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Inflation with cosmological constant and a nonminimally coupled scalar field with mass term

Bachelor Thesis

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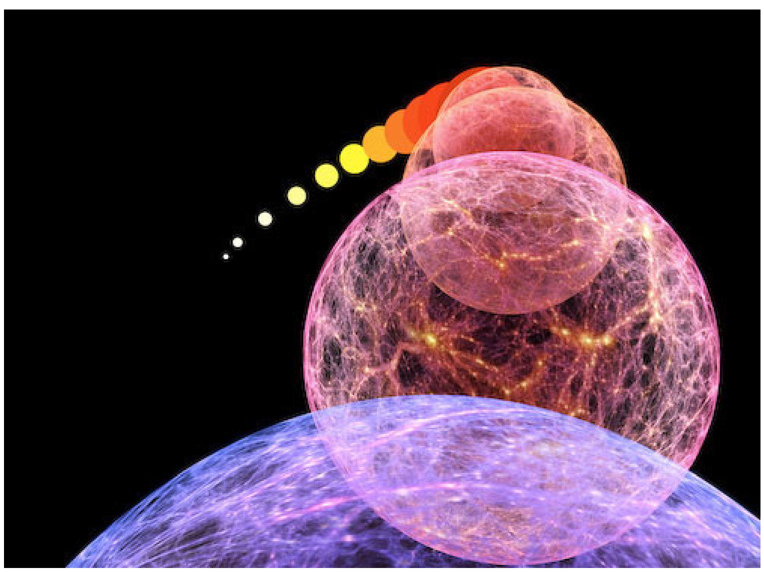

Abstract

We consider inflation in a universe with cosmological constant and a nonminimally coupled scalar field with a mass term inspired by previous work [1]. First the tools will be derived to make a good analysis. Then the model will be analysed. The slow roll parameter converges to 3/4 for a mass term $m^2 > 0$. The spectral index n_s peaks for $m = 0$. The rate of quantum tunneling is highest for small m. However because it is given in units of Hubble time, when we look at a great amout of Hubble volumes it can be possible that for larger m somewhere in the universe the field tunnels. Which is sufficient for inflation to start. . We conclude that for a mass term to be included in this model there must be some kind of phase transition where m^2 turns from positive to negative.

Contents

1 Introduction

One of the most pressing questions of cosmologists of all times is, how did the universe arise? And although we will never know for sure, we can can build models upon theories and test them with our measurements and observations. In this paper we will first explain the concepts of cosmic inflation, describing very early stages of our universe, and why it is important. Then we will present a model realised within a tensor-scalar theory and discuss how to analyse it in the Einstein frame. Furthermore we will derive and provide the tools to test the model by explaining the slow roll approximation, the spectra of scalar and tensor perturbations and the concepts of quantum tunneling. Eventually we will compare our calculated results to observations and discuss the probability of our model being viable.

2 Inflation

2.1 Three Cosmological Problems

In the standard *Hot Big Bang model*, in which the universe was initially very hot and dense and since then has been expanding and cooling, there are three underlying problems [4]. The first is the “ flatness problem”. The results of measurement and observations tells us that the spatial curvature of the prsent universe is nearly flat and was even flatter in the past. However there is no reason for it to be flat. It could just have been strongly curved without violating any laws of physics. We could however just state that the initial conditions just happened to be so that by coincidence they produced a spatially flat universe. However this becomes extremely far fetched when you extrapolate the density parameter for the curvature back in to the past [4]. And thus it would be far more satisfactory to find a physical mechanism for flattening the universe instead of assuming highly far fetched initial conditions.

The second second problem is the “horizon problem”. It states that our observations tell us that that the universe in homogenous and isotropic on very large scales. As convenient as this may be, there is no reason that this should be the case. Consider two antipodal points separated by 180° as seen by an observer on earth. According to the Benchmark model the current proper distance to the last scattering surface is $d_p(t_0) = 0.98 d_{hor}(t_0)$ [4], where $d_{hor}(t_0)$ is the current horizon distance. So distance between these two points would be $1.96 d_{hor}(t_0)$ and thus would not be in causal contact with each other and in particular would not have had time to come in to thermal equilibrium with each other. Nevertheless these two points have the same temperature to within one part in 10^5. Why should regions that were out of causal contact with each other have such identical properties? Again assuming coincidence would seem extremely farfetched.

The third problem is the “monopole problem”. *Grand Unified Theories* predict that in the very early universe the universe underwent a phase transition in which magnetic monopoles were created. These magnetic monopoles would have been so massive and abundant that they would have dominated the energy density of the universe when the temperature had fallen below $T \sim 10^{18}$ [4]. The universe however is definitely not dominated by magnetic monopoles and even more there isn't any strong evidence they exist at all. Every north pole we can find is accompanied by a south pole and vice versa.

2.2 Solution

In the late seventies in the Soviet Union the idea of an exponentially expanding universe existed. Alexei Starobinsky proposed a model where the universe went through an inflationary era which resolved the horizon and flatness problem. Alan Guth then first coined the term *inflation* in 1981 to explain the nonexistence of monopoles. Together they recieved the Kavli Prize in Astrophysics for pioneering the theory of cosmic inflation. Now this theory of inflation is widely accepted because it solves the three cosmological problems at once.

It can be defined as the hypothesis that there was a period in the early universe when the expansion was accelerating outward. Thus this epoch was characterised by $\ddot{a} > 0$, where a is the cosmological scale factor. The acceleration equation is:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\epsilon + 3P), \tag{1}$$

where ϵ is the energy density and P is the pressure. Both are functions of time. For substances of cosmological importance the equation of state can be written in the simple linear from:

$$P = w\epsilon, \tag{2}$$

where w is a dimensionless number which characterises which cosmological substance dominates. Some values of w are of special interest. For $w = 0$ the universe will be dominated by non-relativistic matter. For $w = \frac{1}{3}$ the universe wil be dominated by relativistic matter. And for $w < -\frac{1}{3}$ the universe will be dominated by *dark energy*. One form of dark energy is the cosmological constant, for which $w = -1$. When we look at Eq. (1) and (2) we see that for $w < -\frac{1}{3}$, $\ddot{a} > 0$. And so let us first assume that the energy density would be dominated by the cosmological constant. Then the Friedmann equation would read:

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{\Lambda_i}{3} \tag{3}$$

And so the Hubble parameter would be constant during inflation $H_i = \sqrt{\frac{\Lambda_i}{3}}$. Integrating gives:

$$a(t) \propto e^{H_i t} \tag{4}$$

And so if the duration of inflation would be large compared to the Hubble time during inflation, H_i^{-1}, then the scale factor would grow enormously. The number of e-folds would be $N = H_i(t_e - t_i)$, where t_e and t_i are respectively the end and beginning of inflation. For example if $N = 60$ then the growth in scale factor dring inflaiton was

$$\frac{a(t_e)}{a(t_i)} = e^{60} \approx 10^{26} \tag{5}$$

Such a growth would flatten any universe which was not perfectly flat in the beginning. It would also increase the horizon distance in a postinflationary universe by a factor e^N and thus bringing the entire universe easily in causal contact. Furthermore it would make a magnetic monopole extremely rare to come across even if they were created according to the GUT.
Now that we understand the concept and the importance of inflation, we will apply our model.

3 Model of inflation

3.1 Model

The model we are considering in this paper is a scalar-tensor theory of gravity with a scalar field which has a mass part and a part that couples directly to the curvature scalar. The mechanism driving inflation is not just the cosmological constant as discussed above, but also a scalar field $\phi(t)$ which depends only on time (we assume a homogeneous field). The action can be written as a sum of the coupled scalar-gravitational piece, the pure scalar piece and a cosmological constant piece:

$$S = S_{FR} + S_\phi + S_\Lambda \tag{6}$$

where

$$S_{FR} = \int dx^4 \sqrt{-g} \frac{F(\phi)R}{2} \tag{7}$$

$$S_\phi = \int dx^4 \sqrt{-g} \big(\frac{-1}{2} \partial_\mu \phi \partial_\nu \phi - V(\phi) \big) \ , \tag{8}$$

and

$$S_\Lambda = \int dx^4 - \sqrt{-g} M_P^2 \Lambda \tag{9}$$

These actions and functions are presented in the Jordan frame and from now on we will subscript elements in the Jordan frame with a J. So when we put them together we obtain the total action in the Jordan frame:

$$S_J = \int d^4x \sqrt{-g_J} \big[\frac{1}{2} F(\phi_J) R_J - M_P^2 \Lambda - \frac{1}{2} g_J^{\mu\nu} \partial_\mu \phi_J \partial_\nu \phi_J - V_J(\phi_J) \big], \tag{10}$$

where g =det$[g_{\mu\nu}]$ and Λ is the comsological constant. We assume the following forms for the functions $F(\phi)$ and $V(\phi)$:

$$F(\phi_J) = M_P^2 - \xi_2 \phi_J^2 - \xi_4 \frac{\phi_J^4}{M_P^2} \quad , \quad V_J(\phi_J) = m^2 \phi_J^2, \tag{11}$$

where $M_P^2 = \frac{1}{8\pi G_N}$, G_N is Newtons grsvitational constant. The parameters ξ_2 and ξ_4 are the nonminimal coupling parameters and m is the mass term. In our conventions conformal coupling corresponds to $\xi_2 = 1/6$, $\xi_4 = 0$ and we work with natural units in which $\hbar = c = 1$. However we will be studying the model in which $\xi_2 = 0$, $\xi_4 < 0$ and $m > 0$. For the metric we choose a spatially flat, expanding background:

$$g_{J\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a_J^2(t) & 0 & 0 \\ 0 & 0 & a_J^2(t) & 0 \\ 0 & 0 & 0 & a_J^2(t), \end{pmatrix} \tag{12}$$

where $a_J(t)$ is the scale factor a time t.

3.2 Jordan-Einstein transformation

As mentioned before the model as presented above is in the so called Jordan-frame. However for the analyses in this paper an Einstein-frame is more useful. So we will proceed to make the frame transformation.

To obtain the Einstein-frame one ought perform the following frame transformation:

$$g_{E\mu\nu} = \frac{F(\phi_J)}{M_P^2} g_{J\mu\nu}, \tag{13}$$

In appendix G [2] it is shown that under the transformation $\tilde{g}_{\mu\nu} = \omega^2(x) g_{\mu\nu}$, the Ricci scalar transforms as follows:

$$\tilde{R} = \omega^{-2} R - 2(n-1) g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega) - (n-1)(n-4) g^{\alpha\beta} \omega^{-4} (\nabla_\beta \omega)(\nabla_\alpha \omega), \tag{14}$$

where n is the number of dimensions. In our case $n = 4$. So when we rewrite Eq.(14) and fill in the transformed metric we get:

$$\begin{aligned} R &= \omega^2 \tilde{R} + 6 g^{\alpha\beta} \omega^{-1} (\nabla_\alpha \nabla_\beta \omega) \\ &= \frac{F(\phi_J)}{M_P^2} \tilde{R} + 6 \sqrt{\frac{F(\phi_J)}{M_P^2}} \tilde{g}^{\alpha\beta} \left(\nabla_\alpha \nabla_\beta \sqrt{\frac{F(\phi_J)}{M_P^2}} \right) \\ &= \frac{F(\phi_J)}{M_P^2} \tilde{R} + 6 \sqrt{\frac{F(\phi_J)}{M_P^2}} \tilde{g}^{\alpha\beta} \nabla_\alpha \left(\frac{1}{2} \left(\frac{F(\phi_J)}{M_P^2} \right)^{-\frac{1}{2}} \left(\frac{d}{d\phi_J} \frac{F(\phi_J)}{M_P^2} \right) (\partial_\beta \phi_J) \right) \\ &= \frac{F(\phi_J)}{M_P^2} \tilde{R} + 6 \sqrt{\frac{F(\phi_J)}{M_P^2}} \tilde{g}^{\alpha\beta} \left[\frac{-1}{4} \left(\frac{F(\phi_J)}{M_P^2} \right)^{-\frac{3}{2}} \left(\frac{d}{d\phi_J} \frac{F(\phi_J)}{M_P^2} \right)^2 (\partial_\beta \phi_J)(\partial_\alpha \phi_J) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{F(\phi_J)}{M_P^2} \right)^{-\frac{1}{2}} \nabla_\alpha \left(\left(\frac{d}{d\phi_J} \frac{F(\phi_J)}{M_P^2} \right) (\partial_\beta \phi_J) \right) \right], \end{aligned} \tag{15}$$

where the last term is a surface term so it drops out. Leaving us with:

$$R_J = \frac{F(\phi_J)}{M_P^2} R_E - \frac{3}{2} \left(\frac{F(\phi_J)}{M_P^2} \right)^{-1} g_E^{\mu\nu} \left(\left(\frac{d}{d\phi_J} \frac{F(\phi_J)}{M_P^2} \right)^2 (\partial_\mu \phi_J)(\partial_\nu \phi_J) \right) \tag{16}$$

Now when we plug this back in to Eq.(10)(and keeping in mind that $g_J^{\mu\nu} = \frac{F(\phi_J)}{M_P^2} g_E^{\mu\nu}$) we get:

$$S_E = \int d^4x\sqrt{-g_E} M_p^2 [\frac{1}{2}R_E - \frac{1}{2F^2(\phi_J)}\left(F(\phi)J) + \frac{3}{2}\left(\frac{dF(\phi_J)}{d\phi_J}\right)^2\right) g_E^{\mu\nu}\partial_\mu\phi_J\partial_\nu\phi_J - \frac{M_P^6\Lambda + M_P^4 m^2\phi_J^2}{F^2(\phi_J)}] \tag{17}$$

To complete the transformation we make the following substituion:

$$d\phi_E = \frac{M_p}{F(\phi_J)}\sqrt{F(\phi_J) + \frac{3}{2}\left(\frac{dF(\phi_J)}{d\phi_J}\right)^2} d\phi_J \tag{18}$$

This way we obtain:

$$\begin{aligned} \partial_\mu\phi_E\partial_\nu\phi_E &= \left(\frac{d\phi_E}{d\phi_J}\right)^2 \partial_\mu\phi_J\partial_\nu\phi_J \\ &= \frac{M_P^2}{2F^2(\phi_J)}\left(F(\phi)J) + \frac{3}{2}\left(\frac{dF(\phi_J)}{d\phi_J}\right)^2\right)\partial_\mu\phi_J\partial_\nu\phi_J \end{aligned} \tag{19}$$

And so we arrive in the Einstein frame with the following action:

$$S_E = \int d^4x\sqrt{-g_E}[\frac{M_P^2}{2}R_E - \frac{1}{2}g_E^{\mu\nu}\partial_\mu\phi_E\partial_\nu\phi_E - \frac{M_P^6\Lambda + M_P^4 m^2\phi_J^2(\phi_E)}{F^2((\phi_J(\phi_E))}] \tag{20}$$

With this new action we define an effective potential

$$V_e(\phi_E) = \frac{M_P^4(M_P^2\Lambda + m^2\phi_J(\phi_E)^2)}{F^2(\phi_E)} \tag{21}$$

Which as you can see couples the scalar field to the cosmological constant.

3.3 Slow Roll

The state of the universe before inflation is unclear. It seems rather safe to say that it was expanding, it was in a chaotic state and that the energy-momentum tensor was dominated by field fluctuations. This state can be approximated by the the relativistic matter dominated equation of state, $w \approx 1/3$. In this state the nonminimal couplings do not play a significant role[1]. However as the universe expands the corresponding energy density and pressure will decrease. Eventually they will reach a point when the dark energy(in the form of the cosmological constant and the scalar field) will become dominant. This is when the universe enters inflation. The field will feel a hilltop-like (effective) potential (21) and start rolling down slowly. As it rolls down down the number of e-folds increases and the fluctuations will redshift rapidly. And so the inflation loosens our initial conditions (similar to how the flatness problem is resolved). This is called the *slow roll approximation*. For this approximation to hold, the *slow roll conditions* must be satisfied. These are

$$\epsilon << 1 \qquad\qquad \eta << 1, \tag{22}$$

where

$$\epsilon = \frac{M_P^2}{2}\Big(\frac{V'}{V}\Big)^2 \qquad\qquad \eta = M_P^2\frac{V''}{V}. \tag{23}$$

For small field values, $\phi_J \simeq \phi_E << M_P$ the potential (21) can be approximated by,

$$V(\phi) \simeq M_P^2\Lambda + m^2\phi_E^2 \tag{24}$$

For large field values $\phi_E >> M_P$, ϕ_E (18) gives,

$$\begin{aligned}
d\phi_E &= \frac{M_P}{M_P^2 - \xi_4\frac{\phi_J^4}{M_P^2}}\sqrt{M_P^2 - \xi_4\frac{\phi_J^4}{M_P^2} + \frac{3}{2}\big(M_P^2 - 4\xi_4\frac{\phi_J^3}{M_P^2}\big)^2}d\phi_J \\
&\simeq \frac{M_P}{-\xi_4\frac{\phi_J^4}{M_P^2}}\sqrt{\frac{3}{2}\big(-4\xi_4\frac{\phi_J^3}{M_P^2}\big)^2}d\phi_J \\
&= 4\sqrt{\frac{3}{2}}M_P\frac{d\phi_J}{\phi_J}
\end{aligned} \tag{25}$$

Which gives,

$$\phi_J = \exp\big(\sqrt{\frac{2}{3}}\frac{\phi_E}{4M_P}\big). \tag{26}$$

And also,

$$\begin{aligned}
d\phi_E &= \frac{M_P}{M_P^2 - \xi_4\frac{\phi_J^4}{M_P^2}}\sqrt{M_P^2 - \xi_4\frac{\phi_J^4}{M_P^2} + \frac{3}{2}\big(M_P^2 - 4\xi_4\frac{\phi_J^3}{M_P^2}\big)^2}d\phi_J \\
&\simeq \frac{M_P}{M_P^2 - \xi_4\frac{\phi_J^4}{M_P^2}}\sqrt{\frac{3}{2}\big(M_P^2 - 4\xi_4\frac{\phi_J^3}{M_P^2}\big)^2}d\phi_J \\
&= \frac{M_P}{F(\phi_J)}\sqrt{\frac{3}{2}}\frac{dF(\phi_J)}{d\phi_J} \\
&= M_P\sqrt{\frac{3}{2}}\frac{d}{d\phi_J}\ln\big(F(\phi_J)d\phi_J.
\end{aligned} \tag{27}$$

And so, for large field values,

$$\phi_E = M_P\sqrt{\frac{3}{2}}\ln\Big[\frac{F(\phi_J)}{M_P^2}\Big] \tag{28}$$

Combining (26) and (28) gives the following apporximation for the potential for large field values,

$$\begin{aligned}
V(\phi_E) &= \frac{M_P^4(M_P^2(\Lambda + m^2\phi_E^2)}{F^2(\phi_E)} \\
&= \frac{M_P^6\Lambda + M_P^4 m^2\Big(\exp\big(\sqrt{\frac{2}{3}}\frac{\phi_E}{4M_P}\big)\Big)^2}{\big(M_P^2\exp(\sqrt{\frac{2}{3}}\frac{\phi E}{M_P})\big)^2} \\
&= M_P^2\Lambda\exp\Big(-\sqrt{\frac{8}{3}}\frac{\phi_E}{M_P}\Big) + m^2\exp\Big(-\frac{3}{2}\sqrt{\frac{2}{3}}\frac{\phi_E}{M_P}\Big)
\end{aligned} \tag{29}$$

A plot of the effective potential for different values of m has been given in Figure (1).

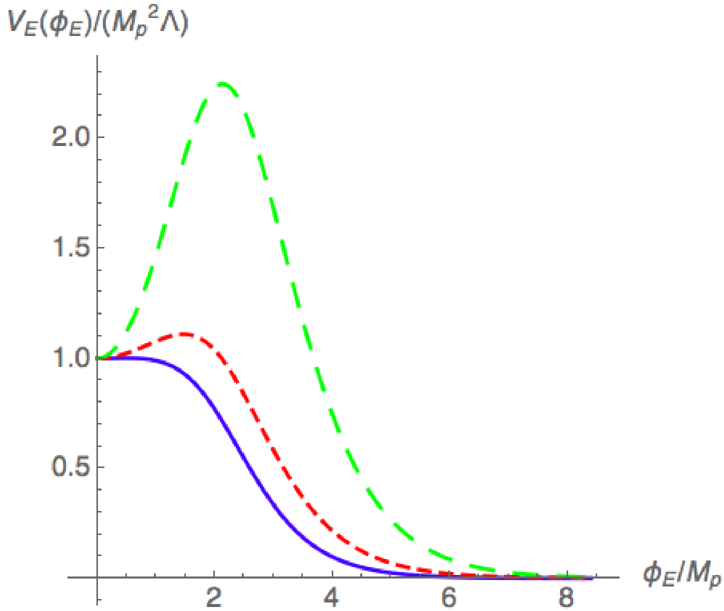

Figure 1: A plot of the effective potential V_E in the Einstein frame as a function of the field value ϕ_E in the Einstein frame. We have chosen a fixed $\xi_4 = -0.01$. We have chosen three values for m_2, rescaled by m^2/Λ: 0.01 (Blue), 0.1 (Red Dashing) and 0.2 (Green, Dashing).

One can see that that the slow roll approximation conditions are easily held for small field values (24). For which the parameters are approximated as,

$$\epsilon = \frac{M_P^2}{2}\Big(\frac{V'}{V}\Big)^2 = \frac{M_P^2}{2}\Big(\frac{2m^2\phi_E}{M_P^2\Lambda + m^2\phi_E^2}\Big)^2 \tag{30}$$

And as the field rolls down and the values become large ϵ approaches unity and so eventually exiting inflation. However if we take the limit for large field values of the slow roll parameter, we see that

$$\begin{aligned}\epsilon &= \frac{M_P^2}{2}\Big(\frac{V'}{V}\Big)^2 \\ &= \frac{M_P^2}{2}\left(\frac{-\sqrt{\frac{8}{3}}M_P^5\Lambda\exp\Big(-\sqrt{\frac{8}{3}}\frac{\phi_E}{M_P}\Big) - \sqrt{\frac{3}{2}}\frac{m^2}{M_P}\exp\Big(-\sqrt{\frac{3}{2}}\frac{\phi_E}{M_P}\Big)}{M_P^6\Lambda\exp\Big(-\sqrt{\frac{8}{3}}\frac{\phi_E}{M_P}\Big) + m^2\exp\Big(-\sqrt{\frac{3}{2}}\frac{\phi_E}{M_P}\Big)}\right)^2 \\ &\to \frac{3}{4}, \quad \text{for} \quad |m| > 0\end{aligned} \tag{31}$$

But if we want to see what happens along the way we will have to derive an exact function for ϵ. We name $G(\phi) = M_P^2\Lambda + m^2\phi^2$, and derive ϵ as follows,

$$\begin{aligned}
\epsilon(\phi_J) &= -\frac{M_P^2}{2}\Big(\frac{G'F^2 - 2GFF'}{F^4}\frac{F^2}{G}\frac{d\phi_J}{d\phi_E}\Big)^2 \\
&= \frac{M_P^2}{2}\Big(\frac{G'F^2 - 2GFF'}{GF^2}\frac{F}{M_P\sqrt{F + \frac{3}{2}F'^2}}\Big)^2 \qquad (32) \\
&= \frac{(G'F - 2GF')^2}{2G^2(F + \frac{3}{2}F'^2)}
\end{aligned}$$

Now we will derive the equations governing this system. Varying the action (20) gives

$$\begin{aligned}
\frac{d}{dt}\Big(\sqrt{-g}\frac{d\phi_E}{dt}\Big) &= -\sqrt{-g}\frac{dV(\phi_E)}{\phi_E} \\
\frac{d}{dt}\Big(a^3\frac{d\phi_E}{dt}\Big) &= -a^3V'(\phi_E) \\
a^3\frac{d^2\phi_E}{dt^2} + 3\dot{a}a^2\frac{d\phi_E}{dt} &= -a^3V'(\phi_E)
\end{aligned}$$

And because $\dot{a}/a = H$, the equation of motion is

$$\ddot{\phi}_E + 3H\dot{\phi}_E + V' = 0. \qquad (33)$$

By definition the energy density is given by the 00 component of the stress energy tensor, given by

$$\begin{aligned}
\rho_\phi &= \partial_0\phi\partial_0\phi - g_{00}\mathcal{L} \qquad (34) \\
&= \frac{1}{2}\dot{\phi}_E^2 + V \qquad (35)
\end{aligned}$$

We can in turn use this in the Friedmann equation for universe dominated by this density.

$$H^2 = \frac{1}{3M_P^2}\Big(\frac{1}{2}\dot{\phi}_E^2 + V(\phi_E)\Big) \qquad (36)$$

Now, when we differentiate equation (36),

$$\begin{aligned}
2H\dot{H} &= \frac{1}{3M_P^2}(\dot{\phi}_E\ddot{\phi}_E + V'(\phi_E)\dot{\phi}_E) \\
&= \frac{\dot{\phi}_E}{M_P^2}(\ddot{\phi}_E + V')
\end{aligned}$$

and use equation (33), we get

$$\dot{H} = -\frac{\dot{\phi}_E^2}{2M_P^2} \qquad (37)$$

Now the equation (33), (36) and (37) are the equations that govern the inflation diynamics and the background geometry. In the *slow-roll approximation* we can neglect

the kinetic term in equation (36) in comparison to the potential. We also neglect the first term ($\ddot{\phi}$) in equation (33) because we assume the field rolls slowly down the potential.
Furthermore, in the slow roll approximation, equation (33) can be written as $\dot{\phi} = -V'/(3H)$. Thus,

$$\begin{aligned}\frac{\dot{\phi}_E}{H} &= \frac{-V'}{3H^2} \\ &= -M_P^2 \frac{V'}{V}.\end{aligned} \tag{38}$$

We measure the number of e-folds from the end of inflation, $\phi(t_e) = \phi_e$, and as discussed before the initial conditions from before inflation can be neglected. So with this in mind and rewriting equation (38) as,

$$\begin{aligned}\frac{1}{H}\frac{d\phi_E}{dt} &= -M_P^2 \frac{V'}{V} \\ H dt &= -\frac{V}{M_P^2 V'} d\phi_E \\ &= -\frac{V}{M_P^2 V'} \frac{\sqrt{F + \frac{3}{2}F'^2}}{F} d\phi_J\end{aligned}$$

we can define the number of e-folds as,

$$\begin{aligned}N(\phi_J) &= \int_t^{t_e} H_E(\tilde{t}) d\tilde{t} \\ &\simeq \int_{\phi_E}^{\phi_{Ee}} d\phi_E \frac{-V}{M_P^2 V'} \\ &= \int_{\phi_J}^{\phi_{Je}} d\phi_J \Big[\frac{-V}{M_P^2 V'} \frac{F}{M_P^2 \sqrt{F + \frac{3}{2}F'^2}}\Big] \\ &= \int_{\phi_J}^{\phi_{Je}} d\phi_J \Big[\frac{G(F + \frac{3}{2}F'^2)}{2GF' - G'F}\Big].\end{aligned} \tag{39}$$

The functions derived in this subsection can be plotted and used to test the model in consideration. This will be done in section 4.

3.4 Perturbations

Although the universe is homogenous and isotropic on large scales there are some small perturbation in temperature (order $\sim 10^-5$). These cosmological perturbations are created by the amplification of quantum fluctuations of matter and metric perturbations during inflation. In our model matter consist only of a scalar field. The spectrum of these perturbations can be calculated as well as measured. So we shall proceed in calculating the power spectra of the scalar (matter) and tensor (metric) perturbations.

We assumed that the field was approximately homogenous with respect to a space-like hypersurface. So it can be decomposed into its condensate and small perturbations:

$$\phi(x) = \phi_0(t) + \varphi(t, \vec{x}) \tag{40}$$

Similarly, the metric tensor can be written as,

$$g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta\hat{g}_{\mu\nu}(\vec{x}), \tag{41}$$

where $\bar{g}_{\mu\nu}(t) = diag\big(-1,\ a^2(t),\ a^2(t),\ a^2(t)\big)$. Now, because these perturbations are due to quantum fluctuations, the obvious step would be to canonically quantise these perturbations. We will restrict ourselves to scalar perturbations as derivation for the tensor perturbations are similar. Even more as you will see they are connected. First we promote the perturbations to operators, $\varphi \to \hat{\varphi}$, and then impose canonical relations,

$$\big[\hat{\varphi}(t, \vec{x}), \hat{\pi}_\varphi(\vec{x}', t)\big] = i\delta^3(\vec{x} - \vec{x}'), \tag{42}$$

where $\hat{\pi}_\varphi = a^2 d\hat{\varphi}/dt$ denotes the canonical momentum of $\hat{\varphi}$. This can be achieved by decomposing the perturbations in Fourier modes,

$$\hat{\varphi}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \big[\varphi(t, k)\hat{a}(\vec{k}) + \varphi^*(t, k)\hat{a}^\dagger(-\vec{k})\big] \tag{43}$$

where $\vec{k}$ is the comoving momentum of the mode, $k = ||\vec{k}||$, $\hat{a}(\vec{k})$ and $\hat{a}^\dagger(\vec{k})$ are the annihilation and creation operators for scalar perturbations, which depend on $\vec{k}$ and satisfy the following commutation relations,

$$\begin{aligned} [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] &= (2\pi)^3\delta^3(\vec{k} - \vec{k}') \\ [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] &= 0 \\ [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] &= 0. \end{aligned} \tag{44}$$

The mode functions φ and φ^* are two linearly independent solutions to the mode function equation, which only depend on the magnitude of $\vec{k}$ and not the direction. The mode function equation can be derived by rescaling for $a\hat{\varphi}$ and then inserting the decomposed perturbations in to the equation of motion (33)[6],

$$\Big(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + \frac{k}{a^2} + \frac{d^2V}{d\phi_0^2}\Big)\varphi(t, k) = 0 \tag{45}$$

From this one can derive two general linearly independent mode functions. These functions however must also obey the *Wronskian normalisation* condition,

$$W[\varphi, \varphi^*] = \varphi_k\Big(\frac{d}{dt}\Big)\varphi_k^* - \Big(\frac{d}{dt}\varphi_k\Big)\varphi_k^* = i. \tag{46}$$

They must also obey the matching conditions at the end of inflation/beginning of radtion,

$$\begin{aligned} \varphi_k\Big|_{t=-H_I^{-1}} &= \varphi_k^{rad}\Big|_{t=-H_I^{-1}} \\ \frac{d\varphi_k}{dt}\Big|_{t=-H_I^{-1}} &= \frac{d\varphi_k^{rad}}{dt}\Big|_{t=-H_I^{-1}}. \end{aligned} \tag{47}$$

With all these conditions we can determine the exact mode functions, φ and φ^*. Once they are determined we can define the power spectrum $\Delta_s^2(t,k)$ as follows [5],

$$\langle 0|\hat{\phi}(\vec{x},t)^2|o\rangle = \frac{1}{a^2}\int \frac{d^3k}{(2\pi)^3}|\varphi_k|^2 \equiv \int \frac{dk}{k}\Delta_s^2(t,k) \tag{48}$$

It can also be shown that during inflation the scalar and tensor perturbations can be calculated in terms of the mode functions,

$$\Delta_s^2(t,k) = \frac{k^3}{8\pi^2\epsilon M_P^2}|\varphi(t,k)|^2 \tag{49}$$

And that the tensor spectrum relates to the scalar spectrum as follows,

$$\Delta_t^2(t,k) = 16\epsilon\Delta_s^2. \tag{50}$$

Now, in the slow roll approximation, the mode functions on super-Hubble scales can be approximated by[5],

$$|\varphi(t,k)|^2 \simeq \frac{H_*^2}{(2k^3)}\Big(\frac{k}{aH_*}\Big)^{n_s-1}. \tag{51}$$

Astronomers usually parametrise the observed spectra as follows,

$$\begin{aligned}\Delta_s^2(k) &= \Delta_s^2(k_*)\Big(\frac{k}{k_*}\Big)^{n_s-1}\\ \Delta_t^2(k) &= \Delta_t^2(k_*)\Big(\frac{k}{k_*}\Big)^{n_t}\end{aligned} \tag{52}$$

where the * denotes the time when the perturbation crosses the Hubble radius and thus becoming super-Hubble during inflation. So $\Delta_s^2(k_*)$ and $\Delta_t^2(k_*)$ are the amplitudes of the spectra evaluated at $k = k_* = aH_*$. Thus,

$$\begin{aligned}\Delta_{s*}^2(k) &= \frac{H_*^2}{8\pi^2\epsilon M_P^2}\\ \Delta_{t*}^2(k) &= \frac{2H_*^2}{M_P^2}\end{aligned} \tag{53}$$

Furthermore n_s and n_t are the spectral indices, which can be determined by variation of Δ_s^2 and Δ_t^2 with respect to k at k_* as follows[1]:

$$\begin{aligned}n_s &= 1 + \lim_{k\to k_*}\frac{\ln[\Delta_s^2(k)/\Delta_s^2(k_*)}{\ln[k/k_*]}\\ &= 1 + \lim_{k\to k_*}\frac{\ln[\Delta_s^2(k)] - \ln[\Delta_s^2(k_*)}{\ln[k] - \ln[k_*]}\\ &= 1 + \lim_{k\to k_*}\frac{d\ln[\Delta_s^2(k)]}{d\ln[k]}\\ &= \frac{dt}{d\ln(Ha)}\frac{d\ln[\Delta_s^2(k_*)]}{dt}\\ &\simeq 1 - 2\epsilon - \eta\end{aligned} \tag{54}$$

$$
\begin{aligned}
n_t &= \lim_{k\to k_*} \frac{\ln[\Delta_t^2(k)/\Delta_t^2(k_*)]}{\ln[k/k_*]} \\
&= \frac{dt}{d\ln(Ha)} \frac{d\ln[\Delta_t^2(k_*)]}{dt} \\
&\simeq -2\epsilon
\end{aligned} \tag{55}
$$

Now we have derived functions for the spectral indices which we can plot against the the parameters m and ξ_4 for different numbers of e-folds. The spectral indices are measurable quantities and thus this way the model can be compared with our observations. This is done in section 4.

3.5 Quantum tunneling

As you can see in Figure (1) the effective potential $V_E(\phi_E)$ has a local minimum for $\phi_E = 0$ and a local maximum for some $\phi_E > 0$, when we choose $\xi_2 = 0$, $\xi_4 < 0$ and $m > 0$. This means that the field will not start rolling downhill from $\phi_E = 0$ simply because it has to go uphill first. In this case a phenomenon called *quantum tunneling* may occur. Quantum tunneling occurs when a particle (or in our case: field) transitions through a classically forbidden energy state. The field then tunnels from the local minimum at $\phi_E = 0$ through the energy barrier to the other side. At this point the field feels the hill-top-like potential and starts rolling down and entering inflation as discussed before. The state of the field at the local minimum is called *false vacuum*, where the state of the field at a lower energy is called *true vacuum*. The quantum tunneling happens in a way where bubbles of true vacuum nucleate in the false vacuum. These bubbles then start to grow to eventually fill the universe. We will now derive a way to find the probability that quantum tunneling may occur.

First let us take a look at the potential V_E (21). The field will be able to "roll down the hill" and enter inflation when it has somehow passed over the maximum $\big($Figure (1)$\big)$. Let us denote ϕ_{Ec} as the value for ϕ_E where the the potential has this local maximum. Then let us consider the quantum mechanical wave function $\psi(\phi_E(\vec{x}))$ for the field. The rate of tunneling of the field to a potential where $\phi_E > \phi_{Ec}$ or $-\phi_E < -\phi_{Ec}$ (due to symmetry), and be able to enter inflation is,

$$
\begin{aligned}
P &= \int_{-\infty}^{-\phi_{Ec}} d\phi_E \ \psi(\phi_E) + \int_{\phi_{Ec}}^{\infty} d\phi_E \ \psi(\phi_E) \\
&= 2\int_{\phi_{Ec}}^{\infty} d\phi_E \ \psi(\phi_E).
\end{aligned} \tag{56}
$$

Which is given in units of Hubble time. In general the wave function $\psi(\phi_E(\vec{x}))$ is complicated. However, we are interested in the infrared sector of the the theory. For the infrared sector we can approximate the wave function by that of the zero mode of the decomposed field, $\varphi(t, 0)$, which is spatially homogenous. The wave function of the zero mode can be approximated by a Gaussian. The gaussian wave function looks like,

$$\psi(\phi) = A \exp\left[\frac{-\phi_E^2}{2\sigma}\right], \tag{57}$$

where A is the normalisation constant and σ is the width of the Gaussian wave function. Integrating (57) from $-\infty$ to ∞, gives $\sqrt{2\sigma\pi}$ (for $\mathrm{Re}(\sigma) > 0$). Thus,

$$A = \frac{1}{\sqrt{2\sigma\pi}} \tag{58}$$

In quantum field theory the general propagator is the probability amplitude of a particle traveling from one space time point to another. This propagator is defined by [7],

$$\imath\Delta(\vec{x};\vec{x}') = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D-1}{2}+\nu_D)\Gamma(\frac{D-1}{2}-\nu_D)}{\Gamma(\frac{D}{2})} \, {}_2F_1\left(\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; 1-\frac{y(\vec{x};\vec{x}')}{4}\right) \tag{59}$$

where $D = 4$ is the number of dimensions, ${}_2F_1$ is the hypergeometric function, $\nu_D = \sqrt{(\frac{D-1}{2})^2 - \frac{m^2}{H^2}}$ and

$$y(x;x') = a(\eta)a(\eta')H^2[-(|\eta-\eta'| - i\delta)^2 + ||\vec{x}-\vec{x}'||]. \tag{60}$$

The width of the the wave function σ is given by the propagator at coincedence, i.e. when $\vec{x} = \vec{x}'$ and $y(\vec{x};\vec{x}') = 0$. This gives,

$$\begin{aligned}\imath\Delta(\vec{x};\vec{x}) &= \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D-1}{2}+\nu_D)\Gamma(\frac{D-1}{2}-\nu_D)}{\Gamma(\frac{D}{2})} \, {}_2F_1\left(\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; 1\right) \\ &= \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{\Gamma(1-\frac{1}{2})\Gamma(\frac{D-1}{2}+\nu_D)\Gamma(\frac{D-1}{2}-\nu_D)}{\Gamma(\frac{1}{2}-\nu_D)\Gamma(\frac{1}{2}+\nu_D)}\end{aligned} \tag{61}$$

However, the function (61) is divergent for $D = 4$. And because $m^2 \ll H^2$, we can expand this function in powers of $\frac{m^2}{H^2}$. Which gives [7],

$$\begin{aligned}\sigma = \imath\Delta(\vec{x};\vec{x}) = \frac{H^{D-2}\Gamma(\frac{D-1}{2}}{(4\pi)^{D/2}} &\left[\psi\Big(\frac{D}{2} - \psi(D-1) - \psi(1-D) - \psi\Big(1-\frac{D}{2}\Big) - \gamma_E + \frac{1}{D-1}\right] \\ &+ \frac{\Gamma(\frac{D+1}{2})}{2\pi^{(D+1)/2}} \frac{H^D}{m^2} + \mathcal{O}\Big(\frac{m^2}{H^2}\Big),\end{aligned} \tag{62}$$

where the first part is again divergent for $D = 4$. We use the second term, assuming that this is the physical part. So we have,

$$\begin{aligned}\sigma &= \frac{\Gamma(\frac{5}{2})}{2\pi^{(5)/2}} \frac{H^4}{m^2} \\ &= \frac{3}{8\pi^2} \frac{H^4}{m^2} \\ &= \frac{\Lambda^2}{24\pi^2 m^2}\end{aligned} \tag{63}$$

and

$$A = \sqrt{\frac{12\pi m^2}{\Lambda^2}}. \tag{64}$$

Now we calculate the value of ϕ_{Ec},

$$\begin{aligned} V'(\phi_E) &= \frac{G'F - 2GF'}{F^3} = 0 \\ &\Rightarrow G'F - 2GF' = 0 \\ 2m^2\phi_E(1 - \xi_4\frac{\phi_E^4}{M_P^4}) - 2(\Lambda M_P^2 + m^2\phi_E^2)(-4\xi_4\frac{\phi_E^3}{M_P^4}) &= 0 \\ 2m^2 + 8\Lambda\xi_4\frac{\phi_E^2}{M_P^2} + 6m^2\xi_4\frac{\phi_E^4}{M_P^2} &= 0 \end{aligned} \tag{65}$$

$$\begin{aligned} \rightarrow \phi_{Ec}^2 &= \frac{-8\frac{\Lambda\xi_4}{M_P^2} + \sqrt{64\Lambda^2\xi_4^2/M_P^4 - 48m^4\xi_4/M_P^2}}{12m^2\xi_4/M_P^4} \\ &= M_P^2\sqrt{\Big(\frac{2\Lambda}{3m^2}\Big)^2 - \frac{1}{3\xi_4}} - \frac{2}{3}\frac{\Lambda M_P^2}{m^2}. \end{aligned} \tag{66}$$

And so,

$$\begin{aligned} \frac{\phi_{Ec}^2}{\sigma} &= \frac{M_P^2\sqrt{\Big(\frac{2\Lambda}{3m^2}\Big)^2 - \frac{1}{3\xi_4}} - \frac{2}{3}\frac{\Lambda M_P^2}{m^2}}{\frac{\Lambda^2}{24\pi^2m^2}} \\ &= 16\pi^2\frac{M_P^2}{\Lambda}\sqrt{1 - \frac{3m^4}{4\xi_4\Lambda^2}} - 1 \\ &\simeq 16\pi^2\frac{M_P^2}{\Lambda}\Bigg(- \frac{3m^4}{8\xi_4\Lambda^2}\Bigg), \quad \text{for } \Big|\frac{3m^4}{4\xi_4\Lambda^2}\Big| < 1 \end{aligned} \tag{67}$$

Thus,

$$\psi(\phi_E) = \sqrt{\frac{12\pi m^2}{\Lambda^2}} \exp\Big[\frac{3\pi^2M_P^2m^4}{\Lambda^3\xi_4}\Big(\frac{\phi_E}{\phi_{Ec}}\Big)^2\Big]. \tag{68}$$

Which gives,

$$P = 2\sqrt{\frac{12\pi m^2}{\Lambda^2}} \int_{\phi_{Ec}}^{\infty} \exp\Big[\frac{3\pi^2M_P^2m^4}{\Lambda^3\xi_4}\Big(\frac{\phi_E}{\phi_{Ec}}\Big)^2\Big]. \tag{69}$$

This resembles the rate of tunneling in units of the Hubble constant and thus can be interpreted as the probability that the field tunnels in Hubble time. As a consequence this tunneling rate does not have to be very large. The field just has to tunnel at least once anywhere in the universe and inflation will start. For example, if you look at 10^5 Hubble volumes, then even if $P = 10^{-5}$ it will tunnel on average once per Hubble time somewhere in the universe. Then inflation will start, can last 65 e-foldings and we can be in that part of the universe.

4 Results

Now that we have derived all the tools that we need to test our model we will present the results. First let us take a look at the slow roll parameter ϵ_E. We use (32) to plot ϵ_E in Figure 2.

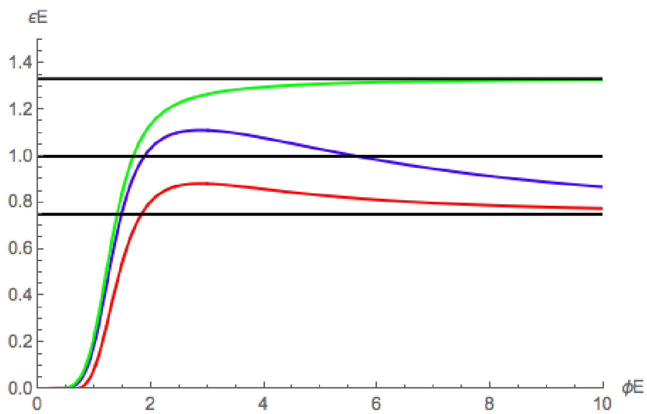

Figure 2: A plot of ϵ_E as a function of the field value ϕ_J for fixed $\xi_4 = -0.1$. Because the dependence on ξ_4 is minimal. Green: $m^2 = 0$, Blue: $m^2 = 0.04$, Red: $m^2 = 0.16$. Als the three black horizontal lines are at respectively, 3/4, 1, 4/3.

Here we see that the for $m^2 = 0$, ϵ_E converges to 4/3. As for when $m^2 > 0$, ϵ_E will converge to 3/4 as calculated before. When m^2 is large ϵ_E will never cross 1 and thus inflation will never end. However if $m^2 > 0$ is small enough it will end inflation but eventually the universe will enter an epoch where $\epsilon_E = 3/4$. Which is not a satisfying outcome.
When we use equation (39), we can plot ϵ_E as a function of the number of e-folds, as is done in the figures 3 and 4.

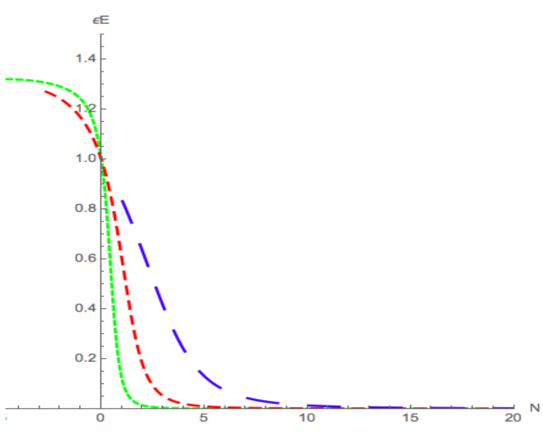

Figure 3: ϵ_E as a function of the number of E-folds for fixed $\frac{m}{\sqrt{\Lambda}} = 0.01$. Green: $\xi_4 = -0.1$, Red: $\xi_4 = -0.01$ and Blue: $\xi_4 = -0.001$.

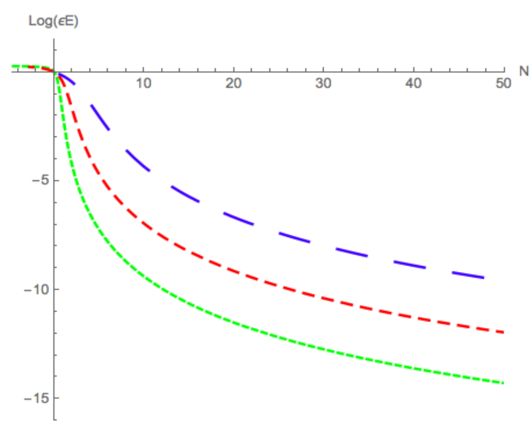

Figure 4: Log(ϵ_E) as a function of the number of E-foldsfor fixed $\frac{m}{\sqrt{\Lambda}} = 0.01$. Green: $\xi_4 = -0.1$, Red: $\xi_4 = -0.01$ and Blue: $\xi_4 = -0.001$.

Because the number of e-folds is measured from the end of inflation the plot is in the opposite direction of figure 2. However we can see that the preferable amount of 60 e-folds can easily be met.
Using equation (32) we can define the end of inflation ϕ_{Ee} as a function of m^2 and ξ_4. Together with equations (39) and (54) we can plot the scalar power spectrum as a function of m and ξ_4 for different amounts of e-folds. This is done in Figure 5 and 6.

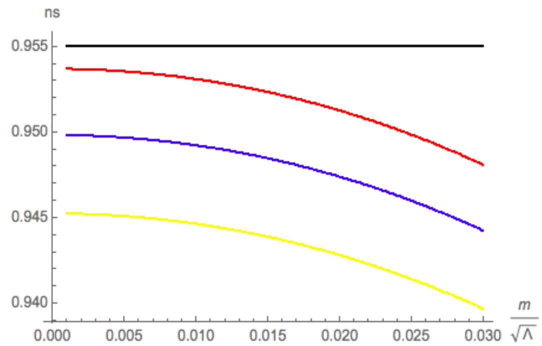

Figure 5: Spectral index n_s as a function of m, for fixed value $\xi_4 = -0.1$. Red: 65 e-fols, Blue: 60 e-folds, Yellow: 55 e-folds.

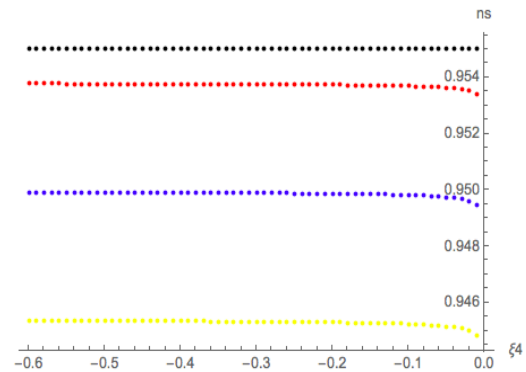

Figure 6: Spectral index n_s as a function of ξ_4, for fixed value $\frac{m}{\sqrt{\Lambda}} = 0.01$. Red: 65 e-folds, Blue: 60 e-folds, Yellow: 55 e-folds.

As you can see the spectrum peaks for $m^2 = 0$. Which indicates that the mass term would be very small or doesn't exist at all. For small mass terms the plot of the spectrum as a function of ξ_4 doesn't peak at all. We chose small $\frac{m}{\sqrt{\Lambda}}$ becasue of the peak in Figure 5. Furthermore, the curve for 60 e-folds is smaller than the central value for n_s obtained by the Planck Collaboration [8], with about 2.2σ. Even the curve for 65 e-folds is below the central value with aobut 1.5σ. While for standard cosmologies the number of e-folds would be at most 62 [1].

Using equation (53) and using $H^2 = \frac{\Lambda}{3}$, we can write,

$$\Lambda = 48\pi^2 \epsilon_E M_P^2 \Delta_{s*}^2, \tag{70}$$

where the COBE normalisation constrains [5],

$$\Delta_{s*} = (2.20 \pm 0.08) * 10^{-9}. \tag{71}$$

And we can approximate $48\pi^2\epsilon_E \sim 1$ before tunneling when the slow roll hasn't started yet. Applying this we can make a plot of the probability (69) as a function of m, as is done in figure 7.

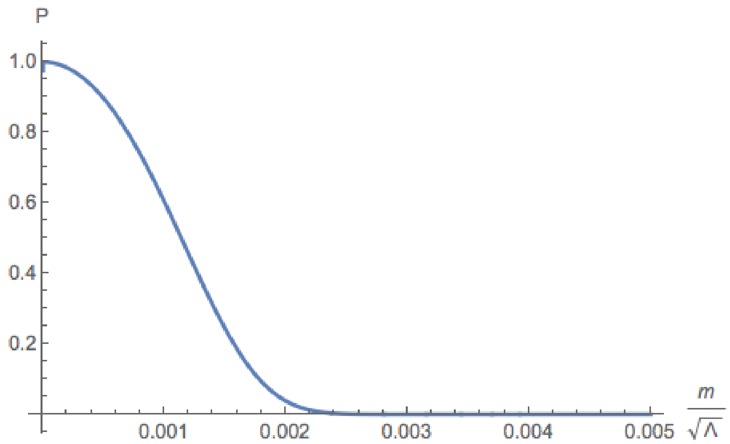

Figure 7: A plot of the probability that the the field will tunnel as a function of m, for fixed $\xi_4 = -0.1$.

We can see in figure 7 that the tunneling rate peaks, for small values of m. However it is still possible for the field to tunnel for larger values of m, as the field only needs to tunnel once to start inflation. When we zoom in (Figure 8) we see that for $\frac{m}{\sqrt{\Lambda}} = 0.0037$ the tunneling rate decreases so much that you must look at more than 10^{12} Hubble volumes.

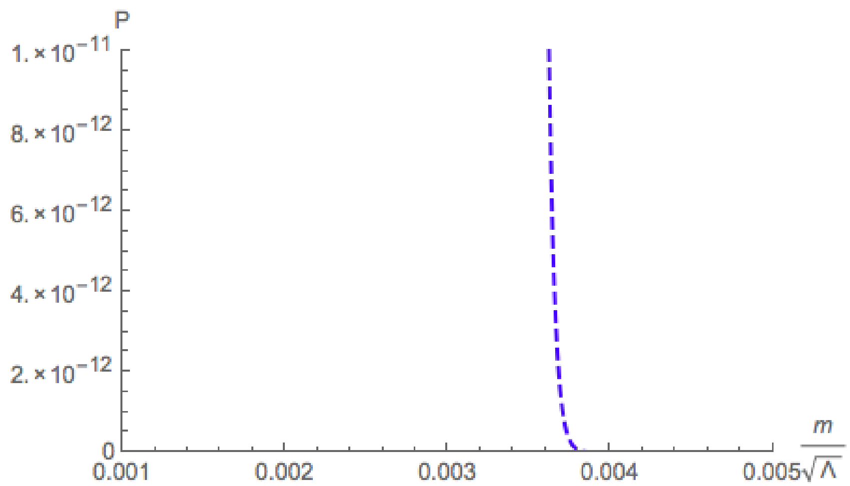

Figure 8: A plot of the tunneling rate as a function of $\frac{m}{\sqrt{\Lambda}}$, for fixed $\xi_4 = -0.1$.

5 Discussion

In this thesis we analysed an inflationary model. In the model the inflation was driven by a cosmological constant and a nonminimally coupled scalar field with a mass term. The model was inspired by previous work [1] where the model did not contain a mass term. The original model had some tuning problems of initial conditions. A positive

mass term creates a local minimum of the potential and thus makes it more likely that the field will start there. However, in this work, it shows that a positive mass term had undesirable consequences. The spectral index decreases, hence making agreement with the CMB data more tenuous. Even for high numbers of e-folds (N=65) the spectral index n_s is still 1.5σ below the central values, thus representing a tension. Furthermore, after the introduction of the mass term, the slow roll parameter ϵ_E converges to 3/4 for $m^2 > 0$ and thus enter an epoch of some strange kind. When we look at the results for quantum tunneling, we see that the plot for the tunneling rate showed a peak for $m = 0$, indicating that the smaller the mass term, the higher the tunneling rate and the more likely it would be that the field entered slow roll inflation. However, the field only has to tunnel once somewhere (anywhere), but as is shown in Figure 8, after $\frac{m}{\sqrt{\Lambda}} = 0.004$ you would need to look at a tremendous amount of Hubble volumes.
A way to still retain the mass term, would be to propose that a phase transition occurs, in which m^2 turns from positive to negative, which can either start or end inflation. This can be achieved for example if m is not just a constant, but instead is composed of two parts, $m^2 = m_0^2 + \xi_\psi \psi^2$, where $\xi_\psi > 0$ is a coupling constant and ψ is a field that couples to ϕ. At early times $\psi > 0$, giving a positive m^2, while at late times $\psi = 0$, resulting in $m^2 \leq 0$. This way it will have the same predictions as the original model, but will require less fine tuning. Perhaps for future work.

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