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Homotopy theory of presheaves

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Erik te Rietmole

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Supervisors:

Prof. Dr. G. Heuts
Utrecht University

Prof. Dr. F.L.M. Meier
Utrecht University

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Abstract

Equipping a category with a model structure, or with the structure of a category of fibrant objects, allows one to perform homotopy theoretic arguments in the given category. In this thesis the classical Kan-Quillen model structure on simplicial sets is generalized to (pre-)sheaves of simplicial sets on a site, following the work of Jardine. This structure has an important application, for it can used to describe sheaf cohomology in homotopy theoretic terms.

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1 Model Categories

Historically, the notion of homotopy was used to study maps between topological spaces. Beside that, homotopies occurred in the context of simplicial sets, cubical sets and chain complexes. In the book [14], Quillen first introduced a more abstract approach to homotopy in order to capture the similarities between these forms of homotopy. He formulated axioms for general categories that give criteria for a class of arrows to behave like the class of homotopy equivalences for the known examples. In fact, he used three classes of morphisms to formulate these axioms, which are called fibrations, cofibrations and weak equivalences. A category that satisfies these axioms for these fixed classes of morphisms is called a model category.

A few years after the publication of this book, Brown wrote his P.h.D. thesis [1] under the supervision of Quillen on a more flexible structure on categories, namely that of a category of fibrant object. This structure involves only two classes of maps; fibrations and weak equivalences. Both Brown's and Quillen's structure admit homotopy categories. These are obtained by formally inverting weak equivalences.

This chapter roughly follows Chapter 7 of [5] and Part I and II of [1]. We will review the basics of model structures and categories of fibrant objects. We will see that every model structure induces a category of fibrant objects. Moreover, we will show that both structures have an well-behaved induced homotopy category. Finally, we discuss a very useful tool to induce a model structure onto another category along an adjunction. This technique is known as a transference of model structure.

1.1 Categorical Axioms

Definition 1.1.1. A model category is a category $\mathcal{E}$ together with a choice of three classes of morphisms in $\mathcal{E}$ called fibrations, cofibrations and weak equivalences. These classes must satisfy:

M1) The category $\mathcal{E}$ has all small limits and colimits.

M2) If two out of three morhpisms $f : X \to Y$, $g : Y \to Z$ and $gf : X \to Z$ are weak equivalences, then so is the third.

M3) The classes of fibrations, cofibrations and weak equivalences are closed under retracts.

M4) For any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array} \tag{1}$$

where i is a cofibration and f a fibration, and at least one of them is also a weak equivalence, there exists a lift $l : B \to Y$ making the diagram commute.

M5) Any morphism $f : X \to Y$ can be factored as a composition of $p : Z \to Y$ after $i : X \to Z$ where i is a cofibration and a weak equivalence and p is a fibration, or alternatively, where i is a cofibration and p is a fibration and a weak equivalence.

If a category $\mathcal{E}$ satisfies these axioms for some choice fibrations, cofibrations and weak equivalences, we say that $\mathcal{E}$ admits a model structure or simply that $\mathcal{E}$ is a model category.

Remark 1.1.2. Axiom **M3)** refers to retracts in the arrow category of $\mathcal{E}$. Explicitly, $f : X \to Y$ is a retract of $g : Z \to W$, if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & W \arrow[r] & Y \end{tikzcd} \tag{2}$$

such that the composites of the upper and lower horizontal arrows are the identities on X and Y, respectively.

We adopt the convention that a morphism is called a trivial fibration if it is a fibration as well as a weak equivalence. Likewise, we say a morphism is a trivial cofibration if it is a cofibration as well as a weak equivalence. Some authors refer to these as acyclic fibrations and cofibrations.

Sometimes it is convenient to formulate axiom **M4)** in terms of lifting properties. We say that a map $f : X \to Y$ has the right lifting property with respect to a class of morphisms C if for any commutative diagram of the form (1) with $i \in C$, there exists a lift. In the same fashion, we say that an arrow $i : A \to B$ has the left lifting property with respect to a class of morphisms C', if for every commutative diagram (1) with $f \in C'$, there exists a lift.

Remark 1.1.3. For a model category $\mathcal{E}$ axiom **M4)** describes a necessary condition for a map to be a (trivial) fibration or (trivial) cofibration. Actually, a retract argument shows that this condition is also sufficient. Indeed, suppose that $f : X \to Z$ has the right lifting property with respect to all trivial cofibrations. Then factor f as a trivial cofibration j followed by a fibration p. By assumption, there exists a lift

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "j"] & X \arrow[d, "f"] \\ Y \arrow[r, "p"] \arrow[ur, dashed, "l"] & Z. \end{tikzcd} \tag{3}$$

Using l, we see that f is a retract of p.

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"] & Y \arrow[r, "l"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Z \arrow[r, "\mathrm{id}"] & Z \arrow[r, "\mathrm{id}"] & Z \end{tikzcd} \tag{4}$$

This shows that f is indeed a fibration. To prove that trivial fibrations are stable under pullback involves showing that these maps have the right lifting property with respect to cofibrations and then constructing a similar retract. The proof for (trivial) cofibrations is dual.

There are many examples of model structures. A few of the classical examples are given below. In further chapters we consider other categories that admits a model structure, where we will verify the axioms.

Example 1.1.4. 1. The category of topological spaces admits a model structure. A map $f : X \to Y$ is a called a Serre fibration if every commutative diagram of the form

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, hook, "{(\mathrm{id},0)}"] & X \arrow[d, "f"] \\ D^n \times I \arrow[r] & Y \end{tikzcd} \tag{5}$$

admits a lift. In this setting, a map $f : X \to Y$ is a weak equivalence if it induces an isomorphism on the homotopy groups. That is, $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism and for every choice of base point $x \in X$, the induced maps $f_* : \pi_n(X, x) \to \pi_n(Y, f(y))$ are isomorphisms. The cofibrations are retracts of cellular extensions.

2. The category of simplicial sets admits the so-called Kan-Quillen model structure. A map $f : X \to Y$ is called a Kan fibration if every diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd} \tag{6}$$

with $0 \leq k \leq n$, admits a lift. A map is called a weak equivalence if it induces an isomorphism on the simplicial homotopy groups. The class of cofibrations is exactly the class of monomorphisms.

3. If $\mathcal{E}$ is a category that is equipped with a model structure, then we get more examples of model categories by considering slice categories $\mathcal{E}/Z$ for objects $Z \in \mathcal{E}$. Say that a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd} \tag{7}$$

in the slice category is a fibration, cofibration or weak equivalence, if f is a fibration, cofibration or weak equivalence, respectively, in $\mathcal{E}$. It is relatively straight forward to check that the axioms from Definition 1.1.1 are satisfied in this case.

△

Proposition 1.1.5. *In any model category fibrations and trivial fibrations are stable under pullback. Dually, cofibrations and trivial cofibrations are stable under pushout.*

Proof. Let $f : X \to Y$ be a fibration and $g : Z \to Y$ be any map. To show that the pullback of f along g is again a fibration, we will show that it has the right lifting property w.r.t. the class of trivial cofibrations. Then the result follows from Remark 1.1.3. Therefore, consider the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & Z \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & Z \arrow[r, "g"] & Y \end{tikzcd} \tag{8}$$

where i is a trivial cofibration. Applying axiom **M4)** to the outer square gives us a lift $B \to X$. By the universal property of the pullback, we obtain a lift $B \to Z \times_Y X$. □

Definition 1.1.6. An object X in a model category $\mathcal{E}$ is said to be fibrant, if the unique map $X \to 1$ from X to the terminal object is a fibration. Dually, X is said to be cofibrant, if the unique map $0 \to X$ from the initial object to X is a cofibration.

The factorization axiom **M5)** provides a way to produce fibrant and cofibrant objects. For any $X \in \mathcal{E}$ consider the factorizations

$$X \to X_f \to 1$$
$$0 \to X_c \to X$$

where the former is a trivial fibration followed by a cofibration and the latter is a cofibration followed by a trivial fibration. We say that X_f is a fibrant replacement for X and X_c a cofibrant replacement for X. This shows that every object is fibrant and cofibrant, up to weak equivalence.

Beside model categories, we also are also interested in categories of fibrant objects.

Definition 1.1.7. A category of fibrant objects is a category $\mathcal{E}$ together with a choice of two classes of morphisms in $\mathcal{E}$ called fibrations and weak equivalences. These classes must satisfy:

N1) If two out of three of the morphisms $f : X \to Y$, $g : Y \to Z$ and $gf : X \to Z$ are weak equivalences, then so is the third.

N2) Fibrations are closed under composition and any isomorphism is a fibration.

N3) If $f : X \to Y$ is a fibration or trivial fibration, then for any map $B \to Y$ the pullback

$$\begin{tikzcd} B \times_Y X \arrow[r] \arrow[d, "f^*"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd} \tag{9}$$

exists and f^* is again a (trivial) fibration.

N4) Every object $X \in \mathcal{E}$ admits a path object. That is, for any X there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[dr, "\Delta"] & X^I \arrow[d, "(d_0, d_1)"] \\ & X \times X \end{tikzcd} \tag{10}$$

where Δ denotes the diagonal map, s is a weak equivalence and (d_0, d_1) is a fibration.

N5) Every object is fibrant, i.e. every map $X \to 1$ is a fibration.

Remark 1.1.8. Categories of fibrant objects are in many ways a generalization of model categories. Namely, categories of fibrant objects do not have a class of cofibrations and therefore lack the lifting property **M5)**. Concretely, given a model category $\mathcal{E}$, then the classes of fibrations and weak equivalences satisfy **N1)** to **N4)**. However, not every object in a model category needs to be fibrant. We conclude that every model category contains a largest subcategory that naturally has the structure of a category of fibrant objects, for the inherited classes of fibrations and weak equivalences. This subcategory consists of the fibrant objects of $\mathcal{E}$, which explains the name.

Lemma 1.1.9 (Brown's Lemma)**.** *Let $\mathcal{E}$ be a category of fibrant objects and let $f : X \to Y$ be a map in $\mathcal{E}$. Then f can be factored as a composite $f = gi$, where g is a fibration, i is split monic and its left inverse is a trivial fibration.*

Proof. Let (Y^I, s, d_0, d_1) be a path object of Y. Consider the pullback square

$$\begin{tikzcd} Z \arrow[r, "p_1"] \arrow[d, "p_0"] & Y^I \arrow[d, "d_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{11}$$

Define $i : X \to Z$ as the unique map with components (id_X, sf) and define $g : Z \to Y$ as the composite $d_1 p_1$. Indeed, this makes i a split monomorphism with right inverse p_0. As p_0 is the pullback of a trivial fibration, it is a trivial fibration as well. To see that g is a fibration, consider the diagram

$$\begin{tikzcd} X \times_Y Y^I \arrow[r, "p_1"] \arrow[d, "(p_0, d_1 p_1)"] & Y^I \arrow[d, "(d_0, d_1)"] \\ X \times Y \arrow[r, "f \times \mathrm{id}"] \arrow[d, "\mathrm{id}"] & Y \times Y \\ X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & 1. \end{tikzcd} \tag{12}$$

The vertical composite $X \times_Y Y^I \to Y$ is the map g. Moreover, both squares are pullbacks. Hence the remark that the maps on the right are fibrations, concludes the proof. ☐

Corollary 1.1.10. *Let $F : \mathcal{E} \to \mathcal{E}'$ be a functor between categories of fibrant objects and suppose that F preserves trivial fibrations. Then F also preserves weak equivalences.*

Proof. Let $f : X \to Y$ be a weak equivalence in $\mathcal{E}$. Then Brown's Lemma (1.1.9) provides a factorization $f = gi$ where g is a local fibration and i is a right inverse to a trivial local fibration. It follows that g is a trivial local fibration by the two-out-of-three property of weak equivalences. Therefore, $F(i)$ and $F(g)$ are weak equivalences as well, showing that $F(f)$ is a weak equivalence. ☐

1.2 General Homotopy Categories

In the classical example of topological spaces, one first defines homotopies between maps of topological spaces and after that the homotopy equivalences. In the context of a category of fibrant objects $\mathcal{E}$, we will reverse this order. That is, we start with a class of maps that are supposed to be homotopy equivalences, namely the class of weak equivalences. Using weak equivalences as a foundation, we will define an abstract notion of homotopy. Likewise, homotopy categories will be defined using these weak equivalences. The main result of this section is stated as Theorem 1.2.11. This theorem gives a description of the homotopy category that involves a quotient by a certain homotopy relation. It is this result that validates the name "homotopy category", for without such a characterization it would be more appropriate to simply refer to it as the localization of $\mathcal{E}$ with respect to weak equivalences.

Definition 1.2.1. Let $\mathcal{E}$ be a category and let $W \subseteq \mathcal{E}_1$ be a class of arrows. Then the localization of the category $\mathcal{E}$ with respect to the class W is a category $\mathcal{E}[W^{-1}]$ together with a functor $\eta : \mathcal{E} \to \mathcal{E}[W^{-1}]$ such that η sends arrows in W to isomorphisms and η is universal with respect to this property. Explicitly, for any functor $F : \mathcal{E} \to \mathcal{D}$ that sends arrows in W to isomorphisms in $\mathcal{D}$, there exists a functor $G : \mathcal{E}[W^{-1}] \to \mathcal{D}$ that is unique (up to natural isomorphism) such that $G\eta$ and F are naturally isomorphic.

Informally, the localization of a category $\mathcal{E}$ is the smallest extension that makes arrows in W invertible.

Remark 1.2.2. Using the results from [3], any class $W \subseteq \mathcal{E}$ admits a localization, given that $\mathcal{E}$ is locally small. They are constructed as follows. Let G be the graph having vertices corresponding to the objects of $\mathcal{E}$. The set of edges of G is given by $\mathcal{E}_1 \cup W^{-1}$ where W^{-1} denotes the set of formal inverses to arrows in W. Let $P(\mathcal{E}, W)$ be the path category associated to G. Then the localization category $\mathcal{E}[W^{-1}]$ is defined as the quotient category obtained from $P(\mathcal{E}, W)$ with respect to the relations generated by:

- $\langle\rangle_X \sim \langle \mathrm{id}_X \rangle$ for every $X \in \mathcal{E}$,
- $\langle g, f \rangle \sim \langle gf \rangle$ for every pair of morphisms $f : X \to Y, g : Y \to Z$ in $P(\mathcal{E}, W)$,
- $\langle\rangle_X \sim \langle w^{-1}, w \rangle$ and $\langle\rangle_Y \sim \langle w, w^{-1} \rangle$ for every arrow $w : X \to Y$ in W.

However, there is no guaranty that $\mathcal{E}[W^{-1}]$ is locally small.

This allows us to define the homotopy category.

Definition 1.2.3. Let $\mathcal{E}$ be a model category or a category of fibrant objects, then the homotopy category $\mathrm{Ho}(\mathcal{E})$ of $\mathcal{E}$ is defined as the localization of $\mathcal{E}$ with respect to the class of weak equivalences. Depending on the context, the homotopy category is sometimes called the derived category.

In many cases, the construction given in 1.2.2 is impractical to work with. Therefore, we will discuss other descriptions of the homotopy category. Depending on whether $\mathcal{E}$ is a model category or a category of fibrant objects, we obtain different descriptions of $\mathrm{Ho}(\mathcal{E})$. From now on we assume $\mathcal{E}$ to be a category of fibrant objects.

Definition 1.2.4. Two arrows $f, g \in \mathrm{Hom}_{\mathcal{E}}(X, Y)$ are said to be homotopic if there exists a commutative diagram of the form

$$\begin{tikzcd} & Y^I \arrow[d, "{(d_0,d_1)}"] \\ X \arrow[ur, "h"] \arrow[r, "{(f,g)}"'] & Y \times Y \end{tikzcd} \tag{13}$$

where Y^I is a path object of Y. Furthermore, h is said to be a homotopy from f to g, which we denote as $f \sim g$.

Proposition 1.2.5. *The homotopy relation on* $\mathrm{Hom}_{\mathcal{E}}(X, Y)$ *defined above is an equivalence relation.*

Proof. Reflexivity and symmetry are trivial. For transitivity, let $f_1, f_2, f_3 \in \mathrm{Hom}_{\mathcal{E}}(X, Y)$ be arrows and suppose there exists a homotopy h from f_1 to f_2 for a path object (Y^I, s, d_0, d_1) and there exists a homotopy h' from f_2 to f_3 for a path object $(Y^{I'}, s', d_0', d_1')$. Form the pullback

$$\begin{tikzcd} Y^{I''} \arrow[r, "e_1"] \arrow[d, "e_0"] & Y^{I'} \arrow[d, "d_0'"] \\ Y^I \arrow[r, "d_1"] & Y \end{tikzcd} \tag{14}$$

and observe that every arrow in the diagram is a trivial fibration. Then we obtain a new path object for Y given by

$$\begin{tikzcd} & Y^{I''} \arrow[d, "{(d_1e_0,e_1d_0')}"] \\ Y \arrow[ur, "s''"] \arrow[r, "\Delta"'] & Y \times Y \end{tikzcd} \tag{15}$$

where s'' is the unique map such that $s = e_0 s''$ and $s' = e_1 s''$. Let h'' be the unique map such that $h = e_0 h''$ and $h' = e_1 h''$, then h'' is a homotopy from f_1 to f_3 for the path object $(Y^{I''}, s'', d_1 e_0, e_1 d_0')$. $\square$

Although the homotopy relation is an equivalence relation, the category obtained from $\mathcal{E}$ by quotienting out by this relation is not in general well-defined. Namely, a homotopy between maps $f, g : X \to Y$ does not induce a homotopy $nf \sim ng$ for arbitrary maps $n : Y \to Z$. To remedy this, we will consider a different relation. First, we need some technical results.

Lemma 1.2.6. *Let* $n : Y \to Z$ *be an arrow,* (Y^I, s, d_0, d_1) *be a path object for* Y *and let* (Z^I, s', d_0', d_1') *be a path object for* Z*. Then there exists another path object* $(Y^{I''}, s'', d_0'', d_1'')$ *for* Y*, a trivial fibration* $t : Y^{I''} \to Y^I$ *and a map* $\bar{n} : Y^{I''} \to Z^I$ *making the following diagram commute:*

$$\begin{tikzcd} & Y \arrow[dl, "s"'] \arrow[d, "s''"] \arrow[r, "n"] & Z \arrow[d, "s'"] \\ Y^I \arrow[dr, "{(d_0,d_1)}"'] & Y^{I''} \arrow[l, "t"'] \arrow[r, "\bar{n}"] \arrow[d, "{(d_0'',d_1'')}"] & Z^I \arrow[d, "{(d_0',d_1')}"] \\ & Y \times Y \arrow[r, "n \times n"] & Z \times Z. \end{tikzcd} \tag{16}$$

Proof. Consider the commutative square

$$\begin{tikzcd} Y \arrow[r, "s'u"] \arrow[d, "s"] & Z^I \arrow[d, "{(d_0',d_1')}"] \\ Y^I \arrow[r, "{(nd_0,nd_1)}"] & Z \times Z. \end{tikzcd} \tag{17}$$

Let W be the pullback of the span $Y^I \to Z \times Z \leftarrow Z^I$. Then there exists a unique map $Y \to W$ making triangles commute, to which we apply Brown's Lemma 1.1.9. Consequently, we obtain a diagram of the form

$$\begin{tikzcd} Y \arrow[dr, bend left, "s''"] \arrow[ddrrr, bend left, "s'u"] \arrow[dddrr, bend right, "s"'] & & & \\ & Y^{I''} \arrow[ul, bend left, "t'"] \arrow[dr, "g"] & & \\ & & W \arrow[r] \arrow[d] & Z^I \arrow[d, "{(d_0',d_1')}"] \\ & & Y^I \arrow[r, "{(nd_0,nd_1)}"] & Z \times Z. \end{tikzcd} \tag{18}$$

We define:

- t as the composite $Y^{I''} \xrightarrow{t'} Y \xrightarrow{s} Y^I$,
- $\bar{n}$ as the composite $Y^{I''} \xrightarrow{g} W \to Z^I$,
- (d_0'', d_1'') as the composite $Y^{I''} \xrightarrow{t} Y \xrightarrow{(d_0,d_1)} Y \times Y$.

This construction does make the diagram in the statement of this Lemma commute and it is straightforward to check that $Y^{I''}$ together with these maps give a path object of Y and that $\bar{n}$ is indeed a trivial fibration. $\square$

Lemma 1.2.7. *Given a diagram of the form $Y \xrightarrow{n} Z \xleftarrow{t} X$, then for any path object (Z^I, s, d_0, d_1) of Z the projection $Y \times_Z Z^I \times_Z X \to Y$ is a fibration. If t is a weak equivalence, then so is the projection.*

Proof. The projection $Y \times_Z Z^I \times_Z X \to Y$ is obtained from $Z^I \times_Z X \to Z$ as the pullback along $n : Y \to Z$. The argument summarized in diagram (12) from the proof of Lemma 1.1.9 may also be used here to prove that $Z^I \times_Z X \to Z$ is a fibration. If t is a weak equivalence, the second result follows from a chase in the pullback diagram for $Y \times_Z Z^I \times_Z X$. $\square$

Proposition 1.2.8. *a) Given any diagram of the form $Y \to Z \xleftarrow{t} X$ where t is a weak equivalence, there exists an extension*

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed, "t'"'] & X \arrow[d, "t"] \\ Y \arrow[r] & Z \end{tikzcd} \tag{19}$$

where t' is a weak equivalence, making the diagram commute up to homotopy.

b) Given a diagram of the form

$$\begin{tikzcd} W \arrow[r, dashed, "t'"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Z \end{tikzcd} \tag{20}$$

with t a weak equivalence such that there exists a homotopy $tf \simeq tg$, there exists a weak equivalence $t' : W \to X$ such that $ft' \simeq gt'$.

Proof. For part a), choose $W = Y \times_Z Z^I \times_Z X$ together with the canonical projections to Y and X. Then the result follows from Lemma 1.2.7.

For part b), let $h : X \to Z^I$ be a homotopy $tf \simeq tg$ for a path object (Z^I, s, d_0, d_1) of Z. Consider the pullback square

$$\begin{tikzcd} X \arrow[dr, dashed, "{(f,h,g)}"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "{(f,g)}"'] & & \\ & Y \times_Z Z^I \times_Z Y \arrow[r, "l"] \arrow[d] & Z^I \arrow[d, "{(d_0,d_1)}"] \\ & Y \times Y \arrow[r, "t\times t"] & Z \times Z. \end{tikzcd} \tag{21}$$

and let $(f,h,g) : X \to Y \times_Z Z^I \times_Z Y$ be the unique map making the diagram commute. Applying Lemma 1.2.7 to the span $Y \xrightarrow{t} Z \xleftarrow{t} Y$ tells us that both projections $Y \times_Z Z^I \times_Z Y \to Y$ are trivial fibrations. Therefore the map $(\mathrm{id}, st, \mathrm{id}) : Y \to Y \times_Z Z^I \times_Z Y$ is a weak equivalence. We may factorize this map using Brown's Lemma 1.1.9 into $Y \xrightarrow{s'} Y^I \xrightarrow{n} Y \times_Z Z^I \times_Z Y$ as a weak equivalence followed by a trivial fibration. This makes $(Y^I, s', d_0 ln, d_1 ln)$ into a path object of Y. Let us denote $A = Y \times_Z Z^I \times_Z Y$, then the pullback

$$\begin{tikzcd} X \times_A Y^I \arrow[r, "pr"] \arrow[d, "t'"] & Y^I \arrow[d, "n"] \\ X \arrow[r, "{(f,h,g)}"] & A, \end{tikzcd} \tag{22}$$

defines t'. By construction t' is a trivial fibration. Moreover, $pr : X \times_A Y^I \to Y^I$ is a homotopy from ft' to gt'. □

This allows us to define the category $\pi\mathcal{E}$ as a quotient category of $\mathcal{E}$. For any pair $f, g \in \mathrm{Hom}_{\mathcal{E}}(X, Y)$ we say that f and g are equivalent and write $f \sim g$, if there exists a weak equivalence $t : W \to X$ together with a homotopy $ft \simeq gt$.

Proposition 1.2.9. *The quotient category $\pi\mathcal{E}$ is well-defined.*

Proof. Clearly the relation $\sim$ is reflexive and symmetric. For transitivity, suppose that $f_1, f_2, f_3 : X \to Y$ are arrows and $t : W \to X$ and $s : W' \to X$ are both weak equivalences admitting homotopies $f_1 t \simeq f_2 t$ and $f_2 s \simeq f_3 s$. Then Proposition 1.2.8.a) applied to the span $W \xrightarrow{t} X \xleftarrow{s} W'$ gives weak equivalences $s' : W'' \to W$ and $t' : W'' \to W'$ such that $ts' = st'$. This reduces the proof to the transitivity of the homotopy relation, which is a consequence of Proposition 1.2.5.

Additionally, we need to verify that the relation is well-behaved with respect to composition of maps. Suppose that $f, g : X \to Y$ are maps, $t : W \to X$ is a weak equivalence such that $ft \simeq gt$, and $m : W' \to X$ is any map. Applying Proposition 1.2.5.a), we obtain a weak equivalence $t' : W'' \to W'$ such that $fmt' \simeq gmt'$ and thus $fm \sim gm$.

Let f, g and t be as before, and let $n : Y \to Z$ be arbitrary. Let $h : X \to Y^I$ be the given homotopy, then we use Lemma 1.2.6 to obtain diagram (16). Then $X \times_{Y^I} Y^{I''} \to Y'' \xrightarrow{\bar{n}}$ is a homotopy $nft \simeq ngt$, showing that $nf \sim ng$. □

Although $\pi\mathcal{E}$ does not inherit the structure of a category of fibrant objects from $\mathcal{E}$, it does inherit a class of weak equivalences W. Namely, if $f \sim g$ then f is a weak equivalence if and only if g is a weak equivalence. Therefore, it makes sense to speak of the localization $\pi\mathcal{E}[W^{-1}]$.

Lemma 1.2.10. *The categories $\mathcal{E}[W^{-1}]$ and $\pi\mathcal{E}[W^{-1}]$ are canonically isomorphic.*

Proof. Let $\eta : \mathcal{E} \to \mathcal{E}[W^{-1}]$ and $\eta' : \pi\mathcal{E} \to \pi\mathcal{E}[W^{-1}]$ denote the localization functors and let $q : \mathcal{E} \to \pi\mathcal{E}$ denote the quotient functor. Define $\gamma : \pi\mathcal{E} \to \mathcal{E}[W^{-1}]$ as the functor that is the identity on objects and sends the equivalence class $[f]$ to the path $\langle f \rangle$ of length one. This is a well-defined functor. Indeed, let $f, g : X \to Y$ be equivalent, then there is a weak equivalence $t : W \to X$ and a homotopy $h : X \to Y^I$ from ft to gt. As $\langle d_0 \rangle = \langle s^{-1} \rangle = \langle d_1 \rangle$, we get

$$\begin{aligned}
\gamma([f]) \circ \gamma([t]) &= \gamma([ft]) \\
&= \langle f, t \rangle \\
&= \langle d_0, h, t \rangle \\
&= \langle d_1, h, t \rangle \\
&= \langle g, t \rangle \\
&= \gamma([gt]) \\
&= \gamma([g]) \circ \gamma([t]).
\end{aligned}$$

It follows that $\gamma([f]) = \gamma([g])$, since $\gamma([t])$ is an isomorphism.

Consider the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "q"] \arrow[d, "\eta"] & \pi\mathcal{E} \arrow[d, "\eta'"] \arrow[dr, "\gamma"] & \\
\mathcal{E}[W^{-1}] \arrow[r, dashed, "L"] & \pi\mathcal{E}[W^{-1}] \arrow[r, dashed, "R"] & \mathcal{E}[W^{-1}].
\end{tikzcd} \tag{23}$$

The composite $\eta' q$ sends weak equivalences in $\mathcal{E}$ to isomorphisms, so by the universal property of localization we obtain a functor $L : \mathcal{E}[W^{-1}] \to \pi\mathcal{E}[W^{-1}]$ making the left square commute (up to natural isomorphism). Likewise, γ sends weak equivalences in $\pi\mathcal{E}$ to isomorphisms in $\mathcal{E}[W^{-1}]$, so there exists a functor $R : \pi\mathcal{E}[W^{-1}] \to \mathcal{E}[W^{-1}]$ making the right triangle commute (up to natural isomorphism). The uniqueness part of the universal property for η gives a natural isomorphism $RL \cong \mathrm{id}_{\mathcal{E}[w^{-1}]}$. Observe that $L\gamma q \cong L\eta \cong \eta' q$. Since q is surjective on objects and full, we have $L\gamma \cong \eta'$. Consequently, we may use the uniqueness property for η' to conclude that the exists a natural transformation $LR \cong \mathrm{id}_{\pi\mathcal{E}[w^{-1}]}$. □

As a direct consequence of Proposition 1.2.8, the category $\pi\mathcal{E}$ admits a calculus of right fractions. This gives us an explicit description of the localization category $\pi\mathcal{E}[W^{-1}]$, and thus a description of $\mathcal{E}[W^{-1}]$.

Theorem 1.2.11. *Let $\mathcal{E}$ be a category of fibrant objects. Then the homotopy category $\mathrm{Ho}(\mathcal{E})$ is a category that has the same objects as $\mathcal{E}$. A morphism in $\mathrm{Ho}(\mathcal{E})$ from X to Y is given by an equivalence class of roofs in $\pi\mathcal{E}$, i.e. an equivalence class of diagrams $X \xleftarrow{t} W \xrightarrow{f} Y$ with t and f morphisms in $\pi\mathcal{E}$ and t a weak equivalence. Two roofs (t^{-1}, f) and (s^{-1}, g) are said to be equivalent if there exists a commutative diagram of the form*

$$\begin{tikzcd}
& & W'' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
& W \arrow[dl, "t"'] \arrow[drrr, "f"] & & W' \arrow[dlll, "s"'] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd} \tag{24}$$

where u is a weak equivalence. The composition of roofs $(t^{-1}, f) : X \to Y$ and $(s^{-1}, g) : Y \to Z$ consists of a choice of a roof $(u^{-1}, h) : W \to W'$ such that

$$\begin{tikzcd}
& & W'' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
& W \arrow[dl, "t"'] \arrow[dr, "f"'] & & W' \arrow[dl, "s"] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd} \tag{25}$$

commutes. We denote the compostion by $((tu)^{-1}, gh) : X \to Z$.

Proof. See [3]. □

1.3 Homotopy Categories for Model Categories

The definition for homotopy category of a category of fibrant objects also makes sense for model categories. Indeed, let $\mathcal{E}$ be a model category. Then we define the homotopy category of $\mathcal{E}$, which we also denote by $\mathrm{Ho}(\mathcal{E})$, as the localization of $\mathcal{E}$ with respect to the class of weak equivalences. Let $\mathcal{E}_f \subseteq \mathcal{E}$ denote the full subcategory on fibrant objects. Recall that every object $X \in \mathcal{E}$ is connected by a weak equivalence to a fibrant object X_f, using fibrant replacements. Considering the diagram

$$\begin{tikzcd}
\mathcal{E}_f \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathrm{Ho}(\mathcal{E}_f) \arrow[r, hook] & \mathrm{Ho}(\mathcal{E}),
\end{tikzcd} \tag{26}$$

we can therefore conclude that the functor $\mathrm{Ho}(\mathcal{E}_f) \to \mathrm{Ho}(\mathcal{E})$ is essentially surjective, in addition to being an inclusion of a subcategory. This makes the functor into an equivalence of categories. As Theorem 1.2.11 gives an explicit definition of $\mathrm{Ho}(\mathcal{E}_f)$, this description of the homotopy category is also accurate for $\mathrm{Ho}(\mathcal{E})$.

As it stands, this would be a natural end to the section. However, the consideration above did not fully use the extra structure that a model category has over a category of fibrant objects. For this reason, we will give yet another description of the homotopy category, this time under the assumption that $\mathcal{E}$ is a model category. This description will be considerably shorter than the one given in Theorem 1.2.11 in addition to being convenient later on.

Definition 1.3.1. A cylinder object for $A \in \mathcal{E}$ is a commutative diagram of the form

$$\begin{tikzcd} & \mathrm{Cyl}(A) \arrow[d, "\varepsilon"] \\ A \amalg A \arrow[ur, "(i_0,i_1)"] \arrow[r, "\bigtriangledown"'] & A \end{tikzcd} \tag{27}$$

where $\bigtriangledown$ is the canonical map induced by the identity on A, (i_0, i_1) is a cofibration and ε is a trivial fibration.

In the context of model categories, path objects are defined as the dual version of cylinder objects. That is, a path object for $X \in \mathcal{E}$ is a commutative diagram of the form

$$\begin{tikzcd} & X^I \arrow[d, "(d_0,d_1)"] \\ X \arrow[ur, "s"] \arrow[r, "\bigtriangleup"] & X \times X \end{tikzcd} \tag{28}$$

where $\bigtriangleup$ is the diagonal map for X, s is a trivial cofibration and (d_0, d_1) is a fibration.

Sometimes we will use the term cylinder object or path object and mean the actual object $\mathrm{Cyl}(A)$ or P^I and leave the rest of the diagram implicit. If ε or s are just weak equivalences, instead of a trivial fibration or trivial cofibration, we call these diagrams weak cylinder objects or weak path objects, respectively.

Note that it directly follows from the two-out-of-three property that the maps i_0, i_1, d_0 and d_1 are weak equivalences. Moreover, every object in $\mathcal{E}$ admits a cylinder object as well as a path object by the factorization axiom.

Definition 1.3.2. A left homotopy from $f : A \to X$ to $g : A \to X$ is a map $h : \mathrm{Cyl}(A) \to X$ for some weak cylinder object $\mathrm{Cyl}(A)$ for A such that $hi_0 = f$ and $hi_1 = g$. Dually, a right homotopy from f to g is a map $h : A \to P^I$ for some weak path object P^I for P such that $d_0h = f$ and $d_1h = g$. We denote the corresponding relations by $f \simeq_L g$ and $f \simeq_R g$ and say that f is left, respectively right, homotopic to g.

Proposition 1.3.3. *Let $f, g : A \to X$ be maps with A cofibrant and X fibrant. Then the following statements hold:*

i) The existence of a left homotopy $h : \mathrm{Cyl}(A) \to X$ from f to g does not depend on the choice of a weak cylinder $\mathrm{Cyl}(A)$. A similar statement holds for right homotopy.

ii) We have $f \simeq_L g$ if and only $f \simeq_R g$.

iii) The relations $\simeq_L$ and $\simeq_R$ on the set $\mathrm{Hom}_{\mathcal{E}}(A, X)$ are equivalence relations.

Proof. For *i)*, we only consider the case of left homotopies, the argument for right homotopies is dual. Let $h : \mathrm{Cyl}(A) \to X$ be a left homotopy for a weak cylinder object

$$A \amalg A \xrightarrow{(i_0,i_1)} \mathrm{Cyl}(A) \xrightarrow{\varepsilon} A.$$

There exists a factorization of ε as a trivial cofibration followed by a trivial fibration, which we denote as $\mathrm{Cyl}(A) \xrightarrow{j} \widetilde{\mathrm{Cyl}}(A) \xrightarrow{\varepsilon'} A$. Then $A \amalg A \xrightarrow{(ji_0,ji_1)} \widetilde{\mathrm{Cyl}}(A) \xrightarrow{\varepsilon'} A$ is a cylinder object for A. Moreover, the left homotopy h induces a left homotopy

$$\begin{tikzcd} \mathrm{Cyl}(A) \arrow[r, "h"] \arrow[d, "j"'] & X \arrow[d] \\ \widetilde{\mathrm{Cyl}}(A) \arrow[r] \arrow[ur, dashed, "h'"] & 1 \end{tikzcd} \tag{29}$$

h' from f to g for the cylinder object $\widetilde{\mathrm{Cyl}}(A)$. Therefore, we may assume without loss of generality that left homotopies originate from cylinder objects instead of weak cylinder objects.

Now suppose that $\mathrm{Cyl}(A)$ and $\widetilde{\mathrm{Cyl}}(A)$ are two different cylinder objects for A. Then the commutative square

$$\begin{tikzcd} A \amalg A \arrow[r] \arrow[d] & \mathrm{Cyl}(A) \arrow[d] \arrow[dl, dashed, <->] \\ \widetilde{\mathrm{Cyl}}(A) \arrow[r] & A \end{tikzcd} \tag{30}$$

as well as its transpose admit a lift. Hence there exists a left homotopy from f to g originating from $\mathrm{Cyl}(A)$ if and only if there exists one originating from $\widetilde{\mathrm{Cyl}}(A)$.

For *ii)*, again suppose that $h : \mathrm{Cyl}(A) \to X$ is a left homotopy from f to g. The canonical inclusions $\iota_0, \iota_1 : A \to A \amalg A$ are cofibrations, since they are pushouts of the cofibration $0 \to A$. Therefore, $i_0 = (i_0, i_1) \circ \iota_0 : A \to \mathrm{Cyl}(A)$ is a cofibration and so is i_1. This shows that i_0 and i_1 are in fact trivial fibrations. To construct a right homotopy from f to g, let $X \xrightarrow{c} X^I \xrightarrow{(d_0,d_1)} X \times X$ be a path object for X and define l as the lift:

$$\begin{tikzcd} A \arrow[r, "cf"] \arrow[d, "i_0"'] & X^I \arrow[d, "(d_0,d_1)"] \\ \mathrm{Cyl}(A) \arrow[r, "(f\varepsilon,h)"] \arrow[ur, dashed, "l"] & X \times X. \end{tikzcd} \tag{31}$$

Then $li_1 : A \to X^I$ is a right homotopy from f to g. Indeed, $d_0li_1 = f\varepsilon i_1 = f$ and $d_1li_1 = hi_1 = g$. The other implication is dual.

The proof of Proposition 1.2.5 also tells us that $\simeq_R$ is an equivalence relation. Therefore, *iii)* follows from this remark and statement *ii)*. $\square$

As a consequence of this proposition, it makes sense to consider the set $[A, X]$ of maps $A \to X$ modulo homotopy, given that A is cofibrant and X fibrant. We will use these homotopy classes of maps to form another description of the homotopy category. That is, define $\widetilde{\mathrm{Ho}}(\mathcal{E})$ as the category whose objects are fibrant and cofibrant objects of $\mathcal{E}$ and has morphism sets given by $\mathrm{Hom}_{\widetilde{\mathrm{Ho}}(\mathcal{E})}(X,Y) := [X,Y]$ for $X,Y \in \widetilde{\mathrm{Ho}}(\mathcal{E})$.

This definition comes with a functor $\eta : \mathcal{E} \to \widetilde{\mathrm{Ho}}(\mathcal{E})$. For every $X \in \mathcal{E}$, choose a fibrant replacement $i_X : X \to X_f$ such that $i_X = \mathrm{id}$ if X is fibrant and a cofibrant replacement $p_X : X_c \to X$ such that $p_X = \mathrm{id}_X$ if X is cofibrant. For objects, we define $\eta(X) = (X_f)_c$. Let $\alpha : X \to Y$ be an arrow in $\mathcal{E}$. Then there exist lifts

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i_X"] & Y \arrow[r, "i_Y"] & Y_f \arrow[d] \\ X_f \arrow[rru, dashed, "\alpha_f"] \arrow[rr] & & 1, \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[rr] \arrow[d] & & (Y_f)_c \arrow[d, "p_{Y_f}"] \\ (X_f)_c \arrow[rru, dashed, "(\alpha_f)_c"] \arrow[r, "p_{X_f}"'] & X_f \arrow[r, "\alpha_f"'] & Y_f. \end{tikzcd} \tag{32}$$

This allows us to define $\eta(\alpha) = [(\alpha_f)_c]$. To show that η is a functor, it suffices to show that the lifts above are unique up to homotopy. In fact, any lift obtained from axiom **M4)** is unique up to left or right homotopy. First consider a diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[ru, dashed, "{k,l}"] \arrow[r, "v"] & Y \end{tikzcd} \tag{33}$$

with i a trivial cofibration and p a fibration, that admits two possible lifts k and l. Then the lift h from

$$\begin{tikzcd} A \arrow[r, "su"] \arrow[d, "i"] & X^I \arrow[d, "{(d_0,d_1)}"] \\ B \arrow[ru, dashed, "h"] \arrow[r, "{(k,l)}"] & X \times X \end{tikzcd} \tag{34}$$

gives a right homotopy from k to l. For the case where i is a cofibration and p a trivial fibration, we can use a similar diagram to obtain a left homotopy from k to l. This shows that $[((\mathrm{id}_X)_f)_c] = [\mathrm{id}_X]$ for each $X \in \mathcal{E}$ and $[(\beta_f)_c \circ (\alpha_f)_c] = [((\beta\alpha)_f)_c]$ for every pair of arrows $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$.

Before we proof that $\widetilde{\mathrm{Ho}}(\mathcal{E})$ describes $\mathrm{Ho}(\mathcal{E})$, we need an intermediate result.

Proposition 1.3.4. *Let A, B be cofibrant objects and X, Y be fibrant objects. Then any trivial fibration $i : A \to B$ and trivial cofibration $p : X \to Y$ induce a commutative diagram of bijections*

$$\begin{tikzcd} {[B,X]} \arrow[r, "p_*"] \arrow[d, "i^*"] & {[B,Y]} \arrow[d, "i^*"] \\ {[A,X]} \arrow[r, "p_*"] & {[A,Y]}. \end{tikzcd} \tag{35}$$

Proof. Let $f, g : B \to X$ be left homotopic maps. Then we can postcompose the homotopy $h : \mathrm{Cyl}(B) \to X$ with p to obtain a homotopy $pf \simeq_L pg$. This shows that $p_* : [B,X] \to [B,Y]$ is well-defined on equivalence classes. For injectivity of $p_* : [B,X] \to [B,Y]$, suppose that

$f, g : B \to X$ are such that $pf \simeq_L pg$ by a homotopy $h : \mathrm{Cyl}(B) \to Y$. Then the lift h' for the diagram

$$\begin{tikzcd} B \amalg B \arrow[r, "{(f,g)}"] \arrow[d, "{(i_0,i_1)}"'] & X \arrow[d, "p"] \\ \mathrm{Cyl}(B) \arrow[r, "h"] \arrow[ur, dashed, "h'"] & Y \end{tikzcd} \tag{36}$$

witnesses $f \simeq_L g$. Surjectivity of $p_* : [B, X] \to [B, Y]$ follows from lifting:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{37}$$

These arguments also directly apply to $p_* : [A, X] \to [A, Y]$. The cases for i^* are dual. $\square$

Theorem 1.3.5. *The functor $\eta : \mathcal{E} \to \widetilde{\mathrm{Ho}}(\mathcal{E})$ is the localization of $\mathcal{E}$ with respect to the class of weak equivalences.*

Proof. First we need to show that η sends weak equivalences to isomorphisms. Let $\alpha : A \to B$ be a weak equivalence. Then $\eta(\alpha) = (\alpha_f)_c$ is a weak equivalence as well, by using the two-out-of-three property in diagrams (32). We may factorize $\eta(\alpha)$ into a trivial cofibration after a trivial fibration $\eta(X) \xrightarrow{i} Z \xrightarrow{p} \eta(Y)$ and we show that both $[i]$ and $[p]$ are isomorphisms in $\widetilde{\mathrm{Ho}}(\mathcal{E})$. It follows directly that Z is fibrant and cofibrant. Therefore, Proposition 1.3.4 tells us $i^* : [Z, \eta(X)] \to [\eta(X), \eta(X)]$ and $i^* : [Z, Z] \to [\eta(X), Z]$ are bijections. The first bijection implies that there exists a map $j : Z \to \eta(X)$ such that $ji \simeq \mathrm{id}_{\eta(X)}$. It follows that $i^*(ij) = iji \simeq i = i^*(\mathrm{id}_Z)$ as maps $\eta(X) \to Z$. Thus the second bijection gives $ij \simeq \mathrm{id}_Z$.

For the universality property, consider a functor $F : \mathcal{E} \to \mathcal{D}$ that sends weak equivalences to isomorphisms in $\mathcal{D}$. Then we need to construct a functor $G : \widetilde{\mathrm{Ho}}(\mathcal{E}) \to \mathcal{D}$. Let $G(X) = F(X)$ for every fibrant and cofibrant object X and let $G([\alpha]) = F(\alpha)$ for morphisms α. This definition does not depend on the choice of representing morphism. Namely, if $\alpha, \alpha' : X \to Y$ are maps such that there exists a homotopy $h : X \to Y^I$ between them, then $F(\alpha) = F(d_0 h) = F(d_1 h) = F(\alpha')$ as a consequence of $F(d_0) = F(s)^{-1} = F(d_1)$.

Furthermore, we need to provide an natural isomorphism $\sigma : F \Rightarrow G\eta$. For every $X \in \mathcal{E}$ we have fixed a diagram $X \xrightarrow{i_X} X_f \xleftarrow{p_{X_f}} \eta(X)$ in $\mathcal{E}$. Applying F to this diagram makes p_{X_f} invertible, so we obtain the isomorphism

$$\sigma_X := F(q_{X_f})^{-1} \circ F(i_X) : F(X) \to F(\eta(X)) = G(\eta(X)).$$

Note that σ is natural in X. Now suppose that $G' : \widetilde{\mathrm{Ho}}(\mathcal{E}) \to \mathcal{D}$ is some functor such that F and $G'\eta$ are naturally isomorphic. The remarks above give a procedure to associate a functor $\widetilde{\mathrm{Ho}}(\mathcal{E}) \to \mathcal{D}$ to any functor $\mathcal{E} \to \mathcal{D}$. However, following this procedure for $G'\eta$ precisely gives us G' as the associated functor. Hence the natural isomorphism between F and $G'\eta$ descends to a natural isomorphism between G and G'. This proves the universality. $\square$

1.4 Transfer of Model Structure

In subsequent chapters, we will equip various categories with a model structure. As it turns out, there are model categories that can transfer their model structure to other categories along a given adjunction. The main theorem of this section gives sufficient conditions for such a transference to occur. Checking these conditions makes proving axioms **M1)** to **M5)** considerably quicker, because it does not involve a so-called small object argument.

Definition 1.4.1. Let $\mathcal{E}$ be a cocomplete category. A class W of morphisms is called saturated if it:

- contains all isomorphisms,
- is closed under retracts,
- is closed under pushouts,
- is closed under composition and
- is closed under transfinite composition.

To clarify the term transfinite composition, let γ be an infinite ordinal viewed as a poset category and let $F : \gamma \to \mathcal{E}$ be a continuous functor. That is, for every limit ordinal $\beta < \gamma$ we have

$$F(\beta) = \lim_{\alpha<\beta} F(\alpha). \tag{38}$$

Then the canonical map $F(0) \to \varinjlim F$ is said to be the transfinite composition of the morphisms in the image of F.

It is a good exercise to check that a class of arrows in $\mathcal{E}$ having the left lifting property with respect to any class $W \subseteq \mathcal{E}_1$, is saturated. In particular, the classes of cofibrations and trivial cofibrations in a model category are always saturated.

If $\mathcal{E}$ is small, then any class W of morphisms in $\mathcal{E}$ admits a smallest class of morphisms in $\mathcal{E}$ that is saturated and contains W. This is proven by transfinite induction. Unsurprisingly, we call this class the saturation of W denoted by $\mathrm{sat}(W)$.

Definition 1.4.2. A model category $\mathcal{E}$ is said to be cofibrantly generated if there exist sets I and J of arrows in $\mathcal{E}$ such that $\mathrm{sat}(I)$ and $\mathrm{sat}(J)$ are the classes of cofibrations and trivial cofibrations, respectively.

Example 1.4.3. The first and second example of 1.1.4 are both cofibrantly generated model categories. Cofibrations in the category of topological spaces are generated by maps $\partial D^n \hookrightarrow D^n$ and trivial cofibrations by maps $(\mathrm{id}, 0) : D^n \hookrightarrow D^n \times I$. Similarly for simplicial sets, cofibrations are generated by boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ and trivial cofibrations by horn inclusions $\bigwedge^n_k \hookrightarrow \Delta^n$. $\triangle$

Theorem 1.4.4. *Let $\mathcal{E}$ be a locally small model category that is cofibrantly generated and let*

$$\begin{tikzcd} \mathcal{E} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "R"] \end{tikzcd} \tag{39}$$

be an adjunction. Assume that $\mathcal{D}$ has all small colimits and limits. We say $X \to Y$ in $\mathcal{D}$ is a fibration or weak equivalence, if $RX \to RY$ is a fibration or weak equivalence in $\mathcal{E}$, respectively. Then $\mathcal{D}$ admits a cofibrantly generated model structure using these classes of fibrations and weak equivalences, under these conditions:

i) Any pushout of a morphism Li in $\mathcal{D}$ is a weak equivalence, if $i : A \to B$ is a generating trivial cofibration in $\mathcal{E}$.

ii) Transfinite compositions of pushout morphisms as in i) are weak equivalences.

iii) There exists a regular cardinal β such that for every transfinite composition of maps $(Z_\alpha \to Z_{\alpha'})_{\alpha<\alpha'<\beta}$ in $\mathcal{D}$ indexed by β, and every map $LA \to Z_\beta$ where A is the domain of a generating (trivial) cofibration in $\mathcal{E}$, there exists a decomposition $LA \to Z_\alpha \to Z_\beta$ for some $\alpha < \beta$.

Proof. Let I and J denote the sets of generating cofibrations and trivial cofibrations in $\mathcal{E}$, respectively. Then we define the class of cofibrations of $\mathcal{D}$ as the saturation of set containing the images of generating cofibrations $i \in I$ under L, i.e. as $\mathrm{sat}(L(I))$. Clearly, **M1)**, **M2)** and **M3)** are satisfied.

For the factorization axiom **M5)**, we will use a classical approach referred to as a small object argument. Let $f : X \to Y$ be an arrow in $\mathcal{D}$. In order to factor f as a trivial cofibration j followed by a fibration p, let us recursively define composites $X \xrightarrow{j_\alpha} Z_\alpha \xrightarrow{p_\alpha} Y$ for α smaller than the fixed cardinal β. Choose $j_0 = \mathrm{id}_X$ and $p_0 = f$. Now suppose that j_α and p_α have been defined for all $\alpha < \gamma$ for some fixed $0 < \gamma \leq \beta$. Then define $\mathcal{S}_\alpha$ to be the set of commutative diagrams S

$$\begin{array}{ccc} LA & \longrightarrow & Z_\alpha \\ \downarrow{\scriptstyle Lj} & & \downarrow{\scriptstyle p_\alpha} \\ LB & \longrightarrow & Y \end{array} \tag{40}$$

in $\mathcal{D}$ where $j : A \to B$ is a generating trivial cofibration. The assumption that $\mathcal{E}$ is locally small ensures that there are no size issues. First suppose that γ is a successor ordinal, i.e. $\gamma = \alpha + 1$ for a fixed α. Define $Z_{\alpha+1}$ as the pushout

$$\begin{array}{ccc} \coprod_{S \in \mathcal{S}_\alpha} LA & \longrightarrow & Z_\alpha \\ \downarrow{\scriptstyle \coprod Lj} & & \downarrow \\ \coprod_{S \in \mathcal{S}_\alpha} LB & \longrightarrow & Z_{\alpha+1}. \end{array} \tag{41}$$

As $\coprod Lj$ is in the saturation of $L(J)$, it follows that $Z_\alpha \to Z_{\alpha+1}$ is a cofibartion. Using assumptions *i)* and *ii)*, we conclude that $Z_\alpha \to Z_{\alpha+1}$ is in fact a trivial cofibration. Specifically, we use that p_α is an element of a subset of the saturation of $L(J)$ obtained by only closing the set $L(I)$ under pushouts and (transfinite) compositions. Reorganizing the colimits involved, $Z_\alpha \to Z_{\alpha+1}$ may in fact be written as a (transfinite) composition of pushouts of arrows in $L(J)$. Define $j_{\alpha+1}$ as the composite $X \xrightarrow{j_\alpha} Z_\alpha \to Z_{\alpha+1}$ and $p_{\alpha+1} : Z_{\alpha+1} \to Y$ as the map obtained from the universal property of the pushout (41). If γ is a limit ordinal, define j_γ as the transfinite composition of j_α with $\alpha < \gamma$ and p_γ as the induced map $Z_\gamma \to Y$.

By construction, $j_\beta : X \to Z_\beta$ is a trivial cofibration. We want to show that $p_\beta : Z_\beta \to Y$ is a fibration. By adjunction, it suffices to show that p_β has the right lifting property with respect to all maps $Lj : LA \to LB$ with $i \in J$. Such lifting problem has the form

$$\begin{tikzcd} LA \arrow[r] \arrow[d, "Lj"] & Z_\beta \arrow[d, "p_\beta"] \\ LB \arrow[r] & Y. \end{tikzcd} \tag{42}$$

Then by assumption *iii)*, there exists an ordinal $\alpha < \beta$ for which there is a decomposition $LA \to Z_\alpha \to Z_\beta$ of the top map. By definition of $Z_{\alpha+1}$ we have a commutative diagram

$$\begin{tikzcd} LA \arrow[r, hook] \arrow[d, "Lj"] & \coprod_{S \in \mathcal{S}_\alpha} LA_S \arrow[r] \arrow[d] & Z_\alpha \arrow[r] \arrow[d] & Z_\beta \\ LB \arrow[r, hook] & \coprod_{S \in \mathcal{S}_\alpha} LB_S \arrow[r] & Z_{\alpha+1} \arrow[ur] & \end{tikzcd} \tag{43}$$

which provides the required lift. For the factorization of $f : X \to Y$ into a cofibration followed by a trivial fibration, we use the same argument where we use the set of generating cofibrations I instead of the set J. This case is slightly easier, in the sense that we do not need the assumptions *i)* and *ii)*.

For axiom **M4)**, consider a commutative diagram in $\mathcal{D}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd} \tag{44}$$

and suppose that i is a cofibration and p a trivial fibration. Every morphism in $L(I)$ has the left lifting property with respect to trivial fibrations in $\mathcal{D}$, so every $i \in \mathrm{sat}(L(I))$ also has this lifting property.

Now suppose that i is a trivial cofibration and p a fibration. As we have already proven the factorization axiom **M5)** for $\mathcal{D}$, we may factorize i as a trivial cofibration $j : A \to C$ followed by a fibration $f : C \to B$. By **M3)**, f is a trivial fibration, and therefore has the right lifting property with respect to cofibrations. Consider the lift

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & C \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "r"] & B. \end{tikzcd} \tag{45}$$

We constructed j in such a way that it has the left lifting property with respect to all fibrations in $\mathcal{D}$. As i is a retract of j, witnessed by the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "i"] & A \arrow[r, "\mathrm{id}"] \arrow[d, "j"] & A \arrow[d, "i"] \\ B \arrow[r, "r"] & B \arrow[r, "f"] & B, \end{tikzcd} \tag{46}$$

i also has the right lifting property with respect to all fibrations. In particular, i has the left lifting property with respect to p. $\square$

2 Simplicial Presheaves

In the previous chapter we have developed the necessary framework to generalize homotopy theory to categories. In this chapter we follow a paper of Jardine [8] to review the category of simplicial presheaves. As the term suggests, this involves working with sheaves as well as the theory of simplicial sets. Readers unfamiliar with these subjects may want to consult a standard reference that explains them. For instance Chapters 2 and 3 of [11] cover sheaves and Section 1.1 of [12] covers simplicial sets.

Throughout this chapter, $\mathcal{C}$ denotes a small Grothendieck site, i.e. a small category equipped with a Grothendieck topology. The category of simplicial presheaves on $\mathcal{C}$ is defined as the functor category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{sSets})$ and we will denote this category by $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ from now on. Alternatively, simplicial presheaves on $\mathcal{C}$ can be thought of as objects in the category $\mathrm{Fun}(\mathcal{C}^{op} \times \mathbf{\Delta}^{op}, \mathrm{Sets})$ where $\mathbf{\Delta}$ denotes the simplex category, or simplicial objects in the category of presheaves $\mathrm{Pre}(\mathcal{C}) = \mathrm{Pre}(\mathcal{C}, \mathrm{Sets})$. Usually we will refer to objects $X \in \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ as simplicial presheaves and leave the site $\mathcal{C}$ implicit.

2.1 The local structure

In this section we want equip (a subcategory of) $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ with the structure of a category of fibrant objects. The most naive approach to this problem would be to perform a transfer argument, starting from the category of simplicial sets equipped with the Kan-Quillen model structure, and then considering the induced subcategory of fibrant objects. Indeed, the evaluation functor $\mathrm{ev}_U : \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \to \mathrm{sSets}$ for $U \in \mathcal{C}$, does admit a left adjoint:

$$\begin{tikzcd} \mathrm{sSets} \arrow[r, bend left, "(-)_U"] \arrow[r, phantom, "\perp"] & \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \arrow[l, bend left, "\mathrm{ev}_U"] \end{tikzcd} \tag{1}$$

where $(-)_U$ is the functor sending a simplicial set A to the simplicial presheaf defined by

$$A_U(V) = \coprod_{\varphi: V \to U} A. \tag{2}$$

At this point we run into several problems. Firstly, even if the conditions for transfer are met for this adjunction, the induced model structure on $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ would involve the choice of an object $U \in \mathcal{C}$ and is therefore not canonical. Secondly, the induced model structure would not depend on the given Grothendieck topology on $\mathcal{C}$.

Jardine suggested a more subtle approach to this problem. He proposed a definition for fibrations in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ that does use the notion of Kan fibration from simplicial sets, but does not come from a transfer argument. In case that the site $\mathcal{C}$ is a set of opens on a topological space, the idea is to state that a map $f : X \to Y$ of simplicial presheaves is a local fibration if and only if every induced map $f_x : X_x \to Y_x$ of stalks is a Kan fibration. In order to extend this definition to arbitrary sites, we require a bit more setup.

Definition 2.1.1. Let $i : A \to B$ be a map of simplicial sets and $p : X \to Y$ be a map of simplicial presheaves. Then a diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X(U) \arrow[d, "p_U"] \\ B \arrow[r, "\beta"] & Y(U) \end{tikzcd} \tag{3}$$

admits a local lift, if there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\varphi : V \to U$ in R, there exists a map $l : B \to X(V)$ making

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "p_V"] \\ B \arrow[r, "\beta"'] \arrow[urr, "l"] & Y(U) \arrow[r, "\varphi^*"'] & Y(V) \end{tikzcd} \tag{4}$$

commute.

Definition 2.1.2. Let $i : A \to B$ be a map of simplicial sets and $p : X \to Y$ be a map of simplicial presheaves. Then we say p has the right lifting property with respect to i (or i has the left lifting property with respect to p) if for every $U \in \mathcal{C}$, every commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X(U) \arrow[d, "p_U"] \\ B \arrow[r, "\beta"] & Y(U) \end{tikzcd} \tag{5}$$

admits a local lift.

We say that a map of simplicial presheaves p has the right local lifting property with respect to a class W of morphisms of simplicial sets, if p has the local right lifting property with respect to each $i \in W$. Similarly, i has the left local lifting property with respect to a class M of morphisms of simplicial presheaves, if i has the local left lifting property with respect to each $p \in M$.

Definition 2.1.3. Let $p : X \to Y$ be a map of simplicial presheaves. Then f is called a local fibration if it has the local right lifting property with respect to every constant map of simplicial presheaves of the form $\bigwedge_k^n \to \Delta^n$ for $0 \leq k \leq n$ and $n \geq 1$.

This property of the map p may also be described as p having the local right lifting property with respect to horn inclusions.

Definition 2.1.4. A simplicial presheaf X is called locally fibrant if the unique map $X \to *$ is a local fibration. Here $*$ denotes the terminal object of $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$, i.e. the constant functor on the simplicial set Δ^0. We write $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f \subseteq \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ for the full subcategory on locally fibrant simplicial presheaves.

Now that we introduced the right notion of fibrations, we will have to work yet again in order to arrive at the definition of weak equivalence in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$. Moreover, we will show that local fibrations defined in 2.1.3 coincide with stalkwise Kan fibrations, given a certain condition on $\mathcal{C}$.

The way to proceed is to investigate the properties of the class of arrows in sSets having the local left lifting property with respect to local fibrations.

Definition 2.1.5. A class W of arrows in sSets is said to be sparsely saturated, if it:

- contains all isomorphisms,
- is closed under pushouts,
- is closed under retracts,
- is closed under finite composition and
- is closed under finite direct sums.

Lemma 2.1.6. *Let M be a set consisting of maps of simplicial presheaves. Then the class W, containing exactly those arrows in* sSets *that have the local left lifting property with respect to maps in M, is sparsely saturated.*

Proof. The arguments for isomorphisms, pushouts and retracts are trivial. For composition, suppose that the maps $A \xrightarrow{i} B \xrightarrow{j} C$ are both in W, let $p : X \to Y$ in M be arbitrary and consider a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "ji"] & X(U) \arrow[d, "p_U"] \\ C \arrow[r, "\beta"] & Y(U) \end{tikzcd} \tag{6}$$

of simplicial maps. As i has the local left lifting property with respect to p, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for each $\varphi : V \to U$, there exists a lift

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "p_V"] \\ B \arrow[r, "\beta j"'] \arrow[urr, dashed, "l"] & Y(U) \arrow[r, "\varphi^*"'] & Y(V). \end{tikzcd} \tag{7}$$

Now we use the assumption that j has the local left lifting property with respect to p, which means that there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for every $\psi : W \to V$ in S_φ, there exists a lift

$$\begin{tikzcd} B \arrow[r, "l"] \arrow[d, "j"] & X(V) \arrow[r, "\psi^*"] & X(W) \arrow[d, "p_W"] \\ C \arrow[r, "\beta"'] \arrow[urr, dashed, "k"] & Y(V) \arrow[r, "\psi^*"'] & Y(W). \end{tikzcd} \tag{8}$$

Let $R \circ S_* \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ be the sieve on U consisting of those composites $W \xrightarrow{\psi} V \xrightarrow{\varphi} U$ with $\varphi \in R$ and $\psi \in S_\varphi$. Then $R \circ S_*$ is a covering sieve. We call $R \circ S_*$ a refinement of R. Moreover, for every choice of a morphism $R \circ S_*$, the map k is a local lift of diagram (6). Thus we have $ji \in W$.

For closure under direct sums, suppose that $i : A \to B$ and $j : A' \to B'$ are in W and $p : X \to Y$ in M. Then the local lifting problem for

$$\begin{array}{ccc} A \amalg A' & \xrightarrow{(\alpha,\alpha')} & X(U) \\ \Big\downarrow{\scriptstyle i \amalg j} & & \Big\downarrow{\scriptstyle p_U} \\ B \amalg B' & \xrightarrow{(\beta,\beta')} & Y(U) \end{array} \tag{9}$$

decomposes into two local lifting problems for the squares expressing $p_U \circ \alpha = \beta \circ i$ and $p_U \circ \alpha' = \beta' \circ j$. These separate squares admit local lifts for covering sieves $S, R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$. Hence (9) has a local lift for the covering sieve $R \cap S$. □

Definition 2.1.7. A morphism $i : A \to B$ in sSets is called a sparsely anodyne extension if it is contained in the smallest sparsely saturated class containing all horn inclusions $\bigwedge_k^n \hookrightarrow \Delta^n$.

Consequently, local fibrations have the local right lifting property with respect to sparsely anodyne extensions.

Lemma 2.1.8. *Suppose that $A \to B$ is a sparsely anodyne extension and $C \to D$ a monomorphism between finite simplicial sets. Then the induced map*

$$(A \times D) \cup (B \times C) \to B \times D \tag{10}$$

is also a sparsely anodyne extension.

The proof of the lemma involves the skeletal filtration of the map (10). This will reduce the proof to showing that the inclusion

$$(\textstyle\bigwedge_k^n \times \Delta^1) \cup (\Delta^n \times \{0\}) \to \Delta^n \times \Delta^1, \tag{11}$$

is a sparsely anodyne extension, which can be checked by hand. For the complete proof, we refer the reader to Proposition 3.1.2.8 from [12]. This is essentially the same statement, except that Lemma 2.1.8 is the version for finite simplicial sets.

Definition 2.1.9. Let K be a simplicial set and X a simplicial presheaf. Then the define the simplicial presheaf X^K by

$$X^K(U) = X(U)^K, \tag{12}$$

using the exponentiation in simplicial sets. Any morphism $V \to U$ in $\mathcal{C}$ induces a section map $X(U) \to X(V)$ in sSets, which in turn defines a map $X^K(U) \to X^K(V)$ by naturality of the exponentiation, making X^K into a functor $\mathcal{C}^{op} \to$ sSets.

Corollary 2.1.10. *Let $p : X \to Y$ be a local fibration and $i : L \to K$ an inclusion of simplicial sets with K finite. Then the canonical map $X^K \to X^L \times_{Y^L} Y^K$ is also a local fibration.*

Proof. Consider the local lifting problem

$$\begin{tikzcd} \bigwedge^n_k \arrow[r] \arrow[d, hook] & X^K(U) \arrow[d, "(i^*,p_*)"] \\ \Delta^n \arrow[r] & X^L \times_{Y^L} Y^K(U). \end{tikzcd} \tag{13}$$

This square transposes to

$$\begin{tikzcd} (\bigwedge^n_k \times K) \cup (\Delta^n \times L) \arrow[r] \arrow[d] & X(U) \arrow[d, "p_U"] \\ \Delta^n \times K \arrow[r] & Y(U), \end{tikzcd} \tag{14}$$

so the result follows from Lemma 2.1.8. ☐

Loosely formulated, the weak equivalences in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$ will be those maps that induce isomorphisms on homotopy groups for each $n \geq 0$. Just like regular homotopy groups of topological spaces, we want homotopy groups of locally fibrant simplicial presheaves to consist of homotopy classes of maps from disks into the object in question, sending the boundary to a chosen base point. In order to formalize this, we first need to introduce the relevant notion of homotopy between maps.

Definition 2.1.11. Let K be a simplicial set and X a simplicial presheaf and consider two maps $f, g : K \to X(U)$ of simplicial sets. Then f and g are called locally homotopic if there exists a covering sieve $R \subseteq \mathrm{Hom}(-, U)$ such that for every $\varphi : V \to U$ in R there exists a commutative diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i_0"] & X(U) \arrow[d, "\varphi^*"] \\ K \times \Delta^1 \arrow[r, "h_\varphi"] & X(V) \\ K \arrow[r, "g"] \arrow[u, "i_1"] & X(U) \arrow[u, "\varphi^*"] \end{tikzcd} \tag{15}$$

where i_0 and i_1 denote the maps induced by the inclusions of the first and last vertex into Δ^1, respectively. We denote this relation by $f \simeq_{loc} g$ or $[f]_{loc} = [g]_{loc}$. If there exists a subcomplex $L \subseteq K$ such that for every φ in R the restriction $h_\varphi|_{L\times\Delta^1} : L \times \Delta^1 \to X(V)$ factors through the projection $L \times \Delta^1 \to L$ (i.e. h_φ is constant on L), then we say that the local homotopy is relative to L.

Lemma 2.1.12. *Let K be a finite simplicial set and X a locally fibrant simplicial presheaf, then the (local) homotopy relation on maps $K \to X(U)$ is an equivalence relation.*

Proof. First consider the case $K = \Delta^0$. Reflexivity follows immediately. For symmetry, suppose that $x, y : \Delta^0 \to X(U)$ are vertices such that $x \simeq_{loc} y$. Then there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\varphi : V \to U$ in R there is a local homotopy

$h_\varphi : \Delta^1 \to X(V)$. As X is locally fibrant, there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for each $\psi : W \to V$ in S_φ there exists a lift

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "{(h_\varphi, \varphi^* x d_1, -)}"] \arrow[d] & X(V) \arrow[r, "\psi^*"] \arrow[d] & X(W) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[urr, dashed, "l_{\varphi,\psi}"] & \Delta^0 \arrow[r] & \Delta^0 \end{tikzcd} \tag{16}$$

where $d_1 : \Delta^1 \to \Delta^0$ denotes the face map projecting to the last vertex. Then for every map $\varphi \circ \psi$ in the covering sieve $R \circ S_* \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ the local homotopy $\Delta^1 \xrightarrow{s_2} \Delta^2 \xrightarrow{l_{\varphi,\psi}} X(W)$ witnesses $y \simeq_{loc} x$.

For transitivity, suppose that $x, y, z : \Delta^0 \to X(U)$ are vertices such that $x \simeq_{loc} y$ and $y \simeq_{loc} z$. Then there exist covering sieves $R, R' \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for each $\varphi : V \to U$ in R and $\varphi' : V' \to U$ in R, there are local homotopies $h_\varphi : \Delta^1 \to X(V)$ and $h'_{\varphi'} : \Delta^1 \to X(V')$. Like before, for each $\varphi : V \to U$ in $R \cap R'$, there exists a covering sieve S_φ such that for each $\psi : W \to V$ in S_φ there exists a lift

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(h'_{\varphi'}, -, h_\varphi)}"] \arrow[d] & X(V) \arrow[r, "\psi^*"] \arrow[d] & X(W) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[urr, dashed, "l_{\varphi,\psi}"] & \Delta^0 \arrow[r] & \Delta^0 \end{tikzcd} \tag{17}$$

Then for every map $\varphi \circ \psi$ in the covering sieve $(R \cap R') \circ S_* \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ the local homotopy $\Delta^1 \xrightarrow{s_1} \Delta^2 \xrightarrow{l_{\varphi,\psi}} X(W)$ witnesses $x \simeq_{loc} z$.

Finally, we will show that local homotopy is an equivalence relation on the set of maps $K \to X(U)$ for general finite simplicial sets K. Consider the restriction $X|_U$ of X to the slice category $\mathcal{C}/U$. The slice category inherits the Grothendieck topology from X, i.e. we say that a set of arrows $(V_i \to V)_{i \in I}$ over U is covering if and only if the set $(V_i \to V)_{i \in I}$ is covering in $\mathcal{C}$. Observe that $X|_U$ being a locally fibrant simplicial presheaf over $\mathcal{C}/U$ is a weaker condition than X being locally fibrant over $\mathcal{C}$, the former only requires local lifts of horn inclusions at objects V for which there exists a map $V \to U$. Therefore $X|_U$ is locally fibrant.

Let $f, g : K \to X(U)$ be maps such that $f|_L = g|_L$ for some subcomplex $i : L \to K$. Then form the pullback

$$\begin{tikzcd} (X|_U)K/L \arrow[r] \arrow[d] & (X|_U)^K \arrow[d, "i^*"] \\ * \arrow[r, "fi"] & (X|_U)^L. \end{tikzcd} \tag{18}$$

Then i^* is a local fibration by Corollary 2.1.10, so $(X|_U)K/L$ is locally fibrant. Note that $(X|_U)K/L$ is constructed in such a way that it has vertices corresponding to f and g. Moreover, f is locally homotopic to g relative to L if and only if the vertices of $(X|_U)K/L$ corresponding to f and g are locally homotopic. □

At this point we have the required machinery to construct homotopy groups in simplicial presheaves, which allows us to define the class of weak equivalences. Suppose that $\mathcal{C}$ has a terminal object t. Then we choose a vertex $x_t \in X(t)_0$ referred to as the base point of X at t. This choice induces a base point for $X(U)$ for every $U \in \mathcal{C}$. Namely, there is a unique natural transformation $x : * \to X$ from the terminal simplicial presheaf to X with the property that $x(t) = x_t : \Delta^0 \to X(t)$. Explicitly, the induced base point $x_U \in X(U)_0$ is the restriction of x_t along the unique map $U \to t$. Let $n \geq 1$, then

$$[(\Delta^n, \partial\Delta^n), (X(U), x_U)]_{loc} \tag{19}$$

denotes equivalences classes of maps $\Delta^n \to X(U)$ that send $\partial\Delta^n$ to the base point x_U, where the relation is given by local homotopy relative to $\partial\Delta^n$. This gives rise to a presheaf

$$\pi_n^p(X, x)(U) := \{[(\Delta^n, \partial\Delta^n), (X(U), x_U)]_{loc}\}. \tag{20}$$

For any $\alpha : V \to U$ in $\mathbb{C}$, the map $\pi_n^p(X, x)(\alpha)$ sends the equivalence class represented by an element $f : \Delta^n \to X(U)$ to the class represented by the composite

$$\Delta^n \xrightarrow{f} X(U) \xrightarrow{X(\alpha)} X(V). \tag{21}$$

The exercise that this construction produces a well-defined simplicial presheaf is left to the reader. For reasons that will be become clear in the next section, we want to work with the sheafification $\pi_n(X, x)$ of $\pi_n^p(X, x)$, instead of working with $\pi_n^p(X, x)$ itself. When working with the homotopy groups $\pi_n(X, x)$, it is good to know that they are obtained from $\pi_n^p(X, x)$ by applying the plus-construction once instead of twice, because $\pi_n^p(X, x)$ is separated. In order to show this, we need the following lemma:

Lemma 2.1.13. *Let $i : A \to B$ be a sparsely anodyne extension and X be a locally fibrant presheaf and $f, f' : A \to X(U)$ be a maps of simplicial sets such that $f \simeq_{loc} f'$. Then for any choice of local lifts*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X(U) \arrow[d, "\varphi^*"] \\ B \arrow[r, dashed, "l_\varphi"] & X(V) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "i"] & X(U) \arrow[d, "(\varphi')^*"] \\ B \arrow[r, dashed, "k_{\varphi'}"] & X(V') \end{tikzcd} \tag{22}$$

corresponding to maps $\varphi : V \to U$ in a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ and maps $\varphi' : V' \to U$ in a covering sieve $R' \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$, we have $l \simeq_{loc} k$.

Proof. As $f \simeq_{loc} f'$, there exists a covering sieve $\bar{R} \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\bar{\varphi} : \bar{V} \to U$ in $\bar{R}$ there exists a local homotopy $h_{\bar{\varphi}} : A \times \Delta^1 \to X(\bar{V})$. By considering their intersection, we may assume that $R = R' = \bar{R}$. By Corollary 2.1.8 the map $(A \times \Delta^1) \cup (B \times \partial\Delta^n) \to B \times \Delta^n$ is sparsely anodyne. Therefore, for every $\varphi \in R$ there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for each $\psi : W \to V$ in S_φ there exists a commutative diagram

$$\begin{tikzcd} (A \times \Delta^1) \cup (B \times \partial\Delta^1) \arrow[r, "h_\varphi \cup (l_\varphi \amalg k_\varphi)"] \arrow[d] & X(V) \arrow[d, "\psi^*"] \\ B \times \Delta^1 \arrow[r, dashed, "L_{\varphi,\psi}"] & X(W). \end{tikzcd} \tag{23}$$

This map $L_{\varphi,\psi}$ is the required local homotopy showing $l_\varphi \simeq_{loc} k_\varphi$. $\square$

Proposition 2.1.14. *The presheaf $\pi_n^p(X,x)$ for $n \geq 1$ is separated.*

Proof. Let $f, g : \Delta^n \to X(U)$ represent classes in $\pi_n^p(X,x)(U)$ and suppose that there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that for each $\varphi : V \to U$ in R we have $\pi_n^p(X,x)(\varphi)([f]_{loc}) = \pi_n^p(X,x)(\varphi)([g]_{loc})$. Unpacking the definitions, we get that for every such φ, there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-,V)$ such that for every $\psi : W \to V$ in S_φ there exists a commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "f"] \arrow[d, "i_0"] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "\psi^*"] \\
\Delta^n \times \Delta^1 \arrow[rr, "h_{\varphi,\psi}"] & & X(W) \\
\Delta^n \arrow[r, "g"] \arrow[u, "i_1"] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[u, "\psi^*"]
\end{tikzcd} \tag{24}$$

where the local homotopy $h_{\varphi,\psi}$ sends $\partial\Delta^n \times \Delta^1$ to the base point x_W. Then this diagram expresses that $f \simeq_{loc} g$ relative to $\partial\Delta^n$ for the refinement $R \circ S_* \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ of R. Hence $\pi_n^p(X,x)$ is separated. □

Explicitly, this proposition implies that $\pi_n(X,x)(U)$ consists of equivalences classes of compatible families of $\pi_n^p(X,x)$ at U, where two such families are considered equivalent if the sieve on which they agree is covering. If $n = 0$, we define $\pi_0^p(X)(U)$ to be the set of equivalence classes of vertices of $X(U)$, where two vertices are related if they are locally homotopic. The proof of Proposition 2.1.14 can be slightly altered to show that $\pi_0^p(X)$ is a separated simplicial presheaf and write $\pi_0(X)$ for its sheafification.

The simplicial presheaf $\pi_n^p(X,x)$ naturally comes with the structure of a group. We define the multiplication as follows. Let $f, g : \Delta^n \to X(U)$ be maps representing elements of $\pi_n^p(X,x)$, then there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that for every $\varphi : V \to U$ in R there exists a commutative diagram

$$\begin{tikzcd}
\bigwedge_n^{n+1} \arrow[r, "{(x_U,\dots,x_U,f,-,g)}"] \arrow[d, hook] & X(U) \arrow[d, "\varphi^*"] \\
\Delta^{n+1} \arrow[r, dashed, "w_\varphi"] & X(V)
\end{tikzcd} \tag{25}$$

using that X is locally fibrant. Moreover, f and g are n-simplices of $X(U)$ that are constant on the boundary, making the map on top well-defined. Thus we have an element $\{[d_n w_\varphi]_{loc}\}_{\varphi \in R} \in \pi_n(X,x)(U)$. Note that for any other choice $f', g' : \Delta^n \to X(U)$ of representatives of $[f], [g] \in \pi_n^p(X,x)(U)$, the relations $f \simeq_{loc} f'$ and $g \simeq_{loc} g'$ induce a local relative homotopy $(x_U, \dots, x_U, f, -, g) \simeq_{loc} (x_U, \dots, x_U, f', -, g')$ as maps $\bigwedge_n^{n+1} \to X(U)$. There is a generalization of Lemma 2.1.13 that considers the relative version of local homotopy. This implies that the construction above does not depend on the choices we made and as of such we have a well-defined map

$$m^p : \pi_n^p(X,x) \times \pi_n^p(X,x) \to \pi_n(X,x). \tag{26}$$

As m^p maps into a sheaf, the universal property of sheafification provides a canonical map

$$m : \pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x) \tag{27}$$

of sheaves. Here we also used that sheafification preserves finite products.

Proposition 2.1.15. *For $n \geq 1$, $\pi_n(X,x)$ is a group object in* $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})$ *and for $n \geq 2$ it is an abelian group object.*

Proof. We want to show that m has the properties of group multiplication on $\pi_n(X,x)$, where we choose the compatible family $[x_V : \Delta^n \to X(V)]_{loc}$ for the maximal sieve $\mathrm{Hom}_{\mathcal{C}}(-,U)$ as identity element of $\pi_n(X,x)(U)$. Suppose that $n \geq 1$. We will work towards an inverse map with respect to the multiplication. Let the compatible family of maps $\{f_\varphi : \Delta^n \to X(V)\}_{\varphi : V \to U \in R}$ for some covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ represent an arbitrary element of $\pi_n(X,x)(U)$. Then for every $\varphi : V \to U$ in R there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-,V)$ such that for each $\psi : W \to V$ in S_φ there exists a diagram

$$\begin{tikzcd}
\Lambda^{n+1}_{n+1} \arrow[r, "{(x_V,\ldots,x_V,f_\varphi,x_V,-)}"] \arrow[d, hook] & X(V) \arrow[d, "\psi^*"] \\
\Delta^{n+1} \arrow[r, dashed, "l_{\psi,\varphi}"] & X(W).
\end{tikzcd} \tag{28}$$

The class $\{[f^{-1}_{\varphi_\psi}]_{loc}\}_{\varphi \circ \psi \in S_* \circ R} := \{[d_{n+1} l_{\psi,\varphi}]_{loc}\}_{\psi \circ \varphi \in S_* \circ R}$ in $\pi_n(X,x)(U)$ acts as a left inverse to the family $\{[f_\varphi]_{loc}\}_{\varphi \in R}$. After refining, we may assume that the original family as well as the constructed left inverse are defined for the same covering sieve, for which we write R by abuse of notation. This construction, ignoring locality, for $n = 1$ corresponds to the filling the horn

$$\begin{tikzcd}
 & 1 \arrow[dr, dashed, "f^{-1}"] & \\
0 \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"] & & 2.
\end{tikzcd} \tag{29}$$

It is useful to have these kind of pictures in mind. Now we need to show that this family of maps also acts as a right inverse, which involves a local lift along a horn inclusion of dimension $n+2$. Formally, we accomplish this by observing that for every $\varphi : U \to V$ in R there exists a covering sieve $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-,V)$ such that for every $\psi : W \to V$ in S_φ there exists diagram

$$\begin{tikzcd}
\Lambda^{n+2}_{n-1} \arrow[r, "{(x_V,\ldots,x_V,-,\sigma,\sigma',\sigma'')}"] \arrow[d, hook] & X(V) \arrow[d, "\psi^*"] \\
\Delta^{n+2} \arrow[r, dashed, "L_{\psi,\varphi}"] & X(W)
\end{tikzcd} \tag{30}$$

where

$$\begin{aligned}\sigma &= \Delta^{n+1} \xrightarrow{d_{n+1}} \Delta^n \xrightarrow{f_\varphi} X(V),\\ \sigma' &= \Delta^{n+1} \xrightarrow{d_{n-1}} \Delta^n \xrightarrow{f_\varphi} X(V),\\ \sigma'' &= \Delta^{n+1} \xrightarrow{l_\varphi} X(V)\end{aligned}$$

and l_* refers to the lift obtained in diagram (28). Then $\{d_{n-1}L_{\psi,\varphi}\}_{\psi\varphi\in S_*\circ R}$ witnesses that $\{[f_{\psi\varphi}^{-1}]_{loc}\}_{\psi\varphi\in S_*\circ R}$ is the right inverse to $\{[f_{\psi\varphi}]_{loc}\}_{\psi\varphi\in S_*\circ R}$. The intuition behind the proof may be summarized in a picture, this time of $\bigwedge_0^3$:

$$\begin{tikzcd} & 2 \arrow[ddr, "f"] & \\ & 1 \arrow[u, "f^{-1}"] \arrow[dr, "\mathrm{id}"] & \\ 0 \arrow[uur, "\mathrm{id}"] \arrow[ur, "f"] \arrow[rr, "f"] & & 3 \end{tikzcd} \tag{31}$$

where we choose the first and second face degenerate and the third face as the simplex described in diagram (29). Here the 0-th face obtained by filling the horn tells us that f^{-1} is a right inverse to f.

Associativity and symmetry of m follow by similar arguments. First try to find a horn lifting that establishes the result in low dimension for simplicial homotopy groups. Then generalize this to higher n and improve it by introducing gruesome notation for the argument that does use locality. □

Recall that in the setting of topological spaces a path from one base point to another induces an isomorphism on their corresponding homotopy groups. A similar result holds for homotopy groups of locally fibrant simplicial presheaves.

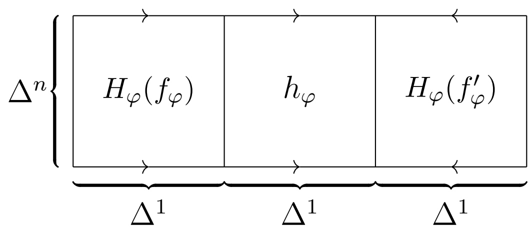
Lemma 2.1.16. *Let X be a locally fibrant simplicial presheaf on a site $\mathcal{C}$ Suppose that $\mathcal{C}$ has a terminal object t and let $x, x' \in X(t)_0$ be two choices of base points. If $x \simeq_{loc} x'$, then there is an induced isomorphism of sheaves $\pi_n(X,x) \to \pi_n(X,x')$.*

Proof. By assumption there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,t)$ such that for every $U \to t$ in R, there exists a local homotopy $h_U : \Delta^1 \to X(U)$ from x_U to x'_U. Let $f : \Delta^n \to X(U)$ represent an element $\pi_n^p(X,x)$. We can form local lift as follows, for every $U \to t$ in R there exists a covering sieve $S \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that for every $\varphi : V \to U$ in S there is a diagram

$$\begin{tikzcd} (\Delta^n \times \Delta^0) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "f\cup(h_U\circ \mathrm{pr}_1)"] \arrow[d] & X(U) \arrow[d, "\varphi^*"] \\ \Delta^n \times \Delta^1 \arrow[r, dashed, "H_\varphi"] & X(V). \end{tikzcd} \tag{32}$$

This makes $\{[H_\varphi \circ s_1]_{loc}\}_{\varphi \in S}$ into an S-compatible family of $\pi_n^p(X, x')(U)$, which represents an element of $\pi_n(X, x')(U)$. Suppose that $f' : \Delta^n \to X(U)$ represents the same element of $\pi_n^p(X, x)(U)$ as f, then there exists a local homotopy $f \simeq_{loc} f'$ which induces a local homotopy $f \cup h_U \simeq_{loc} f' \cup h_U$ as maps $(\Delta^n \times \Delta^0) \cup (\partial\Delta^n \times \Delta^1) \to X(U)$. Then Lemma 2.1.13 tells us that there is an induced local homotopy between the obtained compatible families H_φ. Hence this construction gives a well-defined map $H_*^p : \pi_n^p(X, x) \to \pi_n(X, x')$, which induces a map of simplicial sheaves $H_* : \pi_n(X, x) \to \pi_n(X, x')$.

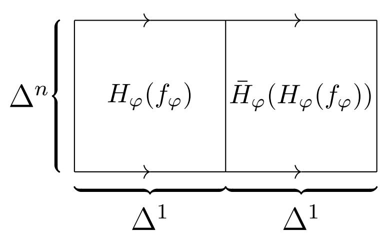
In order to show that H_* is monic, let $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ be a covering sieve such that for all $\varphi : V \to U$ in R there exist compatible families $f_\varphi : \Delta^n \to X(V)$ and $f'_\varphi : \Delta^n \to X(V)$ representing two elements of $\pi_n(X, x)(U)$. We can always intersect covering sieves, so that the compatible families are defined on the same covering sieve. Suppose that $H_*(([f_\varphi]_{loc})_{\varphi \in R}) = H_*(([f'_\varphi]_{loc})_{\varphi \in R})$. Then $f_\varphi \simeq_{loc} f'_\varphi$ for each $\varphi \in R$. After refining, this means that for every $\varphi \in R$ there exists a local homotopy $h_\varphi : \Delta^n \times \Delta^1 \to X(V)$ from f_φ to f'_φ. For each φ we have a map $(\Delta^n \times \Delta^1) \cup (\Delta^n \times \Delta^1) \cup (\Delta^n \times \Delta^1) \to X(V)$ given by



(33)

where leftmost vertical map is $f_\varphi : \Delta^n \to X(V)$ and the rightmost vertical map is $f'_\varphi : \Delta^n \to X(V)$. It is clear from the picture that forming a local lift twice yields a homotopy $f_\varphi \simeq_{loc} f'_\varphi$, proving that the two compatible families represented the same element in $\pi_n(X, x)(U)$.

To show that H_* is epic, recall that the local homotopy relation on vertices is an equivalence relation for locally fibrant X. Therefore, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for each $\varphi : V \to U$ there is a local homotopy $\bar{h}_\varphi : \Delta^1 \to X(V)$ from x'_V to x_V. Following the same procedure as before, we obtain an induced map $\bar{H}_* : \pi_n(X, x') \to \pi_n(X, x)$. After refining, we have a map $(\Delta^n \times \Delta^1) \cup (\Delta^n \times \Delta^1) \to X(V)$ given by



(34)

After one local lift, this diagram provides a local homotopy from f_φ to $\bar{H}_*(H_*(f))$, proving that H_* is epic. □

In order to define a base point for a simplicial presheaf X, we assumed that the site has a terminal object. One way to work around this assumption, is to consider the sites $\mathcal{C}/U$ which has the identity morphism on U as terminal object. This gives rise to the definition of local weak equivalence.

Definition 2.1.17. Let $f : X \to Y$ be a map between locally fibrant simplicial presheaves, then f is a local weak equivalence if

$$f_* : \pi_0(X) \to \pi_0(Y) \tag{35}$$

$$f_* : \pi_n(X|_U, x) \to \pi_n(Y|_U, y) \tag{36}$$

are isomorphisms of simplicial sheaves for every choice of $U \in \mathcal{C}$ and base points $x \in X|_U(\mathrm{id}_U)_0$ and $y \in Y|_U(\mathrm{id}_U)_0$.

Corollary 2.1.18. *Local weak equivalences have the two-out-of-three property.*

Proof. Consider maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ and suppose that f and gf are local weak equivalences, then we will show that g is a local weak equivalence. The other cases are trivial. It follows immediately that g induces an isomorphism $g_* : \pi_0(Y) \to \pi_0(Z)$ of local path components. Choose objects $U \in \mathcal{C}$, $(\varphi : V \to U) \in \mathcal{C}/U$ and a base point $y \in Y|_U(\mathrm{id}_U)_0$. Then we have check that g induces a bijection $g_* : \pi_n(Y|_U, y)(\varphi : V \to U) \to \pi_n(Z|_U, gy)(\varphi : V \to U)$. As $f_* : \pi_0(X) \to \pi_0(Y)$ is surjective, there exists a vertex $x \in X(V)_0$ such that $fx \simeq_{loc} y$. By Lemma 2.1.16 it suffices to show that $g_* : \pi_n(Y|_U, fx)(\varphi : V \to U) \to \pi_n(Z|_U, gfx)(\varphi : V \to U)$ is a bijection. We have a commutative diagram of sets

$$\begin{tikzcd}
\pi_n(X|_U, x)(\varphi : V \to U) \arrow[rr, "f_*"] \arrow[dr, "(gf)_*"'] & & \pi_n(Y|_U, fx)(\varphi : V \to U) \arrow[dl, "g_*"] \\
& \pi_n(Z|_U, gfx)(\varphi : V \to U). &
\end{tikzcd} \tag{37}$$

where f_* and $(gf)_*$ are bijections by assumption, hence g_* is a bijection as well. □

Thus far we have just been concerned with constructions in the category of simplicial presheaves on $\mathcal{C}$ and their immediate properties. Now we will see that the chosen definitions for local fibrations and local weak equivalences lead to a class of trivial local fibrations that admits a very nice characterization in terms of a local lifting property.

Theorem 2.1.19. *A map $p : X \to Y$ in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ between locally fibrant simplicial presheaves is a local fibration and a local weak equivalence if and only if it has the local right lifting property with respect to every inclusion of finite simplicial sets.*

Proof. Using Lemma 2.1.6, this statement is equivalent to saying that the class of trivial local fibrations is the class of maps having the right lifting property with respect to the local saturation of the class of finite inclusions of simplicial sets. It is a general fact that the class of monomorphisms in simplicial sets is the saturation of the set of boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$. Consequently, the class of inclusions of *finite* simplicial sets is the *local* saturation of the set on boundary inclusions. Therefore, it suffices to check that $p : X \to Y$ is a trivial local fibration if and only if it has the local right lifting property with respect to all maps $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$.

Observe that $f : \Delta^n \to Y(V)$ represents the unit of the group $\pi_n(Y|_U, px)(\varphi : V \to U)$ if and only if there exists a covering sieve $S \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for all $\psi : W \to V$ in S there exists a commutative diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "{(x_W,\ldots,x_W,\psi^* f)}"] \arrow[d, hook] & Y(W) \\ \Delta^{n+1}. \arrow[ur, "w_\psi"] & \end{tikzcd} \tag{38}$$

Indeed, any local homotopy $f \simeq_{loc} x_W$ induces such diagram and conversely such diagram expresses that the multiplication in $\pi_n(X|_U, x)(\varphi : V \to U)$ of the unit with the class of f is again equal to the unit.

Suppose that $p : X \to Y$ has the local right lifting property with respect to all maps $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 0$. Then local lifts with respect to $\emptyset \to \Delta^0$ imply that $p_* : \pi_0(X) \to \pi_0(Y)$ is surjective, while local lifts with respect to $\partial\Delta^1 \hookrightarrow \Delta^1$ give injectivity. Let $n \geq 1$, let $x \in X|_U(\mathrm{id}_U)_0$ be a base point and $\varphi : V \to U$ be an object of $\mathcal{C}/U$. Then for any $f : \Delta^n \to Y(V)$ representing an element of $\pi_n(Y|_U, px)(\varphi : V \to U)$ there exists a covering sieve $S \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for all $\psi : W \to V$ in S there exists a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "{(x_W,\ldots,x_W)}"] \arrow[d, hook] & X(W) \arrow[d, "p_W"] \\ \Delta^n \arrow[r, "\psi^* f"] \arrow[ur, dashed, "l_\psi"] & Y(W). \end{tikzcd} \tag{39}$$

This shows $p_* : \pi_n(X|_U, x)(\varphi : V \to U) \to \pi_n(Y|_U, px)(\varphi : V \to U)$ is surjective. For injectivity, let $f : \Delta^n \to X(V)$ represent an element of $\pi_n(X|_U, x)(\varphi : V \to U)$ such that $p_*([f]_{loc})$ is the unit of $\pi_n(Y|_U, px)(\varphi : V \to U)$. Using the map w_φ from diagram (38), we form a local lift. Thus there exists a covering sieve $S' \subseteq \mathrm{Hom}_{\mathcal{C}}(-, W)$ such that for every $\chi : T \to W$ in S' there exists a diagram

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "{(x_T,\ldots,x_T,\chi^*\psi^* f)}"] \arrow[d, hook] & X(T) \arrow[d, "p_T"] \\ \Delta^{n+1} \arrow[r, "\chi^* w_\psi"] \arrow[ur, dashed, "k_\chi"] & Y(T). \end{tikzcd} \tag{40}$$

This shows that f represents the unit of $\pi_n(X|_U, x)(\varphi : V \to U)$ and hence that p_* has a trivial kernel. Hence p is a local weak equivalence.

As horn inclusions are contained the in the local saturation of the maps $\partial\Delta^n \hookrightarrow \Delta^n$, it follows directly that p is a local fibration.

For the converse suppose that p is a local weak equivalence and a local fibration. Consider a commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, hook] & X(U) \arrow[d, "p_U"] \\ \Delta^n \arrow[r, "g"] & Y(U) \end{tikzcd} \tag{41}$$

with $n \geq 1$ that we will abbreviate as $D(f, g, U)$. Then we want to show that $D(f, g, U)$ admits a local lift, i.e. that there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\varphi : V \to U$ in R there is a diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\varphi^* f"] \arrow[d, hook] & X(V) \arrow[d, "p_V"] \\ \Delta^n \arrow[r, "\varphi^* g"'] \arrow[ur, dashed, "l_\varphi"] & Y(V). \end{tikzcd} \tag{42}$$

We claim that if two diagrams $D(f, g, U)$ and $D(f', g', U)$ are locally homotopic, then the former admits a local lift if and only if the latter admits one. To elaborate on this terminology, we say that to diagrams $D(f, g, U)$ and $D(f', g', U)$ are locally homotopic if there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\varphi : V \to U$ in R there exist diagrams

$$\begin{tikzcd} \partial\Delta^n \times \partial\Delta^1 \arrow[r, "(f,f')"] \arrow[d] & X(U) \arrow[d, "\varphi^*"] & \Delta^n \times \partial\Delta^1 \arrow[r, "(g,g')"] \arrow[d] & Y(U) \arrow[d, "\varphi^*"] \\ \partial\Delta^n \times \Delta^1 \arrow[r, dashed, "h_\varphi"] & X(V), & \Delta^n \times \Delta^1 \arrow[r, dashed, "h'_\varphi"] & Y(V) \end{tikzcd} \tag{43}$$

such that $p_V \circ h_\varphi = h'_\varphi \circ i$ where $i : \partial\Delta^n \times \Delta^1 \to \Delta^n \times \Delta^1$ denotes the inclusion. Towards the proof of this claim, suppose $D(f, g, U)$ and $D(f', g', U)$ are locally homotopic and $D(f, g, U)$ admits a local lift such as described in diagram (42). Then there exists a covering sieve $S \subseteq \mathrm{Hom}_{\mathcal{C}}(-, V)$ such that for every $\psi : W \to V$ in S there is a diagram

$$\begin{tikzcd} (\Delta^n \times \{0\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "\psi^* l_\varphi \cup \psi^* h_\varphi"] \arrow[d] & X(W) \arrow[d, "p_W"] \\ \Delta^n \times \Delta^1 \arrow[r, "\psi^* h'_\varphi"'] \arrow[ur, dashed, "L_{\varphi,\psi}"] & Y(W). \end{tikzcd} \tag{44}$$

Then the maps $L_{\varphi,\psi} \circ d_0 : \Delta^n \to X(W)$ for $\varphi \circ \psi \in R \circ S_*$ witness that $D(f', g', U)$ admits a local lift, proving the claim.

Returning to diagram $D(f, g, U)$, where p is a trivial local fibration, we will show that it admits a local lift by providing a locally homotopic diagram that admits a local lift. First, we homotope f to be constant on all but one face of $\partial\Delta^n$. That is, for all $\varphi \in R$ some covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ the diagram $D(\varphi^* f, \varphi^* g, V)$ is locally homotopic to

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "(x_V, \ldots, x_V, f_\varphi)"] \arrow[d, hook] & X(V) \arrow[d, "p_V"] \\ \Delta^n \arrow[r, "\varphi^* g"'] & Y(V) \end{tikzcd} \tag{45}$$

where x_V is the map $\Delta^{n-1} \to X(V)$ whose image is the last vertex of $\varphi^* f : \Delta^n \to X(V)$. This local homotopy of diagrams is induced by the retraction of $\bigwedge_n^n \subseteq \Delta^n$ onto the last vertex. This shows that $p_V f_\varphi$ represents the unit of $\pi_{n-1}(Y|_V, px_V)(\mathrm{id}_V)$. As p is a local weak equivalence, f_φ is the unit of $\pi_{n-1}(X|_V, x_V)(\mathrm{id}_V)$. Hence we may refine further for maps $\psi : W \to V$ to obtain a local homotopy between diagrams $D(\psi^*\varphi^* f, \psi^*\varphi^* g, W)$ and

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "x_W"] \arrow[d, hook] & X(W) \arrow[d, "p_W"] \\ \Delta^n \arrow[r, "\psi^*\varphi^* g"] & Y(W). \end{tikzcd} \tag{46}$$

Finally, we use that $p_* : \pi_n(X|_W, x_W) \to \pi_n(Y|_W, px_W)$ is an epimorphism of sheaves to conclude that this diagram admits a local lift. Local lifts for vertices of Y follow from the isomorphism $p_* : \pi_0(X) \to \pi_0(Y)$. $\square$

Theorem 2.1.20. *The classes of local fibrations and local weak equivalences equip the full subcategory* $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f \subseteq \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ *of locally fibrant simplicial presheaves on* $\mathcal{C}$ *with the structure of a category of fibrant objects.*

Proof. Axiom **N1)** was proofed in Corollary 2.1.18. Axiom **N2)** is an immediate consequence of the definition of local fibrations. For **N3)**, consider a pullback diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd} \tag{47}$$

in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ for locally fibrant simplicial presheaves X, Y and Z. If g is a (trivial) local fibration, then Theorem 2.1.19 tells us that g has a the right local lifting property with respect to maps $\bigwedge^n_k \to \Delta^n$ (respectively to maps $\partial\Delta^n \to \Delta^n$). Then $X \times_Z Y \to X$ also has this right local lifting property and thus is a (trivial) local fibration. This also implies that $X \times_Z Y$ is indeed locally fibrant.

Axiom **N4)** describes the existence of path objects. Exponentiation of the diagram

$$\begin{tikzcd} & \Delta^1 \arrow[d] \\ \partial\Delta^1 \arrow[ur, "i"] \arrow[r] & \Delta^0 \end{tikzcd} \tag{48}$$

induces a diagram

$$\begin{tikzcd} X^{\Delta^1} \arrow[dr, "i^*"] & \\ X^{\Delta^0} \arrow[u] \arrow[r] & X^{\partial\Delta^1} \end{tikzcd} \tag{49}$$

for any locally fibrant simplicial presheaf X. There are isomorphisms $X^{\Delta^0} \cong X$ and $X^{\partial\Delta^1} \cong X \times X$. Therefore we may write the exponential transpose of i as $(d_0^*, d_1^*) : X^{\Delta^1} \to X \times X$. Using adjunction, Lemma 2.1.8 shows that $d_0^* : X^{\Delta^n} \to X$ has the right local lifting property with respect to every finite inclusion of simplicial sets. Therefore it is a trivial local fibration by Theorem 2.1.19. By two-out-of-three $X \to X^{\Delta^1}$ is also a local weak equivalence. As X is locally fibrant, so is $X \times X$. Finally, as $d_0^* : X^{\Delta^1} \to X$ is a local fibration, X^{Δ^1} is locally fibrant and this shows that diagram (49) is a construction for path objects in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$. Axiom **N5)** is clear. $\square$

2.2 The global structure

In this section we will show that the category of simplicial presheaves on a Grothendieck site $\mathcal{C}$ admits a cofibrantly generated model structure, which we will refer to as the global structure. The cofibrations and weak equivalences for the global structure are easy to describe. The majority of work involves finding a description of the induced class of fibrations.

A map of simplicial presheaves is called a cofibration if it is a (pointwise) monomorphism. For any simplicial presheaf X, object $U \in \mathcal{C}$, base point $x \in X(U)_0$ and $n > 0$, define the sheaf $\pi_n^{top}(X|_U, x)$ as the sheafification of the presheaf

$$(\mathcal{C} \downarrow U)^{op} \to \text{Sets} : (\varphi : V \to U) \mapsto \pi_n(|X(V)|, x_V). \tag{50}$$

Here $|X(V)|$ denotes the geometric realization of the simplicial set $X(V)$ and $\pi_n(|X(V)|, x_V)$ stands for the topological n-th homotopy group. For $n = 0$, we define $\pi_0^{top}(X)$ as the sheafification of the presheaf

$$\mathcal{C}^{op} \to \text{Sets} : U \mapsto \pi_0(|X(U)|) \tag{51}$$

where $\pi_0(|X(U)|)$ is the set of path components of $|X(U)|$. We say that a map $f : X \to Y$ of simplicial presheaves is a topological weak equivalence if the induced maps

$$\begin{aligned} f_* &: \pi_n^{top}(X|_U, x) \to \pi_n^{top}(Y|_U, fx) \\ f_* &: \pi_0^{top}(X) \to \pi_0^{top}(Y) \end{aligned} \tag{52}$$

are isomorphisms. A map of simplicial presheaves is called a global fibration if it has the right lifting property with respect to all trivial cofibrations.

As a first step, we will investigate the relation between topological weak equivalences and simplicial homotopy groups. For a Kan complex A with a chosen base point $a \in A_0$, the simplicial homotopy groups $\pi_n^{simp}(A, a)$ consist of equivalence classes of pointed maps $(\Delta^n, \partial\Delta^n) \to (A, a)$ where the equivalence relation is given by simplicial homotopy [10]. Let $\mathcal{C}_t$ be a site with a terminal object t. Suppose X is a simplicial presheaf of Kan complexes on $\mathcal{C}_t$ (meaning $X : \mathcal{C}_t^{op} \to \text{sSets}$ has the property that each $X(U)$ is a Kan complex) and choose a base point $x \in X(t)_0$. The procedure of taking pointwise simplicial homotopy groups of $(X(U), x_U)$ is natural in U and therefore defines a functor $\mathcal{C}_t^{op} \to \text{Sets}$. We will denote the sheafification of this functor by $\pi_n^{simp}(X, x)$. The simplicial sheaf $\pi_0^{simp}(X)$ is defined similarly.

Recall that the geometric realization $|-| : \text{sSets} \to \text{Top}$ functor is defined as the left adjoint of singular complex functor $\text{Sing} : \text{Top} \to \text{sSets}$, see [5] Section 8.6. Also note that any simplicial set of the form $\text{Sing}(Y)$ for $Y \in \text{Top}$ is a Kan complex. Using the adjunction $|-| \dashv \text{Sing}$, we conclude that there is a canonical isomorphism

$$\pi_n^{top}(X|_U, x) \cong \pi_n^{simp}(\text{Sing}(|X|)|_U, x) \tag{53}$$

where $\text{Sing}(|X|)|_U$ is defined on objects $(\varphi : V \to U) \in \mathcal{C} \downarrow U$ as $\text{Sing}(|X(V)|)$. We have proved:

Remark 2.2.1. Every map $f : X \to Y$ of simplicial presheaves is topological weak equivalence if and only if $\text{Sing}(|f|) : \text{Sing}(|X|) \to \text{Sing}(|Y|)$ is a combinatorial weak equivalence.

Hence, we conclude:

Corollary 2.2.2. *The class of topological weak equivalences satisfies the two-out-of-three property.*

This Corollary is part of the main result that we are working towards.

Theorem 2.2.3. *The classes of cofibrations, global fibrations and topological weak equivalences form a cofibrantly generated model structure on the category of simplicial presheaves.*

Proof. Axioms **M1)** and **M3)** are easy to verify and axiom **M2)** is Lemma 2.2.2. Moreover, Axiom **M4)** is a formal consequence of **M5)** in this setting. Indeed, consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd} \tag{54}$$

in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ where i is a cofibration and f is a trivial global fibration. (Lifts in such diagrams where i is a trivial cofibration and f a global fibration exist by definition of global fibrations.) We can construct the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "i"] & X \arrow[rr, "\mathrm{id}_X"] \arrow[dd, "i_*"] \arrow[dr, "ji_*"] & & X \arrow[dd, "f"] \\ & & Z \arrow[ur, dashed, "l"] \arrow[dr, "p"] & \\ B \arrow[r] & B \cup_A X \arrow[ur, "j"] \arrow[rr, "q"] & & Y. \end{tikzcd} \tag{55}$$

Here the left square is a pushout diagram, q is the map induced by the pushout, which is factored as a cofibration j followed by a trivial global fibration p. Note that monomorphisms of simplicial sets are closed under pushout and pushouts in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ are formed pointwise. This means in particular that i_* is a pointwise monomorphism and thus a cofibration. Using the two-out-of-three property of topological weak equivalences, it follows that ji_* is a trivial cofibration. Hence there exists a lift $l : Z \to X$, solving the lifting property. Therefore, we only need to prove the factorization axiom **M5)**. We will do this by providing a set I of cofibrations and J of trivial cofibrations such that $\mathrm{sat}(I)$ and $\mathrm{sat}(J)$ are the collection of cofibrations, respectively trivial cofibrations. This allows us to use a small object argument. Before we arrive at this part of the proof, we need some results that the class of cofibrations (resp. trivial cofibrations) is indeed of the form $\mathrm{sat}(I)$ (resp. $\mathrm{sat}(J)$). □

Let α be an infinite regular cardinal greater than the cardinality of the power set of the set of all morphisms in $\mathcal{C}$. We say that a simplicial presheaf is α-bounded if for every $U \in \mathcal{C}$ and $n \geq 0$ the cardinality of the set $X_n(U)$ is strictly smaller than α. A cofibration $i : A \to B$ of simplicial presheaves is called α-bounded if B is α-bounded.

Lemma 2.2.4. *A map $f : X \to Y$ of simplicial presheaves is a global fibration if it has the right lifting property with respect to all α-bounded trivial cofibrations $i : A \to B$.*

Proof. Let $i : A \to B$ be a trivial cofibration and $j : C \to B$ be a subobject such that C is α-bounded. As a first step towards the proof, we will show that there exists a pullback diagram

$$\begin{tikzcd} C_\omega \cap A \arrow[r] \arrow[d, "i'"] & A \arrow[d, "i"] \\ C_\omega \arrow[r] & B \end{tikzcd} \tag{56}$$

where $C_\omega \subseteq B$ is a subobject containing C such that C_ω is α-bounded and i' is a trivial cofibration. For every $U \in \mathcal{C}$ and $x \in A(U)_0$ there is a long exact sequence of relative homotopy groups

$$\begin{tikzcd} \dots \arrow[r] & \pi_n^{top}(A|_U(\mathrm{id}_U), x) \arrow[r, "i_*"] & \pi_n^{top}(B|_U(\mathrm{id}_U), x) \arrow[r] & \pi_n^{top}(B|_U(\mathrm{id}_U), A|_U(\mathrm{id}_U), x) \arrow[dll, out=-10, in=170] \\ & \pi_{n-1}^{top}(A|_U(\mathrm{id}_U), x) \arrow[r, "i_*"] & \pi_{n-1}^{top}(B|_U(\mathrm{id}_U), x) \arrow[r] & \pi_{n-1}^{top}(B|_U(\mathrm{id}_U), A|_U(\mathrm{id}_U), x) \arrow[dll, out=-10, in=170] \\ & \pi_{n-2}^{top}(A|_U(\mathrm{id}_U), x) \arrow[r, "i_*"] & \dots & \end{tikzcd} \tag{57}$$

where $\pi_n^{top}(B|_U(\mathrm{id}_U), A|_U(\mathrm{id}_U), x)$ denotes the sheaf associated to the presheaf

$$(\mathcal{C} \downarrow U)^{op} \to \mathrm{sSets} : (\varphi : V \to U) \mapsto \pi_n(|B(U)|, |A(U)|, x), \tag{58}$$

evaluated at id_U. As i is a topological weak equivalence, every i_* in the long exact sequence is an isomorphism. This means in particular that for every $\gamma \in \pi_n(|C(U)|, |C \cap A(U)|, x)$ there exists a covering sieve $R \subseteq \mathcal{C}(-, U)$ such that for every $\varphi : V \to U$ in R, the element $\varphi^* j_* \gamma \in \pi_n(|B(U)|, |A(U)|, x)$ is trivial. We know that A is a filtered colimit of its α-bounded subobjects. Moreover, the cardinality of R is strictly less than α. Therefore, there exists an α-bounded subobject $C_\gamma \subseteq B$ containing C with the property that the composite

$$\pi_n(|C(U)|, |C \cap A(U)|, x) \xrightarrow{incl_*} \pi_n(|C_\gamma(U)|, |C_\gamma \cap A(U)|, x) \xrightarrow{\varphi^*} \pi_n(|C_\gamma(V)|, |C_\gamma \cap A(V)|, x_V) \tag{59}$$

is the zero map, for every $\varphi : V \to U$ in R. Let $C_1 = \bigcup C_\gamma$ the colimit ranges over every $\gamma \in \pi_n(|C(U)|, |C \cap A(U)|, x)$ for every object $U \in \mathcal{C}$, base point $x \in C \cap A(U)_0$ and $n \geq 0$. We proceed by recursion and define $C_\omega = \bigcup_{n \geq 1} C_n$. Then C_ω still is α-bounded. By a compactness argument it follows that each $\gamma \in \pi_n(|C_\omega(U)|, |C_\omega \cap A(U)|, x)$ becomes trivial after restriction along some covering sieve, showing that $i' : C_\omega \cap A \to C_\omega$ is a trivial cofibration.

Having proved this technicality, consider a commutative diagram in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd} \tag{60}$$

where i is a trivial cofibration and p has the right lifting property with respect to all α-bounded trivial cofibrations. Let L be a set of commutative diagrams

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "i''"] \arrow[ddr, "i"'] & & X \arrow[dd, "f"] \\ & C \arrow[ur, "\theta"] \arrow[d, "k"] & \\ & B \arrow[r] & Y \end{tikzcd} \tag{61}$$

where i'' is a trivial cofibration, k is monic and $A \neq C$. We equip L with a partial order where a diagram containing $\theta : C \to X$ is less or equal to a diagram containing $\theta' : C' \to X$ if and only if $A \subseteq C \subseteq C' \subseteq B$ as subobjects of B and $\theta'|_C = \theta$. In order to show that L is non-empty, let $C' \to B$ be an α-bounded subobject of B that is not contained in A. (If there are no such subobjects, then i is an isomorphism, so f immediately has the right lifting property with respect to i.) Let $C'_\omega \to B$ be the subobject corresponding to C' obtained from the construction from diagram (56). Then form the diagram

$$\begin{tikzcd} C'_\omega \cap A \arrow[r] \arrow[d, "i''"] & A \arrow[d, "i'"'] \arrow[ddr] & \\ C'_\omega \arrow[r] \arrow[drr] & C \arrow[dr] & \\ & & B \end{tikzcd} \tag{62}$$

where the outer square is a pullback and the inner square is a pushout. Then i'' is a trivial cofibration by construction. Then Proposition 2.2.5, which we have yet to prove, implies that i' is a trivial cofibration as well. It is not hard to confirm that L satisfies the condition for Zorn's Lemma. Hence L has a maximal element. Maximal elements of L correspond to lifts diagram (60). Hence f is indeed a global fibration. $\square$

Proposition 2.2.5. *Trivial cofibrations in* $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ *are stable under pushout.*

Proof. Consider a pushout diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"] & E \arrow[d] \\ B \arrow[r] & F \end{tikzcd} \tag{63}$$

in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ where i is a trivial cofibration. Using the Kan-Quillen model structure on sSets, there are pointwise factorizations of this diagram of the form:

$$\begin{tikzcd} A(U) \arrow[r, "k"] \arrow[d, "i"] & C(U) \arrow[r, "p"] \arrow[d, "i'"] & E(U) \arrow[d] \\ B(U) \arrow[r, "j'"] & D(U) \arrow[r, "p'"] & F(U). \end{tikzcd} \tag{64}$$

Here k is monic and p is a trivial Kan fibration. As monomorphisms are stable under pushout in sSets, i' is monic. Moreover, the Kan-Quillen model structure is left proper, which implies that p' is a weak equivalence. The factorization above is natural in U and pointwise weak equivalences are weak equivalences in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$. Therefore, we reduced the proof to showing that

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"] & C \arrow[d, "i'"] \\ B \arrow[r] & D \end{tikzcd} \tag{65}$$

i' is a trivial cofibration if k is a cofibration. We will do this by showing that for every diagram

$$\begin{tikzcd} {|\partial\Delta^n|} \arrow[r, "\alpha"] \arrow[d, hook] & {|C(U)|} \arrow[d, "|i'|"] \\ {|\Delta^n|} \arrow[r, "\beta"] & {|D(U)|} \end{tikzcd} \tag{66}$$

of topological spaces, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ such that for every $\varphi : V \to U$ in R there are diagrams

$$\begin{tikzcd} {|\partial\Delta^n|} \arrow[r, "d^0"] \arrow[d, hook] & {|\partial\Delta^n| \times |\Delta^1|} \arrow[r, "c_\varphi"] \arrow[d, hook] & {|C(V)|} \arrow[d, "|i'|"] \\ {|\Delta^n|} \arrow[r, "d^0"] & {|\Delta^n| \times |\Delta^1|} \arrow[r, "h_\varphi"] & {|D(V)|} \end{tikzcd} \tag{67}$$

$$\begin{tikzcd} {|\partial\Delta^n|} \arrow[r, "|\varphi^*|\alpha"] \arrow[d, hook] & {|C(V)|} \arrow[d, "|i'|"] \\ {|\Delta^n|} \arrow[r, "h_\varphi d^1"] \arrow[ur, "\theta_\varphi"] & {|D(V)|} \end{tikzcd} \tag{68}$$

where $c_\varphi : |\partial\Delta^n| \times |\Delta^1| \to |C(V)|$ is the constant homotopy on $|\varphi^*|\alpha$ and $h_\varphi d^0 = |\varphi^*|\beta$. Note that β is a singular n-simplex the CW-complex $|D(U)|$, obtained as a pushout of CW-complexes. Therefore, there exists a subdivision $|L| \cong |\Delta^n|$ together with a homotopy $h : |L| \times |\Delta^1| \to |D(U)|$ such that $hd^1 : |L| \to |D(U)|$ sends any simplex of L to a simplex completely contained in the image of $|C(U)|$ or to a simplex contained in the image of $|B(U)|$. Let $K \subseteq L$ be the induced subdivision of $\partial\Delta^n \subseteq \Delta^n$. Then there exists a finite sequence

$$K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m = L \tag{69}$$

where every inclusion corresponds to adjoining a single simplex. Suppose that the diagram

$$\begin{tikzcd} {|K|} \arrow[r, "\alpha'"] \arrow[d, hook] & {|C(U)|} \arrow[d, "|i'|"] \\ {|K_i|} \arrow[r, "\beta'_i"] & {|D(U)|} \end{tikzcd} \tag{70}$$

admits the required local lift up to homotopy. That is, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that for every $\varphi : V \to U$ in R, there exist diagrams

$$\begin{tikzcd} |K| \arrow[r, "d^0"] \arrow[d, hook] & |K| \times |\Delta^1| \arrow[r, "c'_\varphi"] \arrow[d, hook] & |C(V)| \arrow[d, "|i'|"] \\ |K_i| \arrow[r, "d^0"] & |K_i| \times |\Delta^1| \arrow[r, "h_\varphi"] & |D(V)| \end{tikzcd} \tag{71}$$

$$\begin{tikzcd} |K| \arrow[r, "|\varphi^*|\alpha'"] \arrow[d, hook] & |C(V)| \arrow[d, "|i'|"] \\ |K_i| \arrow[r, "h_\varphi d^1"] \arrow[ur, "\theta_\varphi"] & |D(V)| \end{tikzcd} \tag{72}$$

where $c'_\varphi : |K| \times |\Delta^1| \to |C(V)|$ is the constant homotopy on $|\varphi^*|\alpha'$ and $h_\varphi d^0 = |\varphi^*|\beta'_i$, such that for every simplex σ of K_i we have

- $\beta'_i(|\sigma|) \subseteq |C(U)|$ implies that h_φ is constant on $|\sigma|$,
- $\beta'_i(|\sigma|) \subseteq |B(U)|$ implies that $h_\varphi(|\sigma| \times |\Delta^1|) \subseteq |B(V)|$.

Now consider the unique p-simplex σ of K_{i+1} that is not in K_i. If $\beta'_i(|\sigma|) \subseteq |C(U)|$, then h_φ admits an extension $h'_\varphi : |K_{i+1}| \times |\Delta^1| \to |D(V)|$ that is constant on $|\sigma|$. If $\beta'_i(|\sigma|) \subseteq |B(U)|$, then we use a retract map $|\Delta^p \times \Delta^1| \to |(\partial\Delta^p \times \Delta^1) \cup (\Delta^p \times \Delta^0)|$ to extend h_φ to a map $h'_\varphi : |K_{i+1}| \times |\Delta^1| \to |D(V)|$ such that $h'_\varphi(|\sigma| \times |\Delta^1|) \subseteq B(V)$. As $i : A \to B$ is a trivial cofibration, there exist covering sieves $S_\varphi \subseteq \mathrm{Hom}_{\mathcal{C}}(-,V)$ such that for every $\psi : W \to V$ there are diagrams

$$\begin{tikzcd} |K_i| \arrow[r, "d^0"] \arrow[d, hook] & |K_i| \times |\Delta^1| \arrow[r, "c'_{\varphi,\psi}"] \arrow[d, hook] & |C(W)| \arrow[d, "|i'|"] \\ |K_{i+1}| \arrow[r, "d^0"] & |K_{i+1}| \times |\Delta^1| \arrow[r, "h_{\varphi,\psi}"] & |D(W)| \end{tikzcd} \tag{73}$$

$$\begin{tikzcd} |K_i| \arrow[r, "|\psi^*|\theta_\varphi"] \arrow[d, hook] & |C(W)| \arrow[d, "|i'|"] \\ |K_{i+1}| \arrow[r, "h_{\varphi,\psi} d^1"] \arrow[ur, "\theta_{\varphi,\psi}"] & |D(W)| \end{tikzcd} \tag{74}$$

where $c'_{\varphi,\psi} : |K_i| \times |\Delta^1| \to |C(W)|$ is the constant homotopy on $|\psi^*|\theta_\varphi$ and $h_{\varphi,\phi} d^0 = |\psi^*|h'_\varphi$. This finishes the construction of the lift in diagram (68), showing that i' is indeed a trivial cofibration. $\square$

Continuation of the proof of Theorem 2.2.3. Let J be the set of all α-bounded trivial cofibrations $i : A \to B$. In order to prove axiom **M5)**, let $f : X \to Y$ be a map of simplicial presheaves. We use a small object argument, as was demonstrated in the proof of Theorem 1.4.4, to obtain a factorization

$$X \xrightarrow{j_\alpha} Z_\alpha \xrightarrow{p_\alpha} Y \tag{75}$$

of f where j_α is contained in the saturation of J. In order to verify that p_α has the right lifting property with respect to morphisms in J, let $i : A \to B$ be an arrow in J and consider a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & Z_\alpha \arrow[d, "p_\alpha"] \\ B \arrow[r] & Y \end{tikzcd} \tag{76}$$

in $\mathrm{Pre}(\mathbf{C}, \mathrm{sSets})$. As i is α-bounded, so is A. This means that the set of all simplexes of $A(U)$, where U ranges over all objects of $\mathcal{C}$, contains strictly less than α elements. By regularity of α, the map $A \to Z_\alpha$ factors through $Z_{\alpha'} \to Z_\alpha$ for some $\alpha' < \alpha$. The existence of a lift in diagram (76) is now a formal consequence of the construction of p_α.

Clearly j_α is a trivial cofibration and Lemma 2.2.4 ensures that p_α is a global fibration.

The factorization of $f : X \to Y$ into a cofibration followed by a trivial global fibration is also constructed by a small object argument. Consider the adjunction

$$\begin{tikzcd} \mathrm{sSets} \arrow[r, bend left, "(-)_U"] \arrow[r, phantom, "\perp"] & \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \arrow[l, bend left, "\mathrm{ev}_U"] \end{tikzcd} \tag{77}$$

mentioned at the introduction of Section 2.1. Let I be the set of subobjects of Δ^n_U for any choice of $U \in \mathbf{C}$ and $n \geq 0$. Like before, there exists a factorization

$$X \xrightarrow{i_\alpha} Z_\alpha \xrightarrow{q_\alpha} Y \tag{78}$$

such that i_α is in the saturation of I, and thus a cofibration, and q_α has the right lifting property with respect to every arrow in I. An argument involving Zorn's Lemma in the way it was used for Lemma 2.2.4 shows that maps having the right lifting property with respect I, also has the right lifting property with respect to all cofibrations. This means that q_α is a fibration. Moreover, q_α has the right lifting property with respect to every inclusion $\partial\Delta^n_U \subseteq \Delta^n_U$. It follows from the adjunction that every $q_\alpha(U) : Z_\alpha(U) \to Y(U)$ is a trivial Kan fibration. Hence q_α is a pointwise weak equivalence and in particular a topological weak equivalence. This completes axiom **M5)**.

In order to show that the model structure on $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ is cofibrantly generated, it suffices to see that $\mathrm{sat}(I)$ (resp. $\mathrm{sat}(J)$) forms the collection of all cofibrations (resp. trivial cofibrations). The argument given at the last paragraph of the proof of Theorem 1.4.4 provides a way to write any cofibration (resp. trivial cofibration) as a retract of a morphism in $\mathrm{sat}(I)$ (resp. $\mathrm{sat}(J)$), showing that I and J are indeed the generating sets. □

2.3 Simplicial sheaves

The full subcategory $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f \subseteq \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$ on locally fibrant simplicial sheaves inherits the notions of local fibration and combinatorial weak equivalence from the local structure on $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$. As for the global structure on $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$, we say that a map of simplicial sheaves is a cofibration (or topological weak equivalence) if it is a cofibration (or topological weak equivalence) as a map of simplicial presheaves. We define global fibrations of simplicial sheaves to be the maps having the right lifting property with respect to all maps of simplicial sheaves that are topological weak equivalences as well as cofibrations. In this section we will see that these definitions give $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f$ the structure of a category of fibrant objects and $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})$ the structure of a model category. Finally, we will work towards the following result that relates these structures.

Theorem 2.3.1. *The inclusions of full subcategories*

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f \arrow[r, hook] \arrow[d, hook] & \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f \arrow[d, hook] \\ \mathrm{Shv}(\mathcal{C}, \mathrm{sSets}) \arrow[r, hook] & \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \end{tikzcd} \tag{79}$$

induce equivalences of categories

$$\begin{tikzcd} \mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f) \arrow[r] \arrow[d] & \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f) \arrow[d] \\ \mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})) \arrow[r] & \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})) \end{tikzcd} \tag{80}$$

on the respective homotopy categories.

Let us first recall some terminology about sheafification. Let X be a simplicial presheaf and $U \in \mathcal{C}$. Then for any covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-, U)$ we define

$$X(U)_R = \varprojlim_{(\varphi: V \to U) \in R} X(V) \tag{81}$$

and denote the induced map from $X(U)$ into the limit by $\tau_R : X(U) \to X(U)_R$. The set of covering sieves on U, write $J(U)$, is a poset ordered by inclusion. We form the colimit

$$L(X)(U) = \varinjlim_{R \in J(U)} X(U)_R. \tag{82}$$

Note that this construction is functorial in U, and as of such provides a functor $L : \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \to \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$. This functor comes with a canonical natural transformation $\eta : X \to LX$. We denote $\tilde{X} = L(LX)$, which is the sheafification of X or the sheaf associated to X.

With these notational conventions out of the way, we will investigate the relation between local fibrations of simplicial presheaves and sheafification.

Lemma 2.3.2. *Let X be a simplicial presheaf, then the canonical map $X \to \tilde{X}$ is a trivial local fibration.*

Proof. We will show that $\eta : X \to LX$ is a trivial local fibration. Let $U \in \mathcal{C}$ and consider the commutative diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X(U) \arrow[d, "\eta"] \\ \Delta^n \arrow[r] & LX(U) \end{tikzcd} \tag{83}$$

of simplicial sets. As $LX(U)$ is formed as a colimit of simplicial sets, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that $\Delta^n \to LX(U)$ factors through the canonical map $X(U)_R \to LX(U)$. This reduces the proof to a local lifting problem of the form:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X(U) \arrow[d, "\tau_R"] \\ \Delta^n \arrow[r] & X(U)_R. \end{tikzcd} \tag{84}$$

Observe that for each $\varphi : V \to U$ in R there exists a commutative diagram

$$\begin{tikzcd} X(U) \arrow[r, "\varphi^*"] \arrow[d, "\tau_R"] & X(V) \arrow[d, "\tau_{\varphi^* R}"] \\ X(U)_R \arrow[r, "\varphi^*"] & X(V)_{\varphi^* R} \end{tikzcd} \tag{85}$$

expressing the naturality of τ. As $\varphi^* R$ is the full covering sieve on V, it follows that $\tau_{\varphi^* R} : X(V) \to X(V)_{\varphi^* R}$ is an isomorphism. This gives the required local lift. $\square$

Corollary 2.3.3. *The functor* $L : \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \to \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$ *preserves local fibrations.*

Proof. Let $f : X \to Y$ be a local fibration of simplicial presheaves. For $U \in \mathcal{C}$ consider a commutative diagram of the form

$$\begin{tikzcd} \bigwedge^n_k \arrow[r] \arrow[d, hook] & LX(U) \arrow[d, "L(f)"] \\ \Delta^n \arrow[r] & LY(U). \end{tikzcd} \tag{86}$$

As $\emptyset \hookrightarrow \bigwedge^n_k$ is an inclusion of finite simplicial sets and $\eta_U : X(U) \to LX(U)$ is a trivial local fibration by Lemma 2.3.2, there exists a covering sieve $R \subseteq \mathrm{Hom}_{\mathcal{C}}(-,U)$ such that for any $\varphi : V \to U$ in R, there is a lift $l_\varphi : \bigwedge^n_k \to X(V)$ making

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X(U) \arrow[r, "\varphi^*"] & X(V) \arrow[d, "\eta"] \\ \bigwedge^n_k \arrow[r] \arrow[d, hook] \arrow[urr, dashed, "l_\varphi"] & LX(U) \arrow[r, "\varphi^*"] \arrow[d, "L(f)"] & LX(V) \arrow[d, "L(f)"] \\ \Delta^n \arrow[r] & LY(U) \arrow[r, "\varphi^*"] & LY(V) \end{tikzcd} \tag{87}$$

commute. Note that $L(f) \circ \eta = \eta \circ f : X \to LY$, by naturality of η. Hence $L(f) \circ \eta$ is a local fibration. Therefore we obtain local lifts of the form

$$\begin{tikzcd} \bigwedge^n_k \arrow[rr, "l_\varphi"] \arrow[d, hook] & & X(V) \arrow[r, "\psi^*"] & X(W) \arrow[d, "L(f)\circ\eta"] \\ \Delta^n \arrow[r] \arrow[urrr, dashed] & LY(U) \arrow[r, "\varphi^*"] & LY(V) \arrow[r, "\psi^*"] & LY(W). \end{tikzcd} \tag{88}$$

In particular, these local lifts are a solution to the initial lifting problem from diagram 86. $\square$

In particular, it follows from this corollary that the sheaf associated to a locally fibrant simplicial presheaf, is locally fibrant as well.

Proposition 2.3.4. *The category* $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f$ *together with the classes of local fibrations and combinatorial weak equivalences has the structure of a category of fibrant objects. Moreover, the inclusion functor induces an equivalence of categories* $\mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f) \cong \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f)$ *on homotopy categories.*

Proof. Finite limits in $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f$ coincide with with finite limits in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$. Therefore, the only non-trivial axiom to check is **N4)**. If X is a simplicial sheaf and K a finite simplicial set, then X^K is also a simplicial sheaf. Hence the existence of path objects in $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$ directly provides the existence of path objects in $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f$.

Towards proving the second statement, observe that it suffices to show that the sheafification functor L^2 preserves weak equivalences. Indeed, consider the diagram

$$\begin{tikzcd}
\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f \arrow[rr, bend left, "L^2"] \arrow[d, "i"] \arrow[rr, phantom, "\perp"] & & \mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f \arrow[ll, bend left, "incl"] \arrow[d, "j"] \\
\mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f) \arrow[rr, bend left, dashed] & & \mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f) \arrow[ll, bend left, dashed]
\end{tikzcd} \tag{89}$$

The composite $i \circ incl : \mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f \to \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f)$ sends weak equivalences to isomorphisms, so there is an induced functor $\mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f) \to \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f)$. Under the assumption that L^2 preserves weak equivalences, $j \circ L^2$ sends weak equivalences to isomorphisms, which provides the functor $\mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f) \to \mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f)$. Lemma 2.3.2 implies that these functors are pseudo-inverses to each other.

Corollary 1.1.10 reduces the proof to showing that L^2 preserves trivial local fibrations. Let $f : X \to Y$ be map of locally fibrant simplicial presheaves. As a consequence of Theorem 2.1.19 and an argument using adjunction, it follows that f is a trivial local fibration if and only if the induced map

$$X^{\Delta^n} \to X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n} \tag{90}$$

is a degree-wise local epimorphism, i.e. has the local right lifting property with respect to every map $\emptyset \to \Delta^m$ with $m \geq 0$. As $\eta : X \to LX$ is a degree-wise local epimorphism, we conclude that f being a trivial local fibration implies that the induced map

$$L^2(X^{\Delta^n}) \to L^2(X^{\partial\Delta^n} \times_{Y^{\partial\Delta^n}} Y^{\Delta^n}) \tag{91}$$

is a degree-wise local epimorphism. Thus so is

$$\tilde{X}^{\Delta^n} \to \tilde{X}^{\partial\Delta^n} \times_{\tilde{Y}^{\partial\Delta^n}} \tilde{Y}^{\Delta^n}, \tag{92}$$

since sheafification commutes with taking finite limits. Therefore, $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a trivial local fibration. $\square$

Let us now focus defining the global model structure on $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})$.

Lemma 2.3.5. *Let X be a simplicial presheaf, then the canonical map $X \to \tilde{X}$ is a topological weak equivalence.*

Proof. We will show that $\eta : X \to LX$ is a topological weak equivalence for any simplicial presheaf X. For $n \geq 1, U \in \mathcal{C}$ and $x \in X(U)_0$ we want to show that the induced map

$$\eta_* : \pi_n^{top}(X|_U, x) \to \pi_n^{top}(LX|_U, \eta x) \tag{93}$$

is an isomorphism. Let $\sigma : |\Delta^n| \to |LX(U)|$ represent an element of $\pi_n(|LX(U)|, \eta x)$. We have

$$|LX(U)| = |\varinjlim_{R \in J(U)} X(U)_R| \cong \varinjlim_{R \in J(U)} |X(U)_R|, \tag{94}$$

which shows that $|LX(U)|$ is a filtered colimit of CW-complexes. Since $|\Delta^n|$ is a compact topological space, the image of σ is contained in finitely many cells of $|LX(U)|$. Therefore, there exists a finite subset $J'(U) \subseteq J(U)$ such that σ factors through $\varinjlim_{R \in J'(U)} |X(U)_R|$. In particular, σ factors through $|X(U)_{R'}|$ where $R' = \bigcap J'(U)$. This shows that singular n-simplices of $|LX(U)|$ correspond to singular n-simplices of $|X(U)_R|$ for some $R \in J(U)$. Elements of $\pi_n(|X(U)_R|, x)$ locally lift along the covering sieve R. This gives the surjectivity of η_*.

For injectivity, suppose that $\sigma : |\Delta^n| \to |X(U)|$ represents an element of $\pi_n(|X(U)|, x)$ that vanishes in $\pi_n(|LX(U)|, x)$. Then there exists a homotopy

$$|\Delta^n \times \Delta^1| \to |LX(U)| \tag{95}$$

from the map induced by σ to the constant map ηx. By a similar compactness argument, there exists a $R \in J(U)$ such that the homotopy factors through $|X(U)_R|$. As before, the map $|\Delta^n \times \Delta^1| \to |X(U)_R|$ locally lifts to a map into $|X(U)|$, along the covering sieve R. Hence σ represents the trivial element of $\pi_n^{top}(X|_U, x)$. The case for $n = 0$ is similar. ☐

Proposition 2.3.6. *The category $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})$ together with the classes of cofibrations, weak equivalences and global fibrations of simplicial sheaves defined at the introduction of this section, has the structure of a model category.*

Proof. The first three axioms **M1)**, **M2)** and **M3)** are easy to verify. For axiom **M5)** we will make use of the factorization axiom from $\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})$. Indeed, let $f : X \to Y$ be a map of simplicial sheaves. By transfinite recursion, we will define factorizations

$$X \xrightarrow{i_\beta} \tilde{Z}_\beta \xrightarrow{p_\beta} Y \tag{96}$$

for every ordinal $\beta \leq \alpha$. Let $i_0 = \mathrm{id}_X$ and $p_0 = f$. For any ordinal β consider the following construction:

$$\begin{tikzcd} \tilde{Z}_\beta \arrow[r] \arrow[dr, "j_\beta"'] & \tilde{Z}_{\beta+1} \arrow[r, "p_{\beta+1}"] & Y \\ & Z_{\beta+1}. \arrow[u, "\eta"] \arrow[ur, "q_{\beta+1}"'] & \end{tikzcd} \tag{97}$$

Here $q_{\beta+1} \circ j_\beta$ is a factorization of f into a trivial cofibration followed by a fibration, viewed as simplicial presheaves. Moreover, η is the canonical map from $Z_{\beta+1}$ to its associated sheaf and $p_{\beta+1}$ is the induced map making the triangle commute. Now suppose β is a limit ordinal, then define $\tilde{Z}_\beta = \varinjlim_{\gamma<\beta} \tilde{Z}_\gamma$.

By Lemma 2.3.5 every map $\tilde{Z}_\beta \to \tilde{Z}_{\beta+1}$ is a trivial cofibration. Hence $i_\alpha : X \to \tilde{Z}_\alpha$ is a trivial cofibration as well. By construction, $p_\alpha : \tilde{Z}_\alpha \to Y$ has the right lifting property with respect to all α-bounded trivial cofibrations. Therefore p_α is a fibration. Factorizations of maps into a cofibration followed by a trivial fibration are constructed by a similar argument.

One half of the lifting axiom **M4)** is immediate, by definition of fibrations in $\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})$. The other part follows from the trick described in diagram (55). □

In order prove Theorem 2.3.1, we will need to find inverses to the maps shown in diagram (80). There is one obvious choice for the inverses to the horizontal maps, these will be the functors induced by sheafification. For the vertical maps, we will make use of the Kan's Ex^∞ functor.

Remark 2.3.7. The functor $\mathrm{Ex}^\infty : \mathrm{sSets} \to \mathrm{sSets}$ has the following properties [4]:

- For any simplicial set A, $\mathrm{Ex}^\infty A$ is a Kan complex.
- There exists a natural transformation $\nu : A \to \mathrm{Ex}^\infty A$ that is a weak equivalence in the Kan-Quillen model structure.

Besides Kan's Ex^∞ functor, we need a result on presheaves on Kan complexes. Recall that the n-th simplicial homotopy group on a Kan complex A with base point $a \in A_0$ is the set consisting of equivalence classes of pairs of maps $(\Delta^n, \partial\Delta^n) \to (A, a)$, where two such pairs are equivalent if there exists a simplicial homotopy between them, respecting the base point. The assumption that A is a Kan complex ensures that this relation is an equivalence relation, which is not true for general simplicial sets. Simplicial homotopy groups extend naturally to presheaves of Kan complexes. That is, let X be a simplicial presheaf, $U \in \mathcal{C}$ and $x \in X(U)_0$ a base point, then $\pi_n^{simp}(X|_U, x)$ is the functor

$$(\mathcal{C} \downarrow U)^{op} \to \mathrm{Sets} : (\varphi : V \to U) \mapsto [(\Delta^n, \partial\Delta^n), (X(V), x_V)] \tag{98}$$

where the square brackets indicate an equivalences class with respect to relative simplicial homotopy.

Lemma 2.3.8. *Let X be a presheaf of Kan complexes, then there is a canonical isomorphism between the sheaf associated to $\pi_n^{simp}(X|_U, x)$ and $\pi_n(X|_U, x)$ for any choice of $U \in \mathcal{C}$ and base point $x \in X(U)_0$.*

Proof. Let $\alpha, \beta : (\Delta^n, \partial\Delta^n) \to (X(U), x)$ represent two elements of $\pi_n^{simp}(X|_U, x)(\mathrm{id}_U)$. These elements become equal after applying

$$\eta_U : \pi_n^{simp}(X|_U, x)(\mathrm{id}_U) \to L\pi_n^{simp}(X|_U, x)(\mathrm{id}_U), \tag{99}$$

if and only if they are locally homotopic relative to the boundary. Therefore, there exists a factorization

$$\begin{tikzcd} \pi_n^{simp}(X|_U, x) \arrow[rr, "\eta"] \arrow[dr] & & L\pi_n^{simp}(X|_U, x) \\ & \pi_n^p(X|_U, x). \arrow[ur] & \end{tikzcd} \tag{100}$$

After sheafification of this diagram, every arrow becomes an isomorphism. □

Proof of Theorem 2.3.1. First and foremost, a map between locally fibrant simplicial presheaves is a combinatorial weak equivalence if and only if it is a topological weak equivalence. Observe that this claim follows immediately from Remark 2.2.1 once we have shown that for any locally fibrant simplicial presheaf X, the canonical map $X \to \mathrm{Sing}(|X|)$ is a combinatorial weak equivalence. Consider the commutative square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Ex}^\infty X \arrow[d] \\ \mathrm{Sing}(|X|) \arrow[r] & \mathrm{Sing}(|\mathrm{Ex}^\infty X|). \end{tikzcd} \tag{101}$$

top map is a combinatorial weak equivalence by Proposition 1.17 from [8]. Moreover, every map in this diagram induces an isomorphism on path components. Thus $\mathrm{Sing}(|X|) \to \mathrm{Sing}(|\mathrm{Ex}^\infty X|)$ is a combinatorial weak equivalence if and only if for every $n \geq 1$, every $U \in \mathcal{C}$ and $x \in X(U)_0$ the induced map

$$\pi_n^{simp}(\mathrm{Sing}(|X|)|_U, x) \to \pi_n^{simp}(\mathrm{Sing}(|\mathrm{Ex}^\infty X|)|_U, x) \tag{102}$$

is an isomorphism, by Lemma 2.3.8. Hence it is sufficient to show that

$$\mathrm{Sing}(|X(U)|) \to \mathrm{Sing}(|\mathrm{Ex}^\infty X(U)|) \tag{103}$$

is a weak equivalence of simplicial sets, for each $U \in \mathcal{C}$. Using the Quillen equivalence from Theorem 8.65 from [5] and Remark 2.3.7, this map is indeed a weak equivalence. Also note that $\mathrm{Ex}^\infty X \to \mathrm{Sing}(|\mathrm{Ex}^\infty X|)$ is a pointwise weak equivalence and therefore a combinatorial weak equivalence. This proves the claim.

Consequently, the vertical maps in diagram (80) are well-defined. Proposition 2.3.4 states that $\mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets})_f) \to \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f)$ is an equivalence of categories. Lemma 2.3.5 ensures that $\mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})) \to \mathrm{Ho}(\mathrm{Shv}(\mathcal{C}, \mathrm{sSets}))$ induced by sheafification is a pseudo-inverse to functor induced by the inclusion of sheaves into presheaves. Observe that $\mathrm{Sing}(|X|) \to \mathrm{Sing}(|\mathrm{Ex}^\infty X|)$ being a combinatorial weak equivalence in particular means that $X \to \mathrm{Ex}^\infty X$ is a topological weak equivalence. Hence $\mathrm{Ex}^\infty : \mathrm{Pre}(\mathcal{C}, \mathrm{sSets}) \to \mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f$ induces a pseudo-inverse to $\mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets})_f) \to \mathrm{Ho}(\mathrm{Pre}(\mathcal{C}, \mathrm{sSets}))$. □

3 Sheaf Cohomology

Sheaf cohomology is an invariant that is associated to a topological space. Unlike singular cohomology which requires a choice of an abelian group as coefficients, sheaf cohomology is defined by choosing a system of abelian groups. To arrive at the definition of sheaf cohomology, we first need to introduce the setting for these arguments. This involves answering questions such as: what properties must a category have in order to formulate a sensible criterium for exactness of a sequence?

The version of sheaf cohomology given below is the standard one, but there are other forms of cohomology as well that are closely related to this one. For example, all of the below still works when abelian groups are replaced by R-modules for a commutative ring R. One could also make another adaptation by replacing the category of opens on a topological space X by the category $\mathrm{Et}(X)$ of étale morphisms from any scheme into a fixed scheme X. This leads to the definition of étale cohomology, which is an important concept in algebraic geometry. In the upcoming sections we follow the approach from Chapter 4 of [13].

3.1 Abelian Categories

In this section we will introduce a concept that gives a precise meaning to a category behaving like the category of abelian groups or the category of modules over a commutative ring. Observe that in case of the category of abelian groups, the set of arrows $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ has the structure of an abelian group itself by pointwise addition. Moreover, any arrow $f : A \to B$ in Ab has a kernel and a cokernel. These are the properties that we want to generalize.

Definition 3.1.1. A linear category $\mathcal{L}$ is a category that is enriched over the category of abelian groups. That is, every set $\mathrm{Hom}_{\mathcal{L}}(A, B)$ has the structure of an abelian group such that every composition map

$$\circ : \mathrm{Hom}_{\mathcal{L}}(B, C) \times \mathrm{Hom}_{\mathcal{L}}(A, B) \to \mathrm{Hom}_{\mathcal{L}}(A, C) \tag{1}$$

is bilinear. We will use additive notation for the group multiplication on the Hom-sets.

Definition 3.1.2. Let $\mathcal{C}$ be a category. Then an object $0 \in \mathcal{C}$ is a called a zero object if it is both initial and terminal.

Remark 3.1.3. If a linear category $\mathcal{L}$ has a zero object, then for any two objects $A, B \in \mathcal{L}$ there exists a zero map $0_{A,B} : A \to B$. This map is defined as the unique composite $A \to 0 \to B$. It follows directly from the bilinearity of the composition that the zero map is the unit of the group $\mathrm{Hom}_{\mathcal{L}}(A, B)$. Usually, we will just write 0 for the zero map and it will be clear from the context whether it refers to the object in $\mathcal{L}$ or to a morphism.

Definition 3.1.4. A linear category $\mathcal{A}$ is a called additive if it has a zero object and finite products (and therefore also has finite coproducts by the proposition below). A functor $F : \mathcal{A} \to \mathcal{B}$ between linear categories is said to be additive if for every pair $A, A' \in \mathcal{A}$ the induced map $\mathrm{Hom}_{\mathcal{A}}(A, A') \to \mathrm{Hom}_{\mathcal{B}}(F(A), F(A'))$ is a group homomorphism.

Proposition 3.1.5. *In an additive category $\mathcal{A}$ finite products and coproducts coincide.*

Proof. Let $A, B \in \mathcal{A}$. We will show that $\mathrm{id}_{A\amalg B}$ admits a canonical factorization through $A \times B$. Consider the (non-commutative) diagram

$$\begin{tikzcd}
 & A \amalg B \arrow[dl, "{(\mathrm{id},0)}"'] \arrow[dr, "{(0,\mathrm{id})}"] & \\
A \arrow[dr, "{(\mathrm{id},0)}"] \arrow[dd, "\mathrm{id}"'] & & B \arrow[dl, "{(0,\mathrm{id})}"'] \arrow[dd, "\mathrm{id}"] \\
 & A \times B \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\
A \arrow[dr, "i_A"'] & & B \arrow[dl, "i_B"] \\
 & A \amalg B. &
\end{tikzcd} \tag{2}$$

First observe that the two triangles in the diagram do commute. This follows from the Yoneda lemma. Indeed, let $\alpha : P \to A$ be any map in $\mathcal{A}$, then $p_A \circ (\mathrm{id}, 0) \circ \alpha = p_A \circ (\alpha, 0) = \alpha$ and thus $p_A \circ (\mathrm{id}, 0) = \mathrm{id}_A$. Define $\sigma : A \amalg B \to A \times B$ as the sum $(\mathrm{id}, 0) \circ (\mathrm{id}, 0) + (0, \mathrm{id}) \circ (0, \mathrm{id})$ and $\sigma' : A \times B \to A \amalg B$ as the sum $i_A \circ p_A + i_B \circ p_B$. Using bilinearity of composition and the commutativity of the triangles, we obtain

$$\sigma' \circ \sigma = (i_A \circ p_A + i_B \circ p_B) \circ ((\mathrm{id}, 0) \circ (\mathrm{id}, 0) + (0, \mathrm{id}) \circ (0, \mathrm{id})) \tag{3}$$
$$= i_A \circ p_A \circ (\mathrm{id}, 0) \circ (\mathrm{id}, 0) + i_B \circ p_B \circ (0, \mathrm{id}) \circ (0, \mathrm{id}) \tag{4}$$
$$= i_A \circ (\mathrm{id}, 0) + i_B \circ (0, \mathrm{id}). \tag{5}$$

We proceed by using the dual version of the Yoneda lemma. Consider an arbitrary map $(\alpha, \beta) : A \amalg B \to P$ in $\mathcal{A}$. Then

$$(\alpha, \beta) \circ (i_A \circ (\mathrm{id}, 0) + i_B \circ (0, \mathrm{id})) = (\alpha, \beta) \circ i_A \circ (\mathrm{id}, 0) + (\alpha, \beta) \circ i_B \circ (0, \mathrm{id}) \tag{6}$$
$$= \alpha \circ (\mathrm{id}, 0) + \beta \circ (0, \mathrm{id}) \tag{7}$$
$$= (\alpha, 0) + (0, \beta) \tag{8}$$
$$= (\alpha + 0, \beta + 0), \tag{9}$$

and by Remark 3.1.3 the zero map is the unit of $\mathrm{Hom}_{\mathcal{L}}(A \amalg B, P)$. This shows that $\sigma' \circ \sigma = \mathrm{id}_{A\amalg B}$. The argument for $\sigma \circ \sigma' = \mathrm{id}_{A\times B}$ is dual. □

Within an additive category, kernels and cokernels arise as specific limits and colimits, respectively.

Definition 3.1.6. Let $f : A \to B$ be a morphism in an additive category $\mathcal{A}$, then the kernel $i : \ker(f) \to A$ of f is defined by the pullback diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
A \arrow[r, "f"] & B.
\end{tikzcd} \tag{10}$$

Dually, the cokernel $p : B \to \mathrm{coker}(f)$ of f is defined by the pushout diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ B \arrow[r, "p"] & \mathrm{coker}(f). \end{tikzcd} \tag{11}$$

Remark 3.1.7. Beware that in general kernels and cokernels do not need to exist. We only assumed additive categories to have products (and coproducts) and did not require the existence of all finite limits and colimits.

On that note, if a morphism f does have a kernel, then the canonical map $i : \ker(f) \to A$ is monic. To show this, let $\alpha, \beta : A' \to \ker(f)$ be arrows with the property that $i\alpha = i\beta$. Consider the cone

$$\begin{tikzcd} A' \arrow[dr, dashed] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "i\alpha=i\beta"'] & & \\ & \ker(f) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\ & A \arrow[r, "f"] & B, \end{tikzcd} \tag{12}$$

then both choices α and β for the dashed arrow make the obvious triangles commute. Hence the uniqueness part of the universal property of the pullback gives $\alpha = \beta$. A dual argument shows that $p : B \to \mathrm{coker}(f)$ is epic, if it exists.

Remark 3.1.8. Any map $f : A \to B$ in a linear category $\mathcal{L}$ induces exact sequences of abelian groups

$$0 \to \mathrm{Hom}_{\mathcal{L}}(T, \ker(f)) \xrightarrow{i_*} \mathrm{Hom}_{\mathcal{L}}(T, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{L}}(T, B), \tag{13}$$

$$0 \to \mathrm{Hom}_{\mathcal{L}}(\mathrm{coker}(f), T) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{L}}(B, T) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{L}}(A, T). \tag{14}$$

The reader is encouraged to check exactness at each position for him- or herself.

Definition 3.1.9. An additive category $\mathcal{A}$ is said to abelian if the following conditions hold:

i) $\mathcal{A}$ has all finite limits and colimits.

ii) If $f : A \to B$ is monic, then the kernel of the cokernel of f is f itself. That is, if f is monic, then the pushout diagram (10) is also a pullback.

iii) If $f : A \to B$ is epic, then the cokernel of the kernel of f is f itself. That is, if f is epic, then the pullback diagram (11) is also a pushout.

For arbitrary morphisms $f : A \to B$ we construct

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\ B \arrow[r, "p"] & \mathrm{coker}(f) \\ \ker(p) \arrow[r] \arrow[u, "i"] & 0 \arrow[u] \end{tikzcd} \tag{15}$$

where the top square is a pushout and the bottom square a pullback.

Definition 3.1.10. The image of a morphism $f : A \to B$ in an abelian category $\mathcal{A}$ is defined as the object $\ker(p)$ from the diagram above and we denote it by $\operatorname{im}(f)$. Moreover, there exists a canonical map $e : A \to \operatorname{im}(f)$ such that $f = ie$ by the universal property of the pullback.

Proposition 3.1.11. *For every morphism $f : A \to B$ in an abelian category $\mathcal{A}$, the factorization $A \xrightarrow{e} \operatorname{im}(f) \xrightarrow{i} B$ is a canonical way to factorize f into an epimorphism followed by a monomorphism.*

Proof. It follows from Remark 3.1.7 that i is monic. To show that e is epic, let $\alpha, \beta : \operatorname{im}(f) \to T$ be two maps such that $\alpha e = \beta e$. Then $\alpha e - \beta e = (\alpha - \beta)e = 0$. Let $k : \ker(\alpha - \beta) \to \operatorname{im}(f)$ be the kernel of $\alpha - \beta$. Because $(\alpha - \beta)e = 0$, there exists a map $w : A \to \ker(\alpha - \beta)$ such that $kw = e$. Hence we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "w"] \arrow[d, "e"] & \ker(\alpha - \beta) \arrow[r] \arrow[d, "ik"] & 0 \arrow[d] \\ \operatorname{im}(f) \arrow[r, "i"] & B \arrow[r] & \operatorname{coker}(ik). \end{tikzcd} \tag{16}$$

As k and i are both kernels, they are monic by Remark 3.1.7 and thus ik is monic. By condition ii) of Definition 3.1.9, the rightmost square of the diagram above is a pullback in addition to being a pushout. Therefore, there exists a map $v : \operatorname{im}(f) \to \ker(\alpha - \beta)$ such that $ikv = i$. As i is monic, this implies that $kv = \mathrm{id}_{\operatorname{im}(f)}$. Hence $\alpha - \beta$ factors through $\ker(\alpha - \beta)$, showing that $\alpha - \beta = 0$ and consequently $\alpha = \beta$. $\square$

Now that we have defined kernels, cokernels and images in $\mathcal{A}$, we are able to formulate what it means for a sequence $A_\bullet$ of maps

$$\ldots \to A_{n-1} \xrightarrow{d_n} A_n \xrightarrow{d_{n+1}} A_{n+1} \to \ldots \tag{17}$$

to be exact. We say this sequence is exact at n if $\operatorname{im}(d_n) = \ker(d_{n+1})$. The sequence is said to be exact if it is exact at every position.

If the sequence $A_\bullet$ is a chain complex, that is $d \circ d = 0$, then we define its n-th cohomology group as the quotient $\ker(d_{n+1})/\operatorname{im}(d_n)$ and denote it by $H^n(A_\bullet)$. That is, the condition $d_{n+1} \circ d_n = 0$ imposes a canonical map $\operatorname{im}(d_n) \to \ker(d_{n+1})$ and the quotient is defined as the cokernel of this map.

Lemma 3.1.12. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an sequence in an abelian category $\mathcal{A}$. If for every object $T \in \mathcal{A}$ the sequence*

$$\operatorname{Hom}_{\mathcal{A}}(C, T) \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{A}}(B, T) \xrightarrow{f^*} \operatorname{Hom}_{\mathcal{A}}(A, T) \tag{18}$$

is an exact sequence of abelian groups, then $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\mathcal{A}$.

Proof. First consider the case where $T = C$. Then $f^*g^* = 0$, so in particular $f^*g^*(\mathrm{id}_C) = gf = 0$. Decomposing $f = ie$ as in Proposition 3.1.11, we have $gie = 0$ and thus $gi : \mathrm{im}(f) \to C$ is the zero map, using that e is epic. Therefore, there exists a map $v : \mathrm{im}(f) \to \ker(g)$ such that $i = jv$, where $j : \ker(g) \to B$ the inclusion of the kernel. This shows that $\mathrm{im}(f) \subseteq \ker(g)$ as subobjects of B.

Now consider the case $T = \mathrm{coker}(f)$, then we have an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(C, \mathrm{coker}(f)) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(B, \mathrm{coker}(f)) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A, \mathrm{coker}(f)). \tag{19}$$

Let $p : B \to \mathrm{coker}(f)$ denote the cokernel of f. Then we have $0 = pf = f^*(p)$. Using exactness, we conclude that $p \in \ker(f^*) = \mathrm{im}(g^*)$. Hence there exists a morphism $w : C \to \mathrm{coker}(f)$ such that $p = gw$. This implies that $\ker(p) \subseteq \ker(g)$ as subobjects of B and recall that $\ker(p) = \mathrm{im}(f)$, which completes the proof. □

Proposition 3.1.13. *Let $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B} \to \mathcal{A}$ be additive functors between abelian categories that form an adjunction $L \dashv R$. Then L is a right exact functor and R a left exact functor.*

Proof. We will prove that R is left exact, the right exactness of L then follows from duality. Let

$$0 \to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0 \tag{20}$$

be a short exact sequence in $\mathcal{B}$. Then for any $A \in \mathcal{A}$ the adjunction induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{B}}(L(A), B') & \xrightarrow{f_*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathcal{B}}(L(A), B'') \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{A}}(A, R(B')) & \xrightarrow{R(f_*)} & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) & \xrightarrow{R(g_*)} & \mathrm{Hom}_{\mathcal{A}}(A, R(B'')). \end{array} \tag{21}$$

The top row is exact by Remark 3.1.8. As the vertical maps are bijections, the bottom row is also exact. Hence Lemma 3.1.12 tells us that $R(B') \xrightarrow{R(f)} R(B) \xrightarrow{R(g)} R(B'')$ is exact. The same argument applied to the sequence $0 \to B' \to B$ proves exactness at $R(B')$. □

3.2 Right Derived Functors

Say that $F : \mathcal{A} \to \mathcal{X}$ is an additive functor between abelian categories and suppose that F is left exact. Then any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ is sent to an exact sequence

$$0 \to F(A) \to F(B) \to F(C) \tag{22}$$

in $\mathcal{X}$. For such a fixed functor F, one can wonder to what extent F fails to be an exact functor. One way to investigate this, is by asking whether there exists a long exact sequence in $\mathcal{X}$ which starts with (22). As we will see in this section, given some condition on the category $\mathcal{A}$, it is possible to functorially construct such a long exact sequence with the help of right derived functors. In what follows we will first discuss this condition, that of $\mathcal{A}$ having enough injectives. Then we will construct right derived functors, after which we will provide a list of their most important properties, one of them involving this long exact sequence. As a side note, everything in this section can be dualized. This will give the notion of left derived functors, that exist if $\mathcal{A}$ has enough projectives. However, we do not need left derived functors to define sheaf cohomology and we will therefore pay not further attention to them.

Recall that the category of modules over a fixed ring admits a notion of injective modules. The most common definition is already stated in terms of objects and morphisms and therefore generalizes well to arbitrary categories.

Definition 3.2.1. An object I in a category $\mathcal{C}$ is called injective if for every diagram of the form

$$\begin{tikzcd} B \arrow[dr, dashed, "g"] & \\ A \arrow[u, "i"] \arrow[r, "f"] & I \end{tikzcd} \tag{23}$$

where i is monic, there exists an extension $g : B \to I$ such that $gi = f$. Moreover, $\mathcal{C}$ is said to have enough injectives if for every $C \in \mathcal{C}$ there exists a monomorphism $C \to I$ into an injective object I.

Similar to the case of modules over a ring, we can make sense of injective resolutions.

Definition 3.2.2. An injective resolution of an object A of an abelian category $\mathcal{A}$ is an exact sequence

$$0 \to A \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} \dots \tag{24}$$

such that the I_n for $n \geq 0$ are injective objects. We will often abbreviate the exact sequence as $0 \to A \to I_\bullet$.

Remark 3.2.3. Every module over a ring admits an injective resolution and this resolution is unique up to chain homotopy (see [13] Section 1.10). Provided that an abelian category $\mathcal{A}$ has enough injectives, every step of the proof of the existence and uniqueness of injective resolutions can easily be reformulated as an argument within $\mathcal{A}$. As of such, $\mathcal{A}$ has (essentially unique) injective resolutions. Likewise, we can prove the statement that every map $f : A \to B$ in $\mathcal{A}$ with chosen injective resolutions $0 \to A \to I_\bullet$ and $0 \to B \to J_\bullet$ extends uniquely (up to chain homotopy) to a map of chain complexes $f_* : I_\bullet \to J_\bullet$, by simply translating the proof for abelian groups to arbitrary abelian categories.

Definition 3.2.4. Let $F : \mathcal{A} \to \mathcal{X}$ be an additive functor between abelian categories that is left exact and suppose that $\mathcal{A}$ has enough injectives. Let $A \in \mathcal{A}$ and fix an injective resolution $0 \to A \to I_\bullet$. Then we define the right derived functors $R_i(F) : \mathcal{A} \to \mathcal{X}$ for $i \geq 0$ on objects by

$$R_i(F)(A) = H^i(F(I_\bullet)). \tag{25}$$

Let $0 \to A \to I_\bullet$ and $0 \to B \to J_\bullet$ be injective resolutions of A and B, respectively, and consider an arrow $f : A \to B$ in $\mathcal{A}$. We define the right derived functors on arrows by

$$R_i(F)(f) = H^i(F(f_*)) \tag{26}$$

where $f_* : I_\bullet \to J_\bullet$ is the induced chain map.

By Remark 3.2.3 any two choices of injective resolutions $0 \to A \to I_\bullet$ and $0 \to A \to J_\bullet$ for an object $A \in \mathcal{A}$ induce a chain homotopy equivalence between $I_\bullet$ and $J_\bullet$, which in turn induces an isomorphism on the level of cohomology. Therefore, the definition of $R_i(F)(A)$ is independent of the choice of a injective resolution. Furthermore, Remark 3.2.3 ensures that $R_i(F)(\mathrm{id}_A)$ is (canonically isomorphic to) the identity on $H^i(F(I_\bullet))$. This shows the right derived functors of F are well-defined.

Proposition 3.2.6 discusses the key properties of right derived functors. For the proof of that proposition, we first need a rather technical result about an extension of injective resolutions using a short exact sequence.

Proposition 3.2.5. *For any short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{27}$$

in an abelian category $\mathcal{A}$ together with two chosen injective resolution $0 \to A \xrightarrow{d} I_\bullet$ and $0 \to C \xrightarrow{l} K_\bullet$, there exists an injective resolution $0 \to B \xrightarrow{e} J_\bullet$ for B and chain maps f_* and g_* extending f and g, respectively, such that

$$0 \to I_\bullet \xrightarrow{f_*} J_\bullet \xrightarrow{g_*} K_\bullet \to 0 \tag{28}$$

is a split exact sequence at every degree.

Proof. As I_0 is injective and f monic, there exists a map $\alpha : B \to I_0$ such that $d = \alpha f$. Choose $J_0 = I_0 \oplus K_0$, where $\oplus$ denotes the product (and therefore also the coproduct by Proposition 3.1.5). Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "d"] & B \arrow[r, "g"] \arrow[d, dashed, "e"] \arrow[dl, "\alpha"'] & C \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & I_0 \arrow[r, shift right, "i"'] & I_0 \oplus K_0 \arrow[l, shift right, "p"'] \arrow[r, shift left, "q"] & K_0 \arrow[r] \arrow[l, shift left, "j"] & 0
\end{tikzcd} \tag{29}$$

where i and j are the inclusion maps and p and q the projection maps. We define $e : B \to I_0 \oplus K_0$ as the unique map with components (α, lg). Then we have

$$\begin{aligned} ef =&(\mathrm{id}_{J_0})ef = (ip + jq)ef = ipef + jqef \\ =&i\alpha f + jlgf = i\alpha f = id. \end{aligned} \tag{30}$$

In order to check that e is monic, suppose a map $h : E \to B$ has the property that $eh = 0$. Then $qeh = 0$, so by commutativity $lgh = 0$. As l is monic, this yields $gh = 0$. Now we use exactness of the top row of diagram (29) to conclude that there exists an arrow $y : E \to A$ such that $h = fy$. It follows that $0 = eh = efy = idy$ and i and d are both monic. Therefore $y = 0$, which shows $h = 0$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & I_0 \arrow[r, "i"] \arrow[d, "p_1"] & I_0 \oplus K_0 \arrow[r, "q"] \arrow[d, "p_2"] & K_0 \arrow[r] \arrow[d, "p_3"] & 0 \\
0 \arrow[r] & \mathrm{coker}(d) \arrow[r, "i'"] \arrow[d, "d'"] & \mathrm{coker}(e) \arrow[r, "q'"] \arrow[d, dashed, "e'"] & \mathrm{coker}(l) \arrow[r] \arrow[d, "l'"] & 0 \\
0 \arrow[r] & I_1 \arrow[r, "i"] & I_1 \oplus K_1 \arrow[r, "q"] & K_1 \arrow[r] & 0.
\end{tikzcd} \tag{31}$$

Here p_1, p_2 and p_3 are the cokernels of d, e and l, respectively. Then i' and q' are the maps induced by i, respectively q, between the cokernels. Moreover, it follows from the exactness of the injective resolutions $0 \to A \xrightarrow{d} I_\bullet$ and $0 \to C \xrightarrow{l} K_\bullet$ that the maps $I_0 \to I_1$ and $K_0 \to K_1$ admit a factorization $d'p_1$, respectively, $l'p_3$ for some maps $d' : \mathrm{coker}(d) \to I_1$ and $l' : \mathrm{coker}(l) \to K_1$.

We want show that the middle row of diagram (31) is exact. As q and p_3 are epic, so is q'. The exactness at $\mathrm{coker}(d)$ involves another diagram chase. Note that $d'p_1 ief = 0$. From the exactness of the top row of diagram (29), there exists a map $\gamma : C \to I_1$ such that $\gamma g = d'p_1 ie$. As l is monic and I_1 injective, there exists an extension $\beta : K_0 \to I_1$ such that $\gamma = \beta l$. We define $\phi : I_0 \oplus K_0 \to I_1$ by $\phi = d'p_1 p - \beta q$. Using the definition of ϕ, β and γ, we derive

$$\phi e = d'p_1 pe - \beta qe = d'p_1 pe - \beta lg = d'p_1 pe - \gamma g = 0. \tag{32}$$

Consequently, there exists an arrow $\psi : \mathrm{coker}(e) \to I_1$ such that $\psi p_2 = \phi$. It follows that $\psi i' p_1 = \psi p_2 i = \phi i = d'p_1$. As p_1 is epic, this implies $\psi i' = d'$. As d is monic, Proposition 3.1.11 tells us that $d'p_1$ is an epi-mono factorization of the map $I_0 \to I_1$. Now $d' = \psi i'$ being monic implies that i' is monic.

Exactness at $\mathrm{coker}(e)$ comes down to providing two inclusions. Note that $q'i'p_1 = q'p_2 i = p_3 qi = 0$ and p_1 is epic, so $q'i' = 0$. This yields $\mathrm{im}(i') \subseteq \ker(q)$. For the other inclusion, let $h = (h_0, h_1) \in I_0 \oplus K_0$ be an element such that $p_2(h) \in \ker(q')$. (Strictly speaking we should use maps $h : X \to I_0 \oplus K_0$ for elements, but the notation $h \in I_0 \oplus K_0$ is easier to work with and the argument still holds for the general case.) Since $q'p_2(h) = 0$, we also have $p_3 q(h) = p_3(h_1) = 0$ by commutativity. Therefore, $h_1 \in \ker(p_3) = \mathrm{im}(l)$. For notational convenience, we view B, respectively C, as a subobject of $I_0 \oplus K_0$, respectively K_0. Thus $h_1 \in C$. As g is surjective, there exists an $x \in B$ such that $g(x) = h_1$. It follows that $q(h - x) = 0$, which implies that $h - x = i(h_0)$. From this we conclude that

$$p_2(h) = p_2(i(h_0)) = i'(p_1(h_0)) \tag{33}$$

showing that $p_2(h) \in \mathrm{im}(i')$.

The rest of the construction follows by defining $J_n = I_n \oplus K_n$ and inductively constructing morphisms e' analogous to e. Then $0 \to B \to J_\bullet$ is an injective resolution of B and each row $I_n \xrightarrow{i} J_n \xrightarrow{q} K_n$ is split exact. □

Proposition 3.2.6. *Let $F : \mathcal{A} \to \mathcal{X}$ and $\mathcal{A}$ be as in Definition 3.2.4. Then we have:*

i) There is a natural isomorphism $R_0(F) \cong F$.

ii) Any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{34}$$

in $\mathcal{A}$ induces a long exact sequence

$$0 \to R_0(F)(A) \xrightarrow{f_*} R_0(F)(B) \xrightarrow{g_*} R_0(F)(C) \xrightarrow{\delta} R_1(F)(A) \xrightarrow{f_*} \dots \tag{35}$$

in $\mathcal{X}$.

iii) If $A \in \mathcal{A}$ is injective, then $R_i(F)(A) = 0$ for all $i > 0$.

Proof. For *i)*, consider an injective resolution $0 \to A \to I_\bullet$. Then $0 \to F(A) \to F(I_0) \to F(I_1)$ is an exact sequence in $\mathcal{X}$, because F is left exact by assumption. This gives an isomorphism $F(A) \cong \ker(F(d_1))$, which is isomorphic to $H^0(F)(A) = R_0(F)(A)$. The last part of Remark 3.2.3 shows this isomorphism is natural in A.

For *ii)*, choose injective resolutions for A and C. Then we use Proposition 3.2.6 to obtain a diagram $0 \to I_\bullet \xrightarrow{f_*} J_\bullet \xrightarrow{g_*} K_\bullet \to 0$ of which the columns are chain complexes and the rows are split exact sequences. As F is additive, it preserves chain complexes as well as split exact sequences. Therefore, the image of the diagram under F induces a long exact sequence in cohomology by a standard diagram chase.

For *iii)*, it suffices to note that

$$0 \to A \xrightarrow{\mathrm{id}} A \to 0 \to \dots \tag{36}$$

is an injective resolution for A, if A is injective. □

3.3 Sheaf Cohomology

Classical singular cohomology of a topological space requires a choice of an abelian group for the coefficients. For some applications, it is convenient to have a more flexible notion of cohomology which requires a system of local coefficients. That is, instead of choosing a single abelian group for a topological space X, we associate to every open $U \subseteq X$ an abelian group and associate a group homomorphism to every inclusion $V \subseteq U$ of opens in X. These associations are required to satisfy some compatibility conditions that can best be described using presheaves.

Definition 3.3.1. Let X be a topological space. Then a system of local coefficients for X is a functor $A : \mathcal{O}(X)^{op} \to \mathrm{Ab}$ on X. Here $\mathcal{O}(X)$ denotes the poset category on opens of X, ordered by inclusion. We also refer to A as an abelian presheaf. We denote the functor category of all abelian presheaves on a space X by $\mathrm{Pre}(\mathcal{O}(X), \mathrm{Ab})$ or just $\mathrm{Pre}(X, \mathrm{Ab})$. Given an abelian presheaf A on X, an inclusion $V \subseteq U$ of opens in X and an element $x \in A(U)$, we will write $x|_V$ for the element $A(V \subseteq U)(x)$.

We say that an abelian presheaf $A : \mathcal{O}(X)^{op} \to \mathrm{Ab}$ is an abelian sheaf if A becomes a sheaf after postcomposing with the forgetful functor $\mathrm{Ab} \to \mathrm{Sets}$. Let $\mathrm{Shv}(X, \mathrm{Ab}) \subseteq \mathrm{Pre}(X, \mathrm{Ab})$ denote the full subcategory on abelian presheaves. Define the global section functor

$$\Gamma : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab} : A \mapsto A(X) \tag{37}$$

as the evaluation of abelian sheaves at the largest open X. It is this functor that will define sheaf cohomology.

However, before we have a closer look at what properties this functor possesses, we will first investigate how monomorphisms and epimorphisms in $\mathrm{Shv}(X, \mathrm{Ab}) \subseteq \mathrm{Pre}(X, \mathrm{Ab})$ behave. Recall that monomorphisms in a category $\mathcal{C}$ are defined as maps $f : X \to Y$ fulfilling a condition for every parallel pair of arrows into X. Restricting $\mathcal{C}$ to a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ weakens this condition. Therefore, any map f in $\mathcal{C}'$ that is monic in $\mathcal{C}$ is also monic when considered as a map in the subcategory $\mathcal{C}'$, but the reverse implication does not hold in general. A similar argument can be made for epimorphisms.

Definition 3.3.2. Let $f : A \to B$ be a map of abelian presheaves on X and let $U \in \mathcal{O}(X)$, then we say that an element $b \in B(U)$ is locally in the image of f, if there exists an open cover $U = \bigcup_{i \in I} U_i$ such that for every $i \in I$ there exists an element $a_i \in A(U_i)$ with $f_{U_i}(a_i) = b|_{U_i}$.

Lemma 3.3.3. *A morphism $f : X \to Y$ of abelian sheaves on X is monic if and only if it is monic as a morphism of abelian presheaves. Moreover, f is epic if and only if for every open $U \in \mathcal{O}(X)$, every element $b \in B(U)$ locally lies in the image of f.*

Proof. The inclusion functor $\mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Pre}(X, \mathrm{Ab})$ is right adjoint to the sheafification functor and therefore preserves limits. In particular, this means the inclusion preserves monomorphisms.

Moreover, f is epic if and only if for every $x \in X$ the induced map on stalks $f_x : A_x \to B_x$ is epic. That is, for every $x \in X$ and for every $b \in B(U)$ with $x \in U$, there exists an open neighbourhood $V \subseteq U$ of x and an element $a \in A(V)$ such that $f_V(a) = b|_V$. In other words, for every open $U \in \mathcal{O}(X)$ and every $b \in B(U)$, b locally lies in the image of f. □

Theorem 3.3.4. *Let X be a topological space. Then* $\mathrm{Shv}(X, \mathrm{Ab})$ *is an abelian category.*

Proof. For *i)*, note that for any two $A, B \in \mathrm{Shv}(X, \mathrm{Ab})$ the set $\mathrm{Hom}_{\mathrm{Shv}(X,\mathrm{Ab})}(A, B) = \mathrm{Hom}_{\mathrm{Pre}(X,\mathrm{Ab})}(A, B)$ and therefore inherits the group structure. Moreover, the constant abelian presheaf on the trivial group is an abelian sheaf and this is a zero object of $\mathrm{Shv}(X, \mathrm{Ab})$. As sheafification preserves finite products, $\mathrm{Shv}(X, \mathrm{Ab})$ has all finite products. This shows that $\mathrm{Shv}(X, \mathrm{Ab})$ is an additive category.

We define kernels in $\mathrm{Shv}(X, \mathrm{Ab})$ pointwise. That is, let $f : A \to B$ be a map of abelian sheaves, then we define $\ker(f) \in \mathrm{Pre}(X, \mathrm{Ab})$ on objects by $\ker(f)(U) = \ker(f_U)$. For any inclusion $V \subseteq U$ of opens in X, consider the corresponding restriction diagram

$$\begin{tikzcd} \ker(f_U) \arrow[r] \arrow[d, dashed] & A(U) \arrow[r, "f_U"] \arrow[d, "\restriction_V"] & B(U) \arrow[d, "\restriction_V"] \\ \ker(f_V) \arrow[r] & A(V) \arrow[r, "f_V"] & B(V). \end{tikzcd} \tag{38}$$

Then the universal property of the pullback provides an induced map $\ker(f_U) \to \ker(f_V)$. This makes $\ker(f)$ into a well-defined abelian presheaf. To check that this kernel is indeed a sheaf, let $\{a_i \in \ker(f_{U_i})\}_{i \in I}$ be a compatible family for an open cover $U = \bigcup_{i \in I} U_i$. As A is a sheaf, there exists a unique amalgamation $a \in A(U)$ such that $a|_{U_i} = a_i$ for every $i \in I$. Therefore, $f_U(a) \in B(U)$ must be an amalgamation of the compatible family $\{f_{U_i}(a_i) \in B(U_i)\}_{i \in I}$. However, every $f_{U_i}(a_i) = 0$ by assumption. As B is a sheaf, we conclude that $f_U(a) = 0$ by uniqueness of amalgamations in B. This shows that $a \in \ker(f_U)$.

Cokernels in $\mathrm{Shv}(X, \mathrm{Ab})$ are not defined pointwise and require a slightly more subtle approach. Definition 3.3.2 allows us to formulate a looser condition on families than them being compatible. We say that a family $\{b_i \in B(U_i)\}_{i \in I}$ is locally f-compatible, if for every two $i, j \in I$ the element $b_i|_{U_i \cap U_j} - b_j|_{U_i \cap U_j}$ is locally in the image of f.

Define $\mathrm{coker}(f)$ for $f : A \to B$ in $\mathrm{Shv}(X, \mathrm{Ab})$ on an object $U \in \mathcal{O}(X)$ as a set of equivalences classes of pairs $(\{U_i\}, \{b_i\})_{i \in I}$ where the $U_i \in \mathcal{O}(X)$ form an open cover of U and the $b_i \in B(U_i)$ are a locally f-compatible family. Two pairs $(\{U_i\}, \{b_i\})_{i \in I}, (\{V_j\}, \{c_j\})_{j \in J}$

are considered to be equivalent if for every $i \in I$ and $j \in J$ the element $b_i|_{U_i \cap V_j} - c_j|_{U_i \cap V_j}$ is locally in the image of f. We equip $\operatorname{coker}(f)(U)$ with a group structure by imposing

$$[(\{U_i\}, \{b_i\})] + [(\{V_j\}, \{c_j\})] = [(\{U_i \cap V_j\}, \{b_i|_{U_i \cap V_j} + c_j|_{U_i \cap V_j}\})], \tag{39}$$

suppressing the index sets. This is well-defined, because the set of elements $b \in B(U)$ that are locally in the image of f, is closed under addition. Any inclusion $V \subseteq U$ of opens induces a restriction map

$$\operatorname{coker}(f)(U) \to \operatorname{coker}(f)(V) : [(\{U_i\}, \{b_i\})] \mapsto [(\{U_i \cap V\}, \{b_i|_{U_i \cap V}\})]. \tag{40}$$

Therefore $\operatorname{coker}(f)$ is an abelian presheaf.

To check that $\operatorname{coker}(f)$ is a sheaf, let $U \in \mathcal{O}(X)$ and let $b_i \in \operatorname{coker}(f)(U_i)$ form a compatible family for an open cover $U = \bigcup_{i \in I} U_i$. Then for each $i \in I$ we can write each $b_i = [(\{U_{ij}\}, \{b_{ij}\})_{j \in J_i}]$ for some index set J_i depending on i. Consider the element $b = [(\{U_{ij}\}, \{b_{ij}\})_{(i,j) \in \coprod_I J_i}]$ of $\operatorname{coker}(f)(U)$. This element is well-defined, because $\bigcup_{(i,j) \in \coprod_I J_i} U_{ij} = U$ and for every two $(i,j), (i',j') \in \coprod_I J_i$ the element $b_{ij}|_{U_{ij} \cap U_{i'j'}} - b_{i'j'}|_{U_{ij} \cap U_{i'j'}}$ is locally in the image of f. The latter is a direct consequence of the compatibility assumption of the b_i. Finally, note that restricting b along $U_i \subseteq U$ yields b_i again. This shows that b is the required amalgamation.

This construction comes with a map $q : B \to \operatorname{coker}(f)$ given by $q_U : B(U) \to \operatorname{coker}(f)(U) : b \mapsto (\{U\}, \{b\})$. We will now check that this definition does indeed give the cokernel in $\operatorname{Shv}(X, \mathrm{Ab})$. It suffices to check that

$$\begin{tikzcd} A(U) \arrow[r] \arrow[d, "f_U"] & 0 \arrow[d] \\ B(U) \arrow[r, "q_U"] & \operatorname{coker}(f)(U) \end{tikzcd} \tag{41}$$

is a pushout square of abelian groups for each $U \in \mathcal{O}(X)$. For any $a \in A(U)$, we have $q_U(f_U(a)) = [(\{U\}, \{f_U(a)\})] = [(\{U\}, \{0\})]$, because $f_U(a)$ is (locally) in the image of f. Hence the square commutes. Now suppose that $b \in B(U)$ has the property that $q_U(b) = 0$. Then b is locally in the image of f. This is equivalent to the existence of an open cover $\bigcup_{i \in I} U_i = U$ such that for every $i \in I$ there exists an $a_i \in A(U_i)$ with $b|_{U_i} = f(a_i)$. As A is a sheaf, there exists an amalgamation $a \in A(U)$ of the a_i. Note $f(a)$ and b are both amalgamations of the b_i, so we must have $f(a) = b$, as B is a sheaf, showing that the square is indeed a pushout.

Having proved the existence of kernels and cokernels, we still have to check that monomorphisms arise as kernels and epimorphisms arise as cokernels. For the former, suppose $f : A \to B$ in $\operatorname{Shv}(X, \mathrm{Ab})$ is monic. The kernel of the canonical map $q : B \to \operatorname{coker}(f)$ is constructed pointwise. Hence elements of $\ker(q)(U)$ correspond one-to-one with elements $b \in B(U)$ such that $q_U(b) = [(\{U\}, \{b\})] = [(\{U\}, \{0\})]$. That is, elements $b \in B(U)$ that are locally in the image of f. Any such b determines an $a \in A(U)$ such that $f_U(a) = b$, because A is a sheaf. Also, this a is unique by the assumption that f is monic and Lemma 3.3.3. This gives the required isomorphism $\ker(q) \cong A$.

Now suppose that $f : A \to B$ is epic. Let $\iota : \ker(f) \to A$ denote the canonical inclusion. An element of $\operatorname{coker}(\iota)(U)$ is represented by a pair $(\{U_i\}, \{a_i\})_{i \in I}$ where the U_i form an open cover of U and the $a_i \in A(U_i)$ form an ι-compatible family. This compatibility means that for any $i, j \in I$, we have $f(a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j}) = 0$. Hence the $f(a_i) \in B(U_i)$ form a compatible family with a unique amalgamation $b \in B(U)$. This element b does not depend on the choice of representative. That is, any two elements $(\{U_i\}, \{a_i\})_{i \in I}, (\{V_j\}, \{a'_j\})_{j \in J}$ representing the same element of $\operatorname{coker}(\iota)(U)$ admit a common refinement $\{U_i \cap V_j\}_{(i,j) \in I \times J}$ such that $a_i|_{U_i \cap V_j} - a|_{U_i \cap V_j} \in \ker(f)(U_i \cap V_j)$ for each $(i, j) \in I \times J$. Therefore, both families yield the same $b \in B(U)$. This determines a map $\operatorname{coker}(\iota) \to B$. Now suppose we start with an element $b \in B(U)$. Using Lemma 3.3.3, b locally lies in the image. Therefore, there exists an element $(\{U_i\}, \{a_i\})_{i \in I} \in \operatorname{coker}(\iota)(U)$ such that $f(a_i) = b|_{U_i}$ for each $i \in I$. The choices of these elements for each $b \in B(U)$ together form a map $B \to \operatorname{coker}(\iota)$. It is easy to check that the constructed maps are inverses to each other and therefore give the isomorphism $B \cong \operatorname{coker}(\iota)$. □

As an immediate consequence of the explicit constructions in the proof of Theorem 3.3.4, we conclude:

Corollary 3.3.5. *For every open $U \subseteq X$ of a topological space X, the induced evaluation functor $\mathrm{ev}_U : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab} : A \mapsto A(U)$ is additive and left exact.*

Recall that $\Gamma : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$ admits right derived functors, which define sheaf cohomology, if Γ is a left exact functor between abelian categories and $\mathrm{Shv}(X, \mathrm{Ab})$ has enough injectives. The first part follows from Theorem 3.3.4 and Corollary 3.3.5 and last part remains.

Proposition 3.3.6. *The category $\mathrm{Shv}(X, \mathrm{Ab})$ has enough injectives.*

Proof. We will construct an adjunction

$$\begin{tikzcd} \mathrm{Shv}(X, \mathrm{Ab}) \arrow[rr, bend left, "\mathrm{Stalk}"] \arrow[rr, phantom, "\perp"] & & \prod_{x \in X} \mathrm{Ab} \arrow[ll, bend left, "G"] \end{tikzcd} \tag{42}$$

to eventually conclude that $\mathrm{Shv}(X, \mathrm{Ab})$ has enough injectives. For each $X \in X$ let $\mathrm{Stalk}_x : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$ denote the functor that takes the stalk of abelian sheaves at x. These functors together form $\mathrm{Stalk} : \mathrm{Shv}(X, \mathrm{Ab}) \to \prod_{x \in X} \mathrm{Ab}$. Define $G : \prod_{x \in X} \mathrm{Ab} \to \mathrm{Shv}(X, \mathrm{Ab})$ on objects by $G(\{A_x\}_{x \in X})(U) = \prod_{x \in U} A_x$. The restriction maps $G(\{A_x\}_{x \in X})(V \subseteq U)$ are given by projections $\prod_{x \in U} A_x \to \prod_{x \in V} A_x$.

First we have to check that G does take values in abelian sheaves. Fix an object $\prod_{x \in X} A_x \in \prod_{x \in X} \mathrm{Ab}$. Consider an open cover $\{U_i\}_{i \in I}$ of an open $U \in \mathcal{O}(X)$ and let $a^i \in G(\prod_{x \in X} A_x)(U_i)$ be a compatible compatible family at U. Then each a^i has the form $\prod_{x \in U_i} a^i_x$ with $a^i_x \in A_x$. Define $a := \prod_{x \in X} a_x \in G(\prod_{x \in X} A_x)(U)$ where $a_x = a^i_x$ for some $i \in I$. This does not depend on the choice of i, because of the compatibility and therefore defines an amalgamation.

For the adjunction, let $A \in \mathrm{Shv}(X, \mathrm{Ab})$ and $B \in \prod_{x \in X} \mathrm{Ab}$ with components B_x. A morphism $\mathrm{Stalk}(A) \to B$ consists of a collection of maps $A_x \to B_x$ of abelian groups indexed by X. For every $U \in \mathcal{O}(X)$ precomposition with the canonical map $\tau_{U,x} : A(U) \to A_x$ yields a collection of maps that can be organized as a single map into a product, thus a map of the form $A(U) \to \prod_{x \in U} B_x$. This gives the pointwise construction for a map $A \to G(B)$.

Now suppose we start with a map $A \to G(B)$ of sheaves. After application of the stalk functor, we obtain maps $A_x \to G(B)_x$. Note that $G(B)_x = \varinjlim \prod_{y \in U} B_y$ where the colimit ranges over the poset $\mathcal{O}_x(X)$ of all opens U containing x. For every such open there is a map $\prod_{y \in U} B_y \to B_x$ that is simply given by projection. Together these maps form a cocone over the diagram $G(B)_{(-)} : \mathcal{O}_x(X) \to \mathrm{Ab}$. This induces a map $G(X)_x \to B_x$, which we use to form maps $A_x \to G(B)_x \to B_x$. These two construction are inverse to each other and therefore give the required adjunction.

For any commutative ring R, the category of R-modules has enough injectives [2]. In particular, this means that the category of abelian groups has enough injectives. Therefore, $\prod_{x \in X} \mathrm{Ab}$ also has enough injectives. Let $A \in \mathrm{Shv}(X, \mathrm{Ab})$. Then there exists a monomorphism $f : \mathrm{Stalk}(A) \to I$ in $\prod_{x \in X} \mathrm{Ab}$ such that I is injective. As the stalk functor is exact, it preserves monomorphisms. Using the adjunction and preservation of monomorphisms, we can directly check that $G(I)$ is injective. Therefore, we only have to check that the transpose map $\bar{f} : A \to G(I)$ is monic. It suffices to show that $\bar{f}$ is pointwise monic. Let $U \in \mathcal{O}(X)$ and $a \in A(U)$ and suppose that $\bar{f}_U(a) = 0$. This implies that for every $x \in U$ the composite

$$A(U) \xrightarrow{\tau_{U,x}} A_x \xrightarrow{f_x} I_x,$$

where f_x denotes a component of f, sends a to 0. As f_x is monic, we must have $\tau_{U,x}(a) = 0$. Hence there exists an open cover $\{V_x\}_{x \in X}$ of U such that $a|_{V_x} = 0$ for each V_x. As A is a sheaf, we conclude that $a = 0$. □

In conclusion, $\mathrm{Shv}(X, \mathrm{Ab})$ is an abelian category with enough injectives and the global section functor is additive and left exact. Therefore, Γ admits right derived functors $R^n(\Gamma) : \mathrm{Shv}(X, \mathrm{Ab}) \to \mathrm{Ab}$.

Definition 3.3.7. Let X be a topological space and $B \in \mathrm{Shv}(X, \mathrm{Ab})$ be an abelian sheaf. Then the n-th sheaf cohomology group of X with coefficients in B is defined as

$$H^n(X; B) = R^n(\Gamma)(B). \tag{43}$$

4 Sheaf Cohomology with Fibrant Replacements

So far, we have studied simplicial presheaves and simplicial sheaves on a site $\mathcal{C}$. These categories come with the structure of a model category and consequently admit a derived category. In this chapter, we will study presheaves that take values in simplicial abelian groups, non-negatively graded chain complexes and $\mathbb{Z}$-graded chain complexes. We will see that the model structure on these categories allows us to give a characterization of sheaf cohomology that involves fibrant replacements instead of injective resolutions. This allows us to use all the tool from homotopy theory, specifically the ones for model categories, when studying sheaf cohomology.

In comparison to other chapters, this chapter relies more on references to other works. The interested reader can use these to investigate some of the steps in detail. In the sections below we follow the approach of Jardine from Chapter 2 of [9].

4.1 Simplicial abelian presheaves

In order to describe sheaf cohomology using homotopy theory, we will first discuss a model structure on the category $\mathrm{Pre}(X, \mathrm{sAb})$ of simplicial abelian presheaves. This category is defined as the functor category $\mathrm{Fun}(\mathcal{O}(X)^{op}, \mathrm{sAb})$, where $\mathrm{sAb} = \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Ab})$. Once we established this model structure, we will equip $\mathrm{Pre}(X, \mathrm{Ch}_+)$ with the induced model structure. At the end of the section, we will give a characterization of the weak equivalences in this setting.

Consider the forgetful-free adjunction

$$\begin{tikzcd} \mathrm{Sets} \arrow[r, bend left, "\mathbb{Z}"] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend left, "F"] \end{tikzcd} \tag{1}$$

between sets and abelian groups. This induces an adjunction between siplicial sets and simplicial abelian groups, which in turn induces an adjunction between simplicial presheaves and simplicial abelian presheaves.

Theorem 4.1.1. *The adjunction*

$$\begin{tikzcd} \mathrm{Pre}(X, \mathrm{sSets}) \arrow[r, bend left, "\mathbb{Z}"] \arrow[r, phantom, "\perp"] & \mathrm{Pre}(X, \mathrm{sAb}) \arrow[l, bend left, "F"] \end{tikzcd} \tag{2}$$

satisfies the conditions of transfer. Therefore, the global structure on $\mathrm{Pre}(X, \mathrm{sSets})$ *induces a model structure on* $\mathrm{Pre}(X, \mathrm{sAb})$.

Proof. For condition *iii)* of Theorem 1.4.4, let $A \in \mathrm{Pre}(X, \mathrm{sSets})$ be the domain of a generating cofibration or trivial cofibration as described in Theorem 2.2.3. Then A is in particular α-bounded and thus so is $F(\mathbb{Z}A)$. It follows by regularity of α that for any transfinite composition $(Z_\gamma)_{\gamma<\alpha}$ in $\mathrm{Pre}(X, \mathrm{sAb})$ and map $\mathbb{Z}A \to Z_\alpha$, there exists a factorization through some Z_γ with $\gamma < \alpha$.

Note that $\mathbb{Z} : \mathrm{Pre}(X, \mathrm{sSets}) \to \mathrm{Pre}(X, \mathrm{sAb})$ preserves monomorphisms. Therefore, conditions *i)* and *ii)* are fulfilled once we have shown that $\mathbb{Z}$ also preserves weak equivalences. Suppose that $A \to B$ is a weak equivalence in $\mathrm{Pre}(X, \mathrm{sSets})$. Then $\mathrm{Ex}^\infty A \to \mathrm{Ex}^\infty B$ is a combinatorial weak equivalence by Theorem 2.3.1. Hence $(\mathrm{Ex}^\infty A)_x \to (\mathrm{Ex}^\infty B)_x$ is a weak equivalence of simplicial sets for each $x \in X$. As $\mathbb{Z}$ commutes with taking stalks, it follows from Proposition 2.14 from [4] that $F(\mathbb{Z}\,\mathrm{Ex}^\infty A)_x \to F(\mathbb{Z}\,\mathrm{Ex}^\infty B)_x$ is a weak equivalence of simplicial sets for each $x \in X$. The fact that $F \circ \mathbb{Z}$ commutes with colimits and with $\mathrm{Ex} : \mathrm{sSets} \to \mathrm{sSets}$ implies that the maps $\mathrm{Ex}^\infty(F\mathbb{Z}A)_x \to \mathrm{Ex}^\infty(F\mathbb{Z}B)_x$ are weak equivalences of simplicial sets. Therefore, $\mathrm{Ex}^\infty(F\mathbb{Z}A) \to \mathrm{Ex}^\infty(F\mathbb{Z}B)$ is a combinatorial weak equivalence in $\mathrm{Pre}(X, \mathrm{sSets})_f$, showing that $F\mathbb{Z}A \to F\mathbb{Z}B$ is a topological weak equivalence. □

Given a simplicial abelian group A, let MA denote the chain complex where MA_n consists of the n-simplices of A and whose differential is given by

$$\partial : MA_n \to MA_{n-1} : x \mapsto \sum_{i=0}^{n} (-1)^i d_i x. \tag{3}$$

This construction determines a functor $M : \mathrm{sAb} \to \mathrm{Ch}_+$ called the Moore complex functor. Each Moore complex MA contains a certain subcomplex $NA \subseteq MA$ called the normalized chain complex corresponding to A. The inclusion $NA \to MA$ is a chain homotopy equivalence (Theorem II.2.4 from [4]). There is a classical result that states that the normalized chain complex functor admits a pseudo-inverse,

$$\begin{tikzcd} \mathrm{sAb} \arrow[r, bend left, "N"] & \mathrm{Ch}_+ \arrow[l, bend left, "\Gamma"] \end{tikzcd} \tag{4}$$

called the Dold-Kan correspondence (Theorem II.2.3 from [4]). This equivalence of categories induces an equivalence of categories on the level of presheaves, that is, $\mathrm{Pre}(X, \mathrm{sAb}) \cong \mathrm{Pre}(X, \mathrm{Ch}_+)$. Therefore, we can directly conclude that:

Corollary 4.1.2. *The category* $\mathrm{Pre}(X, \mathrm{Ch}_+)$ *admits a model structure by imposing that* $f : A \to B$ *in* $\mathrm{Pre}(X, \mathrm{Ch}_+)$ *is a fibration, weak equivalence or cofibration if and only if* Γf *is one in* $\mathrm{Pre}(X, \mathrm{sAb})$.

The weak equivalences of presheaves of chain complexes admit a useful characterization. Indeed, suppose that A is a presheaf of chain complexes. Then we can form its n-th homology presheaf H_nA by

$$H_nA(U) := \ker(\partial_n : A_n(U) \to A_{n-1}(U)) / \operatorname{im}(\partial_{n+1} : A_{n+1}(U) \to A_n(U)) \tag{5}$$

where we adopt the convention $A_{-1}(U) = 0$ for the moment. We define the n-th homology sheaf of A to be the sheafification of H_nA. Suppose that $f : A \to B$ is a map of presheaves of chain complexes, then we say that f is a quasi-isomorphism if f induces an isomorphism on all homology sheaves.

Proposition 4.1.3. *A map in* $\mathrm{Pre}(X, \mathrm{Ch}_+)$ *is a weak equivalence if and only if it is a quasi-isomorphism.*

Proof. Theorem 2.3.1 shows that each simplicial presheaf is weakly equivalent to a locally fibrant simplicial presheaf. Whether a map of locally fibrant simplicial presheaves is a weak equivalence can be checked stalkwise. Hence the result follows from Corollary II.2.5 from [4]. □

In the model structures that we defined on presheaf categories, we chose the classes of cofibrations and weak equivalences. We did not impose any conditions of the class of fibrations, other than the one stating that fibrations are maps having the right lifting property with respect to the chosen class of trivial cofibrations. Therefore, it is difficult to give a characterization of fibrations that avoids the use of lifting properties. In particular, this is the case for fibrations in $\mathrm{Pre}(X, \mathrm{Ch}_+)$. Nevertheless, we are able to formulate a necessary condition for a map to be a fibration.

Proposition 4.1.4. *Let $p : C \to D$ be a fibration in $\mathrm{Pre}(X, \mathrm{Ch}_+)$. Then $p_n : C_n \to D_n$ in $\mathrm{Pre}(X, \mathrm{Ab})$ is an epimorphism, for each degree $n \geq 1$.*

Proof. Since $p : C \to D$ is a fibration, $\Gamma p : \Gamma C \to \Gamma D$ is a fibration in $\mathrm{Pre}(X, \mathrm{sAb})$. Forgetting the group structure, this means that $\Gamma p : \Gamma C \to \Gamma D$ is a global fibration in $\mathrm{Pre}(X, \mathrm{sSets})$. Using the adjunction $(-)_U \dashv \mathrm{ev}_U$ from Theorem 2.2.3, we conclude that $p_* : \Gamma C(U) \to \Gamma D(U)$ is a Kan fibration for each $U \in \mathcal{O}(X)$. Let $x : \Delta^n \to D(U)$ be an n-simplex with $n \geq 1$. Because $D(U) \cong N\Gamma D(U)$ by the Dold-Kan correspondence, we may assume without loss of generality that $x|_{\Lambda^n_n} = 0$. Consider the commutative diagram

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "0"] \arrow[d, hook] & \Gamma C(U) \arrow[d, "p_*"] \\ \Delta^n \arrow[r, "x"] \arrow[ur, dashed] & \Gamma D(U). \end{tikzcd} \tag{6}$$

This diagram admits a lift, because p_* is a Kan fibration and thus shows that $p_n : C_n \to D_n$ is an epimorphism. □

4.2 Spectra of chain complexes

Although the canonical inclusion $\mathrm{Pre}(X, \mathrm{Ch}_+) \to \mathrm{Pre}(X, \mathrm{Ch})$ of presheaves of chain complexes into presheaves of $\mathbb{Z}$-graded chain complexes does admit a right adjoint, a transfer argument does not give $\mathrm{Pre}(X, \mathrm{Ch})$ a model structure that we are interested in. In particular, we want weak equivalences in $\mathrm{Pre}(X, \mathrm{Ch})$ to be the quasi-isomorphisms. In this section we will introduce spectra (of presheaves of chain complexes) and show that this category of spectra inherits a model structure from $\mathrm{Pre}(X, \mathrm{Ch}_+)$, that we refer to as the strict model structure. The strict model structure induces a stable model structure on the category of spectra, which can be proven by a technique called left Bousfield-localization. The stable structure is used to solve our initial problem of finding a model structure on $\mathrm{Pre}(X, \mathrm{Ch})$ with the expected notion of weak equivalence.

For any $n \in \mathbb{Z}$, the shift functor $(-)[n] : \mathrm{Pre}(X, \mathrm{Ch}) \to \mathrm{Pre}(X, \mathrm{Ch}_+)$ is defined by:

$$A[n]_p = \begin{cases} A_{p+n}, & \text{if } p > 0, \\ \ker(\partial : A_n \to A_{n-1}), & \text{if } p = 0. \end{cases} \tag{7}$$

This also induces a functor $\mathrm{Pre}(X, \mathrm{Ch}_+) \to \mathrm{Pre}(X, \mathrm{Ch}_+)$ by regarding a presheaf of chain complexes as a presheaf of $\mathbb{Z}$-graded chain complexes concentrated in non-negative degrees. By abuse of notation, we will also refer to the latter functor as the shift functor $(-)[n]$. Note that for positive n, the shift functors

$$\begin{tikzcd} \mathrm{Pre}(X, \mathrm{Ch}_+) \arrow[rr, bend left, "(-)[-n]"] \arrow[rr, phantom, "\perp"] & & \mathrm{Pre}(X, \mathrm{Ch}_+) \arrow[ll, bend left, "(-)[n]"] \end{tikzcd} \tag{8}$$

form an adjunction.

Proposition 4.2.1. *The adjunction above is a Quillen adjunction.*

Proof. It suffices to consider the case $n = 1$, because Quillen adjunctions are composable. Clearly shift functors preserve quasi-isomorphisms. Lemma 1.9 from [9] states that $(-)[-1]$ preserves cofibrations. □

Definition 4.2.2. The category $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ of spectra (of presheaves of chain complexes) has objects A that are sequences A^n of presheaves of chain complexes, indexed by $\mathbb{N}$, together with maps $\sigma^n : A^n[-1] \to A^{n+1}$ in $\mathrm{Pre}(X, \mathrm{Ch}_+)$ for each $n \geq 0$. These maps are called the bonding maps of A. A morphism of spectra $f : A \to B$ is a collection of maps $f^n : A^n \to B^n$ of presheaves of chain complexes, respecting the bonding maps. That is, for every $n \geq 0$ the diagram

$$\begin{tikzcd} A^n[-1] \arrow[r, "\sigma^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[d, "f^{n+1}"] \\ B^n[-1] \arrow[r, "\sigma^n"] & B^{n+1} \end{tikzcd} \tag{9}$$

must commute.

Remark 4.2.3. Unravelling the definition above, we can think of spectra as presheaves of diagrams of abelian groups, much like presheaves of chain complexes. That is, a spectrum $A \in \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ is a commutative diagram of presheaves of abelian groups having the form

$$\begin{tikzcd} & & 0 \arrow[d] & A_0^0 \arrow[l] \arrow[d] & A_1^0 \arrow[l] \arrow[d] & \dots \arrow[l] \\ & 0 \arrow[d] & A_0^1 \arrow[l] \arrow[d] & A_1^1 \arrow[l] \arrow[d] & A_2^1 \arrow[l] \arrow[d] & \dots \arrow[l] \\ 0 \arrow[d] & A_0^2 \arrow[l] \arrow[d] & A_1^2 \arrow[l] \arrow[d] & A_2^2 \arrow[l] \arrow[d] & A_3^2 \arrow[l] \arrow[d] & \dots \arrow[l] \\ \dots & \dots & \dots & \dots & \dots & \end{tikzcd} \tag{10}$$

where any composition of two horizontal arrows is the zero map.

We say that a map $f : A \to B$ of spectra is a strict weak equivalence (resp. strict fibration) if each $f^n : A^n \to B^n$ is a weak equivalence (resp. fibration) in $\mathrm{Pre}(X, \mathrm{Ch}_+)$. A cofibration in $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ is a map having the left lifting property with respect to every map that is a strict weak equivalence as well as a strict fibration.

Theorem 4.2.4. *The classes of cofibrations, strict fibrations and strict weak equivalences defined above equip* $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ *with the structure of a model category.*

This strict model structure from the theorem above can be defined in a much more general context. If we start with a cofibrantly generated model category $\mathcal{D}$ (in our case $\mathrm{Pre}(X, \mathrm{Ch}_+)$) together with a left Quillen functor $L : \mathcal{D} \to \mathcal{D}$ (we use a shift functor), the category of spectra on $\mathcal{D}$ consists of sequences $(D^n)_{n \geq 0}$ in $\mathcal{D}$ together with bonding maps $L(D^n) \to D^{n+1}$, with the obvious choice for morphisms. The definition of the strict model structure also makes sense in the general case. The proof of the general version of Theorem 4.2.4 can is given in Chapter 1 of [7]. In the same chapter Hovey also gives a characterization for cofibrations in this setting.

Proposition 4.2.5. *A map $f : A \to B$ of spectra is a cofibration if and only if $f^0 : A^0 \to B^0$ and the canonical maps $A^{n+1} \cup_{A^n[-1]} B^n[-1] \to B^{n+1}$ for $n \geq 0$ are cofibrations in* $\mathrm{Pre}(X, \mathrm{Ch}_+)$.

This result implies that cofibrations are in particular monomorphisms.

Corollary 4.2.6. *If $f : A \to B$ is a cofibration of spectra, then each $f^n : A^n \to B^n$ is a cofibration of presheaves of chain complexes.*

Proof. We proceed by induction. Suppose that $f^n : A^n \to B^n$ is a cofibration of presheaves of chain complexes. Then the induced map $A^n[-1] \to B^n[-1]$ is a cofibration by Lemma 1.9 of [9]. It is easy to see that $\mathrm{Pre}(X, \mathrm{Ch}_+)$ is an abelian category. Therefore, monomorphisms in $\mathrm{Pre}(X, \mathrm{Ch}_+)$ are stable under pushout. Hence $A^{n+1} \to A^{n+1} \cup_{A^n[-1]} B^n[-1]$ is a cofibration. Then Proposition 4.2.5 ensures that $f^{n+1} : A^{n+1} \to B^{n+1}$ is a cofibration as well. □

Now that we have discussed the strict model structure on the category of spectra, let us investigate the relation between spectra and presheaves of $\mathbb{Z}$-graded chain complexes. In particular, we will construct an adjunction between these categories. Let $A \in \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$, then the bonding maps of A form a sequence

$$A^n_m \to A^{n+1}_{m+1} \to A^{n+2}_{m+2} \to \dots \tag{11}$$

in $\mathrm{Pre}(X, \mathrm{Ab})$, using the notation from Remark 4.2.3. We denote the colimit of this sequence by SA^{m-n}. The boundary maps induce morphisms $\partial : SA^{m-n+1} \to SA^{m-n}$. This makes SA into a presheaf of chain complexes and as of such determines a functor

$$S : \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+)) \to \mathrm{Pre}(X, \mathrm{Ch}). \tag{12}$$

There is also a functor

$$T : \mathrm{Pre}(X, \mathrm{Ch}) \to \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+)) \tag{13}$$

in the other direction. This functor takes good truncations in degrees $n \leq 0$ of a presheaf of $\mathbb{Z}$-graded chain complexes and organises it in a spectrum. By this we mean that a presheaf of $\mathbb{Z}$-graded chain complexes

$$\dots \xleftarrow{\partial_{-1}} A_1 \xleftarrow{\partial_0} A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} \dots \tag{14}$$

is sent to the spectrum corresponding to the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & \ker(\partial_0) \arrow[l] \arrow[d] & A_1 \arrow[l] \arrow[d, equal] & \dots \arrow[l] \\
 & 0 \arrow[d] & \ker(\partial_{-1}) \arrow[l] \arrow[d] & A_0 \arrow[l] \arrow[d, equal] & A_1 \arrow[l] \arrow[d, equal] & \dots \arrow[l] \\
0 \arrow[d] & \ker(\partial_{-2}) \arrow[l] \arrow[d] & A_{-1} \arrow[l] \arrow[d, equal] & A_0 \arrow[l] \arrow[d, equal] & A_1 \arrow[l] \arrow[d, equal] & \dots \arrow[l] \\
\dots & \dots & \dots & \dots & \dots &
\end{tikzcd} \tag{15}$$

Note that there is a canonical isomorphism

$$\varepsilon_A : STA \to A \tag{16}$$

for any presheaf of $\mathbb{Z}$-graded chain complexes A and this isomorphism is natural in A. Furthermore, for any spectrum B there is a map

$$\eta_B : B \to TSB \tag{17}$$

which is componentwise induced by the canonical maps $B^n_m \to SB^{m-n}$. Note that η is also natural in B. In order to show that S is left adjoint to T, we consider the triangle identities. For $B \in \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$, it is straight-forward to check the diagram

$$\begin{tikzcd}
SB \arrow[r, "S\eta_B"] \arrow[dr, "\mathrm{id}"'] & STSB \arrow[d, "\varepsilon_{SB}"] \\
 & SB
\end{tikzcd} \tag{18}$$

commutes. Because ε is a natural isomorphism, it follows immediately that the dual triangle equality also holds. Indeed, for any $A \in \mathrm{Pre}(X, \mathrm{Ch})$ consider the diagram:

$$\begin{tikzcd}
 & TSTA \arrow[r, "TS\eta_{TA}"] \arrow[dl, "T\varepsilon_A"'] \arrow[dr, equal] & TSTSTA \arrow[dl, "TST\varepsilon_A"'] \arrow[d, "T\varepsilon_{STA}"] \\
TA \arrow[r, "\eta_{TA}"] \arrow[dr, equal] & TSTA \arrow[d, "T\varepsilon_A"] & TSTA \arrow[dl, "T\varepsilon_A"] \\
 & TA &
\end{tikzcd} \tag{19}$$

Then the triangle in the back is T applied to diagram (18) for the case $B = TA$, and therefore commutes. The top, bottom and right squares are naturality squares, which are commutative.

As every arrow pointing in a downwards-left direction is an isomorphism, it follows that the front triangle commutes. This establishes $S \dashv T$.

We want to define model structures on both $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ and $\mathrm{Pre}(X, \mathrm{Ch})$ such that $S \dashv T$ becomes a Quillen equivalence. Moreover, we want the weak equivalences in $\mathrm{Pre}(X, \mathrm{Ch})$ to be quasi-isomorphisms. As S should detect weak equivalences, this motives the choice of a new form of weak equivalence of spectra; the stable weak equivalence. That is, a map $f : A \to B$ of spectra is a stable weak equivalence if Sf is a quasi-isomorphism. Observe that S sends strict weak equivalences to quasi-isomorphisms. This implies that every strict weak equivalence is in particular a stable weak equivalence. Define stable fibrations as the maps having the right lifting property with respect to all maps that are cofibrations as well as stable weak equivalences. Then it follows immediately that every stable fibration is a strict fibration.

Remark 4.2.7. As $\varepsilon : ST \Rightarrow \mathrm{id}$ is a natural isomorphism, it follows from diagram (18) that $S\eta : S \Rightarrow STS$ is also a natural isomorphism. Hence, $\eta_B : B \to TSB$ is a stable weak equivalence for each spectrum B. Consequently, a map $f : A \to B$ is a stable weak equivalence if and only if the induced map $TSf : TSA \to TSB$ is a strict weak equivalence.

There are many ways to find different model structures on a category that is already equipped with a model structure. These techniques are referred to as localizations, which are not to be confused with the localization used to obtain homotopy categories. One way to equip spectra with the stable model structure using the strict structure, is to proceed by a left Bousfield localization. This is the approach chosen in [7]. It involves showing that the strict structure is left proper, cellular and admits functorial cofibrant replacements. We, however, will arrive at the stable structure in a different way, by making use of the functor $TS : \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+)) \to \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$. Specifically, once we have checked the conditions, we apply Theorem IX.6.1 from [4] to the pair $(TS, \eta : \mathrm{id} \Rightarrow TS)$ to conclude:

Theorem 4.2.8. *The classes of cofibrations, stable fibrations and stable weak equivalences defined above equip* $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ *with the structure of a model category.*

Lemma 4.2.9 and Lemma 4.2.10 together show that the required conditions are met and therefore form a proof of Theorem 4.2.8.

Lemma 4.2.9. *The maps* $\eta_{TSB}, TS(\eta_B) : TSB \to TSTSB$ *are strict weak equivalences.*

Proof. For any $B \in \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$, diagram (18) provides a factorization of the identity on TSB as

$$TSB \xrightarrow{TS\eta_B} TSTSB \xrightarrow{T\varepsilon_{SB}} TSB. \tag{20}$$

Alternatively, diagram (19) allows us to factorize the identity on TSB as

$$TSB \xrightarrow{\eta_{TSB}} TSTSB \xrightarrow{T\varepsilon_{SB}} TSB. \tag{21}$$

By Remark 4.2.7, the map $TS\eta_B$ is a strict weak equivalence. Since ε is a natural isomorphism, it follows from the two factorizations above that η_{TSB} is a strict weak equivalence as well. □

Let $p : C \to D$ be a stable fibration of spectra. Since $\mathrm{Pre}(X, \mathrm{Ch}_+)$ is an abelian category, consider a factorization

$$C^n \xrightarrow{\pi^n} E^n \xrightarrow{j^n} D^n \tag{22}$$

of p^n as a monomorphism followed by an epimorphism. The sequence $(E^n)_{n\geq 0}$ of presheaves of chain complexes inherits bonding maps from D and thus determines a spectrum E. This makes π and j into maps of spectra. Proposition 4.1.4 states that each map $p^n_m : C^n_m \to D^n_m$ is an epimorphism for $m > 0$. Therefore, $j^n_m : E^n_m \to D^n_m$ is an isomorphism for each $m > 0$, which implies that j is a stable weak equivalence.

Let $i : K \to C$ be the (degreewise) kernel of p. Then we obtain a short exact sequence

$$0 \to SK \xrightarrow{i_*} SC \xrightarrow{\pi_*} SE \to 0 \tag{23}$$

of presheaves of $\mathbb{Z}$-graded chain complexes. This in turn induces a long exact sequence of homology sheaves

$$\ldots \to H_{n+1}(SE) \xrightarrow{\partial} H_n(SK) \xrightarrow{i_*} H_n(SC) \xrightarrow{\pi_*} H_n(SE) \xrightarrow{\partial} \ldots \tag{24}$$

The fact that j is a stable weak equivalence means that j induces isomorphisms $H_n(SE) \cong H_n(SD)$. As of such, we also have a long exact sequence

$$\ldots \to H_{n+1}(SD) \xrightarrow{\partial} H_n(SK) \xrightarrow{i_*} H_n(SC) \xrightarrow{p_*} H_n(SD) \xrightarrow{\partial} \ldots \tag{25}$$

Now suppose that $i : A \to B$ is cofibration of spectra. Then Corollary 4.2.6 states that $i^n : A^n \to B^n$ is monomorphism of presheaves of chain complexes. Let $p : B \to C$ be the (degreewise) cokernel of i. Then there also is an induced long exact sequence of homology sheaves

$$\ldots \to H_{n+1}(SC) \xrightarrow{\partial} H_n(SA) \xrightarrow{i_*} H_n(SB) \xrightarrow{p_*} H_n(SC) \xrightarrow{\partial} \ldots \tag{26}$$

Lemma 4.2.10. *Stable weak equivalences are preserved under pullback along stable fibrations. Dually, stable weak equivalences are preserved under pushout along cofibrations.*

Proof. Consider a pullback diagram

$$\begin{tikzcd} A \times_D C \arrow[r, "f_*"] \arrow[d, "g_*"] & C \arrow[d, "g"] \\ A \arrow[r, "f"] & D \end{tikzcd} \tag{27}$$

in $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ where f is a stable weak equivalence and g a stable fibration. Then g_* is a stable fibration as well. Factorize $g = jp$ as an epimorphism followed by a monomorphism and factorize $g_* = j'p'$ similarly. Then there is a sequence of isomorphisms

$$\ker(p) \cong \ker(g) \cong \ker(g_*) \cong \ker(p'). \tag{28}$$

The map of the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker(p') \arrow[r] \arrow[d] & D \arrow[r, "p'"] \arrow[d, "f_*"] & E' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \ker(p) \arrow[r] & C \arrow[r, "p"] & E \arrow[r] & 0 \end{tikzcd} \tag{29}$$

determines a map between the corresponding long exact sequences of homology sheaves. Note that $H_n(S\ker(p)) \cong H_n(S\ker(p'))$. Furthermore, f is a stable weak equivalence by assumption and the maps j and j' are stable weak equivalences by the consideration above this lemma. Hence these maps induce isomorphisms $H_n(SE') \cong H_n(SA) \cong H_n(SD) \cong H_n(SE)$. It follows from the long exact sequences that f_* induces isomorphisms $H_n(SD) \to H_n(SC)$, i.e. f_* is a stable weak equivalence.

The dual statement is proven in a similar fashion. This proof is slightly shorter, because cofibrations are monic and we thus do not need to consider epi-mono factorizations. □

Now that we properly introduced the stable model structure on $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$, we will use it to equip $\mathrm{Pre}(X, \mathrm{Ch})$ with a model structure.

Theorem 4.2.11. *The stable model structure on the category of spectra induces a model structure on* $\mathrm{Pre}(X, \mathrm{Ch})$ *that makes the adjunction* $S \dashv T$ *into a Quillen equivalence.*

Proof. Let f be a map in $\mathrm{Pre}(X, \mathrm{Ch})$. We say that f is a fibration or weak equivalence if Tf is a stable fibration or stable weak equivalence, respectively. Moreover, f is a cofibration if and only if f has the left lifting property with respect to all trivial fibrations in $\mathrm{Pre}(X, \mathrm{Ch})$.

It follows that the class of weak equivalences in $\mathrm{Pre}(X, \mathrm{Ch})$ is exactly the class of quasi-isomorphisms. Note that Tf is a stable weak equivalence if and only if $TSTf$ is a strict weak equivalence. As $\varepsilon : \mathrm{id} \Rightarrow ST$ is a natural isomorphism, this is equivalent to Tf being a strict weak equivalence. Moreover, Theorem IX.6.8 of [4] states that Tf is a stable fibration if and only if Tf is a strict fibration and

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[d, "Tf"] & TSTA \arrow[d, "TSTf"] \\ TB \arrow[r, "\eta_B"] & TSTB \end{tikzcd} \tag{30}$$

is a homotopy cartesian diagram. However, η_{TA} and η_{TB} are isomorphisms, so the square above is always a homotopy cartesian square by Lemma II.9.20.i) from [4]. Hence Tf is a stable fibration if and only if Tf is a strict fibration. The strict model category on $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ is cofibrantly generated by sets I and J of cofibrations and trivial cofibrations, respectively. Hence we conclude f is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all maps Sj for j in J (respectively to maps Si with i in I).

One can easily check the axioms **M1)** to **M3)**. The observation above allows us to perform a small object argument to obtain factorizations. Following the proof of Theorem 1.4.4 **M4)** and **M5)** are checked the standard way.

The functor T preserves weak equivalences and fibrations by definition, so $S \dashv T$ is a Quillen adjunction. Let $C \in \mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$ be cofibrant (in the stable structure) and $D \in \mathrm{Pre}(X, \mathrm{Ch})$ be fibrant. Then a map $f : SC \to D$ is a quasi-isomorphism if and only if $Tf : TSC \to TD$ is a stable weak equivalence. This is equivalent to $Tf \circ \eta_C : C \to TD$ being a stable weak equivalence and $Tf \circ \eta_C$ is the transpose of f. Hence Theorem 8.46 from [5] ensures that $S \dashv T$ is a Quillen equivalence. □

4.3 Injective resolutions and fibrant replacements

Recall that the sheaf cohomology of a topological space X with coefficients in $B \in \mathrm{Shv}(X, \mathrm{Ab})$ involves taking global sections of an injective resolution $0 \to B \to I_\bullet$ for B. Let $B[0] \in \mathrm{Pre}(X, \mathrm{Ch})$ denote the presheaf of chain complexes given by B concentrated in degree 0. Then an injective resolution of B can be reorganized

$$\begin{tikzcd} \dots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \dots \arrow[l] \\ \dots & I_{-2} \arrow[l] & I_{-1} \arrow[l] & I_0 \arrow[l] & 0 \arrow[l] & \dots \arrow[l] \end{tikzcd} \tag{31}$$

as a map $B[0] \to I$ of presheaves of $\mathbb{Z}$-graded chain complexes. We can formulate the conditions for a sequence $0 \to B \to I_\bullet$ to be an injective resolution in terms of the corresponding map $B[0] \to I$. That is, $B[0] \to I$ must be a quasi-isomorphism, I must be concentrated in non-positive degrees and the abelian sheaves I_n must be injective for each $n \in \mathbb{Z}$. As it turns out, the latter two conditions may be substituted for a new condition that involves the model structure on $\mathrm{Pre}(X, \mathrm{Ch})$.

Theorem 4.3.1. *Let B be a sheaf of abelian groups on a topological space X. Suppose that $B[0] \to J$ is a fibrant replacement for $B[0]$ in $\mathrm{Pre}(X, \mathrm{Ch})$. Then there is a natural isomorphism*

$$H^n(X, B) \cong H^n(\Gamma(J)_\bullet). \tag{32}$$

This theorem tells us that sheaf cohomology can alternatively be described by taking global sections of fibrant replacements of $B[0]$. In order to prove this, we need two intermediate results.

Proposition 4.3.2. *The category $\mathrm{Pre}(X, \mathrm{Ch}_+)$ of presheaves of chain complexes has the structure of a category of fibrant objects where weak equivalences are given by quasi-isomorphisms and fibrations are maps that become epimorphic after sheafification.*

Proof. With these definitions, the axioms **N1)**, **N2)** and **N5)** are trivial. For axiom **N3)**, we are required to show that fibrations and trivial fibrations are preserved under pullback. Note that sheafification commutes with taking pullbacks. Hence in the case of fibrations, it suffices to show that epimorphisms in $\mathrm{Shv}(X, \mathrm{Ch}_+)$ are preserved under pullback. This is the case, because $\mathrm{Shv}(X, \mathrm{Ch}_+)$ is an abelian category.

Let $f : A \to B$ be a trivial fibration and $g : C \to B$ any map in $\mathrm{Pre}(X, \mathrm{Ch}_+)$. Let $i : K \to A$ be the kernel of f and $j : K' \to A \times_B C$ be the kernel of $f_* : A \times_B C \to C$. Since sheafification commutes with pullbacks, we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K'} \arrow[r] \arrow[d] & \tilde{A} \times_{\tilde{B}} \tilde{C} \arrow[r] \arrow[d] & \tilde{C} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{K} \arrow[r] & \tilde{A} \arrow[r] & \tilde{B} \arrow[r] & 0 \end{tikzcd} \tag{33}$$

of sheaves of chain complexes where the rows are short exact sequences. Since f is a quasi-isomorphism, the long exact sequence of homology sheaves yields that $H_n(\tilde{K}) = 0$ for each

$n \geq 0$. As $K \cong K'$, we also have $H_n(\tilde{K'}) = 0$, and thus by the long exact sequence of homology sheaves we get that $f_* : A \times_B C \to C$ is a quasi-isomorphism. This proves axiom **N3)**.

Finally, we need to provide path objects. Let $A \in \mathrm{Pre}(X, \mathrm{Ch})$. Using the approach of Hovey in Chapter 2.3 of [6], define $A^I \in \mathrm{Pre}(X, \mathrm{Ch})$ degreewise by

$$A_n^I = A_n \oplus A_n \oplus A_{n+1} \tag{34}$$

and with the differential given by

$$\begin{aligned}\partial_n :& A_n(U) \oplus A_n(U) \oplus A_{n+1}(U) \to A_{n-1}(U) \oplus A_{n-1}(U) \oplus A_n(U) \\ :& (x, y, z) \mapsto (dx, dy, x - y - dz).\end{aligned}$$

This induces a factorization of the diagonal map of A by

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[dr, "\Delta"'] & A^I \arrow[d, "t"] \\ & A \oplus A \end{tikzcd} \tag{35}$$

where $s = (\mathrm{id}_A, \mathrm{id}_A, 0)$ and t is the projection onto the first two components. Clearly t is a fibration. Hovey shows that s is a pointwise quasi-isomorphism of $\mathbb{Z}$-graded chain complexes. Therefore s itself is a quasi-isomorphism of presheaves of $\mathbb{Z}$-graded chain complexes. Applying the shift functor to the diagram above yields a diagram

$$\begin{tikzcd} A[0] \arrow[r, "s"] \arrow[dr, "\Delta"'] & A^I[0] \arrow[d, "t"] \\ & A[0] \oplus A[0] \end{tikzcd} \tag{36}$$

in $\mathrm{Pre}(X, \mathrm{Ch}_+)$ consisting of good truncations. The shift functor preserves quasi-isomorphisms and fibrations. Hence this construction provides the required path objects, proving axiom **N4)**. □

From now on we reserve the term fibration of presheaves of $\mathbb{Z}$-graded chain complexes for the fibrations obtained from Theorem 4.2.11. Unless we specifically refer to Proposition 4.3.2, any fibration in $\mathrm{Pre}(X, \mathrm{Ch})$ will mean a fibration of presheaves of $\mathbb{Z}$-graded chain complexes.

Lemma 4.3.3. *Let $I \in \mathrm{Shv}(X, \mathrm{Ch})$ be a sheaf of $\mathbb{Z}$-graded chain complexes concentrated in non-positive degrees and suppose that $I \to J$ is a fibrant replacement of I in $\mathrm{Pre}(X, \mathrm{Ch})$. Then for every open $U \in \mathcal{O}(X)$, the map $I(U) \to J(U)$ is a quasi-isomorphism of ordinary $\mathbb{Z}$-graded chain complexes.*

Proof. For $A, B \in \mathrm{Pre}(X, \mathrm{Ch}_+)$, let $\pi(A, B)$ denote the set of equivalences classes of maps $A \to B$ where two maps are considered equivalent if they are chain homotopic. Let $[A, B]$ denote the set of maps $A \to B$ in $\mathrm{Ho}(\mathrm{Pre}(X, \mathrm{Ch}_+))$. As homotopic maps become equal in the homotopy category, there is a canonical map $\pi(A, B) \to [A, B]$.

Let $I \to J$ be as in the statement of this lemma. Then there is a commutative diagram sets

$$\begin{array}{ccc} \pi(A, I[-n]) & \longrightarrow & \pi(A, J[-n]) \\ \downarrow & & \downarrow \\ [A, I[-n]] & \longrightarrow & [A, J[-n]]. \end{array} \tag{37}$$

Fix an open $U \in \mathcal{O}(X)$ and an integer $q \geq 0$. Let $A \in \mathrm{Pre}(X, \mathrm{Ch}_+)$ be a complex given by

$$A(V) = \begin{cases} \mathbb{Z}[q] & \text{if } V \subseteq U \\ 0 & \text{if } V \nsubseteq U, \end{cases} \tag{38}$$

where $\mathbb{Z}[q]$ denotes the complex given by $\mathbb{Z}$ concentrated in degree $q \geq 0$. Then A is cofibrant. Moreover, J being fibrant implies that TJ is fibrant in $\mathrm{Spt}(\mathrm{Pre}(X, \mathrm{Ch}_+))$. Therefore, $J[-n]$ is fibrant in $\mathrm{Pre}(X, \mathrm{Ch}_+)$. By Theorem 1.3.5, the right map in diagram (37) is an isomorphism. As the shift functor preserves quasi-isomorphisms, it follows that the bottom map is an isomorphism.

Brown gives a characterization of classes of maps in the homotopy category associated to a category of fibrant objects, see Theorem 1 of [1]. Considering the structure on $\mathrm{Pre}(X, \mathrm{Ch}_+)$ from Proposition 4.3.2, there is an isomorphism

$$\varinjlim_{A' \to A} \pi(A', I[-n]) \to [A, I[-n]] \tag{39}$$

where the colimit ranges over equivalence classes of trivial fibrations into A. The isomorphism is induced by the map sending $f : A' \to I[-n]$ at stage $\varphi : A' \to A$ to the morphism

$$A \xrightarrow{\varphi^{-1}} A' \xrightarrow{f} I[-n] \tag{40}$$

in the homotopy category. In Theorem 2.7 of [9] Jardine provides an argument involving spectral sequences showing that the functor

$$\mathrm{Pre}(X, \mathrm{Ch}) \to \mathrm{Sets} : A' \mapsto \pi(A', I[-n]) \tag{41}$$

sends weak equivalences to isomorphisms. This argument uses the assumptions that I is concentrated in non-positive degrees and that I is a sheaf. Consequently, the colimit from (39) is constant. Therefore, the left map of diagram (37) is an isomorphism.

Now that we have shown that $\pi(A, I[-n]) \to \pi(A, J[-n])$ is an isomorphism, varying $q \geq 0$, it follows that each map $J[-n](U) \to I[-n](U)$ is a quasi-isomorphism of chain complexes. Therefore, $I(U) \to J(U)$ is a quasi-isomorphism of $\mathbb{Z}$-graded chain complexes. □

This lemma directly proves Theorem 4.3.1. Indeed, let $B[0] \to I$ and $B \to J$ be as in the statement of the theorem. Then the diagram

$$\begin{tikzcd} B[0] \arrow[r] \arrow[d] & J \arrow[d] \\ I \arrow[r] \arrow[ur, dashed] & * \end{tikzcd} \tag{42}$$

admits a lift, because $B[0] \to I$ is a trivial cofibration and J is fibrant. By two-out-of-three, the lift $I \to J$ is a weak equivalence. Hence $I \to J$ satisfies the conditions of Lemma 4.3.3, showing that $I(X) \to J(X)$ is a quasi-isomorphism. This establishes

$$H^n(X, B) \cong H^n(\Gamma(J)_\bullet). \tag{43}$$

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