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Master thesis

Random Continued Fraction Expansions

Author:
Margriet Oomen
3855732
Study: Mathematical Sciences
Utrecht University

Supervisor:
Karma Dajani
Utrecht University
Second reader:
Roberto Fernandez
Utrecht University

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Abstract

A 2-continued fraction expansion is a generalisation of the regular continued fraction expansion, where the digits 1 in the numerators are replaced by the natural number 2. Each real number has uncountably many expansions of this form. In this thesis we consider a random algorithm that generates all such expansions. This is done by viewing the random system as a dynamical system, and then using tools in ergodic theory to analyse these expansions. In particular, we use a recent Theorem of Inoue (2012) to prove the existence of an invariant measure of product type whose marginal in the second coordinate is absolutely continuous with respect to Lebesgue measure on the unit interval. Also some dynamical properties of the system are shown and the asymptotic behaviour of such expansions is investigated. Furthermore, we show that the theory can be extended to the 3-random continued fraction expansion.

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Contents

Chapter 1

Introduction

In 2008 Burger, Gell-Redman, Kravitz, Walton, and Yates [2008] introduced the N-continued fraction expansion. Given a $x \in \mathbb{R}$ and a $N \in \mathbb{Z}$ Burger *et al.* [2008] showed that x can be represented in the following way:

$$x = d_0 + \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}}, \tag{1.0.1}$$

where the digits $d_i \in \mathbb{N}$. Anselm and Weintraub [2011] showed that every $x \in \mathbb{R}$ has in fact infinitely many such expansions. Dajani, Kraaikamp, and van der Wekken [2013] obtained the N-continued fraction expansions from transformations of the form

$$T : (0, N] \to (0, N], \qquad T_{N,i}(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor + i, \tag{1.0.2}$$

where $N \in \mathbb{N}$ and $i \in \{0, 1, \cdots, N-1\}$. Approaching the N-continued fraction expansions as a dynamical system Dajani *et al.* [2013] showed that the result obtained by Anselm and Weintraub [2011] is immediate. They also obtained invariant measures for the transformations $T_{N,i}(x)$.

In this thesis we will consider the N-continued fraction expansions by a random dynamical system. A random dynamical system consists of a family of transformations on a state space and a probability distribution on the family of transformations. Then for each iterate a transformation of the family is chosen according to the probability distribution. In this thesis the family of transformations $\{T_{N,i}, i \in \{0, 1, \cdots, N-1\}\}$ where $T_{N,i}$ are given by 1.0.2. The main question is whether we can find an invariant measure for this random transformation. The existence of invariant measures for random systems is frequently studied over the past decades. In this thesis we will use a recent theorem of Inoue [2012] to obtain an invariant measure.

Random dynamical systems can be used to obtain expansions similar to 1.0.1. Defining the dynamical system as a skew product, results from ergodic theory can be used to gain information about the asymptotic behaviour of the expansions. This is done in Kalle, Kempton, and Verbitskiy [2015]. In Dajani and de Vries [2005] more invariant measures for random β-expansions are obtained by constructing an isomorphism between the skew product for the random β-expansion and the digit sequences it induces.

In this thesis we will prove the existence of an invariant measure for the random transformation generating 2- and 3-continued fraction expansions, so expansions of the form 1.0.1 where $N = 2$ and subsequently $N = 3$. We will use the approach of Kalle *et al.* [2015] to show that the obtained measure is equivalent with the Lebesgue measure. We will write the random dynamical system as a skew product to obtain asymptotic properties of expansions like 1.0.1. Constructing an isomorphism between the digit sequences obtained by the random dynamical system and the skew product we will show the existence of invariant measures which are singular to the Lebesgue measure.

This thesis is organized as follows. In chapter 2 the general theory that will be used in the thesis will be stated. In chapter 3 we define the random N-continued fraction transformation. Subsequently we shortly discuss the article of Inoue [2012], we state the formal definition of a random transformation as used in the paper and give the existence theorem of invariant measures for random transformations. At the end of chapter 3 we show how we can apply the existence theorem of Inoue to an induced transformation of the 2-random continued fraction transformation. In chapter 4 we will introduce the notion of a skew product and define the 2-random continued fraction transformation as a skew product. Subsequently we introduce a skew product for the induced system and show that there exists an invariant product measure for this skew product which can be lifted to an invariant measure for 2-random continued fraction transformation. The rest of chapter 4 shows how we use the skew products and their invariant measures to show several ergodic properties of the 2 random-continued fraction transformation as well as the entropy of the skew product. In chapter 5 we construct an isomorphism to obtain more invariant measures. In chapter 6 we show how the theory developed in chapters 3 and 4 can be generalized to the 3-random continued fraction transformation. Finally in chapter 7 we summarize the obtained results and state some questions for further research to N-random continued fractions.

Chapter 2

The Toolbox

2.1 A very short introduction to ergodic theory

Given a probability space X endowed with a σ-algebra $\mathcal{F}$ and a measure μ on $\mathcal{F}$, we define a measurable transformation T,

$$T : X \to X. \tag{2.1.1}$$

Ergodic theory investigates sequences $x, Tx, T^2x, T^3x, \cdots$, so we investigate the orbits $\{T^i x\}_{i\in\mathbb{N}}$ of a point $x \in X$. In ergodic theory we like to find **invariant measures** i.e. $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{F}$. The following theorem gives some sufficient condition, to show that a measure is invariant:

Theorem 2.1.1. *Let $(X, \mathcal{F}, \mu)$ be a probability space and let $T : X \to X$ a measurable transformation. Let $\mathcal{P}$ be a generating π-system, i.e. $\mathcal{P}$ is a family of subsets of X such that if $A, B \in \mathcal{P}$, then $A \cap B \in \mathcal{P}$ and $\sigma(\mathcal{P}) = \mathcal{F}$. If for all $A \in \mathcal{P}$ we have $\mu(A) = \mu(T^{-1}(A))$, then μ is T-invariant.*

The same theorem holds by replacing the π-system $\mathcal{P}$ by a generating semi-algebra $\mathcal{S}$, see Dajani [2014] p 9 and Boyarski and Góra [1997] p 29. If μ is an invariant measure for T, we say that T is stationary with respect to μ.

The system $(X, \mathcal{F}, \mu, T)$ is called a **dynamical system**. We say an dynamical system is **ergodic** if for all $A \in \mathcal{F}$ such that $T^{-1}(A) = A$ we have $\mu(A) = 0$ or 1. One of the main theorems in ergodic theory is the *Pointwise Ergodic Theorem*, also called *Birkhoffs Ergodic Theorem*. Let $\mathcal{L}^1(\mu)$ to denote all μ-integrable functions.

Theorem 2.1.2. *Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then, for any f in $L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = f^*(x) \tag{2.1.2}$$

exists a.e., is T-invariant and $\int_X f d\mu = \int_X f^* d\mu$. If moreover T is ergodic, then f^* is a constant a.e. and $f^* = \int_X f d\mu$.

For a proof see Dajani [2014] p35 or Boyarski and Góra [1997] p40. Suppose μ is an invariant measure and $f = \mathbf{1}_E$ for $E \subset X$, then by the Birkhoff ergodic theorem the number of times the orbit of x is in E, equals the measure of E. So the Birkhoff ergodic theorem tells us that time-average value of $f(T^i(x))$ equals the spacial average of $f(x)$ over X. From the Birkhoff ergodic theorem we can derive an equivalent definition of ergodicity:

Proposition 2.1.3. *Let $(X, \mathcal{F}, \mu)$ be a probability space, and set $\mathcal{S}$ a generating semi-algebra of $\mathcal{F}$. Let $T : X \to X$ be a measure preserving transformation. Then, T is ergodic if and only if for all $A, B \in \mathcal{S}$, one has:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B). \tag{2.1.3}$$

For a proof we again refer to Dajani [2014] p.43. Instead of a generating semi-algebra $\mathcal{F}$ it is enough to know that 2.1.3 holds for a collection $\mathcal{G} \subset \mathcal{F}$, such that for all $B \in \mathcal{F}$ and for all ϵ there exists a $A \in \mathcal{G}$ such that $\mu(A\Delta B) < \epsilon$. Here $A\Delta B$ denotes the symmetric difference, i.e. $A\Delta B = A\backslash B \cup B\backslash A$.

Based on this definition of ergodicity we can define even stronger properties than ergodicity. Hence we say:

Definition 2.1.4. Let $(X, \mathcal{F}, \mu)$ be a probability space, and $T : X \to X$ a measure preserving transformation. Then,

- T is **weakly mixing** if for all $A, B \in \mathcal{F}$, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left|\mu(T^{-i}A \cap B) - \mu(A)\mu(B)\right| = 0. \tag{2.1.4}$$

- T is **strongly mixing** or **mixing** if for all $A, B \in \mathcal{F}$, one has

$$\lim_{n\to\infty} \mu(T^{-i}A \cap B) = \mu(A)\mu(B). \tag{2.1.5}$$

$\diamondsuit$

Note that strongly mixing implies weakly mixing and weakly mixing implies ergodicity.

Furthermore in this thesis we will use the notion of expanding.

Definition 2.1.5. Given an dynamical system $([a, b], \mathcal{B}, \mu, T)$, $[a, b] \subset \mathbb{R}$ an interval and $\mathcal{B}$ the Borel-σ-algebra, we say that T is **expanding** if T is C^1 and $|T'(x)| > 1$. $\diamondsuit$

2.1.1 Induced system

Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system on a probability space. Let $A \in \mathcal{F}$ be a set of positive measure. We define the **return time** τ as follows

$$\tau : A \to \mathbb{N} \tag{2.1.6}$$
$$\tau(x) = \inf\{n \in \mathbb{N} : T^n(x) \in A\}. \tag{2.1.7}$$

The **Poincaré Recurrence Theorem**, tells us that if $\mu(A) > 0$, then $\tau(x) < \infty$ $\mu - a.e.$. We can define

$$T_A : A \to A \tag{2.1.8}$$
$$T_A(x) = T^{\tau(x)}(x). \tag{2.1.9}$$

This transformation is called the **induced transformation**. Note that T_A is measurable. We endow A with the sigma-algebra $\mathcal{F} \cap A$ and define the measure $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$, for $B \in A \cap \mathcal{F}$. Hence we find the induced system $(A, A \cap \mathcal{F}, \mu_A, T_A)$. We have the following theorems:

Theorem 2.1.6. *If μ is an invariant measure for T, then μ_A is an invariant measure for T_A.*

Theorem 2.1.7. *If T is ergodic on $(X, \mathcal{F}, \mu)$ then T_A is ergodic on $(A, \mathcal{F}, \mu)$.*

For both proofs we refer to Dajani [2014] p18, p25.

Let T be a transformation on $(X, \mathcal{F})$ and let T_A, be the induced transformation on $A \in \mathcal{F}$. Suppose μ_A is an invariant measure for T_A. Can we find an invariant measure for transformation T? The answer is yes and it is given by the measure ν.

Proposition 2.1.8. *Let T be a transformation on $(X, \mathcal{F})$. Let $A \in \mathcal{F}$ and suppose the induced transformation T_A has an invariant measure $\mu(A)$. Then the measure ν defined by:*

$$\nu(E) = \frac{1}{\int_A \tau(x) d\mu_A(x)} \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) > n\} \cap T^{-n}(E)) \tag{2.1.10}$$

is an invariant measure for the transformation T.

Proof. We check that ν is indeed a T-invariant measure:

$$\begin{aligned}
\nu(T^{-1}E) =& \frac{1}{\int_A \tau(x) d\mu_A} \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) > n\} \cap T^{-n+1}(E)) \\
=& \frac{1}{\int_A \tau(x) d\mu_A(x)} \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) = n+1\} \cap T^{-n+1}(E)) \\
& + \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) > n+1\} \cap T^{-n+1}(E)) \\
=& \frac{1}{\int_A \tau(x) d\mu_A} \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) = n+1\} \cap T_A^{-1}(E \cap A)) \\
& + \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) > n+1\} \cap T^{-n+1}(E)) \\
=& \frac{1}{\int_A \tau(x) d\mu_A} \mu_A(T_A^{-1}(E \cap A)) \\
& + \sum_{n=1}^{\infty} \mu_A(\{x \in A; \tau(x) > n\} \cap T^{-n}(E)) \\
=& \frac{1}{\int_A \tau(x) d\mu_A} \sum_{n=0}^{\infty} \mu_A(\{x \in A; \tau(x) > n\} \cap T^{-n}(E)) \\
=& \nu(E).
\end{aligned}$$

In the second equation we have used that if $x \in \{x \in A : \tau(x) = n, T^n(x) \in E\}$ then $T^n(x) = T_A(x)$ and $T_A(x) \in E \cap A$. $\square$

Now suppose that the measure μ_A is ergodic with respect to T_A. Then we can use the following theorem, a proof can be found in Boyarski and Góra [1997].

Theorem 2.1.9. *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system and let $A \in \mathcal{F}$ be a set such that $\mu(A) > 0$. Then if $(A, A \cap \mathcal{F}, \mu_A, T_A)$ is ergodic and if $\mu(\bigcup_{n=0}^{\infty} T^{-n} A) = 1$, then also T is ergodic with respect to μ.*

2.1.2 Isomorphism

What does it mean for two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{G}, \nu, S)$ to be the same? The answer yields there exists an isomorphism $\psi : X \to Y$. Such an **isomorphism** has the following properties.

- There exist sets $M \subset X$ and $N \subset Y$ such that $\mu(M) = \nu(N) = 1$ and $\psi : M \to N$ is one-to-one and onto. So there exists an inverse ψ^{-1}.
- ψ is measurable, so $\psi^{-1}G \in \mathcal{F}$ for all $G \in \mathcal{G}$. And also ψ^{-1} is measurable.
- The function ψ preserves measure. This means $\mu \circ \psi^{-1}(A) = \nu(A)$ for all $A \in \mathcal{G}$ and $\nu \circ \psi(B) = \nu(B)$ for all $B \in \mathcal{F}$.
- The function ψ preserves the transformations. This means that $\psi \circ T = S \circ \psi$ for all $x \in M$ and vice versa.

The last property can be shown in a commuting diagram:

$$\begin{array}{ccc} M & \xrightarrow{T} & M \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ N & \xrightarrow{S} & N \end{array}$$

2.1.3 Entropy

This section is based on Dajani [2014]. Most proofs of the theorems below can be found there.

The notion of entropy is used to determine the amount of randomness in a system. In ergodic theory we can interpret this notion as follows. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system and let α be a partition of X. Suppose we know $x \in A$ for $A \in \alpha$. Can we say something about in which $B \in \alpha$ Tx ends up? The randomness about where Tx ends up is determined by the entropy $h(T)$. The entropy is defined such that it is non-negative and it is independent of the partition we use. Moreover higher entropy corresponds to higher randomness. We first define the entropy of a partition.

Definition 2.1.10. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system on a probability space. Let α be a partition of X. By convention we say $0 \log 0 = 0$. We define the **entropy of a partition** by:

$$H(\alpha) = -\sum_{A \in \alpha} \mu(A) \log(\mu(A)). \qquad (2.1.11)$$

$\diamondsuit$

Since we work with probability measures, $H(\alpha)$ will always be non-negative. The value $\log(A)$ for $A \in \alpha$ can be interpreted as the amount of information contained in A. Given a partition α we introduce the partition $\bigvee_{i=0}^{n-1} T^{-i}\alpha$. This is the partition with elements $A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-1}A_{i_n}$, for $A_{i_j} \in \alpha$, $0 \le j \le n$. The entropy of T with respect to a partition α will be defined as follows:

Definition 2.1.11. The **entropy of the measure preserving transformation T with respect to the partition** α is given by:

$$h(\alpha, T) = h_\mu(\alpha, T) := \lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\alpha),$$

where

$$\lim_{n\to\infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} T^{-i}\alpha) = -\frac{1}{n} \sum_{D\in\bigvee_{i=0}^{n-1} T^{-i}\alpha} \mu(D)\log(D).$$

$\diamondsuit$

Finally we define entropy.

Definition 2.1.12. The **entropy** of the transformation T is given by

$$h(T) = h_\mu(T) = \sup_\alpha h(\alpha, T)$$

where the supremum is taken over all finite or countable partitions α of X. $\diamondsuit$

Related to entropy is the information function which is defined as follows.

Definition 2.1.13. Given a dynamical system $(X, \mathcal{F}, \mu, T)$ and α a finite or countable partition of X. We define the information function with respect to α by:

$$I_\alpha : X \to \mathbb{R} \tag{2.1.12}$$

$$I_\alpha(x) = \sum_{A\in\alpha} \mathbf{1}_A(x) \log(\mu_p(A)). \tag{2.1.13}$$

$\diamondsuit$

It is in general not easy to compute the entropy from the definition. Hence we will use the following theorems to compute entropy. A partition α is called a **generator** with respect to the transformation T if $\sigma(\bigvee_{i=0}^{\infty} T) = \mathcal{F}$.

Theorem 2.1.14 (Kolmogorov, Sinai 1958)**.** *If α is a generator with respect to T and $H(\alpha) < \infty$, then $h(T) = h(\alpha, T)$.*

An other important theorem for calculating the entropy is the Shannon-MCMillan-Breiman Theorem which states:

Theorem 2.1.15. *Suppose T is an ergodic measure preserving transformation on a probability space$(X, \mathcal{F}, \mu)$, and let α be a countable partition with $H(\alpha) < \infty$. Then:*

$$\lim_{n\to\infty} \frac{1}{n+1} I_{\bigvee_{i=0}^{n} T^{-i}\alpha}(x) = h(\alpha, T) \ \ a.e. \tag{2.1.14}$$

We also have the following theorem.

Theorem 2.1.16. *Entropy is isomorphism invariant.*

Finally we state Abrahamov's formula for the relation between the entropy of a system and its induced system. The proof of Abrahamov's formula can be found in Petersen [1983].

Theorem 2.1.17. *Let $T : X \to X$ be a measure preserving transformation on a probability space $(X, \mathcal{F}, \mu)$ and $E \in \mathcal{F}$ a set of positive measure. Let $T_E : E \to E$ denote the induced transformation. Then*

$$h(T_E) = \frac{h(T)}{\mu(E)}.$$

2.1.4 Maximal entropy

From a probabilistic point of view we can define the entropy of discrete probability distribution by
$$h(p) = -\sum_{i\geq 1} p_i \log(p_i).$$
Then $h(p)$ tells us something about the amount of information carried in the distribution p. Also here higher entropy means less information. Suppose we are looking for a probability distribution satisfying certain constraints, for example a certain mean or a certain variation. Then it turns out that maximizing the entropy gives us the most probable probability distribution. So it gives us the probability distribution that does not assume any information we do not know. In the article of Conrad the principle of maximum entropy is explained in detail and he derives some properties of the maximal entropy. The theorem we will use is the following:

Theorem 2.1.18. *On $\{k, k+1, k+2, \cdots\}$, for $k \in \mathbb{N}$ the unique probability distribution with a given mean and maximum entropy is the geometric distribution with that mean.*

2.2 The regular continued fraction transformation

This section discusses the regular continued fraction transformation. It is the "basis" of the N-random-continued fraction transformation. Also it serves as an example of a dynamical system as introduced in section 2.1.

The regular continued fraction expansion is generated by the following transformation:
$$T : (0,1] \to (0,1], \tag{2.2.1}$$
$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor. \tag{2.2.2}$$
We define digits a_n by:
$$a_1(x) = k \qquad \text{if } x \in (\frac{1}{k+1}, \frac{1}{k}],\ k \in \mathbb{N}, \tag{2.2.3}$$
$$a_n(x) = a_1(T^{n-1}(x)). \tag{2.2.4}$$
Partitioning $[0,1]$ in intervals $(\frac{1}{k}, \frac{1}{k+1}]$, the digit a_n tells us where x ends up after $n-1$ times applying the transformation T. Using these digits we can write:
$$T(x) = \frac{1}{x} - a_1. \tag{2.2.5}$$
And hence $x = \frac{1}{a_1 + T(x)}$. Applying the transformation T on $T(x)$ we obtain $T(x) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + T^2(x)}}$. Continuing this way we obtain after n iterations of T
$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + T^n(x)}}}}. \tag{2.2.6}$$

We define the n'th partial fractions $\frac{p_n}{q_n}$ by:

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{2.2.7}$$

Now we will show $\lim_{n\to\infty} \frac{p_n}{q_n} = x$, so we can expand x in the following way

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}. \tag{2.2.8}$$

Let A_i be the matrix

$$A_i = \begin{bmatrix} 0 & 1 \\ 1 & a_i \end{bmatrix}. \tag{2.2.9}$$

We define the matrix M_n by $M_n = A_1 A_2 \cdots A_n$. We will use the Moebius transformation to obtain the partial fractions $\frac{p_n}{q_n}$. The **Moebius transformation** is the transformation:

$$M_n : \mathbb{C}^* \to \mathbb{C}^* \tag{2.2.10}$$

$$M_n(z) = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}(z) = \frac{r_n z + p_n}{s_n z + q_n}. \tag{2.2.11}$$

Hence $M_1(0) = A_1(0) = \frac{1}{a_1} = \frac{p_n}{q_n}$ and $M_2(0) = A_2 \cdot A_1(0) = \cfrac{1}{a_2 + \cfrac{1}{a_1}} = \frac{p_2}{q_2}$.

Continuing this way we see

$$M_n(0) = A_1 A_2 A_3 \cdots A_n(0) = \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}(0) = \frac{p_n}{q_n}, \tag{2.2.12}$$

where $\frac{p_n}{q_n}$ is of the form 2.2.7. In the same way we see $M_n(T^n(x))$ gives an expression of x in the from of 2.2.6. Note that from the associativity of matrix multiplication it follows that the Moebius transformation is associative.

To show x can be expressed in the form of 2.2.6 we investigate recurrence relations induced by the Mobius transformation. Note

$$M_n = M_{n-1} A_n \tag{2.2.13}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \tag{2.2.14}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} p_{n-1} & r_{n-1} + a_n p_{n-1} \\ q_{n-1} & s_{n-1} + a_n q_{n-1} \end{bmatrix}. \tag{2.2.15}$$

We obtain the following recurrence relations:

$$p_{-1} = 1 \qquad p_0 = 0 \qquad p_n = p_{n-2} + a_n p_{n-1} \tag{2.2.16}$$

$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_n = q_{n-2} + a_n q_{n-1}. \tag{2.2.17}$$

Hence

$$x = M_{n-1}A_n(T_\omega^n(x)) = \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} (T_\omega^n(x)) \tag{2.2.18}$$

$$= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \left(\frac{1}{a_n + T_\omega^n(x)} \right) \tag{2.2.19}$$

$$= \frac{p_{n-2} + p_{n-1}k_n + p_{n-1}T_\omega^n(x)}{q_{n-2} + q_{n-1}k_n + q_{n-1}T_\omega^n(x)} \tag{2.2.20}$$

$$= \frac{p_n + p_{n-1}T_\omega^n(x)}{q_n + q_{n-1}T_\omega^n(x)}. \tag{2.2.21}$$

It follows that

$$|x - \frac{p_n}{q_n}| = \left| \frac{p_n + p_{n-1}T_\omega^n(x)}{q_n + q_{n-1}T_\omega^n(x)} - \frac{p_n}{q_n} \right| = \left| \frac{(p_{n-1}q_n - p_n q_{n-1})T_\omega^n(x)}{(q_n + q_{n-1}T_\omega^n(x))q_n} \right| \tag{2.2.22}$$

$$= \left| \frac{\det M_n T_\omega^n(x)}{(q_n + q_{n-1}T_\omega^n(x))q_n} \right|. \tag{2.2.23}$$

Note that $\det A_i = \det \begin{bmatrix} 0 & 1 \\ 1 & a_i \end{bmatrix} = -1$, so $\det M_n = (-1)^n$ and therefore:

$$|x - \frac{p_n}{q_n}| = \left| \frac{(-1)^n T_\omega^n(x)}{(q_n + q_{n-1}T_\omega^n(x))\, q_n} \right|. \tag{2.2.24}$$

By the recurrence relations of q_n we find that $q_n > 0$ for all $n \in \mathbb{N}$, q_n is integer valued and $q_n > q_{n-1}$. Hence $\lim_{n\to\infty} q_n = \infty$. Therefore it follows that $\lim_{n\to\infty} |x - \frac{p_n}{q_n}| \leq \lim_{n\to\infty} \frac{1}{q_n^2} = 0$.

Endowing $[0,1]$ with the Borel-σ-algebra, it turns out that the Gauss-measure, which is given by:

$$\mu(A) = \frac{1}{\log(2)} \int_A \frac{1}{x+1} d\lambda \tag{2.2.25}$$

is an invariant measure for the continued fraction transformation. One can also show that the continued fraction transformation T is ergodic with respect to the above measure μ. For more properties of the regular continued fractions and proofs we refer to Dajani [2014], p69.

2.3 Functions of bounded variation

This section is based on Chapter 2.3 in Boyarski and Góra [1997]. We refer the reader to this book for the proofs of the theorems we state in this section. Let $[a,b] \subset \mathbb{R}$ be a non-empty inverval. The variation of a function $f : [a,b] \to \mathbb{R}$ over $[a,b]$ is a measure of the fluctuation f. Let α be a finite partition of $[a,b]$, so $\alpha = \{(x_i, x_{i+1}] : i \in \mathbb{N}, x_0 = a, x_n = b, x_i \leq x_{i+1}\}$. Then we associate a set of endpoints $P = \{x_0, \cdots, x_n\}$ with α. Let $\mathcal{P}$ denote the collection of sets P associated with a partitions α of $[a,b]$. Then the variation of f is defined as follows.

Definition 2.3.1. Let $f : [a,b] \to \mathbb{R}$. Then the **variation** of f on $[a,b]$ is defined as:

$$\bigvee_{[a,b]} f(x) = \sup_{\mathcal{P}} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \tag{2.3.1}$$

We say that f is of **bounded variation** if there exists an $M \in \mathbb{R}$ such that $\bigvee_{[a,b]} f < M$. ♢

We give a few theorems about functions of bounded variation.

Theorem 2.3.2. *Let $f : [a, b] \to \mathbb{R}$ be of bounded variation and assume $c \in (a, b)$. Then f is of bounded variation on $[a, c]$ and on $[c, d]$ and we have:*

$$\bigvee_{[a,b]} f = \bigvee_{[a,c]} f + \bigvee_{[c,d]} f. \tag{2.3.2}$$

Theorem 2.3.3. *Let f and g be functions of bounded variation on $[a, b]$. Then so are their sum, difference and product. Also, we have:*

$$\bigvee_{[a,b]} (f \pm g) \leq \bigvee_{[a,b]} f + \bigvee_{[a,b]} g \tag{2.3.3}$$

and

$$\bigvee_{[a,b]} (f \cdot g) \leq A \bigvee_{[a,b]} f + B \bigvee_{[a,b]} g, \tag{2.3.4}$$

where $A = \sup\{|g(x)| : x \in [a, b]\}$, $B = \sup\{|f(x)| : x \in [a, b]\}$.

Corollary 2.3.4. *Let $f_1, \cdots, f_n$, $f_i : [a, b] \to \mathbb{R}$ be a family of functions which is bounded by C, i.e. $\sup\{|f_i(x)| : x \in [a, b]\} \leq C$ for all $1 \leq i \leq n$. Then*

$$\bigvee_{[a,b]} \prod_{i=1}^{n} f_n \leq C^{n-1} \sum_{i=1}^{N} \bigvee f_i.$$

Proof. Suppose $n = 1$, so we have 1 function $f : [a, b] \to \mathbb{R}$ then clearly the statement holds. Now suppose the statement holds for all $n \leq N$. Then by theorem 2.3.3 we obtain:

$$\bigvee_{[a,b]} \prod_{i=1}^{N+1} f_i \leq C \bigvee \prod_{i=1}^{N} f_i + C^{N-1} \bigvee_{[a,b]} f_{N+1} \tag{2.3.5}$$

$$\leq C^N \sum_{i=1}^{N} \bigvee_{[a,b]} f_i + C^N \bigvee_{[a,b]} f_{N+1} = C^N \sum_{i=1}^{N+1} \bigvee_{[a,b]} f_i. \tag{2.3.6}$$

□

Remark 2.3.5. If $f : [a, b] \to \mathbb{R}$ is a monotone function then

$$\bigvee_{[a,b]} f = |f(b) - f(a)|.$$

♢

We obtain also the following lemma, see Lasota and Mackey [1994] chapter 6:

Lemma 2.3.6. *Let $f : [c, d] \to \mathbb{R}$ and $g : [a, b] \to [c, d]$. Suppose that g is a monotone function and the image of g, $Im(g) = [c, d]$, then $\bigvee_{[a,b]} f \circ g \leq \bigvee_{[c,d]} f$.*

Proof. Assume w.l.o.g. that g is increasing. Suppose we have a partition $x_0 < x_1 < \cdots < x_n$ of $[a, b]$, then $g(x_0) < g(x_1) < \cdots < g(x_n)$ is a partition of $[c, d]$. By the definition of variation we have

$$\sum_{k=1}^{n} |f(g(x_k)) - f(g(x_{k-1}))| \leq \bigvee_{[c,d]} f.$$

This holds for all partitions $x_0 < x_1 < \cdots < x_n$ of $[a, b]$ and hence we conclude

$$\bigvee_{[a,b]} f \circ g \leq \bigvee_{[c,d]} f.$$

□

Finally the space of functions of bounded variation can be made to a Banach space as follows. We define the space of functions of bounded variation on $[a, b]$ by

$$BV([a, b]) = \{f \in \mathcal{L}^1 : \inf_{f_1 = f \ a.e.} \bigvee_{[a,b]} f_1 < +\infty\}.$$

Let $||f||_1 = \int_{[a,b]} |f| d\lambda$ be the norm on $\mathcal{L}^1(\lambda)$. We define a norm on BV by $||f||_{BV} = ||f||_1 + \inf_{\{f_1 = f \ a.e.\}} \bigvee_{[a,b]} f_1$.

2.4 Lower semi-continuity

An important property of a function of bounded variation is that we can modify it on a countable number of points to obtain a lower semi-continuous function. In this section we will introduce lower semi-continuity, give some properties of it and state the theorem mentioned above. The statements in this section are based on Kurdila and Zabarankin [2005] chapter 7 and Boyarski and Góra [1997] chapter 8.

Definition 2.4.1. Let (X, τ) be a topological space and let $f : X \to \overline{\mathbb{R}}$. Then f is lower semi-continuous at x_0 if the inverse image of every half-open set of the form (r, ∞), with $f(x_0) \in (r, \infty)$ contains an open set $U \subset X$ that contains x_0. That is,

$$f(x_0) \in (r, \infty) \Longrightarrow \exists U \in \tau : x_0 \in U \subset f^{-1}(r, \infty). \tag{2.4.1}$$

We say a function f is lower semi-continuous on a topological space X if it is lower semi-continuous at each point in X. ♢

Using this definition we get the following equivalent definition for metric spaces.

Proposition 2.4.2. *Let (X, d) be a metric space. A function $f : X \to \overline{\mathbb{R}}$ is lower semi-continuous at a point $x_0 \in X$ if*

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

Proof. See Kurdila and Zabarankin [2005]. □

By proposition 2.4.2 we can get some intuition what lower semi-continuity means. Since the $\liminf_{x\to x_0} f(x) = \lim_{x\to x_0} f(x)$ if the limit on the right hand side exists, we see that each continuous function is lower semi-continuous. The definition of lower semi-continuity allows a function $f : X \to \mathbb{R}$ to make a jump at x_0 from a to b for $a, b \in \mathbb{R}$. The only restriction is that $f(x_0)$ is always the lower point of the jump so $f(x_0) = \min\{a, b\}$. This is illustrated in figure 2.1

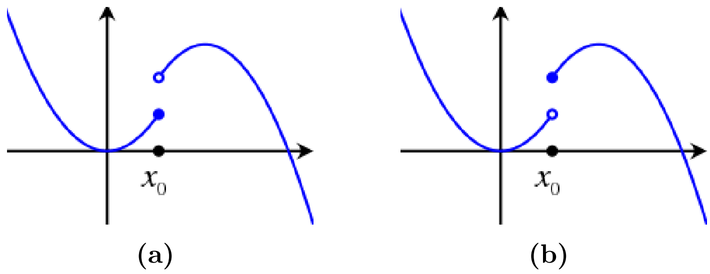

Figure 2.1 – (a) a lower semi-continuous function, (b) a non lower semi-continuous function.[1]

We have the following lemma and theorem.

Theorem 2.4.3. *If f is a lower semi-continuous function on $I = [a, b] \subset R$, then it is bounded from below and assumes its minimum value. For any $a \in \mathbb{R}$ the set $\{x : f(x) > a\}$ is open.*

Lemma 2.4.4. *If f is of bounded variation on I, then it can be redefined on a countable set to become a lower continuous function.*

Proof. A function of bounded variation has at most countably many discontinuities $y_0, y_1, \cdots$. So intuitively at each discontinuity we define $f(y_i)$ to be the "lowest" point of the jump. For a formal proof see Boyarski and Góra [1997] chapter 8. □

2.5 Perron Frobenius Operator

We recall a few definitions from measure theory.

Definition 2.5.1. Let μ, ν be two normalized measures defined on $(X, \mathcal{B})$. We say that μ is **absolutely continuous** with respect to ν, if for all set $A \in \mathcal{B}$ such that $\nu(A) = 0$ we have $\mu(A) = 0$. We writ $\mu << \nu$. ◇

Definition 2.5.2. Let μ, ν two normalized measures defined on $(X, \mathcal{B})$. We say that μ is **equivalent** with to ν if $\mu << \nu$ and $\nu << \mu$. ◇

Theorem 2.5.3. *Let $(X, \mathcal{B})$ be a measure space and let ν and μ be two normalized measures on $(X, \mathcal{B})$. If $\mu << \nu$, then there exists an a.e-unique $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$ such that for every $A \in \mathcal{B}$ we have,*

$$\mu(A) = \int_A f d\nu. \tag{2.5.1}$$

f is called the Randon-Nikodym derivative and it is denoted by $\frac{d\nu}{d\mu}$

[1]figure from https://en.wikipedia.org/wiki/Semi-continuity

Let $([0,1], \mathcal{B}[0,1], \mu)$ be a probability space, where μ is absolutely continuous with the Lebesgue measure, so μ has a density f such that $\mu(A) = \int_A f d\lambda$, for $A \in \mathcal{B}[0,1]$. Suppose T is a transformation on $([0,1], \mathcal{B}[0,1], \mu)$. How should the density function f be changed after applying the transformation T? Therefore we are looking for a density ψ such that $\int_{T^{-1}A} f d\lambda = \int_A \phi d\lambda$. To find such a density we introduce the notion of a **non-singular** transformation.

Definition 2.5.4. Let $(X, \mathcal{B}, \mu)$ be a normalized measure space. Then $T : X \to X$ is said to be **nonsingular** if and only if for any $A \in \mathcal{B}$ such that $\mu(A) = 0$, we have $\mu(T^{-1}A) = 0$. $\diamondsuit$

Intuitively we can describe a **non-singular** transformation as a transformation that does not send mass to a null-set.

Suppose the transformation T on $([0,1], \mathcal{B}[0,1], \mu)$ is non-singular. Then for all sets $A \in \mathcal{B}[0,1]$ such that $\lambda(A) = 0$, we have $\mu(A) = 0$ and since T is non-singular $\mu(T^{-1}(A)) = 0$. Hence we see that the measure given by $\int_{T^{-1}(A)} f d\lambda$ is absolutely continuous with respect to λ and hence by theorem 2.5.3 we find a unique density ϕ such that $\int_{T^{-1}A} f d\lambda = \int_A \phi d\lambda$. Let $P_T : \mathcal{L}^1 \to \mathcal{L}^1$ be the operator that sends $f \to \phi$, where ϕ is the density we find by theorem 2.5.3. We call this operator the **Perron-Frobenius**-operator.

Definition 2.5.5. Let $I = [a,b]$, $\mathcal{B}$ the Borel-σ-algebra restricted to I and λ the normalized Lebesgue measure on I. Let $T : I \to I$ be a non-singular transformation. We define the **Perron-Frobenius operator** $P_T : \mathcal{L}^1 \to \mathcal{L}^1$ as follows. For any $f \in \mathcal{L}^1$, $P_T f$ is the unique (up to a.e. equivalence) function in $\mathcal{L}^1$ such that:

$$\int_A P_T f d\lambda = \int_{T^{-1}(A)} f d\lambda \tag{2.5.2}$$

for any $A \in \mathcal{B}$ $\diamondsuit$

Notice that the Perron-Frobenius operator tells us in some sense whether a measure μ is T invariant.

Proposition 2.5.6. *Let $T : I \to I$ be nonsingular. Then $P_T f = f$ $- a.e.$ if and only if the measure $\mu = f \cdot \lambda$, defined by $\mu(A) = \int_A f d\lambda$, is T-invariant measure.*

Proof. Suppose μ is an invariant measure. Then $\mu(A) = \mu(T^{-1}A)$ for all sets $A \in \mathcal{B}$, so $\int_A f d\lambda = \int_{T^{-1}(A))} f d\lambda$ and hence $\int_A f d\lambda = \int_A P_T f d\lambda$ for all $A \in \mathcal{B}$. We conclude $P_T f = f - a.e.$. On the other hand if $P_T f = f$ we have:

$$\begin{aligned}
\mu(T^{-1}(A)) &= \int_{T^{-1}(A)} f d\lambda \\
&= \int_A P_T f d\lambda \\
&= \int_A f d\lambda \\
&= \mu(A).
\end{aligned}$$

Therefore we conclude that μ is an T-invariant measure if and only if $P_T f = f$ $a.e.$ $\square$

Proposition 2.5.7. *$P_T : \mathcal{L}^1 \to \mathcal{L}^1$ is a linear operator,*

$$P_T(\alpha f + \beta g) = \alpha P_T f + \beta P_T g.$$

Proposition 2.5.8. *$P_T : \mathcal{L}^1 \to \mathcal{L}^1$ is a contraction, $||P_T f||_1 \le ||f||_1$.*

Proposition 2.5.9. *$\int_I P_T f d\lambda = \int_I f d\lambda$.*

Proof. If T is a non-singular transformation then:

$$\int_I P_T f d\lambda = \int_{T^{-1}I} f d\lambda = \int_I f d\lambda.$$

□

Let $\mathcal{L}^\infty$ denote all almost everywhere bounded functions. The norm on L^∞ is defined by $||f||_\infty = \inf\{M : \mu(\{x : f(x > M)\}) = 0\}$.

Proposition 2.5.10. *$P_T(f \circ T \cdot g) = f \cdot P_T g$ a.e. for $f \in \mathcal{L}^1$ and $g \in \mathcal{L}^\infty$.*

Proof. To prove this we use standard machinery. Let A be an arbitrary set in $\mathcal{B}$. Suppose $f = \mathbf{1}_B$ for $B \in \mathcal{B}$,then we obtain:

$$\begin{aligned}\int_A P_T(\mathbf{1}_B(T(x)) \cdot g) d\lambda &= \int_{T^{-1}A} \mathbf{1}_{T^{-1}B}(x) g(x) d\lambda \\ &= \int_{T^{-1}(A \cap B)} g(x) d\lambda \\ &= \int_{A \cap B} P_T g d\lambda \\ &= \int_A \mathbf{1}_B P_T g d\lambda.\end{aligned}$$

Where in the first equality we just use that the definition of the Perron-Frobenius operator. By linearity of the Perron-Frobenius operator and the integral, we obtain the same result for simple functions, $f = \sum_{i=1}^n b_i \mathbf{1}_{B_i}$. Using dominated convergence, this holds for all positive integrable functions f. Hence it holds for all integrable functions f. □

Proposition 2.5.11. *$P_{T \circ T} = P_T(P_T f)$.*

Proposition 2.5.12. *$P_T : \mathcal{L}^1 \to \mathcal{L}^1$ sends the space of functions of bounded variation over the interval to the space of functions of bounded variation.*

The proof of the last proposition can be found in Hofbauer and Keller [1982].

2.5.1 Construction of the Perron Frobenius operator

For a piecewise expanding, C^1 monotone transformation on an interval $I \subset \mathbb{R}$ we can construct the Perron-Frobenius operator explicitly:

Definition 2.5.13. Let $I = [a, b]$. The transformation $T : I \to I$ is called **piecewise monotonic** if there exists a partition of I, $a = a_0 < a_1 < \cdots < a_q = b$, and a number $r \ge 1$ such that:

- $T|_{(a_{i-1}, a_i)}$ is a C^r function, $i = 1, \cdots, q$ which can be extended to a C^r function on $[a_{i-1}, a_i]$, $i = 1, \cdots, q$,
- $|T'(x)| > 0$ on (a_{i-1}, a_i), $i = 1, \cdots, q$.

◊

We show how to construct the Perron-Frobenius operator explicitly for piecewise monotone functions. We know by definition $\int_A P_t f d\lambda = \int_{T^{-1}(A)} f d\lambda$ for an arbitrary set $A \in \mathcal{B}$. Let $B_i = T([a_{i-1}, a_i])$ for $1 \leq i \leq q$, then we can write $A = \bigcup_{i=1}^{q} B_i \cap A$. Let $T_i(x) = T(x)\mathbf{1}_{[a_{i-1}, a_i]}$ denote the restriction of T to $[a_{i-1}, a_i]$, so T_i is monotone and C^1 on $[a_{i-1}, a_i]$. Then we define for each T_i an inverse function $\phi_i : B_i \to I$. Using the definition of the Perron-Frobenius operator and the change of variable formula we write:

$$\begin{aligned}
\int_A P_T f d\lambda &= \sum_{i=1}^{q} \int_{(A \cap B_i)} P_T f d\lambda \\
&= \sum_{i=1}^{q} \int_{\phi_i(A \cap B_i)} f(y) d\lambda \\
&= \sum_{i=1}^{q} \int_{(A \cap B_i)} f(\phi(x)) |\phi'(x)| d\lambda \\
&= \int_A \sum_{i=1}^{q} \frac{f(T_i^{-1}(x))}{T_i'(T_i(x))} d\lambda.
\end{aligned}$$

Since A is an arbitrary set we conclude that $P_T f = \sum_{i=1}^{q} \frac{f(T_i^{-1}(x))}{T_i'(T_i(x))}$ $a.e.$. Therefore we have found an explicit formula for the Perron-Frobenius operator.

Chapter 3

N-random continued fraction expansions and a quest for an invariant measure

3.1 The N-random continued fraction transfromation

In this section we introduce the N-continued random fraction transformation. To do so, we first introduce the non-random N-continued fraction transformations. The N-continued fraction transformations are natural generalizations of the regular continued fraction transformation and are defined as follows.

Definition 3.1.1. Let $N \in \mathbb{N}$, we define for $1 \leq i \leq N$, the transformations T_i:

$$T_i : [0, N] \to [0, N],$$
$$T_i(x) = \begin{cases} \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor + i & \text{if } x \in (0, \frac{N}{i+1}], \\ \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor & \text{if } x \in (\frac{N}{i+1}, N], \\ 0 & \text{if } x = 0. \end{cases}$$

$\diamondsuit$

We depicted the case $N = 5$ in figure 3.1. Note that in the case $N = 1$ we obtain the regular continued fraction. The transformation $T_0(x)$ is called the greedy transformation and the transformation T_{N-1} is called the lazy transformation. To each transformation T_i we associate digits $d_{n,i}(x)$ which are defined by:

$$d_{1,i}(x) = \begin{cases} k - i & \text{if } x \in (\frac{N}{k+1}, \frac{N}{k}],\ k \geq i + 1 \\ k & \text{if } x \in (\frac{N}{k+1}, \frac{N}{K}],\ k \leq i \\ \infty & \text{if } x = 0, \end{cases}$$
$$d_{n,i}(x) = d_{1,i}(T_i^{n-1} x).$$

Using these digits we can write

$$T_i(x) = \begin{cases} \frac{1}{x} - d_{1,i}(x) & \text{if } x \in (0, N] \\ 0 & \text{if } x = 0. \end{cases} \tag{3.1.1}$$

In the same way as we did in the case $N = 1$ we use the transformations T_i to obtain an expansion for $x \in [0.N]$. Since we use only the transformation T_i so

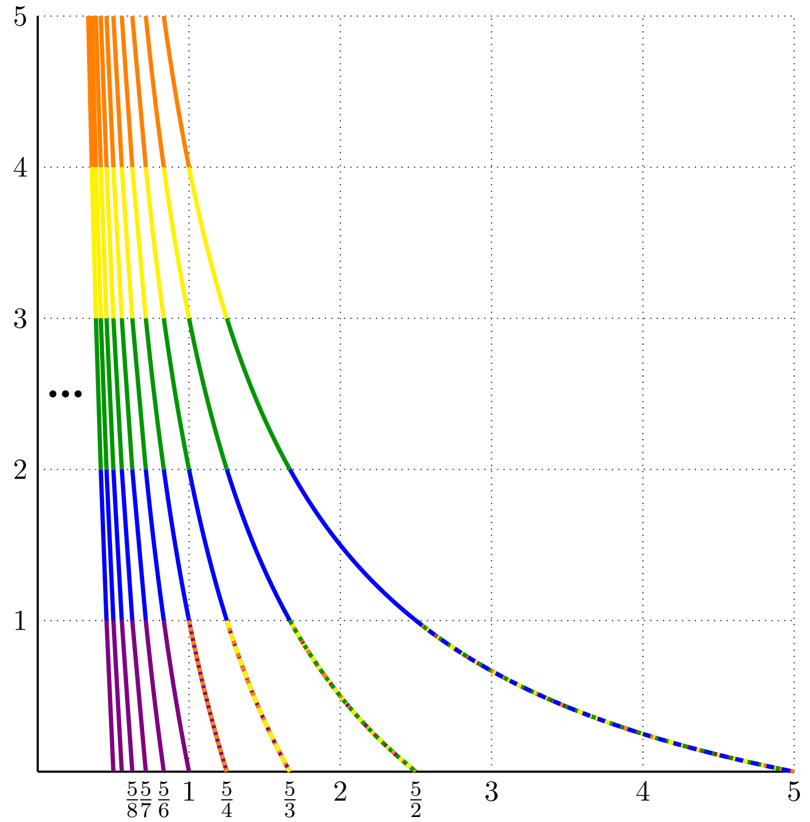

Figure 3.1 – The 5 random continued fraction transformation, violet, blue, green, yellow and orange illustrate the maps T_0, T_1, T_2, T_3, T_4 respectively

the i is fixed we set $d_1 = d_{1,i}$, $d_2 = d_{2,i}$. Then

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \cfrac{N}{\ddots}}}. \tag{3.1.2}$$

In the article of Anselm and Weintraub [2011] is shown that this expansion converges indeed to x. The article of Dajani *et al.* [2013] tells us that if we endow $[0, N]$ with the Borel-σ-algebra each transformation T_i has an invariant measure.

We extend these N-continued fractions to a random transformation. Let $\{0, 1, 2, \cdots, N-1\}$ be our **parameter space**, so it is the space where we choose from to decide which transformation we will use. Let $(p_0, p_1, \cdots, p_{N-1})$ be a probability vector on the parameter space, with probability p_j we choose i equal to j and $\sum_{i=0}^{N-1} p_i = 1$. Let $\{T_0, T_1, \cdots, T_{N-1}\}$ be our family of transformations, as defined above. Each $T_i : [0, N] \to [0, N]$, and hence we define $[0, N]$ to be our **state space**. So loosely saying the **state space**, is the space where the evolution of x, $x, Tx, T^2x, \cdots$ lives in. Now each time we apply the transformation we choose according to our probability vector an i and apply transformation T_i. Applying T_i we obtain a digit $d_{n,i}(x)$ as defined above. Using these digits we expand x like:

$$x = \cfrac{N}{d_{1,i} + \cfrac{N}{d_{2,i} + \cfrac{N}{\ddots}}} \tag{3.1.3}$$

and also this expansion converges to x by Anselm and Weintraub [2011]. Since each time we choose an i in our parameter space, the i is no longer fixed. Note that we can choose our transformations in infinitely many ways, by choosing infinitely many different sequences of i's. Hence we get infinitely many expansions for each irrational $x \in [0, N]$.

Example 3.1.2. We give an example of a 2-random continued fraction transformation. We use the parameter space $\{0, 1\}$ with probability density vector $\{\frac{1}{2}, \frac{1}{2}\}$. We define the transformations T_0, T_1 as follows: $T_0, T_1 : [0, 2] \to [0, 2]$.

$$\begin{aligned} T_0 =& \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor, \\ T_1 =& \begin{cases} \frac{2}{x} - \left(\left\lfloor \frac{2}{x} \right\rfloor - 1\right) & \text{if } x \in [0, 1] \\ \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor & \text{if } x \in (1, 2]. \end{cases} \end{aligned}$$

So our family of transformations is $\{T_0, T_1\}$. We depicted the transformations in figure 3.2. We will refer to T_0 as the lower map and to T_1 as the upper map. Define digits $b_i(x)$ by:

$$d_{1,i}(x) = \begin{cases} 1 & x \in (\frac{2}{1+1}, \frac{2}{1}] \\ k - i & x \in (\frac{2}{k+1}, \frac{2}{k}] \\ \infty & x = 0. \end{cases} \tag{3.1.4}$$

Now we like to expand $x \in [0, 2]$. Each time we iterate, we have to choose which transformation we use according to our probability distribution. This

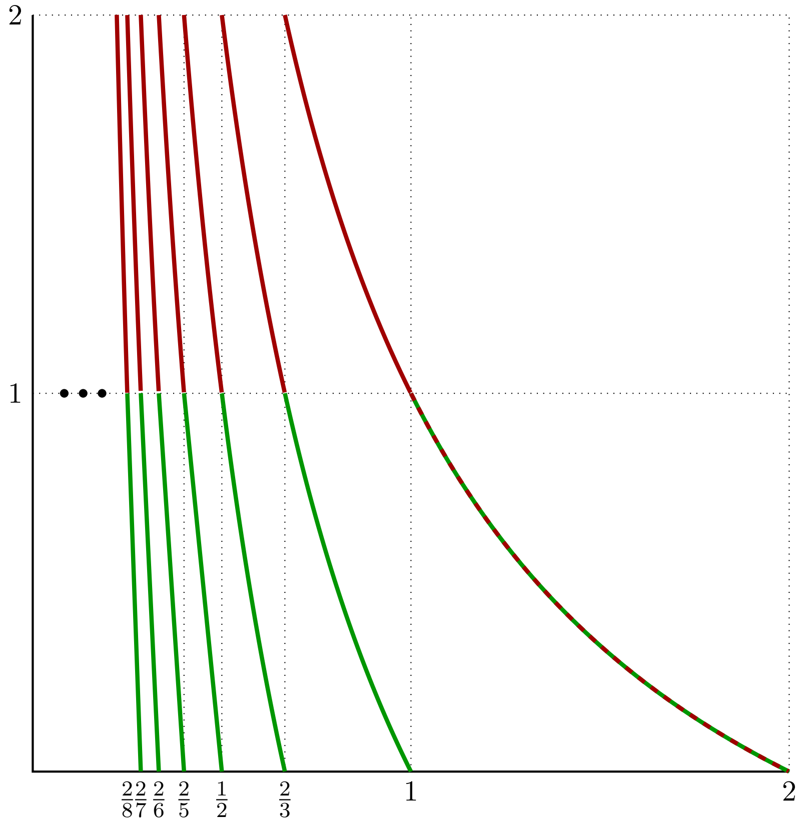

Figure 3.2 – The 2-continued fraction transformation. The map T_0 in green and T_1 in red

can be seen as throwing a fair coin each time we apply the transformation. If we throw tails we use the lower transformation and if we throw heads we use the upper transformation. In fact, if $x \in [1, 2]$, then we do not need to throw the coin, since in this area the upper and lower transformation coincide. Let $x \in [0, 1]$ and suppose we throw tails. Then we get a digit $d_{1,0}(x)$ and can write $x = \frac{2}{d_{1,0}+T_1(x)}$. To obtain an expansion of length two we need another digit. So we throw again our coin to see which transformation we use and obtain a new digit $d_{1,i}(T_i(x)) = d_{2,i}(x)$. Suppose we had thrown heads the first time, then we see from the figure that we enter the region $[1, 2]$, where T_0 and T_1 coincide. Therefore we do not have to throw the coin and $d_{2,i}(x) = 1$. Continuing this way we will obtain an expression for x given by:

$$x = \cfrac{2}{d_{1,i} + \cfrac{2}{d_{2,i} + \cfrac{2}{\ddots}}}. \tag{3.1.5}$$

That we indeed can expand x this way will be explicitly proved in chapter 4 $\triangle$

The question is whether we can find an invariant measure for the N-random continued fraction transformation. If we go back to the definition of an invariant measure, this tells us that $\mu(A) = \mu(T^{-1}(A))$. However using the random transformation T_i could be different each time we apply the transformation. Therefore we will introduce the following definition of **an invariant measure for a random transformation**

3.2 Random transformations and the theorem of Inoue

3.2.1 Random transformations

In this section we will state the theory of random transformations as stated in the article of Inoue [2012].

Definition 3.2.1. Let $(W, \mathcal{B}, \nu)$ be a σ-finite measure space, i.e. there exists an sequence $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{B}$, such that $A_j \subset A_{j+1}$, $\bigcup_{j\in\mathbb{N}} A_j = W$ and $\nu(A_j) < \infty$ for all $j \in \mathbb{N}$. We will use W as our parameter space. Let $(X, \mathcal{A}, \mu)$ be a σ-finite space, X will be our state space. Let $\{T_t : X \to X, t \in W\}$ be a family of non-singular transformations such that for fixed x, $T_t(x)$ is a $\mathcal{B}$-measurable function. Let $p : W \times X \to [0, \infty)$ be probability density function for $t \in W$, so $\int_W p(t,x)d\nu = 1$ for all $x \in X$. Then we define the **random map** $T = \{T_t, p_t(x)\}$ as the Markov process with transition function $P(x, A) = \int_W p(t,x)\mathbf{1}_A(T_t(x))$ where $A \in \mathcal{A}$. ♢

3.2.2 Invariant measures for random transformations

Using the definition of a random transformation we can extend the definition of an invariant measure to random transformations. In the deterministic case we say that μ is an invariant measure for T if $\mu(A) = \mu(T^{-1}(A)) = \int_X \mathbf{1}_A(Tx)d\mu$. We can interpret $\mu(T^{-1}(A))$ as the probability that x ends up in A after applying T. An invariant measure says then that the probability that $x \in A$ equals the probability that $T(x) \in A$. The Markov transition function

$$P(x, A) = \int_W p(t,x)\mathbf{1}_A(T_t(x))$$

see definition 3.2.1, tells us the probability that x ends up in A after applying the random transformation T. Hence we can define the operator P_* on measures μ on X by:

$$\begin{aligned} P_*\mu(A) &= \int_X P(x, A)d\mu && (3.2.1)\\ &= \int_X \int_W p(t,x)\mathbf{1}_A(T_t(x))d\nu d\mu. && (3.2.2)\end{aligned}$$

Note that by the above interpretation $P_*\mu(A) = \mu(T^{-1}A)$ for the random transformation T. Therefore we obtain the following definition of an invariant measure for random transformations.

Definition 3.2.2. If $P_*\mu = \mu$, then we say that μ is **an invariant measure for the random transformation** T. ♢

Example 3.2.3. Lets apply the above definition to the N-random continued fraction transformation. In that case $W = \{0, 1, \cdots, N-1\}$ is our parameter space, endowed with a probability vector $(p_0, p_1, \cdots, p_{N-1})$. In this case p_i does not depend on x. We have the family $\{T_i : [0, N] \to [0, N], i \in W\}$ of measurable non-singular transformations on the state space $[0, N]$. Hence we say that the measure μ is invariant with respect to the random transformation

$T = \{T_i(x), p_i, i \in W\}$ if

$$\mu(A) = P_*\mu(A) \tag{3.2.3}$$

$$= \int_X \int_W p_t \mathbf{1}_A(T_t(x)) d\nu d\mu \tag{3.2.4}$$

$$= \sum_{i=0}^{N-1} p_i \mu(T_i^{-1}(A)). \tag{3.2.5}$$

△

3.2.3 The random Perron-Frobenius operator

Let again $(W, \mathcal{B}, \nu)$ be our parameter space, $(X, \mathcal{A}, \mu)$ be our state space and $T = \{T_t(x), p(t,x)\}$ a random transformation. Suppose that μ is an invariant measure for the random transformation T and that μ admits a density f with respect to the Lebesgue measure λ. Since the maps T_t are non-singular we have for $A \in \mathcal{A}$ that if $\lambda(A) = 0$ then $\mu(A) = 0$ and hence $\mu(T_t^{-1}(A)) = 0$. Since the integral over a null-set is zero we obtain

$$\int_X \mathbf{1}_A(T_t(x)) p(t,x) d\mu = 0$$

and therefore

$$\int_W \int_X \mathbf{1}_A(T_t(x)) p(t,x) d\mu d\nu = 0.$$

Hence by Fubini's theorem $\int_X \int_W \mathbf{1}_A(T_t(x)) p(t,x) d\nu d\mu = 0$ and therefore by theorem 2.5.3 $P_*\mu$ admits a density. Denoting this density by $P_T f$ we can write:

$$P_*\mu(A) = \int_A P_T f d\lambda \tag{3.2.6}$$

$$= \int_X \int_W p(t,x) \mathbf{1}_A(T_t(x)) f(x) d\nu d\lambda. \tag{3.2.7}$$

Suppose W consist of only one element, then $P_*\mu(A) = \int_{T^{-1}(A)} f d\lambda$, so $P_T f$ is the Perron-Frobenius operator. Therefore 3.2.7 gives us a natural generalisation of the random Perron-Frobenius operator. Let P_{T_t} denote the Perron-Frobenius operator with respect to the transformation T_t. Using Fubini's theorem on the last equation of 3.2.7 we see that the random Perron-Frobenius operator $P_T : \mathcal{L}^1 \to \mathcal{L}^1$ is given by:

$$P_T f = \int_W P_{T_t} f(x) p_t(x) d\nu. \tag{3.2.8}$$

3.2.4 Piecewise monotone random transformations on an interval

Let $(W, \mathcal{B}, \nu)$ be a parameter space and $(X, \mathcal{A}, \mu)$ a state space. From now on let X be the interval $[0,1] \subset \mathbb{R}$, $\mathcal{A}$ the Borel-σ-algebra and μ the Lebesgue measure. Let $T = \{T_t, p(t,x)\}$ be a random transformation, thus $T_t : [0,1] \to [0,1]$, and $p(t,x) : W \times [0,1] \to [0, \infty)$. We define for each transformation T_t a countable partition of $[0,1]$. Let Λ be a countable set of indices and let $\Lambda_t \subset \Lambda$ for each $t \in W$. For each t let $\{I_{i,t}\}_{i \in \Lambda_t}$ be such that $\mu([0,1] \backslash \bigcup_{i \in \Lambda_t} I_{i,t}) = 1$ and for $i, j \in \bigcup_{\Lambda_t}$, $i \neq j$ $I_{i,t} \cap I_{j,t} = \emptyset$. For notational reasons we define $I_{i,t} = \emptyset$ if $i \in \Lambda \backslash \Lambda_t$ and define $\emptyset$ to be closed. Let $\mathrm{int}(I_{i,t})$ denote the interior of $I_{i,t}$. We assume two conditions for the random map $\{T_t, p(t,x), \{I_{t,x}\}, t \in W\}$:

1. The restriction of T_t to $\text{int}(I_{i,t})$ is C^1 and monotone for each $i \in \Lambda$ and $t \in W$.

2. Let $T_{t,i}$ be the restriction of T_t to $\text{int}(I_{t,i})$ for each $t \in W$ and $i \in \Lambda$. Put

$$\phi_{t,i}(x) = \begin{cases} T_{t,i}^{-1}(x) & \text{if } x \in T_{t,i}(\text{int}(I_{t,i})) \\ 0 & \text{if } x \in [0,1] \backslash T_{t,i}(\text{int}(I_{t,i})) \end{cases} \tag{3.2.9}$$

   for each $t \in W$ and $i \in \Lambda$. Note that $\phi_{t,i}(x) = 0$ if $i \in \Lambda \backslash \Lambda_t$. We assume that for each $x \in X$ and $i \in \Lambda$, $w_{x,i}(t) := \phi_{t,i}(x)$ is a measurable function of t.

If $\{T_t, p(t,x), \{I_{t,x}\}, t \in W\}$ satisfies 1 and 2 then we call $\{T_t, p(t,x), \{I_{t,x}\}, t \in W\}$ a piecewise monotonic random transformation. Like we did in the deterministic case, we can construct the random-Perron-Frobenius operator explicitly. Let $\phi^*(x) = \phi'(x)\mathbf{1}_{T_{t,i}(\text{int}(I_{t,i}))}(x)$ Using change of variables formula and Fubini's theorem we can write for $A \in \mathcal{A}$:

$$P_*\mu(A) = \int_A P_T f(x) d\lambda \tag{3.2.10}$$

$$= \int_W \int_X p(t,x)\mathbf{1}_A(T_t(x)) f(x) d\lambda d\nu \tag{3.2.11}$$

$$= \int_W \int_A \sum_{i\in\Lambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i}(x)) |\phi^*_{t,i}(x)| d\lambda d\nu. \tag{3.2.12}$$

Applying Fubini's theorem once more we obtain:

$$P_T f(x) = \int_W \sum_{i\in\Lambda} p(t, \phi_{t,i}(x)) f(\phi_{t,i(x)}) |\phi'_{t,i}(x)| \mathbf{1}_{T_t(\text{int} I_{i,t})}(x) d\nu. \tag{3.2.13}$$

In the article of Inoue [2012], the following properties of the random Perron-Frobenius operator are stated:

Lemma 3.2.4. *Let $T = \{T_t, p(t,x) : t \in W\}$ be a random map defined in section 3.2.1, let $P_T : \mathcal{L}^1(\lambda) \to \mathcal{L}^1(\lambda)$ the corresponding Perron-Frobenius operator, and let $f \in \mathcal{L}^1(\lambda)$. Then*

1. *P_T is linear,*
2. *$P_T f \geq 0$ if $f \geq 0$,*
3. *$\int_X P_T f d\lambda = \int_X f d\lambda$,*
4. *$||P_T f||_1 \leq ||f||_1$,*
5. *$P_{T^n} = P_T^n$.*

Like the "deterministic" Perron-Frobenius-operator, also the random Perron-Frobenius-operator gives us a tool to find an invariant measure.

Lemma 3.2.5. *Let $T = \{T_t, p(t,x) : t \in W\}$ be a random map as defined in section 3.2.1. Let $P_T : \mathcal{L}^1(\lambda) \to \mathcal{L}^1(\lambda)$ be the corresponding Perron-Frobenius operator and let f be a probability density function on the measure space $(X, \mathcal{A}, \lambda)$. Set $\mu(A) = \int_A f(x) d\lambda(x)$ for $A \in \mathcal{A}$. Then $P_T f = f$ $\lambda - a.e.$ if an only if μ is an invariant probability measure for T.*

3.2.5 Existence theorem of Inoue

Now we are ready to state the theorem of Inoue [2012].

Theorem 3.2.6. *Let $T = \{T_t, p(t,x), \{I_{i,t}\}_{i\in\Lambda} : t \in W\}$ be a random transformation as defined in 3.2.4. For $t \in W$ and $x \in [0,1]$, put*

$$g(t,x) = \begin{cases} \frac{p(t,x)}{|T_t'(x)|}, & \text{if } x \in \bigcup_i int(I_{t,i}) \\ 0, & \text{if } x \in [0,1] \setminus \bigcup_i int(I_{t,i}). \end{cases} \tag{3.2.14}$$

Assume the following conditions hold:

1. $\sup_{x\in[0,1]} \int_W g(t,x)\nu(dt) < \alpha < 1$, *i.e. the functions T_i are expanding on average.*

2. *There exists a constant M such that $\bigvee_{[0,1]} g(t,\cdot) < M$ for almost all $t \in W$, that is, there exists a ν-measurable set $W_0 \subset W$ such that $\int_{W_0} p(t,x)\nu(dt) = 1$ and $\bigvee_{[0,1]} g(t,\cdot) < M$ for all $t \in W_0$.*

Then T has an invariant probability measure μ_p which is absolutely continuous with respect to the Lebesgue measure. Moreover μ_p admits a probability density function h_p which is of bounded variation and satisfies for all $A \in \mathcal{A}$:

$$\mu_p(A) = \int_X \int_W p(t,x)\mathbf{1}_A(T_t(x))h_p(x)d\nu d\lambda. \tag{3.2.15}$$

3.3 Applying Inoue to the N-continued fraction transformation

Let us see whether we can apply the theorem 3.2.6 to the N-random continued fraction transformation. Unluckily we can not directly apply the theorem to the N-random continued fraction transformation, because this transformation is defined from $[0,N] \to [0,N]$, instead of from $[0,1] \to [0,1]$. Another problem is that the N-random continued fraction transformation is not on average expanding. For each $T_i(x)$ we find $T_i'(x) = \frac{-N}{x^2}$. Therefore on the area $[\sqrt{N}, N]$ we obtain $\sum_{i=1}^{N} g(i,x) \geq 1$.

The question arises if we can solve these problems? It turns out we can at least for the case $N = 2$ as explained in the next chapter. For the moment we turn back to example 3.1.2, to see how we modify the 2-random continued fraction transformation such that it satisfies the conditions of theorem 3.2.6.

3.4 The accelerated 2-random continued fraction transformation

In this section we show how the 2-random continued fraction transformation can be modified in order to satisfy the conditions of theorem 3.2.6 First we reduce the 2-continued random fraction transformation which is defined $[0,2] \to [0,2]$ to a transformation $[0,1] \to [0,1]$. Recall in this case we have two transformations defined by:

$$
\begin{aligned}
S_0, S_1 :& [0,2] \to [0,2] \\
S_0 =& \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor \\
S_1 =& \begin{cases} \frac{2}{x} - \left(\left\lfloor \frac{2}{x} \right\rfloor - 1 \right) & \text{if } x \in [0,1] \\ \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor & \text{if } x \in (1,2] \end{cases}.
\end{aligned}
$$

Note $S_0 = T_0$ from example 3.1.2 and $S_1 = T_1$, so S_0 is the lower transformation and S_1 the upper transformation. Let $p \in (0,1)$ and set $p_0 = p = 1 - p_1$ then we can define the random transformation $S = \{S_i, p_i, i \in \{0,1\}\}$. Looking at figure 3.2 we see that for each point $x \in [0,1]$, $S_1(x) \in [1,2]$. On the other hand if $x \in [1,2]$ then $S_0(x) \in [0,1]$ but also $S_1(x) \in [0,1]$ since S_0 and S_1 coincide on $[0,1]$. This is illustrated in figure 3.3.

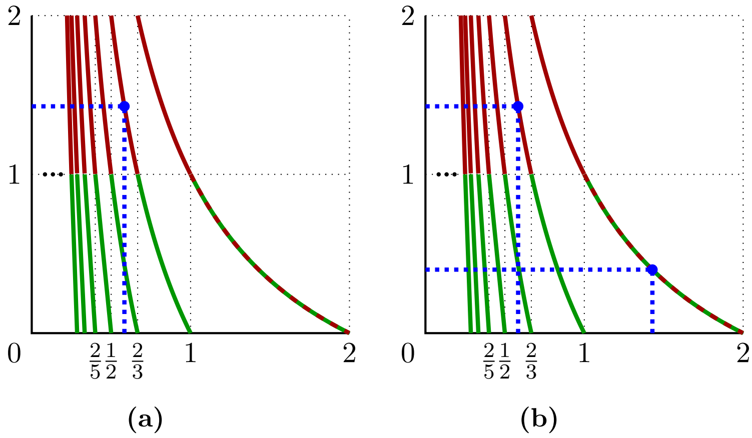

Figure 3.3 – (a) x after one iteration by S, (b) x after two times applying S

Let $x \in [0,1]$ and suppose we have to start with the upper transformation, then applying the random transformation once more we always end up in $[0,1]$. So if we have to start with the upper transformation we can force our random transformation to the area $[0,1]$ by applying S one more time. This suggests to redefine our transformation S_0, S_1 as follows:

$$
\begin{aligned}
T_0, T_1 :& [0,1] \to [0,1] \\
T_0(x) =& S_0(x) = \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor \\
T_1(x) =& S_1 \circ S_1(x) = S_1 \circ S_0 = \frac{2}{\frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor} - 1.
\end{aligned}
$$

Let $p \in (0,1)$, then we use transformation T_0 with probability p, so $p_0 = p$ and we use T_1 with probability $1-p$, so $p_1 = 1-p$. Hence we obtain the random transformation $T = \{T_i, p_i, i \in \{0,1\}\}$. Endow $[0,1]$ with the Borel-σ-algebra. Let us see if we can find an invariant measure by theorem 3.2.6. Notice the theorem 3.2.6 takes in this case the following form.

Theorem 3.4.1. *Given two non-singular maps $T_0, T_1 : [0,1] \to [0,1]$. Let $p \in [0,1]$ and set $p_0 = p$ and $p_1 = (1-p)$. For $i \in \{0,1\}$ let $\{I_{i,k}\}$ be a countable* *partition of* $[0,1]$ *into intervals and use* $int(I_{i,k})$ *to denote the interior of these intervals. Let* $g(i,x)$ *be functions satisfying:*

$$g(i,x) = \frac{p_i}{|T_i'(x)|} \tag{3.4.1}$$

on $\bigcup_k int(I_{i,k})$. *Assume that the following conditions are satisfied:*

1. *The restriction of* T_i *to each interval* $int(I_{i,k})$ *are* C_1 *and monotone.*
2. *The weighted average expansion of* T_i *is uniformly positive for all* x*, i.e.,*

$$\sup_{x\in[0,1]} (g(0,x) + g(1,x)) < 1.$$

3. *For each* $i \in \{0,1\}$ *the functions* $g(i,x) : [0,1] \to \mathbb{R}$ *are of bounded variation.*

Then there exists a probability measure μ_p *on* $[0,1]$ *which is absolutely continuous with respect to the Lebesgue measure* λ *with density function* h_p *that is of bounded variation. Moreover,* μ_p *has the property that*

$$\mu_p(A) = p\mu_p(T_0^{-1}A) + (1-p)\mu_p(T_1^{-1}A)$$

for each Borel measurable set $A \subset [0,1]$.

We check whether our random transformation satisfies 3.4.1.

Proposition 3.4.2. *The maps* T_0, T_1 *with* $p_0 = p = 1 - p_1$*, for* $p \in [0,1]$ *satisfy the conditions of theorem 3.4.1 and therefore we find an invariant density* μ_p *with the above properties.*

Proof. We set $\{I_{0,k}\} = \{I_{1,k}\} = \left\{ \left(\frac{2}{k+1}, \frac{2}{k}\right], k \in \mathbb{N} \right\}$. The derivatives of T_0, T_1 are given by:

$$T_0'(x) = \frac{-2}{x^2} \tag{3.4.2}$$

$$T_1'(x) = \frac{4}{(2-(k-1)x)^2} \qquad \text{for } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]. \tag{3.4.3}$$

Therefore the restriction of T_0 to $(\frac{2}{k+1}, \frac{2}{k}]$ is a continuous monotone decreasing function and the restriction of T_1 to $(\frac{2}{k+1}, \frac{2}{k}]$ is a continuous monotone increasing function, so condition 1 is satisfied. For condition 2 we compute $g(0,x)$ and $g(1,x)$:

$$g(0,x) = \frac{p_0}{\frac{2}{x^2}} = \frac{p}{2}x^2, \tag{3.4.4}$$

$$g(1,x) = \frac{p_1}{\frac{4}{(2-x(k-1))^2}} = \frac{1-p}{4}(2-x(k-1))^2 \qquad \text{for } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]. \tag{3.4.5}$$

Suppose that $x = 0$, then $g(0,0) + g(1,0) = 1 - p < 1$. If $x \in (0,1]$ we have $g(0,x) < \frac{p}{2}$. Therefore it is enough to show that $g(1,x) \leq (1 - \frac{p}{2})$ for $x \in [0,1)$. For $x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]$ we find:

$$\frac{1-p}{4}\left(2 - \frac{2}{k}(k-1)\right)^2 < \frac{1-p}{4}(2-x(k-1))^2 < \frac{1-p}{4}\left(2 - \frac{2}{k+1}(k-1)\right)^2,$$

$$\frac{1-p}{4}\frac{4}{k^2} < \frac{1-p}{4}(2-x(k-1))^2 < \frac{1-p}{4}\frac{16}{(k+1)^2} < \frac{4}{9}(1-p).$$

In the last inequality we used that $k \geq 2$. It follows that:

$$\sup_{x\in[0,1]} (g(0,x) + g(1,x)) < 1 \tag{3.4.6}$$

and condition 2 is satisfied.

Finally we show that the functions $g(i,x) : [0,1] \to \mathbb{R}$ are of bounded variation. Note $g(0,x) = \frac{1}{2}px^2$ and therefore $\bigvee_{[0,1]} g(0,x) = \frac{p}{2}$. Since $g(1,x)$ is a monotone continuous function on $(\frac{2}{k+1}, \frac{2}{k}]$ for each $k \geq 2,\ k \in \mathbb{N}$ we find:

$$\bigvee_{[0,1]} g(1,x) = \frac{1-p}{4} \sum_k \frac{16}{(k+1)^2} - \frac{4}{k^2} < \infty.$$

Therefore all conditions of theorem 3.4.1 are satisfied. We conclude that there exists a probability measure μ_p on $[0,1]$ which is absolutely continuous with respect to the Lebesgue measure λ and has a density function h_p that is of bounded variation. Moreover, μ_p has the property that

$$\mu_p(A) = p\mu_p(T_0^{-1}A) + (1-p)\mu_p(T_1^{-1}A)$$

for each Borel measurable set $A \subset [0,1]$. □

Chapter 4

The 2-random continued fraction expansion

In section 3.4 we found an invariant measure μ_p for the transformation $\{T_0, T_1, p_0, p_1\}$ where $T_0 : (0,1] \to [0,1]$ and $T_1 : (0,1] \to [0,1]$ are defined by:

$$T_0(x) = \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor, \tag{4.0.1}$$

$$T_1(x) = \frac{2}{\frac{2}{x} - \left(\left\lfloor \frac{2}{x} \right\rfloor - 1\right)} - 1, \tag{4.0.2}$$

and $p_0 = p = 1 - p_1$ for $p \in (0,1)$. In this chapter we show how we can use this accelerated 2-random continued fraction transformation to find an invariant measure for the 2-random-continued-fraction transformation. Using this invariant measure we derive properties about the asymptotic behaviour of the expansions induced by the 2-random continued fraction transformation and its entropy. In particular we like to apply the theorems from section 2.1. However all these theorems require deterministic transformations, but we have a random transformation. There is a general solution to de-randomize a random transformation and it is called the skew product. Therefore we start this chapter with introducing the 2-random continued fraction transformation as a skew product.

4.1 The 2-random continued fraction transformation as skew product

Definition 4.1.1. Let $\{\Omega, \mathcal{C}, \nu, \sigma\}$ be a dynamical system on a probability space and suppose that $\{T_\omega, \omega \in \Omega\}$ is a family of measure preserving transformations on another probability space $(X, \mathcal{F}, \mu)$. Assume that $T_\omega(x)$ is $\mathcal{C} \times \mathcal{F}$ measurable. Then a transformation of the form $\mathcal{T}(\omega, x) : \Omega \times X \to \Omega \times X$ defined by

$$\mathcal{T}(\omega, x) = (\sigma(\omega), T_\omega(x)).$$

is called a **skew product** $\diamondsuit$

The transformation σ serves as a function that picks "at random" an $\omega \in \Omega$. Such a transformation can be defined as follows.

Definition 4.1.2. Let $\Omega = \{0,1\}^{\mathbb{N}}$, the **left shift** σ is the function that shifts all coordinates of ω one place to the left. Hence σ is defined by:

$$\sigma : \Omega \to \Omega, \tag{4.1.1}$$

$$\sigma(\omega_1, \omega_2, \omega_3, \cdots) \to (\omega_2, \omega_3, \cdots). \tag{4.1.2}$$

Or equivalently let ω_i denotes the i'th coordinate of ω then $\sigma(\omega_i) = \omega_{i+1}$. ◊

We define the 2-random continued fraction transformation as a skew product. Let $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the left shift and set $\Omega = \{0,1\}^{\mathbb{N}}$. Let

$$\begin{aligned} &S_0, S_1 : \Omega \times [0,2] \to \Omega \times [0,2], \\ &S_0 = \frac{2}{x} - \lfloor \frac{2}{x} \rfloor \\ &S_1 = \begin{cases} \frac{2}{x} - \left(\lfloor \frac{2}{x} \rfloor - 1\right) & \text{if } x \in [0,1] \\ \frac{2}{x} - \lfloor \frac{2}{x} \rfloor & \text{if } x \in (1,2]. \end{cases} \end{aligned}$$

We define the transformation $R(\omega, x) : \Omega \times [0,2] \to \Omega \times [0,2]$ by:

$$R(\omega, x) = \begin{cases} (\omega, S_1 x = S_0 x) & x \in (1,2] \\ (\sigma(\omega), S_{\omega_1}(x)) & x \in [0,1] \\ (\sigma(\omega), 0) & x = 0. \end{cases} \tag{4.1.3}$$

Define the digits of R by:

$$b_1(\omega, x) = \begin{cases} 1 & x \in (\frac{2}{1+1}, \frac{2}{1}] \\ k & x \in (\frac{2}{k+1}, \frac{2}{k}], \quad \omega_1 = 0 \\ k-1 & x \in (\frac{2}{k+1}, \frac{2}{k}], \quad \omega_1 = 1 \\ \infty & x = 0 \end{cases} \tag{4.1.4}$$

and set $b_n(\omega, x) = b_1(R^{n-1}(\omega, x))$. Let π_2 denote the projection on the second coördinate, then we can write

$$\pi_2(R(\omega, x)) = \frac{2}{x} - b_1(\omega, x). \tag{4.1.5}$$

Let $b_i = b_i(\omega, x)$ then we can write:

$$x = \frac{2}{b_1 + \pi_2(R(\omega, x))} = \frac{2}{b_1 + \frac{2}{b_2 + \pi_2(R^2(\omega, x))}} = \cdots = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{1}{\ddots + \cfrac{2}{b_n + \pi_2(R^n(\omega, x))}}}}.$$

Now the question is whether we can expand x like

$$x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{1}{\ddots + \cfrac{2}{b_n + \ddots}}}}. \tag{4.1.6}$$

The answer is yes and we prove it explicitly in the next section.

4.1.1 Convergence of the 2-continued fraction expansion

Again let $b_i = b_i(\omega, x)$ as defined in 4.1.4. Like we did in section 2.2 for the regular continued fractions, we can define matrices M_n and B_n for the transformation $R(\omega, x)$ such that

$$B_n(\omega, x) = B_n = \begin{bmatrix} 0 & 2 \\ 1 & b_n \end{bmatrix} \text{ and } M_n(\omega, x) = M_n = B_1 \cdot B_2 \cdots B_n.$$

Using the Moebius transformation, see section 2.2 we obtain

$$B_1(0) = \frac{2}{b_1},\ B_2 \cdot B_1(0) = \cfrac{2}{b_1 + \cfrac{2}{b_2}}.$$

Therefore the partial fractions $\frac{p_n}{q_n}$ are given by:

$$M_n(0) = B_1 \cdot B_2 \cdots B_n(0) = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \cfrac{1}{\ddots + \cfrac{2}{b_n}}}} = \frac{p_n}{q_n}.$$

We use M_n and B_n to derive recurrence relations for the partial fractions.

$$M_n = M_{n-1} B_n \tag{4.1.7}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & b_n \end{bmatrix} \tag{4.1.8}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} p_{n-1} & 2r_{n-1} + b_n p_{n-1} \\ q_{n-1} & 2s_{n-1} + b_n q_{n-1} \end{bmatrix}. \tag{4.1.9}$$

Hence we obtain we obtain the following recurrence relations:

$$p_{-1} = 1 \qquad p_0 = 0 \qquad p_n = 2p_{n-2} + b_n p_{n-1}, \tag{4.1.10}$$

$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_n = 2q_{n-2} + b_n q_{n-1}. \tag{4.1.11}$$

Using M_n we can also derive an expression for x:

$$x = M_{n-1} B_n(\pi_2(R(\omega, x))) = \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & b_n \end{bmatrix} (\pi_2(R^n(\omega, x))) \tag{4.1.12}$$

$$= \begin{bmatrix} p_{n-2} & p_{n-1} \\ q_{n-2} & q_{n-1} \end{bmatrix} \left(\frac{2}{b_n + \pi_2(R^n(\omega, x)} \right) \tag{4.1.13}$$

$$= \frac{2p_{n-2} + p_{n-1} b_n + p_{n-1}\pi_2(R^n(\omega, x)}{2q_{n-2} + q_{n-1} b_n + q_{n-1}\pi_2(R^n(\omega, x))} \tag{4.1.14}$$

$$= \frac{p_n + p_{n-1}(\pi_2(R^n(\omega, x)))}{q_n + q_{n-1}(\pi_2(R^n(\omega, x)))}. \tag{4.1.15}$$

To show that we can expand x in the form of 4.1.6, we have to show that $\lim_{n\to\infty} |x - \frac{p_n}{q_n}| = 0$. Note that $\det B_i = (-2)$ fo $i \in \mathbb{N}$ and hence $\det M_n = (-2)^n$.

Therefore we write

$$\left|x-\frac{p_n}{q_n}\right| = \left|\frac{(p_{n-1}q_n - p_nq_{n-1})\,\pi_2(R^n(\omega,x))}{(q_n+q_{n-1}R^n(\omega,x))\,q_n}\right| \tag{4.1.16}$$

$$= \left|\frac{\det M_n \cdot R^n(\omega,x)}{(q_n+q_{n-1}R^n(\omega,x))\,q_n}\right| \tag{4.1.17}$$

$$\leq \frac{2^{n+1}}{q_n^2}. \tag{4.1.18}$$

We used in the last equation that for $x \in [0,2]$ we have $\pi_2(R^n(\omega,x)) \in [0,2]$ for all $n \in \mathbb{N}$. Now we need some estimate of q_n.

Proposition 4.1.3. *Define q_n and p_n as above, then $q_n \geq 2^{n-1}$ and $p_n \geq 2^{n-1}$ $\forall n \in \mathbb{N}$.*

Proof. We prove only $q_n \geq 2^{n-1}$ since the proof of $p_n \geq 2^{n-1}$ follows in the same way. We use induction. Since $q_1 = b_1 \geq 1 = 2^0$ the base step is proved. Suppose the result holds $\forall n \leq N$. Using the induction hypothesis and noting $b_{N+1} \in \mathbb{N}$ we write:

$$q_{N+1} = 2q_{N-1} + b_{N+1}q_N \geq 2 \cdot 2^{N-2} + 2^{N-1} = 2^N.$$

Hence $q_{N+1} \geq 2^N$ which concludes the proof. □

Proposition 4.1.4. *If $x \in [0,1]$ then $\lim_{n\to\infty}\left|x-\frac{p_n}{q_n}\right| = 0$.*

Proof. By equation 4.1.18 and proposition 4.1.1 it immediately follows that

$$\lim_{n\to\infty}\left|x-\frac{p_n}{q_n}\right| = 0.$$

□

Finally we show a relation between the numerator p_n and the denominator q_n of the partial fractions.

Proposition 4.1.5. *Let $\frac{p_n}{q_n} = \frac{p_n(\omega,x)}{q_n(\omega,x)}$ denote the partial fractions. Then*

$$p_n(\omega,x) = 2q_{n-1}(R(\omega,x))$$

for all $n \in \mathbb{N}$.

Proof. We use induction to prove the statement. Note that $p_1(\omega,x) = 2$ and $q_0(R(\omega,x)) = 1$. Suppose the result holds true for all $n \leq N$ then

$$p_{N+1}(\omega,x) = 2p_{N-1}(\omega,x) + b_{N+1}(\omega,x)p_N(\omega,x) \tag{4.1.19}$$

$$= 4q_{N-2}(R(\omega,x)) + b_N(R(\omega,x)) \cdot 2 \cdot q_{N-1}(R(\omega,x)) \tag{4.1.20}$$

$$= 2q_N(R(\omega,x)). \tag{4.1.21}$$

□

4.2 The accelerated 2-random continued fraction transformation as skew product

In this section we will define a skew product for the accelerated 2-random continued fraction transformation. We will show this skew product is an induced transformation of $R(\omega, x)$ as defined in section 4.1.1.

Recall that the accelerated 2-random continued fraction transformation was given by $\{T_0, T_1, p_0, p_1\}$ where $T_0 : (0,1] \to [0,1]$ and $T_1 : (0,1] \to [0,1]$ are defined by:

$$T_0(x) = \frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor, \tag{4.2.1}$$

$$T_1(x) = \frac{2}{\frac{2}{x} - \left(\left\lfloor \frac{2}{x} \right\rfloor - 1\right)} - 1. \tag{4.2.2}$$

Let $\Omega : \{0,1\}^{\mathbb{N}}$ and let $\sigma : \Omega \to \Omega$ denote the left shift. We define the skew product K for the 2-random continued fraction transformation by

$$K(\omega, x) : \Omega \times [0,1] \to \Omega \times [0,1], \tag{4.2.3}$$

$$K(\omega, x) = \begin{cases} (\sigma(\omega), T_{\omega_1} x) & \text{if } x \in (0,1] \\ (\sigma(\omega), 0) & \text{if } x = 0. \end{cases} \tag{4.2.4}$$

Notice that

$$K(\omega, x) = (\sigma(\omega), T_{\omega_1}(x)) \tag{4.2.5}$$

$$= \begin{cases} R(\omega, x) & \text{if } \omega_1 = 0, x \in [0,1] \\ R^2(\omega, x) & \text{if } \omega_1 = 1, x \in [0,1]. \end{cases} \tag{4.2.6}$$

Let τ be the first return time defined by:

$$\tau : \Omega \times [0,1] \to \mathbb{N} \tag{4.2.7}$$

$$\tau(\omega, x) = \inf \{n \geq 1 : R^n(\omega, x) \in \Omega \times [0,1]\} \tag{4.2.8}$$

$$= \begin{cases} 1 & \text{if } \omega_1 = 0 \\ 2 & \text{if } \omega_1 = 1. \end{cases} \tag{4.2.9}$$

Then

$$K(\omega, x) = R^{\tau(\omega, x)}(\omega, x) \tag{4.2.10}$$

and we see that K is indeed the induced transformation of R. Notice that for $x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]$, $k \in \mathbb{N}$, $k \geq 2$ we can write:

$$T_0(x) = \frac{2}{x} - k, \tag{4.2.11}$$

$$T_1(x) = \frac{2}{\frac{2}{x} - (k-1)} - 1. \tag{4.2.12}$$

Therefore given (ω, x) we can write x as follows:

$$x = \begin{cases} \dfrac{2}{k + T_0(x)} & \text{if } \omega_1 = 0 \text{ and } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right] \\ \dfrac{2}{(k-1) + \dfrac{2}{1 + T_1(x)}} & \text{if } \omega_1 = 1 \text{ and } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right]. \end{cases} \tag{4.2.13}$$

We define digits a_i as follows:

$$a_1(\omega, x) = \begin{cases} k & \text{if } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right] \text{ and } \omega_1 = 0 \\ (k-1, 1) & \text{if } x \in \left(\frac{2}{k+1}, \frac{2}{k}\right] \text{ and } \omega_1 = 1 \\ \infty & x = 0 \end{cases} \tag{4.2.14}$$

$$a_i = a_1\left(K^{i-1}(\omega, x)\right). \tag{4.2.15}$$

From 4.2.13 we see that the expansion of x induced by using $K(\omega, x)$ is in fact the same as the expansion induced $R(\omega, x)$. The only difference is that if $\omega_1 = 1$ and we use K we obtain the same expansions as using $R(\omega, x)$ twice. To relate the expansions obtained by K and R we introduce the variable $\tilde{n}$:

$$\tilde{n} : \mathbb{N} \times \Omega \times [0,1] \to \mathbb{N} \tag{4.2.16}$$

$$\tilde{n}(n, \omega, x) = \sum_{i=1}^{n} \mathbf{1}_{\{\omega_i = 0\}}(\omega, x) + \mathbf{1}_{\{\omega_i = 1\}}(\omega, x) = \sum_{i=0}^{n-1} \tau(K^i(\omega, x)). \tag{4.2.17}$$

So applying K n times is the same expansion as applying R $\tilde{n}$ times. Therefore the partial fraction $\frac{p_n}{q_n}$ obtained by applying $K(\omega, x)$ n times equals the partial fraction $\frac{p_{\tilde{n}}}{q_{\tilde{n}}}$ obtained by applying $R(\omega, x)$ $\tilde{n}$ times. Hence the partial fractions $\frac{p_n}{q_n}$ for K are just an subsequence of the partial fractions $\frac{p_{\tilde{n}}}{q_{\tilde{n}}}$ for R.

Proposition 4.2.1. *Let $x \in [0,1]$ and let $\frac{p_n}{q_n}$ be the partial fractions of K, then* $\lim_{n\to\infty} |x - \frac{p_n}{q_n}| = 0$.

Proof. Since the partial fraction $\frac{p_n}{q_n}$ for K are a subsequence of those of R, the result is consequence of proposition 4.1.4. □

4.2.1 Length of the fundamental interval

A nice property of the transformation K is that given a block of digits $a_1, \cdots, a_n$ where $a_i \in \{k, (k-1, 1) : k \geq 2\}$ we can find corresponding $\omega \in \Omega$. We set

$$\omega_i = \begin{cases} 0 & \text{if } a_i = k \text{ for some } k \geq 2 \\ 1 & \text{if } a_i = (k-1, 1) \text{ for some } k \geq 2. \end{cases} \tag{4.2.18}$$

We denote this corresponding sequence $\omega_1 \cdots \omega_n$, by $[\omega]_n$. Define

$$\Delta(a_1, \cdots, a_n) = \{x \in [0,1] : \forall \omega \in [\omega]_n, d_1(\omega, x) = a_1, \cdots, d_n(\omega, x) = a_n\}.$$

We will show that $\Delta(a_1, \cdots, a_n)$ is an interval of length $\frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}} + q_{\tilde{n}-1})}$.

Proposition 4.2.2. *$X = \{x \in [0,1] : (\omega, x) \in \Delta(a_1, \cdots, a_n)\}$ is an interval of length $\frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}} + q_{\tilde{n}-1})}$, where $\tilde{n} = \sum_1^n 1_{\{\omega_n = 0\}} + 2 \cdot 1_{\{\omega_n = 1\}} = \sum_{i=0}^{n-1} \tau(K^i(\omega, x))$.*

Proof. Note that we can write $\pi_2(K^n(\omega, x)) = T_\omega(x)$, where the ω stresses that the transformation we take depends on ω. Suppose $x \in \Delta(a_1, \cdots, a_n)$, so the first n digits of x are $a_1 \cdots a_n$. Since each $a_i \in \{k_i, (k_i - 1, 1)\}$, $k_i \in \mathbb{N}$, $k_i \geq 2$ it follows $x \in (\frac{2}{k_1+1}, \frac{2}{k_1}]$, $T_\omega x \in (\frac{2}{k_2+1}, \frac{2}{k_2}]$, $\cdots, T_\omega^{n-1} x \in (\frac{2}{k_n+1}, \frac{2}{k_n}]$. From 4.1.15 we have $x = \frac{p_{\tilde{n}} + p_{\tilde{n}-1} T_\omega^n(x)}{q_{\tilde{n}} + q_{\tilde{n}-1} T_\omega^n(x)}$. We will show this is a monotone function in $T_\omega^n(x)$. Define $f : [0,1] \to [0,1]$, by $f(y) = \frac{p_{\tilde{n}} + p_{\tilde{n}-1} y}{q_{\tilde{n}} + q_{\tilde{n}-1} y}$, so $\frac{df(y)}{dy} = \frac{p_{\tilde{n}-1} q_{\tilde{n}} - q_{\tilde{n}-1} p_{\tilde{n}}}{(q_{\tilde{n}} + q_{\tilde{n}-1} y)^2} = \frac{(-2)^{\tilde{n}}}{(q_{\tilde{n}} + q_{\tilde{n}-1} y)^2}$.

Therefore $f(y)\big|_{[0,1]}$ is a continuous monotone function which is increasing for even $\tilde{n}$ and decreasing for odd $\tilde{n}$. Since $T^n_\omega(x)$ takes values in $[0,1)$ if $\omega_n = 0$ and $T^n_\omega(x) \in (0,1]$ for $\omega_n = 1$ it follows that $x \in \left[\frac{p_{\tilde{n}}}{q_{\tilde{n}}}, \frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}\right)$ for $\tilde{n}$ even and $\omega_n = 0$ and $x \in \left(\frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}, \frac{p_{\tilde{n}}}{q_{\tilde{n}}}\right]$ for $\tilde{n}$ odd and $\omega_n = 0$. In the case $\omega_n = 1$ we find similar results, only the open and closed boundaries of the interval are interchanged.
Now we show by induction that if $z \in \left[\frac{p_{\tilde{n}}}{q_{\tilde{n}}}, \frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}\right)$, then (ω, z) has digits $\Delta(a_1, \cdots, a_n)$ if $\omega \in [\omega]_n$. First we take $N = 1$. We have two cases, a_1 is of the form k_1 and a_1 is of the form $(k_1 - 1, 1)$. Suppose $a_1 = k_1$, so $\omega_1 = 0$ and $z \in \left(\frac{p_{\bar{1}}+p_{\bar{0}}}{q_{\bar{1}}+q_{\bar{0}}}, \frac{p_{\bar{1}}}{q_{\bar{1}}}\right]$. Therefore:

$$\frac{2}{k_1+1} < z \leq \frac{2}{k_1}. \tag{4.2.19}$$

We see $a_1(\omega, z) = k_1$. Now suppose $a_1 = (k_1 - 1, 1)$, so $\omega_1 = 1$. In this case $z \in \left(\frac{p_{\bar{2}}}{q_{\bar{2}}}, \frac{p_{\bar{2}}+p_{\bar{1}}}{q_{\bar{2}}+q_{\bar{1}}}\right]$. Therefore:

$$\frac{2}{(k_1-1)+\frac{2}{1}} < z \leq \frac{2}{(k_1-1)+\frac{2}{1+1}} \tag{4.2.20}$$

$$\frac{2}{(k_1+1} < z \leq \frac{2}{k_1}. \tag{4.2.21}$$

We see that the $a_1(\omega, z) = (k-1, 1)$. So we see that the result holds true for $N = 1$. Suppose we have that the result holds for all $n \leq N$. We consider the case $n = N+1$. In this case we have $z \in \left[\frac{p_{\tilde{n}}}{q_{\tilde{n}}}, \frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}\right)$ where $\tilde{n} = \tilde{n}(\omega, n)$ and $\tilde{n}$ is even. Suppose that $\omega_1 = 0$ and $\omega_{n+1} = 0$. Then we can write

$$\frac{2}{k_1+\cfrac{2}{k_2+\ddots+\cfrac{2}{k_{n+1}}}} \leq z < \frac{2}{k_1+\cfrac{2}{k_2+\ddots+\cfrac{2}{k_{n+1}+1}}}. \tag{4.2.22}$$

Note that $\frac{2}{k_2+\ddots+\frac{2}{k_{n+1}}} \in [0,1]$, see lemma 4.2.3. If $\frac{2}{k_2+\ddots+\frac{2}{k_{n+1}}}$ equals 0 or 1, then there is an $m \leq n+1$ such that $k_m = \infty$ and the interval $\left[\frac{p_{\tilde{n}}}{q_{\tilde{n}}}, \frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}\right)$ becomes empty, since $T^i(T^m(x)) = 0\ \forall i \in \mathbb{N}$. Therefore we can assume:

$$\frac{2}{k_2+\ddots+\cfrac{2}{k_{n+1}}} \in (0,1),$$

$$\frac{2}{k_2+\ddots+\cfrac{2}{k_{n+1}+1}} \in (0,1),$$

so $a_1 = k_1$ and

$$\frac{2}{k_2+\ddots+\cfrac{2}{k_n}} \geq T_\omega(z) > \frac{2}{k_2+\ddots+\cfrac{2}{k_n+1}}. \tag{4.2.23}$$

Applying the induction hypothesis on $T_\omega(x)$ gives digits $a_2, \cdots, a_n$ for z and the result holds. The other cases follow in a similar way.

Finally note that $\lambda\left(\left[\frac{p_{\tilde{n}}}{q_{\tilde{n}}}, \frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}\right)\right) = \frac{2^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}}+q_{\tilde{n}-1})}$. $\square$

Lemma 4.2.3. *Let $\frac{p_n}{q_n}$ denote the partial fractions induced by K, then*

$$\frac{p_n}{q_n} = \cfrac{2}{k_1 + \cfrac{2}{k_2 + \ddots + \cfrac{2}{k_n}}} \in [0,1]$$

for all $x \in [0,1]$.

Proof. We use induction. For the case $\omega_1 = 0$ we have $\frac{p_1}{q_1} = \frac{2}{k_1} \in [0,1]$, since $k_1 \geq 2$. If $\omega_1 = 1$ then $\frac{p_1}{q_1} = \cfrac{2}{(k_1-1) + \cfrac{2}{1}} = \cfrac{2}{k_1+1} \in [0,1]$ since $k_1 \geq 2$. Suppose the result holds for $n \leq N$ then for $N+1$ we see in the case $\omega_1 = 0$ that

$$\frac{p_{N+1}}{q_{N+1}} = \cfrac{2}{k_1 + \cfrac{2}{k_2 + \ddots + \cfrac{2}{k_n}}} = \cfrac{2}{k_1 + \frac{p_N(T_\omega(x))}{q_N(T_\omega(x))}} \in [0,1].$$

Where we used that $T_\omega(x) \in [0,1]$. The case $\omega_1 = 1$ follows the same lines. $\square$

Remark 4.2.4. Notice that for a sequence of digits induced by R the above strategy does not work. Suppose we are given the digit block (1), then we could not find a corresponding ω. For $(\omega, x) \in [1] \times (\frac{2}{3}, 1]$ as well for $(\omega, x) \in [0] \times [1, 2]$ we obtain the digit 1. That it is so easy for K is a consequence of the definition of the digits. We will further study the digit sequences induced by R in chapter 5. $\diamondsuit$

4.3 Invariant measures for R and K

In this section we use the invariant measure for the accelerated 2-random continued fraction expansion, see section 3.4 to define an invariant measure for the skew product $K(\omega, x)$. Using the obtained measure we derive an invariant measure for $R(\omega, x)$.

We set $\Omega = \{0,1\}^{\mathbb{N}}$ and endow $\Omega \times [0,1]$ with $\sigma(\Omega \times [0,1]) = \sigma(\mathcal{C} \times \mathcal{B}[0,1])$, where $\mathcal{C}$ is the σ-algebra generated by the cylinders on $\{0,1\}^{\mathbb{N}}$ and $\mathcal{B}[0,1]$ the Borel σ-algebra restricted to $[0,1]$. Let m_p be the product measure on $\mathcal{C}$ and μ_p the invariant measure obtained in section 3.4 for the accelerated 2-random continued fraction transformation. Then we can prove the following proposition:

Proposition 4.3.1. *The measure $m_p \times \mu_p$ is an invariant measure for the map $K(\omega, x)$.*

Proof. Let $A \in \sigma(\Omega \times [0,1])$, such that $A = B \times [a,b]$, where

$$B = \{\omega \in \Omega : \omega_1 = i_1, \cdots, \omega_n = i_n,\ i_1, \cdots i_n \in \{0,1\}\}$$

is a cylinder set. Hence A is in the set of generators of $\mathcal{C} \times \mathcal{B}[0,1]$. It follows by theorem 3.4.1:

$$\begin{aligned} m_p \times \mu_p(K^{-1}A) =& m_p \times \mu_p\left(\{(x,\omega) : K(\omega,x) \in A\}\right) \\ =& m_p \times \mu_p\left(\{(x,\omega) : \sigma(\omega) \in B, T_{\omega_1}(x) \in [a,b]\}\right) \\ =& m_p \times \mu_p\left(\{(x,\omega) : \omega_1 = 0, \omega_2 = i_1, \cdots, \omega_{n+1} = i_n, T_0(x) \in [a,b]\}\right) \\ &+ m_p \times \mu_p\left(\{(x,\omega) : \omega_1 = 1, \omega_2 = i_1, \cdots, \omega_{n+1} = i_n, T_1(x) \in [a,b]\}\right) \\ =& m_p\left(\{\omega : \omega_1 = 1, \omega_2 = i_1, \cdots, \omega_{n+1} = i_n\}\right) \cdot \mu_p\left(T_0^{-1}([a,b])\right) \\ &+ m_p\left(\{\omega : \omega_1 = 1, \omega_2 = i_1, \cdots, \omega_n = i_{n+1}\}\right) \cdot \mu_p\left(T_1^{-1}([a,b])\right) \\ =& p m_p(B)\mu_p\left(T_0^{-1}([a,b])\right) + (1-p)m_p(B)\mu_p\left(T_1^{-1}([a,b])\right) \\ =& m_p \times \mu_p(A). \end{aligned}$$

□

Since K is an induced measure of R we use proposition 2.1.8 to construct the invariant measure ρ for R. Let $E \in \sigma(\mathcal{C} \times \mathcal{B}[0,2])$ then

$$\begin{aligned} \rho(E) =& \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} \sum_{n=0}^{\infty} m_p \times \mu_p(\{(\omega,x) \in \Omega \times [0,1]; \tau(\omega,x) > n\} \cap R^{-n}(E)) \\ =& \frac{1}{2-p}[m_p \times \mu_p(\Omega \times [0,1] \cap E) + m_p \times \mu_p([1] \times [0,1] \cap R^{-1}(E))]. \end{aligned}$$

We can integrate with respect to ρ in the following way:

$$\int_{\Omega\times[0,2]} f(\omega,x) d\rho = \frac{1}{2-p}\left[\int_{\Omega\times[0,1]} f(\omega,x) dm_p \times \mu_p + \int_{[1]\times[0,1]} f(R(\omega,x)) dm_p \times \mu_p\right].$$

This is seen as follows. Let A be a measurable set in $\sigma(\Omega \times [0,2])$ then we can write:

$$\begin{aligned} \int_{\Omega\times[0,2]} \mathbf{1}_A(\omega,x) d\rho =& \frac{1}{2-p}[m_p \times \mu_p(\Omega \times [0,1] \cap A) \\ &+ m_p \times \mu_p([1] \times [0,1] \cap R^{-1}(A))] \\ =& \frac{1}{2-p}[\int_{\Omega\times[0,1]} \mathbf{1}_A(\omega,x) dm_p \times d\mu_p \\ &+ \int_{[1]\times[0,1]} \mathbf{1}_{R^{-1}A}(\omega,x) dm_p \times d\mu_p] \\ =& \frac{1}{2-p}\int_{\Omega\times[0,1]} \mathbf{1}_A(\omega,x) dm_p \times d\mu_p \\ &+ \int_{[1]\times[0,1]} \mathbf{1}_{\{(\omega,x):R(\omega,x)\in A\}}(\omega,x) dm_p \times d\mu_p \\ =& \frac{1}{2-p}\int_{\Omega\times[0,1]} \mathbf{1}_A(\omega,x) dm_p \times d\mu_p \\ &+ \int_{[1]\times[0,1]} \mathbf{1}_A R(\omega,x) dm_p \times d\mu_p \end{aligned}$$

By linearity of the integral we obtain for finite simple functions f_n,

$$\int_{\Omega\times[0,2]} f_n d\rho = \frac{1}{2-p}\int_{\Omega\times[0,1]} f_n dm_p \times d\mu_p + \int_{[1]\times[0,1]} f_n(R(\omega,x)) dm_p \times d\mu_p$$

Let $f \in \mathcal{L}^1$ and write $f = f^+ - f^-$ where f^+ and f^- are positive functions. We can find a sequence of positive finite simple functions $\{f_n\}$ such that $\lim_{n\to\infty} f_n = f^+$ ρ-a.e. If $\rho(E) = 0$ for $E \in \sigma(\mathcal{C} \times \mathcal{B}[0,3])$ then by the definition of ρ $m_p \times \mu_p(E \cap \Omega \times [0,1]) = 0$ and $m_p \times \mu_p(R^{-1}(E) \cap [1] \times [0,1]) = 0$. $\lim_{n\to\infty} f_n = f^+ \ \rho - a.e.$ means there exists a set $Y \in \mathcal{C} \times \mathcal{B}[0,3]$ such that for all $x \in Y$ we have $\lim_{n\to\infty} f_n(x) = f^+$ and $\rho(\Omega \times [0,3]\backslash Y) = 0$. Then

$$m_p \times \mu_p((\Omega \times [0,3]\backslash Y) \cap \Omega \times [0,1]) = 0$$

and

$$m_p \times \mu_p(R^{-1}(\Omega \times [0,3]\backslash Y) \cap [1] \times [0,1]) = 0.$$

Hence

$$\lim_{n\to\infty} f_n(x) = f^+(x)$$

for all $x \in Y \cap \Omega \times [0,1]$ and we conclude $\lim_{n\to\infty} f_n = f^+ \ m_p \times \mu_p - a.e.$. Note that

$$\lim_{n\to\infty} f_n(R(\omega,x)) = f^+(R(\omega,x))$$

for all $R(\omega,x) \in Y$, so $(\omega,x) \in R^{-1}Y$ and in particular for all

$$(\omega,x) \in R^{-1}Y \cap [1] \times [0,1].$$

Hence

$$\lim_{n\to\infty} f_n(R(\omega,x))\mathbf{1}_{[1]\times[0,1]}(\omega,x) = f^+(R(\omega,x))\mathbf{1}_{[1]\times[0,1]}(\omega,x) \ m_p \times \mu_p - a.e.$$

Using monotone convergence it follows that

$$\begin{aligned}
\rho(f^+) &= \lim_{n\to\infty} \int_{\Omega\times[0,2]} f_n d\rho \\
&= \lim_{n\to\infty} \frac{1}{2-p} \int_{\Omega\times[0,1]} f_n(\omega,x) dm_p \times d\mu + \int_{[1]\times[0,1]} f_n(R(\omega,x)) dm_p \times d\mu \\
&= \frac{1}{2-p} \int_{\Omega\times[0,1]} f^+(\omega,x) dm_p \times d\mu + \int_{[1]\times[0,1]} f^+(R(\omega,x)) dm_p \times d\mu.
\end{aligned}$$

In the same way we find

$$\rho(f^-) = \frac{1}{2-p} \int_{\Omega\times[0,1]} f^-(\omega,x) dm_p \times d\mu + \int_{[1]\times[0,1]} f^-(R(\omega,x)) dm_p \times d\mu.$$

We conclude that we can indeed integrate $f \in \mathcal{L}^1(\rho)$ by

$$\int_{\Omega\times[0,2]} f(\omega,x) d\rho = \frac{1}{2-p} \left[\int_{\Omega\times[0,1]} f(x,\omega) dm_p \times \mu_p + \int_{[1]\times[0,1]} f(R(\omega,x)) dm_p \times \mu_p \right].$$

4.4 More about the invariant measure

In this section we show the measure μ_p, the marginal of the invariant measure for K, is equivalent with the Lebesgue measure λ. The method we use here is the same as the one used in the article Kalle *et al.* [2015].

Proposition 4.4.1. *Let $I \subset [0,1]$ be a non-trivial interval. Then $\forall \omega \in \Omega$, there is a $n \geq 1, n \in \mathbb{N}$ such that $(0,1) \subset (T_{\omega_n} \circ \cdots \circ T_{\omega_1}) I \subset [0,1]$.*

Proof. Let $J \subset [0,1]$ be a non-trivial open interval, then we can write $J = (c,d)$. First assume $\exists k \in \mathbb{N}$ such that $\frac{1}{k} \in J$. Notice $T_0(\frac{2}{k}) = 0$, so $T_0(J) = (b,1) \cup [0,c)$ for some $b,c \in (0,1)$. Then $\exists k \in \mathbb{N}$ such that $(\frac{2}{k+1}, \frac{2}{k}] \subset [0,c)$ and hence $(0,1) \subset T_{\omega_2}(T_{\omega_1}(J)) \subset [0,1]$. If $\omega_1 = 1$, then $T_1(J) = (b,1] \cup (0,c)$ for some $b,c \in (0,1)$, since $T_1(\frac{2}{k}) = 1$. Hence $(0,1) \subset T_{\omega_2}(T_{\omega_1}(J)) \subset [0,1]$.

Suppose $J \subset \left(\frac{2}{k+1}, \frac{2}{k}\right]$ for some $k \in \mathbb{N}$, $k \geq 2$. Then for the lower transformation it follows

$$\lambda(T_0 J) = \frac{2}{c} - k - \left(\frac{2}{d} - k\right) = \frac{2}{c} - \frac{2}{d} = \frac{2(d-c)}{cd} > d - c = \lambda(c,d).$$

For the upper map we get

$$\begin{aligned}
\lambda(T_1 J) &= \frac{2}{\frac{2}{d} - (k-1)} - 1 - \left(\frac{2}{\frac{2}{c} - (k-1)} - 1\right) = \frac{2}{\frac{2}{d} - (k-1)} - \left(\frac{2}{\frac{2}{c} - (k-1)}\right) \\
&= \frac{2\left(\frac{2}{c} - (k-1)\right) - 2\left(\frac{2}{d} - (k-1)\right)}{\left(\frac{2}{c} - (k-1)\right)\left(\frac{2}{d} - (k-1)\right)} = \frac{\frac{4}{c} - \frac{4}{d}}{\left(\frac{2}{c} - (k-1)\right)\left(\frac{2}{d} - (k-1)\right)} \\
&= \frac{4(d-c)}{cd\left(\frac{2}{c} - (k-1)\right)\left(\frac{2}{d} - (k-1)\right)} > \frac{4(d-c)}{4cd} \\
&= \frac{d-c}{cd} > d - c = \lambda(c,d).
\end{aligned}$$

Where we use that $\frac{2}{c} - (k-1), \frac{2}{d} - (k-1) \in [1,2)$. Set $J_1 = T_{\omega_1}(c,d) = (c_1, d_1)$ and $J_i = T_{\omega_i}(c_{i-1}, d_{i-1})$. We claim $\lambda(J_i) \geq \left(\frac{1}{1-(d-c)}\right)^n (d-c)$.We already proved the base step, the case $i = 1$ since $c \leq 1 - (d-c)$ and therefore $cd \leq 1 - (d-c)$. Suppose the result holds for all $n \leq N$. Then

$$\begin{aligned}
\lambda(J_{N+1}) = \lambda(T_{\omega_N}(c_N, d_N)) &> \frac{d_N - c_N}{c_N d_N} \\
&> \frac{1}{1 - (d_N - c_N)} \left(\frac{1}{1-(d-c)}\right)^N (d-c) > \left(\frac{1}{1-(d-c)}\right)^{N+1} (d-c),
\end{aligned}$$

since $d_N - c_N > d - c$ implies $1 - (d_N - c_N) < 1 - (d-c)$. Therefore there exists a $n \in \mathbb{N}$ such that $\frac{2}{k} \in J_n$.
Notice that we only proved the statement for J open. However suppose K is a closed or half-closed interval then there exists an open interval J such that $J \subset K$. Hence $T_{\omega_1}(J) \subset T_{\omega_1}(K)$ and the statement holds for each non-trivial interval $J \subset [0,1]$. □

Lemma 2.4.4 tells us that if f is a function of bounded variation on I, then it can be redefined on a countable set to become a lower semi-continuous function. Theorem 2.4.3 tells us that if f is lower semicontinuous on $I = [a,b] \subset \mathbb{R}$, then it is bounded from below and assumes its minimum value. Using this two statements we can proof the following proposition.

Proposition 4.4.2. *Let h_p be the probability density function from Theorem 3.2. Then $h_p > 0$ for all $x \in (0,1)$.*

Proof. We can redefine h_p on a countable number of points to get a lower-semi-continuous function. From now on assume that h_p is lower-semi-continuous. Lemma 3.2.5 tells us that h_p is a fixed point of the random Perron Frobenius operator as defined in subsection 3.2.3. In our case the Perron Frobenius operator is given by:

$$
\begin{aligned}
P_T f(y) = \sum_{k\in\mathbb{N}} pf\left(T^{-1}_{(0,k)}(y)\right)\left|\frac{1}{T'_{0,k}(T^{-1}_{0,k}(y))}\right| 1_{T_0(\mathrm{int}(\frac{2}{k+1},\frac{2}{k}])} \\
+(1-p)f\left(T^{-1}_{(1,k)}(y)\right)\left|\frac{1}{T'_{1,k}(T^{-1}_{1,k}(y))}\right| 1_{T_1(\mathrm{int}(\frac{2}{k+1},\frac{2}{k}])}.
\end{aligned}
$$

Since h_p is a probability density function of bounded variation and $P_T h_p = h_p$, we know $\exists I \subset [0,1]$ non-trivial interval and $\alpha > 0$ such that $h_p 1_I > \alpha$, see lemma 4.4.3. Therefore

$$
\begin{aligned}
h_p(y) &= P_T h_p(y) \\
&> \alpha P_T 1_I(y) \\
&= \alpha \sum_{k\in\mathbb{N}} p 1_I\left(T^{-1}_{(0,k)}(y)\right)\left|\frac{1}{T'_{0,k}(T^{-1}_{0,k}(y))}\right| 1_{T_0(\mathrm{int}(\frac{2}{k+1},\frac{2}{k}])}(y) \\
&+ (1-p) 1_I\left(T^{-1}_{(1,k)}(y)\right)\left|\frac{1}{T'_{1,k}(T^{-1}_{1,k}(y))}\right| 1_{T_1(\mathrm{int}(\frac{2}{k+1},\frac{2}{k}])}(y) \\
&= \alpha \sum_{(\omega_1)\in\Omega} \sum_{x\in T^{-1}_{\omega_1}\{y\}} p_{\omega_1} 1_I(x)\left|\frac{1}{T'_{\omega_1}(x)}\right|.
\end{aligned}
$$

In the article of Inoue Inoue [2012] is proved that $P_{T\circ S} = P_T \circ P_S$ so therefore also $P_{T^n} = \underbrace{P_T \circ \cdots \circ P_T}_{n-times}$, see 3.2.4 . Since $h_p = P_T h_p$ we have $h_p = P_{T^n} h_p$. Using this we can write the above as:

$$
h_p(y) = P_{T^n} h_p(y) > \alpha \sum_{(\omega_1,\cdots,\omega_n)\in\Omega} \sum_{x\in\left(T_{\omega_1}\circ\cdots\circ T_{\omega_n}\right)^{-1}\{y\}} 1_I(x)\left|\frac{p_{\omega_1}\cdots p_{\omega_n}}{\left(T_{\omega_1}\circ\cdots\circ T_{\omega_n}\right)'(x)}\right|.
$$

By proposition 4.4.1 we know there exists a n such that $T_{\omega_n} \circ \cdots \circ T_{\omega_n}(I) = (0,1)$ and therefore there exists a $x \in I$ such that $T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x) = y$. We conclude

$$
h_p(y) > 0 \quad \text{for all } y \in (0,1).
$$

□

Lemma 4.4.3. *If f is a probability density function with respect to the Lebesgue measure, which is of bounded variation on $[0,1]$, then there exists a non-trivial interval I and an $\alpha \in (0,1)$ such that $f\mathbf{1}_I > \alpha$.*

Proof. Assume the contrary, that there is not such an I. So for all $I \subset [0,1]$ exists a $x \in I$ such that $f(x) < \alpha$. Let $r \in (\alpha,1)$, we show there are at least countably many $y \in [0,1]$ such that $f(y) > r$. To see this, suppose there are only finitely many such y then

$$
\begin{aligned}
\int_0^1 f(x)dx &= \int_{\{f>r\}} f(x)d\lambda + \int_{f(x)\le r} f(x)d\lambda \\
&\le \sum_{\{y:f(y)>r\}} f(y)d\lambda(y) + \int_0^1 r d\lambda \\
&= r < 1.
\end{aligned}
$$

This is a contradiction with the fact that f is a probability density function on $[0, 1]$. We conclude there are at least countable many y such that $f(y) > r$.

Since we assumed there does not exist a non-trivial interval $I \subset [0, 1]$ such that $f(x) > \alpha$ for all $x \in I$, we can find for each $I \in [0, 1]$ a x such that $f(x) < \alpha$. Therefore we can make partitions with countable many atoms, which endpoints alternate between x such that $f(x) < \alpha$ and y such that $f(y) > r$. Hence we find $\bigvee f > M(r - \alpha)$ for all $M \in \mathbb{N}$ and we conclude that f is not of bounded variation which is a contradiction. □

Proposition 4.4.4. *The density function h_p is bounded from above and from below.*

Proof. Since $[0, 1]$ is a closed and bounded subset in $\mathbb{R}$ and h_p is of bounded variation, h_p is bounded from above. We can redefine h_p on a countable set to get a lower semi-continuous function. A lower semi-continuous functions attains its minimum on [0,1]. By proposition 4.4.2 we see that $h_p > 0$ on $(0, 1)$. Therefore we are left to show that $h_p(1) > 0$ and $h_p(0) > 0$. Let $\epsilon > 0$ and look at $T_0^{-1}(1-\epsilon, 1)$. Note that for $k \geq 2$, $k \in \mathbb{N}$

$$(\frac{2}{1+k}, \frac{2}{1-\epsilon+k}) \subset T_0^{-1}(1-\epsilon, 1)$$

and

$$\lambda\left((\frac{2}{1+k}, \frac{2}{1-\epsilon+k})\right) = \frac{2\epsilon}{(1+k)(1-\epsilon+k)}.$$

Hence

$$\frac{k^2}{2}\lambda\left((\frac{2}{1+k}, \frac{2}{1-\epsilon+k})\right) < \lambda((1-\epsilon, 1)) < \frac{(k+1)^2}{2}\lambda\left((\frac{2}{1+k}, \frac{2}{1-\epsilon+k})\right).$$

Therefore

$$\begin{aligned}
\lim_{x\uparrow 1} h_p(x) &= \lim_{\epsilon\to 0} \frac{1}{\lambda((1-\epsilon, 1))}\int_{1-\epsilon}^{1} h_p(x)dx \\
&= \lim_{\epsilon\to 0} \frac{\mu_p((1-\epsilon, 1))}{\lambda((1-\epsilon, 1))} \\
&= \lim_{\epsilon\to 0} \frac{p\mu_p(T_0^{-1}(1-\epsilon, 1)) + (1-p)\mu_p(T_1^{-1}(1-\epsilon, 1))}{\lambda((1-\epsilon, 1))} \\
&\geq \lim_{\epsilon\to 0} \frac{p\mu_p((\frac{2}{1+k}, \frac{2}{1-\epsilon+k}))}{\frac{(k+1)^2}{2}\lambda\left((\frac{2}{1+k}, \frac{2}{1-\epsilon+k})\right)} \\
&\geq \lim_{\epsilon\to 0} \frac{2p}{(k+1)^2} \inf_{x\in(\frac{2}{k+1}, \frac{2}{1-\epsilon+k})} h_p(x) \\
&= \frac{2p}{(k+1)^2} h_p\left(\frac{2}{k+1}\right) > 0.
\end{aligned}$$

The case $h_p(0) > 0$ follows in the same way, choosing $(0, \epsilon)$ as starting interval and taking $\lim_{x\downarrow 0}$. □

Corollary 4.4.5. *The measure μ_p is equivalent to the Lebesgue measure and there exists a $c \in \mathbb{R}$ such that for all $B \in \mathcal{B}$ we have $c\lambda(B) < \mu_p(B) < \frac{1}{c}\lambda(B)$.*

4.5 Ergodic properties

In this section will show the measure $m_p \times \mu_p$ is mixing with respect to K. We choose a suitable partition of $\Omega \times [0,1]$. Let P be a countable partition of $[0,1]$ with atoms D_m. We define the diameter of an atom D_m by

$$\text{diam} D_m = \max_{x,y \in D_m} |x-y|.$$

The diameter of a partition P is given by

$$\text{diam} P = \max_{n \in \mathbb{N}} D_m.$$

Theorem 4.5.1. *Let $\{P_n\}_n$ be a sequence of countable partitions, which is diameter reducing i.e. $\lim_{n\to\infty}$ diam$\{P_n\}_n = 0$, then the atoms $\{D_{n,m}\}$ of the partitions in the sequence generate the Borel-σ-algebra $\mathcal{B}$.*

Proof. Let $\mathcal{D}$ denote the collection of atoms $\{D_n, m\}$ of the sequence of partitions $\{P_n\}_n$. Since $\forall X \in \mathcal{D}$ we have $X \in \mathcal{B}$ it follows $\sigma(\mathcal{D}) \subset \mathcal{B}$. We know that the open intervals of the form (a,b), $a < b$ $a,b \in \mathbb{R}$ generate $\mathcal{B}$ and hence it is enough to show that $(a,b) \in \sigma(\mathcal{D})$ for all $a,b \in [0,1]$. We can write $(a,b) = \bigcup_{k\in\mathbb{N}}[a+\frac{1}{k}, b-\frac{1}{k}]$ and show that for all $a,b \in \mathbb{R}, k \in \mathbb{N}$ there exists at most countable $A_i \in D_{n,m}$ such that $[a+\frac{1}{k}, b-\frac{1}{k}] \subset \bigcup A_i \subset (a,b)$. $\lim_{n\to\infty} \text{diam}\{P_n\}_n = 0$ implies for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $\text{diam}\{P_N\} < \epsilon$. Let $\epsilon = \frac{1}{2k}$, so we find a $N \in \mathbb{N}$ such that $\{D_{N,m}\} < \frac{1}{2k}$ for all $m \in \mathbb{N}$. Take all elements A_i from $\{D_{N,m}\}$ such that $D_{N,m} \cap [a+\frac{1}{k}, b-\frac{1}{k}] \neq \emptyset$. This are at most countable many A_i, since $\{D_{N,m}\}$ is a countable set. Now $[a+\frac{1}{k}, b-\frac{1}{k}] \subset \bigcup A_i \subset (a,b)$ and $\bigcup A_i \in \sigma(\mathcal{D})$. Doing so for all $k \in \mathbb{N}$ we can write $(a,b) = \bigcup_{k\in\mathbb{N}} \bigcup A_{i,k}$, which is a countable union of elements in $\mathcal{D}$. We conclude $(a,b) \in \sigma(\mathcal{D})$. □

The function K is defined on $\{0,1\}^{\mathbb{N}} \times [0,1]$ endowed with $\sigma(\mathcal{C} \times \mathcal{B}[0,1])$, the product sigma algebra of the cylinders on $\{0,1\}^{\mathbb{N}}$, denoted $\mathcal{C}$ and the Borel-σ-algebra restricted to $[0,1]$, $\mathcal{B}[0,1]$. Consider the cylinders

$$\begin{aligned}
[\bar{\omega}]_n \times \Delta_n a =& [\bar{\omega_1}, \cdots, \bar{\omega_n}] \times \Delta(a_1, \cdots, a_n)_{\bar{\omega_1}, \cdots, \bar{\omega_n}} \\
=& \{(\omega, x) : \omega_1 = \bar{\omega_1}, \cdots, \omega_n = \bar{\omega_n}, d_1(\omega,x) = a_1, \cdots, d_n(\omega,x) = a_1\} \\
=& \{(\omega, x) : \omega_1 = \bar{\omega_1}, \cdots, \omega_n = \bar{\omega_n}, x \in \bigcap_{i=1}^{n} (T_{\omega_{i-1}} \circ \cdots \circ T_{\omega_1})^{-1}(\frac{2}{k_i+1}, \frac{2}{k_i}]\}.
\end{aligned}$$

Here were the k_i's in the last line are the k_i associated with a_i, i.e. if $a_i = k_i$ or $a_i = (k_i - 1, 1)$ we use in both cases the interval $\left(\frac{2}{k_i+1}, \frac{2}{k_i}\right]$. Clearly the cylinders $[\bar{\omega_1}, \cdots, \bar{\omega_n}]$ generate $\mathcal{C}$. For each $\omega \in \Omega$ we have that $\Delta(a_1, \cdots, a_n)_{\omega_1, \cdots, \omega_n}$ gives a partition $\{P_n\}_n$ such that $\lim_{n\to\infty} \text{diam}\{P_n\} = 0$, see section 4.2.1. Hence for each $\omega \in \Omega$, $\mathcal{B}[0,1]$ is generated in the second coordinate. Note that $|\{P_N\}| = |\mathbb{N}^N| = |\mathbb{N}|$. Hence we conclude that the cylinders generate $\sigma(\mathcal{C} \times \mathcal{B}[0,1])$.

Let $A \triangle B = A\backslash B \cup B \backslash A$ denote the symmetric difference. Recall that

$$\mu(A \triangle B) \leq \mu(A\backslash B) + \mu(B\backslash A),$$

$$A \triangle B = A^c \triangle B^c,$$

and that symmetric difference is associative i.e.

$$(A \triangle B) \triangle C = A \triangle (B \triangle C).$$

Hence $A \triangle C = (A \triangle B) \triangle (B \triangle C)$.

Lemma 4.5.2. *Let $\mathcal{A} = \{\bigcup_{i=1}^n [\bar{\omega}_{m(i)}] \times \Delta_{m(i)} a, n \in \mathbb{N}\}$, so $\mathcal{A}$ are all finite unions of cylinders. Then for each $\epsilon > 0$ and $X \in \sigma(\mathcal{C} \times \mathcal{B})$, we have that $\exists A \in \mathcal{A}$ such that $m_p \times \mu_p(X \triangle A) < \epsilon$.*

Proof. Let $\mathcal{D} = \{B \in \sigma(\mathcal{C} \times \mathcal{B})$ s.t. $\exists C \in \mathcal{A}, m_p \times \mu_p(B \triangle A) < \epsilon\}$.

First we show that $\Omega \times [0,1] \in \mathcal{D}$. We can write

$$\Omega \times [0,1] = \bigcup_{k \in \mathbb{N}, k > 2} [0] \times (\frac{2}{k+1}, \frac{2}{k}] \cup [1] \times (\frac{2}{k+1}, \frac{2}{k}].$$

Therefore $\Omega \times [0,1]$ is a disjoint union and

$$\begin{aligned} m_p \times \mu_p(X \triangle \bigcup_{k=1}^n [0] \times (\frac{2}{k+1}, \frac{2}{k}] \cup [1] \times (\frac{2}{k+1}, \frac{2}{k}]) \\ \leq m_p \times \mu_p(X \setminus \bigcup_{k=1}^n [0] \times (\frac{2}{k+1}, \frac{2}{k}] \cup [1] \times (\frac{2}{k+1}, \frac{2}{k}]) \\ = m_p \times \mu_p [0] \times [0, \frac{2}{n+1}] \cup [1] \times [0, \frac{2}{n+1}] \\ \leq C\lambda[0, \frac{2}{n+1}] \\ \leq \frac{2C}{n+1}, \end{aligned}$$

where we used that $\mu_p \leq C\lambda$. Choosing n large enough it follows that for all $\epsilon > 0$ we find an $B \in \mathcal{A}$ such that $m_p \times \mu((\Omega \times [0,1]) \triangle B) < \epsilon$. To prove that for each $(A_n)_n \subset \mathcal{D}$ there exists a $C \in \mathcal{D}$ such that $m_p \times \mu_p(\bigcup_n A_n \triangle C) < \epsilon$ we refer to Dajani [2014], p9. Finally suppose $A \in \mathcal{D}$, we show that $A^c \in \mathcal{D}$. $A^c \triangle B^c = A \triangle B$, hence we like to show that $B^c \in \mathcal{D}$. We know $B = \bigcup_{i=1}^n [\bar{\omega}]_{m(i)} \times \Delta_{m(i)} a$, for $n \in \mathbb{N}$. Therefore $B^c = \bigcap_{i=1}^n ([\bar{\omega}_{m(i)}] \times \Delta_{m(i)} a)^c$, so we can not immediately conclude that $B^c \in \mathcal{D}$. However the set of cylinders of length $n+1$ is a refinement of the cylinders of length n. So taking $N = \max\{m(i), 1 \leq i \leq n\}$, we can express the complement B^c in terms of $[\bar{\omega}]_N \times \Delta_N(a)$. Hence we get a union over all 2^N possible $[\bar{\omega_N}]$. For each $[\bar{\omega_N}]$ we have a union consisting of at most a countable number of elements of $\mathcal{A}$, which gives us just a countable union of elements of $\mathcal{A}$. Therefore we can find a $C \in \mathcal{A}$ such that $m_p \times \mu_p(C \triangle B^c) \leq \frac{\epsilon}{2}$. Using $A^c \triangle C = (A^c \triangle B^c) \triangle (B^c \triangle C)$ it follows that $m_p \times \mu_p(A \triangle C) \leq \epsilon$. Hence $\mathcal{D}$ is a σ-algebra containing all cylinders. We conclude $\mathcal{D} = \sigma(\mathcal{C} \times \mathcal{B})$. □

We will use the set $\mathcal{A}$ as defined in lemma 4.5.2 to show that K is mixing. Recall a dynamical system $(X, \mathcal{F}, \mu, T)$ is called mixing if for all $A, B \in \mathcal{F}$ $\lim_{n \to \infty} \mu(T^{-i} A \cup B) = \mu(A)\mu(B)$.

Proposition 4.5.3. *If it holds for all cylinders that*

$$\begin{aligned} \lim_{l \to \infty} (m_p \times \mu_p)(K^{-l}([w]_n \times \Delta_n(a)) \cap [v]_m \times \Delta_m(b)) \\ = (m_p \times \mu_p)([w]_n \times \Delta_n(a)(m_p \times \mu_p)([v]_m \times \Delta_m(b) \end{aligned}$$

then K is mixing.

Proof. By double induction we show that the statement holds for a finite union of cylinders. The case $(n,m) = (1,1)$ is already in the statement. Note that given the union of two cylinders A, B we have either the union is disjoint or $A \subset B$ or

$B \subset A$. Now suppose that the result holds for a unions of (n, m) cylinders $n \leq N$ and $m \leq M$ and let $\bigcup_{i=1}^{N+1} [w]_{n(i)} \times \Delta_{n(i)}(a_i)$ be a union of $N+1$ cylinders then we consider

$$\lim_{l \to \infty} m_p \times \mu_p[K^{-l}(\bigcup_{i=1}^{N+1} [w]_{i,n(i)} \times \Delta_{n(i)}(a_i)) \cap \bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}] - \tag{4.5.1}$$

$$m_p \times \mu_p(\bigcup_{i=1}^{N+1} [w]_{i,n(i)} \times \Delta_{n(i)}(a_i)) m_p \times \mu_p(\bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}). \tag{4.5.2}$$

$$\tag{4.5.3}$$

We assume that the $N+1$'th cylinder is the "most refined" cylinder, so

$$N + 1(i) = \max\{n(i), 1 \leq i \leq N+1\}.$$

If this is not the case we can just rewrite the union in this way. If

$$[w]_{N+1,n(N+1)} \times \Delta(a_{N+1}) \subset \bigcup_{i=1}^{N} [w]_{i,n(i)} \times \Delta(a_i)_{n(i)})$$

then the result follows from the induction hypothesis. Therefore suppose

$$[w]_{N+1,n(N+1)} \times \Delta(a_{N+1}) \not\subset \bigcup_{i=1}^{N+1} [\omega]_{i,n(i)} \times \Delta(a_i)_{n(i)}),$$

so we have a disjoint union. Therefore we write:

$$\lim_{n \to \infty} m_p \times \mu_p[K^{-n}(\bigcup_{i=1}^{N} [w]_{i,n(i)} \times \Delta(a_i)_{n(i)}) \cap \bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}] \tag{4.5.4}$$

$$+ m_p \times \mu_p(K^{-n}([w]_{N+1,n(N+1)}) \times \Delta(a_{N+1}) \cap \bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}) - \tag{4.5.5}$$

$$m_p \times \mu_p(\bigcup_{i=1}^{N} [w]_{i,n(i)} \times \Delta(a_i)_{n(i)}) m_p \times \mu_p(\bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}) - \tag{4.5.6}$$

$$m_p \times \mu_p([w]_{N+1,n(N+1)} \times \Delta(a_{N+1})) m_p \times \mu_p(\bigcup_{j=1}^{M} [v]_{j,m(j)} \times \Delta(b_j)_{m(j)}) = 0. \tag{4.5.7}$$

The last equality follows from the induction hypothesis. In the same way we obtain the result $(N+1, M+1)$ and we conclude that the result holds for all finite unions of cylinders. By a standard argument, see for example Dajani [2014], we can show that the result holds for arbitrary sets in our σ-algebra. □

So we are left to show that

$$\lim_{l \to \infty} (m_p \times \mu_p)(K^{-l}([w]_n \times \Delta_n(a)) \cap [v]_m \times \Delta_m(b))$$
$$= (m_p \times \mu_p)([w]_n \times \Delta_n(a)(m_p \times \mu_p)([v]_m \times \Delta_m(b),$$

for all cylinders. To do this we use the result of Aimino, Nicol, and Vaienti [2015] Recall $\mathcal{L}^\infty$ denote all almost everywhere bounded functions and the norm on L^∞ is defined by $||f||_\infty = \inf\{M : \mu(\{x : f(x > M)\}) = 0\}$. Further for a function of bounded variation f on $[0,1]$, $||f||_{BV} = ||f||_1 + \inf_{\{f_1 = f \ a.e.\}} \bigvee_{[0,1]} f_1$

Proposition 4.5.4. *(See Aimino et al. [2015] proposition 3.1) There exist constants $C \geq 0$ and $\rho < 1$ such that for all functions f of bounded variation and all $g \in L^\infty(\lambda)$,*

$$\lim_{n\to\infty} \left| \int_{[0,1]} P_{T^n} f \cdot g d\mu_p - \int_{[0,1]} f d\mu_p \int_{[0,1]} g d\mu_p \right| \leq C\rho^n ||f||_{BV} ||g||_\infty .$$

Here $P_T f$ is the random Perron-Frobenius operator as defined in section 3.2.3. Note that in our case we have by equation 3.2.8

$$P_T f = pP_{T_0} + (1-p)P_{T_1}.$$

Since $P_{T^n} = P_T^n$ we obtain So we can write:

$$P_{T^n} f = \sum_{(\omega_1,\cdots,\omega_n)\in\Omega^n} m_p([\omega_1,\cdots,\omega_n]) P_{T_{\omega_n}\circ\cdots\circ T_{\omega_n}} f.$$

Theorem 4.5.5. *The map K is mixing with respect to $m_p \times \mu_p$.*

Proof. By proposition 4.5.3 it is enough to show that for the cylinder sets it holds that

$$\lim_{l\to\infty} (m_p \times \mu_p)(K^{-l}([w]_n \times \Delta_n(a)) \cap [v]_m \times \Delta_m(b)) = \\ (m_p \times \mu_p)([w]_n \times \Delta_n(a)(m_p \times \mu_p)([v]_m \times \Delta_m(b)).$$

Let $\Omega_l = \{0,1\}^l$ and set $\Omega_{l-m} = \{0,1\}^{l-m}$, then

$$\begin{aligned}
&K^{-l}([w]_n \times \Delta_n(a)) \cap ([v]_m \times \Delta_m(b)) \\
&\quad = \{(\omega,x) : K^n(\omega,x) \in [w]_n \times \Delta_n(a)\} \cap ([v]_m \times \Delta_m(b)) \\
&\quad = \{(\omega,x) : (\sigma^n\omega, T_{\omega_n}\circ\cdots\circ T_{\omega_1}(x)) \in [w]_n \times \Delta_n(a)\} \cap ([v]_m \times \Delta_m(b)) \\
&\quad = \bigcup_{(\omega_1,\cdots,\omega_l)\in\Omega^l} (\{(\omega,x) : (\sigma^n\omega, T_{\omega_l}\circ\cdots\circ T_{\omega_1}(x)) \in [w]_n \times \Delta_n(a)\} \cap ([v]_m \times \Delta_m(b)) \\
&\quad = \bigcup_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} [v_1,\cdots,v_m,\omega_{m+1},\cdots,\omega_l,w_1,\cdots,w_n] \\
&\qquad\quad \times (T_{w_n}\circ\cdots\circ T_{\omega_{m+1}}\circ T_{v_m}\circ\cdots\circ T_{v_1})^{-1}\Delta_n(a) \cap \Delta_m(b).
\end{aligned}$$

Note that the unions are disjoint. Therefore:

$$\lim_{l\to\infty}(m_p\times\mu_p)(K^{-l}([w]_n\times\Delta_n(a))\cap[v]_m\times\Delta_m(b))$$
$$=\lim_{l\to\infty} m_p\times\mu_p(\bigcup_{(\omega_m,\cdots,\omega_l)\in\Omega^{l-m}}[v_1,\cdots,v_m,\omega_{m+1},\cdots,\omega_l,w_1,\cdots,w_n]$$
$$\times(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}}\circ T_{v_m}\circ\cdots\circ T_{v_1})^{-1}\Delta_n(a)\cap\Delta_m(b))$$
$$=\lim_{l\to\infty}\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[v_m]m_p[w_n]m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\mu_p((T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}}\circ T_{v_m}\circ\cdots\circ T_{v_1})^{-1}\Delta_n(a)\cap\Delta_m(b))$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]}\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}}\circ T_{v_m}\circ\cdots\circ T_{v_1})^{-1}\Delta_n(a)}(x)\mathbf{1}_{\Delta_m(b)}(x)d\mu_p$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]}\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}(T_{v_m}\circ\cdots\circ T_{v_1}(x))\mathbf{1}_{\Delta_m(b)}(x)d\mu_p.$$

We use that $\int_{[0,1]} P_T f d\mu_p = \int_{[0,1]} f d\mu_p$, see proposition 2.5.9 and subsequently that $P_T(f\circ T\cdot g) = f\cdot P_T g\mu_p - a.e.$ for $f\in\mathcal{L}^1$ and $g\in\mathcal{L}^\infty$, see proposition 2.5.10. Hence we can write:

$$\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]}\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}(T_{v_m}\circ\cdots\circ T_{v_1}(x))\mathbf{1}_{\Delta_m(b)}(x)d\mu_p$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]} P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}(T_{v_m}\circ\cdots\circ T_{v_1}(x))\mathbf{1}_{\Delta_m(b)}(x))d\mu_p$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]}\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}(x)\cdot P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x))d\mu_p.$$

Using the random Perron Frobenius operator we write:

$$\lim_{l\to\infty} m_p[v_m]m_p[w_n]\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\int_{[0,1]}\mathbf{1}_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}(x)\cdot P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x))d\mu_p$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\int_{(T_{\omega_n}\circ\cdots\circ T_{\omega_{m+1}})^{-1}\Delta_n(a)}\sum_{(\omega_{m+1},\cdots,\omega_l)\in\Omega^{l-m}} m_p[\omega_{m+1},\cdots,\omega_l]$$
$$\cdot P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x))d\mu_p$$
$$=\lim_{l\to\infty} m_p[v_m]m_p[w_n]\int_{\Delta_n(a)} P_{T^{l-m}}(P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x)))d\mu_p.$$

By 2.5.12 the Perron Frobenius operator sends the space of functions of bounded variation to itself. Since $\mathbf{1}_{\Delta b_n}$ is of bounded variation we have that $P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta b_n}(x))$ is of bounded variation. Therefore by proposition 4.5.4

$$\begin{aligned}
&\lim_{l\to\infty}(m_p\times\mu_p)(K^{-l}([w]_n\times\Delta_n(a))\cap[v]_m\times\Delta_m(b))-\\
&\quad(m_p\times\mu_p)(([w]_n\times\Delta_n(a))m_p\times\mu_p([v]_m\times\Delta_m(b))\\
&\quad=\lim_{l\to\infty}m_p[v_m]m_p[w_n]\int_{\Delta a_n}P_{T^{l-m}}(P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x)))d\mu_p\\
&\quad-m_p[w_n]m_p[v_n]\mu_p(\Delta_n(a))\mu_p(\Delta_m(b))\\
&\quad=\lim_{l\to\infty}m_p[v_m]m_p[w_n]\\
&\quad(\int_{\Delta_n(a)}P_{T^{l-m}}(P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x)))d\mu_p-\int_{[0,1]}\mathbf{1}_{\Delta_n(a)}d\mu_p\int_{[0,1]}\mathbf{1}_{\Delta_m(b)}d\mu_p)\\
&\quad=\lim_{l\to\infty}m_p[v_m]m_p[w_n]\\
&\quad(\int_{\Delta_n(a)}P_{T^{l-m}}(P_{T_{v_m}\circ\cdots\circ T_{v_1}}(\mathbf{1}_{\Delta_m(b)}(x)))d\mu_p-\int_{[0,1]}\mathbf{1}_{\Delta_m(b)}d\mu_p\int_{[0,1]}P_{T_{v_m}\circ\cdots\circ T_{v_1}}\mathbf{1}_{\Delta_n(a)}d\mu_p)\\
&=0.
\end{aligned}$$

It follows that K is mixing. □

Using 2.1.9 we can show that ρ is ergodic.

Proposition 4.5.6. *The measure ρ is ergodic with respect to the transformation R.*

Proof.

$$\begin{aligned}
R^{-1}(\Omega\times[0,1])=&\{(\omega,x):R(\omega,x)\in\Omega\times[0,1]\}\\
=&\{(\omega,x):(\omega,x)\in[0]\times[0,1]\cup\Omega\times[1,2]\}\\
=&[0]\times[0,1]\cup\Omega\times[1,2]
\end{aligned}$$

Hence $R^{-1}(\Omega\times[0,1])\cup\Omega\times[0,1]=\Omega\times[0,2]$ so indeed we have $\rho\left(\bigcup_{k\geq 0}R^{-k}(\Omega\times[0,1])\right)=1$ and by 2.1.9 we conclude that $R(\omega,x)$ is ergodic. □

4.6 Existence of $\lim_{n\to\infty}\frac{1}{n}\log q_n$

In this section we investigate the asymptotic behaviour of the expansions induced by R and K.

Proposition 4.6.1. $\lim_{n\to\infty}\frac{1}{n}\log q_n$ *exists for the function R.*

Proof. By the recurrence relations and 4.1.19 for any irrational $x\in[0,1)$ one has

$$\begin{aligned}
\frac{1}{q_n(\omega,x)}&=\frac{1}{q_n(\omega,x)}\frac{p_n(\omega,x))}{q_{n-1}(R(\omega,x))}\frac{p_{n-1}(R(\omega,x))}{q_{n-2}(R^2(\omega,x))}\cdots\frac{p_2(R^{n-2}(\omega,x))}{q_1(R^{n-1}(\omega,x)}\frac{p_1(R^{n-1}(\omega,x))}{q_0(R^n(\omega,x)}\cdot\left(\frac{1}{2}\right)^{n-1}\\
&=\frac{p_n(\omega,x)}{q_n(\omega,x)}\frac{p_{n-1}(R(\omega,x))}{q_{n-1}(R(\omega,x))}\cdots\frac{p_1(R^{n-1}(\omega,x))}{q_1(R^{n-1}(\omega,x))}\cdot\left(\frac{1}{2}\right)^{n-1}.
\end{aligned}$$

Taking logarithm yields

$$-\log q_n(\omega,x) = \log\frac{p_n(\omega,x)}{q_n(\omega,x)} + \log\frac{p_{n-1}(R(\omega,x))}{q_{n-1}(R(\omega,x))} + \cdots + \log\frac{p_1(R^{n-1}(\omega,x))}{q_1(R^{n-1}(\omega,x))} + (n-1)\log\frac{1}{2}.$$

We know that $\frac{p_n(\omega,x)}{q_n(\omega,x)}$ is a number close to x and therefore we can write:

$$-\log q_n(\omega,x) = \log x + \log(\pi_2 R(\omega,x)) + \cdots + \log(\pi_2(R^{n-1}(\omega,x)) + r(n,\omega,x),$$

where π_2 is the projection on the second coordinate and $r(n,\omega,x)$ is the rest term:

$$r(n,\omega,x) = \log\frac{p_n(\omega,x)}{q_n(\omega,x)} - \log x + \log\frac{p_{n-1}(R(\omega,x))}{q_{n-1}(R(\omega,x))} - \log(\pi_2(R(\omega,x))) + \cdots + \log\frac{p_1(R^{n-1}(\omega,x))}{q_1(R^{n-1}(\omega,x))} - \log(\pi_2(R^{n-1}(\omega,x))) + (n-1)\log\frac{1}{2}.$$

In case n is even we can find by the mean value theorem we a $\xi \in (x, \frac{p_n}{q_n})$ such that:

$$0 < \left(\log x - \log\frac{p_n(\omega,x)}{q_n(\omega,x)}\right) = \left(x - \frac{p_n(\omega,x)}{q_n(\omega,x)}\right)\frac{1}{\xi} \le \frac{2^n}{q_n^2}\frac{q_n}{p_n} = \frac{2^n}{p_n}\frac{1}{q_n} \le \frac{2}{q_n}.$$

In the case n is odd we obtain in the same way:

$$0 > \left(\log x - \log\frac{p_n(\omega,x)}{q_n(\omega,x)}\right) = \left(x - \frac{p_n(\omega,x)}{q_n(\omega,x)}\right)\frac{1}{\xi} \ge -\frac{2^n}{q_n^2}\frac{q_n}{p_n} \ge -\frac{2}{q_n}.$$

Let $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \cdots$ denote the Fibonacci sequence. From the recursion relations it follows that $\frac{1}{q_n} \le \frac{1}{\mathbf{F_n}}$. Since $\sum_{i=1}^{\infty}\frac{1}{\mathbf{F_i}} = \mathrm{C} < \infty$ we can estimate the rest term $r(n,\omega,x)$ by

$$|r(n,\omega,x)| \le 2\sum_{i=1}^{n}\frac{1}{\mathbf{F_i}} + (n-1)\log\frac{1}{2} \le \mathrm{C} + (n-1)\log\frac{1}{2}.$$

Hence for each x such that

$$\lim_{n\to\infty}\frac{1}{n}\left(\log x + \log(\pi_2(R(\omega,x))) + \log(\pi_2(R^2(\omega,x))) + \cdots + \log(\pi_2(R^{n-1}(\omega,x)))\right)$$

exists, we have that $-\lim_{n\to\infty}\frac{q_n(\omega,x)}{n}$ exists and equals the above limit up to a term $\log\frac{1}{2}$. If we can show that $\int_{\Omega\times[0,2]}\log(\pi_2(\omega,x))d\rho < \infty$, then we can use the Birkhoff ergodic theorem 2.1.2 to obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log(\pi_2(R^i(\omega,x))) = \int_{\Omega\times[0,2]}\log(\pi_2(\omega,x))d\rho.$$

Note that $\log(\pi_2(\omega, x)) = \log x$ and compute the integral:

$$\begin{aligned}
\int_{\Omega\times[0,2]} \log(\pi_2(\omega,x))d\rho &= \int_{\Omega\times[0,1]} \log(\pi_2(\omega,x))dm_p \times \mu_p + \\
&\quad \int_{[1]\times[0,1]} \log(\pi_2(R(\omega,x)))dm_p \times \mu_p \\
&= \int_{[0,1]}\int_{\Omega} \log(\pi_2(\omega,x))dm_p d\mu_p + \\
&\quad \int_{[0,1]}\int_{[1]} \log(\pi_2(R(\omega,x)))dm_p d\mu_p \\
&= \int_{[0,1]} \log(x)d\mu + p\int_{[0,1]} \log(S_1 x)d\mu_p \\
&= \int_{[0,1]} \log(x)d\mu + p\int_{[0,1]} \log(\frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor + 1)d\mu_p.
\end{aligned}$$

We know μ is equivalent to the Lebesgue measure, so we can estimate the integral. For the first term we get:

$$-c = c\int_{[0,1]} \log(x)d\lambda > \int_{[0,1]} \log(x)d\mu > C\int_{[0,1]} \log(x)d\lambda = -C. \tag{4.6.1}$$

For the second term note $1 \le \frac{2}{x} - \lfloor \frac{2}{x} \rfloor + 1 < 2$. Therefore we can write

$$0 < \int_{[0,1]} p\log(\frac{2}{x} - \left\lfloor \frac{2}{x} \right\rfloor + 1)d\mu < C\int_{[0,1]} p\log(2)d\lambda. \tag{4.6.2}$$

We conclude by the sqeeze-theorem that the sum converges and

$$\lim_{n\to\infty} \frac{\log q_n}{n} < \infty.$$

$\square$

Using proposition 4.6.1 we can prove that the $\lim_{n\to\infty} \frac{\log q_n}{n}$ for the transformation $K(\omega, x)$.

Proposition 4.6.2. $\lim_{n\to\infty} \frac{1}{n} \log q_n$ *exists for the transformation* $K(\omega, x)$.

Proof. Define $\tilde{n} = \sum_{i=1}^n \tau(K^i(\omega, x))$ see in section 4.2.1. Then

$$q_n(K(\omega,x)) = q_{\tilde{n}}(R(\omega,x)).$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log q_n(K(\omega,x))}{n} &= \lim_{n\to\infty} \frac{\log q_{\sum_{i=1}^n \tau(K^i(\omega,x))}(R(\omega,x))}{n} &\quad (4.6.3)\\
&= \lim_{n\to\infty} \frac{\log q_{\sum_{i=1}^n \tau(K^i(\omega,x))}}{\sum_{i=1}^n \tau(K^i(\omega,x))} \frac{\sum_{i=1}^n \tau(K^i(\omega,x))}{n}. &\quad (4.6.4)
\end{aligned}$$

We know by proposition 4.6.1 that $\lim_{n\to\infty} \frac{\log q_{\sum_{i=1}^n \tau(K^i(\omega,x))}}{\sum_{i=1}^n \tau(K^i(\omega,x))}$ exists. The existence of $\lim_{n\to\infty} \frac{\sum_{i=1}^n \tau(K^i(\omega,x))}{n}$ is shown by the Birkhoff ergodic theorem 2.1.2 since τ is an integrable function, so

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \tau(K^i(\omega,x))}{n} = \int_{\Omega\times[0,1]} \tau(\omega,x)dm_p \times \mu_p = (2-p). \tag{4.6.5}$$

Hence we conclude that $\lim_{n\to\infty} \frac{1}{n} \log q_n$ exists and is smaller than ∞. $\square$

Let $I_{\mathcal{C}}$ be the information function, as defined in 2.1.13, with respect to the cylinder $\mathcal{C}$ as defined in section 4.5. Then $I_{\mathcal{C}}$ assigns to (ω, x) the log of the measure of the cylinder to which (ω, x) belongs. Hence we can write

$$I_{\mathcal{C}} = \log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n)(\omega, x)),$$

where $[\omega]_n \times \Delta_n(a_n)(\omega, x)$ denotes the cylinder set to which (ω, x) belongs. Now we are able to calculate the $\lim_{n\to\infty} \frac{1}{n} \log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n))(\omega, x))$.

Proposition 4.6.3. $\lim_{n\to\infty} \frac{1}{n} \log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n)))$ *exists and is finite.*

Proof.

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log(m_p \times \mu_p([\omega]_n \times \Delta_n(a_n))) &= \lim_{n\to\infty} \frac{1}{n} \log(m_p([\omega]_n)(\omega) \cdot \mu_p(\Delta a_n)(x)) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(m_p([\omega]_n(\omega))) + \lim_{n\to\infty} \frac{1}{n} \log(\mu_p(\Delta a_n)(x)).\end{aligned}$$

Calculating the first limit we obtain the following:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log(m_p([\omega]_n(\omega))) &= \lim_{n\to\infty} \frac{1}{n} \log(p^{\sum_{i=1}^n \mathbf{1}_{\{\omega_\mathbf{i}=\mathbf{0}\}}(\omega)}(1-p)^{n-\sum_{i=1}^n \mathbf{1}_{\{\omega_\mathbf{i}=\mathbf{0}\}}(\omega)}) \\ &= \lim_{n\to\infty} \frac{1}{n} \log(\prod_{i=1}^n p^{\mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega))}(1-p)^{\mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega))}) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \log(p^{\mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega))}) + \log((1-p)^{\mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega))}) \\ &= \int_\Omega \mathbf{1}_{\{\omega_1=0\}}(\omega)\log(p) + \mathbf{1}_{\{\omega_1=1\}}(\omega)\log((1-p))dm_p \\ &= p\log p + (1-p)\log(1-p).\end{aligned}$$

The last step follows by applying the Birkhoff Ergodic theorem, which can be done since $\sigma(\omega)$, the left shift is ergodic with respect to m_p.

For the second limit we use $c\lambda(\Delta a_n) < \mu_p(\Delta a_n) < C\lambda(\Delta a_n)$, and therefore

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu_p(\Delta a_n)(x)) = \lim_{n\to\infty} \frac{1}{n} \log(\lambda(\Delta a_n)) = \lim_{n\to\infty} \frac{1}{n} \log(\frac{2^{\tilde{n}}}{q_n(q_n + q_{n-1})}).$$

So we can write:

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{2^{\tilde{n}}}{2q_n^2}\right) < \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{2^{\tilde{n}}}{q_n(q_n+q_{n-1})}\right) < \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{2^{\tilde{n}}}{q_n^2}\right) \\ &\lim_{n\to\infty} \frac{1}{n}(\log(2^{\tilde{n}-1}) - \log\left(q_n^2\right)) < \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{2^{\tilde{n}}}{q_n(q_n+q_{n-1})}\right) < \\ &\lim_{n\to\infty} \frac{1}{n}(\log(2^{\tilde{n}}) - \log\left(q_n^2\right)).\end{aligned}$$

Note that by the Birkhoff ergodic theorem we get:

$$\lim_{n\to\infty} \frac{1}{n} \log(2^{\tilde{n}}) = \lim_{n\to\infty} \frac{\tilde{n}}{n} \log(2) = \lim_{n\to\infty} \frac{\sum_{i=1}^n \tau(K^i(\omega, x))}{n} \log(2) = (2-p)\log 2.$$

We conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log(\mu_p(\Delta a_n)(x) = (2-p)\log 2 + \lim_{n\to\infty} \frac{1}{n+1} 2\log(q_n) = 2\alpha < \infty,$$

where we used proposition 4.6.2 in the last step.

□

4.7 Entropy

In order to calculate the entropy we will use the Shannon-McMillan-Breiman Theorem, see section 2.1.4. If α is generating partition we can use proposition 2.1.14 to strengthen the conclusion to $h(\alpha)$ a.e.. We prove the following theorem.

Theorem 4.7.1. *The function $K(\omega, x)$ has finite entropy.*

Proof. Let $\alpha = \{[\omega_i]_1 \times \Delta_1(a_i), \omega_i \in \{0,1\}, a_i \in \{k_i, (k_i - 1, 1) : k_i \in \mathbb{N}\}\}$. Denote an atom of α, so a cylinder of length 1 as defined in chapter 4.5, by $[\omega]_{1,i} \times \Delta_1(a_i)$. First we note that the cylinders of the form $[\omega]_1 \times \Delta_n(a_n)$ form a generating partition. Denoting by k_n the first coördinate of a_i we can write:

$$
\begin{aligned}
&[\omega_{i_0}]_1 \times \Delta_1(a_{i_0}) \cap K^{-1}([\omega_{i_1}]_1 \times \Delta_1(a_{i_1})) \cap \cdots \cap K^{-n}([\omega_{i_n}]_1 \times \Delta_1(a_{i_n})) \\
&\qquad = \{(\omega, x) \in \Omega \times [0,1] : \\
&\qquad\qquad \omega_1 = \omega_{i_0}, x \in (\frac{2}{k_{i_0}}, \frac{2}{k_{i_0}+1}], \omega_2 = \omega_{i_1}, T_{\omega_0} x \in (\frac{2}{k_{i_1}+1}, \frac{2}{k_{i_1}}], \\
&\qquad\qquad \cdots \omega_{n+1} \omega_{i_n}, T_{\omega_{i_{n-1}}} \circ \cdots \circ T_{\omega_{i_1}}(x) \in (\frac{2}{k_{i_n}+1}, \frac{2}{k_{i_n}}]\} \\
&\qquad = [\omega]_n \times \Delta_n(a_n).
\end{aligned}
$$

In section 4.5 we proved that cylinders of the form $[\omega]_n \times \Delta_n(a_n)$ generate the $\sigma(\mathcal{C} \times [0,1])$, α is a generating partition. To apply the Schannon-McMillan-Breiman theorem we have to check that $H(\alpha) < \infty$:

$$
\begin{aligned}
H(\alpha) &= -\sum_{[\omega]_1 \times \Delta_1(a), \omega \in \{0,1\}, a \in \mathbb{N}} m_p \times \mu([\omega]_1 \times \Delta_1(a)) \log(m_p \times \mu([\omega]_1 \times \Delta_1(a))) \\
&= -\sum_{[\omega]_1 \times \Delta_1(a), \omega \in \{0,1\}, a \in \mathbb{N}} m_p([\omega]_1)\mu_p(\Delta_1(a)) \log(m_p([\omega]_1)\mu_p(\Delta_1(a))) \\
&= -p\log p - (1-p)\log(1-p) - \sum_{a \in \mathbb{N}} \mu_p(\Delta a) \log(\mu_p(\Delta a)) \\
&< \infty.
\end{aligned}
$$

The convergence of the sum can be seen as follows. First note that

$$
\begin{gathered}
c\lambda(\Delta a) \log \mu_p(\Delta a) < \mu_p(\Delta a) \log \mu_p(\Delta a) < C\lambda(\Delta a) \log \mu_p(\Delta a) \\
c\lambda(\Delta a) \log(c\lambda(\Delta a)) < \mu_p(\Delta a) \log \mu_p(\Delta a) < C\lambda(\Delta a) \log(C\lambda(\Delta a))
\end{gathered}
$$

and

$$
\begin{aligned}
\sum_{k=1}^{n} C\lambda(\Delta a) \log(C\lambda(\Delta a)) &= \sum_{k=1}^{n} C(\frac{2}{k(k+1)}) \log(\frac{2C}{k(k+1)}) \\
&= 2C \sum_{k=1}^{n} \frac{1}{k(k+1)} (\log(2C) - \log(k) - \log(k+1)).
\end{aligned}
$$

The first term converges since $\sum_{k \in \mathbb{N}} \frac{1}{k^2}$ converges. The convergence of the second term can be seen as follows: $\int_1^\infty \frac{\log(x)}{x(x+1)} d\lambda$ is finite, $\lim\limits_{k \to \infty} \frac{\log(k)}{k(k+1)} = 0$ by l'Hopital and $\frac{\log(k)}{k(k+1)} > 0 \; \forall k \in \mathbb{N}$. Moreover computing the derivative:

$$
\frac{d}{dx} \frac{\log x}{x(x+1)} = \frac{(x+1) - (2x+1)\log(x)}{(x(x+1))^2}
$$

we see that $\frac{\log(k)}{k(k+1)}$ is decreasing from a certain point onwards. Therefore from a certain k on we can estimate the sum by its integral and hence it converges. The same strategy could be used to show convergence for the third term. We conclude that α has finite entropy and therefore we can apply the Shannon-McMillan-Breiman Theorem. We calculate $\lim_{n\to\infty} \frac{1}{n+1} I_{\bigvee_{i=0}^{n} T^{-i}\alpha}(x)$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n+1} I_{\bigvee_{i=0}^{n} T^{-i}\alpha}(x) &= \lim_{n\to\infty} \frac{-1}{n+1} \log(m_p \times \mu_p(\bigvee_{i=0}^{n} T^{-i}\alpha(\omega, x))) \\
&= \lim_{n\to\infty} \frac{-1}{n+1} \log(m_p([\omega]_n)(\omega) \cdot \mu_p(\Delta a_n)(x)) \\
&= \lim_{n\to\infty} \frac{-1}{n+1} \log(m_p([\omega]_n(\omega))) \\
&\quad + \lim_{n\to\infty} \frac{-1}{n+1} \log(\mu_p(\Delta a_n)(x)) \\
&< \infty.
\end{aligned}$$

Where the convergence follows from proposition 4.6.3. Therefore we conclude that $h(K(\omega, x)) = \beta < \infty$, for some $\beta \in \mathbb{R}$. □

Now we know the entropy for K we can find the entropy of R.

Proposition 4.7.2. *The transformation R has finite entropy.*

Proof. This is a direct consequence of theorem 2.1.17. □

4.8 Convergence of the digits

In this section we will proof some properties of the digits b_i induced by the function $R(\omega, x)$.

Proposition 4.8.1. *Let b_i be the digits induced by R as defined in 4.1.1, then for $\rho - a.e.(\omega, x) \in \Omega \times [0, 1]$, we have*

$$1 < \lim_{n\to\infty} (b_1(x, \omega), \cdots, b_n(x, \omega))^{\frac{1}{n}} < \infty$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} b_i(x, \omega)}{n} = \infty.$$

Proof. We start with the first inequality.
If we can show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(b_i) < \infty,$$

then also

$$\lim_{n\to\infty} e^{\frac{1}{n}\sum_{i=1}^{n} \log(b_i)} = (b_1 \cdots b_n)^{\frac{1}{n}} < \infty.$$

By the Birkhoff ergodic theorem we obtain.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(b_i) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log(b_1(R^{i-1}(\omega, x))) = \int_{\Omega\times[0,1]} \log(b_1(\omega, x)) d\rho.$$

Calculating the integral gives:

$$\begin{aligned}\int_{\Omega\times[0,2]}\log(b_1(\omega,x))d\rho &= \int_{\Omega\times[0,1]}\log(b_1(\omega,x))dm_p\times\mu_p + \int_{[1]\times[0,1]}\log(b_1(R(\omega,x)))dm_p\times\mu_p\\ &= \int_0^1 p\log(b(0\omega,x)) + (1-p)\log(b(1\omega,x))d\mu_p + \int_{[1]\times[0,1]}\log(1)dm_p\times\mu_p\\ &= \int_0^1\sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]} + (1-p)\log(k-1)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]}d\mu_p.\end{aligned}$$

We estimate the integral by:

$$\begin{aligned}&c\int_0^1\sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]} + (1-p)\log(k-1)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]}d\lambda\\ &<\int_0^1\sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]} + (1-p)\log(k-1)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]}d\mu_p\\ &< C\int_0^1\sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]} + (1-p)\log(k-1)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]}d\lambda.\end{aligned}$$

By monotone convergence we can switch the integral and sum and we obtain:

$$\begin{aligned}&\int_0^1\sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]} + (1-p)\log(k-1)\mathbf{1}_{(\frac{2}{k+1},\frac{2}{k}]}d\lambda\\ &\qquad = \sum_{k\geq 2,k\in\mathbb{N}} p\log(k)\frac{2}{k(k+1)} + (1-p)\log(k-1)\frac{2}{k(k+1)} < \infty.\end{aligned}$$

Where convergence of the sum is in the same way as in theorem 4.7.1.

We continue with $\lim_{n\to\infty}\frac{\sum_{i=1}^n b_i(x,\omega)}{n} = \infty$ which we also can write as $\lim_{n\to\infty}\frac{\sum_{i=1}^n b_1(K^i(x,\omega))}{n}$. Let $b_N(\omega,x) = b_1(\omega,x)\mathbf{1}_{(\frac{2}{N+1},1]}$. Then b_N is bounded and therefore it is in $\mathbf{L_1}$, so integrable. We apply the Birkhoff ergodic theorem:

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum_{i=1}^n b_N(R^i(x,\omega))}{n} &= \int_{\Omega\times[0,1]} b_N(\omega,x)d\rho\\ &= \int_{\Omega\times[0,1]} b_N(\omega,x)dm_p\times\mu_p + \int_{[1]\times[0,1]} b_N(R(\omega,x))dm_p\times\mu_p\\ &= \int_{\Omega\times[0,1]} b_N(\omega,x)dm_p\times\mu_p + \int_{[1]\times[0,1]}\mathbf{1}_{(\frac{2}{N+1},1]}dm_p\times\mu_p.\end{aligned}$$

Note that $b_N(\omega,x)\geq\frac{2}{x}-2>0$ for $(\omega,x)\in\Omega\times(\frac{2}{N+1},1]$, therefore:

$$\begin{aligned}\int_{\Omega\times[0,1]} b_N(\omega,x)dm_p\times d\mu_p &> \int_{\Omega\times[0,1]}(\frac{2}{x}-2)\mathbf{1}_{(\frac{2}{N+1},1]}dm_p\times d\mu_p\\ &> c\int_{[0,1]}(\frac{2}{x}-2)\mathbf{1}_{(\frac{2}{N+1},1]}d\lambda\\ &= [2\log x - 2x]^1_{\frac{2}{N+1}}\\ &= -2-2\log(\frac{2}{N+1})-\frac{4}{N+1}.\end{aligned}$$

Since $\lim_{N\to\infty} b_N(\omega, x) = b_1(\omega, x)$ and b_N is an increasing sequence of functions we can write:

$$\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} b_i(x,\omega)}{n} \geq \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} b_N(R^i(x,\omega))}{n}. \tag{4.8.1}$$

The above result holds true for all N and hence:

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} b_i(x,\omega)}{n} &\geq \sup_N \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} b_N(R^i(x,\omega))}{n} \\
&= \lim_{N\to\infty} \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} b_N(R^i(x,\omega))}{n} \\
&= \lim_{N\to\infty} \int_{\Omega\times[0,1]} b_N(\omega,x)\, dm_p \times d\mu_p \\
&= \lim_{N\to\infty} -2 - 2\log(\frac{2}{N+1}) - \frac{4}{N+1} = \infty.
\end{aligned}$$

□

What does this proposition tells us about the digits induced by the transformation K? The digits of K are almost the same as the digits of R. The only difference is that for K we take *"two digits together"* if we use the upper transformation, so we put brackets. Therefore we could not formulate a proposition as we did for R since $1 < \lim_{n\to\infty}(a_1(x,\omega), \cdots, a_n(x,\omega))^{\frac{1}{n}} < \infty$ and $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i(x,\omega)}{n} = \infty$ would not make sense if some a_i are of the form $(k-1, 1)$ and some of the form k. However the expansion K induces is exactly the expansion R induces. Hence looking at expansion induced by K, we find an exactly equal proposition.

Chapter 5

More invariant measures

In chapter 4 we obtained results for the 2-continued fraction transformation. A lot of these results where only existence results, since we do not know the explicit form of the invariant measure. Moreover the results concerning digits and entropy were $m_p \times \mu_p$ almost everywhere. So we do not know anything about the behaviour of the 2-random continued fraction transformation on the $m_p \times \mu_p$ null-sets. Therefore it is interesting to look for other invariant measures for the dynamical system $(\Omega \times [0,1], \sigma(\mathcal{C} \times \mathcal{B}[0,1]), T)$. We do this in the next section by constructing a commuting diagram between our space $\Omega \times [0,2]$ and $\mathbb{N}^{\mathbb{N}}$, the space where the digits sequences induced by R live. Constructing invariant measures on $\mathbb{N}^{\mathbb{N}}$ will give us an invariant measure on $\Omega \times [0,2]$.

5.1 Commuting diagram

In this section we want to approach the function R by the digit sequences that it induces. Therefore we would like to find an isomorphism such that

$$\psi : \Omega \times [0,2] \to (\mathbb{N} \cup \{\infty\})^{\mathbb{N}} \tag{5.1.1}$$

$$\psi(\omega, x) = (b_1(\omega, x), b_2(\omega, x), \cdots). \tag{5.1.2}$$

Notice that for some values in $\Omega \times [0,2]$, for example $((0,0,\cdots), \frac{2}{k})$ we get only a finite digit sequence and $\psi((0,0,\cdots), \frac{2}{k}) \notin (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. Therefore we will use the following subsets of full measure. Let

$$M \subset \Omega \times [0,2],\ M = \{(\omega, x) : \pi_2(R^n(\omega, x)) \neq 0\ \forall n \in \mathbb{N}\}$$

and

$$N \subset (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}, N = \mathbb{N}^{\mathbb{N}}.$$

We will show that ϕ is indeed an isomorphism.

In order to do this we like to make a commuting diagram:

$$\begin{tikzcd} M \arrow[r, "R"] \arrow[d, "\psi"] & M \arrow[d, "\psi"] \\ N \arrow[r, "\sigma"] & N \end{tikzcd}$$

Here σ denotes the left shift. We show that ψ is indeed a bijection. To do this, we first prove the following proposition:

Proposition 5.1.1. *Let $x \in M$ and let $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ with $b_i \in \mathbb{N}$. Then there exists an $\omega \in \Omega$ such that $b_i = d_i(\omega, x)$.*

To prove the above proposition we will first prove a helpful lemma.

Lemma 5.1.2. *For all $k \in \mathbb{N}$ let I_k denote the interval $(\frac{2}{k+1}, \frac{2}{k}]$. Then:*

1. *If $x \in I_1$ we have $b_1 = 1$.*
2. *If $x \in I_k$ for $k \geq 2$ we have $b_1 \in \{k-1, k\}$.*

Proof. Since $b_1 \in \mathbb{N}$ we have that if $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ then, $0 < x < \frac{2}{b_1} \leq 2$, so $x \in (0, 2)$. For case 1 suppose $x \in I_1 = (1, 2]$ and $b_1 > 1$. Then it follows that $x \leq \cfrac{2}{2 + \cfrac{2}{b_2 + \ddots}} < \frac{2}{2} = 1$, which is a contradiction. We conclude that $b_1 = 1$. For case 2 we assume first that $x \in I_k$ and $b_1 < k-1$. Then it follows that $x \geq \cfrac{2}{k - 2 + \cfrac{2}{b_2 + \ddots}} > \cfrac{2}{k-2+2} = \cfrac{2}{k}$. So $x \notin I_k$ and hence $b_1 \geq k-1$. Now suppose $b_1 > k$, then $x \leq \cfrac{2}{k + 1 + \cfrac{2}{b_2 + \ddots}} < \cfrac{2}{k+1}$, so $x \notin I_k$. We conclude $b_1 \leq k$ and therefore $b_1 \in \{k-1, k\}$.

□

Now we will prove proposition 5.1.1.

Proof. Denote by $x_n = \cfrac{2}{b_n + \cfrac{2}{b_{n+1} + \ddots}}$. Let $l_n(x)$ be a variable which counts the number of times $x_i \in [0, 1]$ for $1 \leq i \leq n$. We will show by induction that for each $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$ we can find $\omega \in \Omega$ such that $d_i(\omega, x) = b_i$ for all $i \in \mathbb{N}$. For the base step, note the following:

1. If $x \in I_1 = (1, 2]$ then by lemma 5.1.2 we have $b_1 = d_1(x, \omega) = 1$ for all $\omega \in \Omega$. Since $x \in (1, 2]$, $l_1(x) = 0$ and $[\omega]_{l_1} = \Omega$.
2. If $x \in I_k$, for $k \geq 2$ then we have by lemma 5.1.2 that $b_1 \in \{k-1, k\}$.
   - If $b_1 = k$, then we have $d_1(\omega, x) = k$ for all $\omega \in [0]$, so we set $\omega_1 = 0$, $l_1(x) = 1$, since $x \in [0, 1]$ and $[\omega]_{l_1} = [0]$.

- If $b_1 = k-1$, then for all $\omega \in [1]$ we have $d_1(\omega, x) = k-1$, hence we set $\omega_1 = 1$. Again $l_1(x) = 1$, since $x \in [0,1]$, but $[\omega]_{l_1} = [1]$.

Therefore we have found an cylinder $[\omega]_{l_1}$ such that $\forall \omega \in [\omega]_{l_1}$, $d_1(\omega, x) = b_1$, where the cylinder $[\omega]_0 = \Omega$. Suppose we have found a cylinder $[\omega]_{l_n}$ such that $\forall \omega \in [\omega]_{l_n}$ we have

$$(d_1(\omega, x), d_2(\omega, x), \cdots, d_n(\omega, x)) = (b_1, b_2, \cdots, b_n).$$

Consider x_{n+1} and note that b_{n+1} is b_1 for x_{n+1}. If $x_{n+1} \in [0,1]$ we find by the above procedure an cylinder $[\omega]_{l_{n+1}}$, such that $[\omega]_{l_{n+1}} \subset [\omega]_{l_n}$ and

$$(d_1(\omega, x), d_2(\omega, x), \cdots, d_n(\omega, x), d_{n+1}(\omega, x)) = (b_1, b_2, \cdots, b_n, b_{n+1}).$$

If $x_{n+1} \in (1,2]$, then $l_{n+1} = l_n$ so $[\omega]_{l_n} = [\omega]_{l_{n+1}}$ and we do not refine the cylinder. Notice each time $x_n \in (1,2]$ we know that $x_{n+1} \in [0,1]$, so $l_n \geq \frac{n}{2}$. Therefore if $n \to \infty$, then $l_n \to \infty$ and $[\omega]_{l_{n+1}} \subset [\omega]_{l_n}$. Finally we have that $\bigcap_n [\omega]_{l_n} = \{\omega\}$, for some $\omega \in \Omega$. This concludes the lemma. □

Remark 5.1.3. The proof of proposition 5.1.1 shows that for any continued fraction expansion $(b_1, b_2, \cdots, b_n)$ of x there exists an unique $\omega \in \Omega$, such that

$$(d_1(\omega, x), d_2(\omega, x), \cdots) = (b_1, b_2, \cdots).$$

◊

Now we are able to show that $\psi : (\Omega \times [0,2], \sigma(\Omega \times [0,2]), \mu, R) \to (\mathbb{N}^{\mathbb{N}}, \mathcal{C}, \nu, \sigma)$ is indeed an isomorphism. Recall from section 4.5 that $\sigma(\Omega \times [0,2])$ is the product σ-algebra generated by cylinder sets of the form $[\omega]_n \times \Delta a_n$, which where defined in terms of the digits induced by the transformation K. The same kind of cylinder sets we define for the transformation R. We start with the partition:

$$\mathcal{P} = \{\Omega \times I_1, [0] \times I_k, [1] \times I_k, \ k \in \mathbb{N}\},$$

which we call the time-0-partition. Let

$$\mathcal{P}_n = \mathcal{P} \vee R^{-1}\mathcal{P} \vee \cdots \vee R^{-(n-1)}\mathcal{P}$$

be the time-n-partition, an element of $C \in \mathcal{P}_n$ is then of the form

$$C = A_1 \vee R^{-1}A_2 \vee \cdots \vee R^{-n-1}A_n$$

for $A_i \in \mathcal{P}$. For each $(\omega, x) \in C$, the value $l_n(\omega, x) = \sum_{i=0}^{n-1} \mathbf{1}_{(\Omega \times [0,1])}(R^i(\omega, x))$ is the same, as well as $(\omega_1, \omega_2, \cdots, \omega_{l_n})$ and the first n digits in the expansion. The elements of $\mathcal{P}$ are the cylinders of length 1 and the elements of $\mathcal{P}_n$ are the cylinders of length n.

In contrast to the cylinders we have defined for the transformation K, we do not work here with digits b_i. This is because for each digit induced by the function K we know precisely which ω is used. For the function R we do not know this. For example a digit $b_1 = k, \ k \geq 2$ could be induced by R if $(\omega, x) \in [0] \times (\frac{2}{k+1}, \frac{2}{k}]$ or if $(\omega, x) \in [1] \times (\frac{2}{k+2}, \frac{2}{k+1}]$. Clearly the cylinder sets for R induce for each ω a partition on $[0,2]$ and the size of an atom of this partition goes to 0 if $n \to \infty$, therefore they generate $\sigma(\mathcal{C} \times \mathcal{B})$.

We want to investigate R by its digits sequences. In order to do this we define $\mu = \nu \circ \psi$, where ν is a product measure on the cylinder-σ-algebra in $\mathbb{N}^{\mathbb{N}}$. Later in this section we will treat different measures ν.

Proposition 5.1.4. *Let M be the subset defined in the begin of this section. The function*

$$\psi : (M, \sigma(\Omega \times [0,2]), \mu, R) \to (\mathbb{N}^{\mathbb{N}}, \mathcal{C}, \nu, \sigma) \tag{5.1.3}$$
$$\psi(\omega, x) = (d_1(\omega, x), d_2(\omega, x), \cdots) \tag{5.1.4}$$

is an isomorphism.

Proof. To show that ψ is an isomorphism we have to prove the following properties:

1. ψ is one-to-one and onto a.e.,
2. ψ and ψ^{-1} are measurable,
3. ψ preserves the measures,
4. ψ preserves the dynamics of T and S.

Recall $M \subset \Omega \times [0,2]$, $M = \{(\omega, x) : R^n(\omega, x) \neq 0, \forall n \in \mathbb{N}\}$ and $N \subset (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$, $N = \mathbb{N}^{\mathbb{N}}$. First we show that ψ is one-to-one and onto from M to N. For all measures we treat later on, we shall show that $\nu(N) = \mu(M) = 1$. By proposition 5.1.1 we have that ψ is onto. Now we construct $\psi' : \mathbb{N}^{\mathbb{N}} \to \Omega \times [0,2]$. Given a sequence $(b_1, b_2, \cdots)$ we can show that we only have one possible value of (ω, x). Note that given $(b_1, b_2, \cdots)$ we can write $r_n = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots + \cfrac{2}{b_n}}}$. Using the Moebius transformation we see that $r_n = A_1 \cdot A_2 \cdots A_n(0)$, where $A_i = \begin{bmatrix} 0 & 2 \\ 1 & b_i \end{bmatrix}$ and therefore $r_n = \frac{p_n}{q_n}$. Hence

$$r_n = \sum_{i=1}^{n} \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \sum_{i=1}^{n} \frac{-(-2)^n}{q_n q_{n-1}} \tag{5.1.5}$$

and $\lim_{n\to\infty} r_n = \sum_{i=1}^{\infty} \frac{-(-2)^n}{q_n q_{n-1}}$. Since $\frac{|(-2)^n|}{q_n q_{n-1}}$ decreases monotonically, see section 4.1.1, we have by the alternating series test that the series 5.1.5 converges. We conclude that for each series $(b_1, b_2, \cdots)$ there exists a x such that $x = \lim_{n\to\infty} r_n$. On the other hand we have already proved in proposition 5.1.1 that there exists an unique ω, such that $x = \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}$.

To show that we have constructed an inverse of ψ we have to show that

$$\psi \circ \psi' = \psi' \circ \psi = \mathrm{id},$$

where id denotes the identity function. Consider

$$\psi \circ \psi'(b_1, b_2, \cdots) = \psi(\omega, \cfrac{2}{b_1 + \cfrac{2}{b_2 + \ddots}}) = (b_1, b_2, \cdots),$$

where the last equation follows just by construction of ω and the fact that $b_i = d_i(\omega, x)$
On the other hand

$$\psi' \circ \psi(\omega, x) = \psi'(d_1(\omega, x), d_2(\omega, x), \cdots) = (\omega, \cfrac{2}{d_1 + \cfrac{2}{d_2 + \ddots}}).$$

Since ω is the unique element in Ω generating the digit sequence $(d_1(\omega, x), d_2(\omega, x)\cdots)$, we have $\psi' \circ \psi(\omega, x) = (\omega, x)$

We prove that ψ is a measurable bijection. First we proof that $\psi : \Omega \times [0, 1] \to \infty$ is measurable. To see this notice the following:

- If $b_1 \neq 1$ then
$$\begin{aligned} \psi^{-1}([b_1]) =& \{(\omega, x) : d_1(\omega, x) = b_1\} && (5.1.6) \\ =& ((\frac{2}{b_1+1}, \frac{2}{b_1}] \times [0] \cup (\frac{2}{b_1+2}, \frac{2}{b_1+1}] \times [1]) \cap M, && (5.1.7) \end{aligned}$$
so $\psi^{-1}([b_1]) \in \sigma(\mathcal{C} \times \mathcal{B})$.

- If $b_1 = 1$ then
$$\begin{aligned} \psi^{-1}([1]) = & \quad \{(\omega, x) : d_1(\omega, x) = 1\} && (5.1.8) \\ = & \quad ((\tfrac{2}{3}, 1] \times [1] \cup (1, 2] \times \Omega) \cap M, && (5.1.9) \end{aligned}$$
so $\psi^{-1}([1]) \in \sigma(\mathcal{C} \times \mathcal{B})$.

By induction we show that the function is measurable. So suppose that the result holds for cylinders of length n. Then we obtain:

$$\begin{aligned} &\psi^{-1}[b_1, b_2, \cdots, b_n, b_{n+1}] \\ =& \{(\omega, x) : d_1(\omega, x) = b_1, d_2(\omega, x) = b_2, \cdots, d_n(\omega, x) = b_n, d_{n+1}(\omega, x) = b_{n+1}\} \\ =& \{(\omega, x) : d_1(\omega, x) = b_1, d_2(\omega, x) = b_2, \cdots, d_n(\omega, x) = b_n\} \cap \{(x, \omega) : d_{n+1}(\omega, x) = b_{n+1}\} \\ =& \{(x, \omega) : d_1(\omega, x) = b_1, d_2(\omega, x) = b_2, \cdots, d_n(\omega, x) = b_n\} \cap \{(x, \omega) : d_1(R^n(\omega, x)) = b_1\} \\ =& \psi^{-1}[b_1, b_2, \cdots, b_n] \cup R^{-n}(\psi^{-1}[b_1]) \cap M. \end{aligned}$$

The last line is measurable since R is a measurable function.

Now we show that $\psi^{-1} = \psi'$ is a measurable function. Let $A \in \sigma(\mathcal{C} \times \mathcal{B})$, then since ψ is a bijection it follows.

$$\begin{aligned} \psi'^{-1}A =& \{(b_1, b_2, \cdots) : \psi'(b_1, b_2, \cdots) \in A\} \\ =& \bigcup_{y \in A} \{(b_1, b_2, \cdots) : \psi'(b_1, b_2, \cdots) \in \{y\}\} \\ =& \bigcup_{y \in A} \{(b_1, b_2, \cdots) : \psi'(b_1, b_2, \cdots) = y\} \\ =& \bigcup_{y \in A} \{(b_1, b_2, \cdots) : (b_1, b_2, \cdots) = \psi(y)\} \\ =& \mathrm{Im}\psi(A). \end{aligned}$$

Hence to show the measurability of ψ' it is enough to check that $\mathrm{Im}(\psi[\omega]_{l_n} \times (\frac{3}{k+1} \frac{3}{k}])$ is in the σ-algebra generated by the cylinder sets on $\mathbb{N}$. Define

$A_{y,i} = \bigcup_{n\geq 2}[y, \underbrace{1,\cdots,1}_{2i+1 \text{ times}}, n]$ and $A_i = \bigcup_{n\geq 2}[\underbrace{1,\cdots,1}_{2i+1 \text{ times}}, n]$, $B_{y,i} = \bigcup_{n\geq 2}[y, \underbrace{1,\cdots,1}_{2i \text{ times}}, n]$ and $C_{y,1} = (y, \underbrace{1,1,1,\cdots}_{\text{infinitely many 1's}})$. Then

$$\psi'^{-1}([\omega]_{l_1} \times \Delta k_1) = \psi([\omega]_{l_1} \times (\frac{2}{k_1+1}, \frac{2}{k_1}]) \tag{5.1.10}$$

$$= \begin{cases} \bigcup_{i\in\mathbb{N}_0} B_{k_1,i} \cup C_{y,1} & \text{if } k_1 \geq 2,\ [\omega]_{l_1} = [0] \\ \bigcup_{i\in\mathbb{N}} A_{k_1-1,i} \cup C_{y,1} & \text{if } k_1 \geq 2,\ [\omega]_{l_1} = [1] \\ \bigcup_{i\in\mathbb{N}} A_i \cup C_{y,1} & \text{if } k_1 = 1. \end{cases} \tag{5.1.11}$$

Hence $\psi'^{-1}([\omega]_{l_1} \times \Delta k_1) \in \mathcal{C}$. Now suppose the result holds for sets of length n so $\psi'^{-1}([\omega]_{l_n} \times \Delta k_n) \in \mathcal{C}$, we show that $\psi'^{-1}([\omega]_{l_{n+1}} \times \Delta k_{n+1}) \in \mathcal{C}$. First define:

$$A_{n,y,i} = \bigcup_{(b_1,\cdots,b_n)\in\mathbb{N}^n} \bigcup_{m\geq 2} [b_1, \cdots b_n, y, \underbrace{1,\cdots,1}_{2i+1 \text{ times}}, m] \tag{5.1.12}$$

$$A_{n,i} = \bigcup_{(b_1,\cdots,b_n)\in\mathbb{N}^n} \bigcup_{m\geq 2} [b_1, \cdots b_n, \underbrace{1,\cdots,1}_{2i+1 \text{ times}}, m] \tag{5.1.13}$$

$$B_{n,y,i} = \bigcup_{(b_1,\cdots,b_n)\in\mathbb{N}^n} \bigcup_{m\geq 2} [b_1, \cdots b_n, y, \underbrace{1,\cdots,1}_{2i \text{ times}}, m] \tag{5.1.14}$$

$$C_{n,y,1} = \bigcup_{(b_1,\cdots,b_n)} (b_1, \cdots, b_n, y, \underbrace{1,1,1,\cdots}_{\text{infinitely many 1's}}). \tag{5.1.15}$$

So we can write,

$$\begin{aligned} \psi'^{-1}([\omega]_{l_{n+1}} \times \Delta k_{n+1}) &= \psi'^{-1}([\omega]_{l_n} \times \Delta k_n) \cap \psi'^{-1}(\{(x,\omega) : R^n(x,\omega) \in [\omega_{l_{n+1}}] \times (\frac{2}{k_n}, \frac{2}{k_{n+1}}]\}) \\ &= \psi([\omega]_{l_n} \times \Delta k_n) \cap \psi(\{(x,\omega) : R^n(x,\omega) \in [\omega_{l_{n+1}}] \times (\frac{2}{k_n}, \frac{2}{k_{n+1}}]\}) \\ &= \begin{cases} C \cap (\bigcup_{i\in\mathbb{N}_0} B_{k_n,i} \cup C_{n,y,1}) & \text{if } k_n \geq 2,\ [\omega_{l_n}] = [0] \\ C \cap (\bigcup_{i\in\mathbb{N}} A_{k_n-1,i} \cup C_{n,y,1}) & \text{if } k_n \geq 2,\ [\omega_{l_n}] = [1] \\ C \cap (\bigcup_{i\in\mathbb{N}} A_{n,i} \cup C_{n,y,1}) & \text{if } k_n = 1, \end{cases} \end{aligned}$$

where $C \in \mathcal{C}$ is the element $\psi'^{-1}([\omega]_{l_n} \times \Delta k_n)$. We conclude that ψ' is indeed a measurable function.
The fact that ψ preserves measure is immediately since we defined $\mu = \nu \circ \psi$.
Finally we have to show that $\sigma \circ \psi$ is the same operation as $\psi \circ R$. Since σ is the left shift we have

$$\sigma \circ \psi(\omega, x) = \sigma(d_1(\omega,x), d_2(\omega,x), \dagger_3(\omega,x), \cdots) = (d_2(\omega,x), d_3(\omega,x), \cdots)$$

On the other hand

$$\psi \circ R(\omega, x) = (d_1(R(\omega,x)), d_2(R(\omega,x)), \cdots) = (d_2(\omega,x), d_3(\omega,x), \cdots)$$

so indeed $\sigma \circ \psi = \psi \circ R$. We conclude that ψ is indeed an isomorphism.

□

Now we can define measures on $\Omega \times [0,2]$ by defining measures on $\mathbb{N}^{\mathbb{N}}$, with the property that $\nu\{\infty\} = 0$, so $\nu(\mathbb{N}^{\mathbb{N}} = 1)$. Such measures can be constructed by using some probability vector $(p_0, p_1, \cdots)$ on $\mathbb{N}$ and by defining ν on the cylinder sets as a product measure. So for all $b_1, b_2, \cdots, b_n \in \mathbb{N}$,

$$\nu_p(\{(y_1, y_2, \cdots) \in \mathbb{N}^{\mathbb{N}} : y_1 = b_1, \cdots, y_n = b_n\}) = p_{b_1} p_{b_2} \cdots p_{b_n}.$$

By theorem of Caratheodory, see for example Schilling [2005] we can extend this measure uniquely to a measure on $\mathcal{C}$. Examples of probability-measures we can choose on $\mathbb{N}$ to construct the product measure ν are:

- $p_i = \frac{1}{i(i+1)}$,
- $p_i = \frac{e^{-\lambda}\lambda^i}{i!}$, Poisson distribution,
- $p_i = r^i$, Geometric Distribution.

Since each product measure on the cylinder sets in $\mathbb{N}^{\mathbb{N}}$ is ergodic with respect to the left shift, the above probability distributions induce an ergodic measure ν. Therefore the measure $\mu = \nu \circ \psi$ on $\Omega \times [0, 2]$ is also ergodic. However since all the measures ν are different it follows from a standard theorem in ergodic theory that they are singular with respect to each other. Also the measure ρ, that we find with the help of the article of Inoue is singular with respect to these measures. This is seen since the mean of the digits with respect to ρ is infinite and the mean of the digits with respect to the measures ν constructed by the above probability distributions are all finite. In general we see thus that a finite arithmetic mean, seems to be a generic behaviour on ρ-null sets. Considering the entropy, we see that for measures ν which give finite mean digit sequences, the geometric measure with this mean is the measure of maximal entropy, see subsection 2.1.4. Whether the product measure $m_p \times \mu_p$ has maximum entropy is not known.

Chapter 6

Case $N = 3$

6.1 The 3-random continued fraction transformation

We will show that the theory developed in chapter 4 can be extend to 3-random continued fractions. We define three functions:

$$S_0, S_1, S_2 : (0,3] \to (0,3] \tag{6.1.1}$$

$$S_0 : x \to \frac{3}{x} - \left\lfloor \frac{3}{x} \right\rfloor \tag{6.1.2}$$

$$S_1 : x \to \begin{cases} \frac{3}{x} - \lfloor \frac{3}{x} \rfloor + 1 & \text{if } x \in (0, 1\frac{1}{2}] \\ \frac{3}{x} - \lfloor \frac{3}{x} \rfloor & \text{if } x \in (1\frac{1}{2}, 3] \end{cases} \tag{6.1.3}$$

$$S_2 : x \to \begin{cases} \frac{3}{x} - \lfloor \frac{3}{x} \rfloor + 2 & \text{if } x \in (0, 1] \\ \frac{3}{x} - \lfloor \frac{3}{x} \rfloor & \text{if } x \in (1, 3]. \end{cases} \tag{6.1.4}$$

We will refer to S_0 as the lower map, to S_1 as the middle map and S_2 as the upper map. Let $\Omega = \{0, 1, 2\}^{\mathbb{N}}$ and define the function R as follows:

$$R : \Omega \times [0,3] \to \Omega \times [0,3], \tag{6.1.5}$$

$$R(\omega, x) = \begin{cases} (\sigma\omega, S_{\omega_1}(x)) & \text{if } x \in (0, 1\frac{1}{2}] \\ (\omega, S_{\omega_1}(x)) & \text{if } x \in (1\frac{1}{2}, 3] \\ (\sigma\omega, 0) & \text{if } x = 0. \end{cases} \tag{6.1.6}$$

Figure 6.1 illustrates the transformation R. Notice R does not shift ω, if $x \in (1\frac{1}{2}, 3]$. In the area $(1\frac{1}{2}, 3]$ the three maps S_0, S_1 and S_2 coincide and therefore we do not have to choose which map we use. Like we did in the case $N = 2$, we will define the transformation $K : \Omega \times [0,1] \to \Omega \times [0,1]$ as the induced function $R_{\Omega \times [0,1]}(\omega, x)$. In order to do this we introduce the return time

$$\begin{aligned} \tau : \Omega \times [0,1] &\to \mathbb{N}, \\ (\omega, x) &\to \inf\{n \in \mathbb{N} : R^n(\omega, x) \in \Omega \times [0,1]\}. \end{aligned}$$

First let us have a look at the function R. The return time τ will be treated in the next section. As before denote the digits of R by:

$$b_1(\omega, x) = \begin{cases} 1 & x \in (1\frac{1}{2}, 3] & \\ k & x \in (\frac{3}{k+1}, \frac{3}{k}] & \omega_1 = 0,\ k \in \mathbb{N}\ , k \geq 2 \\ k-1 & x \in (\frac{3}{k+1}, \frac{3}{k}] & \omega_1 = 1,\ k \in \mathbb{N},\ k \geq 2 \\ k-2 & x \in (\frac{3}{k+1}, \frac{3}{k}] & \omega_1 = 2,\ k \in \mathbb{N},\ k \geq 3 \\ 2 & x \in (1, 1\frac{1}{2}] & \omega_1 = 2 \end{cases} \tag{6.1.7}$$

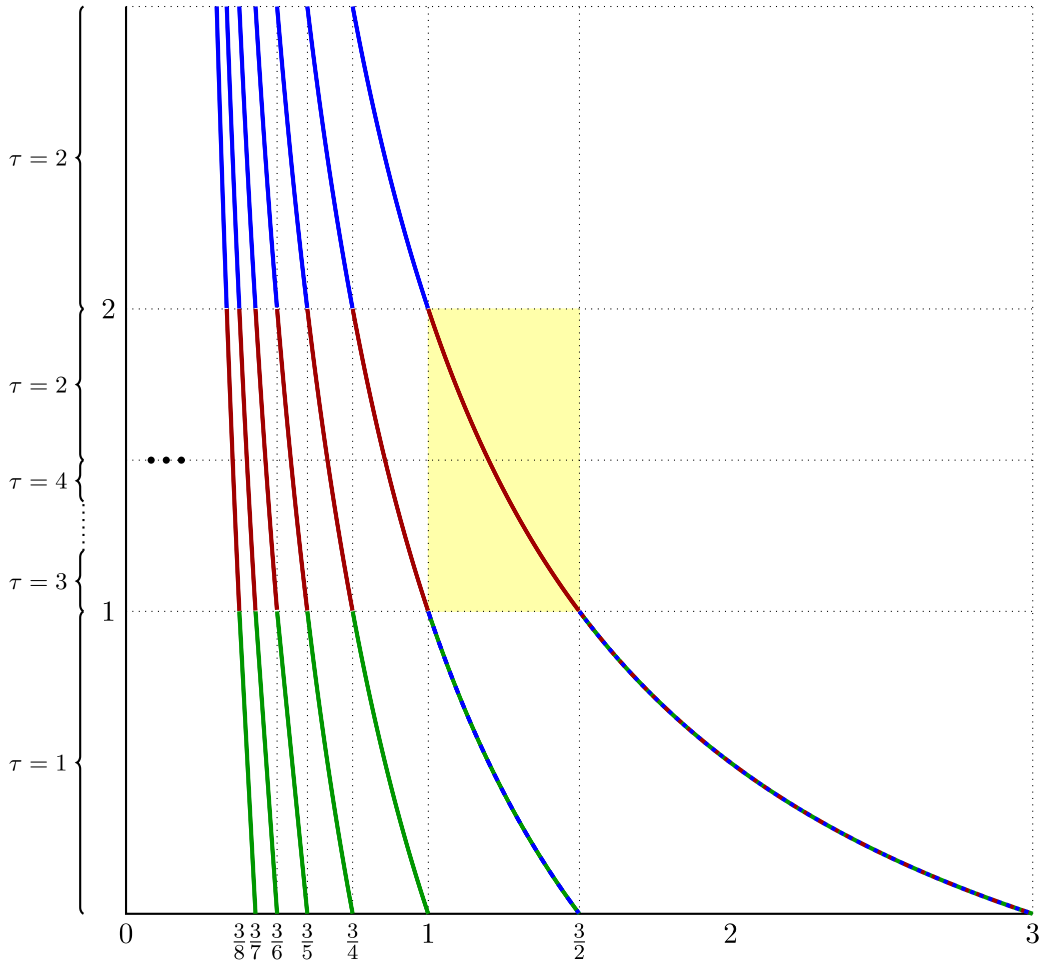

Figure 6.1 – Map T_0 in green, T_1 in red and T_2 in blue.

And we define $b_n(\omega,x) = b_1(R^{n-1}(\omega,x))$. We will show that we can use R to expand x in the following way.

$$x = \cfrac{3}{b_1 + \cfrac{3}{b_2 + \cfrac{3}{b_3 + \cfrac{3}{\ddots}}}} \tag{6.1.8}$$

Let π_2 denote the projection on the second coordinate, then we can write

$$\pi_2(R(\omega,x)) = \frac{3}{x} - b_1(\omega,x). \tag{6.1.9}$$

Hence,

$$x = \frac{3}{b_1 + \pi_2(R(\omega,x))}. \tag{6.1.10}$$

Continuing this way we find

$$x = \cfrac{3}{b_1 + \cfrac{3}{b_2 + \cfrac{3}{b_3 + \cfrac{3}{\ddots + \cfrac{3}{b_n + \pi_2(R^n(\omega,x))}}}}}.$$

Let $\frac{p_n}{q_n}$ denote the partial quotients of x, so

$$\frac{p_n}{q_n} = \cfrac{3}{b_1 + \cfrac{3}{b_2 + \cfrac{3}{b_3 + \cfrac{3}{\ddots + \cfrac{3}{b_n}}}}}.$$

Like we did for K, we can define matrices M_n and B_n for the transformation $R(\omega,x)$.
Let $B_n = \begin{bmatrix} 0 & 3 \\ 1 & b_n \end{bmatrix}$ and $M_n = B_1 \cdot B_2 \cdots B_n$. So

$$M_n = M_{n-1}B_n \tag{6.1.11}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & b_n \end{bmatrix} \tag{6.1.12}$$

$$\begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix} = \begin{bmatrix} p_{n-1} & 3r_{n-1} + b_n p_{n-1} \\ q_{n-1} & 3s_{n-1} + b_n q_{n-1} \end{bmatrix}. \tag{6.1.13}$$

We obtain the following recurrence relations:

$$p_{-1} = 1 \qquad p_0 = 0 \qquad p_n = 3p_{n-2} + b_n p_{n-1}, \tag{6.1.14}$$

$$q_{-1} = 0 \qquad q_0 = 1 \qquad q_n = 3q_{n-2} + b_n q_{n-1}. \tag{6.1.15}$$

Using the recursion relation we can derive some estimates for p_n and q_n.

Proposition 6.1.1. *Let p_n and q_n be defined as above. If $n = 2k + 1$ for $k \in \mathbb{N}$, $k \geq 2$ then $p_n \geq 4^k$ and $q_n \geq 4^k$. If $n = 2k$ then $p_n \geq 4^k$ and $q_n \geq 4^k$.*

Proof. We prove this statement by induction. Note $p_1 = 3$, $p_2 \geq 3$, $p_3 \geq 12$ and $p_4 \geq 21$ and $q_1 = 1 \geq 1$, $q_2 \geq 4$, $q_3 \geq 7$ and $q_4 \geq 19$. Hence we have the basestep. Now suppose the result holds true for all $n \leq N$ and let N be odd. Then:

$$\begin{aligned} p_{N+1} = p_{2k} \geq & 3p_{2(k-1)} + p_{2(k-1)+1} \\ = & 3 \cdot 4^{k-1} + 4^{k-1} \\ = & 4 \cdot 4^{k-1} = 4^k. \end{aligned}$$

In the case N is even we obtain:

$$\begin{aligned} p_{N+1} = p_{2k+1} \geq & 3p_{2(k-1)+1} + p_{2k} \\ = & 3 \cdot 4^{k-1} + 4^k \\ \geq & 4^k. \end{aligned}$$

We conclude that $p_n \geq 4^n$ for all $n \in \mathbb{N}$. The proof for q_n follows in the same way. $\square$

Proposition 6.1.2. $\lim_{n\to\infty} |x - \frac{p_n}{q_n}| = 0$.

Proof. Like we did in the case $N = 2$ we can use the Moebius transformations to express x as follows,

$$\begin{aligned} x &= M_n(\pi_2(R^n(\omega, x))) \\ &= \frac{p_n + p_{n-1}(\pi_2(R^n(\omega, x)))}{q_n + q_{n-1}(\pi_2(R^n(\omega, x)))}. \end{aligned}$$

Notice that $\det B_n = -3$ and hence $\det M_n = (-3)^n$. Using this and proposition 6.1.1 we obtain:

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{\det M_n \cdot R^n(\omega, x)}{q_n(q_n + q_{n-1}(\pi_2(R^n(\omega, x))))} \right| \tag{6.1.16}$$

$$\leq \frac{3^{n+1}}{q_n^2} \tag{6.1.17}$$

$$\leq 27 \cdot (\frac{3}{4})^{n-2}. \tag{6.1.18}$$

Hence we see that $\lim_{n\to\infty} \left| x - \frac{p_n}{q_n} \right| = 0$ and we can expand x like equation 6.1.8. $\square$

Finally we prove a proposition about the relation between p_n and q_n.

Proposition 6.1.3. *Let p_n and q_n be defined as above, then $p_n(\omega, x) = 3q_{n-1}(R(\omega, x))$ for $n \in \mathbb{N}$, $n \geq 2$.*

Proof. Note

$$p_1(\omega, x) = 3p_{-1}(\omega, x) + b_1(\omega, x)p_0 = 3$$

and

$$q_0(R(\omega, x)) = 1.$$

Furthermore

$$p_2(\omega, x) = 3p_0(\omega, x) + b_2(\omega, x)p_1(\omega, x) = 3b_2(\omega, x).$$

Now

$$q_1(R(\omega, x)) = 3q_{-1}(R(\omega, x)) + b_1(R(\omega, x))q_0(R(\omega, x)) = b_1(R(\omega, x)) = b_2(\omega, x)$$

so the induction step is fine. Suppose the result holds for $n \le N$ then we find:

$$\begin{aligned} p_{N+1}(\omega, x) =& 3p_{N-1}(\omega, x) + p_N(\omega, x)b_{N+1}(\omega, x) && (6.1.19)\\ =& 3q_{N-2}(R(\omega, x)) + q_{N-1}(R(\omega, x))b_N(R(\omega, x)) && (6.1.20)\\ =& q_N(R(\omega, x)) && (6.1.21)\end{aligned}$$

and therefore the proposition is proved. □

6.1.1 The return time

To derive an explicit expression for the induced function $K : \Omega\times[0, 1] \to \Omega\times[0, 1]$ we need an explicit expression for the return time τ. Recall

$$\begin{aligned} \tau : \Omega \times [0, 1] &\to \mathbb{N},\\ (\omega, x) &\to \inf\{n \in \mathbb{N} : R^n(\omega, x) \in \Omega \times [0, 1]\}.\end{aligned}$$

Looking at figure 6.1 we see for points in $[0, 1]$ that if we use the lower transformation we end up in $[0, 1]$. If we use the upper transformation we end up in $[2, 3]$. But from figure 6.1 it is also clear that $S_0[2, 3] = S_1[2, 3] = S_2[2, 3] = [0, \frac{1}{2}]$. So starting with the upper transformation on $\Omega \times [0, 1]$ we will be back in $\Omega \times [0, 1]$ in two steps. We obtain:

$$\tau(\omega, x) = \begin{cases} 1 & \text{if } \omega_1 = 0\\ 2 & \text{if } \omega_1 = 2.\end{cases} \tag{6.1.22}$$

For $\omega_1 = 1$, we start with the middle transformation and $\tau(\omega, x) = n$ for some $n \in \mathbb{N}$, $n \ge 2$. Note that when (ω, x) enters the region $(1, 1\frac{1}{2})$, it can stay there for a very long time if we use only the middle transformation S_1, i.e. $\omega_i = 1$ for $1 \le i \le n$, $n \in \mathbb{N}$ large. We can get an idea how long a point (ω, x) will stay in $(1, 1\frac{1}{2})$. Suppose $(\omega, x) \in \Omega \times (1, 1\frac{1}{2})$, then there are two ways for (ω, x) to leave $\Omega \times (1, 1\frac{1}{2})$. The first way is using the upper or lower transformation instead of the middle transformation. Let ω_{n+1} be the smallest coordinate of ω such that $\omega_{n+1} \ne 1$, so $\pi_1(R^n(\omega, x)) = \omega_{n+1} \ne 1$. Then

$$R^{n+1}(\omega, x) = (\sigma(\pi_1 R^n(\omega, x), S_0(\pi_2 R^n(\omega, x))) = (\sigma(\pi_1 R^n(\omega, x), S_2(\pi_2 R^n(\omega, x))).$$

Since $S_0(0, 3] = (0, 1]$ and $S_2(1, 3] = (0, 1]$ it follows that $R^{n+1}(\omega, x) \in \Omega \times [0, 1]$. Recall we assumed that $\pi_2(R^n(\omega, x)) \in (1, 1\frac{1}{2})$ for all $0 \le i \le n$, hence we see $\tau(\omega, x) = n + 1$ For the second option notice that $S_1(1, 1\frac{1}{2}] = (1, 2]$ and $S_i[1\frac{1}{2}, 2] \subset [0, 1]$ for $i \in \{0, 1, 2\}$. Hence if $R^n(\omega, x) = (\sigma^n(\omega), S_1^n(x))$ it could occur that $S_1^i(\omega, x) \in \Omega \times [1, 1\frac{1}{2}]$ for $1 \le i \le n - 1$, $S_1^n(x) \in [1\frac{1}{2}, 2]$. Then $R^{n+1}(\omega, x) \in \Omega \times [0, 1]$. Note that the transformation S_1 has one fixed point in $(1, 1\frac{1}{2}]$, namely $x = -\frac{1}{2} + \frac{1}{2}\sqrt{13}$. Also note that $|\frac{dS_1(x)}{dx}| = \frac{3}{x^2} \in (1\frac{1}{3}, 3)$ if $x \in (1\frac{1}{2}, 2]$, so $S_1(x)$ is expanding. Therefore intuitively for each $x \in (1\frac{1}{2}, 2]\backslash\{-\frac{1}{2} + \frac{1}{2}\sqrt{13}\}$ there exists a $n \in \mathbb{N}$ such that $S_1^n(x) \in [1\frac{1}{2}, 2]$ and

hence $S_1^{n+1}(x) \in [0,1]$.
From the above discussion we see that the function τ has the following properties:

$$\tau(\omega,x) = \begin{cases} 1 & \text{if } \omega_1 = 0,\ x \in [0,1] \\ 2 & \text{if } \omega_1 = 2,\ x \in [0,1] \\ n+1 & \text{if } (\omega_1,\omega_2,\cdots,\omega_{n-1},\omega_n,\omega_{n+1}) \in \{\underbrace{(1,1,\ldots,1}_{n\ times},0),\underbrace{(1,1,\ldots,1}_{n\ times},2)\} \\ & \text{and } x \in \cap_{i=1}^{n} S_1^{-i}(1,1\frac{1}{2}) \cap [0,1] \\ n+1 & \text{if } (\omega_1,\omega_2,\cdots,\omega_{n-1},\omega_n) = \underbrace{(1,1,\ldots,1)}_{n\ times} \\ & \text{and } x \in \cap_{i=1}^{n-1} S_1^{-i}(1,1\frac{1}{2}) \cap S_1^{-n}(1\frac{1}{2},2] \cap [0,1] \end{cases}$$

Now we like to know what the sets

$$\cap_{i=1}^{n} S_1^{-i}(1,1\frac{1}{2}) \cap [0,1] \tag{6.1.23}$$

and

$$\cap_{i=1}^{n-1} S_1^{-i}(1,1\frac{1}{2}) \cap S_1^{-n}(1\frac{1}{2},2] \cap [0,1] \tag{6.1.24}$$

look like. Like we did in section 4.1.1, we can write

$$x = \frac{p_n + p_{n-1}S_1^n(x)}{q_n + q_{n-1}S_1^n(x)}.$$

Recall that

$$\frac{p_n + p_{n-1}S_1^n(x)}{q_n + q_{n-1}S_1^n(x)}$$

is a monotone function in $S_1^n(x)$. Since $S_1^n(x) \in (1,1\frac{1}{2})$ if $x \in \cap_{i=1}^{n} S_1^{-i}(1,1\frac{1}{2}) \cap [0,1]$, it follows that x is in an interval with endpoints

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} \text{ and } \frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}}.$$

If n is odd this is the interval

$$\left(\frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right)$$

and if n is even we find

$$\left(\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}}\right).$$

On the other hand when x is in such an interval, then x has digits

$$(b_1,b_2,\cdots,b_n) = (k-1,1,\cdots,1)$$

if we only us the middle transformation S_1. Hence we can use the recursion relations to find the intervals 6.1.23 and 6.1.24. First we write:

$$\cap_{i=1}^{n} S_1^{-i}(1,1\frac{1}{2}) = S_1^{-1}(\cap_{i=0}^{n-1} S_1^{-i}(1,1\frac{1}{2})) = S_1^{-1}I_n,$$

where $I_n = (\cap_{i=0}^{n-1} S_1^{-i}(1, 1\frac{1}{2}))$. By the discussion above we know for some $p_n,\ p_{n-1},\ q_n,\ q_{n-1}$ which satisfy the recursion relations,

$$I_n = \left(\frac{p_{n-1} + 1\frac{1}{2}p_{n-2}}{q_{n-1} + 1\frac{1}{2}q_{n-2}}, \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right)$$

if $n-1$ is odd, so n even and

$$I_n = \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}}, \frac{p_{n-1} + 1\frac{1}{2}p_{n-2}}{q_{n-1} + 1\frac{1}{2}q_{n-2}} \right)$$

if n is odd. Therefore if $x \in I_n$, then $x, S_1(x), \cdots, S_1^{n-1}(x)$ are all in $(1, 1\frac{1}{2})$. If $(\omega, x) \in [\underbrace{1, \cdots, 1}_{n\ times}] \times S_1^{-1}I_n$, then $\tau(\omega, x) > n$. Suppose

$$(\omega, x) \in [\underbrace{1, \cdots, 1}_{n+1\ times}] \times S_1^{-1}(I_n \backslash I_{n+1}),$$

then $S_1^{n+1}(\omega, x) \in (1\frac{1}{2}, 2)$ and hence $\tau(\omega, x) = n+2$. Let J_n denote the interval $J_n = I_n \backslash I_{n+1}$. Then $S_1^{-1}J_n = S_1^{-1}(I_n \backslash I_{n+1})$ and

$$J_n = \left(\frac{p_{n-1} + 1\frac{1}{2}p_{n-2}}{q_{n-1} + 1\frac{1}{2}q_{n-2}}, \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right) \Big\backslash \left(\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}} \right)$$

when n is even and

$$J_n = \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}}, \frac{p_{n-1} + 1\frac{1}{2}p_{n-2}}{q_{n-1} + 1\frac{1}{2}q_{n-2}} \right) \Big\backslash \left(\frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right)$$

when n is odd. If $x \in I_n \times [\underbrace{1, \cdots, 1}_{n\ times}]$ then $(b_1, b_2, \cdots, b_n) = (1, 1, \cdots, 1)$. Therefore the recursion relations satisfy

$$p_n + p_{n-1} = 3p_{n-2} + 2p_{n-1} = 2(p_{n-1} + 1\frac{1}{2}p_{n-2})$$

and in the same way

$$q_n + q_{n-1} = 2(q_{n-1} + 1\frac{1}{2}q_{n-2}).$$

Hence

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}} = \frac{p_{n-1} + 1\frac{1}{2}p_{n-2}}{q_{n-1} + 1\frac{1}{2}q_{n-2}}.$$

so

$$J_n = \left(\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}}, \frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}} \right]$$

for n is odd and

$$J_n = \left[\frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}}, \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \right)$$

for n is even. For example

n	p_n	q_n	I_n	J_n
-1	1	0	-	-
0	0	1	-	$[1\frac{1}{2}, 2]$
1	3	1	$(1, 1\frac{1}{2})$	$(1, \frac{6}{5}]$
2	3	4	$(\frac{6}{5}, \frac{3}{2})$	$[\frac{15}{11}, \frac{3}{2})$
3	12	7	$(\frac{6}{5}, \frac{15}{11})$	$(\frac{6}{5}, \frac{33}{26}]$
4	21	19	$(\frac{33}{26}, \frac{15}{11})$	$(\frac{78}{59}, \frac{15}{11}]$

Hence for $(\omega, x) \in [1] \times (S_1^{-1} J_n \cap [0,1])$ we have that

$$\tau(\omega, x) = \min\{n+2, \inf\{i : \omega_i \neq 1\}\}.$$

So determining $S_1^{-1} J_n \cap [0,1]$ we know $\tau(\omega, x)$ exactly for each $x \in \Omega \times [0,1]$. Note that if $x \in (\frac{3}{k+1}, \frac{3}{k}]$ then $S_1(x) = \frac{3}{x} - (k-1)$ and hence for n is even:

$$\begin{aligned}
S_1^{-1}(J_n) \cap \left(\frac{3}{k+1}, \frac{3}{k}\right] &= \left\{x \in \left(\frac{3}{k+1}, \frac{3}{k}\right] : \frac{3}{x} - (k-1) \in J_n\right\} \\
\frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}} &< \frac{3}{x} - k + 1 < \frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} \\
\frac{p_n + 1\frac{1}{2}p_{n-1} + (k-1)(q_n + 1\frac{1}{2}q_{n-1})}{q_n + 1\frac{1}{2}q_{n-1}} &< \frac{3}{x} < \frac{p_{n-1} + p_{n-2} + (k-1)(q_{n-1} + q_{n-2})}{q_{n-1} + q_{n-2}} \\
\frac{3(q_n + 1\frac{1}{2}q_{n-1})}{p_n + 1\frac{1}{2}p_{n-1} + (k-1)(q_n + 1\frac{1}{2}q_{n-1})} &> x > \frac{3(q_{n-1} + q_{n-2})}{p_{n-1} + p_{n-2} + (k-1)(q_{n-1} + q_{n-2})} \\
\frac{3}{\frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}} + (k-1)} &> x > \frac{3}{\frac{p_{n-1} + p_{n-2}}{q_{n-1} + q_{n-2}} + (k-1)} \\
\frac{p_n + p_{n-1}}{q_n + q_{n-1}} &< x < \frac{p_{n+1} + 1\frac{1}{2}p_n}{q_{n+1} + 1\frac{1}{2}q_n}
\end{aligned}$$

From this we see each interval $(\frac{3}{k+1}, \frac{3}{k}]$ contains exactly one sub-interval with return time $\tau = n+2$ when $\omega = (\underbrace{1, \cdots, 1}_{n-1 \ times}, \cdots)$, i.e. when we only use the middle transformation S_1. Notice that for each $x \in [0,1]$, $x \in S_1^{-1} J_i$ for some i. Therefore the intervals $S_1^{-1} J_i \cap (\frac{3}{k+1}, \frac{3}{k}]$ give a partition of $[0,1]$. Moreover we have some more structure. Consider the intervals $S_1^{-1} I_n$, they contain the $x \in [0,1]$ for which at least $S_1(x), \cdots, S_1^i(x) \in [1, 1\frac{1}{2}]$. So by definition of $S_1^{-1}(I_n)$ we see

$$S_1(S_1^{-1}(I_n)) = S_1^{-1} I_{n-1},$$

$$S_1^2(S_1^{-1}(I_n)) = S_1^{-1}(S_1^{-1}(I_{n-1})) = S_1^{-1}(I_{n-2})$$

and so on. In the same way we see $S_1(J_n) = J_{n-1}$ and in general $S_1^k(J_n) = J_{n-l}$ for $1 \leq l \leq n$. Notice that on $S_1^{-1} J_n \cap (\frac{3}{k+1}, \frac{3}{k}]$ we have that S_1 is monotone on $S_1^{-1} J_n \cap (\frac{3}{k+1}, \frac{3}{k}]$ for all $n \in \mathbb{N}_{n \geq 0}$. Hence also S_1^l for $l \leq n$ is monotone on $S_1^{-1} J_n \cap (\frac{3}{k+1}, \frac{3}{k}]$, since a composition of monotone functions is monotone. The same holds for S_1^l on the intervals I_n, $l \leq n$.

6.1.2 The induced transformation

Let $K(\omega, x)$ be the induced function of R, so we define $K : \Omega \times [0,1] \to \Omega \times [0,1]$ by $(\omega, x) \to R^{\tau(\omega,x)}(\omega, x)$. Let us see what $K(\omega, x)$ looks like. If $\omega_1 = 0$ then

$\tau(\omega,x)=1$ and hence $K(\omega,x)=(\sigma(\omega),S_0(x))$. If $\omega_1=2$ then $\tau(\omega,x)=2$ and therefore $K(\omega,x)=(\sigma(\omega),S_0\circ S_2(x))$. In fact in this case we have

$$S_0\circ S_2(x)=S_1\circ S_2(x)=S_2^2(x)$$

and hence we shift ω only once. When $\omega_1=1$ the situation becomes more difficult. In order to give an explicit formula for $K(\omega,x)$ we will first define functions T_i.

$$T_i:[0,1]\to[0,1]$$
$$T_0(x)=S_0(x)=\frac{3}{x}-\left\lfloor\frac{3}{x}\right\rfloor$$
$$T_1(x)=S_0\circ S_2(x)=\frac{3}{\frac{3}{x}-\lfloor\frac{3}{x}\rfloor+2}-1$$
$$T_{(2,i)}(x)=\begin{cases}S_0\circ S_1^{i+1}(x) & \text{if } x\in S_1^{-1}J_i \text{ for } i\in\mathbb{N}\cup\{0\}\\ 2x & \text{otherwise}\end{cases}$$
$$T_{(3,i)}(x)=\begin{cases}S_0\circ S_1^{i}(x) & \text{if } x\in S_1^{-1}I_i \text{ for } i\in\mathbb{N}\\ 2x & \text{otherwise}\end{cases}$$

Note that

$$S_0\circ S_2(x)=S_1\circ S_2(x)=S_2\circ S_2(x),$$
$$S_0\circ S_1^{i+1}(x)=S_1\circ S_1^{i+1}(x)=S_2\circ S_1^{i+1}(x)$$

if $x\in S_1^{-1}J_i$ for $i\in\mathbb{N}\cup\{0\}$ and

$$S_0\circ S_1^{i}(x)=S_2\circ S_1^{i}(x)$$

if $x\in S_1^{-1}I_i$ for $i\in\mathbb{N}$. We let $2x$ occur in both $T_{(2,i)}$ and $T_{3,i}$ with probability 0. So this transformation $2x$ will in fact never occur, but it just helpful to satisfy the conditions of Inoue's theorem, see section 3.2. Note that we can write for the last two transformations:

$$T_{(2,i)}(x)=\begin{cases}\dfrac{3}{S_1^{i+1}(x)}-1 & \text{if } x\in S_1^{-1}J_i\\ 2x & \text{otherwise}\end{cases}$$
$$T_{(3,i)}(x)=\begin{cases}\dfrac{3}{S_1^{i}(x)}-2 & \text{if } x\in S_1^{-1}I_i\\ 2x & \text{otherwise}\end{cases}$$

Given a probability vector (p_0,p_1,p_2) on$\{0,1,2\}$ we define a probability vector for the transformations $T_0,T_1,T_{(2,i)},T_{(3,i)},i\in\mathbb{N}$. We set:

$$\mathbb{P}(T_0)=p_0 \tag{6.1.25}$$
$$\mathbb{P}(T_1)=p_2 \tag{6.1.26}$$
$$\mathbb{P}(T_{(2,i)})(x)=p_1^{i+1}\mathbf{1}_{S_1^{-1}J_i}(x) \tag{6.1.27}$$
$$\mathbb{P}(T_{(3,i)})(x)=p_1^{i}(1-p_1)\mathbf{1}_{S_1^{-1}I_i}(x) \tag{6.1.28}$$

Now let $x\in J_i$, so $x\in I_k$ for $1\le k\le i$. Then it follows that

$$p_0+p_2+p_1^{i+1}+(1-p_1)\sum_{j=1}^{i}p_1^j=p_0+p_2+p_1^{i+1}+p_1-p_1^{i+1}=1.$$

Hence we see that for each $x \in [0,1]$ we get a well defined probability vector. Now we can give an explicit expression for the transformation K.

$$K(\omega,x) = \begin{cases} (\sigma(\omega), T_0(x)) & \text{if } \omega_1 = 0 \\ (\sigma^2(\omega), T_1(x)) & \text{if } \omega_1 = 2 \\ (\sigma^{i+1}(\omega), T_{(2,i)}(x)) & \text{if } \omega_j = 1\ \forall 1 \le j \le i+1 \text{ and } x \in S_1^{-1}(J_i) \\ (\sigma^{i+1}(\omega), T_{(3,i)}(x)) & \text{if } \omega_j = 1\ \forall 1 \le j \le i,\ \omega_{i+1} \in \{0,2\} \text{ and } x \in S_1^{-1} I_n \end{cases} \tag{6.1.29}$$

.

We can define digits for K in a similar way we did in chapter 2, namely:

$$d_i(\omega,x) = \begin{cases} k & \text{if } \omega_1 = 0 \text{ and } x \in (\frac{3}{k+1}, \frac{3}{k}] \\ (k-2,1) & \text{if } \omega_1 = 2 \text{ and } x \in (\frac{3}{k+1}, \frac{3}{k}] \\ (k-1, \underbrace{1,\cdots,1}_{n\ times}, 1) & \text{if } \omega \in [\underbrace{1,\cdots,1}_{n\ times}] \text{ and } x \in (\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} J_n \\ (k-1, \underbrace{1,\cdots,1}_{n-1\ times}, 2) & \text{if } \omega \in [\underbrace{1,\cdots,1}_{n\ times}, 0] \cup [\underbrace{1,\cdots,1}_{n\ times}, 2] \\ & \text{and } x \in (\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} I_n \end{cases} \tag{6.1.30}$$

Notice that it makes no sense to define fundamental intervals by its digit sequences. Suppose we have a digit $(3,1)$, then we do not know if this is $(5-2,1)$, so $x \in [2] \times (\frac{3}{6}, \frac{3}{5}]$ or $(4-1,1)$,so $x \in [1] \times (\frac{3}{5}, \frac{3}{4}] \cap S_1^{-1}(J_0)$. Hence we can not deduce which ω is used. However we could define a generating partition as we shall see in section 6.3.

6.1.3 Existence of an invariant measure

Again we will apply theorem 3.2.6 in section 3.2. We start with defining interval partitions for each maps $T_0, T_1, T_{2,i}, T_{3,i}$. We set

$$\{I_{0,k}\} = \{I_{2,k}\} = \{(\frac{3}{k}, \frac{3}{k+1}], k \in \mathbb{N}, k \ge 3\}$$

and

$$\{I_{(2,i),k}\} = \{(\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} J_i, (\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} J_i)^c,\ k \in \mathbb{N}, k \ge 3\},$$

$$\{I_{(3,i),k}\} = \{(\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} I_i, (\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} I_i)^c,\ k \in \mathbb{N}, k \ge 3\}.$$

Where $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} I_i)^c$ and $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} J_i)^c$ denote the intervals in the complement of $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} I_i)$ and $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} J_i)$. We check whether the functions are monotone and C_1 on the intervals. To do this we compute the derivatives of the T_i. We find:

$$\frac{dT_0}{dx} = \frac{-3}{x^2}$$

$$\frac{dT_1}{dx} = \frac{9}{(3-(k-2)x)^2}$$

$$\frac{dT_{(2,i)}}{dx} = \begin{cases} \frac{dS_0 \circ S_1^{i+1}(x)}{dx} = \frac{d}{dx} \frac{3}{S_1^{i+1}(x)} - 1 = \frac{-3}{(S_1^{i+1}(x))^2} \frac{dS_1^{i+1}(x)}{dx} & \text{if } x \in S_1^{-1} J_i \\ 2 & \text{otherwise} \end{cases}$$

$$\frac{dT_{(3,i)}}{dx} = \begin{cases} \frac{dS_0 \circ S_1^{i}(x)}{dx} = \frac{-3}{S_1^{i}(x)^2} \frac{dS_1^{i}(x)}{dx} & \text{if } x \in S_1^{-1} I_i \\ 2 & \text{otherwise.} \end{cases}$$

From this it immediately follows that T_0 is monotone and C_1 on the intervals $\{I_0, k\}$. The same holds for T_1 since for each $k \geq 3, k \in \mathbb{N}$, we see that $T_1'(x) > 0$ and continuous on $(\frac{3}{k+1}, \frac{3}{k}]$. $T_{(2,i)}(x)$ is a composition of monotone functions on $J_{(2,i),k}$ and hence monotone, see the end of section 6.1.1. The derivative is a product of compositions of continuous function on $J_{(2,i),k}$ and therefore continuous. By the same reasoning we see that the functions $T_{(3,i)}$ are continuously differentiable and monotone on the intervals $\{I_{(3,i),k}\}$.

For condition 2 we have to show that the functions

$$g(t,x) = \frac{p_t}{|T_t(x)'|},$$

for $t \in \{0, 1, (2,i), (3,i), i \in \mathbb{N}\}$ satisfy

$$\sup_{x \in [0,1]} \sum_t g(t,x) < \infty.$$

Since

$$\frac{dT_0}{dx} = \frac{-3}{x^2}$$

it follows that

$$g(0,x) = \frac{p_0 x^2}{3} < \frac{p_0}{3}$$

for all $x \in [0,1]$. For $t = 1$,

$$g(1,x) = \sum_{k \geq 3}^{\infty} \frac{p_2(3-(k-2)x)^2}{9} \mathbf{1}_{(\frac{3}{k+1}, \frac{3}{k}]}$$

and hence for $x \in (\frac{3}{k+1}, \frac{3}{k})$ we find using the monotonicity of T_2 on $(\frac{3}{k+1}, \frac{3}{k}]$:

$$\begin{aligned}
\frac{p_2(3-\frac{3(k-2)}{k})^2}{9} &< g(1,x) < \frac{p_2(3-\frac{3(k-2)}{k+1})^2}{9} \\
\frac{p_2(\frac{6}{k})^2}{9} &< g(1,x) < \frac{p_2(\frac{9}{k+1})^2}{9} \\
\frac{4p_2}{k^2} &< g(1,x) < \frac{9p_2}{(k+1)^2} \\
& \quad\; g(1,x) < \frac{9}{16} p_2.
\end{aligned}$$

Where in the last equation we used that $k \geq 3$ if $x \in [0,1]$.

Now we look at the functions $g((2,i),x)$, which are given b.

$$g((2,i),x) = \begin{cases} \frac{p_1^{i+1}}{|\frac{d}{dx}\frac{3}{S_1^{i+1}(x)} - 1|} & \text{if } x \in S_1^{-1} J_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$\frac{d}{dx}\frac{3}{S_1^{i+1}(x)} - 1 = \frac{-3}{(S_1^{i+1}(x))^2}\frac{d}{dx}S_1^{i+1}(x) \tag{6.1.31}$$

$$= \frac{-3}{(S_1^{i+1}(x))^2}\frac{-3}{(S_1^{i}(x))^2}\frac{d}{dx}S_1^{i}(x) \tag{6.1.32}$$

$$\vdots \tag{6.1.33}$$

$$= \frac{-3}{(S_1^{i+1}(x))^2} \cdots \frac{-3}{(S_1(x))^2}\frac{d}{dx}S_1(x) \tag{6.1.34}$$

$$= \frac{-3}{(S_1^{i+1}(x)))^2} \cdots \frac{-3}{(S_1(x))^2}\frac{-3}{x^2}. \tag{6.1.35}$$

Hence

$$g((2,i),x) = \frac{p_1^{i+1}(S_1^{i+1}(x))^2 \cdots (S_1(x))^2 x^2}{3^{i+2}}.$$

Since $x \in S_1^{-1}J_i \cap [0,1]$, $S_1^j(x) \in (1, 1\frac{1}{2})$ for $1 \leq j \leq i$ and $S_1^{i+1} \in (1\frac{1}{2}, 2)$ we obtain

$$g((2,i),x) < \frac{4p_1^{i+1} \cdot (\frac{9}{4})^i}{3^{i+2}} = \frac{4p_1^{i+1}}{9}\left(\frac{3}{4}\right)^i.$$

Finally we consider

$$g((3,i),x) = \frac{p_1^i(1-p_1)}{|\frac{d}{dx}\frac{3}{S_1^i(x)} - 2|}\mathbf{1}_{I_i}.$$

Note that

$$\frac{d}{dx}\frac{3}{S_1^i(x)} - 2 = \frac{-3}{(S_1^i(x))^2} \cdot \frac{-3}{(S_1^{i-1})^2} \cdots \frac{-3}{(S_1(x))^2}\frac{-3}{x^2}$$

and again $S_1^i \in (1, 1\frac{1}{2})$ and $x \in [0,1]$. Hence:

$$\begin{aligned} g((3,i),x) &= \frac{p_1^i(1-p_1)(S_1^i(x))^2 \cdots (S_1(x))^2 \cdot x^2}{3^{i+1}} \\ &< \frac{p_1^i(1-p_1)(\frac{9}{4})^i}{3^{i+1}} \\ &< \frac{p_1^i(1-p_1)}{3}\left(\frac{3}{4}\right)^i. \end{aligned}$$

Therefore we can compute $\sup_{x\in[0,1]} \sum_t g(t,x)$,

$$\begin{aligned} \sup_{x\in[0,1]} \sum_t g(t,x) = \sup_{x\in[0,1]} &(g(0,x) + g(1,x) + \sum_{i\in\mathbb{N}} g((2,i),x)\mathbf{1}_{J_i}(x) \\ &+ \sum_{i\in\mathbb{N}} g((3,i),x)\mathbf{1}_{I_i}(x)) \\ <&\frac{p_0}{3} + \frac{9}{16}p_2 + \sup_{i\geq 0}\frac{4p_1^{i+1}}{9}\left(\frac{3}{4}\right)^i + \sup_{i\geq 1}\sum_{j=1}^{i}\frac{p_1^j(1-p_1)}{3}\left(\frac{3}{4}\right)^j \\ =&\frac{p_0}{3} + \frac{9}{16}p_2 + \frac{4p_1}{9} + \sup_{i\geq 1}\frac{1-p_1}{3}\frac{3p_1}{4}\frac{1-(\frac{3p_1}{4})^i}{1-\frac{3p_1}{4}} \\ <&\frac{p_0}{3} + \frac{9}{16}p_2 + \frac{4p_1}{9} + \frac{p_1}{3} \\ <&\frac{1}{3}p_0 + \frac{9}{16}p_2 + \frac{7}{9}p_1 \\ <&1. \end{aligned}$$

Finally we have to check that the functions $g(t,x)$ are of bounded variation and that their variation can be bound uniformly. Since $g(0,x)$ is monotone on $[0,1]$, it is of bounded variation. For $g(1,x)$ we already saw for $x \in (\frac{3}{k+1}, \frac{3}{k})$ that

$$\frac{4p_2}{k^2} < \quad g(1,x) \quad < \frac{9p_2}{(k+1)^2} \tag{6.1.36}$$

Since $g(1,x)$ is continuous and monotone on $(\frac{3}{k+1}, \frac{3}{k})$ we obtain:

$$\bigvee_{[0,1]} g(1,x) = \sum_{k=3}^{\infty}\frac{9p_2}{(k+1)^2} - \frac{4p_2}{k^2} < \infty. \tag{6.1.37}$$

Where the convergence follows from the convergence of $\sum_{n\in\mathbb{N}} \frac{1}{n^2}$.

We will proof that we can bound the variation of the functions $g_{(2,i)}(x)$ uniformly. Recall

$$g((2,i),x) = \frac{p_1^{i+1}(S_1^{i+1}(x))^2 \cdots (S_1(x))^2 x^2}{3^{i+1}} \mathbf{1}_{S_1^{-1}J_i}.$$

We will apply theorem 2.3.3 which tells us

$$\bigvee_{[a,b]} (f \cdot g) \leq A \bigvee_{[a,b]} f + B \bigvee_{[a,b]} g \tag{6.1.38}$$

where $A = \sup\{|g(x)| : x \in [a,b]\}$ and $B = \sup\{|f(x)| : x \in [a,b]\}$. We set

$$f = \frac{p_1^i(S_1^i(x))^2 \cdots (S_1(x))^2}{3^{i+1}} \text{ and let } h(x) = x^2,$$

so $g((2,i),x) = f(x) \cdot h(x)$. By 6.1.38 it follows that:

$$\bigvee_{[0,1]} g_{(2,i)}(x) = \sum_{k\geq 2} \bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} g_{(2,i)}(x) \tag{6.1.39}$$

$$\leq \sum_{k\geq 2} A \bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} f(x) + B \bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} h(x) \tag{6.1.40}$$

So here

$$A = \sup\left\{|x^2| : x \in \overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}\right\} \leq \frac{9}{k^2}$$

and

$$B = \sup\left\{|f(x)| : x \in \overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}\right\}.$$

Note that $(S_1^l(x))^2 \leq \frac{9}{4}$ for $1 \leq l \leq i$ and $S_1^{i+1}(x) \leq 4$ since we are in J_i. Therefore

$$\sup\left\{\left|\frac{p_1^i(S_1^i(x))^2 \cdots (S_1(x))^2}{3^{i+1}}\right| : x \in \overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}\right\} \leq \frac{4 \cdot (\frac{9}{4})^i \cdot p_1^i}{3^{i+1}} < \frac{4}{3}.$$

We also know

$$\bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} h(x) < \frac{9}{k^2}$$

since $h(x) = x^2$ is monotone on $\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}$. Hence our estimate is reduced to

$$\bigvee_{S_1^{-1}J_i} g_{(2,i)}(x) \leq \quad \sum_{k\geq 2} \frac{9}{k^2} \bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} f(x) + \frac{4}{3}\frac{9}{k^2}. \tag{6.1.41}$$

Therefore we only have to show that we can bound $\bigvee_{\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}} f(x)$ uniformly for all i. To do so we use lemma 2.3.6. $S_1(x)$ is a monotone function on $\overline{S_1^{-1}J_i \cap (\frac{3}{k+1},\frac{3}{k}]}$, since it is monotone on $(\frac{3}{k+1},\frac{3}{k}]$. We write

$$f(x) = f(S_1(x)) = \frac{p_1^{i+1}(S_1^i(S_1(x)))^2 \cdots (S_1(x))^2}{3^{i+2}}.$$

Let $S_1(x) = y$ then it follows that

$$\bigvee_{S^{-1}J_i \cap (\frac{3}{k+1}, \frac{3}{k}]} f(x) \leq \bigvee_{J_i} f(y) = \bigvee_{J_i} \frac{p_1^{i+1}(S_1^i(y))^2 \cdots (y)^2}{3^{i+2}}.$$

Therefore we see that we can estimate $\bigvee_{S^{-1}J_i \cap (\frac{3}{k+1}, \frac{3}{k}]} f(x)$ independent of k.

Recall that by definition of J_i we have

$$S_1(J_i) = J_{i-1},\ S_1^2(J_i) = S_1^1(J_{i-1}) = J_{i-2}$$

and so on. So $S_1^l(J_i) = J_{n-l}$. $S_1(x)$ is monotone on $[1, \frac{3}{2}]$ and therefore $S_1^l(x)$ is monotone on J_i, since it is a composition of continuous functions and $S_1^l(J_i) \subset [1, \frac{3}{2}]$ for all $1 \leq l \leq i-1$. Therefore it also follows that $(S_1^l(y))^2 \leq \frac{9}{4}$ for all $y \in J_i$ and $0 \leq l \leq n-1$. Now we are going to use

$$\bigvee \prod_{l=1}^{i} f_l \leq C \sum_{l=1}^{i} \bigvee f_l,$$

for a finite family of uniformly bounded f_l and $C \in \mathbb{R}$, see lemma 2.3.4. Let

$$f_l = (S_1^{i-1}(y))^2$$

for $1 \leq l \leq i$. We have $S_1^l(J_i) = J_{i-l}$, and S_1^l is monotone and positive on J_i. Let a, b be the endpoints of the interval J_{i-l} , then we obtain

$$\bigvee_{J_i} (S_1^l)^2 = |a^2 - b^2| = |a-b||a+b| \leq 3\lambda(J_i).$$

Hence we need to know $\lambda(J_i)$.

Lemma 6.1.4. *For the intervals I_n as defined before we have $\lambda(I_n) \leq 3 \cdot (\frac{3}{4})^n$ and $\lambda(J_n) \leq 3 \cdot (\frac{3}{4})^{n-1}$.*

Proof. Recall I_n is an interval with endpoints $\frac{p_n+p_{n-1}}{q_n+q_{n-1}}$ and $\frac{p_n+1\frac{1}{2}p_{n-1}}{q_n+1\frac{1}{2}q_{n-1}}$. Hence

$$\lambda(I_n) = \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n + 1\frac{1}{2}p_{n-1}}{q_n + 1\frac{1}{2}q_{n-1}} \right| \tag{6.1.42}$$

$$= \left| \frac{(p_n + p_{n-1})(q_n + 1\frac{1}{2}q_{n-1}) - ((p_n + 1\frac{1}{2}p_{n-1})(q_n + q_{n-1}))}{(q_n + q_{n-1})(q_n + 1\frac{1}{2}q_{n-1})} \right| \tag{6.1.43}$$

$$= \left| \frac{p_{n-1}q_n + p_n 1\frac{1}{2}q_{n-1} - p_n q_{n-1} - 1\frac{1}{2}p_{n-1}q_n}{(q_n + q_{n-1})(q_n + 1\frac{1}{2}q_{n-1})} \right| \tag{6.1.44}$$

$$= \frac{\frac{1}{2}|\det M_n|}{(q_n + q_{n-1})(q_n + 1\frac{1}{2}q_{n-1})} \leq 3 \cdot \frac{3^{n-1}}{4^{n-1}}. \tag{6.1.45}$$

Where in the last step we used that $\det M_n = (-3)^n$ and $q_n \geq 4^{\lfloor \frac{n}{2} \rfloor}$. Hence we conclude $I_n \leq 3 \cdot (\frac{3}{4})^{n-1}$. Since $J_n = I_n \backslash I_{n+1}$ it follows immediatly that $\lambda(J_n) \leq 3 \cdot (\frac{3}{4})^n$. □

Therefore $\bigvee_{J_i}(S_1^{i-1}(y))\cdots(S_1(y))^2y^2 \leq (\frac{9}{4})^{i-1}\sum_{l=1}^{i-1} 9\cdot(\frac{3}{4})^l$. We have

$$f = \left(\frac{p_1^{i+1}(S_1^i(y))^2}{3^{i+2}}\right)\cdot((S_1^{i-1}(y))^2\cdots(S_1(y))^2y^2)$$

and

$$\sup_{J_i}((S_1^{i-1}(y))\cdots(S_1(y))^2y^2) \leq (\frac{9}{4})^{i-1}$$

and

$$\sup_{J_i}\left(\frac{p_1^{i+1}(S_1^i(y))^2}{3^{i+2}}\right) \leq \left(\frac{4p_1^{i+1}}{3^{i+2}}\right)$$

and by monotonicity also

$$\bigvee_{J_i}\left(\frac{p_1^{i+1}(S_1^i(y))^2}{3^{i+2}}\right) \leq \left(\frac{4p_1^{i+1}}{3^{i+2}}\right).$$

Using 6.1.38 again we find:

$$\bigvee_{J_i} f \leq 4\cdot\frac{p_1^{i+1}}{3^{i+1}}(\frac{9}{4})^{i-1} + 4\frac{p_1^{i+1}}{3^{i+1}}(\frac{9}{4})^{i-1}\sum_{l=1}^{i-1} 9\cdot(\frac{3}{4})^l \tag{6.1.46}$$

$$\leq 4\cdot\frac{p_1^{i+1}}{3^{i+1}}(\frac{9}{4})^{i-1} + 4\frac{p_1^{i+1}}{3^{i+1}}(\frac{9}{4})^{i-1}\cdot\frac{9}{1-\frac{3}{4}}. \tag{6.1.47}$$

Hence if $i\to\infty$ we see that $\bigvee_{J_i} f \to 0$. Therefore we can bound the function $(g(2,i))$ uniformly by a constant M.

In the same way we can bound the variation of the functions $g(3,i)$ uniformly and therefore all conditions of the Inoue theorem are satisfied. Hence we find an invariant measure μ_p for K, which is absolutely continuous with respect to the Lebesgue measure and which is of bounded variation. In the next section we will consider this measure μ_p.

6.2 Properties of the invariant measure

We will show now some properties of the invariant measure in the case $N = 3$. We will mostly copy the results of the case $N = 2$ to obtain the results for the case $N = 3$.

By the article of Inoue Inoue [2012] we find an absolute continuous density h_p with respect to the Lebesque measure, so we can write,

$$\begin{aligned}
\mu_p(A) := &\int_{[0,1]} (p_0\mathbf{1}_A(T_0(x)) + p_2\mathbf{1}_A(T_1(x)) + p_1\mathbf{1}_{S_1-1J_0}(x)\mathbf{1}_A(T_{(2,0)}(x)) \\
&+ \sum_{i=1}^{\infty}\left[p_1^{i+1}\mathbf{1}_{S_1^{-1}J_i}(x)\mathbf{1}_A(x)(T_{(2,i)}(x)) + (1-p_1)p_1^{i+1}\mathbf{1}_{S_1^{-1}I_i}(x)\mathbf{1}_A(T_{(3,i)}(x))\right] h_p d\lambda \\
= &\, p_0\mu_p(T_0^{-1}A) + p_2\mu_p(T_1^{-1}A) + p_1\mu_p(T_{(2,0)}^{-1}A\cap S_1^{-1}J_0) \\
&+ \sum_{i=1}^{\infty}\left[p_1^{i+1}\mu_p(T_{(2,i)}^{-1}A\cap S_1^{-1}J_i) + (1-p_1)p_1^i\mu_p(T_{(3,i)}^{-1}A\cap S_1^{-1}I_i)\right].
\end{aligned}$$

Where we have used monotone convergence to switch the sum and integral. Let $[\omega_1, \cdots, \omega_n]$ be a cylinder in Ω and $(a,b) \subset [0,1]$ an interval. The cylinders $[\omega_1, \cdots, \omega_n] \times (a,b)$ generate $\sigma(\mathcal{C} \times \mathcal{B}([0,1]))$. Note that

$$\begin{aligned}
K^{-1}([\omega_1, \cdots, \omega_n] \times (a,b)) = & [0, \omega_1, \cdots, \omega_n] \times T_0^{-1}(a,b) \cup [2, \omega_1, \cdots, \omega_n] \times T_1^{-1}(a,b) \\
& \cup [1, \omega_1, \cdots, \omega_n] \times T_{(2,0)}^{-1}(a,b) \cap S_1^{-1} J_0 \\
& \cup \bigcup_{i=1}^{\infty} [\underbrace{1, \cdots, 1}_{i+1 \ times}, \omega_1, \cdots, \omega_n] \times T_{(2,i)}^{-1}((a,b)) \cap S_1^{-1} J_i \\
& \cup [\underbrace{1, \cdots, 1}_{i \ times}, 0, \omega_1, \cdots, \omega_n] \times T_{(3,k)}^{-1}((a,b)) \cap S_1^{-1} I_i \\
& \cup [\underbrace{1, \cdots, 1}_{i \ times}, 2, \omega_1, \cdots, \omega_n] \times T_{(3,k)}^{-1}((a,b)) \cap S_1^{-1} I_i .
\end{aligned}$$

In the same way as in section 4.4 we find:

$$\begin{aligned}
& (m_p \times \mu_p) \left(K^{-1}([\omega_1, \cdots, \omega_n] \times (a,b)) \right) \\
& \quad = p_0 m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_0^{-1}(a,b)) + p_2 m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_1^{-1}(a,b)) \\
& \qquad + p_1 m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_1^{-1}(a,b) \cap S_1^{-1} J_1) \\
& \qquad + \sum_{i=0}^{\infty} p_1^{i+1} m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_{(2,i)}^{-1}(a,b) \cap S_1^{-1} J_i) \\
& \qquad + \sum_{i=0}^{\infty} p_0 p_1^k m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_{(3,k)}^{-1}(a,b) \cap S_1^{-1} J_i) \\
& \qquad + \sum_{i=0}^{\infty} p_2 p_1^k m_p([\omega_1, \cdots, \omega_n]) \mu_p(T_{(3,k)}^{-1}(a,b) \cap S_1^{-1} J_i) \\
& \quad = m_p([\omega_1, \cdots, \omega_n]) \times m_p((a,b)).
\end{aligned}$$

To obtain more properties of the measure $m_p \times \mu_p$ we construct the random Perron Frobenius operator with respect to the random transformation T. Recall the interval partitions we made.

$$\{I_{0,k}\} = \{I_{2,k}\} = \{(\frac{3}{k}, \frac{3}{k+1}], k \in \mathbb{N}, k \geq 3\}$$

and

$$\{I_{(2,i),k}\} = \{(\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} J_i, (\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} J_i)^c, \ k \in \mathbb{N}, k \geq 3\},$$

$$\{I_{(3,i),k}\} = \{(\frac{3}{k+1}, \frac{3}{k}] \cap S_1^{-1} I_i, (\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} I_i)^c, \ k \in \mathbb{N}, k \geq 3\}.$$

Where $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} I_i)^c$ and $(\frac{3}{k+1}, \frac{3}{k}] \cap (S_1^{-1} J_i)^c$ denote the intervals in the complement. Let $J_{(i,k)} = S_1^{-1} J_i \cap (\frac{3}{k+1}, \frac{3}{k})$ and $I_{(i,k)} = S_1^{-1} I_i \cap (\frac{3}{k+1}, \frac{3}{k})$ Using these intervals we obtain the following Perron-Frobenius operator,

$$\begin{aligned} P_T f(y) = & \sum_{k=3}^{\infty} p_0 f(T_{(0,k)}^{-1}(y)) \left| \frac{1}{T'_{(0,k)}(T_{(0,k)}^{-1}(y))} \right| \\ & + p_0 f(T_{(1,k)}^{-1}(y)) \left| \frac{1}{T'_{(1,k)}(T_{(1,k)}^{-1}(y))} \right| \mathbf{1}_{[0,\frac{1}{2}]}(y) \\ & + \sum_{i=0}^{\infty} \mathbf{1}_{J_{i,k}}(T_{((2,i),k)}^{-1}(y)) p_1^{i+1} f(T_{((2,i),k)}^{-1}(y)) \left| \frac{1}{T'_{(1,k)}(T_{(1,k)}^{-1}(y))} \right| \mathbf{1}_{T_{(2,i)}(\mathrm{int}(J_{i,k}))}(y) \\ & + \sum_{i=0}^{\infty} \mathbf{1}_{I_{i,k}}(T_{((3,i),k)}^{-1}(y)) p_1^{i}(1-p_1) f(T_{((3,i),k)}^{-1}(y)) \left| \frac{1}{T'_{(1,k)}(T_{(1,k)}^{-1}(y))} \right| \mathbf{1}_{T_{(3,i)}(\mathrm{int}(I_{i,k}))}(y). \end{aligned}$$

By theory of Perron Frobenius operators we can also construct $P_{T^n} f(y)$. Working out $P_{T^n} f(y)$ would be rather tedious, and in fact we do not need a explicit construction of $P_{T^n} f(y)$. What we need is that for a positive function f,

$$P_{T^n} f(y) = \sum_{k=3}^{\infty} p_0^n f(T_{(0,k)}^{-n}(y)) \left| \frac{1}{T_{(0,k)}^{n'}(T_{(0,k)}^{-n}(y))} \right| + \textit{other positive terms} \quad (6.2.1)$$

Let us see whether we can use the proofs of chapter 2. We change proposition 4.4.1 a little bit to obtain the following proposition.

Proposition 6.2.1. *Let $I \subset [0,1]$ be a non-trivial open interval. Then there exists an $\omega \in \Omega$, such that $(0,1) \subset (T_{\omega_n} \circ \cdots \circ T_{\omega_1})I \subset [0,1]$.*

Proof. Let $J \subset [0,1]$ be a non-trivial open interval and write $J = (c,d)$. Suppose $\exists k \in \mathbb{N}$, such that $\frac{1}{k} \in (c,d)$. Then $T_0(c,d) = [0,a) \cup (b,1)$ and $T_0^2(c,d) = [0,1)$. Therefore it is enough to show that for $J \subset [0,1]$ a non-trivial open interval there exists an $n \in \mathbb{N}$ such that $\frac{1}{k} \in T^n(c,d)$. Notice that

$$\lambda(T_0(c,d)) = \frac{3}{c} - \frac{3}{d} = \frac{3(c-d)}{cd} > \lambda(c,d).$$

Hence in the same way as in proposition 4.4.1 of chapter 4 we find an $n \in \mathbb{N}$ such that $T_0^n(c,d) = [0,1)$. Therefore each $\omega \in [\underbrace{0,\cdots,0}_{n \textit{ times}}]$ can be the ω of the proposition. ☐

Using the properties of the Perron Frobenius operator we copy the proof of proposition 4.4.2 in chapter 4 to obtain the following result.

Proposition 6.2.2. *Let h_p be the probability density function from Theorem 3.2.6, then $h_p > 0$.*

We also obtain again proposition 4.4.4.

Proposition 6.2.3. *The density function h_p is bounded from above and form below.*

And therefore we conclude.

Corollary 6.2.4. *The measure μ_p is equivalent to the Lebesgue measure and there exists a $c \in \mathbb{R}$ such that for all$B \in \mathcal{B}$ we have $c\lambda(B) < \mu_p(B) < \frac{1}{c}\lambda(B)$.*

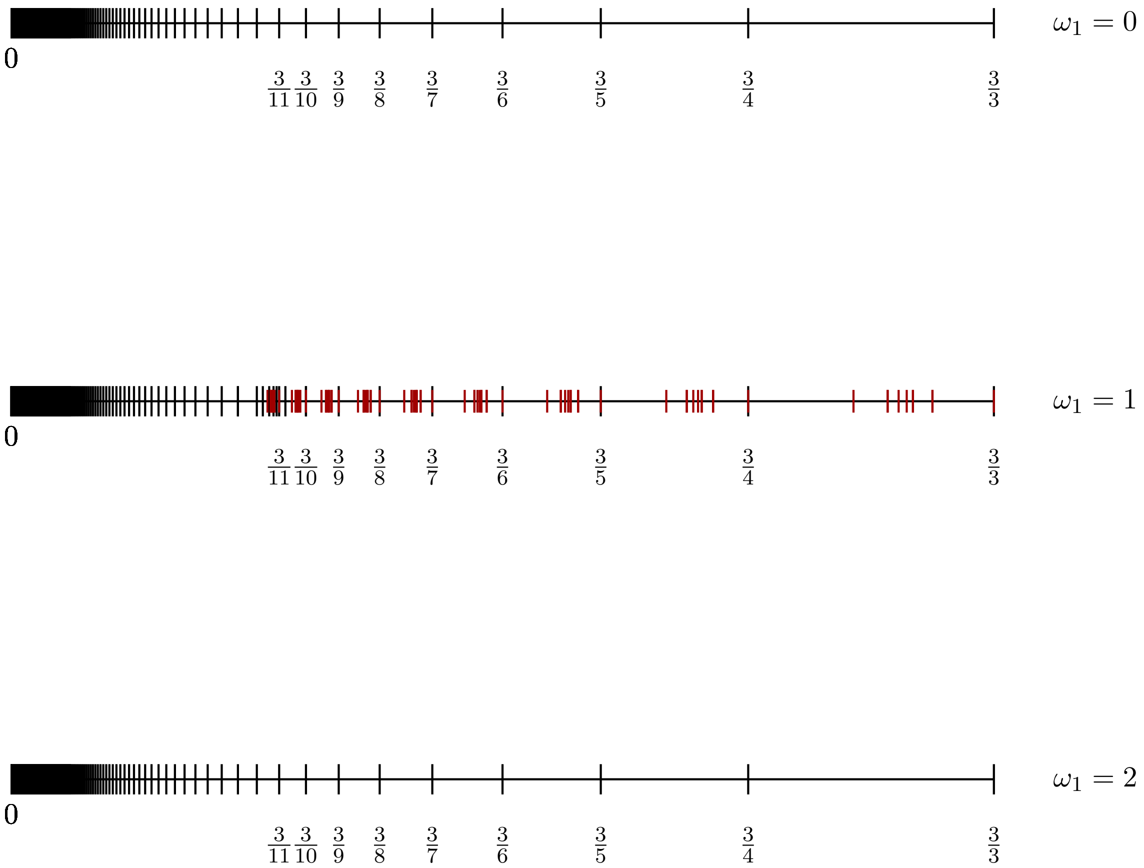

Figure 6.2 – Partition of $\Omega \times [0, 3]$

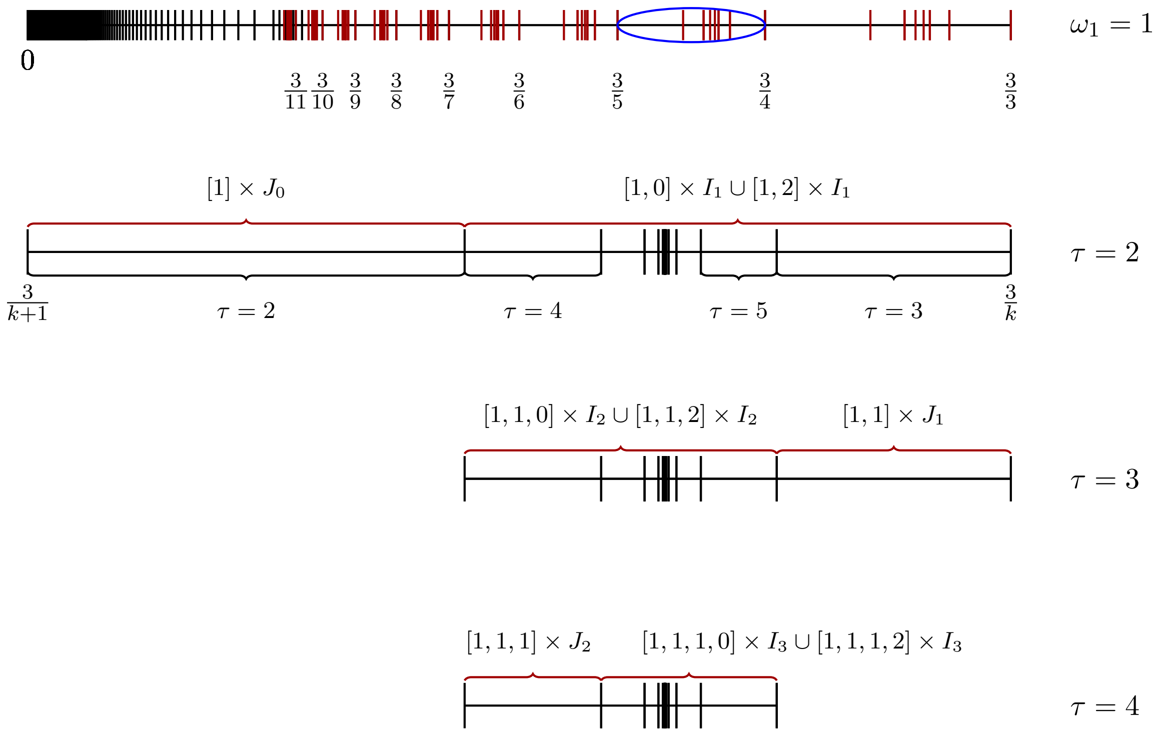

Figure 6.3 – A more detailed picture of the partition of $[1] \times [0, 1]$, we enlarged the area in the blue circle

6.3 Ergodic properties in the case $N = 3$

Let us see whether we can define fundamental intervals as we did in the case $N = 2$. In case $N = 2$ we defined these intervals by the digits of K and we could find ω from the digits of K. Suppose we like to do this for the case $N = 3$ and we encounter an fundamental interval $(3, 1)$. Now we never know whether we started with $(\omega, x) \in [2] \times (\frac{3}{6}, \frac{3}{5}]$ or we started in $(\omega, x) \in [1] \times (\frac{3}{5}, \frac{3}{4}] \cap S_1^{-1} J_0$. Therefore these intervals do not give a nice generating partition of our space $\Omega \times [0, 1]$. Hence we will define another "fundamental" partition $\mathcal{P}$ to generate our space. Our partition $\mathcal{P}$ is defined as follows,

$$\begin{aligned}
\mathcal{P} =& \{[0] \times (\frac{3}{k+1}, \frac{3}{k}], k \in \mathbb{N},\ k \geq 3\} \cup \{[2] \times (\frac{3}{k+1}, \frac{3}{k}], k \in \mathbb{N},\ k \geq 3\} \\
& \cup \{\underbrace{[1, 1, \cdots, 1]}_{i+1\ times} \times S_1^{-1} J_i \cap (\frac{3}{k+1}, \frac{3}{k}], i \in \mathbb{N}_{n \geq 0}, k \in \mathbb{N},\ k \geq 3\} \\
& \cup \{[\underbrace{1, \cdots, 1}_{i\ times}, 0] \times S_1^{-1} I_i \cap (\frac{3}{k+1}, \frac{3}{k}], i \in \mathbb{N}, k \in \mathbb{N},\ k \geq 3\} \\
& \cup \{[\underbrace{1, \cdots, 1}_{i\ times}, 2] \times S_1^{-1} I_i \cap (\frac{3}{k+1}, \frac{3}{k}], i \in \mathbb{N}, k \in \mathbb{N},\ k \geq 3\}.
\end{aligned}$$

This partition is shown in figure 6.2. Note that

$$|\mathcal{P}| = |\mathbb{N}| + |\mathbb{N}| + |\mathbb{N} \times \mathbb{N}| + |\mathbb{N} \times \mathbb{N}| + |\mathbb{N} \times \mathbb{N}| = \mathbb{N},$$

so we have a countable partition. Using this partition we define the cylinders as follows. Cylinders of length one are the elements of $\mathcal{P}$, the cylinders of length n are the elements of:

$$\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P} = \mathcal{P} \vee K^{-1}\mathcal{P} \vee \cdots \vee K^{n-1}\mathcal{P} \tag{6.3.1}$$

An element of $\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}$ is of the form $A_0 \cap K^{-1} A_1 \cap \cdots \cap K^{n-1} A_{n_1}$, where $A_i \in \mathcal{P}$ for $0 \leq i \leq n-1$. Notice that on each set $A_i \in \mathcal{P}$ we have that K is monotone and C^1. Also $\tau(\omega, x)$, takes only one value. In fact this partition just incorporates the interval partitions $\{I_{(0,k)}\}$, $\{I_{(1,k)}\}$, $\{I_{(2,i),k}\}$ and $\{I_{(3,i),k}\}$. Moreover we have that each $A_i \in \mathcal{P}$ has the form of a product set $[\omega]_m \times (a, b)$ for $a, b \in [0, 1]$ and $m \in \mathbb{N}$. So m denotes the number of coordinates of $[\omega]$ that are fixed. For example if $A = [0] \times (\frac{3}{k+1}, \frac{3}{k}]$ then $m = 1$, but if $A = [1, 1, 1, 1, 1] \times S_1^{-1}(J_4) \cap (\frac{3}{k+1}, \frac{3}{k}]$ then $m = 5$. Let B_n be a set of the form $A_0 \cap K^{-1} A_1 \cap \cdots \cap K^{-n} A_n$, where each $A_i \in \mathcal{P}$. For each $(\omega, x) \in A_i$, $A_i \in \mathcal{P}$ the return time $\tau(\omega, x)$ is fixed and also $\tau(\omega, x)$ digits are fixed. Suppose we know $(\omega, x) \in B_n$, what do we know about the digits of (ω, x) induced by K? Notice these are the same digits as induced by R only with some parentheses. If for example $(\omega, x) \in [2] \times (\frac{3}{k+1}, \frac{3}{k}]$, then there are two digits "fixed", namely $d_1(\omega, x) = k - 2$ and $d_2(\omega, x) = 1$. If $(\omega, x) \in [2] \times (\frac{3}{k_1+1}, \frac{3}{k_1}) \cap K^{-1}([1, 0] \times (S_1^{-1} I_1 \cap (\frac{3}{k_2+1}, \frac{3}{k_2}]))$, then $d_1(\omega, x) = k_1 - 2$, $d_2(\omega, x) = 1$, $d_3(\omega, x) = k_2 - 1$ and $d_4(\omega, x) = 2$, so 4 digits are fixed. In general for each $(\omega, x) \in B_n$ $\tilde{n}(\omega, x) = \sum_{i=0}^{n-1} \tau(K^i(\omega, x))$, is fixed and hence $\tilde{n}$ digits are fixed. This is because each time we apply R we obtain a digit and $\tau(\omega, x)$ is just the number of times we apply R. Finally we notice that B_n is an intersection of unions of product sets. However since A_0 is a product set and K is monotone on A_0, so we see B_n is a product set itself. Hence we can write $B_n = [\omega]_m \times (a, b)$ for $a, b \in [0, 1]$. Can we say something about m? At least we know that the number of ω fixed is more then the number of times K is applied, because K shifts at least one time. On the other hand it is smaller then the number of times R is applied, because R does not shift ω each time. Hence we obtain $n \leq m \leq \tilde{n}$.

6.3.1 Length of the interval

Let B_n be defined as above. If $B_n = [\omega]_m \times [a, b]$, can we say something about $[a, b]$? We already saw that B_n fixes $\tilde{n}$ digits. Hence using the transformation R and the recursion relations we can see how large the interval $[a, b]$ is with respect to the Lebesgue measure. Since $K(\omega, x) \in [0, 1]$ we obtain these intervals are intervals with endpoints $\frac{p_{\tilde{n}}}{q_{\tilde{n}}}$ and $\frac{p_{\tilde{n}}+p_{\tilde{n}-1}}{q_{\tilde{n}}+q_{\tilde{n}-1}}$. So the Lebesgue measure of such an interval will be

$$\left| \frac{p_{\tilde{n}}}{q_{\tilde{n}}} - \frac{p_{\tilde{n}} + p_{\tilde{n}-1}}{q_{\tilde{n}} + q_{\tilde{n}-1}} \right| = \frac{3^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}} + q_{\tilde{n}-1})}.$$

Notice

$$\lim_{\tilde{n}\to\infty} \frac{3^{\tilde{n}}}{q_{\tilde{n}}(q_{\tilde{n}} + q_{\tilde{n}-1})} = 0,$$

since the $q_{\tilde{n}} \geq 4^{\lfloor \frac{\tilde{n}}{2} \rfloor}$ sequence. Therefore the intervals associated with K will go to zero if $n \to \infty$.

6.3.2 Generating properties of the partition

In this section we will use some shorter notation for sets of our partition $\mathcal{P}$. We denote by $I_{i,k} = S_1^{-1}(I_i) \cap (\frac{3}{k+1}, \frac{3}{k}]$ and by $J_{i,k} = S_1^{-1}(J_i) \cap (\frac{3}{k+1}, \frac{3}{k}]$. Let $\mathcal{C}$ denote the cylinder sets in $\{0, 1, 2\}^{\mathbb{N}}$ and $\mathcal{B}[0, 1]$ the Borel-σ-algebra restricted to $[0, 1]$. Note that $\mathcal{C} \times [0, 1] \subset \sigma(\bigvee_{i=0}^{\infty} K^{-i}\mathcal{P})$, by taking suitable unions. By theorem 4.5.1, we also have $\Omega \times \mathcal{B}[0, 1] \subset \sigma(\bigvee_{i=0}^{\infty} K^{-i}\mathcal{P})$. Using that σ-algebra's are closed under taking intersections it follows that $\mathcal{C} \times \mathcal{B} \subset \sigma(\bigvee_{i=0}^{\infty} K^{-i}\mathcal{P})$. Since $(\bigvee_{i=0}^{\infty} K^{-i}\mathcal{P}) \subset \sigma(\mathcal{C} \times \mathcal{B})$ we conclude that $\bigvee_{i=0}^{\infty} K^{-i}\mathcal{P}$ is a generating partition. Like lemma 4.5.2, we can estimate the elements of $\sigma(\mathcal{C} \times \mathcal{B})$ by finite unions of $\bigvee_{i=0}^{\infty} K^{-1}\mathcal{P}$. We show that $X = \Omega \times [0, 1]$ can be estimated for some $I, K \in \mathbb{N}$ by

$$\bigcup_{k\geq 3}^{K} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}]$$
$$\cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1 times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{i times}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{i times}, 2] \times I_{i,k}.$$

To see this, let $\epsilon > 0$.

$$
\begin{aligned}
&m_p \times \mu_p(\Omega \times [0,1] \backslash \bigcup_{k\geq 3}^{\infty} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}] \\
&\qquad \cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1 times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 2] \times I_{i,k}) \\
&= m_p \times \mu_p(\bigcup_{k\geq 3}^{\infty} \bigcup_{i=I}^{\infty} \underbrace{[1, \cdots, 1, 1]}_{i+1 times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 2] \times I_{i,k}) \\
&= \sum_{i=I}^{\infty} m_p \times \mu_p(\bigcup_{k\geq 3}^{\infty} \underbrace{[1, \cdots, 1, 1]}_{i+1 times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 2] \times I_{i,k}) \\
&\leq \sum_{i=I}^{\infty} p_1^i
\end{aligned}
$$

Since $p_1 < 1$ the sum $\sum_{i=1}^{\infty} p_1^i$ converges and therefore we can choose I such that $\sum_{i=I}^{\infty} p_1^i < \epsilon/4$. Hence

$$
\begin{aligned}
&m_p \times \mu_p(\Omega \times [0,1] \Delta \bigcup_{k\geq 3}^{\infty} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}] \\
&\qquad \cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1 times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{itimes}, 2] \times I_{i,k}) < \epsilon/4
\end{aligned}
$$

Now we can find K as desired in the following way

$$
\begin{aligned}
m_p \times \mu_p(\bigcup_{k\geq 3}^{\infty} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}]
&\\
\cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1\ times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 2] \times I_{i,k}
&\\
\Delta \bigcup_{k\geq 3}^{K} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}]
&\\
\cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1\ times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 2] \times I_{i,k})
&\\
= m_p \times \mu_p(\bigcup_{K}^{\infty} [0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}]
&\\
\cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1\ times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 2] \times I_{i,k})
&\\
\leq \sum_{K}^{\infty} m_p \times \mu_p([0] \times (\frac{3}{k+1}, \frac{3}{k}] \cup [2] \times (\frac{3}{k+1}, \frac{3}{k}]
&\\
\cup \bigcup_{i=1}^{I} \underbrace{[1, \cdots, 1, 1]}_{i+1\ times} \times J_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 0] \times I_{i,k} \cup \underbrace{[1, \cdots, 1}_{i\ times}, 2] \times I_{i,k})
&\\
\leq \sum_{k=K}^{\infty} 3\mu_p((\frac{3}{k+1}, \frac{3}{k})
&\\
= 3\mu_p[0, \frac{3}{K}].
\end{aligned}
$$

Since the μ_p is equivalent to the Lebesgue measure it follows that by choosing K large enough, we get $\mu_p[0, \frac{3}{K}] < \epsilon/4$. Finally using

$$m_p \times \mu_p(A\Delta C) \leq m_p \times \mu_p(A\Delta B) + m_p \times \mu_p(B\Delta C),$$

we see that we indeed can estimate $\Omega \times [0, 1]$ by a finite union. Estimating the same way, we can proceed the proof like we did in lemma 4.5.2.

Also lemma 4.5.3 holds true by the same proof. Hence we are left to show that the cylinders are mixing. However the proof in case $N = 2$ can also be generalized to $N = 3$. Since the essence of the proof is the same we state just the result here.

Proposition 6.3.1. *The map K is mixing.*

6.4 Invariant measure for R

Like we did in the case $N = 2$ we can find an invariant measure for R in terms of the invariant measure for K. Recall this measure is given by.

$$\rho(E) = \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega, x) dm_p \times \mu_p(\omega, x)} \sum_{n=0}^{\infty} m_p \times \mu(\{(\omega, x) \in \Omega \times [0, 1]; \tau(\omega, x) > n\} \cap R^{-n}(E))$$

We check whether this measure is well defined. Using the product property of $m_p \times \mu_p$ and $\mu_p(J) \leq 1$ for $J \in \mathcal{B}[0,1]$ we can write:

$$\begin{aligned}
&\int_{\Omega\times[0,1]} \tau(\omega,x)dm_p \times \mu_p \\
&\quad= \int_{[0]\times[0,1]} \tau(\omega,x)dm_p \times \mu_p + \int_{[1]\times[0,1]} \tau(\omega,x)dm_p \times \mu_p \\
&\qquad+ \int_{[2]\times[0,1]} \tau(\omega,x)dm_p \times \mu_p \\
&\quad= p_0 + 2p_2 + \int_{[1]\times[0,1]} \tau(\omega,x)dm_p \times \mu_p \\
&\quad= p_0 + 2p_2 + \sum_{n=2}^{\infty} n \cdot (m_p \times \mu_p(S_1^{-1}J_{n-2} \times [\underbrace{1,\cdots,1}_{n-1\ times}]) \\
&\qquad m_p \times \mu_p(S_1^{-1}I_{n-1} \times [\underbrace{1,\cdots,1}_{n-1\ times},0]) + m_p \times \mu_p(S_1^{-1}I_{n-1} \times [\underbrace{1,\cdots,1}_{n-1\ times},2])) \\
&\quad= p_0 + 2p_2 + \sum_{n=2}^{\infty} np_1^{n-1}\mu_p(S_1^{-1}J_{n-2}) + n(1-p_1)p_1^{n-1}\mu_p(S_1^{-1}I_{n-1}) \\
&\quad\leq p_0 + 2p_2 + \sum_{n=2}^{\infty} np_1^{n-1} + n(1-p_1)p_1^{n-1} \\
&\quad< \infty.
\end{aligned}$$

Therefore $\rho(E)$ is well defined. We will show that ρ is ergodic.

Proposition 6.4.1. *The measure ρ is ergodic with respect to the transformation R.*

Proof. By theorem 2.1.9 we have to prove that $\rho\left(\bigcup_{k\geq 0} R^{-k}(\Omega \times [0,1])\right) = 1$. Each time $(\omega, x) \in \Omega \times [0, \frac{3}{2}]$, R shifts ω one coordinate to the left, but if $(\omega, x) \in [\frac{3}{2}, 3]$ R does not shift ω. When we apply R j times, the number of times ω is shifted equals $m(j) = \sum_{k=0}^{j-1} \mathbf{1}_{\Omega\times[0,\frac{3}{2}]}(R^k(\omega,x))$. Since $R^{k-1}(\omega,x) \in \Omega \times [\frac{3}{2}, 3]$ implies $R^k(\omega,x) \in \Omega \times [0,1]$ we obtain $\frac{j}{2} \leq \tilde{n}(j) \leq j$.
Consider the set $\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}$. We have

$$m_p(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}) \leq (1-p_0)^n$$

for all $n \in \mathbb{N}$, so $\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}$ is a m_p-null set.
Let $(\omega, x) \in \{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}^c \times [0,3]$, so there exists a smallest i such that $\omega_i = 0$. Then there exists a j such that $m(j) = i$. Hence

$$R^j(\omega,x) = (\sigma^i(\omega), S_0(\pi_2(R(\omega,x))) \in \Omega \times [0,1],$$

since $S_0[0,3] = [0,1]$. We conclude $(\omega,x) \in \bigcup_{k\geq 0} R^{-k}(\Omega \times [0,1])$ and hence

$$\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}^c \times [0,3] \subset \bigcup_{k\geq 0} R^{-k}(\Omega \times [0,1]).$$

Now

$$\begin{aligned}
&\rho(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\}^c \times [0,3]) \\
&\quad = 1 - \rho(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3]) \\
&\quad = 1 - \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x)dm_p \times \mu_p(\omega,x)}. \\
&\sum_{n=0}^{\infty} m_p \times \mu_p(\{(\omega,x) \in \Omega \times [0,1]; \tau(\omega,x) > n\} \cap R^{-n}(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3])).
\end{aligned}$$

Note

$$\begin{aligned}
R^{-n}(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3]) &= \{(\omega,x) : R^n(\omega,x) \in \{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3]\} \\
&= \{(\omega,x) : \omega_i \neq 0 \text{ for } i > m(n),\ x \in [0,3]\}.
\end{aligned}$$

Hence

$$m_p \times \mu_p(R^{-n}(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3]))) = 0$$

and therefore by definition of ρ we see $\rho(\{\omega : \forall\ i \in \mathbb{N}\ \omega_i \neq 0\} \times [0,3]) = 0$. We conclude that $\rho\left(\bigcup_{k\geq 0} R^{-k}(\Omega \times [0,1])\right) = 1$ and hence that ρ is ergodic with respect to R.

□

Finally we show that we can integrate by ρ as follows:

$$\begin{aligned}
\rho(f) = &\frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x)dm_p \times \mu_p(\omega,x)}[\int_{\Omega\times[0,1]} f(x)dm_p \times d\mu_p \\
&+ \int_{[1]\cup[2]\times[0,1]} f(R(\omega,x))dm_p \times \mu_p \\
&+ \sum_{n=2}^{\infty} \int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}I_n} f(R^n(\omega,x))dm_p \times d\mu_p].
\end{aligned}$$

Like we did in the case $N = 2$ we will prove this by "standard machinary". Let

f be an indicator function then:

$$
\begin{aligned}
\rho(E) &= \int_{\Omega\times[0,3]} \mathbf{1}_E d\rho \\
&= \frac{\sum_{n=0}^{\infty} m_p \times \mu_p(\{(\omega,x) \in \Omega\times[0,1]; \tau(\omega,x) > n\} \cap R^{-n}(E))}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} \\
&= \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} [\int_{\Omega\times[0,1]} \mathbf{1}_E(x) dm_p \times d\mu_p \\
&\quad + \int_{[1]\cup[2]\times[0,1]} \mathbf{1}_E(R(\omega,x)) dm_p \times d\mu_p \\
&\qquad + \sum_{n=2}^{\infty} \int_{\{(\omega,x):\tau(\omega,x)>n\}} \mathbf{1}_E(R^n(\omega,x)) dm_p \times d\mu_p] \\
&= \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} [\int_{\Omega\times[0,1]} \mathbf{1}_E(x) dm_p \times d\mu_p \\
&\quad + \int_{[1]\cup[2]\times[0,1]} \mathbf{1}_E(R(\omega,x)) dm_p \times \mu_p \\
&\qquad + \sum_{n=2}^{\infty} \int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1} I_n} \mathbf{1}_E(R^n(\omega,x)) dm_p \times d\mu_p].
\end{aligned}
$$

In particular the sum in the last equation is finite, since the integrals in the sum can be dominated by p_1^n. By linearity of the integral the result holds also for finite simple functions. Now let $f \in \mathcal{L}^1(\rho)$ be a positive function, then we can find a sequence of positive finite simple functions such that $f_{k+1} \geq f_k$ and $\lim_{k\to\infty} f_k = f$. Using the same strategy as in the case $N = 2$ we obtain by monotone convergence:

$$
\begin{aligned}
\infty > \rho(f) &= \lim_{k\to\infty} \rho(f_k) \\
&= \lim_{k\to\infty} \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} [\int_{\Omega\times[0,1]} f_k(x) dm_p \times d\mu_p \\
&\quad + \int_{[1]\cup[2]\times[0,1]} f_k(R(\omega,x)) dm_p \times \mu_p \\
&\quad + \sum_{n=2}^{\infty} \int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1} I_n} f_k(R^n(\omega,x)) dm_p \times d\mu_p] \\
&= \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} [\int_{\Omega\times[0,1]} f(x) dm_p \times d\mu_p \\
&\quad + \int_{[1]\cup[2]\times[0,1]} f(R(\omega,x)) dm_p \times \mu_p \\
&\quad + \lim_{k\to\infty} \sum_{n=2}^{\infty} \int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1} I_n} f_k(R^n(\omega,x)) dm_p \times d\mu_p].
\end{aligned}
$$

Hence we are left to show:

$$\lim_{k\to\infty}\sum_{n=2}^{\infty}\int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} f_k(R^n(\omega,x))dm_p \times d\mu_p$$
$$= \sum_{n=2}^{\infty}\int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} f(R^n(\omega,x))dm_p \times d\mu_p.$$

Since $f \to f_k$ $\rho - a.e$, there exists an set $Y \subset \Omega \times [0,3]$ such that for all $(\omega,x) \in Y$ we have $\lim_{k\to\infty} f_k(\omega,x) = f(\omega,x)$ and $\rho(Y) = 1$. Note that for all $n \in \mathbb{N}$ and for all $(\omega,x) \in R^{-n}(Y^c)$ we have $\lim_{k\to\infty} f_k(R(\omega,x)) = f(R(\omega,x))$. Since ρ is an R-invariant measure $\rho(Y) = \rho(R^{-n}Y) = 1$. Therefore $\lim_{k\to\infty} f_k \circ R^n = f \circ R^n \rho - a.e.$. If $\rho(Y) = 1$, then $\rho(Y^c) = 0$ so

$$\frac{1}{\int_{\Omega\times[0,1]}\tau(\omega,x)dm_p \times \mu_p(\omega,x)}\sum_{n=0}^{\infty} m_p\times\mu_p(\{(\omega,x)\in\Omega\times[0,1];\tau(\omega,x)>n\}\cap R^{-n}(Y^c)) = 0.$$

Hence

$$m_p \times \mu_p([\underbrace{1,\cdots,1}_{n\ times}] \times S_1^{-1}(I_n) \cap R^{-n}(Y^c)) = 0$$

for all $n \in \mathbb{N}$. Therefore

$$\lim_{n\to\infty} f_k(\omega,x) = f(\omega,x)\ \rho - a.e.$$

implies

$$\lim_{k\to\infty} f_k(R^n(\omega,x))\mathbf{1}_{[\underbrace{1,\cdots,1}_{n\ times}\times S_1^{-1}(I_n)]}$$
$$= f(R^n(\omega,x))\mathbf{1}_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)}\ m_p \times \mu_p - a.e.$$

Let ν denote the counting measure. Then we can write:

$$\lim_{k\to\infty}\sum_{n=2}^{\infty}\int_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} f_k(R^n(\omega,x))dm_p \times d\mu_p$$
$$= \lim_{k\to\infty}\int_{\mathbb{N}}\int_{\Omega\times[0,1]} f_k(R^n(\omega,x))\mathbf{1}_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} dm_p \times \mu_p d\nu.$$

Let

$$G_k = \int_{\Omega\times[0,1]} f_k(R^n(\omega,x))\mathbf{1}_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} dm_p \times \mu_p$$

and

$$G = \int_{\Omega\times[0,1]} f(R^n(\omega,x))\mathbf{1}_{[\underbrace{1,\cdots,1}_{n\ times}]\times S_1^{-1}(I_n)} dm_p \times \mu_p.$$

Since $f_k \uparrow f \ m_p \times \mu_p - a.e.$ we have by monotone convergence that $G_k \uparrow G$. Applying monotone convergence once more yields

$$\lim_{k\to\infty} \int_{\mathbb{N}} G_k(n) d\nu = \int_{\mathbb{N}} G(n) d\nu. \tag{6.4.1}$$

Inserting the definition of G we obtain:

$$\lim_{k\to\infty} \sum_{n=2}^{\infty} \int_{\underbrace{[1,\cdots,1]}_{n\ times}\times S_1^{-1}(I_n)} f_k(R^n(\omega,x)) dm_p \times d\mu_p \tag{6.4.2}$$

$$= \sum_{n=2}^{\infty} \int_{\underbrace{[1,\cdots,1]}_{n\ times}\times S_1^{-1}(I_n)} f(R^n(\omega,x)) dm_p \times d\mu_p. \tag{6.4.3}$$

Suppose $f \in \mathcal{L}^1$, then we can write $f = f^+ - f^-$, for f^+ and f^- both positive functions. By linearity of the integral we obtain the result for f. Therefore we conclude that

$$\begin{aligned}\rho(f) =& \frac{1}{\int_{\Omega\times[0,1]} \tau(\omega,x) dm_p \times \mu_p(\omega,x)} [\int_{\Omega\times[0,1]} f(x) dm_p \times d\mu_p \\ &+ \int_{[1]\cup[2]\times[0,1]} f(R(\omega,x)) dm_p \times \mu_p \\ &+ \sum_{n=2}^{\infty} \int_{\underbrace{[1,\cdots,1]}_{n\ times}\times S_1^{-1} I_n} f(R^n(\omega,x)) dm_p \times d\mu_p].\end{aligned}$$

6.5 Existence of the $\lim_{n\to\infty} \log q_n$

We show that also in the case $N = 3$, the limit $\lim_{n\to\infty} \frac{\log q_n}{n}$ exists.

Proposition 6.5.1. *Let q_n be the denominators of the partial fraction $\frac{p_n}{q_n}$ induced by the transformation R, see section 6.1. Then $\lim_{n\to\infty} \frac{\log q_n}{n} < \infty$ exists and is finite.*

Proof. We can proof this exactly in the same way as we did for the case $N = 2$, since we have propositions 6.1.1 and 6.1.2. The only difference is that we do not estimate the terms of the rest term $r(n,\omega,x)$ by the Fibonnacci sequence, but we estimate them by the geometric series with common ratio $\frac{3}{4}$. Hence we are left to compute the integral $\int_{\Omega\times[0,3]} \log(\pi_2(\omega,x)) d\rho$.

$$\begin{aligned}&\int_{\Omega\times[0,3]} \log(\pi_2(\omega,x)) d\rho = \\ &\int_{\Omega\times[0,1]} \log(\pi_2(\omega,x)) dm_p \times d\mu_p + \int_{[1]\cup[2]\times[0,1]} \log(\pi_2(R(\omega,x))) dm_p \times d\mu_p \\ &\quad + \sum_{n=1}^{\infty} \int_{\underbrace{[1,\cdots,1]}_{n\ times}\times S_1^{-1}(I_n)} \log(\pi_2(R^n(\omega,x))) dm_p \times d\mu_p \\ &= \int_{\Omega\times[0,1]} \log(x) d\mu_p + \int_{[0,1]} p_1 \log(S_1(x)) + p_2 \log(S_2(x)) d\mu_p \\ &\quad + \sum_{n=1}^{\infty} \int_{S_1^{-1} I_n} p_1^n \log(S_1^n(x)) d\mu_p.\end{aligned}$$

Notice that if $x \in [0,1]$ then $S_1(x) \in [1,2]$ and $S_2(x) \in [2,3]$, therefore $0 \leq \log(S_1(x)) \leq \log 2$ and $\log 2 \leq \log(S_2(x)) \leq \log 3$. If $x \in I_n$ then $S_1^n(x) \in [1, 1\frac{1}{2}]$ and therefore $0 \leq \log(S_1^n(x)) \leq \log 1\frac{1}{2}$. Using this and $c\lambda < \mu_p < C\lambda$, we see:

$$\begin{aligned}
-C &\leq \int_{\Omega\times[0,3]} \log(\pi(\omega,x))d\rho \\
&\leq -c + p_1 \log 2 + p_2 \log 3 + C\sum_{n=1}^{\infty} \int_{S_1^{-1}I_n} p_1^n \log\left(\frac{3}{2}\right) d\lambda \\
&\leq -c + p_1 \log 2 + p_2 \log 3 + C\sum_{n=1}^{\infty} p_1^n \log\left(\frac{3}{2}\right) \\
&< \infty.
\end{aligned}$$

Where in the last equation we used $p_1 < 1$, thus $\sum_{n=1}^{\infty} p_1^n \log(\frac{3}{2})$ converges. □

Proposition 6.5.2. *Let q_n be the denominators of the partial fractions $\frac{p_n}{q_n}$ induced by the transformation K, see section 6.1. Then $\lim_{n\to\infty} \frac{\log q_n}{n} < \infty$.*

Proof. Since $\int_{\Omega\times[0,1]} \tau(\omega,x)d\rho$ converges, the proof is identically to the proof of the case $N = 2$. □

Let $\mathcal{P}$ be the partition as defined in section 6.3. We define the information function $I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}}$ by:

$$I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}} : \Omega \times [0,1] \to \mathbb{R} \tag{6.5.1}$$

$$I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}}(\omega,x) = \sum_{A\in\bigvee_{i=0}^{n-1} K^{-i}\mathcal{P}} \mathbf{1}_A(\omega,x)\log(m_p \times \mu_p(A)). \tag{6.5.2}$$

Then we can proof the following proposition:

Proposition 6.5.3. $\lim_{n\to\infty} I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}}(\omega,x)$ *exists.*

Again this proof follows in the same way as the proof of 4.6.3 in the case $N = 2$. We already know that an element of $A \in \bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}$, can be written as $A = [\omega]_m \times (a,b)$, where (a,b) is an interval of length $\frac{3^{\tilde{n}}}{q_n(q_n+q_{n-1})}$ and $\tilde{n} = \tilde{n}(\omega,x) = \sum_{i=0}^{n-1} \tau(K^i(\omega,x))$ for $(\omega,x) \in A$. Therefore we can write

$$\lim_{n\to\infty} I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}}(\omega,x) = \lim_{n\to\infty} \log m_p([\omega]_m(\omega)) + \log(\mu_p(a,b)(x)). \tag{6.5.3}$$

For the first term we see:

$$\begin{aligned}
\lim_{n\to\infty} \log m_p([\omega]_m(\omega)) = \lim_{n\to\infty} \frac{m}{n}\frac{1}{m}\sum_{i=0}^{m} &\log(p_0^{\mathbf{1}_{(\omega_1=0)}(\sigma^i(\omega)}) + \log(p_1^{\mathbf{1}_{(\omega_1=1)}(\sigma^i(\omega)}) \\
&+ \log(p_2^{\mathbf{1}_{(\omega_1=2)}(\sigma^i(\omega))}).
\end{aligned}$$

Since $n \leq m \leq \tilde{n}$ we see that $1 < \lim_{n\to\infty} \frac{m}{n} < \lim_{n\to\infty} \frac{\tilde{n}}{n}$. By the Birkhoff ergodic theorem we have

$$\frac{\tilde{n}(\omega,x)}{n} = \frac{1}{n}\sum_{i=0}^{n-1} \tau(K^i(\omega,x)) = \int_{\Omega\times[0,1]} \tau(\omega,x)d(\omega,x) < \infty.$$

Notice that since $n \leq m \leq \tilde{n}$, we have that if $n \to \infty$, then $m \to \infty$. Therefore by the Birkhoff ergodic theorem we see that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m} \log(p_0^{\mathbf{1}_{(\omega_1=0)}(\sigma^i(\omega))}) + \log(p_1^{\mathbf{1}_{(\omega_1=1)}(\sigma^i(\omega))}) + \log(p_2^{\mathbf{1}_{(\omega_1=2)}(\sigma^i(\omega))})$$
$$= \int_\Omega \mathbf{1}_{\{\omega_1=0\}}(\sigma^i(\omega)) \log(p_0) + \mathbf{1}_{\{\omega_1=1\}}(\sigma^i(\omega)) \log(p_1) + \mathbf{1}_{\{\omega_1=2\}}(\sigma^i(\omega)) \log(p_2) dm_p$$
$$= p_0 \log(p_0) + p_1 \log(p_1) + p_2 \log(p_2).$$

The proof of existence of $\lim_{n\to\infty} \log(\mu_p(a,b))$ is the same as the proof of the case $N = 2$ and hence we refer to 4.6.3. Therefore we conclude that

$$\lim_{n\to\infty} \frac{1}{n} I_{\bigvee_{i=0}^{n-1} K^{-1}\mathcal{P}}(\omega, x)$$

exists and is finite.

6.6 Entropy

Using the Shannon-McMillan-Breiman Theorem, 2.1.15 with our generating partition $\mathcal{P}$, see section 6.3 and proposition 6.5.3 we can compute the entropy of the transformation K. We already saw that the partition $\mathcal{P}$ is countable and generates $\sigma(\mathcal{C} \times \mathcal{B}[0,1])$. Hence we have to show that $H(\mathcal{P}) < \infty$.

Proposition 6.6.1. *Let $\mathcal{P}$ be defined as in section 6.1, then $H(\mathcal{P}) < \infty$.*

Proof. Let $I_{i,k} = S_1^{-1}(I_i) \cap (\frac{3}{k+1}, \frac{3}{k}]$ and $J_{i,k} = S_1^{-1}(J_i) \cap (\frac{3}{k+1}, \frac{3}{k}]$. Then

$$H(\mathcal{P}) = -\sum_{[\omega]_m \times (a,b) \in \mathcal{P}} m_p \times \mu_p([\omega]_m \times (a,b)) \log(m_p \times \mu_p([\omega]_m \times (a,b)))$$
$$= -(\sum_{k=1}^{\infty} p_0 \mu_p((\frac{3}{k+1}, \frac{3}{k}]) \log(\mu_p((\frac{3}{k+1}, \frac{3}{k}]))$$
$$+ \sum_{k=3}^{\infty} p_2 \mu_p((\frac{3}{k+1}, \frac{3}{k}]) \log(\mu_p((\frac{3}{k+1}, \frac{3}{k}]))$$
$$+ \sum_{i=0}^{\infty} \sum_{k=3}^{\infty} p_1^{i+1} \mu_p(J_{i,k}) \log(p_1^{i+1} \mu_p(J_{i,k}))$$
$$+ \sum_{i=0}^{\infty} \sum_{k=3}^{\infty} p_1^i p_0 \mu_p(I_{i,k}) \log(p_1^i p_0 \mu_p(I_{i,k}))$$
$$+ \sum_{i=1}^{\infty} \sum_{k=3}^{\infty} p_1^i p_2 \mu_p(I_{i,k}) \log(p_1^i p_2 \mu_p(I_{i,k}))).$$

Using the equivalence of μ_p with the Lebesgue measure we can show

$$\sum_{k\geq 3}^{\infty} \frac{3p_0}{k(k+1)} \log(\frac{3p_0}{k(k+1)}) + \sum_{k\geq 3}^{\infty} \frac{3p_2}{k(k+1)} \log(\frac{3p_2}{k(k+1)}) < \infty$$

in the same way we did for the case $N = 2$, see the proof of proposition 4.7.1. Therefore we are left to show the convergence of the last three sums. Recall that

$$J_{i,k} = \left(S_1^{-}1 J_i \cap (\frac{3}{k+1}, \frac{3}{k}]\right) = \left(\frac{3}{\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}} + (k-1)}, \frac{3}{\frac{p_n+1\frac{1}{2}p_{n-1}}{q_n+1\frac{1}{2}q_{n-1}} + (k-1)}\right)$$

for n even and the other way around for n odd, see section 6.1. Hence by equivalence of μ_p with λ, we can estimate $\mu_p(J_{i,k})$ by

$$
\begin{aligned}
&C \cdot \lambda\left(\frac{3}{\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}} + (k-1)}, \frac{3}{\frac{p_n+1\frac{1}{2}p_{n-1}}{q_n+1\frac{1}{2}q_{n-1}} + (k-1)}\right) \\
&= C \cdot \frac{3(\frac{p_n+1\frac{1}{2}p_{n-1}}{q_n+1\frac{1}{2}q_{n-1}} + (k-1)) - 3(\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}} + (k-1))}{(\frac{p_n+1\frac{1}{2}p_{n-1}}{q_n+1\frac{1}{2}q_{n-1}} + (k-1))(\frac{p_{n-1}+p_{n-2}}{q_{n-1}+q_{n-2}} + (k-1))} \\
&\leq \frac{3C \cdot \lambda(J_n)}{(k-1)^2} < \frac{3 \cdot (\frac{3}{4})^n \cdot C}{(k-1)^2}.
\end{aligned}
$$

Where in the last step we used lemma 6.1.4. Computing the third sum we obtain:

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{k=3}^{\infty} p_1^{i+1}\mu_p(J_{i,k})\log(p_1^{i+1}\mu_p(J_{i,k})) \\
&\leq C \cdot \sum_{k=2}^{\infty}\sum_{i=0}^{\infty} p_1^{i+1}\frac{3 \cdot (\frac{3}{4})^{i+1}}{(k-1)^2}\log(p_1^{i+1}\frac{3 \cdot (\frac{3}{4})^{i+1}}{(k-1)^2}) \\
&= C \cdot \sum_{k=2}^{\infty}\sum_{i=0}^{\infty} p_1^{i+1}\frac{3 \cdot (\frac{3}{4})^{i+1}}{(k-1)^2} \cdot [(i+1)\log(p_1) + \log(3) \\
&\qquad + (i+1)\log(\frac{3}{4}) - 2\log(k-1)] \\
&= C \cdot \sum_{k=2}^{\infty}\frac{1}{(k-1)^2}\sum_{i=0}^{\infty} 3(i+1)\log(p_1) \cdot (\frac{3p_1}{4})^{i+1} \\
&\quad + C \cdot \sum_{k=2}^{\infty}\frac{1}{(k-1)^2}\sum_{i=0}^{\infty} 3\log(3) \cdot (\frac{3p_1}{4})^{i+1} \\
&\quad + C \cdot \sum_{k=2}^{\infty}\frac{1}{(k-1)^2}\sum_{i=0}^{\infty} 3(i+1)\log(\frac{3}{4}) \cdot (\frac{3p_1}{4})^{i+1} \\
&\quad + C \cdot \sum_{k=2}^{\infty}\frac{-2\log(k-1)}{(k-1)^2}\sum_{i=0}^{\infty} 3 \cdot (\frac{3p_1}{4})^{i+1} \\
&= C \cdot \sum_{k=2}^{\infty}\frac{D_1+D_2+D_3}{(k-1)^2} + \frac{-2\log(k-1)}{(k-1)^2} < \infty.
\end{aligned}
$$

Where D_1, D_2, D_3 are the first three sums over i in the second line. Hence we have found convergence of the third sum. The convergence of the fourth and fifth sum can be seen in the same way. Therefore we conclude that our partition $\mathcal{P}$ has finite entropy. □

Theorem 6.6.2. *The transformations K and R have finite entropy.*

Proof. Since $H(\mathcal{P}) < \infty$ we can apply Shannon-McMillan-Breiman. Therefore by proposition 6.5.3 we obtain that the entropy of K is finite. Using Abramov's formula we find the same result for the transformation R. □

6.7 Convergence of the digits

Finally we show some results about the digits induced by the transformation R. It turns out that exactly the same results hold for the case $N = 3$ as for the case $N = 2$.

Proposition 6.7.1. *For $\rho - a.e.(\omega, x) \in \Omega \times [0,1]$, we have*

$$1 < \lim_{n\to\infty} (b_1(x,\omega), \cdots, b_n(x,\omega))^{\frac{1}{n}} < \infty$$

and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} b_i(x,\omega)}{n} = \infty.$$

We start with the geometric mean. From the proof of proposition 4.8.1 we know it is enough to show that $\int_{\Omega\times[0,3]} \log(b_1(\omega,x))d\rho < \infty$. Computing the integral gives:

$$\begin{aligned}
\int_{\Omega\times[0,3]} \log(b_1(\omega,x))d\rho &= \int_{\Omega\times[0,1]} \log(b_1(\omega,x))dm_p \times \mu_p \\
&\quad + \int_{[1]\cap[2]\times[0,1]} \log(b_1(R(\omega,x)))dm_p \times \mu_p \\
&\quad + \sum_{n=2}^{\infty} \int_{\underbrace{[1,1,\cdots,1]}_{n\ times} \times S_1^{-1}(I_n)} \log(b_1(R^n(\omega,x)))dm_p \times d\mu_p \\
&= \int_{\Omega\times[0,1]} \log(b_1(\omega,x))dm_p \times \mu_p \\
&\quad + \int_{[1]\cup[2]\times[0,1]} \log(b_2(\omega,x))dm_p \times \mu_p \\
&\quad + \sum_{n=2}^{\infty} \int_{\underbrace{[1,1,\cdots,1]}_{n\ times} \times S_1^{-1}(I_n)} \log(b_n(\omega,x))dm_p \times d\mu_p.
\end{aligned}$$

For $x \in ([1]\cup[2] \times [0,1])$ we have $b_2(\omega,x) = 1$ and hence second integral becomes zero. For $x \in \underbrace{[1,1,\cdots,1]}_{n\ times} \times S_1^{-1}(I_n)$ we have that $b_i(\omega,x) = 1$, for $2 \le i \le n$ and hence also the last sum of integrals is zero. Therefore it is enough to show the first integral is finite.

$$\begin{aligned}
\int_{\Omega\times[0,3]} \log(b_1(\omega,x))d\rho &= \int_{\Omega\times[0,1]} \log(b_1(\omega,x))dm_p \times \mu_p \\
&= \sum_{k=3}^{\infty} [p_0 \log(k) + p_1 \log(k-1) + p_2 \log(k-2)]\mu_p((\frac{3}{k+1}, \frac{3}{k}]) \\
&\le \sum_{k=3}^{\infty} [p_0 \log(k) + p_1 \log(k-1) + p_2 \log(k-2)]\frac{3\cdot C}{k(k+1)}.
\end{aligned}$$

We see that the geometric mean of the digits induced by R is finite. The proof of $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} b_i(x,\omega)}{n} = \infty$ follows exactly the same way as in the case $N = 2$, see proposition 4.8.1.

Chapter 7

Conclusion

7.1 Conclusion

We have found an invariant measure for the 2-random continued fraction transformation. The powerful tool we used to do this, was constructing an induced transformation which satisfies the conditions of Inoue's existence theorem for invariant measures. In this way we proved that there exists an invariant measure μ_p for the accelerated 2-random continued fraction transformation. We also have shown that μ_p is equivalent with the Lebesgue measure by applying already existent methods form literature to the accelerated 2-random continued fraction transformations. However we did not obtain an explicit form of μ_p. We defined the accelerated 2-random continued fractions as the skew product K on $\{0,1\}^{\mathbb{N}} \times [0,1]$. Endowing $\{0,1\}^{\mathbb{N}}$ with the σ-algebra generated by the cylinders and a product measure m_p, it turned out that $m_p \times \mu_p$ is an invariant product measure for K. Having the measure $m_p \times \mu_p$ for the induced transformations, we lifted $m_p \times \mu_p$ to an invariant measure ρ for the skew product of the original 2-random continued fraction transformation by standard ergodic theory. Having two invariant measures in our toolbox we were able to use Birkhoffs ergodic theorem and the Shannon-Mc-Millan Breiman theorem. Therefore we were in the position to mimic the proofs of several properties of the regular continued fraction transformation to obtain similar results for the 2 -random continued fraction transformation. The method turned out to be generalizable to the 3-continued fractions and we obtained similar results.

So we obtained a lot of results, but there is also a downside. Since we only proved the existence of an invariant measure μ_p and not the explicit form, we only obtained existence results about the asymptotic properties of the 2- and 3-random fraction expansions. Also the obtained results were $m_p \times \mu_p$ almost everywhere or ρ almost everywhere. Hence we did not get information about the behaviour of the 2 and 3-continued fractions on the $m_p \times \mu_p$ and ρ-null sets. However constructing a commuting diagram between the skew product for the 2-random continued fractions and the digit sequences it induces we found the existence of many other invariant measures for the system.

7.2 Further Research

The most obvious question to ask is whether we can extend the theory for the 2- and 3- random continued fractions to a general theory for N-random continued fractions. Therefore there must be investigated whether it is really needed to find an explicit form of the return-time and of one is always able to find such an explicit form. Then it is needed to show that these functions are expanding on average. Of course also the other three conditions of Inoue must be satisfied in this general N-case.

Furthermore we like to know an explicit form of the density μ_p. To get an idea of this, one can do simulations of the density. An other way could be looking to natural extensions of the random system.

Finally one can ask some adjacent question. Let $\Omega = \{0, 1, 2, \cdots, N-1\}^M$ for $M \in \mathbb{N}$ and T_i for $i \in \{0, 1, 2, \cdots, N-1\}$ denote the different N-continued fractions. Can one obtain an invariant measure for the transformation $T_{\omega_1} \circ \cdots \circ T_{\omega_n}$?

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