Random maps and random β-expansions in two dimensions

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Abstract

In this thesis, we show a method to dynamically generate expansions of numbers in an arbitrary base $\beta > 1$ using maps called the lazy and greedy maps. We introduce concepts such as the Frobenius-Perron operator, which we then use to find the unique absolutely continuous invariant measure for the greedy map in the case where the base is equal to the golden mean. We provide some intuition about most concepts and results as they are introduced. We introduce a two-dimensional random map K which simultaneously generates two random β-expansions and show that it can be essentially identified with the left shift. We then find an invariant measure of maximal entropy for K. We introduce a skew product transformation based on K and prove that there exists an absolutely continuous invariant measure. We prove some properties of digit sequences that give a simultaneous expansion of two numbers x and y in bases β_1 and β_2. Finally, we introduce a random map G that generates these sequences, after which we show that it can be essentially identified with the left shift.

Table of contents

List of figures

Chapter 1

Introduction

In this thesis, we consider the theory of β-expansions and random maps. This thesis is structured as follows: chapter 2 introduces the theory of expansions of numbers and gives a method to dynamically generate two special β-expansions called the greedy and lazy expansions. These will be generated by the greedy and lazy maps, respectively. Chapter 3 introduces concepts from measure theory and ergodic theory such as absolutely continuous invariant measures, the Frobenius-Perron operator, measurable isomorphisms and the Ergodic Theorem. These concepts will be applied at the end of Chapter 3 to find the unique absolutely continuous invariant measure for the greedy map in the case where the base is equal to the golden mean. We then use this measure to find the frequency of the digit 0 by applying the Ergodic Theorem. Chapter 4 is the main chapter of this thesis, in which we return to the theory of β-expansions and show a method to dynamically generate all β-expansions by using a random map based on the greedy and lazy maps. The main goal of this thesis is to generalise this method to two dimensions for the case $1 < \beta < 2$. We show that the two-dimensional random map K which generates these random β-expansions can be essentially identified with the left shift and find an invariant measure of maximal entropy for K. We then introduce a skew product transformation based on K and prove that there exists an absolutely continuous invariant measure. We prove some properties of digit sequences that give a simultaneous expansion of two numbers x and y in bases β_1 and β_2. Finally, we introduce a random map G that generates these sequences, after which we show that it can be essentially identified with the left shift.

Much of this thesis is based on scientific publications. One of our aims is to lower the barrier to entry of the theory of random maps and random β-expansions. To do this, we follow the following principles:

- The text is mostly self-contained. assuming a prior knowledge of measure theory but not ergodic theory. We do refer to proofs of theorems in articles if they are too long to be included.
- We give further details that were omitted from the scientific papers we used as references.
- We aim to provide some intuition about concepts and theorems as they are introduced.
- We give an overview of equivalent definitions or representations that differ from one paper to another.
- We discuss a variety of methods found in the literature, for instance several methods of finding so-called absolutely continuous invariant measures.

Chapter 2

β-expansions

2.1 Representations of numbers

The number e has multiple representations or definitions. For example, we can define e as the unique number such that $\frac{d}{dx}e^x = e^x$ for all $x \in \mathbb{R}$. There are also several ways to define it using limits:

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

$$e = \sum_{k=0}^{\infty}\frac{1}{k!} := \lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}$$

$$e = 2+\frac{7}{10}+\frac{1}{10^2}+\frac{8}{10^3}+\frac{2}{10^4}+\frac{8}{10^5}+\frac{1}{10^6}+...$$

In this thesis, we are particularly interested in limits of the last form. This representation of e is usually written as

$$e = 2.718281...$$

In other words, it is the decimal expansion of e. For some numbers, such a decimal expansion is unique, while for others there are two. For instance, we can write the number 1 in two ways:

$$\begin{aligned}1 &= 1\cdot 10^0+\sum_{k=1}^{\infty}\frac{0}{10^k}\\ &= 0\cdot 10^0+\sum_{k=1}^{\infty}\frac{9}{10^k}\end{aligned}$$

In the usual notation, this is written as

$$1 = 1.0000000... = 0.9999999...$$

This means that there are two different ways to approach the number 1 arbitrarily closely by a sum of the form $\sum_{k=0}^{n} \frac{d_k}{10^k}$, i.e. there are two ways to write the number 1 as a limit that has the special form $\sum_{k=0}^{\infty} \frac{d_k}{10^k}$.

The number 10 here is not of particular interest[1]; we could take a different basis like the number 2 instead. After choosing appropriate digits, we obtain the binary expansion.

Note that here we are considering the real numbers to be defined independently of any representation. If we tried to define all real numbers in terms of expansions, we would run into the following problem: suppose we have defined a number x by an expansion $x = \sum_{k=0}^{\infty} \frac{d_k}{r^k}$. The digits and the basis are integers, which are themselves represented in some basis. This only makes sense if the integers are already defined. To define these integers, one might try to give a decimal or binary expansion, but to do this we again need the integers to already be defined. Essentially, an expansion is a way of writing a number in terms of other numbers. This means that as a starting point we need some numbers that are not defined by an expansion. This is resolved by considering the real numbers as defined independently of any representation, for instance using set theory and Dedekind cuts.

We will need the following definition:

Definition 2.1.1. *The floor function is the function* $\lfloor \cdot \rfloor : \mathbb{R}^+ \to \mathbb{N}$,

$$\lfloor x \rfloor = \max\{m \in \mathbb{N} : m \leq x\}.$$

Note that if we can write $x = d_0.d_1d_2d_3...$, then it is not necessarily the case that $\lfloor x \rfloor = d_0$. For example, in base 10,

$$\lfloor 1.9999999... \rfloor = 2 \neq 1.$$

In this thesis, we will consider non-integer bases and focus on so-called β-expansions. These are expansions of the form $x = \sum_{k=1}^{\infty} \frac{d_k}{\beta^k}$, where $\beta > 1$ is a non-integer and $d_k \in$

[1] the ubiquity of the decimal system is often attributed to the fact that humans have 10 fingers; the word 'digit' is a synonym for 'finger'.

$\{0, 1, ..., \lfloor\beta\rfloor\}$. Taking all digits as large as possible (i.e. equal to $\lfloor\beta\rfloor$), we find $\sum_{n=1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^n} = \frac{\lfloor\beta\rfloor}{\beta-1}$. Hence the largest number that can be represented in this way is $\lfloor\beta\rfloor/(\beta-1)$. The smallest number representable this way is 0, which is found by taking all digits equal to 0.

Note that in the integer case, $\lfloor n\rfloor = n$, but the largest digit allowed is $n-1$. This differs from the non-integer case, where the digit $\lfloor\beta\rfloor < \beta$ is allowed. This difference is only superficial: if we write the largest permissible digits as $\lceil n\rceil - 1$ and $\lceil\beta\rceil - 1$, the difference disappears. In principle, many of the results in this thesis also work for integer bases, but the results for integers are well-known and there is often an easier way to prove the results. To emphasise that these results are mainly interesting for the non-integer case, we will henceforth consider β to be a non-integer.

While a number can have at most two expansions in an integer base, in non-integer bases there can be infinitely many. Out of the possibly infinitely many β-expansions for a given number x, two are particularly interesting: the greedy and the lazy expansions. As we will see, these are respectively the lexicographically largest and smallest β-expansions. We can generate the greedy and lazy expansions using measurable maps respectively called the greedy and the lazy maps. We shall see in Chapter 4 that any β-expansion can be generated by a random map based on these two maps.

2.2 The greedy and lazy expansions

This section is based on [3], [4] and [5].

In this section, we introduce the greedy and lazy maps. Furthermore, we show how they each generate a β-expansion, which we respectively call the greedy and lazy expansions. Finally, we give an alternative way to obtain the greedy and lazy expansions which explains their names.

Definition 2.2.1. *Let $\beta > 1$ be a non-integer. The greedy map is the map $T_\beta : \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \to \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$,*

$$T_\beta(x) = \begin{cases} \beta x \pmod 1 & \textit{if } \; 0 \le x < 1 \\ \beta x - \lfloor\beta\rfloor & \textit{if } \; 1 \le x \le \lfloor\beta\rfloor/(\beta-1) \end{cases}$$

Equivalently,

$$T_\beta(x) = \begin{cases} \beta x - d & \text{if } x \in C(d) := \left[\frac{d}{\beta}, \frac{d+1}{\beta}\right), \ d \in \{0,1,...,\lfloor\beta\rfloor - 1\} \\ \beta x - \lfloor\beta\rfloor & \text{if } x \in C(\lfloor\beta\rfloor) := \left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \end{cases}$$

Consider the first definition of the greedy map. The map is piecewise linear, and the second part of the map extends the final line piece of the first part. To see this, consider the interval $[\frac{\lfloor\beta\rfloor}{\beta}, 1)$. On this interval, βx will have $\lfloor\beta\rfloor$ subtracted from it by the mod function, which is the same number subtracted on $[1, \lfloor\beta\rfloor/(\beta-1)]$. As the extension has the same slope, we conclude that that T_β is differentiable at $x = 1$, and the final line piece is indeed extended. In the second definition of the greedy map, instead of defining the final line piece in two parts, we immediately define it on the entire interval $\left[\frac{\lfloor\beta\rfloor}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ (see Figure 2.1).

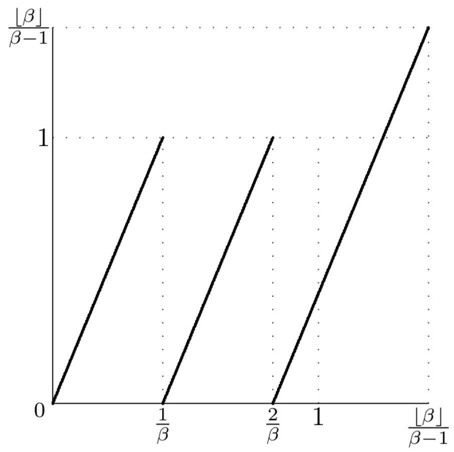

Fig. 2.1 Plot of the greedy map T_β with $\beta = \sqrt{2}+1 \approx 2.41421$. Source: [3].

Definition 2.2.2. *Let $\beta > 1$ be a non-integer. The lazy map is the map $S_\beta : \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \to \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$,*

$$S_\beta(x) = \begin{cases} \beta x & \text{if } x \in \Delta(0) := \left[0, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)}\right] \\ \beta x - d & \text{if } x \in \Delta(d) := \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d-1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{d}{\beta}\right], \ d \in \{1,2,...,\lfloor\beta\rfloor\} \end{cases}$$

(see Figure 2.2)

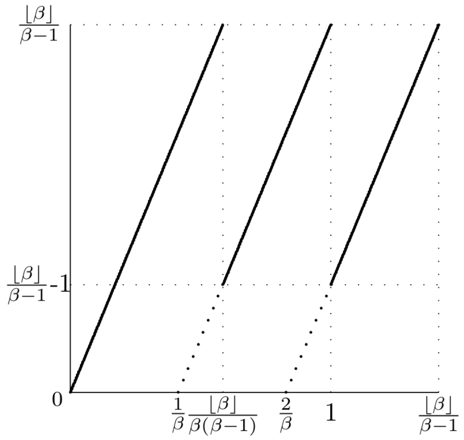

Fig. 2.2 Plot of the lazy map S_β with $\beta = \sqrt{2}+1 \approx 2.41421$. Source: [3].

We now show how to generate the greedy and lazy expansions using iterations of these maps. The digits of the greedy expansion of $x \in [0, \frac{1}{\beta-1}]$ are defined as $d_i(x) = d$ if and only if $T_\beta^{i-1}(x) \in C(d)$. Equivalently, we set $d_i(x) = \lfloor \beta T_\beta^{i-1}(x) \rfloor$.

Similarly, the digits of the lazy expansion of $x \in [0, \frac{1}{\beta-1}]$ are defined as $d_i'(x) = d'$ if and only if $S_\beta^{i-1} \in \Delta(d')$.

We now show that the greedy digits indeed define a β-expansion of x. We can write

$$T_\beta(x) = \beta x - d_1(x) \text{ and } d_n = d_1(T_\beta^{n-1}(x)).$$

Rewriting the first equation, we obtain

$$x = \frac{d_1(x)}{\beta} + \frac{T_\beta(x)}{\beta}.$$

Applying this equation to $T_\beta(x)$, we find

$$T_\beta(x) = \frac{d_1(T_\beta(x))}{\beta} + \frac{T_\beta^2(x)}{\beta} = \frac{d_2(x)}{\beta} + \frac{T_\beta^2(x)}{\beta}.$$

By the first equation,

$$x = \frac{d_1(x)}{\beta} + \frac{d_2(x)}{\beta^2} + \frac{T_\beta^2(x)}{\beta^2}.$$

Repeating this argument, we obtain for all $n \geq 1$:

$$x = \sum_{i=1}^{n} \frac{d_i(x)}{\beta^i} + \frac{T_\beta^n(x)}{\beta^n}.$$

As $0 \leq T_\beta^n(x) \leq \frac{\lfloor \beta \rfloor}{\beta - 1}$ and $\beta > 1$, the remainder term

$$x - \sum_{i=1}^{n} \frac{d_i(x)}{\beta^i} = \frac{T_\beta^n(x)}{\beta^n} \to 0 \text{ for } n \to \infty.$$

We conclude that $x = \sum_{i=1}^{\infty} \frac{d_i(x)}{\beta^i}$.

This proof also works for the lazy expansion, mutatis mutandis.

Example 2.2.3. *Let* $x = \frac{2}{3}$ *and* $\beta = 1.5$. *Then* $\Delta(0) = [0, \frac{1}{\beta(\beta-1)}] = [0, \frac{4}{3}]$ *and* $\Delta(1) = [\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}] = (\frac{4}{3}, 2]$. *Then*

$$\begin{aligned}
x &= 2/3 \in \Delta(0) \\
S_\beta(x) &= 1 \in \Delta(0) \\
S_\beta^2(x) &= 1.5 \in \Delta(1) \\
S_\beta^3(x) &= 1.25 \in \Delta(0) \\
S_\beta^4(x) &= 1.875 \in \Delta(1) \\
S_\beta^5(x) &= 1.8125 \in \Delta(1) \\
S_\beta^6(x) &= 1.71875 \in \Delta(1) \\
S_\beta^7(x) &= 1.57813 \in \Delta(1) \\
S_\beta^8(x) &= 1.36719 \in \Delta(1) \\
S_\beta^9(x) &= 1.05078 \in \Delta(0) \\
S_\beta^{10}(x) &= 1.57617 \in \Delta(1).
\end{aligned}$$

Hence the lazy expansion of $\frac{2}{3}$ *in base 1.5 is 0.00101111101...*

We now consider what the effect is on the digits when two numbers are mapped to the same number by the greedy map. The i-th digit of the greedy expansion is determined by the $i-1$-th iteration of T_β, where $T_\beta^0 x = x$. If for some numbers x, y we have $T_\beta(x) = T_\beta(y)$ and $x \neq y$, then x and y are in different intervals, i.e. $x \in C(d) \neq C(d') \ni y$, so the first digit assigned to x will differ from that assigned to y. Any digit assigned afterwards to x (by the next iterations of T_β) will be equal to the digit assigned to y. Hence the greedy expansion of x and y will only differ in the first digit. An analogous argument works for the lazy expansion.

There is an alternative way to obtain the greedy and lazy expansions, which makes it more obvious in what sense their names are appropriate. The method is recursive: the digits of the greedy expansion are given by

$$d_n = d \in \{0,1,...,\lfloor\beta\rfloor - 1\} \iff \sum_{k=1}^{n-1} \frac{d_k}{\beta^k} + \frac{d}{\beta^n} \le x < \sum_{k=1}^{n-1} \frac{d_k}{\beta^k} + \frac{d+1}{\beta^n}$$

and

$$d_n = \lfloor\beta\rfloor \iff \sum_{k=1}^{n-1} \frac{d_k}{\beta^k} + \frac{\lfloor\beta\rfloor}{\beta^n} \le x.$$

For $n = 1$, the first term is the empty sum, which is defined to be zero. The process starts with the first digit d_1, which is then used to determine the second digit, and so on. In each step, the largest possible digit is taken, i.e. if we took a larger digit then either $\sum_{k=1}^{\infty} \frac{d_k}{\beta^k}$ would be greater than x even if we took the subsequent digits as small as possible (i.e. equal to 0), or the digit would be larger than $\lfloor\beta\rfloor$. Neither of these cases would allow us to obtain a β-expansion of x. In this sense, the greedy expansion is indeed greedy.

We can do something similar for the lazy expansion. The method is again recursive: the digits of the lazy expansion are given by

$$d'_n = 0 \iff x \le \sum_{k=1}^{n-1} \frac{d'_k}{\beta^k} + \sum_{j=n+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^j}$$

and

$$d'_n = d' \in \{1,...,\lfloor\beta\rfloor\}$$

$$\iff \sum_{k=1}^{n-1} \frac{d'_k}{\beta^k} + \frac{d'-1}{\beta^n} + \sum_{j=n+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^j} < x \le \sum_{k=1}^{n-1} \frac{d'_k}{\beta^k} + \frac{d'}{\beta^n} + \sum_{j=n+1}^{\infty} \frac{\lfloor\beta\rfloor}{\beta^j}.$$

Intuitively, we postpone taking a larger digit for as long as possible, i.e. if we did not take a larger digit then $\sum_{k=1}^{\infty} \frac{d'_k}{\beta^k}$ would be less than x even if we took the largest possible digit in all subsequent steps, which would not allow us to obtain a β-expansion of x. In this sense, the lazy expansion is indeed lazy. We will return to these expansions in chapter 3.

One might wonder why we chose the particular form $\sum_n \frac{d_n}{\beta^n}, d_n \in \{0, ..., \lfloor\beta\rfloor\}$ to represent numbers. In fact, it is possible to generalise this form and let the digits be in an arbitrary (but fixed) finite set of real numbers. These can also be generated dynamically, see [7]. It is also possible to use a so-called mixed radix system, where we do not use the same base for all positions, but allow the base to vary from one position to another. However, these expansions cannot in general be generated dynamically.

This still leaves open the question why we take the denominator to be a polynomial. We can write

$$e = 1 \cdot 2! + 0 \cdot 1! + \sum_{i=2}^{\infty} \frac{1}{n!}$$

If we use a notation similar to the decimal expansion, we could write $e = 10.111111111111...$. The difference here is that the denominator is a factorial instead of the power of a base.

It turns out that there is an alternative number system called the factorial number system, which differs from the usual number system in that there is no base. Another difference lies in what digits are allowed: the digit d_n corresponding to $n!$ must be an integer less than or equal to n. The integers are represented in the form $\sum_{n=1}^{\infty} d_n n!$, and fractional values in the form $\sum_{n=2}^{\infty} \frac{d_n}{n!}$. For example, the number 210 in the factorial number system is equal to $2 \cdot 3! + 1 \cdot 2! + 0 \cdot 1! = 14$.

Note that we cannot use the same (finite) digit set to represent all numbers. Furthermore, the set of permissible digits depends on the position, so it is not possible to generate the digits dynamically as with β-expansions. An advantage of the factorial number system is that it makes it easy to represent permutations. The conclusion here is that there are other possibilities to represent numbers, which each have their pros and cons. The main advantage of β-expansions is that they can be generated dynamically.

Chapter 3

Measure theory and ergodic theory

3.1 Useful technical definitions and results

This section is based on [6] and [9].

From now on, we will consider sets such as $X = \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ to be part of a measure space $(X, \mathscr{F}, \mu)$. This will enable us to determine additional properties of β-expansions, such as the frequency of a certain digit. We will introduce concepts such as *absolutely continuous invariant measure* (abbreviated as *acim*) and *ergodicity*, after which we introduce a useful result called the Ergodic Theorem.

Definition 3.1.1. *Let μ and ν be measures on a measurable space $(X, \mathscr{F})$. We say that ν is absolutely continuous w.r.t. μ if for all $A \in \mathscr{F}$ such that $\mu(A) = 0$, we also have $\nu(A) = 0$. We denote this by $\nu \ll \mu$. If μ is the Lebesgue measure, we say that ν is absolutely continuous.*

Definition 3.1.2. *Let $(X, \mathscr{F}, \mu)$ be a probability space. We say that a measurable map $T : X \to X$ is measure-preserving with respect to μ if $\mu\left(T^{-1}(A)\right) = \mu(A)$ for all $A \in \mathscr{F}$. Equivalently, we say that μ is T-invariant.*

There are several reasons why we are interested in absolutely continuous invariant measures. Firstly, they allow us to compute the measure of sets using Riemann integrals because of the equality between proper Riemann integrals and Lebesgue integrals. Secondly, the Lebesgue measure is the natural measure in the sense that it is used to calculate areas and volumes. Thirdly, if an acim exists, then it is often the case that computer simulations have histograms which approach the corresponding density, see [1]. Finally, acims are useful because they allow us to apply results from ergodic theory, as we shall see at the end of this chapter.

The following theorem states that almost every point in a set of positive measure eventually returns to this set after sufficiently many iterations of a measure-preserving map. In fact, the theorem implies that this happens infinitely often.

Theorem 3.1.3 (Poincaré Recurrence Theorem)**.** *Let T be a measure-preserving transformation on a probability space $(X,\mathscr{F},\mu)$ and let $A \in \mathscr{F}$ be such that $\mu(A) > 0$. Then for a.e. $x \in A$ there exists a number $k \geq 1$ such that $T^k(x) \in A$.*

Proof. See [6]. □

Definition 3.1.4. *Let $(X,\mathscr{F},\mu)$ be a measure space. We say that μ is σ-finite if $\mathscr{F}$ contains an increasing sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ such that $\cup_{j\in\mathbb{N}} A_j = X$ and $\mu(A_j) < \infty$* for all $j \in \mathbb{N}$.

Note that any probability measure is σ-finite; we can simply take the constant sequence $A_j = X$ for all j.

Theorem 3.1.5 (Radon-Nikodym)**.** *Let μ and ν be measures on a measurable space $(X,\mathscr{F})$, where μ is σ-finite.*

There exists an a.e. unique non-negative measurable function f such that

$$\nu(A) = \int_A f d\mu \qquad \text{for all } A \in \mathscr{F}$$

if and only if

$$\nu \ll \mu.$$

Proof. See [9]. □

In other words, ν has a density w.r.t. μ if and only if ν is absolutely continuous w.r.t μ.

Lemma 3.1.6. *Let $(X,\mathscr{F},\mu)$ be a measure space and let $f,g \in L^1(\mu)$. If*

$$\int_A f d\mu = \int_A g d\mu$$

for all measurable sets A, then $f = g$ a.e.

Proof. We prove this by contradiction. Assume to the contrary that $\mu(\{f \neq g\}) > 0$. Let $A = \{f > g\}$, $B = \{f < g\}$ and $C_n = \{f - g \geq \frac{1}{n}\}$. As f and g are measurable, these sets are measurable. As $A \cup B = \{f \neq g\}$, either $\mu(A) > 0$ or $\mu(B) > 0$. Assume without loss of generality that $\mu(A) > 0$. As $A = \cup_{n=1}^{\infty} C_n$, there exists an $m \in \mathbb{N}$ such that $\mu(C_m) > 0$. Hence

$$\begin{aligned}\int_{C_m} f d\mu - \int_{C_m} g d\mu &= \int_{C_m} f - g d\mu \\ &\geq \int_{C_m} \frac{1}{m} d\mu \\ &= \frac{1}{m}\mu(C_m) > 0.\end{aligned}$$

This is a contradiction. We conclude that $f = g$ a.e. □

Definition 3.1.7. *Let $(X, \mathscr{F}, \mu)$ be a probability space. We say that a measurable map $T : X \to X$ is non-singular if for all $A \in \mathscr{F}$ such that $\mu(A) = 0$ we have $\mu(T^{-1}(A)) = 0$.*

Note that any measure-preserving map is non-singular.

Definition 3.1.8. *Let $(X, \mathscr{F}, \mu)$ be a probability space. We say that a measure-preserving map $T : X \to X$ is ergodic if for all $A \in \mathscr{F}$ such that $T^{-1}(A) = A$ we have $\mu(A) \in \{0, 1\}$.*

We have the following generalisation of the Strong Law of Large Numbers:

Theorem 3.1.9 (Birkhoff's Ergodic Theorem, theorem 2.1.1 in [6]). *Let $(X, \mathscr{F}, \mu)$ be a probability space and let $T : X \to X$ be a measure-preserving map. Then for all $f \in L^1(\mu)$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) =: f^*(x)$$

exists for almost every $x \in X$, is T-invariant and $\int_X f d\mu = \int_X f^* d\mu$. If additionally T is ergodic, then $f^* \equiv \int_X f d\mu$ a.e. In particular, f^* is constant a.e.

Proof. see [6]. □

Definition 3.1.10. *Let $(X, \mathscr{F}, \mu)$ be a probability space and let T be a measure-preserving transformation. We say that $(X, \mathscr{F}, \mu, T)$ is a dynamical system.*

The following definition states that two dynamical systems are the same if there is a map preserving both the measure structures on each space given by the σ-algebras and the probability measures, and the dynamical structures given by the measure-preserving transformations.

Definition 3.1.11 (based on definition 3.1.1 in [6]). *The dynamical systems $(X, \mathscr{F}, \mu, T)$ and $(Y, \mathscr{C}, \nu, S)$ are called isomorphic there exists a map $\phi : X \to Y$ satisfying the following:*

(i) *ϕ is bijective a.e., i.e. there exist measurable sets $N \subset X$ and $M \subset Y$ with $\mu(X \setminus N) = \nu(Y \setminus M) = 0$, $T(N) \subset N, S(M) \subset M$ such that the restriction $\phi' : N \to M$ is a bijection.*

(ii) *ϕ' is $\mathscr{F} \cap N/\mathscr{C} \cap M$-measurable and its inverse ϕ^{-1} is $\mathscr{C} \cap M/\mathscr{F} \cap N$-measurable.*

(iii) *ϕ' preserves the measures, i.e. $\nu(C) = \mu(\phi^{-1}(C))$ for all $C \in \mathscr{C} \cap M$. In other words, $\nu = \mu \circ \phi^{-1}$.*

(iv) *ϕ' preserves the dynamics of T and S, i.e. $\phi' \circ T = S \circ \phi'$.*

We say that ϕ is a measurable isomorphism.

3.2 The Frobenius-Perron operator

This section is based on chapter 4 of [2].

In this section, we introduce a useful technical tool called the Frobenius-Perron operator. It will allow us to find acims explicitly for some special cases. We first provide a motivating argument leading up to its definition, after which we prove several of its properties.

Let $I = [a, b]$ and consider the probability space $(I, \mathscr{B}(I), \lambda)$, where λ is the normalised Lebesgue measure. Let X be distributed according to the probability density function f, i.e. for any measurable set A,

$$\mathbb{P}(X \in A) = \int_A f d\lambda.$$

Let T be a non-singular measurable map, so that $T(X)$ is another random variable. One could ask what the probability density function of $T(X)$ is. We compute

$$\mathbb{P}(T(X) \in A) = \mathbb{P}(X \in T^{-1}(A)) = \int_{T^{-1}(A)} f d\lambda.$$

To obtain the probability density function of $T(X)$, we need to write the previous integral in the form

$$\int_A \phi d\lambda$$

for some function ϕ, which will then be the probability density function of $T(X)$. We now prove that this is always possible, i.e. that such a function ϕ exists. Define the measure

$$\mu(A) := \int_{T^{-1}(A)} f d\lambda.$$

Let A be a measurable set such that $\lambda(A) = 0$. As T is non-singular, $\lambda(T^{-1}(A) = 0)$, so $\mu(A) = 0$. Therefore, $\mu \ll \lambda$. By the Radon-Nikodym theorem, $T(X)$ has a probability density function ϕ with respect to λ, which is a.e. uniquely defined.

Starting with a probability density function f, we have obtained a new probability density function $P_T f := \phi$, where P_T is an operator from the space of probability density functions to itself. Concluding the previous discussion, we have the following definition:

Definition 3.2.1. *Let $T : I \to I$ be a non-singular measurable map on the probability space $(I, \mathscr{B}(I), \lambda)$, where λ is the normalised Lebesgue measure and $I = [a,b]$. The Frobenius-Perron operator is defined as $P_T : \mathscr{L}^1 \to \mathscr{L}^1$, $P_T f$ is the a.e. unique function in $\mathscr{L}^1$ such that for all measurable sets A,*

$$\int_A P_T f d\lambda = \int_{T^{-1}(A)} f d\lambda.$$

This is well-defined: as we have seen, the Radon-Nikodym Theorem implies existence and a.e. uniqueness of $P_T f$. This definition can be extended to more general measure spaces. Although we have defined the operator for all integrable functions, in practice we will only use it for non-negative functions with integral 1, i.e. probability density functions.

For $A = [a,x] \subset I$, we have by the equality of proper Riemann integrals and Lebesgue integrals:

$$\int_a^x P_T f(y) dy = \int_{T^{-1}([a,x])} f d\lambda$$

Differentiating w.r.t. x gives the formula

$$P_T f(x) = \frac{d}{dx} \int_{T^{-1}([a,x])} f d\lambda.$$

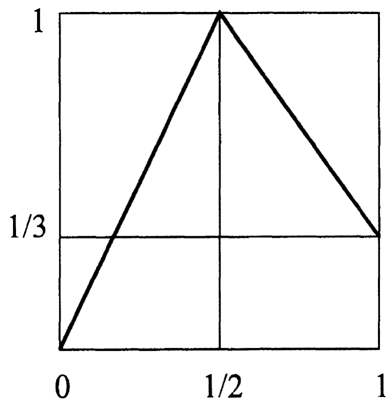

Fig. 3.1 Plot of T. Source: [2].

Example 3.2.2 (example 4.1.2 in [2]).

Let $I = [0,1]$ and define the piecewise linear map $T : I \to I$,

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2} \\ -\frac{4}{3}x + \frac{5}{3} & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

(see Figure 3.1)

Then

$$T^{-1}([0,x]) = \left[0, \frac{1}{2}x\right] \qquad \text{for } x < \frac{1}{3}$$

and

$$T^{-1}([0,x]) = \left[0, \frac{1}{2}x\right] \cup \left[\frac{5}{4} - \frac{3}{4}x, 1\right] \text{ for } x \ge \frac{1}{3}$$

So for any probability density function f on $[0,1]$,

$$\int_{T^{-1}([0,x])} f d\lambda = \begin{cases} \int_0^{\frac{1}{2}x} f d\lambda & \text{if } 0 \le x < \frac{1}{3} \\ \int_0^{\frac{1}{2}x} f d\lambda + \int_{\frac{5}{4}-\frac{3}{4}x}^{1} f d\lambda & \text{if } \frac{1}{3} \le x \le 1 \end{cases}$$

Differentiating, we obtain

$$P_T f(x) = \begin{cases} \frac{1}{2} f(\frac{1}{2}x) & \text{if } 0 \le x < \frac{1}{3} \\ \frac{1}{2} f(\frac{1}{2}x) + \frac{3}{4} f(\frac{5}{4} - \frac{3}{4}x) & \text{if } \frac{1}{3} \le x \le 1 \end{cases}$$

Using an indicator function, we can write this more compactly:

$$P_T f(x) = \frac{1}{2} f(\frac{1}{2}x) + \frac{3}{4} f(\frac{5}{4} - \frac{3}{4}x) \mathbb{1}_{[\frac{1}{3},1]}(x).$$

We will later generalise this result. We now prove several properties of the Frobenius-Perron operator.

Proposition 3.2.3 (Linearity)**.** *P_T is a linear operator.*

Proof. Let A be a measurable set, let α, β be constants and let $f, g \in L^1$. Then

$$\begin{aligned}
\int_A P_T(\alpha f + \beta g) d\lambda &= \int_{T^{-1}(A)} (\alpha f + \beta g) d\lambda \\
&= \alpha \int_{T^{-1}(A)} f d\lambda + \beta \int_{T^{-1}(A)} g d\lambda \\
&= \alpha \int_A P_T f d\lambda + \beta \int_A P_T g d\lambda \\
&= \int_A \alpha P_T f + \beta P_T g d\lambda
\end{aligned}$$

By Lemma 3.1.6, we conclude that $P_T(\alpha f + \beta g) = \alpha P_T f + \beta P_T g$, i.e. P_T is linear. □

Proposition 3.2.4 (Non-negativity)**.** *Let $f \in L^1$ be such that $f \geq 0$. Then $P_T f \geq 0$.*

Proof. Let A be a measurable set. Then

$$\int_A P_T f d\lambda = \int_{T^{-1}(A)} f d\lambda \geq 0.$$

Similarly to the proof of Lemma 3.1.6, we conclude that $P_T f \geq 0$. □

Proposition 3.2.5 (P_T preserves integrals)**.** *Let I be an interval and let $(I, \mathscr{B}(I), \mu)$ be a measure space. Then*

$$\int_I P_T f d\lambda = \int_I f d\lambda$$

Proof. As $T^{-1}(I) = I$,

$$\int_I P_T f d\lambda = \int_{T^{-1}(I)} f d\lambda = \int_I f d\lambda.$$

□

In particular, if f is a probability density function, then $P_T f$ also has integral 1. Combining this with with Proposition 3.2.4, we find that $P_T f$ is also a probability density function.

Proposition 3.2.6 (Composition property)**.** *Let $S : I \to I$ and $T : I \to I$ be non-singular measurable maps. Then $P_{S\circ T} = P_S \circ P_T$.*

Proof. We first show that $P_{S\circ T}$ exists. For this we only need to show that $S \circ T$ is non-singular. Let N be a measurable set such that $\lambda(N) = 0$. As S and T are non-singular, $\lambda\left((S\circ T)^{-1}(N)\right) = \lambda\left(T^{-1}\left(S^{-1}(N)\right)\right) = 0$. Therefore, $S \circ T$ is non-singular and $P_{S\circ T}$ exists. Now let $f \in L^1$ and let A be a measurable set. Repeatedly applying the definition of the Frobenius-Perron operator, we obtain

$$\begin{aligned}
\int_A P_{S\circ T} f \, d\lambda &= \int_{(S\circ T)^{-1}(A)} f \, d\lambda \\
&= \int_{T^{-1}\left(S^{-1}(A)\right)} f \, d\lambda \\
&= \int_{S^{-1}(A)} P_T f \, d\lambda \\
&= \int_A P_S(P_T f) d\lambda
\end{aligned}$$

By Lemma 3.1.6, we conclude that $P_{S\circ T} f = P_S(P_T f)$ a.e. □

Note: if we take $T = S$, then it follows by induction that $P_{S^n} = P_S^n$ for $n \geq 1$.

Finally, we have the following relationship between fixed points of P_T and invariant measures:

Proposition 3.2.7 (Proposition 4.2.7 in [2])**.** *Let $T : I \to I$ be a non-singular measurable map and let $f \in L^1$. Then $P_T f = f$* a.e. *if and only if the measure μ defined by $\mu(A) = \int_A f d\lambda$ is T-invariant.*

Proof. " $\Longleftarrow$ ": assume μ is T-invariant, i.e. $\mu(T^{-1}(A)) = \mu(A)$ for any measurable set A. By definition of μ, this means that

$$\int_{T^{-1}(A)} f \, d\lambda = \int_A f \, d\lambda.$$

By definition of the Frobenius-Perron operator, this implies that

$$\int_A P_T f \, d\lambda = \int_A f \, d\lambda.$$

By Lemma 3.1.6, we conclude that $P_T f = f$ a.e.

" $\implies$ ": assume $P_T f = f$ a.e. Then

$$\begin{aligned}\mu(A) &= \int_A f d\lambda \\ &= \int_A P_T f d\lambda \\ &= \int_{T^{-1}(A)} f d\lambda \\ &= \mu(T^{-1}(A)).\end{aligned}$$

□

In other words, f is a fixed point of P_T if and only if $\mu(A) = \int_A f d\lambda$ is an acim.

3.3 Representations of the Frobenius-Perron operator

This section is based on chapter 4.3 and 9 of [2].

In this section, we will assume conditions on T which allow us to obtain explicit expressions for the Frobenius-Perron operator. We first consider piecewise monotonic transformations, after which we further specialise to piecewise linear Markov transformations. In the latter case, the Frobenius-Perron operator reduces to a matrix. Finally, we apply these results to find an acim for the greedy map in the case that β is equal to the golden mean.

Definition 3.3.1 (based on definition 4.3.1 in [2])**.** *Let $I = [a,b]$. The map $T : I \to I$ is called piecewise monotonic if there exists a partition*

$$\mathscr{P} = \{I_i\}_{i=1}^n = \{[a_0,a_1),[a_1,a_2),...,[a_{n-2},a_{n-1}),[a_{n-1},a_n]\}$$

of I, where $a = a_0 < a_1 < ... < a_n = b$, such that for $1 \le i \le n$,

- *$T|_{I_i}$ is a C^1 function which can be extended to a C^1* function on $\overline{I_i}$.
- $|T'(x)| > 0$ for all $x \in I_i$, *where the derivative on the endpoint(s) is a one-sided derivative.*

If additionally $|T'(x)|$ is positively bounded away from 1, we say that T is piecewise monotonic and expanding. In other words, if T is piecewise monotonic and there exists a constant α such that $|T'(x)| \ge \alpha > 1$ for all $x \in I$.

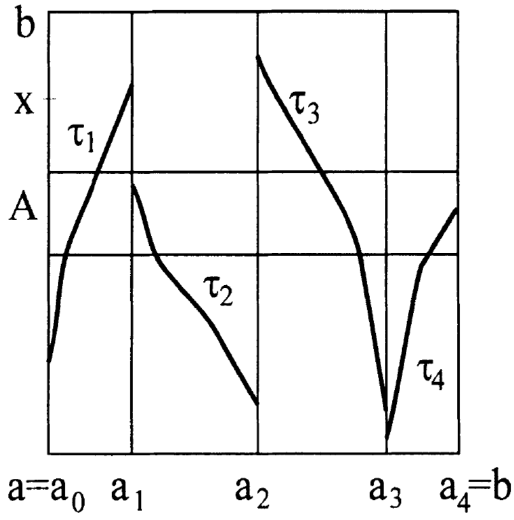

Fig. 3.2 Plot of a piecewise monotonic transformation. Source: [2].

The name 'piecewise monotonic' is appropriate because $|T'(x)| > 0$ on on each interval I_i implies that T is monotonic on I_i. Note that we interpret monotonicity in a strict sense here, i.e. T is either strictly increasing or strictly decreasing on each interval I_i. In particular, T is invertible on each of these intervals. Let $T_i : I_i \to T(I_i), T_i := T|_{I_i}$ and define its inverse $\phi_i : T(I_i) \to I_i, \phi_i := T_i^{-1}$.

We will now derive an explicit expression of P_T for piecewise monotonic maps T. Let A be a measurable set and let $f \in L^1$. We can write $T^{-1}(A)$ as a union of disjoint sets: $T^{-1}(A) = \cup_{i=1}^n \phi_i(A \cap T(I_i))$. Hence

$$\begin{aligned}
\int_A P_T f d\lambda &= \int_{T^{-1}(A)} f d\lambda \\
&= \sum_{i=1}^n \int_{\phi_i(A\cap T(I_i))} f d\lambda \\
&= \sum_{i=1}^n \int_{A\cap T(I_i)} f(\phi_i)|\phi_i'| d\lambda \qquad \text{(by the change of variables formula)} \\
&= \int_A \sum_{i=1}^n f(\phi_i)|\phi_i'| \mathbb{1}_{T(I_i)} d\lambda \\
&= \int_A \sum_{i=1}^n \frac{f(T_i^{-1})}{|T_i'(T_i^{-1})|} \mathbb{1}_{T(I_i)} d\lambda
\end{aligned}$$

By Lemma 3.1.6, we obtain

$$P_T f = \sum_{i=1}^{n} \frac{f(T_i^{-1})}{|T_i'(T_i^{-1})|} \mathbb{1}_{T(I_i)}$$

More compactly, we can write

$$P_T f(x) = \sum_{z \in T^{-1}(x)} \frac{f(z)}{|T'(z)|}$$

These expressions are well-defined: the denominators are non-zero because of the assumption that $|T_i'| > 0$.

Example 3.3.2 (example 4.3.1 in [2]).

Let $T : [0,1] \to [0,1], T(x) = rxe^{-bx}$, where $r = 5e, b = 5$ (see Figure 3.3). Define $I_1 = [0, \frac{1}{b}), I_2 = [\frac{1}{b}, 1], T_1 = T|_{I_1}, T_2 = T|_{I_2}$. By the previous result, for $f \in L^1$,

$$\begin{aligned} P_T f &= \frac{f(T_1^{-1})}{|T_1'(T_1^{-1})|} \mathbb{1}_{T(I_1)} + \frac{f(T_2^{-1})}{|T_2'(T_i^{-1})|} \mathbb{1}_{T(I_2)} \\ &= \frac{f(T_1^{-1})}{|T_1'(T_1^{-1})|} + \frac{f(T_2^{-1})}{|T_2'(T_i^{-1})|} \mathbb{1}_{[re^{-b},1]}. \end{aligned}$$

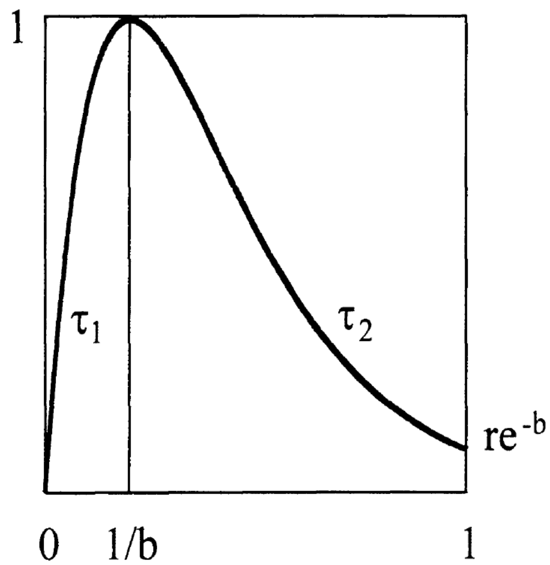

Fig. 3.3 Plot of T. Source: [2].

We now consider a special case of piecewise monotonic transformations: piecewise linear Markov transformations.

Definition 3.3.3. *A map $T : X \to Y$ is called a homeomorphism if*

- *T is bijective (hence it has an inverse T^{-1}), and*
- *T and T^{-1} are continuous.*

Definition 3.3.4. *A map $T : I \to I$ is called a piecewise linear Markov transformation if there exists a partition*

$$\mathscr{P} = \{I_i\}_{i=1}^n = \{[a_0,a_1),[a_1,a_2),...,[a_{n-2},a_{n-1}),[a_{n-1},a_n]\}$$

of I, where $a = a_0 < a_1 < ... < a_n = b$, such that for $1 \leq i \leq n$,

- *$T|_{I_i}$ is a homeomorphism from I_i to a connected union of intervals in $\mathscr{P}$*
- $|T'(x)| > 0$ for all $x \in I_i$, *where the derivative on the endpoint(s) is a one-sided derivative.*
- *$T|_{I_i}$ is linear on I_i.*

If additionally there exists a constant α such that $|T'(x)| \geq \alpha > 1$ for all $x \in I$, we say that T is an expanding piecewise linear Markov transformation.

We will use the result for piecewise monotonic transformations to derive a matrix representation for P_T in the case that T is a piecewise linear Markov transformation. First we introduce some useful concepts and notation.

Definition 3.3.5 (definition 9.1.1 in [2])**.** *Let $T : I \to I$ be a piecewise monotonic transformation and let $\mathscr{P} = \{I_i\}_{i=1}^n$ be a partition of I. The incidence matrix $A_T = (a_{ij})_{1 \leq i,j \leq n}$ induced by T and $\mathscr{P}$ is defined by its entries*

$$a_{ij} = \begin{cases} 1 & \text{if } I_j \subset T(I_i) \\ 0 & \text{otherwise} \end{cases}$$

Definition 3.3.6. *Let $\mathscr{P} = \{I_i\}_{i=1}^n$ be a partition of I. A function $f : I \to I$ is called piecewise constant on $\mathscr{P}$ if it is constant on each I_i, i.e. if we can write $f = \sum_{i=1}^n f_i \mathbb{1}_{I_i}$ for some constants $f_1,...,f_n$. We denote the column vector obtained from f by $\pi^f = (f_1,...,f_n)^\intercal$, where the superscript $\intercal$ denotes transposition.*

Theorem 3.3.7 (theorem 9.2.1 in [2]). *Let $T : I \to I$ be a piecewise linear Markov transformation on a partition $\mathscr{P} = \{I_i\}_{i=1}^n$. There exists an $n \times n$ matrix M_T such that $P_T f = M_T \pi^f$ for all functions f that are piecewise constant on $\mathscr{P}$. The matrix M_T has entries*

$$m_{ij} = \frac{a_{ji}}{|T_j'|} \qquad 1 \le i, j \le n.$$

where $(a_{ij})_{1 \le i,j \le n}$ are the entries of the incidence matrix induced by T and $\mathscr{P}$.

Note that T is a piecewise linear Markov transformation, so each denominator $|T_i'|$ is a non-zero constant. Also note that the matrix is the same for all piecewise constant functions f, and that $P_T f$ is piecewise constant.

Proof. First assume that $f = \mathbb{1}_{I_k}$ for some $1 \le k \le n$. Then

$$\begin{aligned}
P_T f &= \sum_{i=1}^n \frac{f(T_i^{-1})}{|T_i'(T_i^{-1})|} \mathbb{1}_{T(I_i)} && \text{(by the result for piecewise monotonic transformations)} \\
&= \sum_{i=1}^n \frac{\mathbb{1}_{I_k}(T_i^{-1})}{|T_i'(T_i^{-1})|} \mathbb{1}_{T(I_i)} \\
&= \frac{\mathbb{1}_{I_k}(T_k^{-1})}{|T_k'(T_k^{-1})|} \mathbb{1}_{T(I_k)} && \text{(the range of } T_i^{-1} \text{ is } I_i, \text{ so } T_i^{-1}(x) \notin I_k \text{ for } i \neq k, \forall x \in I_i) \\
&= \frac{1}{|T_k'(T_k^{-1})|} \mathbb{1}_{T(I_k)} \\
&= \frac{1}{|T_k'|} \mathbb{1}_{T(I_k)} && \text{(} T \text{ is piecewise linear, so the denominator } |T_k'| \text{ is a constant)}
\end{aligned}$$

Now let $f = \sum_{k=1}^n f_k \mathbb{1}_{I_k}$ be piecewise constant function. By Proposition 3.2.3, P_T is a linear operator, so

$$P_T f = \sum_{k=1}^n \frac{f_k}{|T_k'|} \mathbb{1}_{T(I_k)}$$

For all k, $T(I_k)$ is a union of sets in $\mathscr{P}$. By the formula above, $P_T f$ is piecewise linear on $\mathscr{P}$. Let $(d_1, \ldots, d_n)^T$ be the column vector obtained from $P_T f$. We now show that we can write P_T as a matrix. Let $x \in I_j$, so that $P_T f(x) = d_j$. For $1 \le k \le n$, the term $\frac{f_k}{|T_k'|} \mathbb{1}_{T(I_k)}$ is equal to $\frac{f_k}{|T_k'|}$ if and only if $x \in T(I_k)$. As T is a piecewise linear Markov transformation, this is the case if and only if the entire interval $I_j \subseteq T(I_k)$, i.e. if and only if $a_{kj} = 1$. We conclude that

$$d_j = \sum_{k=1}^n f_k \frac{a_{kj}}{|T_k'|} = \sum_{k=1}^n f_k m_{jk}$$

and

$$P_T f = M_T \pi^f.$$

□

Example 3.3.8.

Consider the greedy transformation T_β for $\beta = \frac{1+\sqrt{5}}{2}$, so that β satisfies $\beta(\beta-1)=1$. We will show that in this case, T_β is a piecewise linear Markov transformation. The greedy map is obviously piecewise linear with non-zero derivatives. Let

$$I_1 = [0, \frac{1}{\beta}),\ I_2 = [\frac{1}{\beta}, 1) \text{ and } I_3 = [1, \frac{1}{\beta-1}].$$

Then

$$T_\beta(I_1) = I_1 \cup I_2,\ T_\beta(I_2) = I_1 \text{ and } T_\beta(I_3) = I_2 \cup I_3.$$

Every part is a continuous bijection with a continuous inverse. Therefore, T_β is a piecewise linear Markov transformation. We now apply Theorem 3.3.7 to find its Frobenius-Perron operator. The (i,j)-th entry of the Frobenius-Perron matrix is found by checking whether $I_j \subset T_\beta(I_i)$ and taking $|T'_{\beta,i}| = \beta$. By the previous theorem, for any piecewise constant function $f = (f_1, f_2, f_3)^\intercal$ on $\mathscr{P}$ we can write

$$P_{T_\beta} f = M_{T_\beta} \pi^f = \begin{pmatrix} \frac{2}{1+\sqrt{5}} & \frac{2}{1+\sqrt{5}} & 0 \\ \frac{2}{1+\sqrt{5}} & 0 & \frac{2}{1+\sqrt{5}} \\ 0 & 0 & \frac{2}{1+\sqrt{5}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

We now find the piecewise constant absolutely continuous invariant density of T_β by solving the system of equations $P_{T_\beta} f = f, \int f(x)dx = 1$ for f. Explicitly, we solve

$$\begin{cases} \begin{pmatrix} \frac{2}{1+\sqrt{5}} & \frac{2}{1+\sqrt{5}} & 0 \\ \frac{2}{1+\sqrt{5}} & 0 & \frac{2}{1+\sqrt{5}} \\ 0 & 0 & \frac{2}{1+\sqrt{5}} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \\ \frac{1}{\beta} f_1 + (1 - 1/\beta) f_2 + (\frac{1}{\beta-1} - 1) f_3 = 1 \end{cases}$$

and obtain the piecewise constant absolutely continuous invariant density

$$(f_1, f_2, f_3) = (\frac{5+3\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10}, 0).$$

In different notation:

$$f(x) = \begin{cases} \frac{5+3\sqrt{5}}{10} & \text{if } x \in [0, \frac{1}{\beta}) \\ \frac{5+\sqrt{5}}{10} & \text{if } x \in [\frac{1}{\beta}, 1) \\ 0 & \text{if } x \in [1, \frac{1}{\beta-1}] \end{cases}$$

The intuition for f_3 being 0 is as follows: after iterating T_β, all points in I_3 except $x = \frac{1}{\beta-1}$ will eventually end up in $I_1 \cup I_2$ and stay there for all subsequent iterations. Points in $I_1 \cup I_2$ are never mapped to I_3. Hence any T_β-invariant measure must assign measure 0 to I_3.

As $|T'_\beta| = \beta > 1$, this map is also expanding. One can show that for expanding piecewise linear Markov transformations, every invariant density is piecewise constant (see [2]). Therefore, the density we found is the only invariant density.

To define the digits of the greedy expansion, we iterated the greedy map. This iterative structure, combined with the acim we have found, allows us to apply the Ergodic Theorem to find the frequency of the digit 0 in the greedy expansion with $\beta = \frac{1+\sqrt{5}}{2}$. By Proposition 3.2.7, the measure $\nu_\beta(A) := \int_A f d\lambda$ is T_β-invariant. One can show that T_β is ergodic w.r.t. ν_β. As in [3], the following holds for a.e. x:

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le i \le n : d_i(x) = 0\} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{[0,1/\beta)} \left(T_\beta^i(x)\right)$$

$$= \nu_\beta([0, 1/\beta)) = \int_0^{\frac{1}{\beta}} \frac{5+3\sqrt{5}}{10} dx = \frac{5+\sqrt{5}}{10}$$

Note that the quantity on the left-hand side does not have anything to do with measure theory or ergodic theory, but that introducing these concepts was useful to prove our result. This example suggests the following general proof strategy:

- Given an object of interest defined on a set X, introduce a σ-algebra $\mathscr{F}$ and consider the measurable space $(X, \mathscr{F})$.
- Write the object in the form $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(T^i(x)\right)$ for some measurable maps T and f on X.

- Find an invariant measure for T.
- Use the Ergodic Theorem to find an expression for the object which holds a.e.

The acim in the case where β is equal to the golden mean was first found by Rényi. The following theorem states that there is in fact a unique acim for the greedy map for all values of $\beta > 1$. This measure is called the extended Parry measure.

Theorem 3.3.9. *Let $\beta > 1$ be a non-integer. There exists a unique T_β-invariant measure equivalent to the Lebesgue measure with density $h_\beta : [0, \lfloor \beta \rfloor/(\beta-1)] \to \mathbb{R}$,*

$$h_\beta(x) = \begin{cases} \frac{1}{F(\beta)} \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{[0, T_\beta^n(1))}(x) & \textit{if } 0 \le x < 1 \\ 0 & \textit{if } 1 \le x \le \lfloor \beta \rfloor/(\beta-1) \end{cases}$$

where $F(\beta)$ is a normalising constant.

Proof. See [8]. □

Chapter 4

Random maps and random β-expansions

In this chapter, we first prove some additional properties of the lazy and greedy expansions. We then introduce the theory of random maps and random β-expansions. We will show how to obtain β-expansions other than the lazy and greedy expansions using a random map based on the lazy and greedy maps. We extend these results to a two-dimensional random map and find an invariant measure of maximal entropy for this map. We introduce a variant of the two-dimensional random map called the skew product transformation, and prove the existence of an acim. Finally, we prove some properties of digit sequences that simultaneously give an expansion of two numbers x and y in bases β_1 and β_2 and introduce a map which generates these sequences.

4.1 The lazy and greedy expansions revisited

We now return to the lazy and greedy expansions. For completeness, we mention the following definition:

Definition 4.1.1. *Let* $0.d_1d_2...$ *and* $0.e_1e_2...$ *be two* β*-expansions. We say that* $0.d_1d_2...$ *is smaller than* $0.e_1e_2...$ *in lexicographical order, written*

$$0.d_1d_2... <_{\text{lex}} 0.e_1e_2...$$

if there exists an integer n *such that* $d_i = e_i$ *for all* $i < n$ *and* $d_n < e_n$.

Note that if we have two β-expansions such that $x = 0.d_1d_2... <_{\text{lex}} 0.e_1e_2... = y$, then it is possible that $y < x$. This conflicts with the usual intuition for binary or decimal expansions that for instance $0.01111111... < 0.1$ (in base 10). This is a consequence of the fact that for

$1<\beta<2$, $\sum_{k=n+1}^{\infty}\frac{1}{\beta^k}>\frac{1}{\beta^n}$. For example, for $\beta=1.5$,

$$\frac{4}{3}=\frac{0}{1.5^1}+\sum_{k=2}^{\infty}\frac{1}{1.5^k}=0.01111111\ldots \text{ (in base 1.5)}$$

while

$$\frac{2}{3}=\frac{1}{1.5^1}+\sum_{k=2}^{\infty}\frac{0}{1.5^k}=0.10000000\ldots \text{ (in base 1.5)}.$$

Furthermore, if we have a β-expansion that is lexicographically between the lazy expansion of a number x and the greedy expansion of that same number x, then it is not necessarily a β-expansion of x. For example, the lazy expansion of $\frac{2}{3}$ in base 1.5 is 0.00101111101... , so

$$0.00101111101\ldots <_{\text{lex}} 0.01111111\ldots <_{\text{lex}} 0.10000000\ldots$$

but

$$\frac{2}{3}=0.00101111101\ldots=0.10000000\ldots<0.01111111\ldots=\frac{4}{3}.$$

However, greedy expansions do have the monotonicity property: if $x<y$, then the greedy expansion of x is lexicographically smaller than the greedy expansion of y.

We now show that the lazy expansion is lexicographically smaller than the greedy expansion (this inequality is not strict; they can be equal). The first interval in the definition of the lazy map and the last interval of the greedy map have length $\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}$. All other intervals of both maps have length $\frac{1}{\beta}$ (see Figure 4.1). As $\lfloor\beta\rfloor>\beta-1$, we have $\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}>\frac{1}{\beta}$.

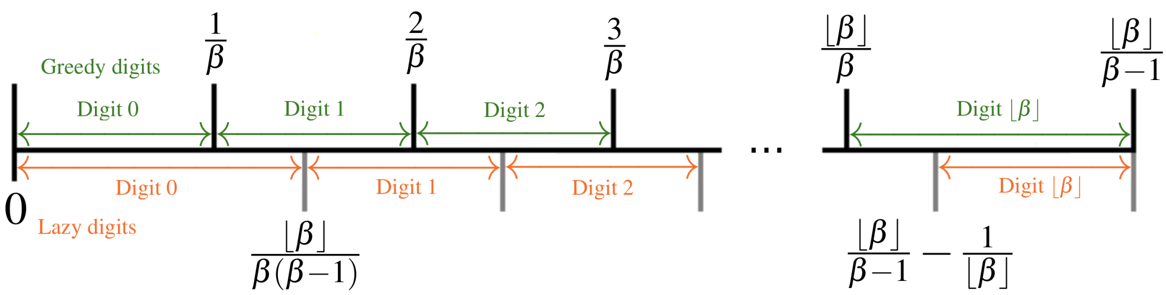

Fig. 4.1 The intervals of the greedy and lazy maps with their assigned digits.

(In Figure 4.1, the first interval of the lazy map is shorter than the union of the first and second intervals of the greedy map. This is only true for certain values of β; the first interval of the lazy map can be longer.)

Therefore, the first interval of the lazy map is longer than the first interval of the greedy map. As the subsequent lazy intervals are shorter than or of the same length as the subsequent greedy intervals, the first lazy digit will be smaller than or equal to the first greedy digit.

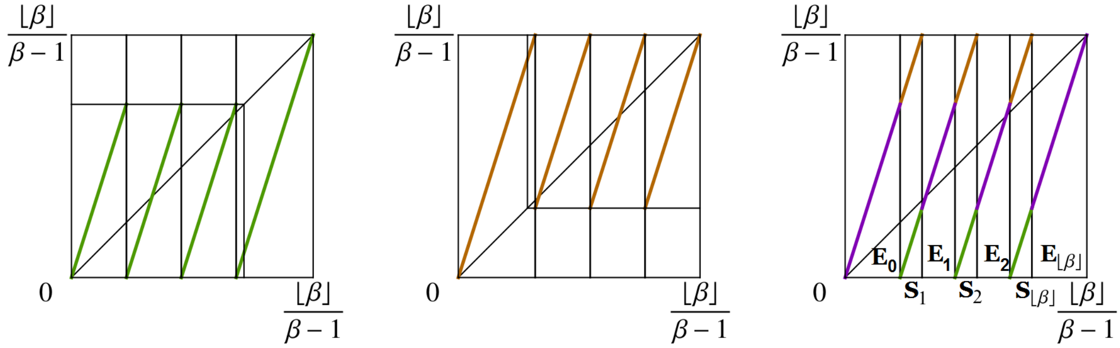

Fig. 4.2 Plot of the greedy map (left) and the lazy map (centre) for $\beta = \pi$, and the two maps superimposed (right). Source: [7], with slightly modified notation.

The larger $T(x)$ is, the larger the second assigned digit will be (above certain thresholds). In Figure 4.2 we see that it is possible that the lazy map maps x to a larger number than the greedy map, so the second lazy digit may be larger than the second greedy digit. However, this only happens if x is in one of the so-called switch regions, denoted by S_i. The lazy digit assigned to x in such a region is strictly smaller than the greedy digit (they differ by exactly 1). Hence, if the second lazy digit is larger than the second greedy digit, then the first greedy digit must be larger than the first lazy digit. In general, assume that the n-th lazy digit is larger than the n-th greedy digit, where n is the smallest number with this property. Then $T_\beta^{n-1}(x) \neq S_\beta^{n-1}(x)$, so there is a smallest number $m \leq n-1$ such that $T_\beta^m(x) \neq S_\beta^m(x)$, which implies that $T_\beta^{m-1}(x) = S_\beta^{m-1}(x)$ is in a switch region. Consequently, the m-th greedy digit is larger than the m-th lazy digit. By the choice of n we conclude that the lazy expansion is lexicographically smaller than the greedy expansion.

4.2 Random β-expansions

This section is based on [4] and [5].

We now introduce a way to dynamically generate any β-expansion of a number $x \in [0, \frac{1}{\beta-1}]$. In the previous section it was shown that the lazy expansion is lexicographically smaller than the greedy expansion. These expansions were generated by repeatedly applying the lazy map and the greedy map, respectively. If at each iteration we randomly choose which of these maps to apply and choose the digit corresponding to the chosen map, we obtain a random β-expansion of x. We shall now make this precise.

The greedy map is piecewise-defined on a partition of $[0, \frac{1}{\beta-1}]$, and on each interval in this partition a certain digit is assigned. The same is true for the lazy map. If we superimpose the greedy and lazy maps, we obtain a finer partition (see the right-hand side in Figure 4.2). This partition is made up of two categories of intervals. Firstly, there are the equality regions E_k, where the greedy and lazy maps assign the same digit k. Secondly, there are the switch regions S_k, where the greedy map assigns digit k and the lazy map assigns digit $k-1$. Explicitly,

$$E_0 = \left[0, \frac{1}{\beta}\right),$$

$$E_k = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta}\right), \quad k = 1, \ldots, \lfloor\beta\rfloor - 1,$$

$$E_{\lfloor\beta\rfloor} = \left(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{\lfloor\beta\rfloor - 1}{\beta}, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$$

and

$$S_k = \left[\frac{k}{\beta}, \frac{\lfloor\beta\rfloor}{\beta(\beta-1)} + \frac{k-1}{\beta}\right], \quad k = 1, \ldots, \lfloor\beta\rfloor.$$

Let $\Omega = \{0,1\}^{\mathbb{N}}$, representing the set of possible outcomes of infinitely many coin tosses, and let $\sigma : \Omega \to \Omega$, $\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots)$ be the left shift. Define the map

$$K_\beta : \Omega \times \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right] \to \Omega \times \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right],$$

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k) & \text{if } x \in E_k, & k = 0, 1, \ldots, \lfloor\beta\rfloor \\ (\sigma(\omega), \beta x - k) & \text{if } x \in S_k \text{ and } \omega_1 = 1, & k = 1, \ldots, \lfloor\beta\rfloor \\ (\sigma(\omega), \beta x - k + 1) & \text{if } x \in S_k \text{ and } \omega_1 = 0, & k = 1, \ldots, \lfloor\beta\rfloor \end{cases}$$

We assign the first digit as follows:

$$d_1 = d_1(\omega, x) = \begin{cases} k & \text{if } \; x \in E_k, \qquad\qquad k = 0, 1, ..., \lfloor \beta \rfloor \\ & \text{or } \; x \in S_k, \omega_1 = 1, \quad k = 1, 2, ..., \lfloor \beta \rfloor \\ k-1 & \text{if } \; x \in S_k, \omega_1 = 0, \quad k = 1, 2, ..., \lfloor \beta \rfloor \end{cases}$$

and the n-th digit as

$$d_n = d_n(\omega, x) = d_1\left(K_\beta^{n-1}(\omega, x)\right).$$

Every time the orbit of x under K_β hits a switch region, we flip a coin to determine which map is applied, and hence which digit is chosen. A 1 ('heads') means that the greedy map will be applied, and a 0 ('tails') means that the lazy map will be applied. On the equality regions, the greedy and the lazy map are the same, so the choice does not matter and we do not flip a coin.

Let $\pi_2 : \Omega \times \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right] \to \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$ be the projection onto the second coordinate.

We can write $\pi_2\left(K_\beta(\omega, x)\right) = \beta x - d_1(\omega, x)$, so

$$\begin{aligned} \pi_2\left(K_\beta^2(\omega, x)\right) &= \beta^2 x - \beta d_1(\omega, x) - d_1(K_\beta(\omega, x)) \\ &= \beta^2 x - \beta d_1(\omega, x) - d_2(\omega, x) \\ &= \beta^2 x - \beta d_1 - d_2 \end{aligned}$$

and in general,

$$\pi_2\left(K_\beta^n(\omega, x)\right) = \beta^n x - \sum_{i=1}^{n} \beta^{n-i} d_i \ ,$$

which we can rewrite as

$$x = \frac{d_1}{\beta} + ... + \frac{d_n}{\beta^n} + \frac{\pi_2\left(K_\beta^n(\omega, x)\right)}{\beta^n}.$$

As $\pi_2\left(K_\beta^n(\omega, x)\right)$ is bounded, we find that for all $\omega \in \Omega$ and $x \in \left[0, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$,

$$x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = \sum_{i=1}^{\infty} \frac{d_i(\omega, x)}{\beta^i}.$$

So far we have only used the coin tosses as a motivating argument, but have not actually needed any probability measure. For all $\omega \in \Omega$ we have derived an algorithm which produces a β-expansion. The following theorem states that all β-expansions of x can be generated this way:

Theorem 4.2.1. *Let $x \in \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ and suppose we can write $x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$, $a_i \in \{0,1,...,\lfloor\beta\rfloor\}$. Then there exists an $\omega \in \Omega$ such that $a_i = d_i(\omega,x)$ for all $i \geq 1$.*

Proof. See [4]. □

Theorem 4.2.2 (theorem 1 in [4])**.** *Let $\omega, \omega' \in \Omega$ be such that $\omega <_{\text{lex}} \omega'$. Then*

$$0.d_1(\omega,x)d_2(\omega,x)... \leq_{\text{lex}} 0.d_1(\omega',x)d_2(\omega',x)...$$

Proof. Let i be the first index such that $\omega_i \neq \omega_i'$. As $\omega <_{\text{lex}} \omega'$, we must have $\omega_i = 0$ and $\omega_i' = 1$. Let $0 \leq r_i \leq \infty$ be the time of the i-th visit to the switch region $\Omega \times (\cup_i S_i)$ of the orbit of (ω,x) under K_β. Then $\pi_2(K_\beta^j(\omega,x)) = \pi_2(K_\beta^j(\omega',x))$ for all $0 \leq j \leq r_i$, so $d_j(\omega,x) = d_j(\omega',x)$ for all $1 \leq j \leq r_i$ (if $r_i = 0$ then the previous statement is vacuous).

If $r_i = \infty$, then by the above, $d_j(\omega,x) = d_j(\omega',x)$ for all j.

If $r_i < \infty$, then $K_\beta^{r_i}(\omega,x), K_\beta^{r_i}(\omega',x) \in \Omega \times (\cup_i S_i)$. As $\omega_i = 0$ and $\omega_i' = 1$, we have $d_{r_i+1}(\omega',x) = d_{r_i+1}(\omega,x) + 1$ so that

$$0.d_1(\omega,x)d_2(\omega,x)... <_{\text{lex}} 0.d_1(\omega',x)d_2(\omega',x)...$$

We conclude that in either case,

$$0.d_1(\omega,x)d_2(\omega,x)... \leq_{\text{lex}} 0.d_1(\omega',x)d_2(\omega',x)...$$

□

Theorem 4.2.2 is a consequence of the fact that we did not apply the left shift to ω for x in the equality regions. If we applied the left shift in every iteration (even on equality regions), the statement in this theorem would no longer be true. This is because if i is the first index such that $\omega_i \neq \omega_i'$, then this difference does not affect any digit if the second coordinate of $K_\beta^{i-1}(\omega,x)$ is in an equality region.

Note that the lazy map corresponds to the lexicographically smallest member of Ω, namely $\omega = (0,0,0,...)$, and the greedy map corresponds to the lexicographically largest member of Ω, namely $\omega = (1,1,1,...)$. Hence the combination of Theorem 4.2.1 and 4.2.2

provides another proof that the lazy expansion is lexicographically smaller than or equal to the greedy expansion.

By Theorem 4.2.1 and 4.2.2, all β-expansions of x can be generated by this process, and they are all lexicographically between the lazy and greedy expansions of x. As we have seen, at least for $1 < \beta < 2$, not all β-expansions lexicographically between the lazy and greedy expansions of x are necessarily also β-expansions of x. Therefore, the orbit of every such x must eventually hit an equality region for all $\omega \in \Omega$. To ensure that the resulting expansion is an expansion of x, the algorithm imposes two restrictions. We are only free to choose a digit in the switch regions, and if we choose the greedy digit, we must then also apply the greedy map. The combination of these two requirements is restrictive enough to limit the resulting β-expansion to expansions of x, as intended. The restrictions are as mild as possible in the sense that the algorithm generates all β-expansions of x.

These results do not say anything about how many β-expansions there are of x. As the choice of ω only matters in the switch regions, how many expansions there are depends on how often the orbit of x hits a switch region. In [4] it was shown that there is a unique β-expansion if and only if the orbit of x under the greedy map T_β always stays in the equality regions, in which case the lazy and greedy expansions coincide. A result in the opposite direction was shown in [10], namely that almost every number has a continuum of β-expansions, i.e. the cardinality of the set of β-expansions is the same as the cardinality of $\mathbb{R}$. In particular, almost every x has infinitely many β-expansions.

The intuition for a number having infinitely many β-expansions follows from the fact that there are two different β-expansions, namely the lazy and greedy expansions. For almost every x, there are infinitely many expansions lexicographically in between these two expansions, so it is intuitively plausible that at least some of these expansions are also expansions of x.

This is in contrast to expansions in integer bases: if there are two different expansions in an integer base, then there are no expansions lexicographically strictly in between them, and hence there can be at most two expansions. For instance, there are no expansions lexicographically strictly between 0.9999999... and 1.0000000... As these are respectively the lazy and greedy expansions of 1 in base 10, there are no other expansions of 1 in base 10.

4.3 A two-dimensional random map

In this section, we define a two-dimensional random map in a manner analogous to the one-dimensional map from the previous section. We define this map by applying the one-dimensional random map to both coordinates individually. We then show that K can be essentially identified with the left shift on $\mathbf{D} = \{(0,0),(0,1),(1,0),(1,1)\}^{\mathbb{N}}$ and find an invariant measure of maximal entropy.

Let $1 < \beta_i < 2$ be non-integers, $i = 1,2$, and consider the random map from the previous section. In this case, any digit will be in $\{0,1\}$ and the equality regions reduce to

$E_0^{\beta_i} = [0, \frac{1}{\beta_i})$, where digit 0 is assigned by both the greedy and lazy maps, and

$E_1^{\beta_i} = (\frac{\lfloor\beta_i\rfloor}{\beta_i(\beta_i - 1)} + \frac{\lfloor\beta_i\rfloor - 1}{\beta_i}, \frac{\lfloor\beta_i\rfloor}{\beta_i - 1}] = (\frac{1}{\beta_i(\beta_i - 1)}, \frac{1}{\beta_i - 1}]$, where digit 1 is assigned by both maps.

Similarly, the switch regions reduce to

$S^{\beta_i} = [\frac{1}{\beta_i}, \frac{1}{\beta_i(\beta_i - 1)}]$, where digit 1 is assigned by the greedy map, and digit 0 by the lazy map.

As before, on the switch regions we randomise the choice of map, and so the choice of digit. We flip two coins, the first coin deciding the digit with respect to base β_1, and the second coin deciding the digit w.r.t. base β_2. A 1 ('heads') on the first coin means that the greedy map will be applied to the x coordinate, and a 0 ('tails') means that the lazy map will be applied to the x coordinate. Similarly, the second coin toss determines what map is applied to the y coordinate. We shall see that this process simultaneously generates two expansions, respectively in base β_1 and β_2. We now make this precise.

Let σ be the left shift. We define the random map

$$K : \Omega \times \Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}] \to \Omega \times \Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}],$$

$$K(\omega,\omega',x,y)=\begin{cases}(\omega,\omega',\beta_1 x,\beta_2 y) & \text{if } (x,y)\in E_{00}\\ (\omega,\omega',\beta_1 x-1,\beta_2 y) & \text{if } (x,y)\in E_{10}\\ (\omega,\omega',\beta_1 x,\beta_2 y-1) & \text{if } (x,y)\in E_{01}\\ (\omega,\omega',\beta_1 x-1,\beta_2 y-1) & \text{if } (x,y)\in E_{11}\\ (\sigma(\omega),\omega',\beta_1 x-\omega_1,\beta_2 y) & \text{if } (x,y)\in S_{\bullet 0}\\ (\omega,\sigma(\omega'),\beta_1 x,\beta_2 y-\omega_1') & \text{if } (x,y)\in S_{0\bullet}\\ (\sigma(\omega),\sigma(\omega'),\beta_1 x-\omega_1,\beta_2 y-\omega_1') & \text{if } (x,y)\in S_{\bullet\bullet}\\ (\sigma(\omega),\omega',\beta_1 x-\omega_1,\beta_2 y-1) & \text{if } (x,y)\in S_{\bullet 1}\\ (\omega,\sigma(\omega'),\beta_1 x-1,\beta_2 y-\omega_1') & \text{if } (x,y)\in S_{1\bullet}\end{cases}$$

where we denote the four equality regions:

$$E_{00}=E_0^{\beta_1}\times E_0^{\beta_2}$$

$$E_{10}=E_1^{\beta_1}\times E_0^{\beta_2}$$

$$E_{01}=E_0^{\beta_1}\times E_1^{\beta_2}$$

$$E_{11}=E_1^{\beta_1}\times E_1^{\beta_2}$$

and the five switch regions:

$$S_{\bullet 0}=S^{\beta_1}\times E_0^{\beta_2}$$

$$S_{0\bullet}=E_0^{\beta_1}\times S^{\beta_2}$$

$$S_{\bullet\bullet}=S^{\beta_1}\times S^{\beta_2}$$

$$S_{\bullet 1}=S^{\beta_1}\times E_1^{\beta_2}$$

$$S_{1\bullet}=E_1^{\beta_1}\times S^{\beta_2}$$

The notation denotes what digits are dependent on the outcome of the coin tosses, and what digits are fixed. On the switch regions $S_{\bullet 0}, S_{0\bullet}, S_{\bullet 1}$ and $S_{1\bullet}$, only one coin toss determines where (x,y) is mapped under K. For instance, in $S_{\bullet 1}$, the digit chosen in the x coordinate (the β_1 digit) depends on ω_1, while the digit chosen in the y coordinate (the β_2 digit) is always 1 and does not depend on ω_1'. The regions are shown in Figure 4.3, along with the possible digits that can be assigned by K.

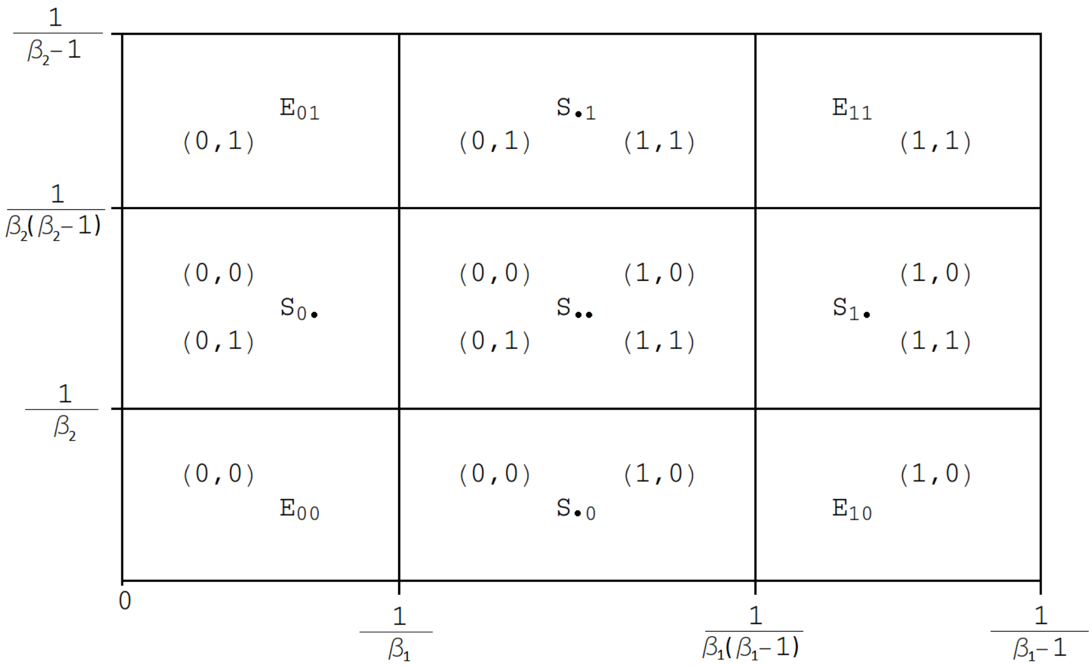

Fig. 4.3 The regions with their possible digits assigned by K.

We can use a similar method as in the previous section to simultaneously obtain a β_1-expansion of x and a β_2-expansion of y. We assign the first digits of x and y as follows:

$$\mathbf{d}_1 = \mathbf{d}_1(\omega,\omega',x,y) = \begin{cases} (k,l) & \text{if } (x,y) \in E_{kl}, & k,l \in \{0,1\} \\ & \text{or } (x,y) \in S_{\bullet l}, \omega_1 = k, & k,l \in \{0,1\} \\ & \text{or } (x,y) \in S_{\bullet\bullet}, \omega_1 = k, \omega_1' = l, & k,l \in \{0,1\} \\ & \text{or } (x,y) \in S_{k\bullet}, \omega_1' = l, & k,l \in \{0,1\} \end{cases}$$

and the n-th digit as

$$\mathbf{d}_n = \mathbf{d}_n(\omega,\omega',x,y) = \mathbf{d}_1\left(K^{n-1}(\omega,\omega',x,y)\right) \qquad n \geq 1.$$

We can write $\pi_{1,3}(K(\omega,\omega',x,y)) = K_{\beta_1}(\omega,x)$, where K_{β_1} is the one-dimensional random map defined in the previous section. Similarly, $\pi_{2,4}(K(\omega,\omega',x,y)) = K_{\beta_2}(\omega',y)$. We now prove by induction that $\pi_{1,3}(K^n(\omega,\omega',x,y)) = K^n_{\beta_1}(\omega,x)$ and $\pi_{2,4}(K^n(\omega,\omega',x,y)) = K^n_{\beta_2}(\omega',y)$ for all $n \geq 1$. The base case has already been shown, so assume that the statement is true for $n-1$; we now show that it is true for n. Then

$$\begin{aligned}\pi_{1,3}\left(K^n(\omega,\omega',x,y)\right) &= \pi_{1,3}\left(K^{n-1}\circ K(\omega,\omega',x,y)\right)\\ &= K^{n-1}_{\beta_1}\left(\pi_{1,3}(K(\omega,\omega',x,y)\right)\\ &= K^n_{\beta_1}(\omega,x)\end{aligned}$$

By the principle of induction we conclude that $\pi_{1,3}(K^n(\omega,\omega',x,y)) = K^n_{\beta_1}(\omega,x)$ for $n \geq 1$. Similarly, $\pi_{2,4}(K^n(\omega,\omega',x,y)) = K^n_{\beta_2}(\omega',y)$.

Furthermore, $\mathbf{d}_1(\omega,\omega',x,y) = \left(d_1(\pi_{1,3}(\omega,\omega',x,y)), d_1'(\pi_{2,4}(\omega,\omega',x,y))\right)$, where d_1 is the first digit of the β_1-expansion of x corresponding to ω defined in the previous section, and d_1' is the first digit of the β_2-expansion of y corresponding to ω'. Therefore, every time we iterate K, we obtain the pair of digits

$$\begin{aligned}\mathbf{d}_n(\omega,\omega',x,y) &= \mathbf{d}_1\left(K^{n-1}\left(\omega,\omega',x,y\right)\right)\\ &= \left(d_1\left(\pi_{1,3}\left(K^{n-1}\left(\omega,\omega',x,y\right)\right)\right), d_1'\left(\pi_{2,4}\left(K^{n-1}\left(\omega,\omega',x,y\right)\right)\right)\right)\\ &= \left(d_1\left(K^{n-1}_{\beta_1}(\omega,x)\right), d_1'\left(K^{n-1}_{\beta_2}(\omega',y)\right)\right)\\ &= \left(d_n(\omega,x), d_n'(\omega',y)\right).\end{aligned}$$

Applying Theorem 4.2.1 to both coordinates, we find that all β_1-expansions of x and all β_2-expansions of y can be obtained by this algorithm. If we take $\beta := \beta_1 = \beta_2$, we can also interpret this as an algorithm to generate β-expansions of $z = x + iy \in [0, \frac{1}{\beta-1}]^2 \subset \mathbb{C}$.

Let K_β be the one-dimensional random map on $\Omega \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}]$ defined earlier in this chapter. In [4] it was shown that K_β can be essentially identified with the left shift $\hat{\sigma}$ on $D = \{0, ..., \lfloor\beta\rfloor\}^{\mathbb{N}}$. More precisely, it was shown that the map $\hat{\varphi}(\omega,x) = (d_1(\omega,x), d_2(\omega,x), ...)$ is a measurable isomorphism from $(\Omega \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}], \hat{\mathscr{A}} \times \hat{\mathscr{B}}, \nu_\beta, K_\beta) \to (D, \hat{\mathscr{D}}, \mathbb{P}, \hat{\sigma})$, where $\hat{\mathscr{A}}$ and $\hat{\mathscr{D}}$ are the product σ-algebra on Ω and D, respectively, $\hat{\mathscr{B}}$ is the Borel σ-algebra on $[0, \frac{\lfloor\beta\rfloor}{\beta-1}]$, $\mathbb{P}$ is the uniform product measure on D and ν_β is the unique K_β-invariant measure of maximal entropy on $\hat{\mathscr{A}} \times \hat{\mathscr{B}}$. For a definition of entropy and a discussion of its properties, we refer to chapter 4 in [6].

To see why this is intuitively plausible, let $\beta > 1$ and suppose we can write $x = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = 0.a_1a_2a_3...$ for some $a \in D$. By Theorem 4.2.1, there exists an $\omega \in \Omega$ such that $a = d(\omega, x)$. We can write $\pi_2(K_\beta(\omega,x)) = \beta x - d_1(\omega,x)$. Then

$$\begin{aligned}
\pi_2(K_\beta(\omega,x)) &= \beta \sum_{i=1}^{\infty} \frac{d_i(\omega,x)}{\beta^i} - d_1(\omega,x) \\
&= d_1(\omega,x) + \sum_{i=2}^{\infty} \frac{d_i(\omega,x)}{\beta^{i-1}} - d_1(\omega,x) \\
&= \sum_{i=1}^{\infty} \frac{d_{i+1}(\omega,x)}{\beta^i} \\
&= 0.a_2a_3a_4...
\end{aligned}$$

Hence $\pi_2(K_\beta(\omega, 0.a_1a_2a_3...)) = 0.a_2a_3a_4...$ In other words, a is left-shifted after applying K_β for an appropriate choice of ω. Furthermore, $\mathbb{P}$ assigns the same probability to all digits, which is intuitively the 'most random' measure, and indeed entropy is maximised by $\mathbb{P}$. As isomorphic systems have the same (maximal) entropy, the measure ν_β corresponding to $\mathbb{P}$ should also be of maximal entropy.

We will show that a similar result is true for K. Let

$$\mathbf{D} = \{(0,0),(0,1),(1,0),(1,1)\}^{\mathbb{N}}$$

and let $\sigma : \mathbf{D} \to \mathbf{D}$, $\sigma\big((a_1,a_1'),(a_2,a_2'),(a_3,a_3'),...\big) = \big((a_2,a_2'),(a_3,a_3'),(a_4,a_4')...\big)$ be the left shift on $\mathbf{D}$. We will show that K can be essentially identified with σ by giving an explicit measurable isomorphism, after which we find a K-invariant measure of maximal entropy. The proof will be analogous to the proof in [4]. Let $1 < \beta_i < 2, i = 1,2$. Define the function $\phi : \Omega \times \Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}] \to \mathbf{D}$,

$$\phi(\omega,\omega',x,y) = \big(\mathbf{d}_1(\omega,\omega',x,y), \mathbf{d}_2(\omega,\omega',x,y),...\big).$$

We will show that ϕ is a measurable isomorphism from $(\Omega \times \Omega \times [0, \frac{1}{\beta_1-1}] \times [0, \frac{1}{\beta_2-1}], \mathscr{A} \times \mathscr{B}, \nu, K) \to (\mathbf{D}, \mathscr{D}, \mathbb{P} \times \mathbb{P}, \sigma)$, where $\mathscr{A}$ and $\mathscr{D}$ are the product σ-algebra on $\Omega \times \Omega$ and $\mathbf{D}$, respectively, $\mathscr{B}$ is the Borel σ-algebra on $[0, \frac{1}{\beta_1-1}] \times [0, \frac{1}{\beta_2-1}]$, $\mathbb{P} \times \mathbb{P}$ is the uniform product measure on $\mathbf{D}$ and ν is a K-invariant measure of maximal entropy on $\mathscr{A} \times \mathscr{B}$.

We first show that ϕ is surjective. Let $\mathbf{a} = \big((a_1,a_1'),(a_2,a_2'),(a_3,a_3'),...\big) \in \mathbf{D}$. Then a is a β_1-expansion of x and a' is a β_2-expansion of y, where $x = \sum_{i=1}^{\infty} \frac{a_i}{\beta_1^i}$ and $y = \sum_{i=1}^{\infty} \frac{a_i'}{\beta_2^i}$. As we have seen, Theorem 4.2.1 implies that there exist ω,ω' such that $(a_i,a_i')=\mathbf{d}_i(\omega,\omega',x,y)$ for all i. Hence $\phi(\omega,\omega',x,y)=\mathbf{a}$. We conclude that ϕ is surjective. We will see that if we restrict ϕ to an appropriate K-invariant subset $\mathbf{Z}$, then the restriction ϕ' is invertible.

Let

$$\mathbf{Z}_1=\Big\{(\omega,\omega',x,y)\in\Omega\times\Omega\times[0,\tfrac{1}{\beta_1-1}]\times[0,\tfrac{1}{\beta_2-1}]:\\ K^n(\omega,\omega',x,y)\in\Omega\times\Omega\times S^{\beta_1}\times[0,\tfrac{1}{\beta_2-1}]\text{ for infinitely many } n\Big\},$$

$$\mathbf{Z}_2=\Big\{(\omega,\omega',x,y)\in\Omega\times\Omega\times[0,\tfrac{1}{\beta_1-1}]\times[0,\tfrac{1}{\beta_2-1}]:\\ K^n(\omega,\omega',x,y)\in\Omega\times\Omega\times[0,\tfrac{1}{\beta_1-1}]\times S^{\beta_2}\text{ for infinitely many } n\Big\},$$

$$\mathbf{D}_1'=\left\{\big((a_1,a_1'),(a_2,a_2'),\ldots\big)\in\mathbf{D}:\sum_{i=1}^{\infty}\frac{a_{n+i-1}}{\beta_1^i}\in S^{\beta_1}\text{ for infinitely many } n\right\},$$

$$\mathbf{D}_2'=\left\{\big((a_1,a_1'),(a_2,a_2'),\ldots\big)\in\mathbf{D}:\sum_{i=1}^{\infty}\frac{a'_{n+i-1}}{\beta_2^i}\in S^{\beta_2}\text{ for infinitely many } n\right\}.$$

Let $\mathbf{Z}=\mathbf{Z}_1\cap\mathbf{Z}_2$ and let $\mathbf{D}'=\mathbf{D}_1'\cap\mathbf{D}_2'$. Then $\phi(\mathbf{Z})=\mathbf{D}'$, $K^{-1}(\mathbf{Z})=\mathbf{Z}$ and $\sigma^{-1}(\mathbf{D}')=\mathbf{D}'$.

Let ϕ' be the restriction of ϕ to $\mathbf{Z}$. We will show that $\phi':\mathbf{Z}\to\mathbf{D}'$ is bijective. Let $\big((a_1,a_1'),(a_2,a_2'),\ldots\big)\in\mathbf{D}'$ and recursively define

$$r_1=\min\left\{n\ge 1:\sum_{i=1}^{\infty}\frac{a_{n+i-1}}{\beta_1^i}\in S^{\beta_1}\right\},$$

$$r_k=\min\left\{n> r_{k-1}:\sum_{i=1}^{\infty}\frac{a_{n+i-1}}{\beta_1^i}\in S^{\beta_1}\right\},$$

$$s_1=\min\left\{n\ge 1:\sum_{i=1}^{\infty}\frac{a'_{n+i-1}}{\beta_2^i}\in S^{\beta_2}\right\},$$

$$s_k=\min\left\{n> s_{k-1}:\sum_{i=1}^{\infty}\frac{a'_{n+i-1}}{\beta_2^i}\in S^{\beta_2}\right\}.$$

For $k \geq 1$ let $(\omega_k, \omega'_k) = (a_{r_k}, a'_{s_k})$. Define

$$\phi^{-1}\big((a_1,a'_1),(a_2,a'_2),\ldots\big) = \left(\omega,\omega',\sum_{i=1}^{\infty}\frac{a_i}{\beta_1^i},\sum_{i=1}^{\infty}\frac{a'_i}{\beta_2^i}\right).$$

Then ϕ' and ϕ^{-1} are measurable, and ϕ^{-1} is the inverse of ϕ'.

Lemma 4.3.1. $\mathbb{P}\times\mathbb{P}\left(\boldsymbol{D}'\right) = 1$.

Proof. We first show that $\mathbb{P}\times\mathbb{P}\left(\mathbf{D}'_1\right) = 1$. For $\mathbf{a} = ((a_1,a'_1),(a_2,a'_2),\ldots) \in \mathbf{D}$ and $m \geq 1$, define

$$x_m = \frac{1}{\beta_1} + \sum_{i=1}^{\infty}\frac{a_i}{\beta_1^{m+i}}.$$

Then

$$\frac{1}{\beta_1} \leq x_m \leq \frac{1}{\beta_1} + \sum_{i=1}^{\infty}\frac{1}{\beta_1^{m+i}} = \frac{1}{\beta_1}\left(1+\frac{1}{\beta_1^{m-1}(\beta_1-1)}\right).$$

As $1<\beta_1<2$ and $1+\frac{1}{\beta_1^{m-1}(\beta_1-1)} \downarrow 1$ as $m\to\infty$, there exists an integer $N>0$ such that for all $m>N$,

$$\frac{1}{\beta_1} \leq x_m \leq \frac{1}{\beta_1(\beta_1-1)}.$$

So $x_m \in S^{\beta_1}$ for all $m>N$. Note that N does not depend on $\mathbf{a}$.

Let

$$\mathbf{D}''_1 = \{\big((a_1,a'_1),(a_2,a'_2),\ldots\big) \in \mathbf{D} : a_n a_{n+1}\ldots a_{n+N-1} = 1\underbrace{00\ldots0}_{N-1\text{ times}} \text{ for infinitely many } n\}.$$

By the above, $\mathbf{D}''_1 \subseteq \mathbf{D}'_1$. The second Borel-Cantelli lemma implies that $\mathbb{P}\times\mathbb{P}\left(\mathbf{D}''_1\right) = 1$, so $\mathbb{P}\times\mathbb{P}\left(\mathbf{D}'_1\right) = 1$. Similarly, $\mathbb{P}\times\mathbb{P}\left(\mathbf{D}'_2\right) = 1$. As the intersection of sets of measure 1 also has measure 1, we conclude that $\mathbb{P}\times\mathbb{P}\left(\mathbf{D}'\right) = 1$. □

Define the measure ν on $\mathscr{A}\times\mathscr{B}$ by $\nu(A) = \mathbb{P}\times\mathbb{P}(\phi(\mathbf{Z}\cap A))$. Then

$$\nu(\mathbf{Z}) = \mathbb{P}\times\mathbb{P}(\phi(\mathbf{Z})) = \mathbb{P}\times\mathbb{P}\left(\mathbf{D}'\right) = 1.$$

So far we have seen that conditions (i) and (ii) of Definition 3.1.11 is satisfied. We now show that condition (iii) is satisfied. For all $D \in \mathscr{D} \cap \mathbf{D}'$,

$$\begin{aligned}\mathbb{P}\times\mathbb{P}(D) &= \mathbb{P}\times\mathbb{P}\left(\phi'\left(\phi^{-1}(D)\right)\right)\\ &= \nu(\phi^{-1}(D)),\end{aligned}$$

where we used that ϕ' is surjective in the first step, and that $\phi^{-1}(D) \subseteq \mathbf{Z}$ in the first and second step. Therefore, condition (iii) is satisfied.

We now verify condition (iv). We have

$$\begin{aligned}\phi\circ K(\omega,\omega',x,y) &= \left(\mathbf{d}_1\left(K(\omega,\omega',x,y)\right),\mathbf{d}_2\left(K(\omega,\omega',x,y)\right),\ldots\right)\\ &= \left(\mathbf{d}_2(\omega,\omega',x,y),\mathbf{d}_3(\omega,\omega',x,y),\ldots\right)\\ &= \sigma\circ\phi(\omega,\omega',x,y).\end{aligned}$$

Hence condition (iv) is satisfied.

Lemma 4.3.2. *Let $B \in (\mathscr{A}\times\mathscr{B})\cap\mathbf{Z}$. Then*

$$\phi'\left(K^{-1}(B)\right) = \sigma^{-1}\left(\phi'(B)\right).$$

Proof. By condition (iv), $\phi'\circ K = \sigma\circ\phi'$. Hence

$$\left(\phi'\circ K\right)^{-1}(B) = K^{-1}\circ\phi^{-1}(B) = \phi^{-1}\circ\sigma^{-1}(B) = \left(\sigma\circ\phi'\right)^{-1}(B).$$

As ϕ' is bijective, the above implies that

$$K^{-1}(B) = \phi^{-1}\circ\sigma^{-1}\circ\phi'(B).$$

We conclude that

$$\phi'\left(K^{-1}(B)\right) = \phi'\circ\phi^{-1}\circ\sigma^{-1}\circ\phi'(B) = \sigma^{-1}\left(\phi'(B)\right).$$

□

It remains to show that $(\Omega\times\Omega\times[0,\frac{1}{\beta_1-1}]\times[0,\frac{1}{\beta_2-1}],\mathscr{A}\times\mathscr{B},\nu,K)$ and $(\mathbf{D},\mathscr{D},\mathbb{P}\times\mathbb{P},\sigma)$ are dynamical systems. Clearly, $\mathbb{P}\times\mathbb{P}$ is σ-invariant. Let $B \in (\mathscr{A}\times\mathscr{B})\cap\mathbf{Z}$. Then

$$\begin{aligned}
\nu(K^{-1}(B)) &= \mathbb{P}\times\mathbb{P}\left(\phi'\left(K^{-1}(B)\right)\right) && (\text{as } K^{-1}(\mathbf{Z})=\mathbf{Z})\\
&= \mathbb{P}\times\mathbb{P}\left(\sigma^{-1}\left(\phi'(B)\right)\right) && (\text{by Lemma 4.3.2})\\
&= \mathbb{P}\times\mathbb{P}\left(\phi'(B)\right) && (\text{as } \mathbb{P}\times\mathbb{P} \text{ is } \sigma\text{-invariant})\\
&= \nu(B).
\end{aligned}$$

By the previous discussion, we have the following theorem:

Theorem 4.3.3 (cf. theorem 4 in [4]). *Let $1<\beta_i<2, i=1,2$ be non-integers. The map ϕ :$(\Omega\times\Omega\times[0,\frac{1}{\beta_1-1}]\times[0,\frac{1}{\beta_2-1}],\mathscr{A}\times\mathscr{B},\nu,K)\to(\boldsymbol{D},\mathscr{D},\mathbb{P}\times\mathbb{P},\sigma)$ is a measurable isomorphism.*

The uniform product measure $\mathbb{P}\times\mathbb{P}$ on 4 symbols is the unique measure of maximal entropy on $\mathbf{D}$, with entropy equal to $\log(4)$. We conclude that ν is the unique measure of maximal entropy that has support $\mathbf{Z}$, with entropy equal to $\log(4)$.

4.4 Skew products

In this section, we consider a variant of the map K from the previous sections. We apply the left shift in every iteration, even on the equality regions.

We define the random map

$$R:\Omega\times\Omega\times[0,\frac{1}{\beta_1-1}]\times[0,\frac{1}{\beta_2-1}]\to\Omega\times\Omega\times[0,\frac{1}{\beta_1-1}]\times[0,\frac{1}{\beta_2-1}],$$

$$R(\omega,\omega',x,y)=\begin{cases}
(\sigma(\omega),\sigma(\omega'),\beta_1x,\beta_2y) & \text{if } (x,y)\in E_{00}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-1,\beta_2y) & \text{if } (x,y)\in E_{10}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x,\beta_2y-1) & \text{if } (x,y)\in E_{01}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-1,\beta_2y-1) & \text{if } (x,y)\in E_{11}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-\omega_1,\beta_2y) & \text{if } (x,y)\in S_{\bullet 0}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x,\beta_2y-\omega_1') & \text{if } (x,y)\in S_{0\bullet}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-\omega_1,\beta_2y-\omega_1') & \text{if } (x,y)\in S_{\bullet\bullet}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-\omega_1,\beta_2y-1) & \text{if } (x,y)\in S_{\bullet 1}\\
(\sigma(\omega),\sigma(\omega'),\beta_1x-1,\beta_2y-\omega_1') & \text{if } (x,y)\in S_{1\bullet}
\end{cases}$$

We call R the skew product transformation. Note that the proof of Theorem 4.3.3 does not work for this map, because ϕ is no longer injective even if S^{β_1} and S^{β_2} are each hit infinitely often. Furthermore, the statement in Theorem 4.2.2 no longer holds.

The projection of R onto the x and y coordinates has four possible realisations, which we denote by R_{00}, R_{10}, R_{01} and R_{11}. Here the subscript corresponds to the outcomes of the two coin flips, i.e. $\pi_{3,4} \circ R(\omega, \omega', \cdot, \cdot) = R_{ij}(\cdot, \cdot)$ if and only if $\omega_1 = i$ and $\omega_1' = j$. These realisations only differ on the switch regions. For instance, there is the realisation

$$R_{10}(x,y) = \begin{cases} (\beta_1 x, \beta_2 y) & \text{if } (x,y) \in E_{00} \\ (\beta_1 x - 1, \beta_2 y) & \text{if } (x,y) \in E_{10} \\ (\beta_1 x, \beta_2 y - 1) & \text{if } (x,y) \in E_{01} \\ (\beta_1 x - 1, \beta_2 y - 1) & \text{if } (x,y) \in E_{11} \\ (\beta_1 x - 1, \beta_2 y) & \text{if } (x,y) \in S_{\bullet 0} \\ (\beta_1 x, \beta_2 y) & \text{if } (x,y) \in S_{0 \bullet} \\ (\beta_1 x - 1, \beta_2 y) & \text{if } (x,y) \in S_{\bullet\bullet} \\ (\beta_1 x - 1, \beta_2 y - 1) & \text{if } (x,y) \in S_{\bullet 1} \\ (\beta_1 x - 1, \beta_2 y) & \text{if } (x,y) \in S_{1 \bullet} \end{cases}$$

Let $p_1 = \mathbb{P}(\omega_1 = 1)$ and $p_2 = \mathbb{P}(\omega_1' = 1)$ denote the probabilities of heads. The realisations occur with the following probabilities:

$$\begin{aligned} \mathbb{P}(\pi_{3,4} \circ R = R_{00}) &= (1-p_1)(1-p_2) \\ \mathbb{P}(\pi_{3,4} \circ R = R_{10}) &= p_1(1-p_2) \\ \mathbb{P}(\pi_{3,4} \circ R = R_{01}) &= (1-p_1)p_2 \\ \mathbb{P}(\pi_{3,4} \circ R = R_{11}) &= p_1 p_2 \end{aligned}$$

These probabilities are the same for all (x,y); in other words, the probabilities are not position-dependent.

For $0 < p < 1$, let m_p be the Bernoulli measure on $\Omega = \{0,1\}^{\mathbb{N}}$, i.e.

$$m_p\left(\{\omega_1 = k_1, ..., \omega_n = k_n\}\right) = p^{\sum_{j=1}^{n} k_j}(1-p)^{n-\sum_{j=1}^{n} k_j}.$$

Lemma 4.4.1. *Let μ be a probability measure on $[0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}]$. Then $\theta := m_{p_1} \times m_{p_2} \times \mu$ is R-invariant if and only if*

$$\mu = (1-p_1)(1-p_2)\cdot \mu \circ R_{00}^{-1} + p_1(1-p_2)\cdot \mu \circ R_{10}^{-1} + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1} + p_1 p_2 \cdot \mu \circ R_{11}^{-1}.$$

Proof. Let $A = C \times B \in \mathscr{A} \times \mathscr{B}$ and define $C_{i,j} = \{\omega_1 = i, \omega_1' = j\} \cap \sigma^{-1}(C)$. Then

$$R^{-1}(A) = C_{0,0} \times R_{00}^{-1}(B) \cup C_{1,0} \times R_{10}^{-1}(B) \cup C_{0,1} \times R_{01}^{-1}(B) \cup C_{1,1} \times R_{11}^{-1}(B).$$

So

$$\begin{aligned} \theta\left(R^{-1}(A)\right) = &(1-p_1)(1-p_2)\cdot m_{p_1} \times m_{p_2}(C)\cdot \mu \circ R_{00}^{-1}(B) \\ &+ p_1(1-p_2)\cdot m_{p_1} \times m_{p_2}(C)\cdot \mu \circ R_{10}^{-1}(B) \\ &+ (1-p_1)p_2 \cdot m_{p_1} \times m_{p_2}(C)\cdot \mu \circ R_{01}^{-1}(B) \\ &+ p_1 p_2 \cdot m_{p_1} \times m_{p_2}(C)\cdot \mu \circ R_{11}^{-1}(B), \end{aligned} \tag{4.1}$$

where we used that $m_{p_1} \times m_{p_2}$ is σ-invariant.

" $\Longrightarrow$ ": assume that θ is R-invariant. Let $B \in \mathscr{B}$. By (4.1), for $C = \Omega \times \Omega$,

$$\begin{aligned} \mu(B) = \theta(\Omega \times \Omega \times B) = &(1-p_1)(1-p_2)\cdot \mu \circ R_{00}^{-1}(B) + p_1(1-p_2)\cdot \mu \circ R_{10}^{-1}(B) \\ &+ (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1}(B) + p_1 p_2 \cdot \mu \circ R_{11}^{-1}(B). \end{aligned}$$

" $\Longleftarrow$ ": assume that

$$\mu = (1-p_1)(1-p_2)\cdot \mu \circ R_{00}^{-1} + p_1(1-p_2)\cdot \mu \circ R_{10}^{-1} + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1} + p_1 p_2 \cdot \mu \circ R_{11}^{-1}.$$

We now show that $\theta = \theta \circ R^{-1}$. It suffices to verify that the measures coincide on sets of the form $A = C \times B \in \mathscr{A} \times \mathscr{B}$, as these sets form a generating π-system. By (4.1),

$$\begin{aligned} \theta\left(R^{-1}(A)\right) &= m_{p_1} \times m_{p_2}(C)\cdot \big((1-p_1)(1-p_2)\cdot \mu \circ R_{00}^{-1}(B) + p_1(1-p_2)\cdot \mu \circ R_{10}^{-1}(B) \\ &\quad + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1}(B) + p_1 p_2 \cdot \mu \circ R_{11}^{-1}(B)\big) \\ &= m_{p_1} \times m_{p_2}(C)\cdot \mu(B) \\ &= \theta(A). \end{aligned}$$

Hence θ is R-invariant. □

Note that $\pi_{3,4} \circ K = \pi_{3,4} \circ R$. The following lemma states that a product measure of the form $m_{p_1} \times m_{p_2} \times \mu$ is K-invariant if and only if it is R-invariant.

Lemma 4.4.2 (cf. lemma 1 in [5]). *Let μ be a probability measure on $[0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}]$. Then*

$$m_{p_1} \times m_{p_2} \times \mu \circ K^{-1} = m_{p_1} \times m_{p_2} \times \mu \circ R^{-1} = m_{p_1} \times m_{p_2} \times \nu,$$

where

$$\nu = (1-p_1)(1-p_2) \cdot \mu \circ R_{00}^{-1} + p_1(1-p_2) \cdot \mu \circ R_{10}^{-1} + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1} + p_1 p_2 \cdot \mu \circ R_{11}^{-1}.$$

Proof. Let $C = \{\omega_m = k, \omega'_n = l\}$. It suffices to verify that the measures coincide on sets of the form $A = C \times B \in \mathscr{A} \times \mathscr{B}$, as these sets form a generating π-system. Then

$$\begin{aligned} K^{-1}(A) = \bigcup_{i,j} \Big(& \{\omega_1 = i, \omega'_1 = j, \omega_m = k, \omega'_n = l\} \times \Big(R_{ij}^{-1}(B) \quad \cap \quad E\Big) \\ & \cup \{\omega_1 = i, \omega'_1 = j, \omega_{m+1} = k, \omega'_n = l\} \quad \times \Big(R_{ij}^{-1}(B) \quad \cap \quad S_{\bullet 0} \cup S_{\bullet 1}\Big) \\ & \cup \{\omega_1 = i, \omega'_1 = j, \omega_m = k, \omega'_{n+1} = l\} \quad \times \Big(R_{ij}^{-1}(B) \quad \cap \quad S_{0\bullet} \cup S_{1\bullet}\Big) \\ & \cup \{\omega_1 = i, \omega'_1 = j, \omega_{m+1} = k, \omega'_{n+1} = l\} \times \Big(R_{ij}^{-1}(B) \quad \cap \quad S_{\bullet\bullet}\Big) \Big). \end{aligned}$$

If $m, n > 1$, it immediately follows that

$$\begin{aligned} m_{p_1} \times m_{p_2} \times \mu \circ K^{-1}(A) = m_{p_1} \times m_{p_2}(C) \cdot \Big(& (1-p_1)(1-p_2) \cdot \mu \circ R_{00}^{-1}(B) + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1}(B) \\ & + p_1(1-p_2) \cdot \mu \circ R_{10}^{-1}(B) + p_1 p_2 \cdot \mu \circ R_{11}^{-1}(B) \Big) \\ = m_{p_1} \times m_{p_2} \times \nu(A). \end{aligned}$$

Now assume that $m = n = 1$. Then

$$\begin{aligned} K^{-1}(A) = & \{\omega_1 = k, \omega'_1 = l\} \times \big(R_{kl}^{-1}(B) \quad \cap \quad E\big) \cup \\ & \bigcup_{i=0}^{1} \{\omega_1 = i, \omega'_1 = l, \omega_{m+1} = k\} \times \big(R_{il}^{-1}(B) \quad \cap \quad S_{\bullet 0} \cup S_{\bullet 1}\big) \cup \\ & \bigcup_{j=0}^{1} \{\omega_1 = k, \omega'_1 = j, \omega'_{n+1} = l\} \times \Big(R_{kj}^{-1}(B) \quad \cap \quad S_{0\bullet} \cup S_{1\bullet}\Big) \cup \\ & \bigcup_{i,j} \{\omega_1 = i, \omega'_1 = j, \omega_{m+1} = k, \omega'_{n+1} = l\} \times \Big(R_{ij}^{-1}(B) \quad \cap \quad S_{\bullet\bullet}\Big) \Big). \end{aligned}$$

On the equality regions, the realisations of $\pi_{3,4}\circ R$ coincide, so can write

$$E\cap R_{00}^{-1}(B)=E\cap R_{10}^{-1}(B)=E\cap R_{01}^{-1}(B)=E\cap R_{11}^{-1}(B).$$

Hence

$$\begin{aligned}\mu\left(E\cap R_{kl}^{-1}(B)\right)&=(1-p_1)(1-p_2)\cdot\mu\left(E\cap R_{00}^{-1}(B)\right)+p_1(1-p_2)\cdot\mu\left(E\cap R_{10}^{-1}(B)\right)\\&\quad+(1-p_1)p_2\cdot\mu\left(E\cap R_{01}^{-1}(B)\right)+p_1p_2\cdot\mu\left(E\cap R_{11}^{-1}(B)\right).\end{aligned}$$

Similarly,

$$\begin{aligned}S_{\bullet 0}\cap R_{00}^{-1}(B)&=S_{\bullet 0}\cap R_{01}^{-1}(B),\\S_{\bullet 1}\cap R_{01}^{-1}(B)&=S_{\bullet 1}\cap R_{00}^{-1}(B),\\S_{0\bullet}\cap R_{00}^{-1}(B)&=S_{0\bullet}\cap R_{10}^{-1}(B),\\S_{1\bullet}\cap R_{10}^{-1}(B)&=S_{1\bullet}\cap R_{00}^{-1}(B),\\S_{\bullet 0}\cap R_{10}^{-1}(B)&=S_{\bullet 0}\cap R_{11}^{-1}(B),\\S_{\bullet 1}\cap R_{11}^{-1}(B)&=S_{\bullet 1}\cap R_{10}^{-1}(B),\\S_{0\bullet}\cap R_{01}^{-1}(B)&=S_{0\bullet}\cap R_{11}^{-1}(B),\\S_{1\bullet}\cap R_{11}^{-1}(B)&=S_{1\bullet}\cap R_{01}^{-1}(B),\end{aligned}$$

so

$$\begin{aligned}\mu(R_{il}^{-1}(B)\quad\cap\quad S_{\bullet 0}\cup S_{\bullet 1})&=(1-p_2)\cdot\mu(R_{i0}^{-1}(B)\quad\cap\quad S_{\bullet 0}\cup S_{\bullet 1})+p_2\cdot\mu(R_{i1}^{-1}(B)\quad\cap\quad S_{\bullet 0}\cup S_{\bullet 1}),\\\mu(R_{kj}^{-1}(B)\quad\cap\quad S_{0\bullet}\cup S_{1\bullet})&=(1-p_1)\cdot\mu(R_{0j}^{-1}(B)\quad\cap\quad S_{0\bullet}\cup S_{1\bullet})+p_1\cdot\mu(R_{1j}^{-1}(B)\quad\cap\quad S_{0\bullet}\cup S_{1\bullet}).\end{aligned}$$

Therefore,

$$\begin{aligned}m_{p_1}\times m_{p_2}\times\mu\circ K^{-1}(A)&=m_{p_1}\times m_{p_2}(C)\cdot\Big((1-p_1)(1-p_2)\cdot\mu\circ R_{00}^{-1}(B)+(1-p_1)p_2\cdot\mu\circ R_{01}^{-1}(B)\\&\quad+p_1(1-p_2)\cdot\mu\circ R_{10}^{-1}(B)+p_1p_2\cdot\mu\circ R_{11}^{-1}(B)\Big)\\&=m_{p_1}\times m_{p_2}\times\nu(A).\end{aligned}$$

The cases $m=1,n>1$ and $m>1,n=1$ are analogous. Hence

$$m_{p_1}\times m_{p_2}\times\mu\circ K^{-1}=m_{p_1}\times m_{p_2}\times\nu.$$

By (4.1),

$$m_{p_1} \times m_{p_2} \times \mu \circ R^{-1} = m_{p_1} \times m_{p_2} \times \nu.$$

We conclude that

$$m_{p_1} \times m_{p_2} \times \mu \circ K^{-1} = m_{p_1} \times m_{p_2} \times \mu \circ R^{-1} = m_{p_1} \times m_{p_2} \times \nu.$$

$\square$

Theorem 4.4.3. *There exists an absolutely continuous R-invariant measure of the form* $m_{p_1} \times m_{p_2} \times \mu$.

Proof. Let

$$R_{\beta_1}(\omega,x) = \begin{cases} \big(\sigma(\omega), T_{\beta_1}(x)\big) & \text{if } \omega_1 = 1 \\ \big(\sigma(\omega), S_{\beta_1}(x)\big) & \text{if } \omega_1 = 0 \end{cases}$$

and

$$R_{\beta_2}(\omega',y) = \begin{cases} \big(\sigma(\omega'), T_{\beta_2}(y)\big) & \text{if } \omega_1' = 1 \\ \big(\sigma(\omega'), S_{\beta_2}(y)\big) & \text{if } \omega_1' = 0 \end{cases}$$

where T_β and S_β are the greedy and lazy maps w.r.t. base β, respectively. Up to a permutation,

$$R(\omega,\omega',x,y) = R_{\beta_1}(\omega,x) \times R_{\beta_2}(\omega',y).$$

We can write

$$R_{00}(x,y) = \big(S_{\beta_1}(x), S_{\beta_2}(y)\big),$$

$$R_{10}(x,y) = \big(T_{\beta_1}(x), S_{\beta_2}(y)\big),$$

$$R_{01}(x,y) = \big(S_{\beta_1}(x), T_{\beta_2}(y)\big),$$

$$R_{11}(x,y) = \big(T_{\beta_1}(x), T_{\beta_2}(y)\big).$$

In [5] it was shown that there exists an R_{β_1}-invariant measure $m_{p_1} \times \mu_{\beta_1}$ and an R_{β_2}-invariant measure $m_{p_2} \times \mu_{\beta_2}$, where μ_{β_1} and μ_{β_2} are absolutely continuous. Hence μ is absolutely continuous. Furthermore,

$$\mu_{\beta_1} = p_1 \cdot \mu_{\beta_1} \circ T_{\beta_1}^{-1} + (1-p_1) \cdot \mu_{\beta_1} \circ S_{\beta_1}^{-1},$$

$$\mu_{\beta_2} = p_2 \cdot \mu_{\beta_2} \circ T_{\beta_2}^{-1} + (1-p_2) \cdot \mu_{\beta_2} \circ S_{\beta_2}^{-1}.$$

We now show that $m_{p_1} \times m_{p_2} \times \mu_{\beta_1} \times \mu_{\beta_2}$ is R-invariant. Let $\mu = \mu_{\beta_1} \times \mu_{\beta_2}$ and let $I_1 \times I_2 \in \mathscr{B}$. Then

$$\begin{aligned}
\mu(I_1 \times I_2) &= \mu_{\beta_1}(I_1) \cdot \mu_{\beta_2}(I_2) \\
&= \Big(p_1 \cdot \mu_{\beta_1} \circ T_{\beta_1}^{-1}(I_1) + (1-p_1) \cdot \mu_{\beta_1} \circ S_{\beta_1}^{-1}(I_1)\Big) \\
&\quad \cdot \Big(p_2 \cdot \mu_{\beta_2} \circ T_{\beta_2}^{-1}(I_2) + (1-p_2) \cdot \mu_{\beta_2} \circ S_{\beta_2}^{-1}(I_2)\Big) \\
&= (1-p_1)(1-p_2) \cdot \mu_{\beta_1} \circ S_{\beta_1}^{-1}(I_1) \cdot \mu_{\beta_2} \circ S_{\beta_2}^{-1}(I_2) \\
&\quad + p_1(1-p_2) \cdot \mu_{\beta_1} \circ T_{\beta_1}^{-1}(I_1) \cdot \mu_{\beta_2} \circ S_{\beta_2}^{-1}(I_2) \\
&\quad + (1-p_1)p_2 \cdot \mu_{\beta_1} \circ S_{\beta_1}^{-1}(I_1) \cdot \mu_{\beta_2} \circ T_{\beta_2}^{-1}(I_2) \\
&\quad + p_1 p_2 \cdot \mu_{\beta_1} \circ T_{\beta_1}^{-1}(I_1) \cdot \mu_{\beta_2} \circ T_{\beta_2}^{-1}(I_2) \\
&= (1-p_1)(1-p_2) \cdot \mu \circ R_{00}^{-1}(I_1 \times I_2) + p_1(1-p_2) \cdot \mu \circ R_{10}^{-1}(I_1 \times I_2) \\
&\quad + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1}(I_1 \times I_2) + p_1 p_2 \cdot \mu \circ R_{11}^{-1}(I_1 \times I_2).
\end{aligned}$$

As these sets form a generating π-system,

$$\mu = (1-p_1)(1-p_2) \cdot \mu \circ R_{00}^{-1} + p_1(1-p_2) \cdot \mu \circ R_{10}^{-1} + (1-p_1)p_2 \cdot \mu \circ R_{01}^{-1} + p_1 p_2 \cdot \mu \circ R_{11}^{-1}.$$

By Lemma 4.4.1, we conclude that $m_{p_1} \times m_{p_2} \times \mu_{\beta_1} \times \mu_{\beta_2}$ is R-invariant. □

By Lemma 4.4.2, μ is also K-invariant.

4.5 Simultaneous expansions

In this section, we prove some properties of digit sequences that give a simultaneous expansion of two numbers x and y in bases β_1 and β_2. We then introduce a random map G which generates these sequences, after which we show that G can be essentially identified with the left shift and find an acim for G.

For $1 < \beta < 2$ and $x \in [0, \frac{1}{\beta-1}]$, let

$$D_x^\beta = \{d \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = x\}$$

be the set of β-expansions of x.

Proposition 4.5.1. *Let $1 < \beta < 2$ be fixed. The sets D_x^β form a partition of $\{0,1\}^{\mathbb{N}}$, i.e.*

$$\bigcup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta = \{0,1\}^{\mathbb{N}},$$

where the left-hand side is a disjoint union.

Proof. "$\subseteq$": let $d \in \cup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta$. By the definition of set unions there exists an $x \in [0, \frac{1}{\beta-1}]$ such that $d \in D_x^\beta$. The set D_x^β is defined as a subset of $\{0,1\}^{\mathbb{N}}$, so $d \in \{0,1\}^{\mathbb{N}}$. We conclude that $\cup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta \subseteq \{0,1\}^{\mathbb{N}}$.

"$\supseteq$": let $d \in \{0,1\}^{\mathbb{N}}$ and let $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$. By definition of D_x^β we have $d \in D_x^\beta$. As $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \in [0, \frac{1}{\beta-1}]$, we have $d \in \cup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta$. We conclude that $\cup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta \supseteq \{0,1\}^{\mathbb{N}}$.

Hence

$$\bigcup_{x\in[0,\frac{1}{\beta-1}]} D_x^\beta = \{0,1\}^{\mathbb{N}}.$$

It remains to show that this is a disjoint union. Assume to the contrary that there exists a $d \in D_x^\beta \cap D_y^\beta$, where $x \neq y$. Then $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} = y$, which is a contradiction. Therefore, the union is disjoint. □

Let $1 < \beta_i < 2, i = 1,2$ and let $x \in [0, \frac{1}{\beta_1-1}]$, $y \in [0, \frac{1}{\beta_2-1}]$. Consider $D_x^{\beta_1} \cap D_y^{\beta_2}$, which is the set of sequences that simultaneously give an expansion of x and y, but in different bases.

So

$$D_x^{\beta_1} \cap D_y^{\beta_2} = \left\{ (d_1, d_2, ...) \in \{0,1\}^{\mathbb{N}} : x = \sum_{i=1}^{\infty} \frac{d_i}{\beta_1^i}, y = \sum_{i=1}^{\infty} \frac{d_i}{\beta_2^i} \right\}.$$

Let $d \in \{0,1\}^{\mathbb{N}}$. By Proposition 4.5.1 there exist unique $x \in [0, \frac{1}{\beta_1 - 1}], y \in [0, \frac{1}{\beta_2 - 1}]$ such that $d \in D_x^{\beta_1}$ and $d \in D_y^{\beta_2}$, namely $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta_1^i}, y = \sum_{i=1}^{\infty} \frac{d_i}{\beta_2^i}$. The same digits are permissible for β_1 and β_2 because $\lfloor \beta_1 \rfloor = \lfloor \beta_2 \rfloor = 1$. Hence d is simultaneously a β_1-expansion of x and a β_2-expansion of y. We have the following corollary:

Corollary 4.5.2. *Let* $1 < \beta_i < 2, i = 1,2$ *be fixed. The sets* $D_x^{\beta_1} \cap D_y^{\beta_2}$ *form a partition of* $\{0,1\}^{\mathbb{N}}$*, i.e.*

$$\bigcup_{(x,y) \in [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}]} D_x^{\beta_1} \cap D_y^{\beta_2} = \{0,1\}^{\mathbb{N}},$$

where the left-hand side is a disjoint union.

As we have seen, almost all x have a continuum of β-expansions. Hence if $x = y$ and $\beta_1 = \beta_2$, then for almost all x, $D_x^{\beta_1} \cap D_y^{\beta_2} = D_x^{\beta_1}$ has the same cardinality as $\mathbb{R}$. There is a set of measure 0 of numbers x for which $D_x^{\beta_1}$ may have a smaller cardinality than $\mathbb{R}$ (it cannot have a greater cardinality because $\{0,1\}^{\mathbb{N}}$ has the same cardinality as $\mathbb{R}$). The cardinality of these sets is a matter of active research. If $x = y \neq 0$ and $\beta_1 \neq \beta_2$, then $D_x^{\beta_1} \cap D_y^{\beta_2}$ is empty as $\sum_{i=1}^{\infty} \frac{d_i}{\beta_1^i} \neq \sum_{i=1}^{\infty} \frac{d_i}{\beta_2^i}$. If $x = y = 0$, then $D_x^{\beta_1} \cap D_y^{\beta_2}$ has 1 element, namely $d = (0,0,...)$. If $x \neq y$ and $\beta_1 = \beta_2$, then $D_x^{\beta_1} \cap D_y^{\beta_2}$ is again empty by Proposition 4.5.1.

The remaining case is $x \neq y, \beta_1 \neq \beta_2$. We have the following proposition:

Proposition 4.5.3. *Let* $x \in [0, \frac{1}{\beta_1 - 1}], y \in [0, \frac{1}{\beta_2 - 1}]$*, let* $d = (d_1, d_2, ...)$ *be the greedy expansion of* x *in base* β_1 *and let* $d' = (d'_1, d'_2, ...)$ *be the lazy expansion of* y *in base* β_2*. If*

$$0.d_1 d_2 ... <_{\text{lex}} 0.d'_1 d'_2 ...$$

then $D_x^{\beta_1} \cap D_y^{\beta_2}$ *is empty.*

Proof. Assume to the contrary that there exists a $\tilde{d} \in D_x^{\beta_1} \cap D_y^{\beta_2}$. All β_2-expansions of y are lexicographically greater than or equal to the lazy expansion in the same basis, so

$$0.d'_1 d'_2 ... \leq_{\text{lex}} 0.\tilde{d}_1 \tilde{d}_2 ...$$

By assumption, $0.d_1d_2... <_{\text{lex}} 0.d'_1d'_2...$, so

$$0.d_1d_2... <_{\text{lex}} 0.\tilde{d}_1\tilde{d}_2...$$

This is a contradiction, because all β_1-expansions of x are lexicographically smaller than or equal to the greedy expansion in the same basis. We conclude that $D_x^{\beta_1} \cap D_y^{\beta_2}$ is empty. □

Example 4.5.4. *The greedy expansion of $x = \frac{1}{2}$ in base $\beta_1 = 1.25$ is 0.000100000010... and the lazy expansion of $y = \frac{2}{3}$ in base $\beta_2 = 1.5$ is 0.00101111101... By Proposition 4.5.3, any expansion of $\frac{1}{2}$ in base 1.25 is not an expansion of $\frac{2}{3}$ in base 1.5, and vice versa.*

We now consider when the situation of Proposition 4.5.3 occurs in the algorithm from section 4.3. Note that on $E_{00} \cup E_{11} \cup S_{0\bullet} \cup S_{\bullet 1}$, the greedy β_1-digit and the lazy β_2-digit are equal (here $\omega = (1,1,1,...)$ and $\omega' = (0,0,0,...)$). The region E_{01} is the only region where the lazy β_2-digit is greater than the greedy β_1-digit. In the remaining regions $S_{\bullet\bullet} \cup S_{\bullet 0} \cup S_{1\bullet} \cup E_{10}$, the lazy β_2-digit is smaller than the greedy β_1-digit. Hence this situation occurs in the algorithm if we take $\omega = (1,1,...), \omega' = (0,0,...)$ and (x,y) such that the orbit under K hits E_{01} before it hits $S_{\bullet\bullet} \cup S_{\bullet 0} \cup S_{1\bullet} \cup E_{10}$.

To generate elements of $D_x^{\beta_1} \cap D_y^{\beta_2}$ using K, we should choose appropriate values of ω and ω'. If $K^{n-1}(\omega, \omega', x, y) \in E_{00} \cup E_{11}$, then the assigned n-th digits will be equal. If $K^{n-1}(\omega, \omega', x, y) \in S_{\bullet 0} \cup S_{0\bullet} \cup S_{\bullet 1} \cup S_{1\bullet}$, then we are forced to choose ω or ω' so that the digits match. If $K^{n-1}(\omega, \omega', x, y) \in S_{\bullet\bullet}$, we can either choose ω and ω' such that the n-th digits are both 0 or both 1. If at any iteration the orbit of (x,y) under K enters $E_{01} \cup E_{10}$, we are forced to choose differing n-th digits and the resulting expansion will not be an element of $D_x^{\beta_1} \cap D_y^{\beta_2}$.

Note that the only freedom of choice we have is on $S_{\bullet\bullet}$. Here we are forced to choose the digits to be equal, so a single coin flip is sufficient. We now introduce a new random map which generates all elements of $D_x^{\beta_1} \cap D_y^{\beta_2}$.

We define the random map

$$G : \Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}] \to \Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}],$$

$$G(\omega,x,y)=\begin{cases}(\omega,\beta_1x,\beta_2y) & \text{if } (x,y)\in E_{00}\\ (\omega,\beta_1x-1,\beta_2y) & \text{if } (x,y)\in E_{10}\\ (\omega,\beta_1x,\beta_2y-1) & \text{if } (x,y)\in E_{01}\\ (\omega,\beta_1x-1,\beta_2y-1) & \text{if } (x,y)\in E_{11}\\ (\omega,\beta_1x,\beta_2y) & \text{if } (x,y)\in S_{\bullet 0}\\ (\omega,\beta_1x,\beta_2y) & \text{if } (x,y)\in S_{0\bullet}\\ (\sigma(\omega),\beta_1x-\omega_1,\beta_2y-\omega_1) & \text{if } (x,y)\in S_{\bullet\bullet}\\ (\omega,\beta_1x-1,\beta_2y-1) & \text{if } (x,y)\in S_{\bullet 1}\\ (\omega,\beta_1x-1,\beta_2y-1) & \text{if } (x,y)\in S_{1\bullet}\end{cases}$$

We assign the first digit as follows:

$$\tilde{d}_1=\tilde{d}_1(\omega,x,y)=\begin{cases}0 & \text{if } (x,y)\in E_{00}\cup S_{\bullet 0}\cup S_{0\bullet}\cup E_{01}\\ k & \text{if } (x,y)\in S_{\bullet\bullet},\omega_1=k, \qquad k\in\{0,1\}\\ 1 & \text{if } (x,y)\in E_{11}\cup S_{\bullet 1}\cup S_{1\bullet}\cup E_{10}\end{cases}$$

and the n-th digit as

$$\tilde{d}_n=\tilde{d}_n(\omega,x,y)=\tilde{d}_1\left(G^{n-1}(\omega,x,y)\right) \qquad n\geq 1.$$

(see Figure 4.4)

The digit we assign on E_{10} and E_{01} ensures that the resulting expansion is always a β_1-expansion of x. However, the resulting expansion is not always a β_2-expansion of y. This is because on E_{10} and E_{01}, all β_2-expansions of y must be assigned digit 0 and 1, respectively. We now show that the resulting expansion is a simultaneous expansion of x and y if and only if the orbit never hits $E_{10}\cup E_{01}$.

Theorem 4.5.5. *The set $D_x^{\beta_1}\cap D_y^{\beta_2}$ is generated by*

$$\tilde{\Omega}=\tilde{\Omega}(x,y):=\left\{\omega\in\Omega:\pi_{2,3}\circ G^n(\omega,x,y)\notin E_{10}\cup E_{01}\quad \forall n\geq 0\right\},$$

i.e.

$$D_x^{\beta_1}\cap D_y^{\beta_2}=\left\{\left(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\ldots\right):\tilde{\omega}\in\tilde{\Omega}\right\}.$$

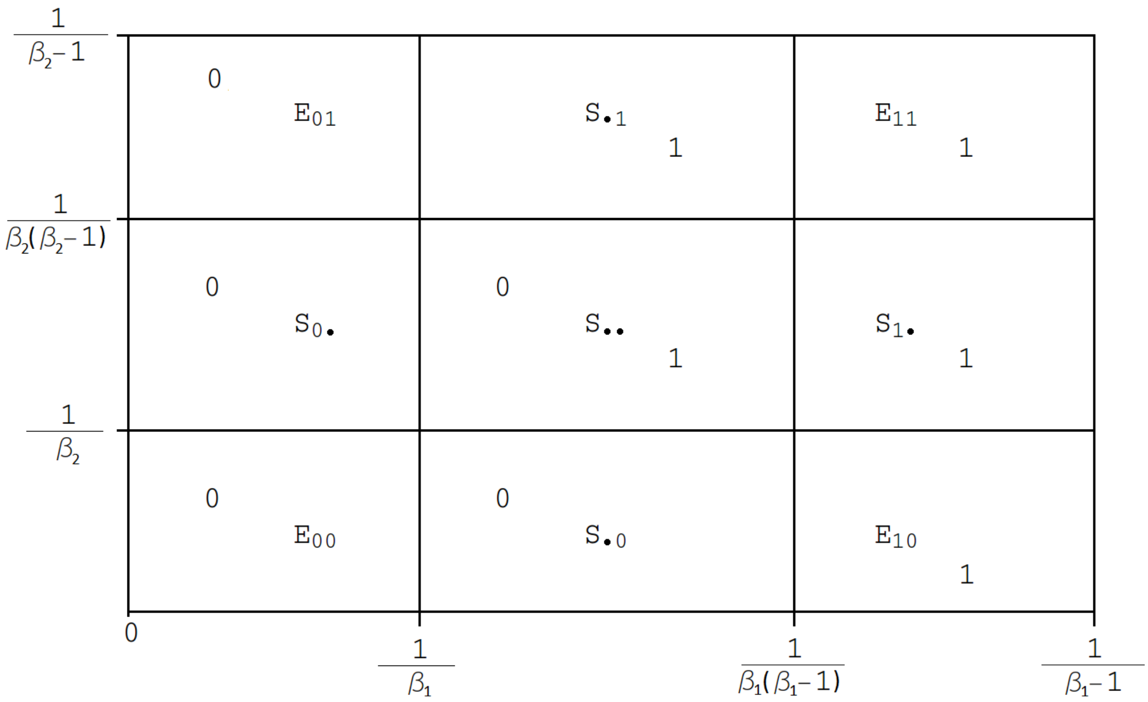

Fig. 4.4 The regions with the digits assigned by G.

Proof. "$\subseteq$": let $a=(a_1,a_2,\ldots)\in D_x^{\beta_1}\cap D_y^{\beta_2}$. The map K from earlier in this chapter generates all expansions of x and y, so there exist $(\omega,\omega')\in\Omega\times\Omega$ such that $((a_1,a_1),(a_2,a_2),\ldots)=\mathbf{d}(\omega,\omega',x,y)$, where $\mathbf{d}_n(\omega,\omega',x,y)=(d_n(\omega,x),d'_n(\omega',y))$. As $d_n(\omega,x)=d'_n(\omega',y)=a_n$ for all $n\geq 1$, the orbit under K never hits $E_{10}\cup E_{01}$, and on the switch regions the digits are chosen to be equal. For $i\geq 1$ let $r_i\geq 0$ be the time index of the i-th time the orbit under K hits the switch region $S_{\bullet\bullet}$ and define $\tilde{\omega}_i=\pi_1\circ K^{r_i}(\omega,\omega',x,y)$ (if $S_{\bullet\bullet}$ is hit only finitely many times, $\tilde{\omega}_i$ can be chosen arbitrarily from a certain point on). As $\pi_{2,3}\circ G^n(\tilde{\omega},x,y)=\pi_{3,4}\circ K^n(\omega,\omega',x,y)$, we have $\tilde{d}(\tilde{\omega},x,y)=d(\omega,x)=d'(\omega',y)$. Hence $\tilde{d}(\tilde{\omega},x,y)=a$. Finally, $\pi_{2,3}\circ G^n(\tilde{\omega},x,y)=\pi_{3,4}\circ K^n(\omega,\omega',x,y)\notin E_{10}\cup E_{01}$ for all $n\geq 0$. Therefore,

$$D_x^{\beta_1}\cap D_y^{\beta_2}\subseteq\{\big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\ldots\big):\tilde{\omega}\in\tilde{\Omega}\}.$$

"$\supseteq$": let $a\in\{\big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\ldots\big):\tilde{\omega}\in\tilde{\Omega}\}$. There exists an $\tilde{\omega}\in\tilde{\Omega}$ such that $(a_1,a_2,\ldots)=\big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\ldots\big)$.

For $i\geq 1$, let $r_i\geq 0$ be time index of the i-th time the orbit under G hits $\tilde{S}_1:=S_{\bullet 0}\cup S_{\bullet 1}\cup S_{\bullet\bullet}$ and let $k_i=\#\{j<r_i:\pi_{2,3}\circ G^j(\tilde{\omega},x,y)\in S_{\bullet\bullet}\}$ be the number of times $S_{\bullet\bullet}$ is hit before

time r_i. Let $\omega_i = 0$ if $\pi_{2,3} \circ G^{r_i}(\tilde{\omega},x,y) \in S_{\bullet 0}$, let $\omega_i = 1$ if $\pi_{2,3} \circ G^{r_i}(\tilde{\omega},x,y) \in S_{\bullet 1}$ and let $\omega_i = \tilde{\omega}_{1+k_i}$ if $\pi_{2,3} \circ G^{r_i}(\tilde{\omega},x,y) \in S_{\bullet\bullet}$. Similarly, let $s_i \geq 0$ be time index of the i-th time the orbit under K hits $\tilde{S}_2 := S_{0\bullet} \cup S_{1\bullet} \cup S_{\bullet\bullet}$ and let $l_i = \#\{j < s_i : \pi_{2,3} \circ G^j(\tilde{\omega},x,y) \in S_{\bullet\bullet}\}$. Let $\omega_i' = 0$ if $\pi_{2,3} \circ G^{s_i}(\tilde{\omega},x,y) \in S_{0\bullet}$, let $\omega_i' = 1$ if $\pi_{2,3} \circ G^{s_i}(\tilde{\omega},x,y) \in S_{1\bullet}$ and let $\omega_i' = \tilde{\omega}_{1+l_i}$ if $\pi_{2,3} \circ G^{s_i}(\tilde{\omega},x,y) \in S_{\bullet\bullet}$. If $\tilde{S}_1$ or $\tilde{S}_2$ is hit only finitely many times, ω_i or ω_i' can be chosen arbitrarily from a certain point on. Then by the choice of ω, ω',

$$\mathbf{d}_1(\omega,\omega',x,y) = \big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_1(\tilde{\omega},x,y)\big) \text{ for all } (x,y) \notin E_{10} \cup E_{01}$$

and

$$\pi_{2,3} \circ G^n(\tilde{\omega},x,y) = \pi_{3,4} \circ K^n(\omega,\omega',x,y) \text{ for all } n \geq 0.$$

As $G^n(\omega,x,y) \notin E_{10} \cup E_{01}$ for all $n \geq 0$, we have

$$\mathbf{d}_n(\omega,\omega',x,y) = \big(\tilde{d}_n(\tilde{\omega},x,y),\tilde{d}_n(\tilde{\omega},x,y)\big) \text{ for all } n \geq 1.$$

Therefore,

$$\mathbf{d}(\omega,\omega',x,y) = ((a_1,a_1),(a_2,a_2),\dots).$$

All sequences generated by K are expansions of x and y, so a is a β_1-expansion of x and a β_2-expansion of y. Hence $a \in D_x^{\beta_1} \cap D_y^{\beta_2}$, so

$$D_x^{\beta_1} \cap D_y^{\beta_2} \supseteq \{\big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\dots\big) : \tilde{\omega} \in \tilde{\Omega}\}.$$

We conclude that

$$D_x^{\beta_1} \cap D_y^{\beta_2} = \{\big(\tilde{d}_1(\tilde{\omega},x,y),\tilde{d}_2(\tilde{\omega},x,y),\dots\big) : \tilde{\omega} \in \tilde{\Omega}\}.$$

□

We now show that G can be essentially identified with the left shift σ on $D = \{0,1\}^{\mathbb{N}}$. By [4], this implies that G can also be essentially identified with K_{β_1} and K_{β_2}.

Let $\hat{\mathscr{A}}$ and $\hat{\mathscr{D}}$ be the product σ-algebra on Ω and D, respectively, let $\mathscr{B}$ be the Borel σ-algebra on $[0,\frac{1}{\beta_1-1}] \times [0,\frac{1}{\beta_2-1}]$ and let $\mathbb{P}$ be the uniform product measure on D.

Let $\tilde{\phi} : \Omega \times [0,\frac{1}{\beta_1-1}] \times [0,\frac{1}{\beta_2-1}] \to D$,

$$\tilde{\phi}(\omega,x,y) = \big(\tilde{d}_1(\omega,x,y),\tilde{d}_2(\omega,x,y),\dots\big)$$

and let

$\tilde{Z}_1 = \Big\{(\omega,x,y) \in \Omega \times [0, \dfrac{1}{\beta_1 - 1}] \times [0, \dfrac{1}{\beta_2 - 1}] : G^n(\omega,x,y) \in \Omega \times S_{\bullet\bullet} \text{ for infinitely many } n\Big\},$

$\tilde{Z}_2 = \Big\{(\omega,x,y) \in \Omega \times [0, \dfrac{1}{\beta_1 - 1}] \times [0, \dfrac{1}{\beta_2 - 1}] : G^n(\omega,x,y) \notin \Omega \times (E_{10} \cup E_{01}) \text{ for all } n \geq 0\Big\},$

$\tilde{Z} = \tilde{Z}_1 \cap \tilde{Z}_2,$

$\tilde{D}' = \Big\{\big(a_1,a_2,...\big) \in D : \Big(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\Big) \in S_{\bullet\bullet} \text{ for infinitely many } n\Big\}.$

Then $\tilde{\phi}(\tilde{Z}) = \tilde{D}'$, $G(\tilde{Z}) \subseteq \tilde{Z}$ and $\sigma^{-1}(\tilde{D}') = \tilde{D}'$.

We will need the following lemma:

Lemma 4.5.6. $\Big\{\big(a_1,a_2,...\big) \in D : \Big(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\Big) \notin E_{10} \cup E_{01} \text{ for all } n \geq 1\Big\} = D.$

Proof. "$\subseteq$": this inclusion is obvious.

"$\supseteq$": let $d = (d_1,d_2,...) \in D$. Then d is a β_1-expansion of $x = \sum_{i=1}^{\infty} \frac{d_i}{\beta_1^i}$ and a β_2-expansion of $y = \sum_{i=1}^{\infty} \frac{d_i}{\beta_2^i}$. By Theorem 4.5.5, there exists an $\omega \in \Omega$ such that $\pi_{2,3} \circ G^n(\omega,x,y) \notin E_{10} \cup E_{01}$ for all $n \geq 0$. As $\pi_{2,3} \circ G^{n-1}(\omega,x,y) = \Big(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\Big)$, we conclude that $\Big(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\Big) \notin E_{10} \cup E_{01}$ for all $n \geq 1$. So

$$\left\{\big(a_1,a_2,...\big) \in D : \left(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\right) \notin E_{10} \cup E_{01} \text{ for all } n \geq 1\right\} \supseteq D.$$

□

The remainder of the proof is analogous to the proof in [4] and section 4.3. Let $\tilde{\phi}'$ be the restriction of $\tilde{\phi}$ to $\tilde{Z}$:

$\tilde{\phi}' : \tilde{Z} \to \tilde{D}'$,

$$\tilde{\phi}(\omega,x,y) = \big(\tilde{d}_1(\omega,x,y), \tilde{d}_2(\omega,x,y), ...\big)$$

We will show that $\tilde{\phi}'$ is bijective. Let $\left(a_1, a_2, ...\right) \in \tilde{D}'$ and recursively define

$$r_1 = \min\left\{n \geq 1 : \left(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\right) \in S_{\bullet\bullet}\right\},$$

$$r_k = \min\left\{n > r_{k-1} : \left(\sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_{n+i-1}}{\beta_2^i}\right) \in S_{\bullet\bullet}\right\}.$$

For $k \geq 1$ let $\omega_k = a_{r_k}$. Define

$$\tilde{\phi}^{-1}\left(a_1, a_2, ...\right) = \left(\omega, \sum_{i=1}^{\infty} \frac{a_i}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_i}{\beta_2^i}\right).$$

Then $\tilde{\phi}'$ and $\tilde{\phi}^{-1}$ are measurable. By Lemma 4.5.6, $\left(\omega, \sum_{i=1}^{\infty} \frac{a_i}{\beta_1^i}, \sum_{i=1}^{\infty} \frac{a_i}{\beta_2^i}\right) \in \tilde{Z}$. Furthermore, $\tilde{\phi}' \circ \tilde{\phi}^{-1}(a) = a$. As $\tilde{\phi}$ is injective, we conclude that $\tilde{\phi}^{-1}$ is the inverse of $\tilde{\phi}'$.

Then $\tilde{\phi}' \circ G = \sigma \circ \tilde{\phi}'$. By the second Borel-Cantelli lemma, $\mathbb{P}(\tilde{D}') = 1$. Define the measure $\tilde{\nu}$ on $\tilde{\mathscr{A}} \times \mathscr{B}$ by $\tilde{\nu}(A) = \mathbb{P}\left(\tilde{\phi}(\tilde{Z} \cap A)\right)$. Then $\tilde{\nu}(\tilde{Z}) = \mathbb{P}\left(\tilde{\phi}(\tilde{Z})\right) = \mathbb{P}\left(\tilde{D}'\right) = 1$.

We have the following theorem:

Theorem 4.5.7. *The map* $\tilde{\phi}(\omega, x, y) = \left(\tilde{d}_1(\omega, x, y), \tilde{d}_2(\omega, x, y), ...\right)$ *is a measurable isomorphism from* $(\Omega \times [0, \frac{1}{\beta_1 - 1}] \times [0, \frac{1}{\beta_2 - 1}], \tilde{\mathscr{A}} \times \mathscr{B}, \tilde{\nu}, G) \to (D, \tilde{\mathscr{D}}, \mathbb{P}, \sigma)$.

The uniform product measure $\mathbb{P}$ on 2 symbols is the unique measure of maximal entropy on D, with entropy equal to $\log(2)$. We conclude that $\tilde{\nu}$ is the unique measure of maximal entropy that has support $\tilde{Z}$, with entropy equal to $\log(2)$.

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