Bachelor Thesis

Congestion control for the Internet: Modeling and Analysis

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August 29, 2016

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Abstract

We consider a fluid flow model of network behavior (as in the papers of Johari [1] and Massoulié [2]) to investigate how congestion control can achieve efficient use of network resources. The aim of the project is to find criteria for local stability for simple resource-user networks in the presence of communication delays. Both a continuous model and a discrete model are discussed. In order to determine the stability of a more complex continuous model, we make use of the direct method of Lyapunov-Kraskovski.

Contents

1 Introduction

The problem of congestion is known among travelers in the form of congested roads. The same problem exists for other networks, such as the Internet. Congestions cause a lot of delays which annoys the users of the network. It is therefore important to investigate how these congestions can be reduced, and investigate how local stability can be achieved. Stability is very important for keeping delays as small as possible during changes in the network.

Internet traffic includes end-users, resources, connections between these two and packages which are send between all of the devices in the network. These end-users are for instance the computers used for surfing the Web and resources includes places where sites and files are stored. Congestion occurs when the number of packages at a node in the network (in this network nodes are end-users and resources) is larger than its capacity. The end-users experience these congestions as a communication delay.

The communication delay of the Internet consists of two elements: propagation delay and queuing delay. The propagation delay is the physical delay in transmission of data along a length of fiber or through space. The queuing delay is caused by data waiting to be processed at resources. Kelly [3] argues that queuing delays are small in comparison with propagation delays, while speeding up the stream of packages and multiplexing streams decreases queuing delays. Propagation delays, on the other hand, are fixed by the distance between nodes in a network and by the speed of light. While the transmission speed is bounded by the speed of light. Therefore we assume queuing delays are very small and focus only at propagation delays.

We consider a fluid flow model for the behavior of packages at the Internet. This means that packages flow along the route of the lowest resistance. In our model, this resistance is caused by queuing delays at recourses. By using a fluid flow approximation queuing delays will be very small, all new packages rather flow through other routes instead of waiting at the resource. As discussed above we assumed that queuing delays are really small, so with a fluid flow approximation we can apply this into our model.

To find a solution for the delays on the Internet, we could look at a solution from a central point of view. The only problem is the large size of the Internet which needs to be covered, while it is growing every day. So we need a solution at a decentral level. This can be done by efficient decision making from end- users about the quantities of packages send on the Internet at a certain moment in time. For this, we need information about current delays at the network. Ping-messages are our solution. These are very small messages send almost constantly between every device using the Internet, to check for their existence. We will use these messages to determine the transmission time and delay between devices.

Let the set of *resources* be represented by J and *route* r be a nonempty subset of J. We denote the set of all routes by R. For our model, we give every route a number which corresponds with a specific user, thus set R is both a family of subsets of J and a set of users. During this thesis, users and routes will both be used to refer to the set R. In order to make an efficient decision of quantities send on the Internet by a user, we apply the following model with strictly positive parameter κ_r:

$$\dot{x}_r(t) = \kappa_r[w_r - x_r(t) \sum_{l \in r} f_l(\sum_{s:l \in s} x_s(t))], \quad r \in R \tag{1}$$

In this model, $x_r(t)$ is the amount of packages user r sends on the Internet at time t. The increase of these packages is a factor κ_r of the scaled costs of the route r. Here, w_r denotes the costs user r is prepared to pay for sending all of his packages over his route and $f_l(y)$ denotes a real cost function for sending one package through recourse l with a total flow of y through this resource.

Note that (1) does not have any delays in it's equation, while these are very important for our investigation. Therefore we introduce a round trip delay D_r and divide it into two parts: a forward delay $\overrightarrow{D}_{l,r}$ from user r to recourse l and a backward delay $\overleftarrow{D}_{l,r}$ from the recourse l back to user r. The model with communication delay is shown by:

$$\dot{x}_r(t) = \kappa_r[w_r - x_r(t - D_r) \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(x_s(t - \overleftarrow{D}_{l,r} - \overrightarrow{D}_{l,s})))], \quad r \in R \tag{2}$$

In this thesis, we are interested in pursuing conditions which each individual user must satisfy to ensure the stability of the system. We will consider conditions of the following form:

$$\kappa_r D_r < \text{ route-dependent constant}, \quad r \in R \tag{3}$$

Note that such a condition is only dependent of the user, so we can choose a κ_r satisfying the stability condition to determine the amount of packages that is a solution of (2) for each user.

In this introduction, we only discussed the continuous model of network behavior. In order to discuss the discrete model, we discuss in Chapter 2 how we may convert these two models into each other. Furthermore, we will show in Chapter 3 the criteria for which a simple discrete model and a more complex discrete model are stable. These criteria will also be proven in this thesis. For the continuous models we show and proof the stability criteria in Chapter 4. Only for the more complex model we use the direct method of Lyapunov Kraskovski. This method is discussed further in Chapter 4 together with the usage of the method for the complex discrete model. At last the basics of determining stability for differential equations will be discussed in the appendix.

2 Continuous v.s. discrete model

As discussed in the introduction, model (2) (also showed below) is used.

$$\dot{x}_r(t) = \kappa_r[w_r - x_r(t - D_r) \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(x_s(t - \overleftarrow{D}_{l,r} - \overrightarrow{D}_{l,s})))], \quad r \in R$$

Notice that this is a continuous model. When sending messages with high speed and within a small time period, this model is remarkably easy rewritten into a discrete model on the condition that the time periods are small enough.

In order to have a discrete model, we need to rewrite some of the variables in (2). At first we approximate $\dot{x}_r(t)$ with the Taylor expansion, which yields:

$$x_r(t+h) = x_r(t) + h\kappa_r[w_r - x_r(t - D_r) \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(x_s(t - \overleftarrow{D}_{l,r} - \overrightarrow{D}_{l,s})))] + R(h), \quad r \in R$$

where $R(h)$ is an error $R(h) = \frac{h}{2}\ddot{x}_r(\xi)$ with $t < \xi < t + h$.
Introducing the following four variables,

$$\tau = \tfrac{t}{h} \qquad \Lambda_r = \tfrac{D_r}{h}$$

$$\widetilde{\kappa}_r = h\kappa_r \qquad \widetilde{x}_r(t) = x_r(ht)$$

the equation will be as follows:

$$\widetilde{x}_r(\tau+1) = \widetilde{x}_r(\tau) + \widetilde{\kappa}_r[w_r - \widetilde{x}_r(\tau - \Lambda_r) \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(\widetilde{x}_s(\tau - \overleftarrow{\Lambda}_{l,r} - \overrightarrow{\Lambda}_{l,s})))] + R(h), \quad r \in R$$

Note that in order to make a good approximation with Taylor, h needs to be very small. From this requirement follows that we also need that $\tau \gg t$, $\Lambda_r \gg D_r$ and $\widetilde{\kappa}_r \ll \kappa_r$.

From now on the discrete system will be presented as:

$$x[t+1] = x[t] + \kappa_r[w_r - x_r[t - D_r] \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(x_s[t - \overleftarrow{D}_{l,r} - \overrightarrow{D}_{l,s}]))], \quad r \in R$$

For this system it still applies that the delay is greater then the delay of the continuous model, and the κ_r is smaller then its continuous counterpart.

Notice that the discreet model is a system of difference equations, while it is discreet. The continuous model we use, is a system of differential equations.

3 Stability of discrete models

The first kind of models we will discuss are the discrete models. The general discrete model looks like:

$$x[t+1] = x[t]+\kappa_r[w_r - x_r[t-D_r] \cdot \sum_{l\in r} f_l(\sum_{s:l\in s}(x_s[t-\overleftarrow{D}_{l,r}-\overrightarrow{D}_{l,s}]))], \quad r \in R \quad (4)$$

For this model we first discuss the simple case of one resource and one user, later we also discuss the case of multiple users and resources.

3.1 One user and one resource

To begin, we consider a simple network of one user and one resource. The difference equation of the system is:

$$x[t+1] = x[t] + \kappa(w - x[t-D]f(x[t-D])) \quad (5)$$

We will linearize (5) to determine the stability. Let the stable point be $\overline{x}$, and $p = f(\overline{x})$ with $w = \overline{x}p$. We assume that f is increasing, nonnegative, not identically zero and differentiable at the stable point with derivative $f'(\overline{x}) = p'$. Linearizing with $x[t] = \overline{x} + y[t]$ shows us the following linearized system,

$$y[t+1] = y[t] - \kappa(p + \overline{x}p')y[t-D]$$

when neglecting higher order terms. Now we try a solution of the form $y[t] = e^{\mu t}$, this gives us the characteristic equation:

$$e^{\mu(D+1)} - e^{\mu D} + \kappa(p + \overline{x}p') = 0 \quad (6)$$

To determine the stability, we are interested in the roots of the characteristic equation. Make sure that the system is stable if and only if all roots of (6) have real parts smaller than zero. For more information and an explanation see the appendix.

Before discussing the stability criterion for this system, we will first show that the maximum real part of the roots of equation (6) is continuous in κ. Secondly, we will show that the same equation has only negative real parts of all roots for κ near the origin. Finally, we will look at the smallest κ such that (6) has a root of zero real part.

Lemma 3.1. *Let $p(\mu, \kappa)$ be a polynomial in e^μ, with coefficients which are continuous functions of κ (where κ may be a vector). Then the maximum real part of the roots μ of $p(\mu, \kappa) = 0$ is continuous in κ.*

Proof. It is known that the roots of any polynomial are continuous functions of the coefficients [5]. If the coefficients are continuous functions of κ, then the roots e^μ are continuous in κ, and so are the real parts of the roots μ. $\square$

Lemma 3.2. *All roots of equation (6) has negative real parts if κ is sufficiently small.*

To prove Lemma 3.2 we use the Implicit Function Theorem.

Definition 3.3 (Implicit Function Theorem)**.** This theorem is for a function $f : A \to \mathbb{R}^n$ with A an open subset of $\mathbb{R}^{n+k}$, which we can write into $f(x, y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$.
Let (a, b) be a point in A such that $f(a, b) = 0$. If the determinant with elements of the derivative of f with respect to y is not equal to zero, then there exists a neighborhood B of $a \in \mathbb{R}^k$ and a function $g : B \to \mathbb{R}^n$, such that $g(a) = b$ and $f(x, g(x)) = 0$ for $x \in B$.

Proof. Define a polynomial $p(\mu, \kappa)$ by:

$$p(\mu, \kappa) = e^{\mu(D+1)} - e^{\mu D} + \kappa a$$

where $a = p + \overline{x}p'$ is strictly positive. Then the characteristic equation (6) is the same as $p(\mu, \kappa) = 0$. In the case that $\kappa = 0$, the root is $\mu = 0$. From the following derivative,

$$\frac{\partial}{\partial \mu} p(\mu, \kappa) = (D + 1)e^{\mu D} - De^{\mu(D-1)}$$

we see that the derivative is nonzero for $\mu = 0$, $\kappa = 0$. With the Implicit Function Theorem we can find an open interval $(-\epsilon, \epsilon)$ and a differentiable complex-valued function $g(\kappa)$ such that $g(0) = 0$ and $g(\kappa) = \mu$ satisfies $p(\mu, \kappa) = 0$ for $-\epsilon < \kappa < \epsilon$. Differentiating $p(\mu, \kappa) = 0$ with respect to κ, for κ within the interval, we have:

$$\frac{d}{d\kappa} p(g(\kappa), \kappa) = (D + 1)e^{g(\kappa)D} g'(\kappa) - De^{g(\kappa)(D-1)} g'(\kappa) + a = 0$$

Make sure that $g'(0) = -a$ when we evaluate the derivative through $\kappa = 0$. Now we use the Taylor expansion of $g(\kappa)$ around $\kappa = 0$. This shows us the following:

$$g(\kappa) = -\kappa a + O(\kappa)$$

From this result, we can conclude that if κ increases away from zero towards the positive boundary of the interval, the real parts of the roots decrease from zero. Hence, for sufficiently small κ all roots have negative real parts. $\square$

Since we have proven the Lemma's, we will discuss the main theorem for the discrete model of one resource and one user.

Theorem 3.4. *System (5) is locally stable if:*

$$\kappa(p + \overline{x}p') < 2\sin(\frac{\pi}{2(2D+1)})$$

Proof. This result is easy to check for $D = 0$, in that case we get the following equation:

$$y[t+1] = (1 - \kappa(p + \overline{x}p'))y[t]$$

We see that if $\kappa(p + \overline{x}p') < 2$ the modulus of the right-hand side term in front of $y[t]$ is smaller than 1, so the system is stable.

Now we assume that $D \geq 1$. By Lemma 3.1, the maximum real part of the roots varies continuously with κ. We also know that the system is locally stable for small κ from Lemma 3.2. Therefore, it is sufficient to find the smallest κ such that equation (6) has a root with zero real part. So presume that the root is of the form $\mu = 2i\theta$, then our characteristic equation can be rewritten as:

$$\begin{aligned}\kappa(p + \overline{x}p') &= -(e^{2i\theta(D+1)} - e^{2i\theta D}) \\ &= (-2i)\frac{e^{i\theta} - e^{-i\theta}}{2i}\; e^{i(2D+1)\theta} \\ &= e^{-\frac{1}{2}\pi}\; 2\sin(\theta)\; e^{i(2D+1)\theta}\end{aligned}$$

When $a = \kappa(p + \overline{x}p')$ the equation of above will result into the following equation:

$$2\sin(\theta)\; e^{i((2D+1)\theta - \frac{1}{2}\pi)} = a \tag{7}$$

Hence, we conclude that:

$$2|\sin\theta| = a \text{ and it can be verified that } \theta = \frac{\pi}{2(2D+1)} + \frac{2\pi n}{2D+1}$$

where n is an integer. Since we are looking for the smallest positive a such that μ has an real negative part, we take $n = 0$. This means that $2|\sin(\frac{\pi}{2(2D+1)})| = a$. We may notice that there are no solutions of n such that θ is a solution of $2|\sin\theta| = a$ if $a < 2\sin(\frac{\pi}{2(2D+1)})$. So the smallest a such that equation (6) has a root with zero real part is $a = 2\sin(\frac{\pi}{2(2D+1)})$, from which we can find the smallest κ. Make sure that the left-hand side of (7) decreases if $\Re(\mu) < 0$, meaning that the system is stable if $a < 2\sin(\frac{\pi}{2(2D+1)})$. □

It can been shown that the system of (5) is unstable if κ satisfies $\kappa(p + \overline{x}p') > 2\sin(\frac{\pi}{2(2D+1)})$. This means that the criterion of Theorem 3.4 is strong. The proof of instability may be found in the article of R. Johari and D. Tan [1, Theorem 3], it is however not discussed in this thesis.

Notice that we have found a criterion such that the user of the network is able to choose a κ which makes the system stable. The choice is only dependent on $\overline{x}$, p, p' and D which are known parameters of the user.

3.2 Multiple users and resources

For a more complex network, we will use a similar way of determining the criterion in which the system is stable. At first, we will show the system for three users and two resources. Make sure that two resources corresponds with three unique possible routes, which is similar to three users. For that system we determine the characteristic equation.

Let the set of users be $R = \{1, 2, 3\}$, the set of resources $J = \{a, b\}$ and $A_{jr} = 0$ if resource j does not lie on route r, and $A_{jr} = 1$ if j does. Then the system looks like:

$$\begin{aligned} x_r[t+1] = x_r[t] + \kappa_r(w_r - x_r[t-D_r] \cdot [\ f_a(\sum_{s\in\{1,2,3\}} A_{as}(x_s[t - \overleftarrow{D}_{a,r} - \overrightarrow{D}_{a,s}])) \\ +f_b(\sum_{s\in\{1,2,3\}} A_{bs}(x_s[t - \overleftarrow{D}_{b,r} - \overrightarrow{D}_{b,s}]))\ ]\ ), \quad r \in R \end{aligned} \tag{8}$$

when neglecting higher order terms. Similar to the linearization of the system with one user and one resource, we linearize system (8) about the stable point $\overline{x} = (\overline{x}_r, r \in R)$. Let us define $p_j = f_j(\sum_{j\in s} \overline{x}_s)$ for all resources $j \in J$ and assume that p_j has a derivative $p'_j = f'_j(\sum_{j\in s} \overline{x}_s)$. For the linearization, let us take $y_r[t] = \frac{x_r[t]-\overline{x}_r}{\sqrt{\kappa_r \overline{x}_r}}$ and make sure that $\sum_{j\in J} A_{jr}p_j = w_r\overline{x}_r^{-1}$ just as in the more simple case before. All of this results in the following linearized system:

$$\begin{aligned} y_r[t+1] = y_r[t] - \kappa_r w_r \overline{x}_r^{-1} y_r[t - D_r] \\ - \sum_{s\in\{1,2,3\}} A_{ar}\sqrt{\kappa_r\overline{x}_r} A_{as}\sqrt{\kappa_s\overline{x}_s} p'_a y_s[t - \overleftarrow{D}_{a,r} - \overrightarrow{D}_{a,s}] \\ - \sum_{s\in\{1,2,3\}} A_{br}\sqrt{\kappa_r\overline{x}_r} A_{bs}\sqrt{\kappa_s\overline{x}_s} p'_b y_s[t - \overleftarrow{D}_{b,r} - \overrightarrow{D}_{b,s}] \\ , r \in R \end{aligned} \tag{9}$$

Now we are interested if there exists a (possibly complex) vector $\alpha = (\alpha_r, r \in R)$ such that $y_r[t] = \alpha e^{\mu t}$ for fixed $\mu \in \mathbb{C}$ is a solution of system (9). Remember that $D_r = \overleftarrow{D}_{j,r} + \overrightarrow{D}_{j,r}$ for all $j \in J$. Then by substituting our solution $y_r[t]$ into (9) and multiplying by $e^{\mu(D_r - t)}$, we get the following system of equations:

$$\begin{aligned}\alpha_r e^{\mu(D_r+1)} - \alpha_r e^{\mu D_r} + \kappa_r w_r \overline{x}_r^{-1}\alpha_r \\ + \sum_{s\in\{1,2,3\}} A_{ar}\sqrt{\kappa_r \overline{x}_r} e^{\mu(\overleftarrow{D}_{a,r})} A_{as}\sqrt{\kappa_s \overline{x}_s} e^{\mu(-\overleftarrow{D}_{a,s})} p'_a \alpha_s \\ + \sum_{s\in\{1,2,3\}} A_{br}\sqrt{\kappa_r \overline{x}_r} e^{\mu(\overleftarrow{D}_{b,r})} A_{bs}\sqrt{\kappa_s \overline{x}_s} e^{\mu(-\overleftarrow{D}_{b,s})} p'_b \alpha_s \\ = 0, \quad r \in R\end{aligned} \tag{10}$$

The system of above may be rewritten into a system of matrices. In order to do so, we define some diagonal matrices $\kappa, W, \overline{X}$ and P' with diagonal coefficients of respectively $\kappa_r, w_r, \overline{x}_r$ and p'_j for $r \in R$ and $j \in J$. Let also $A(\mu)$ be a matrix with coefficients $A_{jr}e^{\overleftarrow{D}_{j,r}}$ for $j \in J$ and $r \in R$ and let β refer to the diagonal matrix of $\kappa W \overline{X}^{-1}$. Then we may express the system into:

$$(\mathrm{diag}(e^{\mu(D_r+1)} - e^{\mu D_r}, r \in R) + \beta + \kappa^{\frac{1}{2}} X^{\frac{1}{2}} A(-\mu)^T P' A(\mu) X^{\frac{1}{2}} \kappa^{\frac{1}{2}})\alpha = 0,\ r \in R$$

In order to satisfy the equation above, it is necessary for the determinant of the matrix premultiplied by α, to be zero. So the following equation holds:

$$\det(\mathrm{diag}(e^{\mu(D_r+1)} - e^{\mu D_r}, r \in R) + \beta + \kappa^{\frac{1}{2}} X^{\frac{1}{2}} A(-\mu)^T P' A(\mu) X^{\frac{1}{2}} \kappa^{\frac{1}{2}}) = 0,\ r \in R \tag{11}$$

The original system (8) is stable if and only if the real parts of all roots of the characteristic equation (11) are negative. To make it ourself easier, we define $C(\mu, \kappa) = \beta + \kappa^{\frac{1}{2}} X^{\frac{1}{2}} A(-\mu)^T P' A(\mu) X^{\frac{1}{2}} \kappa^{\frac{1}{2}})$ and we will refer to the left-hand side of (11) by $p(\mu, \kappa)$ such that:

$$p(\mu, \kappa) = \det(\mathrm{diag}(e^{\mu(D_r+1)} - e^{\mu D_r}, r \in R) + C(\mu, \kappa)) = 0, \quad r \in R$$

is the characteristic equation we would like to be satisfied.

Below we have the theorem in which the stability criteria for the system is given for the case of $D_r = D$ for all $r \in R$.

Theorem 3.5. *When $D_r = D$ for all $r \in R$. The system of (8) is locally stable if for all $r \in R$:*

$$\kappa_r(\sum_{j\in r} p_j + \sum_{j\in r} p'_j \sum_{s:j\in s} \overline{x_s}) < 2\sin(\frac{\pi}{2(2D+1)}) \tag{12}$$

In order to proof this theorem we first need some definitions about matrices.

Definition 3.6 (Hermitian Matrix)**.** A matrix A is Hermitian if $A = \overline{A}^T$, this means that the matrix stays the same after taking the conjugate transpose of itself.

Definition 3.7 (Unitary Matrix)**.** A matrix U is Unitary if $\overline{U}^T = U^{-1}$, this means that the conjugate transpose of U is equal to the inverse matrix of U.

Note that a Hermitian matrix with only real values is the same as a real symmetric matrix. A Hermitian matrix has some known properties, one of them is the following [6, Chapter 7]:

Property 3.8. *A Hermitian matrix is unitary diagonalizable. This means that for any Hermitian matrix A, there exists an unitary matrix U such that* $\overline{U}^T AU = D$ *where D is a complex diagonal matrix.*

Furthermore, we define the absolute row sum by:

Definition 3.9 (Absolute row sum)**.** The absolute row sum of matrix A of row i is given by:

$$\sum_{j=1}^{n} |a_{ij}|$$

where a_{ij} is the element of A at row i and column j, with $j = 1, .., n$.

Since we defined all important terms which we use for the proof, we start with proving Theorem 3.5.

Proof. For this proof we will give an overview of the 5 steps discussed in the proof of Theorem 9 in paper [1].

Step 1. For $0 < a < 2\sin(\frac{\pi}{2(2D+1)})$ *the equation* $e^{\mu(D+1)} - e^{\mu D} + a = 0$ *has no roots with real parts equal to zero.*

This conclusion follows directly from Theorem 3.4.

Step 2. The maximum real value of the roots μ *of* $p(\mu, \kappa) = 0$ *is continuous in* κ*.*

This statement follows from Lemma 3.1. Remark that $p(\mu, \kappa)$ is not instantly a polynomial equation, while it contains terms which are powers of $e^{-\mu}$. Still this is easily corrected by multiplying by a large enough power of e^{μ}.

Step 3. Any κ *which satisfies the hypothesis of Theorem 3.5, ensures that* $p(\mu, \kappa)$ *has no roots with real parts equal to zero.*

To show this we let κ satisfy the condition. Then we compute the absolute row sum of $c(\mu,\kappa) = \beta + \kappa A(-\mu)^T P' A(\mu) X$. Suppose that there exist a root $\mu = i\theta$ with $0 \leq \theta \leq 2\pi$ for $p(\mu,\kappa)$. In that case we may find that $\kappa_r(\sum_{j\in r} p_j + \sum_{j\in r} p_j' \sum_{s:j\in s} \overline{x_s})$ is greater than the absolute row sum of C. Since κ satisfies the condition of Theorem 3.5, $2\sin(\frac{\pi}{2(2D+1)})$ is an upper bound for the absolute row sum.

According to Horn and Johnson [5, Chapter 8] the maximum absolute eigenvalue of any square matrix is bounded by its maximum absolute row sum. So the maximum absolute eigenvalue of $C(\mu,\kappa)$ is also bounded by $2\sin(\frac{\pi}{2(2D+1)})$.

We have the following characteristic equation if $D_r = D$:

$$\det((e^{\mu(D+1)} - e^{\mu D})I + C(\mu,\kappa)) = 0 \tag{13}$$

Furthermore, if $\mu = i\theta$ then the matrix $C(\mu,\kappa)$ is a Hermitian matrix. From Property 3.8 we know that this matrix is unitary diagonalizable.

Therefore we may write $C(\mu,\kappa) = \Gamma\Phi\overline{\Gamma}^T$ where Γ is unitary and Φ is the diagonal matrix of eigenvalues of $C(\mu,\kappa)$. The eigenvalues are strictly positive and the determinant of matrix C is nonzero, as $C(\mu,\kappa)$ has nonnegative values. Since we found the upper bound of the maximum absolute eigenvalues, we know for the eigenvalues ϕ_r that $\phi_r < 2\sin(\frac{\pi}{2(2D+1)})$ for all $r \in R$. Because Γ is unitary, the determinant of C is equal to the determinant of Φ, this yields:

$$\det((e^{\mu(D+1)} - e^{\mu D} + \phi_r, r \in R)) = 0 \tag{14}$$

Remark that we take the determinant of a diagonal matrix, thus we must obtain:

$$e^{\mu(D+1)} - e^{\mu D} + \phi_r = 0$$

for an $r \in R$ with $0 < \phi_r < 2\sin(\frac{\pi}{2(2D+1)})$. This contradicts with *Step 1*, so *Step 3* is proven.

Step 4. There exists a κ satisfying the hypothesis of the theorem, such that all roots μ of $p(\mu,\kappa) = 0$ have real parts less then zero.

We will proof the existence of such a κ by induction of the users. Let $R = \{1, 2, ...N\}$ and we define $R_n = \{1, 2, ..., n\}$. Then we denote the characteristic equation on the subset of routes R_n by $p_n(\mu,\kappa)$.

For $n = 1$ we have already proven the existence of such a κ for $p_1(\mu,\kappa) = 0$ by Theorem 3.4.

For $n \geq 1$ let there be a κ as above for $p_{n-1}(\mu,\kappa)$. Then we may write $p_n(\mu,k)$ into the following with $k = (\kappa_1, ..., \kappa_{n-1}, 0)$

$$p_n(\mu,k) = (e^{\mu(D+1)} - e^{\mu D})p_{n-1}(\mu,\kappa)$$

This follows from decomposing the determinant inside p_n. Make sure that p_{n-1} has all roots with negative parts less then zero and from $e^{\mu(D+1)} - e^{\mu D} = e^{\mu D}(e^{\mu} - 1)$ we see that p_n has a root of $\mu = 0$ with multiplicity 1. Now we need to show that the root $\mu = 0$ of p_n will decrease if the n^{th}- element of k increases from zero to κ_n, from which we can conclude that all roots of $p_n(\mu, \kappa) = 0$ have negative real parts.

Furthermore we may write p_n into the following form:

$$p_n(\mu, \kappa) = (e^{\mu(D+1)} - e^{\mu D})p_{n-1}(\mu, \kappa) + \kappa_n q_{n-1}(\mu, \kappa)$$

For this function we use the Implicit Function Theorem. By means of this theorem we can write μ as a function of κ, as $h(\kappa) = \mu$. Moreover we may find the derivative of the function h at $\kappa = 0$ by computing the derivative of $p(g(\kappa), \kappa)$ with respect to κ. That is $h'(0) = -q_{n-1}(0, \kappa)/p_{n-1}(0, \kappa) = -a_n$ with $a_n > 0$. Taking the Taylor expansion of this derivative through zero yields:

$$h'(0) = -\kappa_n a_n + O(\kappa_n)$$

From this we see that if κ increases from zero that μ is decreasing from zero. So for any n, $p_n(\mu, \kappa) = 0$ with all elements of κ small and greater then zero, has roots with negative real parts. This also applies for $n = N$ such that $p_n = p$, thus *Step 4* is proven.

Step 5. Completion of the proof

Suppose there exists a κ satisfying the hypothesis of the theorem for which the equation $p(\mu, \kappa) = 0$ has at least one root with real part greater then zero. Consider the path $\kappa(t) = t\kappa^* + (1-t)\kappa$ for $0 \leq t \leq 1$ with κ^* the κ of p in *Step 4*. While $\kappa(1) = \kappa^*$ we know that $p(\mu, \kappa(1)) = 0$ has only roots of negative real parts. From *Step 2* we know that the maximum real value of the roots are continuous, so there must exist a t such that $p(\mu, \kappa(t)) = 0$ has a root with real part equal to zero. Since $\kappa(t)$ is a convex combination of κ^* and κ, which both satisfies the hypothesis, $\kappa(t)$ also satisfies the hypothesis. From *Step 3* we conclude that $\kappa(t)$ may not have any roots with real part equal to zero, this is a contradiction to the result of *Step 2*, so the theorem is proven.

□

With proving Theorem 3.5, we have found a solution for how all users of the network how may choose their quantity of packages in order to retain the stability of the system. Their quantity of packages depends on the decision for κ_r which must satisfy:

$$\kappa_r < 2\sin(\frac{\pi}{2(2D+1)})\frac{1}{(\sum_{j\in r} p_j + \sum_{j\in r} p'_j \sum_{s:j\in s} \overline{x_s})}$$

Make sure that the decision making only depends on known parameters, so any user is able to make an efficient choice.

4 Stability of continuous models

Previously we discussed two different cases of the discrete model. We continue by discussing the same two cases for the continuous model. The general continuous model is:

$$\dot{x}_r(t) = \kappa_r[w_r - x_r(t - D_r) \cdot \sum_{l \in r} f_l(\sum_{s:l \in s}(x_s(t - \overleftarrow{D}_{l,r} - \overrightarrow{D}_{l,s})))], \quad r \in R \tag{15}$$

4.1 One user and one resource

We start again with the simple case of one user and one resource. This gives us the following equation out of equation (15):

$$\dot{x}(t) = \kappa[w - x(t - D)f(x(t - D))] \tag{16}$$

To determine the stability of this system, we will need a linear equation. So we linearize around the equilibrium point $\overline{x}$ through

$$y(t) = \frac{x(t) - \overline{x}}{\sqrt{\kappa \overline{x}}}.$$

Taking $p = f(\overline{x})$ and $p' = f'(\overline{x})$, with the assumption that f is increasing, nonnegative and not equal to zero. We obtain the following linearization by neglecting higher order terms:

$$\dot{y}(t) = -\kappa(p + \overline{x}p')y(t - D)) \tag{17}$$

We find the characteristic equations by substituting $y(t) = e^{\lambda t}$ into (17):

$$\lambda = -\kappa(p + \overline{x}p')e^{-\lambda D} \tag{18}$$

Finally, when the substitution $\mu = \lambda D$ we find the following reduced equation:

$$-\kappa(p + \overline{x}p')D - \mu e^{\mu} = 0 \tag{19}$$

Hayes discussed in paper [7] the following interesting Lemma, which we use for determining the stability of the system.

Lemma 4.1. *The roots s of $f(s) = ce^s - s$ have all $\Re(s) \geq K$ if and only if $K < 1$ and $Ke^{-K} < c < e^{-K}\sqrt{V^2 + K^2}$ where $V = V(K)$ is the unique root of $K\tan(v) = v$ for $0 < v < \pi$.*

Proof. We can see that for $K \geq 1$ not all of the roots have $\Re(s) \geq K$, since for $c < 0$ there exist a root $\epsilon_1 < 0$. For $0 < c < e^{-1}$ we can find a real root ϵ_2 with $0 < \epsilon_2 \leq 1$. Furthermore, for $e^{-1} < c$ there are complex roots. Let us consider the complex roots in the upper half plane only. By writing $s = re^{i\theta} = u + iv$ we see that $r = ce^u$ and $\theta = v$, and from the angle θ of the complex number, we know that $\tan(\theta) = \frac{v}{u}$. Therefore, the roots for $e^{-1} < c$ lie on the line of $u \tan(v) = v$ with $0 < v < \pi$ and $u \leq 1$. Thus K is not greater or equal to 1.

First we consider the greatest value of c such that the complex roots with $-\pi < v < \pi$ satisfy $\Re(s) \geq K$. Suppose that these roots lie on the line $\Re(s) = K$. Then these roots are $K \pm iV$ with $K \tan(V) = V$. We take $V = \sqrt{c^2 e^{2K} - K^2}$ such that $c = e^{-K}\sqrt{V^2 + K^2}$. For c smaller than this value, we know that the roots are on the right side of $\Re(s) = K$.

If $K < 0$ we still need a c such that the roots lie on the right side of $\Re(s) = K$. For that, the negative real root ϵ_1 must have real value greater than K, since all other roots lie on the right side of the root ϵ_1. It is easy to see that $s \leq ce^s$ if $s \leq \epsilon_1$, while ϵ_1 is a root. Now we conclude that in order to have $K < \epsilon_1$ we need $K < ce^K$, which is the same as $Ke^{-K} < c$.

Finally, V is a unique root of $K \tan(v) = v$ since the function $g(v) = \frac{v}{\tan(v)}$ has the derivative $\frac{d}{dv} g(v) = \frac{\cos(v)\sin(v) - v}{\sin^2(v)}$ which is smaller then zero for $0 < v < \pi$, thus is decreasing for that interval. Furthermore $g(0) = 1$ and $g(\pi) = \infty$, so there is one unique solution V for $g(v) = K$ for the interval. □

From this we can deduce the following result of Hayes, taken from Bellman and Cooke [8, Theorem 13.8].

Lemma 4.2. *All the roots of* $pe^\mu + q - \mu e^\mu = 0$ *with* p *and* q *real numbers, have negative real parts if and only if:*

(a) $p < 1$, *and*

(b) $p < -q < \sqrt{a_1^2 + p^2}$,

Here a_1 *is the root of* $a = p \tan(a)$ *such that* $0 < a < \pi$. *If* $p = 0$, *we take* $a_1 = \frac{\pi}{2}$.

Proof. This follows directly from Lemma 4.1. If we take $q = -ce^p$ and $\mu = p - \lambda$ we have the equation:

$$(-\lambda e^{-\lambda} + c)e^p = 0$$

which is the same as $ce^{\lambda} - \lambda = 0$, while e^p is nonzero. Remark that this is the same form as in Lemma 4.1. Take $K = p$ and $c = -qe^{-p}$, then we use the previous lemma. Make sure that $\Re(\mu) < 0$ when $\Re(\lambda) > p$. For $p = 0$ we take $a_1 = \frac{\pi}{2}$ while $0 < \frac{\pi}{2} < \pi$ and $\frac{\frac{1}{2}\pi}{\tan(\frac{1}{2}\pi)} = 0$. □

Theorem 4.3. *System (16) is locally stable if:*

$$\kappa(p + \overline{x}p') < \frac{\pi}{2D} \tag{20}$$

Proof. This is a direct application of Lemma 4.2 to equation (19) by taking $p = 0$, $q = -\kappa(p+\overline{x}p')D$ and $a_1 = \frac{\pi}{2}$. From the basics of stability (Appendix) we know that if all roots of an equation have real negative parts, the given system is stable. So the criteria for stability is $0 < \kappa(p+\overline{x}p')D < \frac{\pi}{2}$ for which we already knew that $0 < \kappa(p + \overline{x}p')D$ while $\kappa, \overline{x}, p, p', D > 0$. □

Remark that criteria (20) could be rewritten into:

$$\kappa D < \frac{\pi}{2(p + \overline{x}p')} \tag{21}$$

For which the right hand-side is constant for the one and only user in this case. So for the most simple case of one user and one resource we have found the condition with the form we wanted and which has been shown in the introduction.

4.2 Multiple users and resources

For the more complex case of multiple users and resources for the continuous model, we use the Direct method of Lyapunov-Kraskovski. Before we apply this method on the model, we first discuss the method itself.

4.2.1 Direct method of Lyapunov-Kraskovski

In the Appedix the stability of a simple linear differential equation is discussed. Unfortunately those equations are not always as simple and linear as in that case. So we need another method when dealing with more difficult differential equations. For strongly non-linear cases we will use the direct method of Lyapunov.

Before explaining what this direct method looks like, we first consider some important definitions about different kinds of stability first. The first and weakest stability we discuss here is the Lyapunov Stability.

Definition 4.4 (Lyapunov Stability). Consider the regular system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$, let $\mathbf{x}^*(t)$ be a given real or complex solution of this system. Then $\mathbf{x}^*(t)$ is *Lyapunov stable* on $t \geq t_0$ if,
for any small $\epsilon > 0$ there exists a $\delta(\epsilon, t_0)$ such that

$$\|\mathbf{x}(t_0) - \mathbf{x}^*(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{x}^*(t)\| < \epsilon, \qquad \forall t \geq t_0 \tag{22}$$

where $\mathbf{x}$(t) is any other solution.

The uniform stability is a little stronger than Lyapunov.

Definition 4.5 (Uniform Stability). A solution $\mathbf{x}^*(t)$ is *uniform stable* for $t \geq t_0$ if the solution is Lyapunov stable, for which δ is independent of t_0.

When a solution is uniformly stable as well as attractive, we call this solution asymptotically stable. The exact definition is shown below.

Definition 4.6 (Asymptotic Stability). Let $\mathbf{x}^*(t)$ be a uniformly stable solution for $t \geq t_0$. This solution is *asymptotically stable* when it is additionally attractive, this means there exists $\eta(t_0)$ such that

$$\|\mathbf{x}(t_0) - \mathbf{x}^*(t_0)\| < \eta \Rightarrow \lim_{t \to \infty} \|\mathbf{x}(t) - \mathbf{x}^*(t)\| = 0. \tag{23}$$

Furthermore we show some definitions about characteristics of functions that are of importance of understanding the theorem of Lyanpunov we discuss in this chapter.

First we need the definitions of a positive or negative definite and positive or negative semidefinite function.

Definition 4.7. Function V(x) is positive (or negative) *definite* in a neighbourhood $\mathcal{N}$ of the origin if $V(x) > 0$ ($V(x) < 0$) for all $x \neq 0$ in $\mathcal{N}$, and V(0)=0.

Definition 4.8. Function V(x) is positive (or negative) *semidefinite* in a neighbourhood $\mathcal{N}$ of the origin if $V(x) \geq 0$ ($V(x) \leq 0$) for all $x \neq 0$ in $\mathcal{N}$, and V(0)=0.

This means that a functions described as $V(x, y) = x^2 + y^2$ is positive definite on the whole plane, while the function $V(x, y) = -y^2(x^2 - 1)$ is negative semidefinite on the strip $x^2 < 1$.

Definition 4.9. A function is described as definite (or semidefinite) in case the function is either positive or negative definite (semidefinite).

Now we end up at the direct method of Lyapunov.

Theorem 4.10. *Consider the regular system* $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$ *and let* $\mathbf{x}^*(t) = \mathbf{0}$*,* $t \geq t_0$ *be the zero solution where* $\mathbf{X}(\mathbf{0}) = \mathbf{0}$*. Then* $\mathbf{x}^*(t)$ *is uniformly stable for* $t \geq t_0$ *if there exists a function* $V(\mathbf{x})$ *with the following properties in the neigbourhood of* $\mathbf{x} = \mathbf{0}$*:*

(i) $V(\mathbf{x})$ *and its partial derivatives are continuous;*

(ii) $V(\mathbf{x})$ *is definite;*

(iii) $\dot{V}(\mathbf{x})$ *is semidefinite for the given system and is of opposite sign to* $V(\mathbf{x})$*.*

Theorem 4.11. *Consider the same system and conditions as in Theorem 4.10. Now the zero solution* $\mathbf{x}^*(t)$ *is uniformly and asymptotically stable if there exists a function* $V(\mathbf{x})$ *satisfying the conditions (i), (ii) and the following condition:*

*(iii)** $\dot{V}(\mathbf{x})$ *is definite for the given system and is of opposite sign to* $V(\mathbf{x})$*.*

A function $V(\mathbf{x})$ satisfying the conditions of Theorem 4.10 is called a weak Lyapunov function. Functions satisfying all of the criteria of Theorem 4.11 are strong Lyapunov functions.

From the direct method of Lyapunov we only need to find a function V for a given system with the properties shown above, to determine the stability of the zero solution to the system. In order to show this, we will discuss some examples of using Lyapunov's theorem.

Example 4.12. Consider the differential equation:

$$\ddot{x}(t) + f(x(t)) = 0 \tag{24}$$

We can replace (24) by the following system through letting $y = \dot{x}$.

$$y(t) = \dot{x}(t), \qquad \dot{y}(t) = -f(x(t)) \tag{25}$$

We will assume that

$$f(0) = 0, \quad x(t)f(x(t)) > 0 \text{ for } x(t) \neq 0. \tag{26}$$

Multiplying (25) by $\dot{x}(t)$ and integrating the whole equation to $x(t)$, we get:

$$\frac{y^2(t)}{2} + F(x(t)) = c \tag{27}$$

where c is a constant and $F(x(t)) = \int_0^{x(t)} f(x_1(t))\, dx_1(t)$. Now let us define the function:

$$V(x(t), y(t)) = \frac{y^2(t)}{2} + F(x(t)) \tag{28}$$

We will show that V is a Lyapunov function. First it is easy to see that $V(0,0) = 0$ and $V(x(t), y(t)) > 0$, so V is positive definite. Furthermore we will need the derivative of function V for checking the third condition of Theorem (4.10).

$$\begin{aligned} \frac{d}{dt}V(x(t), y(t)) &= y(t)\ \dot{x}(t) + \dot{F}(x(t))\ \dot{x}(t) \\ &= -f(x(t))\ \dot{x}(t) + f(x(t))\ \dot{x}(t) = 0 \end{aligned} \tag{29}$$

It follows that $\dot{V} = 0$, thus $\dot{V}$ is semidefinite, and the function $V(x(t), y(t))$ is a (weak) Lyapunov function. As we can see, V is constant over time, so the point $(x(t), y(t))$ cannot recede from the origin. A point $(x(t), y(t))$ starting near the origin will remain close to the origin, this means that system (25) is stable.

For higher dimensions it is not always possible to produce the Lyapunov function as above. Therefore we discuss the following example.

Example 4.13. Let we have a system $\dot{\mathbf{x}} = A\mathbf{x}$ with matrix A. Suppose that the solution of the system is asymptotically stable, which implies $\Re(\lambda_i) < 0$ for all n eigenvalues λ_i $(i = 1, 2, ..., n)$ of matrix A. We construct a strong Lyapunov function for the system of the quadratic form:

$$V(\mathbf{x}) = \mathbf{x}^t K \mathbf{x} \tag{30}$$

for which we first determine K to make V positive definite. We also want $\dot{V}$ to be negative definite. For that we need the derivative of (30).

$$\dot{V}(\mathbf{x}) = \mathbf{x}^t(A^t K + K^t A)\mathbf{x} \tag{31}$$

It would be convenient if:

$$A^t K + K^t A = -I \tag{32}$$

because in that case $\dot{V}$ is negative definite. This is easy to see when we rewrite: $\dot{V}(\mathbf{x}) = \mathbf{x}^t(-I)\mathbf{x} = -\sum_{i=1}^{n} x_i^2$.

To satisfy (32) we consider the product of $e^{A^t t}e^{At}$. For the derivative we have

$$\frac{d}{dt}(e^{A^t t}e^{At}) = A^t e^{A^t t}e^{At} + e^{A^t t}e^{At}A. \tag{33}$$

When $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable all eigenvalues have negative real parts. Then from Jordan [9, Section 10.5] we know that for any γ such that

$\gamma > max_{1\leq i\leq n}(\lambda_i)$ there exist a constant $c > 0$ such that $\|e^{At}\| \leq ce^{\gamma t}$. Since A and A^t have the same eigenvalues, we can chose c such that

$$\|e^{At}\|, \|e^{A^t t}\| \leq ce^{\gamma t}, \quad c > 0, \quad \gamma < 0. \tag{34}$$

This ensures that the following integral is convergent.

$$\int_0^\infty \frac{d}{dt}(e^{A^t t}e^{At})dt = \lim_{R\to\infty}\int_0^R \frac{d}{dt}(e^{A^t t}e^{At})dt = \lim_{R\to\infty} e^{A^t R}e^{AR} - I = -I \tag{35}$$

By writing out the derivative inside the integral, we have also:

$$\int_0^\infty \frac{d}{dt}(e^{A^t t}e^{At})dt = A^t\int_0^\infty (e^{A^t t}e^{At})dt + \int_0^\infty (e^{A^t t}e^{At})dt\ A \tag{36}$$

When comparing the results of (35) and (36) with the desired result of (32) we see that

$$K = \int_0^\infty (e^{A^t t}e^{At})dt \tag{37}$$

ensures that $\dot{V}$ is negative definite. Further from Jordan [9, Section 10.5] we also know that K is symmetrical because $(e^{At})^t = e^{A^t t}$ for any nonsingular $n \times n$-matrix A. By (30) we have

$$\begin{aligned} V(\mathbf{x}) &= \int_0^\infty (\mathbf{x}^t e^{A^t t})(e^{At}\mathbf{x})dt \\ &= \int_0^\infty (e^{At}\mathbf{x})^t(e^{At}\mathbf{x})dt \end{aligned}$$

which is a quadratic form. Therefore V is positive definite.

4.2.2 Lyapunov stability of the model

We are interested in the stability of system (15), shown at the beginning of this chapter. Let the set of resources be given by $J = \{1, 2, ..., L\}$. Then we can linearize this system like we did with other systems before, where $y_r(t) = (x_r(t) - \overline{x}_r)(\sqrt{\kappa_r\overline{x}_r})^{-1}$. The linearized system of (15) is then shown by:

$$\dot{y}_r(t) = -(\beta_r y_r(t - D_r) + \sum_{s\in R} M_{sr}y_s(t - \overleftarrow{d}_s - \overrightarrow{d}_r)), \quad r \in R \tag{38}$$

where $\beta_r = \kappa_r w_r \overline{x}_r^{-1}$, the vector M_{sr} is given by

$$M_{sr} = \sqrt{\kappa_r}\sqrt{\overline{x}_r}\sqrt{\kappa_s}\sqrt{\overline{x}_s}\begin{pmatrix} A_{1r}A_{1s}p'_1 & A_{2r}A_{2s}p'_2 & \cdots & A_{Lr}A_{Ls}p'_L \end{pmatrix}$$

and the vector $y_s(t - \overleftarrow{d}_s - \overrightarrow{d}_r)$ by:

$$y_s(t - \overleftarrow{d}_s - \overrightarrow{d}_r) = \begin{pmatrix} y_s(t - \overleftarrow{D}_{1s} - \overrightarrow{D}_{1r}) \\ y_s(t - \overleftarrow{D}_{2s} - \overrightarrow{D}_{2r}) \\ \vdots \\ y_s(t - \overleftarrow{D}_{Ls} - \overrightarrow{D}_{Lr}) \end{pmatrix}$$

We will leaving out the difference in delay for all users though writing $\overleftarrow{D}_{js} - \overrightarrow{D}_{jr} = D$ for all $r, s \in R$ and $j \in J$. Now we can write the system shown above in the following form:

$$\dot{y}(t) = -(\beta + M)y(t - D) \tag{39}$$

Where β is a diagonal matrix with diagonal elements β_r, M is a symmetric matrix with elements M_{sr} and $y(t-D)$ a vector of elements $y_r(t-D)$. Notice that matrix $\beta + M$ contains only positive elements, since $\kappa_r, w_r, \overline{x}_r$ and p_l' are strictly positive parameters.

Many stability theorems are known for delayed systems of the form $\dot{x}(t) = Ax(t - \tau)$. One of them is the following theorem, presented by Buslowicz and used by Schoen [10, Theorem 3.14]:

Theorem 4.14. *The system $\dot{x}(t) = Ax(t - \tau)$ is asymptotically stable if and only if for all eigenvalues $\lambda_i(A)$ with $i = \{1, ..., n\}$ of matrix A,*

$$\Re(\lambda_i(A)) < 0 \quad \text{and} \quad \tau < \frac{\arctan(\frac{\Re(\lambda_i(A))}{\Im(\lambda_i(A))})}{|(\lambda_i(A))|}$$

However, we are interested in finding the stability criterion with using the direct method of Lyapunov-Kraskovski. Marshal Slemrod [11, Section 3] discusses a theorem of stability, using a Lyapunov functional for the linear system of the form:

$$\dot{x}(t) + A\dot{x}(t - r) + Bx(t) + Cx(t - r) = 0$$

This theorem is relevant for us although A and B are zero matrices in our case.

For our system Marshal Slemrod uses a Lyapunov functional V of the form:

$$V(\psi) = \psi(0)^T P\psi(0) + \int_{-r}^{0} \psi^T(\theta)Q\psi(\theta)d\theta \tag{40}$$

in which $\psi(t)$ is a function $\psi(t) : [-r, 0] \to \mathbb{R}^n$. Now we differentiate the functional and since we know that $\dot{\psi}(0) = -C\psi(-r)$, as it satisfies the form of the linear system with A and B zero matrices, then $\dot{V}(\psi)$ is given by:

$$\begin{aligned} \dot{V}(\psi) = -\psi(-r)^T C^T P \psi(0) - \psi(0)^T P C \psi(-r) \\ +\psi(0)^T Q \psi(0) - \psi(-r)^T Q \psi(-r) \end{aligned} \tag{41}$$

We choose the matrices P and Q such that they satisfy the hypothesis of the theorem 4.17. Before discussing the theorem we first give some definitions.

Definition 4.15. A *functional difference operator* D is an operator $D : C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ with

$$D\phi = \phi(0) - \sum_{k=1}^{N} P_k \phi(-\tau_k)$$

where P_k are constant $n \times n$-matrices with $k = 1, .., N$ and $0 < \tau_k \leq r$ for real number τ_k. If $N > 1$ then τ_k/τ_j are rational numbers.

From now on we consider the system

$$\frac{d}{dt} D(x_t) = f(x_t) \text{ for } t \geq 0 \text{ and with } x_0 = \phi.$$

For this system we define also:

Definition 4.16. A subset $\Gamma \subseteq C([-r, 0], \mathbb{R}^n)$ is *invariant* with respect to solutions of the system of above, if there exist a function $g(\phi) : (-\infty, \infty) \to \mathbb{R}^n$ of every $\phi \in \Gamma$, for which $g_0(\phi) = \phi$ and

$$Dg_{t+\sigma}(\phi) = Dg_\sigma(\phi) + \int_0^t f(g_{\sigma+s}(\phi))ds, \quad t \geq 0$$

with $g_\sigma(\phi) \in \Gamma$ for all $\sigma \in (-\infty, \infty)$.

Now we may discuss the theorem for which the matrices P and Q of the Lyapunov functional should satisfy the hypothesis.

Theorem 4.17. *Consider the system* $\frac{d}{dt} D(x_t) = f(x_t)$ *as above. Let* D *be a stable difference operator and* V *be a Lyapunov function on a set of* G_l *with* $G_l = \{\phi \in C([-r, 0], \mathbb{R}^n) : V(\phi) < l\}$.
If there is a constant K_l *such that when* $\phi \in G_l$ *that* $|D\phi| < K_l$*, then for* $t \to \infty$ *any solution* $x_t(\phi)$ *with* $\phi \in G_l$*, approaches* Γ*. For this* Γ *is the largest set* $\{\phi \in \overline{G} : \dot{V}(\phi) \leq 0\}$ *which is invariant to the system, where* $\overline{G}$ *is the closure of* G*.*

The main theorem from Marshall Slemrod, we discuss, for a system of the form $\dot{x}(t) + Cx(t-r) = 0$ with C an constant $n \times n$-matrix, is the following:

Theorem 4.18. *[11, Theorem 3.1] With* $\mathbf{0}$ *the* $n \times n$ *zero matrix, if* $M = \begin{pmatrix} \mathbf{0} & C \\ C^T & \mathbf{0} \end{pmatrix}$ *is positive definite, then the solution* $x(t) = 0$ *is asymptotically stable.*

Proof. Let we take $P = I$ the unity matrix and $Q = \mathbf{0}$ the zero matrix. Then it is easy to see from (41) that we have $\dot{V}(\psi) = -[\psi(0), \psi(-r)]^T M[[\psi(0), \psi(-r)]$. Furthermore, $D\psi = \psi(0)$ while $\dot{x}(t)$ is the only derivative in the differential equation, thus $D\psi$ is a constant difference operator. We can also derive from the form (40) and the choices for P and Q that $V(\psi) = |D\psi|^2 \geq 0$ and $\dot{V}(\psi) < -\mu|\psi(0)|^2$ for any $\mu > 0$.

Then by noticing that $V(\psi) \leq \alpha^2\|\psi\|^2$ with $\|\psi\| = \sup_{-r\leq t\leq 0} |\psi(t)|$ for some $\alpha > 0$. We find that for any $b > 0$ with $\|\psi\| < b$ we have $|D\psi| < \alpha b$. When we take $l = \alpha^2 b^2$ and $K_l = \alpha b$ we can apply Theorem 4.17. This theorem says that any solution $x_t(\phi)$ with $\|\phi\| < b$ approaches Γ when t goes to infinity. In this case $\Gamma = \{0\}$, thus any solution will go to to zero. □

From Theorem 4.18 we can conclude that system (39) is asymptotically stable for the solution $y(t) = 0$, since $(\beta + M)$ is positive definite. This means that all equilibrium solutions $\overline{x}_r$ of system (2) are asymptotically stable if all kinds of round-trip delays are equal to D.

5 Conclusion

In this thesis we discussed the stability of the equilibrium solutions of four different models. For the discrete models we could determine the stability almost directly from determining when all roots of the characteristic equation have negative real parts. However, for the continuous model we used multiple theorem, like the Direct method of Lypanuov Kraskovski.

For both difference and differential systems, the criteria of the simple models of one resource and one user were relatively easily proven. Unfortunately these models where also the most uninteresting ones of those four, while we are interested in stability of a large network. Nevertheless we did find that if κ satisfies

$$\kappa < \frac{2}{p + \overline{x}p'} \sin(\frac{\pi}{2(2D+1)})$$

the equilibrium solution of the simple discreet model is stable. Likewise the equilibrium of the simple continuous model is stable if κ satisfies:

$$\kappa < \frac{\pi}{2D(p + \overline{x}p')}$$

Notice that the κ and D of the continuous models are different of those of the discrete models. So denote the κ and D of the discrete models again by $\Lambda = \frac{D}{h}$ and $\widetilde{\kappa} = h\kappa$, such as in chapter 2. Then $\kappa D = \overline{\kappa}\Lambda$, from this we conclude that both criteria should be almost similar. This can easily be seen if Λ (Discrete D) is very large, in that case $2\sin(\frac{\pi}{2(2\Lambda+1)})$ is almost equal to $\frac{\pi}{2\Lambda+1}$. From that if Λ is great, we see that it is almost the same as $\frac{\pi}{2\Lambda}$. This gives us the same criterion for the discrete model as for the continuous model. Make sure that Λ is large, while h is very small relatively to the small delay D.

The criteria of the more complex systems were a bit harder to find and prove. For the complex discreet model we found a criterion similar to the criterion of the simple discrete model. This can be rewritten as:

$$\kappa_r < 2\sin(\frac{\pi}{2(2D+1)})(\sum_{j\in r} p_j + \sum_{j\in r} p_j' \sum_{s:j\in s} \overline{x_s})^{-1}$$

in order to choose a κ_r for which the equilibrium solution is stable. Note that we made the assumption that the round trip delay is constant over all possible routes, so $D_r = D$ for all $r \in R$. If we make the even stronger assumption that all combinations of forward and backward delays are constant such that $\overrightarrow{D}_{l,r} + \overleftarrow{D}_{l,s} = D$ for all $l \in J$ and $r, s \in R$, we find that the equilibrium solution of the complex continuous model is always stable. For further research it is interesting to look for stability criteria of the complex continuous model using Lyapunov functionals for which weaker assumptions are necessary.

A Appendix: Basics of stability

For simple differential equations we can determine stability as follows.
Let us take the following linear differential equation:

$$x''(t) = \alpha x'(t) + \beta x(t), \qquad x(t_0) = x_0 \text{ and } x'(t_0) = y_0 \tag{42}$$

Make sure that $x(t) = 0$ is a solution for this equation. We will start with finding a characteristic equation by using the solution $x(t) = e^{\mu t}$ with $\mu \in \mathbb{C}$. Inserting the solution into (42) and dividing by $e^{\mu t}$ we get:

$$\mu^2 = \alpha\mu + \beta. \tag{43}$$

Now we can solve (43) for μ so that we have two solutions for $x(t)$. Let μ_1 and μ_2 be the solutions of (43) then the general solution of (42) will be like: $x(t) = Ae^{\mu_1 t} + Be^{\mu_2 t}$.
Notice that for the solution $x(t) = 0$ holds that $A = B = 0$. This solution is stable if A or B, or both, are non-zero and we still have $\lim_{t\to\infty} x(t) = 0$. Make sure that this holds when $|e^{\mu t}| < 1$ for every μ solution of the characteristic equation.
Now the question is for which μ we have $|e^{\mu t}| < 1$. We know that $\mu \in \mathbb{C}$ so we can write $\mu = a + bi$ and we know that $e^{\phi i} = \cos(\phi) + i\sin(\phi)$. So we get:

$$|e^{\mu t}| = |e^a(\cos(b) + i\sin(b))| = |e^a| = e^{\Re(\mu)} < 1$$

From this follows that the solution $x(t) = 0$ is stable if $\Re(\mu) < 0$ for each μ solution of the characteristic equation of the differential equation (42).

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