-1\\3\end{pmatrix},\begin{pmatrix}0\\0\\-1\\3\end{pmatrix},\begin{pmatrix}0\\0\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right\}\right) \tag{7.37a}$$

$$\Delta^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\-2\\-1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}\right) \tag{7.37b}$$

We find the following Hodge diamond for the Calabi-Yau threefold $X = \{W_\Delta = 0\}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 165 & & 165 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,. \tag{7.38}$$

For its mirror $X^* = \{W_{\Delta^*} = 0\}$ we find:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 165 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 165 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}\,. \tag{7.39}$$

The fundamental group $\pi_1(\Delta)$ is given by $\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_2$.

Singular locus

The rational map of interest again looks the same:

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2, \tag{7.40}$$

which is not defined on the locus $S = \{x_0 = x_1 = x_2 = 0\}$. We have,

$$\begin{aligned} S &= \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\} \\ &= \mathbb{P}^1(3, 6) \\ &= \mathbb{P}^1(1, 2). \end{aligned}$$

So, S is a line in $\mathbb{P}$.

Next, let us again find the singular locus of $\mathbb{P}$. Of all the toric strata in the fundamental simplex Δ of $\mathbb{P}$, the following harbour cyclic quotient singularities:

- $P_3 \longleftrightarrow \frac{1}{3}(1, 1, 1, 6)$.
- $P_4 \longleftrightarrow \frac{1}{6}(1, 1, 1, 3)$.
- $P_3P_4 \longleftrightarrow \frac{1}{3}(1, 1, 1)$.

We thus find that the coordinate hyperplane P_3P_4 given by

$$P_3P_4 = \{p \in \mathbb{P} | x_0(p) = x_1(p) = x_2(p) = 0\}, \tag{7.41}$$

harbours all singularities. We find again that:

$$S = \mathbb{P}_{\text{sing}}. \tag{7.42}$$

Define $X \subset \mathbb{P}$ to be a generic smooth hypersurface defined by $X = \{f = 0\}$ with Calabi-Yau condition

$$\deg f = d = \sum_{i=0}^{4} a_i = 1 + 1 + 1 + 3 + 6 = 12. \tag{7.43}$$

Note that in this example $\deg x_0 = 1$, $\deg x_1 = 1$, $\deg x_2 = 1$, $\deg x_3 = 3$ and $\deg x_4 = 6$. So, we can write f as:

$$f(x_0, ..., x_4) = \sum_i b_i g_i(x_0, x_1, x_2) h_i(x_3, x_4) + c_0 x_3^4 + c_1 x_4^2 + c_2 x_3^2 x_4, \tag{7.44}$$

with b_i, c_0, c_1, c_2 constants and for all i: $\deg(g_i(x_0, x_1, x_2) h_i(x_3, x_4)) = 12$ and g_i again not depending trivially on all of x_0, x_1, x_2. Let us now calculate how many singular points X has. Starting from a point $s \in S$, we know that $x_0(s) = x_1(s) = x_2(s) = 0$. So,

$$f(s) = c_0 x_3^4(s) + c_1 x_4^2(s) = 0 \Longleftrightarrow x_4^2 = c x_3^4,$$

with c a constant and $c_2 = 0$ in this example. If we choose a value for x_3: $x_3 = c^{-4}$, then $x_4 = \pm 1$. However, $(0, 0, 0, c^{-3}, 1)$ and $(0, 0, 0, c^{-3}, -1)$ are again not in the same equivalence class, since

$$(0, 0, 0, c^{-4}, 1) \sim (0, 0, 0, \lambda^3 c^{-4}, \lambda^6),$$

and there is no λ that reaches the other solution. So,

$$X \cap S = \{[0:0:0:c^{-4}:1], [0:0:0:c^{-4}:-1]\}.$$

Next, we would like to blow-up the ambient space $\mathbb{P}$ in its singular locus, so that the pre-image of X under the blow-up map will have no singularities. We are then left with a Calabi-Yau manifold as we shall see. Define ψ to be the blow-up map

$$\psi : \tilde{\mathbb{P}} \to \mathbb{P}, \tag{7.45}$$

where $\tilde{\mathbb{P}} = \bar{\Gamma}_\phi$ with Γ_ϕ the closure of the graph of the rational map ϕ defined above. So, if

$$\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2 : [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [\xi_0 : \xi_1 : \xi_2], \tag{7.46}$$

then for $x = [x_0 : x_1 : x_2 : x_3 : x_4], \xi = [\xi_0 : \xi_1 : \xi_2]$ we have

$$\begin{aligned}
\Gamma_\phi &= \{(x, \xi(x)) \in \mathbb{P} \times \mathbb{P}^2 | x \in \mathbb{P}\}, \\
\bar{\Gamma}_\phi &= \{(x, \xi(x)) \in \Gamma_\phi | x_i \xi_j = x_j \xi_i \text{ for } i, j \in \{0, 1, 2\}\}.
\end{aligned}$$

Then $\tilde{X} = \psi^{-1}(X \setminus S)$ (where we take the closure) is the proper transform of X.

This space correspondents with the Calabi-Yau manifold from the Batyrev construction following the MPCP desingularisation, because of the following. Then fan Σ_F corresponding to Δ_F^* is given by the cones:

$$\begin{aligned}
\sigma_0 &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_1 &= \left\langle \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_2 &= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}},
\end{aligned}$$

of which only σ_1 is singular. If we blow-up in the singularity of $\mathbb{P}^2(1,1,2)$, we thus blow-up the cone σ_1 and get:

$$\sigma_1 \longrightarrow \{\sigma_{11}, \sigma_{12}\},$$

with

$$\begin{aligned}
\sigma_{11} &= \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}, \\
\sigma_{12} &= \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{R}_{\geq 0}}.
\end{aligned}$$

We thus obtain Σ'_F, corresponding to a maximal projective triangulation of Δ_F. Using *PolyTori*, we find that the set of integral points of Δ is given by:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Since the singularity of $\mathbb{P}$ is the embedding of the singularity $\mathbb{P}^1(1,2) \subset \mathbb{P}^2(1,1,2)$ in $\mathbb{P}$ and there are no other integral points other then the vertices and the ones obtained by triangulation of Δ_F^*, we can conclude that blowing-up in S is equivalent to the MPCP-desingularization corresponding to the maximal projective triangulation given by the set of integral points above.

Elliptic structure

We now again turn our attention to the elliptic structure on $\tilde{X}$. We have a map

$$\pi : \tilde{\mathbb{P}} \to \mathbb{P}^2 : \tilde{X} \to \mathbb{P}^2, \tag{7.47}$$

defined by projection on the $\mathbb{P}^2$-part in $\mathbb{P} \times \mathbb{P}^2$. Let us look at a generic element $p \in \mathbb{P}^2$. Take $\xi_0(p) \neq 0$ and define $X_1(p) = \frac{\xi_1(p)}{\xi_0(p)}$, $X_2(p) = \frac{\xi_2(p)}{\xi_0(p)}$. Then for $\tilde{\mathbb{P}} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{\mathbb{P}}_p$: $x_1 = x_0 X_1(p)$, $x_2 = x_0 X_2(p)$ and $x_1 \xi_2(p) = x_2 \xi_1(p)$. Write $x_0 = z$, $x_3 = x$, $x_4 = y$. f was a function on $\mathbb{P}$, whose locus defined X. When $q \in X \setminus S$, $\phi(q)$ is well-defined and we defined the proper transform as

$$\tilde{X} = \{(q, \phi(q)) | q \in X \setminus S, x_i(q)\xi_j(\phi(q)) = x_j(q)\xi_i(\phi(q))\}.$$

Since $q \in X \setminus S \subset X$, $f(q) = 0$. We can rewrite f as:

$$f = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^4 + c_1 y^2. \tag{7.48}$$

For $f = 0$, we thus find $\tilde{X} \cap (\mathbb{P} \times \{p\}) \equiv \tilde{X}_p$. Acting with π on this set will give p. X_1, X_2 are functions on $\mathbb{P}^2$, they are fixed for a given choice of $p \in \mathbb{P}^2$. So, we are left with a function f that depends on the coordinates x, y and z. The coordinate z has degree one, the coordinate x has degree three and the coordinate y has degree six. The function f can thus be seen as a homogeneous equation of degree 12 in $\mathbb{P}^2(1,3,6) \simeq \mathbb{P}^2(1,1,2)$. Furthermore, $\tilde{\mathbb{P}}_p \simeq \mathbb{P}^2(1,1,2)$ and $\tilde{X}_p \simeq \mathbb{P}^2(1,1,2) \cap \{f = 0\}$. We would like to show that the fibers are elliptic curves in $\mathbb{P}^2(1,1,2)$. Taking $f = 0$, we find

$$0 = \sum_i b_i g_i(z, zX_1, zX_2) h_i(x, y) + c_0 x^4 + c_1 y^2,$$

which corresponds to $\tilde{X} \cup E$ with E the exceptional divisor. Taking $z = 0$ then corresponds to the exceptional divisor. We are left with:

$$c_0 x^4 + c_1 y^2 = 0. \tag{7.49}$$

Since the solution of this was two distinct points, we know that the closure of $(\tilde{X} \cup E) \setminus E$ will be the smooth manifold $\tilde{X}$.

Next, let us proof in the same way as before that the sections we found are the only ones. We are dealing with a defining function f of degree 12, so there are 78 different equations determining the system. We found that $x \in |\mathcal{O}(3)|$ and $y \in |\mathcal{O}(6)|$. So, there are respectively 6 and 21 variables to determine, which leads to a total of 27. The system again is overdetermined and we have proven that the sections we have found are indeed the only ones.

7.3.4 Base $\mathbb{F}_1$, fiber $\mathbb{P}^2$

Construction

Let us start by giving the initial data:

$$\Delta_B = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.50a}$$

$$\Delta_B^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.50b}$$

$$\Delta_F = \mathrm{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix}\right\}\right) \tag{7.50c}$$

$$\Delta_F^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right) \tag{7.50d}$$

We obtain for the four-dimensional polyhedra:

$$\Delta = \mathrm{conv}\left(\left\{\begin{pmatrix}3\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}3\\-3\\-1\\-1\end{pmatrix}, \begin{pmatrix}-3\\-3\\-1\\-1\end{pmatrix}, \begin{pmatrix}-3\\6\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\2\end{pmatrix}\right\}\right) \tag{7.51a}$$

$$\Delta^* = \mathrm{conv}\left(\left\{\begin{pmatrix}1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}\right) \tag{7.51b}$$

In this case it is hard to see what the ambient space exactly is. However, we know that the corresponding Calabi-Yau threefold will have a map $X_\Delta \to \mathbb{F}_1$ with as generic fiber space $\mathbb{P}^2$.

Elliptic Structure

Using [12] we can immediately talk about the defining equation for the Calabi-Yau as well as its elliptic fiber. Since Σ_B and Σ_F do not have cones which harbour singularities, we can immediately write down W_Δ:

$$\begin{aligned} W_\Delta &= a_0 Y_0^6 Y_1^3 Y_4^3 + a_1 Y_0^6 Y_3^3 Y_4^3 + a_2 Y_2^6 Y_3^9 Y_4^3 + a_3 Y_1^9 Y_2^6 Y_4^3 + a_4 Y_0^3 Y_1^3 Y_2^3 Y_3^3 Y_4^3 \\ &\quad + a_5 Y_5^3 + a_6 Y_6^3 \\ &= h(Y_B) z^3 + a_5 x^3 + a_6 y^3, \end{aligned}$$

with the numbering of the coordinates chosen corresponding to the points as in the definition of Δ^* above. We have relabelled $Y_4 = z$, $Y_5 = x$ and $Y_6 = y$. At $z = 0$ we get a single section, since all the dependence on the base coordinates is in the first term. Setting $z = 0$ we obtain:

$$a_5 x^3 + a_6 y^3 = 0. \tag{7.52}$$

The generic fiber $\mathcal{E}$ is given by X_{Δ_F}, given by:

$$W_{\Delta_F} = a_0 z^3 + a_1 x^3 + a_2 y^3, \tag{7.53}$$

in $\mathbb{P}^2$.

7.3.5 Base $\mathbb{P}^1 \times \mathbb{P}^1$, fiber $\mathbb{P}^2(1,6,9)$

The next example we will study has as base $\mathbb{P}^1 \times \mathbb{P}^1$, and as fiber $\mathbb{P}^2(1,6,9)$.

Construction

We again begin with giving the toric initial data:

$$\Delta_B = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\right\}\right), \tag{7.54a}$$

$$\Delta_B^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\-1\end{pmatrix}\right\}\right), \tag{7.54b}$$

$$\Delta_F = \operatorname{conv}\left(\left\{\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right), \tag{7.54c}$$

$$\Delta_F^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-2\\-3\end{pmatrix}\right\}\right). \tag{7.54d}$$

Following the construction outlined in this chapter, we obtain the following four dimensional polyhedra:

$$\Delta = \operatorname{conv}\left(\left\{\begin{pmatrix}-3\\9\\-1\\-1\end{pmatrix},\begin{pmatrix}-3\\-3\\-1\\-1\end{pmatrix},\begin{pmatrix}3\\-3\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}\right), \tag{7.55a}$$

$$\Delta^* = \operatorname{conv}\left(\left\{\begin{pmatrix}-2\\-1\\-1\\-1\end{pmatrix},\begin{pmatrix}-3\\-3\\-1\\-1\end{pmatrix},\begin{pmatrix}3\\-3\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\2\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}\right). \tag{7.55b}$$

Again it will be hard to see what the ambient space exactly is. However, we know that the corresponding Calabi-Yau threefold will have a map $X_\Delta \to \mathbb{P}^1 \times \mathbb{P}^1$ with as generic fiber space $\mathbb{P}^2(1,6,9)$.

Elliptic structure

We already desingularized the fan of $\mathbb{P}^2(1,6,9)$, we get two extra points:

$$\begin{pmatrix}0\\0\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\-2\end{pmatrix}.$$

We thus obtain for our Laurent polynomial:

$$\begin{aligned}W_\Delta &= a_0Y_0^{12}Y_1^{12}Y_4^6 + a_1Y_0^{12}Y_3^{12}Y_4^6 + a_2Y_2^{12}Y_3^{12}Y_4^6 + a_3Y_1^{12}Y_2^{12}Y_4^6 + a_4Y_0^6Y_1^6Y_2^6Y_3^6Y_4^6\\ &\quad + a_5Y_5^3 + a_6Y_6^2 + a_7Y_0^3Y_1^3Y_2^3Y_3^3Y_4^3Y_6 + a_8Y_0^4Y_1^4Y_2^4Y_3^4Y_4^4Y_5\\ &= h_0(Y_B)z^6 + h_1(Y_B)z^3y + h_2(Y_B)z^4x + a_5x^3 + a_6y^2.\end{aligned}$$

Here, we renamed $Y_4 = z$, $Y_5 = x$, $Y_6 = y$ and regrouped all the terms with base variables Y_B. Since all the terms with base variables have a factor of z in them, when taking $z = 0$ we obtain a section. We are left with:

$$a_5 x^3 + a_6 y^2 = 0. \tag{7.56}$$

This has a unique solution in $\mathbb{P}^2(1,6,9)$. We thus get a fibration map $X_\Delta \to \mathbb{P}^1 \times \mathbb{P}^1$ with generic fibre the elliptic curve $\mathcal{E} = X_{\Delta_F}$ given by:

$$a_0 z^2 + a_1 x^3 + a_2 y^2 + a_3 z^3 y + a_4 z^4 x = 0 \tag{7.57}$$

in $\mathbb{P}^2(1,6,9)$.

7.3.6 $\mathbb{P}^4(1,1,2,2,6)$

As a final example, let us take a look at a weighted projective space obtained from our two-dimensional starting data, where the singular locus of the rational map and the singularities of the weighted projective space do not match. Here, we need to use the same technique as in the previous two examples, since presently we cannot describe the Calabi-Yau space with a direct calculation.

Construction

We begin by giving the initial data:

$$\Delta_B = \operatorname{conv}\left(\left\{\begin{pmatrix}-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\3\end{pmatrix}\right\}\right), \tag{7.58a}$$

$$\Delta_B^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-2\\-1\end{pmatrix}\right\}\right), \tag{7.58b}$$

$$\Delta_F = \operatorname{conv}\left(\left\{\begin{pmatrix}2\\-1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\}\right), \tag{7.58c}$$

$$\Delta_F^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-2\\-3\end{pmatrix}\right\}\right). \tag{7.58d}$$

Following the construction outlined in this chapter, we then obtain the following four dimensional polyhedra:

$$\Delta = \operatorname{conv}\left(\left\{\begin{pmatrix}-6\\-6\\-1\\-1\end{pmatrix},\begin{pmatrix}6\\-6\\-1\\-1\end{pmatrix},\begin{pmatrix}-6\\18\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\-1\end{pmatrix},\begin{pmatrix}0\\0\\2\\-1\end{pmatrix},\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\right\}\right), \tag{7.59a}$$

$$\Delta^* = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\0\\-2\\-3\end{pmatrix},\begin{pmatrix}0\\1\\-2\\-3\end{pmatrix},\begin{pmatrix}-2\\-1\\-2\\-3\end{pmatrix},\begin{pmatrix}0\\0\\-2\\-3\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}\right). \tag{7.59b}$$

Using *PolyTori*, we find the following Hodge diamond for $X = \{W_\Delta = 0\}$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 5 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 5 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{7.60}$$

For its mirror $X^* = \{W_{\Delta^*} = 0\}$ we find:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{7.61}$$

The fundamental group $\pi_1(\Delta)$ is given by $\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_2$.

Elliptic structure

As we explained above, the singular locus and the locus where the rational map to the base is undefined are not equal in this example. We can thus not do a direct calculation as with the previous examples that were weighted projective spaces. We can however directly compute the Laurent polynomial like we did in the previous two examples. We obtain:

$$\begin{aligned} W_\Delta &= a_0 Y_2^{24} Y_3^6 + a_1 Y_0^{12} Y_3^6 + a_2 Y_1^{24} Y_3^6 + a_3 Y_0^6 Y_1^6 Y_2^6 Y_3^6 \\ &\quad + a_4 Y_4^3 + a_5 Y_5^2 \\ &\quad = h(Y_B) z^6 + a_4 x^3 + a_5 y^2 \end{aligned}$$

and for $z = 0$ we get a single section.

Chapter 8

Strings from D3-branes

In this chapter we study an explicit example of a theory which lives in the mathematical spaces we constructed and studied in the previous chapters. This theory was analysed in [8]. In chapter 4 we studied $\mathcal{N} = 4$ super Yang-Mills. In the first section of this chapter we will show that this theory is equal to type IIB superstring on a D3-brane, we will follow a discussion in [20]. We then discuss and give an example of a wrapping procedure. After this, we follow work by Martucci in [18] and Kapustin in [15] on the topological duality twist. This is a different sort of twist from the one we encountered in chapter 3, although it is based on the same principles. After we have performed this twist, we look at the resulting fields and determine the central charges in both the left- and right sector.

8.1 Correspondence between $\mathcal{N} = 4$ SYM and type IIB superstring on D3-brane

In this chapter we will discuss the following set-up. We are working on a target manifold of dimension ten. Target space indices are Greek indices $\mu, \nu, ... = 0, ..., 9$. The worldsheet indices are σ, τ. The worldsheet is embedded into the target space by $X^\mu(\tau, \sigma)$. We want a D3-brane, so the following conditions apply to $X^\mu(\tau, \sigma)$:

$$\partial^\sigma X^\mu(\tau, 0) = 0 = \partial^\sigma X^\mu(\tau, l), \qquad \text{for } \mu = 0, 1, 6, 7; \tag{8.1a}$$

$$\delta X^\mu(\tau, 0) = 0 = \delta X^\mu(\tau, l), \qquad \text{for } \mu = 2, 3, 4, 5, 8, 9; \tag{8.1b}$$

with the first line being the Neumann condition and the second line the Dirichlet boundary condition. Since we have Neumann boundary conditions in three spatial- and one time direction, the endpoints of the string are constrained to a $3 + 1$ dimensional D3-brane. Let us use a as an index for the spatial directions which have the Neumann conditions, and i for the spatial directions which have the Dirichlet boundary conditions. If we want to make the D3-brane dynamical, we have to write down an action which governs its worldvolume, minimizing it. In a way reminiscing the Nambu-Goto action given in equation 1.1, the action is given by:

$$S_3 = -T_3 \int d^4\xi \sqrt{-\det(h_{ab})}, \tag{8.2}$$

with the worldsheet metric given by

$$h_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} g_{\mu\nu}, \tag{8.3}$$

with $g_{\mu\nu}$ the metric of the ambient space. This is thus the generic action for a D3-brane coupled to the metric on the ambient manifold. Usually in string theory, instead of the symmetric metric $g_{\mu\nu}$, we use $g_{\mu\nu} + \alpha' B_{\mu\nu}$, with B the anti-symmetric part of the physical massless states. Filling this in into our action 8.2 we find:

$$S_3 = -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}(g_{\mu\nu} + \alpha' B_{\mu\nu})\right)}.$$

This action has coordinate reparametrization invariance on the coordinates X^μ, we can fix this invariance by choosing a gauge:

$$X^a = \xi^a, \qquad a = 0,1,6,7 \qquad\qquad \text{(static gauge)}. \tag{8.4}$$

We can now redefine the remaining spatial coordinates and rewrite the action:

$$X^i(\xi^a) \equiv \frac{\phi^i(\xi^a)}{\sqrt{T_3}}, \tag{8.5}$$

such that:

$$\begin{aligned}
S_3 &= -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}(g_{\mu\nu} + \alpha' B_{\mu\nu})\right)} \\
&= -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial X^c}{\partial \xi^a}\frac{\partial X^d}{\partial \xi^b}(g_{cd} + \alpha' B_{cd}) + \frac{\partial X^i}{\partial \xi^a}\frac{\partial X^j}{\partial \xi^b}(g_{ij} + \alpha' B_{ij})\right)} \\
&= -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial \xi^c}{\partial \xi^a}\frac{\partial \xi^d}{\partial \xi^b}(g_{cd} + \alpha' B_{cd}) + \frac{1}{T_3}\frac{\partial \phi^i}{\partial \xi^a}\frac{\partial \phi^j}{\partial \xi^b}(g_{ij} + \alpha' B_{ij})\right)}
\end{aligned}$$

Writing $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_N h_{\mu\nu}$ and setting $h, B = 0$, we find:

$$\begin{aligned}
S_3 &= -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial \xi^c}{\partial \xi^a}\frac{\partial \xi^d}{\partial \xi^b}(g_{cd} + \alpha' B_{cd}) + \frac{1}{T_3}\frac{\partial \phi^i}{\partial \xi^a}\frac{\partial \phi^j}{\partial \xi^b}(g_{ij} + \alpha' B_{ij})\right)} \\
&= -T_3 \int d^4\xi \sqrt{-\det\left(\frac{\partial \xi^c}{\partial \xi^a}\frac{\partial \xi^d}{\partial \xi^b}\eta_{cd} + \frac{1}{T_3}\frac{\partial \phi^i}{\partial \xi^a}\frac{\partial \phi^j}{\partial \xi^b}\eta_{ij}\right)} \\
&= -T_3 \int d^4\xi \sqrt{-\det\left(\delta^c_a \delta^d_b \eta_{cd} + \frac{1}{T_3}\frac{\partial \phi^i}{\partial \xi^a}\frac{\partial \phi^j}{\partial \xi^b}\eta_{ij}\right)} \\
&= -T_3 \int d^4\xi \sqrt{-\det\left(\eta_{ab} + \frac{1}{T_3}\partial_a \phi^i \partial_b \phi_i\right)}
\end{aligned}$$

Let us expand the determinant:

$$
\begin{aligned}
-\det\left(\eta_{ab}+\frac{1}{T_3}\partial_a\phi^i\partial_b\phi_i\right) &= \frac{1}{4!}\epsilon^{a_1\dots a_4}\epsilon^{b_1\dots b_4}\prod_{j=1}^{4}\left(\eta_{a_jb_j}+\frac{\partial_{a_j}\phi^i\partial_{b_j}\phi_i}{T_3}+\dots\right)\\
&= \frac{1}{4!}\left(\underbrace{\epsilon^{a_1\dots a_4}\epsilon_{a_1\dots a_4}}_{=4!}+4\underbrace{\epsilon^{a_1a_2a_3a_4}\epsilon^{b_4}_{\ a_1a_2a_3}}_{=3!\eta^{a_4b_4}}\frac{1}{T_3}\partial_{a_4}\phi^i\partial_{b_4}\phi_i+\dots\right)\\
&= \frac{1}{4!}\left(4!+4!\frac{1}{T_3}\partial_a\phi^i\partial^a\phi_i+\dots\right)\\
&= 1+\frac{1}{T_3}\partial_a\phi^i\partial^a\phi_i+\dots
\end{aligned}
$$

where we did not show terms with higher order in ϕ. When we now expand the square root, we find:

$$
\begin{aligned}
S_3 &= -T_3\int d^4\xi\sqrt{-\det\left(\eta_{ab}+\frac{1}{T_3}\partial_a\phi^i\partial_b\phi_i\right)}\\
&= -T_3\int d^4\xi\sqrt{1+\frac{1}{T_3}\partial_a\phi^i\partial^a\phi_i+\dots}\\
&= -T_3\int d^4\xi\left(1+\frac{1}{2T_3}\partial_a\phi^i\partial^a\phi_i+\dots\right)\\
&= -\int d^4\xi\left(T_3+\frac{1}{2}\partial_a\phi^i\partial^a\phi_i+\dots\right).
\end{aligned}
$$

So, up to first order we find a kinetic term for the scalar fields ϕ^i. We also look at the case when $h_{\mu\nu}$ is not equal to zero. Then we find:

$$
\begin{aligned}
S_3 &= -T_3\int d^4\xi\sqrt{-\det\left(\eta_{ab}+2\kappa_N h_{ab}+\frac{1}{T_3}\frac{\partial\phi^i}{\partial\xi^a}\frac{\partial\phi^j}{\partial\xi^b}(\eta_{ij}+2\kappa_N h_{ij})\right)}\\
&= -T_3\int d^4\xi\sqrt{-\det\left(\eta_{ab}+\frac{1}{T_3}\partial_a\phi^i\partial_b\phi_i+2\kappa_N h_{ab}+\frac{2\kappa_N}{T_3}\partial_a\phi^i\partial_b\phi^j h_{ij}\right)}.
\end{aligned}
$$

Again expanding the determinant as before, gives us the same terms for the first two terms in the determinant. We also get a coupling term between the graviton and the scalar fields:

$$
S_3 = -\int d^4\xi\left(T_3+\frac{1}{2}\partial_a\phi^i\partial^a\phi_i+2\kappa_N\sqrt{T_3}h_{ai}\partial^a\phi^i+\dots\right). \tag{8.6}
$$

This last term thus gives a coupling between the closed string mode $g_{\mu\nu}$, which can live anywhere in spacetime, and the open string mode ϕ^i, which lives on the worldvolume of the D3-brane. The NS-NS sector has another mode, the dilaton ϕ. The closed string action has a factor of $1/g_s^2$ in front, with g_s the string coupling equal to $e^{\langle\phi\rangle}$. The D3-brane action is an open string action, which has a factor of g_o^2 in front. Now, $g_s = g_o^2$, so the D3-brane action gets a factor of $e^{-\langle\phi\rangle}$ in front. In the D3-brane direction we also expect, since we have Neumann boundary conditions, that there is a massless vector A_a. In the weak field limit (so after we have expanded the determinant and gotten rid of higher order terms) we know that these massless vector fields will have dynamics determined by the usual Maxwell action term $-\frac{1}{4}F_{ab}F^{ab}$. Of course, the action will have fermions as well; the terms governing the dynamics of fermions and their coupling to the bosonic fields are determined by imposing supersymmetry. We thus expect the bosonic action to be

$$S_3 = -2 \int d^4\xi \ \mathrm{tr} \left(-\frac{1}{4}\partial_a \phi^i \partial^a \phi_i - \frac{1}{4} F_{ab}F^{ab} \right), \tag{8.7}$$

which is precisely the bosonic part of the action in equation 4.2. When we impose supersymmetry, we will uncover precisely the full $\mathcal{N} = 4$ SYM action for which we studied the dimensional reduction on a torus in chapter 4.4. Namely, we have six bosonic degrees of freedom from the scalar fields and in four dimensions two bosonic degrees of freedom from the gauge field. So this gives us eight bosonic degrees of freedom in total. Now, a four dimensional Majorana fermion has two degrees of freedom, so we would need four Majorana fermions to have an equal number of fermionic degrees of freedom as bosonic degrees of freedom. So the total field content is ϕ^i, A_a and Ψ^I, matching the $\mathcal{N} = 4$ SYM field content. Also, the D3-brane action will be invariant under four supersymmetries in four dimensions, leading to 16 different supercharges. This is a different number from the 32 supercharges in type IIB superstring theory. This is because we have an extra condition on the supercharges when working with D-branes. Namely $\Gamma_0 \cdots \Gamma_4 \epsilon = \epsilon$ for a supercharge ϵ, halving the number of supercharges. We thus showed that the action for a D3-brane in ten dimensions in type IIB can be rewritten in the weak field limit as the $\mathcal{N} = 4$ SYM action.

8.2 Wrapping

In this short section, we would like to explain what is meant by "wrapping a certain object around another object". Remember that a Dp-brane is an extended object in p spatial dimensions. Its trajectory, or worldvolume, is a $p + 1$ dimensional manifold. In superstring theory, spacetime has ten dimensions. A Dp-brane in this superstring theory will be a submanifold of the ambient space manifold. So, some examples are:

- $\mathbb{R}^{1,p} \subset \mathbb{R}^{1,9}$, with $0 \leq p \leq 9$.
- $\mathbb{R}^{1,3} \times Y \subset \mathbb{R}^{1,3} \times X$, with $Y \subset X$ a submanifold.

In the last example, we say that the Dp-brane is wrapped around Y. So, for example if X contains a Riemann surface, then we could impose that Y is this Riemann surface. If the brane is wrapped around Y and we look at the limit where Y becomes small, one obtains an effective theory in the remaining non-compact dimensions of the brane. We will need this definition of wrapping in our theory, since we are going to wrap the D3-brane around a particular Riemann curve of genus $g > 0$.

8.3 $\mathcal{N}=4$ SYM group structure

The field content of the $\mathcal{N}=4$ SYM consists of a gauge field A_μ, the six scalar fields ϕ^i with $i=1,...,6$ and the eight fermion fields Ψ^I_A, $\bar{\Psi}_{I\dot{A}}$ with $I=1,...,4$ and A respectively $\dot{A}$ the left- respectively right-moving Weyl indices. The internal R-symmetry of the D3-brane worldvolume is given by group of rotations in the transversal dimensions to the D3-brane, so $SO(6)\equiv SU(4)_R$. If we now perform a Wick rotation so that we are in the Euclidean case, the combined Euclidean total space symmetry group is $SO(4)\times SU(4)_R$. Now, on the level of Lie algebras $\mathfrak{so}(4)=\mathfrak{su}(2)\otimes\mathfrak{su}(2)$ and we have a double cover $SU(2)\times SU(2)\cong SO(4)$. Under the combined Euclidean group $SU(2)\times SU(2)\times SU(4)_R$, the fields of $\mathcal{N}=4$ SYM transform as follows:

$$A_\mu \in (\mathbf{2},\mathbf{2},\mathbf{1}); \qquad \phi^i\in(\mathbf{1},\mathbf{1},\mathbf{6}_v); \qquad \Psi^I_A\in(\mathbf{2},\mathbf{1},\mathbf{4}); \qquad \bar{\Psi}_{I\dot{A}}\in(\mathbf{1},\mathbf{2},\bar{\mathbf{4}}).$$

The sixteen supercharges in the theory transform as:

$$Q_{AI}\in(\mathbf{2},\mathbf{1},\bar{\mathbf{4}}); \qquad \bar{Q}^I_{\dot{A}}\in(\mathbf{1},\mathbf{2},\mathbf{4}).$$

Let us explain the notation here. $(\cdot_1,\cdot_2,\cdot_3)$ denotes the way the field transforms under $SU(2)\times SU(2)\times SU(4)_R$, so $\cdot_1$ tells us something about how the field transforms under $SU(2)$, $\cdot_2$ gives us information about how the field transforms under the other $SU(2)$ and $\cdot_4$ tells us how the field transforms under $SU(4)_R$. More precisely, it tells us in which representation the field in question lies. Here $\mathbf{1}$ is the trivial representation, $\boldsymbol{i}_v$ the i-dimensional vector representation of $SO(6)$, $\boldsymbol{i}$ the i-dimensional Weyl representation with chirality $+1$, $\bar{\boldsymbol{i}}$ the i-dimensional Weyl representation with chirality -1.

8.4 $SL(2,\mathbb{Z})$-duality

The $\mathcal{N}=4$ SYM is self-dual under the $SL(2,\mathbb{Z})$ duality group. If we take an element

$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z}), \tag{8.8}$$

so $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$, we have that the complexified coupling constant

$$\tau=\frac{\theta}{2\pi}+\frac{4\pi}{g^2}i \tag{8.9}$$

transforms as

$$\tau\to\gamma\cdot\tau=\frac{a\tau+b}{c\tau+d}. \tag{8.10}$$

Normally, we take τ constant along the four-dimensional space time. However, we would like to allow τ's value to change along the manifold and undergo non-trivial $SL(2,\mathbb{Z})$-monodromies. If our four-dimensional space time is a Kähler manifold, τ will have to depend holomorphically on the coordinates of this Kähler space. We will discuss all of this. Let us start by looking at the transformation of the supercharges of the theory under $SL(2,\mathbb{Z})$ transformations.

$G=U(N)$ has Langlands dual group $\mathcal{L}_G=U(N)$. According to the S-duality conjecture, there exists a quantum symmetry S of the theory that inverts τ, interchanges

G with $\mathcal{L}_G$ and exchanges the magnetic and electrical charges in the theory. Since the Langlands group is in this case the original group, the S symmetry sends $U(N)$ to $U(N)$. Note that $\tau \in \mathbb{H}$ since its imaginary part is strictly positive. We have a classical symmetry of the system in $T : \tau \mapsto \tau + 1$, which is just a change in angle of $\theta \mapsto \theta + 2\pi$, for any gauge group G. S, T will generate an infinite discrete subgroup of $SL(2,\mathbb{Z})$. If G is simply laced, we find a nice expression for it:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8.11}$$

This is true in our case, since $U(N)$ is simply laced for any N. We also see that

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{8.12}$$

when we look at how an element of $SL(2,\mathbb{Z})$ acted on τ. Note that

$$\langle S, T \rangle = SL(2,\mathbb{Z}). \tag{8.13}$$

Let us look how the Montonen-Olive duality acts on the supersymmetries. Apart from T, $SL(2,\mathbb{Z})$ does not consist of symmetries of the classical theory. Consider an element γ like in equation 8.8. Then it acts on τ as in equation 8.10. It will act on the supersymmetries by an automorphism. This automorphism on the supersymmetries is not strictly determined, since we could combine it with a symmetry of the $\mathcal{N} = 4$ SYM theory. However, note that due to the Montonen-Olive conjecture γ will commute with the Poincaré group. We can also define it to commute with the global R-symmetry group $SU(4)_R$. Combining these facts, we find that γ acts as a scalar multiplication on the supersymmetry charges, in the following way:

$$\gamma \cdot Q_{AI} \equiv \exp(i\hat{\phi}) Q_{AI}, \tag{8.14a}$$

$$\gamma \cdot \bar{Q}^I_{\dot{A}} \equiv \exp(-i\hat{\phi}) \bar{Q}^I_{\dot{A}}. \tag{8.14b}$$

This is because the Poincaré- and R-symmetry group act on the charges, transforming them. So, if the $SL(2,\mathbb{Z})$ element must commute with group elements of these groups, it has to act as a scalar on the charges. We would like to preserve the real structure of the algebra in question, so we ask that $\hat{\phi} \in \mathbb{R}$. These symmetries are called the $U(1)$-chiral rotations. We will denotes this $U(1)$ group with $U(1)_{\mathcal{D}}$. Note that since γ commutes with the R-symmetry group, it is defined up to the center of $SU(4)_R$. The center of $SU(4)_R$ is generated by an element $\mathcal{I}$, which acts like i on the $\mathbf{4}$ and like $-i$ on the $\bar{\mathbf{4}}$ of $SU(4)_R$. Since $\exp(i\frac{\pi}{2}) = i, \exp(-i\frac{\pi}{2}) = -i$, the $\hat{\phi}(\gamma)$ is defined up to $\hat{\phi} \mapsto \hat{\phi} + \frac{\pi}{2}$. Note that for extended supersymmetry theories:

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}, \tag{8.15}$$

with AB the central charge. Now, for $\mathcal{N} = 2$, we have the result that

$$Z^{AB} \equiv Z^{AB}_{q_1} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix}, \tag{8.16}$$

while for $\mathcal{N} > 2$ but still even,

$$Z^{AB} = \begin{pmatrix} Z^{ab}_{q_1} & & & \\ & Z^{ab}_{q_2} & & \\ & & \dots & \\ & & & Z^{ab}_{q_{\frac{\mathcal{N}}{2}}} \end{pmatrix}. \tag{8.17}$$

Looking at this last equation, we see that we can look at a $\mathcal{N} = 2$ subalgebra of the theory, so that the relevant anti-commutation relation becomes:

$$\{Q^i_{\dot{A}}, Q^j_{\dot{B}}\} = \epsilon_{\dot{A}\dot{B}}\epsilon^{ij} Z. \tag{8.18}$$

Let us denote the expectation value of the $\mathcal{N} = 2$ super partner of the gauge field by $\vec{\phi}$, and the electric and magnetic charges by $\vec{\mathfrak{n}}, \vec{\mathfrak{m}}$ respectively. We know that S-duality exchanges the magnetic and electric charges. Note that we can express Z in terms of these quantities:

$$Z = \sqrt{\frac{2}{\mathrm{im}\tau}} \begin{pmatrix} \vec{\mathfrak{m}} & \vec{\mathfrak{n}} \end{pmatrix} \begin{pmatrix} \tau\vec{\phi} \\ \vec{\phi} \end{pmatrix} = \sqrt{\frac{2}{\mathrm{im}\tau}} \vec{\phi} \cdot (\vec{\mathfrak{n}} + \tau\vec{\mathfrak{m}}). \tag{8.19}$$

Now, since we are working with an Abelian gauge group, we can write the action of an γ element (as given above) of the $SL(2,\mathbb{Z})$ duality group on the charges and the supersymmetric partner field as:

$$\begin{aligned} \vec{\phi} &\mapsto \vec{\phi} \\ \begin{pmatrix} \vec{\mathfrak{m}} & \vec{\mathfrak{n}} \end{pmatrix} &\mapsto \begin{pmatrix} d\vec{\mathfrak{m}} - c\vec{\mathfrak{n}} & -b\vec{\mathfrak{m}} + a\vec{\mathfrak{n}} \end{pmatrix} \end{aligned}$$

So, Z changes under $SL(2,\mathbb{Z})$ as:

$$\begin{aligned} Z &\mapsto \sqrt{\frac{2}{\frac{\mathrm{im}\tau}{|c\tau+d|^2}}} \vec{\phi} \cdot \left(-b\vec{\mathfrak{m}} + a\vec{\mathfrak{n}} + \frac{ac|\tau|^2 + ad\tau + bc\bar{\tau} + bd}{|c\tau+d|^2} (d\vec{\mathfrak{m}} - c\vec{\mathfrak{n}}) \right) \\ &= \sqrt{\frac{2}{\mathrm{im}\tau}} |c\tau + d| \vec{\phi} \cdot \left(\frac{c|\tau|^2\vec{\mathfrak{m}} + d\tau\vec{\mathfrak{m}} + d\vec{\mathfrak{n}} + c\tau\vec{\mathfrak{n}}}{|c\tau+d|^2} \right) \\ &= \frac{|c\tau+d|}{c\tau+d} \sqrt{\frac{2}{\mathrm{im}\,\tau}} \vec{\phi} \cdot \frac{c\bar{\tau}+d}{c\bar{\tau}+d} (\vec{\mathfrak{n}} + \tau\vec{\mathfrak{m}}) \\ &= \frac{|c\tau+d|}{c\tau+d} Z. \end{aligned}$$

Now, if we look at equation 8.18 and analyze how the left- and right-hand side of the equation transform under the $SL(2,\mathbb{Z})$-duality, we notice that:

$$\begin{aligned} \{Q^I_{\dot{A}}, Q^J_{\dot{B}}\} &\mapsto \exp(-2i\hat{\phi}(\gamma))\{Q^I_{\dot{A}}, Q^J_{\dot{B}}\}, \\ \epsilon_{\dot{A}\dot{B}}\epsilon^{ij} Z &\mapsto \epsilon_{\dot{A}\dot{B}}\epsilon^{ij} \frac{|c\tau+d|}{c\tau+d} Z. \end{aligned}$$

So, we can conclude that

$$\exp(-2i\hat{\phi}(\gamma)) = \frac{|c\tau+d|}{c\tau+d}. \tag{8.20}$$

In other words, writing $\alpha(\gamma) = \frac{1}{2}\hat{\phi}(\gamma)$, we find:

$$\exp(i\alpha(\gamma)) = \frac{c\tau + d}{|c\tau + d|}. \tag{8.21}$$

Since every pair of supercharges has its own $\mathcal{N} = 2$ superalgebra, we can generalize this statement to $\mathcal{N} = 4$. Now defining the $U(1)_{\mathcal{D}}$ charge as being the number $q_{\mathcal{D}}$ for which the object in question changes by a phase $\exp(iq_{\mathcal{D}}\alpha(\gamma))$, we find that the supercharges Q_{AI} and $\bar{Q}^I_{\dot{A}}$ have a $q_{\mathcal{D}}$-charge of $\frac{1}{2}$ and $-\frac{1}{2}$ respectively. Then also Ψ^I_A has a $q_{\mathcal{D}}$-charge of $\frac{1}{2}$ and $\bar{\Psi}_{I\dot{A}}$ a $q_{\mathcal{D}}$-charge of $-\frac{1}{2}$. Notice that the supercharges transform non-trivially under the $SL(2,\mathbb{Z})$ duality! This is important for the topological duality twist that will follow shortly. First let us turn our attention to a natural connection which will occur on our manifold in the next section.

8.5 The $U(1)_{\mathcal{D}}$ connection

The moment we opt to make τ depend on the coordinates of the underlying manifold, we can use it to construct a connection $\mathcal{A}$ for the group $U(1)_{\mathcal{D}}$, given by

$$\mathcal{A} = \frac{1}{2\mathrm{im}(\tau)} d(\mathrm{re}(\tau)) \tag{8.22}$$

The connection $\mathcal{A}$ then defines a $U(1)_{\mathcal{D}}$ line bundle $L_{\mathcal{D}}$. If fields like Ψ above transform with a certain $q_{\mathcal{D}}$-charge under $U(1)_{\mathcal{D}}$ then these fields can be seen as sections of $L_{\mathcal{D}}^{q_{\mathcal{D}}}$. Furthermore, we can show that the first Chern class of $c_1(L_{\mathcal{D}}) = c_1(B)$, with B the Kähler manifold of complex dimension two lying in the $X^6, ..., X^9$ directions and $C \subset B$ the curve around which the D3-brane we are studying is wrapped. Let us delve a bit more into this and see how this works exactly.

Let us start by rewriting $\mathcal{A}$ in an interesting way. Note that τ, as the axion-dilaton field is by definition

$$\tau = C_0 + ie^{-\phi}. \tag{8.23}$$

Thus, looking at the definition for $\mathcal{A}$, we can write:

$$\mathcal{A} = \frac{1}{2} e^{\phi} d(C_0). \tag{8.24}$$

If we assume τ to be holomorphic for now (we will come back to this later), then

$$d\tau = (\partial + \bar{\partial})\tau = \partial\tau, \tag{8.25}$$

and

$$\frac{1}{2}e^{\phi}d\tau = \frac{1}{2}e^{\phi}\partial\tau = \frac{1}{2}e^{\phi}d(C_0 + ie^{-\phi}) = \frac{1}{2}e^{\phi}(dC_0 - ie^{-\phi}d\phi) = \mathcal{A} - \frac{i}{2}d\phi. \tag{8.26}$$

So,

$$\mathcal{A} = \frac{1}{2}e^{\phi}d\tau + \frac{i}{2}d\phi. \tag{8.27}$$

Now,

$$\frac{1}{2}e^{\phi}d\tau = -i\partial\phi, \tag{8.28}$$

where we used the holomorphicity of τ. So,

$$\mathcal{A} = \frac{1}{2}e^{\phi}d\tau + \frac{i}{2}d\phi = -i\partial\phi + \frac{i}{2}(\partial + \bar{\partial})\phi = -\frac{i}{2}(\partial - \bar{\partial})\phi. \tag{8.29}$$

If we now take the covariant derivative of the connection one-form, we obtain:

$$\begin{aligned}
\mathcal{R}_{\mathcal{A}} &= d\mathcal{A} \\
&= d\left(-\frac{i}{2}(\partial - \bar{\partial})\phi\right) \\
&= \frac{i}{2}(\partial\bar{\partial} - \bar{\partial}\partial)\phi) \\
&= \frac{i}{2}(\partial\bar{\partial} + \partial\bar{\partial})\phi) \\
&= i\partial\bar{\partial}\phi \\
&= i\partial_I\bar{\partial}_{\bar{J}}\phi dz^I \wedge d\bar{z}^{\bar{J}} \\
&= i\nabla_I\bar{\nabla}_{\bar{J}}\phi dz^I \wedge d\bar{z}^{\bar{J}},
\end{aligned}$$

where in the last equality we used that ϕ is a scalar field. We thus found an expression for the curvature two-tensor belonging to the connection $\mathcal{A}$. Now, we will show that from supersymmetry considerations the Einstein equation takes the form:

$$R^B_{I\bar{J}} = \nabla_I\bar{\nabla}_{\bar{J}}\phi. \tag{8.30}$$

Note that this actually gives us that the Ricci form

$$\mathcal{R}_B = R^B_{I\bar{J}}dz^I \wedge d\bar{z}^{\bar{J}}, \tag{8.31}$$

is equal to the curvature 2-form $\mathcal{R}_{\mathcal{A}}$ from $\mathcal{A}$. In particular, this implies that

$$c_1(L_{\mathcal{D}}) \equiv \frac{1}{2\pi}[\mathcal{R}_{\mathcal{A}}] = \frac{1}{2\pi}[\mathcal{R}_B] \equiv c_1(B). \tag{8.32}$$

Note that the Calabi-Yau condition on B is directly related to the non-triviality of the line bundle $L_{\mathcal{D}}$. Note that this Kähler manifold B lies inside a Calabi-Yau manifold X of dimension three and that actually the τ structure defines an elliptic fibration on this Calabi-Yau threefold. The τ structure on B is actually the restriction of the axion-dilaton field to the Kähler space B. This Calabi-Yau threefold is precisely the object that we discussed in the preceding mathematical chapters. We showed how to construct these spaces and give them the appropriate elliptic fibration, and they will now serve as toy-models in these physical theories. The geometry of these spaces determines the underlying physics.

Now, fields are sections of $L_{\mathcal{D}}^{q_{\mathcal{D}}}$. If τ changes by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the field in question changes with a factor

$$e^{iq_{\mathcal{D}}\arg(c\tau+d)}.$$

So, if the background undergoes a $SL(2,\mathbb{Z})$ transformation when going from one local patch to another on B or on X, we get a corresponding $U(1)_{\mathcal{D}}$ transition function for the field. Now, if y^m are internal coordinates, then we can write

$$\mathcal{A} = \mathcal{A}_m dy^m = \frac{1}{2} e^{\phi} dC_0 = -\frac{i}{2}(\partial - \bar{\partial})\phi, \tag{8.33}$$

then the covariant derivative is given by

$$\nabla_m^{\mathcal{A}} \Phi = (\nabla_m - iq_{\mathcal{D}} \mathcal{A}_m)\Phi, \tag{8.34}$$

for Φ a field and ∇_m the covariant derivative that we obtain from the metric. So, $\mathcal{A}$ can be used to construct $SL(2,\mathbb{Z})$ covariant derivatives that allow us to obtain manifestly $SL(2,\mathbb{Z})$-covariant quantities.

Let us show the remaining parts of this proof that we have not touched upon yet. We follow the observations in [2]. We look at the two-dimensional real case and we consider the solutions to the equations of motion for which τ depends only x^8, x^9. The manifold we are working in has the form $\mathcal{M} = \mathbb{R}^{1,7} \times X$. In the case that X is orientable, an ansatz for the metric is

$$ds^2 = dx^M dx_M + e^{\phi}(z,\bar{z}) dz d\bar{z}. \tag{8.35}$$

We look at the action

$$S = \int d^n x \sqrt{-g} \left(-\frac{1}{2} R - \frac{1}{4} \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{(\mathrm{im}(\tau))^2} + \cdots \right). \tag{8.36}$$

Let us start by deriving the equations of motion for τ:

$$\begin{aligned}
\delta_{\bar{\tau}} S &= S[\bar{\tau} + \delta\bar{\tau}] - S[\bar{\tau}] \\
&= \int d^n x \sqrt{-g} \left[-\frac{1}{4} \frac{\partial_\mu \tau \partial^\mu (\bar{\tau} + \delta\bar{\tau})}{\left(\frac{\tau - (\bar{\tau} + \delta\bar{\tau})}{2i}\right)^2} + \frac{1}{4} \frac{\partial_\mu \tau \partial^\mu \bar{\tau}}{\left(\frac{\tau - \bar{\tau}}{2i}\right)^2} \right] \\
&= -\int d^n x \sqrt{-g} \left[-e^{-\phi} \frac{\partial\tau \bar{\partial}(\bar{\tau} + \delta\bar{\tau}) + \bar{\partial}\tau \partial(\bar{\tau} + \delta\bar{\tau})}{(\tau - \bar{\tau} - \delta\bar{\tau})^2} + e^{-\phi} \frac{\partial\tau \bar{\partial}\bar{\tau} + \bar{\partial}\tau \partial\bar{\tau}}{(\tau - \bar{\tau})^2} \right] \\
&= -\int d^n x \sqrt{-g} e^{-\phi} \left[-\frac{\partial\tau \bar{\partial}(\bar{\tau} + \delta\bar{\tau}) + \bar{\partial}\tau \partial(\bar{\tau} + \delta\bar{\tau})}{(\tau - \bar{\tau})^2} \left(1 + 2\frac{\delta\bar{\tau}}{\tau - \bar{\tau}} \right) + \frac{\partial\tau \bar{\partial}\bar{\tau} + \bar{\partial}\tau \partial\bar{\tau}}{(\tau - \bar{\tau})^2} \right] \\
&= -\int d^n x \sqrt{-g} e^{-\phi} \left[\frac{\bar{\partial}\partial\tau}{(\tau - \bar{\tau})^2} \delta\bar{\tau} - 2\frac{\partial\tau}{(\tau - \bar{\tau})^3} (\bar{\partial}\tau - \bar{\partial}\bar{\tau}) \delta\bar{\tau} + \frac{\bar{\partial}\partial\tau}{(\tau - \bar{\tau})^2} \delta\bar{\tau} \right. \\
&\quad \left. -2\frac{\bar{\partial}\tau}{(\tau - \bar{\tau})^3} (\partial\tau - \partial\bar{\tau}) \delta\bar{\tau} \right] \\
&= -\int d^n x \sqrt{-g} e^{-\phi} \left[2\frac{\bar{\partial}\partial\tau}{(\tau - \bar{\tau})^2} - 4\frac{\partial\tau \bar{\partial}\tau}{(\tau - \bar{\tau})^3} \right] \delta\bar{\tau}.
\end{aligned}$$

So, the equation of motion for τ is:

$$\partial\bar{\partial}\tau - 2\frac{\partial\tau \bar{\partial}\tau}{\tau - \bar{\tau}} = 0. \tag{8.37}$$

A solution for this equation is $\tau = \tau(z)$, for $z = x_8 + ix_9$. This solution also preserves the 16 supercharges of the supersymmetric theory.

Using the metric given in equation 8.35, we have that

$$\det g = e^{2\phi}.$$

So, using the fact from complex geometry that

$$\mathcal{R}_X = \partial\bar{\partial}\log(\sqrt{\det g}), \tag{8.38}$$

we find that

$$\mathcal{R}_X = \partial\bar{\partial}\phi. \tag{8.39}$$

It also follows from the Einstein equation, using the action given above. Varying with respect to the metric we find precisely this equation of motion using the same principles as in the previous calculation. We can now straightforwardly generalize these results to higher dimensions and we find the missing links in the proof we gave above.

8.6 Topological duality twist: general procedure

We already saw in chapter 3 an example of a topological twisting. Let us repeat the necessity for it here. When we couple supersymmetry to curved space, the supersymmetry gets broken due to the non-trivial holonomy when moving over the curved manifold, making the supercharges not well-defined. We thus have a failure to transport geometrical data around closed loops on curved manifolds. The standard topological twist tries to solve this problem by accompanying the non-trivial holonomy under the Lorentz group by a non-trivial holonomy under the $\mathcal{R}$-symmetry group. One or more supercharges in the twisted theory are singlets under the combined action and can then be globally defined. Note that we can define different twistings, depending on the way in which $U(1)$ is embedded in $SU(4)$.

We can now try to generalize this procedure, by including a non-constant τ that can undergo non-trivial $SL(2,\mathbb{Z})$-dualities as one moves around the manifold. This is not so straightforward however, since we saw that the supercharges transform non-trivially under $SL(2,\mathbb{Z})$. They have a non-zero $q_{\mathcal{D}}$-charge. The solution to this problem is to compensate the non-trivial $U(1)_{\mathcal{D}}$ transformations by the corresponding $\mathcal{R}$-symmetry transformations. In the next section, we will perform this twist in a specific example and calculate the central charges of the resulting theory.

8.7 Topological duality twist: $\mathcal{N} = 4$ SYM preserving $(2,2)$ SUSY

We study the example given in [18], which was the original example of this topological duality twisting procedure given. We start with S a Kähler manifold and j the Kähler form. The holonomy group is restricted to $SU(2)_L \times U(1)_J$ with $U(1)_J \subset SU(2)_R$. The $\mathcal{R}$-symmetry group splits as

$$SU(4)_{\mathcal{R}} \simeq SU(2)_A \times SU(2)_B \times U(1)_{\mathcal{R}}.$$

Not forgetting the $U(1)_{\mathcal{D}}$ group, we find for the space rotation group:

$$G = SU(2)_L \times SU(2)_A \times SU(2)_B \times U(1)_J \times U(1)_{\mathcal{R}} \times U(1)_{\mathcal{D}}. \tag{8.40}$$

We have already seen to what representation the supercharges belong. However, this time the space rotation group is a little bit different because of the $U(1)_J \subset SU(2)_R$. The representation splits as

$$\begin{aligned} Q_{AI} &\in (\mathbf{2},\mathbf{1},\bar{\mathbf{4}}) \to (\mathbf{2},\mathbf{2},\mathbf{1})_{0,\frac{1}{2},\frac{1}{2}} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{0,-\frac{1}{2},\frac{1}{2}}, \\ \bar{Q}^I_{\dot{A}} &\in (\mathbf{1},\mathbf{2},\mathbf{4}) \to (\mathbf{1},\mathbf{2},\mathbf{1})_{1,-\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{2},\mathbf{1})_{-1,-\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{1,\frac{1}{2},-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-1,\frac{1}{2},-\frac{1}{2}}. \end{aligned}$$

The subscript gives the charges under the respective $U(1)$ groups. The $U(1)_{\mathcal{D}}$ charges we already calculated in this chapter. Since Q_{AI} is in the trivial representation of $SU(2)_R$, it will also have charge zero under $U(1)_J \subset SU(2)_R$. The $SU(4)_{\mathcal{R}}$ work together to create a right-moving spinor. For $\bar{Q}^I_{\dot{A}}$, $U(1)_J$ will transform no longer with a zero charge since the charge is not in the trivial representation of $SU(2)_R$. The $SU(4)_{\mathcal{R}}$ work together to create a left-moving spinor.

We can now twist the $SU(2)_L \times U(1)_J \times U(1)_{\mathcal{D}}$ with $U(1)_{\mathcal{R}}$. We keep the $Spin(4)_{\mathcal{R}} \simeq SU(2)_A \times SU(2)_B$ as the rigid symmetry group. Take J, R, D to be the generators of their respective $U(1)$-groups. We can now define the twisted generators as follows:

$$J' \equiv J + 2R, \qquad\qquad D' \equiv D + R. \tag{8.41}$$

So, the space rotation group after the twist will be:

$$G = SU(2)_L \times SU(2)_A \times SU(2)_B \times U(1)'_J \times U(1)'_{\mathcal{D}}. \tag{8.42}$$

Let us look at what happens to the charges after the twist:

$$\begin{aligned} Q_{AI} &\to (\mathbf{2},\mathbf{2},\mathbf{1})_{1,1} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{-1,0}, \\ \bar{Q}^I_{\dot{A}} &\to (\mathbf{1},\mathbf{2},\mathbf{1})_{0,-1} \oplus (\mathbf{1},\mathbf{2},\mathbf{1})_{-2,-1} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{2,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{0,0}. \end{aligned}$$

Note that the only part of the representation after the twist which generates supercharges which are in the trivial representation of the twisted $U(1)'_J$ and $U(1)'_{\mathcal{D}}$ is the $(\mathbf{1},\mathbf{1},\mathbf{2})_{0,0}$ part of $\bar{Q}^I_{\dot{A}}$. It transforms as a right-spinor on $Spin(4)_{\mathcal{R}}$ and as singles under $SU(2)_L \times U(1)'_J \times U(1)'_{\mathcal{D}}$. We thus obtain a chiral $(0,2)$ twisted supersymmetry. Note that $(\mathbf{1},\mathbf{2},\mathbf{1})_{0,-1} \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{0,0}$ would have sufficed if we had only twisted the $U(1)_J$ generator. Since it is the $U(1)_{\mathcal{D}}$ twist that makes the resulting theory chiral, the resulting theory if we only twisted $U(1)_J$ would have been a $(2,2)$ non-chiral topological twisted supersymmetry. We will now take a look at the main example of this chapter, where we perform the topological duality twist for type IIB superstring theory on a D3-brane.

8.8 IIB on D3-brane

We briefly mention the set-up again for clarity, as is given in [8]. We work on a ten dimensional manifold $\mathcal{M} = \mathbb{R}^{1,5} \times B$, with B a four dimensional Kähler manifold. We have a D3-brane $\mathbb{R}^2_{\|} \times C$, with $C \subset B$ a genus $g > 0$ complex curve, non-degenerate. We can thus write our manifold as $\mathcal{M} = \mathbb{R}^2_{\|} \times \mathbb{R}^4_{\perp} \times B$. Schematically, we have the portrayed the situation in table 8.1. Here $\mathbb{R}^2_{\|}$ lies in the $0-1$ direction, $\mathbb{R}^4_{\perp}$ in the $2-5$ direction, C in the $6-7$ direction and B in the $6-9$ direction.

	X^0	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9
D3	×	×	-	-	-	-	×	×	-	-

Table 8.1: Schematic overview of the set-up for this section.

8.8.1 Group structure

The $\mathcal{R}$-symmetry of the D3-brane worldvolume is given by the rotation group of the perpendicular directions to the D3-brane. So,

$$SO(6)_{\mathcal{R}} = SU(4)_{\mathcal{R}} \to SO(4)_{\mathcal{R}} \times U(1)_{\mathcal{R}}. \tag{8.43}$$

Here, $SO(4) \simeq SU(2)_L \times SU(2)_R$ is a double cover. We denote the Lorentz symmetry of the two-dimensional dimensional worldvolume $\mathbb{R}^2_{\|}$ of the strings that live in the theory by $U(1)_{\|}$. The canonical line bundle at a point on the curve C is given by the two-form $dX^6 \times dX^7$ times a factor. This gives rise to a $U(1)_C$-symmetry in the following way:

$$\begin{pmatrix} X_6' \\ X_7' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_6 \\ X_7 \end{pmatrix} = \begin{pmatrix} \cos\theta X_6 - \sin\theta X_7 \\ \sin\theta X_6 + \cos\theta X_7 \end{pmatrix},$$

so

$$\begin{aligned} dX_6' \wedge dX_7' &= (\cos\theta dX_6 - \sin\theta dX_7) \wedge (\sin\theta dX_6 + \cos\theta dX_7) \\ &= \cos^2\theta dX_6 \wedge dX_7 + \sin^2\theta dX_6 \wedge dX_7 \\ &= dX_6 \wedge dX_7. \end{aligned}$$

So, indeed there is a $U(1)_C$-symmetry. We already noted that for D3-branes in F-theory there is the $U(1)_{\mathcal{D}}$ symmetry group. The full space rotation group will thus be:

$$G = SU(2)_L \times SU(2)_R \times U(1)_{\|} \times U(1)_C \times U(1)_{\mathcal{R}} \times U(1)_{\mathcal{D}}. \tag{8.44}$$

We already discussed how the field content will transform under the combined Euclidean group $SO(4) \times SU(4)_{\mathcal{R}}$. What is left is to determine how the supercharges transform under the space rotation group and to perform the topological duality twist to make the theory well-defined on curved space.

8.8.2 Topological duality twist

The supercharges transform under the space rotation group G as:

$$\begin{aligned} Q_{AI} &\in (\mathbf{2},\mathbf{1},+,+,+,+) \oplus (\mathbf{1},\mathbf{2},+,+,-,+) \oplus (\mathbf{2},\mathbf{1},-,-,+,+) \oplus (\mathbf{1},\mathbf{2},-,-,-,+), \\ \bar{Q}^I_{\dot{A}} &\in (\mathbf{2},\mathbf{1},+,-,-,-) \oplus (\mathbf{1},\mathbf{2},+,-,+,-) \oplus (\mathbf{2},\mathbf{1},-,+,-,-) \oplus (\mathbf{1},\mathbf{2},-,+,+,-). \end{aligned}$$

Here the $\pm$ stands for a $\pm\frac{1}{2}$ charge under the respective $U(1)$ group. Again the $U(1)_{\mathcal{D}}$ charge was calculated before. The $SU(4)_{\mathcal{R}}$ is making sure we get a combination which lies in $\mathbf{4}$ and $\bar{\mathbf{4}}$ respectively. Defining the new generators as follows:

$$T_C' = T_C + T_{\mathcal{R}}, \qquad T_{\mathcal{D}}' = T_{\mathcal{D}} + T_{\mathcal{R}}, \tag{8.45}$$

under the twisted group

$$G' = SU(2)_L \times SU(2)_R \times U(1)_{\parallel} \times U(1)'_C \times U(1)'_{\mathcal{D}}, \tag{8.46}$$

the supercharges will transform as:

$$\begin{aligned}
Q_{AI} &\in (\mathbf{2},\mathbf{1},\frac{1}{2},1,1) \oplus (\mathbf{1},\mathbf{2},\frac{1}{2},0,0) \oplus (\mathbf{2},\mathbf{1},-\frac{1}{2},0,1) \oplus (\mathbf{1},\mathbf{2},-\frac{1}{2},-1,0),\\
\bar{Q}^I_{\dot{A}} &\in (\mathbf{2},\mathbf{1},\frac{1}{2},-1,-1) \oplus (\mathbf{1},\mathbf{2},\frac{1}{2},0,0) \oplus (\mathbf{2},\mathbf{1},-\frac{1}{2},0,-1) \oplus (\mathbf{1},\mathbf{2},-\frac{1}{2},1,0).
\end{aligned}$$

The surviving supercharges which are neutral under the twisted $U(1)'_C \times U(1)'_{\mathcal{D}}$ are

$$(\mathbf{1},\mathbf{2},+,+,-,+) \text{ and } (\mathbf{1},\mathbf{2},+,-,+,-),$$

giving us an effective two-dimensional theory on the string worldsheet $\mathbb{R}^2_{\parallel}$ with $(0,4)$ supersymmetry. The supercharges transform as singlets under $SU(2)_L$ and as doublets under $SU(2)_R$. We can do the same calculation for the scalar fields and the fermions. Under the original group G, we have that:

$$\begin{aligned}
\Psi^I_A &\in (\mathbf{2},\mathbf{1},+,+,-,+) \oplus (\mathbf{2},\mathbf{1},-,-,-,+) \oplus (\mathbf{1},\mathbf{2},+,+,+,+) \oplus (\mathbf{1},\mathbf{2},-,-,+,+),\\
\bar{\Psi}_{I\dot{A}} &\in (\mathbf{1},\mathbf{2},+,-,-,-) \oplus (\mathbf{1},\mathbf{2},-,+,-,-) \oplus (\mathbf{2},\mathbf{1},+,-,+,-) \oplus (\mathbf{2},\mathbf{1},-,+,+,-),\\
\phi^i &\in \{\phi^{\alpha\dot{\beta}} \in (\mathbf{2},\mathbf{2},0,0,0,0)\} \oplus \{\sigma \in (\mathbf{1},\mathbf{1},0,0,1,0)\} \oplus \{\bar{\sigma} \in (\mathbf{1},\mathbf{1},0,0,-1,0)\}.
\end{aligned}$$

After the twist, we find:

$$\begin{aligned}
\Psi^I_A &\in (\mathbf{2},\mathbf{1})_{\frac{1}{2},1,1} \oplus (\mathbf{2},\mathbf{1}_{\frac{1}{2},0,0} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2},0,1} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2},-1,0},\\
\bar{\Psi}_{I\dot{A}} &\in (\mathbf{1},\mathbf{2})_{\frac{1}{2},-1,-1} \oplus (\mathbf{1},\mathbf{2})_{\frac{1}{2},0,0} \oplus (\mathbf{2},\mathbf{1})_{-\frac{1}{2},0,-1} \oplus (\mathbf{2},\mathbf{1})_{-\frac{1}{2},1,0},\\
\phi^i &\in \{\phi^{\alpha\dot{\beta}} \in (\mathbf{2},\mathbf{2})_{0,0,0}\} \oplus \{\sigma \in (\mathbf{1},\mathbf{1})_{0,1,1}\} \oplus \{\bar{\sigma} \in (\mathbf{1},\mathbf{1})_{0,-1,-1}\}.
\end{aligned}$$

Note that a dimensional reduction of this theory to $\mathbb{R}^2_{\parallel}$, so a reduction along C, the components of the gauge field along C will like we saw in the chapter on toroidal reduction (i.e. genus one curve) transform like the scalars $\sigma, \bar{\sigma}$ and will thus pair up with these scalars.

8.8.3 Central charges

Let us now find an expression for the left- and right central charges. We do this by considering the dimensional reduction of the theory. The zero modes of the fields on the Riemann curve C correspond to the zero sections of the line bundles to which these fields are sections of. Fields with charge vectors q_C and $q_{\mathcal{D}}$ transform as sections of the line bundle $\mathcal{L}(K_C)^{q_C} \otimes L_{\mathcal{D}}^{q_{\mathcal{D}}}$. Here $\mathcal{L}K_C$ is the canonical bundle on C. We will handle all the possible values for the charges that we encountered in the previous section. First, denote $L_{\mathcal{D}} = \mathcal{L}(D)$, with divisor D. Let us start with $q_C, q_{\mathcal{D}} = (1,1)$:

$$\begin{aligned}
&\chi(C, \mathcal{L}(K_C) \otimes \mathcal{L}(D)) = \deg(\mathrm{ch}(\mathcal{L}(K_C) \otimes \mathcal{L}(D)).\mathrm{td}(\mathcal{T}_C))_1 \\
&= \deg((1 + c_1(\mathcal{L}(K_C) \otimes \mathcal{L}(D))).\mathrm{td}(\mathcal{T}_C))_1 \\
&= \deg((1 + c_1(\mathcal{L}(K_C)) + c_1(\mathcal{L}(D))).\mathrm{td}(\mathcal{T}_C))_1 \\
&= \deg\left((1 + K_C + D).\left(1 + \frac{1}{2}c_1(\mathcal{L}(-K_C))\right)\right)_1 \\
&= \deg\left((1 + K_C + D).\left(1 - \frac{1}{2}K_C\right)\right)_1 \\
&= \deg\left(1 - \frac{1}{2}K_C + D - \frac{1}{2}D.K_C + K_C - \frac{1}{2}K_C.K_C\right)_1 \\
&= \deg\left(\frac{1}{2}K_C + D\right)
\end{aligned}$$

Now,

$$\begin{aligned}
\chi(C, \mathcal{L}(D)) &= \deg((1 + D)(1 - \frac{1}{2}K_C))_1 \\
&= \deg(D - \frac{1}{2}K_C),
\end{aligned}$$

so if $D = 0$ then $1 - g = -\frac{1}{2}\deg(K_C)$, so $\deg(K_C) = 2g - 2$. Thus,

$$\begin{aligned}
&\chi(C, \mathcal{L}(K_C) \otimes \mathcal{L}(D)) = \deg\left(\frac{1}{2}K_C + D\right) \\
&= g - 1 + \deg(D) \\
&= g - 1 + \deg(c_1(L_{\mathcal{D}})) \\
&= g - 1 + \deg(c_1(B)) \\
&= g - 1 + c_1(B).C.
\end{aligned}$$

We now need Kodaira's vanishing's theorem:

Theorem 5. *$\mathcal{M}$ a compact Kähler manifold of complex dimension n, L a holomorphic line bundle on $\mathcal{M}$ that is positive and $K_{\mathcal{M}}$ the canonical line bundle, then:*

$$H^q(\mathcal{M}, K_{\mathcal{M}} \otimes L) = 0, \qquad \forall q > 0. \tag{8.47}$$

Furthermore, by Serre duality, we obtain:

$$H^q(\mathcal{M}, L^{-1}) = 0, \qquad \forall q < n. \tag{8.48}$$

In our case $\mathcal{M} = C$ and $n = 1$, thus:

$$h^1(C, \mathcal{L}(K_C) \otimes \mathcal{L}(D)) = 0. \tag{8.49}$$

We thus find, using $\chi = h^0 - h^1$, that:

$$h^0(C, \mathcal{L}(K_C) \otimes \mathcal{L}(D)) = g - 1 + c_1(B).C. \tag{8.50}$$

Furthermore, we can conclude that

$$h^0(C, \mathcal{L}(D)^{-1}) = 0. \tag{8.51}$$

Furthermore, we see that

$$h^0(C, \mathcal{L}(K_C)^{-1} \otimes \mathcal{L}(D)^{-1}) = 0, \tag{8.52}$$

since this is an ample line bundle and negative powers of ample line bundles have no non-zero sections. The same conclusion befalls upon

$$h^0(C, \mathcal{L}(K_C)^{-1}). \tag{8.53}$$

Using Kodaira's vanishing theorem, we also obtain that

$$h^0(C, \mathcal{L}(D)) = 0. \tag{8.54}$$

We now use Serre duality to conclude the following:

$$0 = h^1(C, \mathcal{L}(K_C) \otimes \mathcal{L}(D)) = h^0(C, \mathcal{L}(D)) = 0, \tag{8.55}$$

and

$$g = h^1(C, \mathcal{O}_C) = h^0(C, \mathcal{L}(K_C), \tag{8.56}$$

since the left-side calculates the genus of the curve in question and finally, because:

$$\chi(C, \mathcal{O}_C) = \chi(C, \mathcal{L}(K_C)^0) = \deg(1.(1 - \frac{1}{2}K_C))_1 = 1 - g,$$

and $g = h^1(C, \mathcal{O}_C)$, we find $h^0(C, \mathcal{L}(K_C)^0) = 1$. Let us now for each field check where the non-zero contributions come from. For Ψ:

$$\begin{aligned}
&(\mathbf{1}, \mathbf{2})_{\frac{1}{2},1,1} \in \Gamma(K_C \otimes L_{\mathcal{D}}) \to 2(g - 1 + c_1(B).C),\\
&(\mathbf{2}, \mathbf{1})_{\frac{1}{2},0,0} \in \Gamma(\mathcal{O}_C) \to 1.
\end{aligned}$$

For $\bar{\Psi}$:

$$\begin{gathered}
(\mathbf{2}, \mathbf{1})_{\frac{1}{2},0,0} \in \Gamma(\mathcal{O}_C) \to 1,\\
(\mathbf{2}, \mathbf{1})_{-\frac{1}{2},1,0} \in \Gamma(K_C) \to 2g.
\end{gathered}$$

For the bosons we do the same thing. Let us start with ϕ^i:

$$\begin{aligned}
&(\mathbf{2}, \mathbf{2})_{0,0,0} \in \Gamma(\mathcal{O}_C) \to 1,\\
&(\mathbf{1}, \mathbf{1})_{0,1,1} \in \Gamma(K_C \otimes L_{\mathcal{D}}) \to 2(g - 1 + c_1(B).C).
\end{aligned}$$

For A_μ we get the last of the two contributions from the scalar fields, since after reduction we obtain $\sigma, \bar{\sigma}$ as fields. Thus, we obtain table 8.2. Note that only the right-moving bosons and fermions come in equal number, a reflection of the $(0, 4)$ supersymmetry on the worldsheet after the twisting. Counting the contributions from all fields in the left- and right-moving sector, we obtain the following values for the central charges:

$$c_L = 6g + 4c_1(B).C, \qquad c_R = 6g + 6c_1(B).C. \tag{8.57}$$

	bosons	fermions
L	$1 \times (2,2)$	$2g \times (2,1)$
	$4(g-1+c_1(B).C) \times (1,1)$	
R	$1 \times (2,2)$	$2 \times (2,1)$
	$4(g-1+c_1(B).C) \times (1,1)$	$2(g-1+c_1(B).C) \times (1,2)$

Table 8.2: The left- and right-moving sector for fermions and bosons respectively.

We did not count the contribution from left-chiral bosons localized at points where D7-branes pierce the Riemann surface. For each D7-brane intersecting our D3-brane we expect a left-moving contribution. How this count is done in practice is not known. A way to obtain the full contribution is to consider compactifying the theory on a circle instead of wrapping the string around a circle. In that case we are describing a dual M-theory on the same elliptic threefold with the same string arising from wrapping a M5-brane on $\hat{C}$, with $\hat{C}$ being the total space of the elliptic fibration over C. This calculation was performed in [24], were a value for the left central charge of

$$c_L = 6g + 12c_1(B).C, \tag{8.58}$$

was obtained.

We can also relate this story to the toroidal dimensional reduction of $\mathcal{N}=4$ SYM theory done in section 4. We saw there when working on $\mathcal{M} = \mathbb{R}^2 \times T^2$ with the $\mathcal{N}=4$ SYM action given by:

$$S = \int d^4x \ \mathrm{tr}\left(-\frac{1}{2g^2}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} + \frac{\theta_I}{8\pi^2}F_{\hat{\mu}\hat{\nu}}\tilde{F}^{\hat{\mu}\hat{\nu}} - i\bar{\Psi}^I\bar{\sigma}^{\hat{\mu}}D_{\hat{\mu}}\Psi_I - D_{\hat{\mu}}\phi^i D^{\hat{\mu}}\phi^i + gC_i^{IJ}\Psi_I[\phi^i,\Psi_J] + g\bar{C}_{iIJ}\bar{\Psi}^I[\phi^i,\bar{\Psi}^J] + \frac{g^2}{2}[\phi^i,\phi^j]^2\right),$$

but since we are studying the case where the gauge group is $G = U(1)$, we can simplify to:

$$S = \int d^4x \ \mathrm{tr}\left(-\frac{1}{2g^2}F_{\hat{\mu}\hat{\nu}}F^{\hat{\mu}\hat{\nu}} + \frac{\theta_I}{8\pi^2}F_{\hat{\mu}\hat{\nu}}\tilde{F}^{\hat{\mu}\hat{\nu}} - i\bar{\Psi}^I\bar{\sigma}^{\hat{\mu}}D_{\hat{\mu}}\Psi_I - D_{\hat{\mu}}\phi^i D^{\hat{\mu}}\phi^i\right).$$

Now, in order to couple this to what we saw in this chapter, we identify $\mathcal{M} = \mathbb{R}^2 \times T^2$ with $\mathbb{R}^2_{\|} \times C$ since the fields here depend on the brane direction. This means we are making a particular choice for the genus of the complex curve C: $g = 1$. In this manner, we are left with a torus. One extra thing is now required for F-theory applications. Namely, we place C (or T^2 in this case) inside a Kähler manifold B and we define a holomorphic line bundle $L_{\mathcal{D}}$ on B. In this way we can perform the correct topological duality twist. After the twist and the toroidal reduction, we will be left with the same results for both left- and right-moving sectors as well as for the central charges. So, we find:

$$c_L = 6 + 4c_1(B).T^2, \qquad c_R = 6 + 6c_1(B).T^2.$$

Note that the inclusion of T^2 in B is important. We borrow the τ-field from this since it is a section of the line bundle on B. Furthermore, we already know that there is a $SL(2,\mathbb{Z})$-duality and this translates neatly into what we know from this chapter on type IIB superstring theory on a D3-brane.

Chapter 9

Conclusion

In this work we described the process of topological twistings in two settings. Firstly, in the setting of topological string theory, we described $\mathcal{N} = (2, 2)$ SUSY and performed a topological twist to obtain the A- and B-model. Secondly, in the setting of $\mathcal{N} = 4$ SYM theory in four dimensions or equally in type IIB superstring theory on a D3-brane, we performed a topological duality twist. We also looked at toroidal dimensional reduction and explained its importance in this story. Finally we calculated the left- and right-moving sector for fermions and bosons in this theory respectively, using algebraic geometric techniques.

In both cases, the topological twist performed solved the problem of defining a global symmetry on a curved space due to holonomy effects. In practice, this was done by changing the generators of the symmetry groups in question by an appropriate R-symmetry generator, so that the supercharges transform trivially under the new generators.

On the mathematical side, we discussed Calabi-Yau spaces and some of their properties. In particular, we discussed Batyrev's construction to produce multiple examples of Calabi-Yau mirror pairs. We came across these mirror pairs in topological string theory. Furthermore, we discussed a particular construction of obtaining elliptically fibered Calabi-Yau spaces. These spaces were of particular importance in our discussion of type IIB superstring theory on a D3-brane, since the elliptic structure of τ on the base manifold was needed in the description of this theory. We have constructed a number of spaces in different bases that precisely have this structure.

Furthermore, we have written a C++ library that can be used by the reader to perform calculations in toric geometry and derive the Hodge numbers of Calabi-Yau manifolds with an elliptic fibration as constructed by the methods explained in [12]. Next in our study would be things like Gromov-Witten invariants, Gopakumar-Vafa invariants,... in order to further study the properties of the spaces under consideration and apply this knowledge in the string theoretical framework.

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