Universiteit Utrecht

Quantum fluctuations and magnon-magnon interactions in antiferromagnets

Bachelor thesis

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Abstract

We derive relaxation times for magnon-magnon interactions in antiferromagnetic spin configurations with easy-axis anisotropy and an external magnetic field. We apply the Holstein-Primakoff transformation to the Heisenberg exchange Hamiltonian to calculate the ground state energy. It turns out that the antiferromagnetic configuration is not an eigenstate of the Heisenberg exchange Hamiltonian. We go on to describe quantum fluctuations in antiferromagnetic spin configurations with easy-axis anisotropy in an external magnetic field. Also we derive spin wave dispersion using Landau-Lifshitz-Gilbert phenomenology. We calculate the scattering amplitudes of magnon-magnon interactions and derive the scale of relaxation times of out-of-equilibrium antiferromagnetic configurations, from which we find $\frac{1}{\tau} \propto T^2$. Thermal magnons have a relaxation rate $\frac{1}{\tau} \propto T$. For Gilbert damping it is found that the relaxation rate goes as $\frac{1}{\tau} \propto T$.

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1 Introduction

Magnetism is a physical phenomenon exhibited by materials in which neighbouring electrons have a parallel spin alignment (figure 1a) such that their individual magnetic moment adds up to some net *magnetization*. This magnetization give rise to many interesting physical phemomena. Materials with a net magnetization go by the name of *ferromagnets*. Ferromagnets are characterized by their *Curie temperature*; below this temperature the material exhibits long range parallel alignment of neighbouring electron spins, whereas above this temperature all alignment vanishes due to thermal fluctuations [1]. Some materials are known to also exhibit some of these magnetic phenomena, while having no net magnetization. These *antiferromagnets* or *AFM* are magnetically ordered such that neighbouring spins have anti-parallel spin alignment (figure 1b). Antiferromagnetic spin configurations have no net magnetization such that in equilibrium they don't show the same magnetic behaviour as ferromagnets, and are characterized by their *Néel temperature* above which all magnetic ordering vanishes [1]. Magnetic materials find applications in many electronic devices, in particular in the form of magnetic memories.

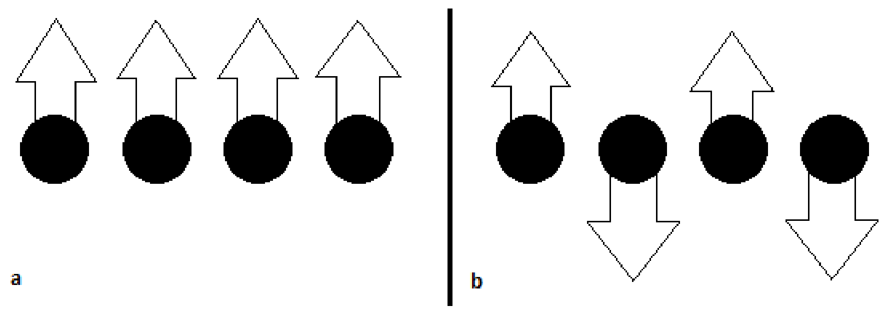

Figure 1: schematic representation of spin configurations. a) ferromagnet b) antiferromagnet

While many magnetic materials are also electrical conductors it is found that certain materials posses magnetic ordering while at the same time being an insulator, therefore having bound electrons. It has been found that disturbances of their magnetic ordering result in *spin waves* propagating through the material [1], as can be seen in figure 2. In a quantum mechanical framework these spin waves turn out to be well described by harmonic oscillators of which the quanta are quasiparticles named *magnons*. As with photons, it was assumed that they can be used to transport information. Since normal electronic information transport depends on electrons being conducted through the material, which scatter with all kinds of impurities therefore giving rise to resistance, information transport using spin waves would be a more efficient way of transport due to less scattering. Current research focuses on creating a Bose-Einstein condensate of magnons to create the room temperature equivalent of a low-temperature electronic superconductor [2]. Recently the first electric circuit with a ferromagnetic insulator (Yttrium Iron Garnet or YIG) has been realised [3]. However, since antiferromagnetic materials exhibit no net magnetization they may be more convenient for certain electric circuits. In this thesis we will look into the subject of spin waves in antiferromagnets. We will consider the ground state and quantum fluctuations in the form of spin waves. As a first step towards a theory of transport and dynamics of spin waves, we will consider spin wave collisions and their relaxation times.

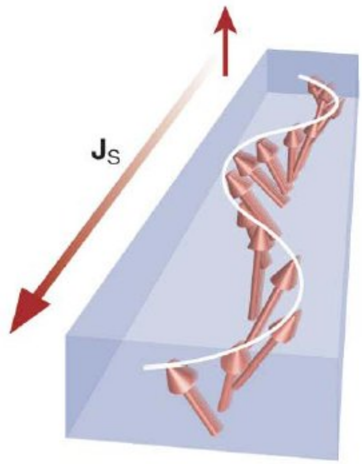

Figure 2: schematic drawing of the propagating disturbance in a ferromagnet: the spin wave (From [4])

In the first two chapters we apply both semiclassical and quantum mechanical techniques to derive the dispersion relation of spin waves in ferromagnets and antiferromagnets. Chapters 2-3.2 are based on previous research [5] and serve mostly as an introduction to the subject. Next there will be a short introduction into Landau-Lifshitz-Gilbert phenomenology, which was done in collaboration with H. Snijder, who went on to apply it to bulk transport calculations in ferromagnets [6]. In Chapter 4 we look into a more complex antiferromagnetic configuration and derive again a dispersion relation. Finally we turn our attention towards magnon-magnon interactions and approximate relaxation times for all possible interactions in both the low temperature regime as well as the high temperature regime. We will compare these results to relaxation due to Gilbert damping.

2 Ferromagnets

In this section we will get ourselves acquainted with the general techniques required to analyze the properties of magnetic spin configurations. The first configuration we look at is the ferromagnet, since it is the simplest configuration.

2.1 Semiclassical regime

The Hamiltonian for magnetic systems is taken to be the Heisenberg exchange Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{1}$$

Where the summation runs over the nearest neighbour pairs and $\mathbf{S}_i$ refers to the spin operator at the i'th site. This Hamiltonian describes the interaction of two neighbouring spins through a coupling constant J which is positive in the case of a ferromagnet. From this we can see that the system is in its lowest energy state (ground state) when all spins are pointing in the same direction. We now want to find the equations of motion of the three spin components S^x,S^y and S^z, by making use of Ehrenfest's theorem, which is given by:

$$\frac{d}{dt}\langle \mathbf{S}_k \rangle = -\frac{i}{\hbar}\langle [\mathbf{S}_k, H] \rangle. \tag{2}$$

Here the square brackets denote the commutator. Rewriting equation 1 so that the summation only runs over single sites instead of pairs, we find:

$$\begin{aligned} H = -J \sum_i \mathbf{S}_i \mathbf{S}_{i+1} + \mathbf{S}_i \mathbf{S}_{i-1} \\ = -J \sum_i [S_i^x S_{i+1}^x + S_i^x S_{i-1}^x + S_i^y S_{i+1}^y + S_i^y S_{i-1}^y + S_i^z S_{i+1}^z + S_i^z S_{i-1}^z]. \end{aligned} \tag{3}$$

Which we can now insert into equation 2, leading to:

$$\begin{aligned} \frac{d}{dt}\langle \mathbf{S}_k \rangle = J\frac{i}{\hbar}\langle [\mathbf{S}_k, \sum_i S_i^x S_{i+1}^x + S_i^x S_{i-1}^x] + [\mathbf{S}_k, \sum_i S_i^y S_{i+1}^y + S_i^y S_{i-1}^y] \\ + [\mathbf{S}_k, \sum_i S_i^z S_{i+1}^z + S_i^z S_{i-1}^z] \rangle. \end{aligned} \tag{4}$$

This expression can we worked out using the commutation relations for the spin operator:

$$\left[S^i, S^j\right] = i\hbar S^k \varepsilon_{ijk} \quad ; \quad i,j,k \in \{x,y,z\}. \tag{5}$$

Applying this commutation relation and remembering that two spins $\mathbf{S}_i$ and $\mathbf{S}_k$ always commute when $k \neq i$ the expression for the first component reduces to:

$$\begin{aligned} \frac{d}{dt}\langle S_k^x \rangle = J\frac{i}{\hbar}\langle [S_k^x, S_k^y S_{k+1}^y + S_k^y S_{k-1}^y] + [S_k^x, S_k^z S_{k+1}^z + S_k^z S_{k-1}^z] \rangle \\ = J\langle S_k^y (S_{k+1}^z + S_{k-1}^z) - S_k^z (S_{k+1}^y + S_{k-1}^y) \rangle. \end{aligned} \tag{6}$$

Since we make use of the commutation relation we can obtain our other two equations by cyclic permutation (due to the Levi-Cevita Tensor). We drop the ensemble notation and we end up with these three equations of motion:

$$\begin{aligned}
\frac{d}{dt}S_k^x &= JS_k^y(S_{k+1}^z + S_{k-1}^z) - JS_k^z(S_{k+1}^y + S_{k-1}^y), \\
\frac{d}{dt}S_k^y &= JS_k^z(S_{k+1}^x + S_{k-1}^x) - JS_k^x(S_{k+1}^z + S_{k-1}^z), \\
\frac{d}{dt}S_k^z &= JS_k^x(S_{k+1}^y + S_{k-1}^y) - JS_k^y(S_{k+1}^x + S_{k-1}^x).
\end{aligned} \tag{7}$$

These equations can be summarized as

$$\frac{d\mathbf{S}_k}{dt} = -\mathbf{S}_k \times \frac{\partial H}{\partial \mathbf{S}_k}. \tag{8}$$

If we choose all spins of the system to be in the z-direction we can look at small deviations in direction. Therefore we take $S^z = \hbar S$ and $S^x, S^y \ll S^z$ so that equation 7 reduces to:

$$\begin{aligned}
\frac{d}{dt}S_k^x &\approx JS\hbar(2S_k^y - S_{k+1}^y - S_{k-1}^y), \\
\frac{d}{dt}S_k^y &\approx JS\hbar(S_{k+1}^x + S_{k-1}^x - 2S_k^x), \\
\frac{d}{dt}S_k^z &\approx 0.
\end{aligned} \tag{9}$$

These are the equations of motion we have to solve. The third equation is a restatement of the spin being deviated around the z-axis.

In order to solve these equations we can make a guess to what is the solution. A good guess turns out to be the plane wave: $S_j^x = Ae^{i\mathbf{k}\cdot\mathbf{R}_j - i\omega t}$ and $S_j^y = Be^{i\mathbf{k}\cdot\mathbf{R}_j - i\omega t}$, where $R_k = k \cdot a\hat{x}$ and a the distance between two sites, since we are looking at a one dimensional model. Inserting this ansatz into the equations above we find:

$$\begin{aligned}
i\omega A &= -2SJ\hbar B(1 - \cos(k_x a)), \\
i\omega B &= 2SJ\hbar A(1 - \cos(k_x a)).
\end{aligned} \tag{10}$$

This set of equation can be written in matrix form:

$$\begin{pmatrix} i\omega & 2SJ\hbar B(1 - \cos k_x a) \\ 2SJ\hbar B(1 - \cos k_x a) & -i\omega \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{11}$$

From this expression we can extract the dispersion relation for a one dimensional ferromagnet by finding the zeroes of the determinant of this first matrix. Performing the calculation yields:

$$\omega_k = 2SJ\hbar(1 - \cos k_x a). \tag{12}$$

Next we calculate the eigenvectors of the system, which are given by:

$$\nu_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \nu_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}. \tag{13}$$

The first eigenvector corresponds to a state where all spins point only in the z-direction, this is the ground state of the system. However, plugging the second eigenvector into our planewave equations and dropping the imaginary part we find:

$$\begin{aligned} S_j^x &= \frac{1}{\sqrt{2}} \sin{(k_x ja - \omega t)}, \\ S_j^y &= \frac{1}{\sqrt{2}} \cos{(k_x ja - \omega t)}, \end{aligned} \tag{14}$$

These equations describe precession of the x and y components of $\mathbf{S}_j$ around the z-axis. Note that for each consecutive site the components pick up a phase shift of $k_x a$. It is this phase shift that gives rise to a wave like behavior of the spins in the system. Like the vibration of a lattice in matter can be described as *quasiparticles* (phonons), these *spin waves* can also be described as *quasiparticles*, called *magnons*. In the limit of small k_x we find that the dispersion relation reduces to:

$$\omega_k \approx 2JS\hbar(k_x a)^2. \tag{15}$$

This means that a *magnon* with wavevector k_x requires an energy $\hbar\omega_k > 0$. For this reason a ferromagnet in its groundstate does not have any *magnons*.

2.2 Quantum mechanical regime: Holstein-Primakoff transformation

In the previous part we derived our results from a semiclassical approach. However for an accurate description we need to also take into account the existence of the *magnons* we found previously. One way to do this turns out to be the Holstein-Primakoff transformation [8]. The first step in performing this transformation is rewriting the familiar rising and lowering operators for spin, $\mathbf{S}^-$ and $\mathbf{S}^+$, in terms of the *bosonic creation and annihilation operators* a_i and $a_i^\dagger$. These operators obey the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$. The operator $\hat{n}_i = a_i^\dagger a_i$ is called the number operator; it counts the number of *bosons* at a lattice site i. The spin raising and lowering operators are given by:

$$\begin{aligned} S_j^+ &= \hbar\sqrt{2S - a_i^\dagger a_i} ai, \\ S_j^- &= \hbar a_i^\dagger \sqrt{2S - a_i^\dagger a_i}, \\ S_j^z &= \hbar S - \hbar a_i^\dagger a_i. \end{aligned} \tag{16}$$

From these expressions we can deduce that $\hat{n}_i \leq 2S$, in order for our results to be real and therefore physical. For $S \ll \hat{n}_i$ we can approximate the first two of these using a Taylor expansion, yielding:

$$\begin{aligned} S_j^+ &= \hbar\sqrt{2S} a_i, \\ S_j^- &= \hbar a_i^\dagger \sqrt{2S}, \\ S_j^z &= \hbar S - \hbar a_i^\dagger a_i. \end{aligned} \tag{17}$$

Also we would like to write the Heisenberg Hamiltonian in terms of these spin operators. Since we know $S_j^+ = S_j^x + iS_j^y$ and $S_j^- = S_j^x - iS_j^y$ and rewriting $\mathbf{S}_i\mathbf{S}_{i+1} + \mathbf{S}_i\mathbf{S}_{i-1}$ as $\sum_\delta \mathbf{S}_i\mathbf{S}_{i+\delta}$ where $i+\delta$ is a vector connecting site i with its nearest neighbours, we end up with the following Hamiltonian:

$$H = -J \sum_i \sum_\delta S_i^z S_{i+\delta}^z + \frac{1}{2} S_i^+ S_{i+\delta}^- + \frac{1}{2} S_i^- S_{i+\delta}^+. \tag{18}$$

The important thing about this Hamiltonian are the last two terms. Effectively, these move spins to different sites, which can be seen as a *spin wave*. We can now insert the *Holstein-Primakoff transformation* from equations 16 and 17 into this Hamiltonian. From the Taylor expansion leading up to equation 17 we can see that terms like $a_i^\dagger a_i a_i/S$ will become very small with increasing S. Therefore we only keep terms up to quadratic order:

$$H = -J\hbar^2 \sum_i \sum_\delta S^2 - Sa_i^\dagger a_i - a_{i+\delta}^\dagger a_{i+\delta} + Sa_{i+\delta}^\dagger a_i + Sa_i^\dagger a_{i+\delta}. \tag{19}$$

Since we want to find the eigenstates of the system we diagonalize the matrix using a Fourier transformation to the $\mathbf{k}$-space, given by:

$$\begin{aligned} a_j &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}_j} a_{\mathbf{k}}, \\ a_j^\dagger &= \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} a_{\mathbf{k}}^\dagger. \end{aligned} \tag{20}$$

The result of the rather cumbersome calculation is given below. Here z denotes the number of nearest neighbours given by $z = 2d$ where d is number of dimensions our system has.

$$H = \underbrace{-\frac{1}{2} J\hbar^2 S^2 N z}_{E_0} + J\hbar^2 S z \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} (1 - \gamma_{\mathbf{k}}) = E_0 + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \tag{21}$$

With $\omega_{\mathbf{k}} = JS\hbar z(1 - \gamma_{\mathbf{k}})$, $\gamma_{\mathbf{k}} = \frac{2}{z}\sum_\delta \cos(\mathbf{k}\delta)$. In the one dimensional case we find $\gamma_{\mathbf{k}} = \cos k_x a$ such that $\omega_{\mathbf{k}} = JS\hbar z(1 - \cos(k_x a))$, which is the same dispersion relation we found using the semiclassical approach. The Hamiltonian is diagonal in $\mathbf{k}$-space and we see that the system can be described as a sum of harmonic oscillators with different wavevectors $\mathbf{k}$. The quantum of this harmonic oscillator is the *magnon*. Note that the ground state of the ferromagnet does not have any quantum corrections and thus has the same energy as the classical ground state.

3 Antiferromagnets

We can derive similar equations for the antiferromagnet in a way completely analogous to the approach discussed in section 2. From a classical point of view we can state that the ground state of an antiferromagnet corresponds to that in which neighbouring spins point in opposite directions (see figure 1b). One convenient way to describe these systems is by introducing two sublattices A and B which correspond to the spins pointing up and down respectively. The Heisenberg Hamiltonian for an antiferromagnet is given by:

$$H = -J \sum_{i,j} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{22}$$

with negative J, such that the energy is minimized if neighbouring spins point in opposite direction. In this chapter we will first look at an antiferromagnetic system from a semiclassical point of view. Next we will look at quantum mechanical effects again using a Holstein-Primakoff transformation. Finally we will look at a phenomonological approach to describing the antiferromagnet, using the Néel vector and total magnetization of the system.

3.1 Semiclassical regime

Back in chapter 2 we found a general equation of motion 8:

$$\frac{d\mathbf{S}_k}{dt} = -\mathbf{S}_k \times \frac{\partial H}{\partial \mathbf{S}_k}.$$

Here H denotes the Heisenberg Hamiltonian for antiferromagnets given in equation22, which can be rewritten in terms of the sublattices A and B where neighbouring spins are always in the other sublattice (see figure 3) and to avoid overcounting we devide by 2:

$$H = \frac{J}{2}\sum_{j\in A} \mathbf{S}_j^A \cdot [\mathbf{S}_{j+1}^B + \mathbf{S}_{j-1}^B] + \frac{J}{2}\sum_{j\in B} \mathbf{S}_j^B \cdot [\mathbf{S}_{j+1}^A + \mathbf{S}_{j-1}^A]. \tag{23}$$

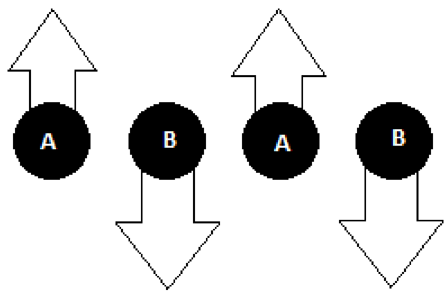

Figure 3: Schematic representation of the sublattices in an antiferromagnet

These two equations yield us the equations of motion of the system:

$$\begin{aligned}\dot{\mathbf{S}}_j^A &= -J\mathbf{S}_j^A \times [\mathbf{S}_{j-1}^B + \mathbf{S}_{j+1}^B],\\ \dot{\mathbf{S}}_j^B &= -J\mathbf{S}_j^B \times [\mathbf{S}_{j-1}^A + \mathbf{S}_{j+1}^A].\end{aligned} \tag{24}$$

Again we take the z direction to be the general direction of orientation, meaning we look at small deviatians about the z-axis:

$$\mathbf{S}_j^A = \begin{pmatrix} \delta S_j^{A,x} \\ \delta S_j^{A,y} \\ \hbar S \end{pmatrix}, \qquad \mathbf{S}_j^B = \begin{pmatrix} -\delta S_j^{B,x} \\ -\delta S_j^{B,y} \\ -\hbar S \end{pmatrix}. \tag{25}$$

Using the equations of motion 24 and the one dimensional plane wave ansatz: $S_j^{A,x} = \delta A_x e^{i(k_x a - \omega t)}$ and similar expressions for the other components we yield the following matrix equation:

$$\begin{pmatrix} i\omega & 2J\hbar S & 0 & 2J\hbar S\cos(k_x a) \\ -2J\hbar S & i\omega & -2J\hbar S\cos(k_x a) & 0 \\ 0 & 2J\hbar S\cos(k_x a) & -i\omega & 2J\hbar S \\ -2j\hbar S\cos(k_x a) & 0 & -2J\hbar S & -i\omega \end{pmatrix} \begin{pmatrix} \delta A_x \\ \delta A_y \\ \delta B_x \\ \delta B_y \end{pmatrix} = 0. \tag{26}$$

Here the zero of the determinant again gives us the dispersion relation $\omega_k = 2J\hbar S\sin(k_x a)$ which has linear k_x dependence in case of small k_x. The four eigenvectors of the Hamiltonian correspond

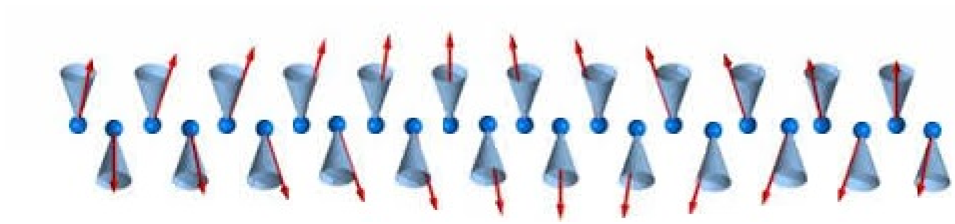

Figure 4: Schematic representation of a spinwave in an antiferromagnet (From [7])

to four different types of wave-like motion: in both sublattices there is clockwise motion, one in phase and one with a phase difference of π between the sublattices and an anti-clockwise motion in both sublattices, again with and without a phase difference of π. A schematic representation is given in figure 4.

3.2 Holstein-Primakoff transformation

As in the case of the ferromagnet we would like to also include quantum effects in our analysis. In order to do this we again want to make use of the Holstein-Primakoff transformation to map our spin operators in terms of the bosonic creation and annihilation operators. The transformed spin operators are given in equation 16 and 17. However, since we now have two sublattices A and B, we should consider the fact that one of these lattices, say B, is *flipped* with respect to lattice A, therefore the spin operators should be swapped in the following way: $S^z \to -S^z$ and $S^+ \leftrightarrow S^-$. We now have a set of operators, one for sublattice A and one for sublattice B:

$$\begin{aligned} S_j^{A,+} &= \hbar\sqrt{2S}a_j, & S_j^{B,+} &= \hbar\sqrt{2S}b_j^\dagger, \\ S_j^{A,-} &= \hbar\sqrt{2S}a_j^\dagger, & S_j^{B,-} &= \hbar\sqrt{2S}b_j, \\ S_j^{A,z} &= \hbar S - \hbar a_j^\dagger a_j, & S_j^{B,z} &= -\hbar S + \hbar b_j^\dagger b_j. \end{aligned} \tag{27}$$

Similar to 18 we can now write down the Hamiltonian for the antiferromagnet in terms of these operators:

$$\begin{aligned} H &= \frac{J}{2}\sum_{j\in A}\sum_{\delta} S_j^{A,+}S_{j+\delta}^{B,-} + S_j^{A,-}S_{j+\delta}^{B,+} + 2S_j^{A,z}S_{j+\delta}^{B,z} \\ &+\frac{J}{2}\sum_{k\in B}\sum_{\delta} S_k^{B,+}S_{k+\delta}^{A,-} + S_k^{B,-}S_{k+\delta}^{A,+} + 2S_k^{B,z}S_{k+\delta}^{A,z}. \end{aligned} \tag{28}$$

Plugging equation 27 into this Hamiltonian yields up to quadratic order:

$$\begin{aligned} H = J\hbar^2 &\sum_{j\in A}\sum_{\delta} S(a_j b_{j+\delta} + a_j^\dagger b_{j+\delta}^\dagger) - S^2 + S(a_j^\dagger a_j + b_{j+\delta}^\dagger b_{j+\delta}) \\ +J\hbar^2 &\sum_{k\in B}\sum_{\delta} S(b_k a_{k+\delta} + b_k^\dagger a_{k+\delta}^\dagger) - S^2 + S(b_k^\dagger b_k + a_{k+\delta}^\dagger a_{k+\delta}). \end{aligned} \tag{29}$$

We now use a Fourier transform of the form:

$$a_j = \frac{1}{\sqrt{N_a}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j}a_{\mathbf{k}}, \qquad b_i = \frac{1}{\sqrt{N_b}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_i}a_{\mathbf{k}}. \tag{30}$$

Applying this Fourier transform to the Hamiltonian yields:

$$H = -J\hbar^2 N \frac{z}{2} S^2 + J\hbar^2 Sz \sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}) + J\hbar^2 Sz \sum_{\mathbf{k}} \gamma_{\mathbf{k}} (a_{\mathbf{k}} b_{-\mathbf{k}} + a^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}}), \tag{31}$$

where $\gamma_{\mathbf{k}} = \frac{2}{z}\sum_\delta \cos(\mathbf{k}\delta) = \cos(|\mathbf{k}|a)$. We now see that performing a fourier transform does not diagonalize the Hamiltonian as it did in the case of the ferromagnet. This means that the bosonic creation and annihilation operators we used are not good operators for the antiferromagnetic Hamiltonian. Instead one should make use of an additional transformation called the Bogoliubov transformation. This transformation implies introducing new operators α and β for which the Hamiltonian will diagonalize. These operators are defined as:

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} - v_{\mathbf{k}} b^\dagger_{-\mathbf{k}}, \qquad \beta_{\mathbf{k}} = u_{\mathbf{k}} b_{\mathbf{k}} - v_{\mathbf{k}} a^\dagger_{-\mathbf{k}}. \tag{32}$$

Here $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ denote some real function of $\mathbf{k}$. Since these transformations are canonical they still obey the commutationrelations $[\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{k}'}] = [\beta_{\mathbf{k}}, \beta^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$ and we see that therefore $u^2_{\mathbf{k}} - v^2_{\mathbf{k}} = 1$. From this we can simply guess these functions to be $u_{\mathbf{k}} = \cosh(\theta_{\mathbf{k}})$ and $v_{\mathbf{k}} = \sinh(\theta_{\mathbf{k}})$ and since $\alpha_{\mathbf{k}}$ and $\beta^\dagger_{\mathbf{k}'}$ do commute we also find that $u_{\mathbf{k}} v_{-\mathbf{k}} = u_{-\mathbf{k}} v_{\mathbf{k}}$ which can be solved for $u_{\mathbf{k}} = u_{-\mathbf{k}}$ and $v_{-\mathbf{k}} = v_{\mathbf{k}}$. The inverse transformations are given by:

$$a_{\mathbf{k}} = u_{\mathbf{k}} \alpha_{\mathbf{k}} + v_{\mathbf{k}} \beta^\dagger_{-\mathbf{k}}, \qquad b_{\mathbf{k}} = u_{\mathbf{k}} \beta_{\mathbf{k}} + v_{\mathbf{k}} \alpha^\dagger_{-\mathbf{k}}. \tag{33}$$

If we now express the Hamiltonian in terms of these new operators α and β we find:

$$\begin{aligned} H = H_0 + J\hbar^2 Sz \sum_{\mathbf{k}} [u^2_{\mathbf{k}} + v^2_{\mathbf{k}} + 2\gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}](\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}}) + 2(v^2_{\mathbf{k}} + \gamma_{\mathbf{k}} v_{\mathbf{k}} u_{\mathbf{k}}) \\ + [\gamma_{\mathbf{k}}(u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) + 2u_{\mathbf{k}} v_{\mathbf{k}}](\beta_{-\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{-\mathbf{k}} \alpha^\dagger_{\mathbf{k}}). \end{aligned} \tag{34}$$

Here $H_0 = -J\hbar^2 N \frac{z}{2} S^2$. If we now take $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ to cancel the terms in the last brackets of the Hamiltonian, we find that for $u_{\mathbf{k}} = \cosh(\theta_{\mathbf{k}})$ and $v_{\mathbf{k}} = \sinh(\theta_{\mathbf{k}})$, $\gamma_{\mathbf{k}} = -\tanh(2\theta_{\mathbf{k}})$. From this we can write the Hamiltonian as:

$$\begin{aligned} H = H_0 + J\hbar^2 Sz \sum_{\mathbf{k}} [u^2_{\mathbf{k}} + v^2_{\mathbf{k}} + 2\gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}](\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} + 1) \\ + J\hbar^2 Sz \sum_{\mathbf{k}} (v^2_{\mathbf{k}} - u^2_{\mathbf{k}}). \end{aligned} \tag{35}$$

Which can be rearranged to form the diagonal Hamiltonian:

$$H = E_0 + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} (\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}}), \tag{36}$$

with $\omega_{\mathbf{k}} = J\hbar Sz\sqrt{1 - \gamma^2_{\mathbf{k}}}$ (note that this reduces again to the dispersion relation found in the semiclassical approach for the onedimensional case) and $E_0 = -\frac{1}{2} N J\hbar^2 (S^2 + S)z + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}$. From this we see that we have quantum corrections on the ground state, with energy $E_{qc} = -\frac{1}{2} N J\hbar^2 Sz + \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}$, which lowers the ground state energy. Thus we see that the classical ground state of the antiferromagnet is not an eigenstate of the Heisenberg exchange Hamiltonian and is corrected with quantum fluctuations.

3.3 Phenomenological approach

3.3.1 Introducing the Néel vector

In addition to the semiclassical approach as seen in the previous chapter, one can approach the problem phenomenologically. We start of with a antiferromagnetic system, consisting of two sublattices A and B with spins pointing up and down respectively. In the continuous limit one can then describe the magnetization of the system by introducing $\mathbf{m}(\mathbf{r},t) = \mathbf{m}_A(\mathbf{r},t) + \mathbf{m}_B(\mathbf{r},t)$ as the total magnetization, with $\mathbf{m}_A(\mathbf{r},t)$ and $\mathbf{m}_B(\mathbf{r},t)$ the magnetic moments of both sublattices. Besides this we also introduce an antiferromagnetic order parameter $\mathbf{l}(\mathbf{r},t) = \mathbf{m}_A(\mathbf{r},t) - \mathbf{m}_B(\mathbf{r},t)$ and its unit vector, called the Néel vector: $\mathbf{n}(\mathbf{r},t) = \mathbf{l}(\mathbf{r},t)/l(\mathbf{r},t)$. From a purely phenomenological point of view one can then give the antiferromagnetic free energy as [9]:

$$U = \int d\mathbf{r} \left[\frac{a}{2}\mathbf{m}^2 + \frac{A}{2}(\nabla\mathbf{n})^2 - \mathbf{H}\cdot\mathbf{m} - \frac{\kappa_z}{2}n_z^2 \right]. \tag{37}$$

Here a and A are respectively the homogeneous and inhomogeneous exchange constants, $\mathbf{H}$ is some external magnetic field and $\frac{\kappa_z}{2}n_z^2$ is an easy-axis anisotropy in the z-direction. This equation roughly translates as: there is an energy contribution for spins to take some orientation at all as well as an energy contribution due to a gradient in the antiferromagnetic order. If an external field is present, then there will be an energy subtraction depending on the orientation of the magnetization with respect to this field, meaning that having a magnetization parallel to the external field minimizes this energy. The anisotropy term minimizes the free energy when $\mathbf{n}$ is parallel to the z-axis. Together with the constraints $\mathbf{m}\cdot\mathbf{n} = 0$ and $|\mathbf{n}| = 1$ one obtains the following equations of motion:

$$\dot{\mathbf{n}} = (\gamma\mathbf{f}_m - G_1\dot{\mathbf{m}}) \times \mathbf{n}, \tag{38}$$

$$\dot{\mathbf{m}} = (\gamma\mathbf{f}_n - G_2\dot{\mathbf{n}}) \times \mathbf{n}. \tag{39}$$

Where γ is the gyromagnetic ratio, also $\mathbf{f}_n = -\delta U/\delta\mathbf{n} = A\mathbf{n}\times(\nabla^2\mathbf{n}\times\mathbf{n}) - \mathbf{m}(\mathbf{H}\cdot\mathbf{n}) - \kappa_z\mathbf{n}$ and $\mathbf{f}_m = -\delta U/\delta\mathbf{m} = -a\mathbf{m} + \mathbf{n}\times(\mathbf{H}\times\mathbf{n})$ denote the *effective fields* corresponding to the free energy.

3.3.2 Equivalence to the semiclassical approach

In order to see if this phenomenological approach properly describes the behaviour of the antiferromagnetic system we will now show that for the one dimensional case one fiends the same dispersion relation as seen before in [5], which showed us a linear dispersion relation (in contrast to the quadratic dispersion relation of the ferromagnet). Taking our main orientation axis to be the z-axis we can look at small deviations of $\mathbf{n}$ about this axis, meaning that $\mathbf{n} = \hat{z} + \delta\mathbf{n}$ and $\mathbf{m} = \delta\mathbf{m}$. In the absence of an external field, anisotropy and damping the equations of motion are then given by:

$$\dot{\mathbf{n}} = -\gamma a\mathbf{m} \times \mathbf{n}, \tag{40}$$

$$\dot{\mathbf{m}} = \gamma A[\mathbf{n}\times(\nabla^2\mathbf{n}\times\mathbf{n})] \times \mathbf{n} + \mathbf{T}, \tag{41}$$

where T denotes the higher order terms we will not be taking into account. Since we expect similar behaviour of the system as we have seen in Chapter 3.1 the way to solve the equations of motion will again be using a plane wave ansatz.

$$\delta\mathbf{n} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ -\frac{\delta n_x^2}{2} - \frac{\delta n_y^2}{2} \end{pmatrix} e^{i(k_x x - \omega t)},$$

$$\delta\mathbf{m} = \begin{pmatrix} \delta m_x \\ \delta m_y \\ \delta n_x \delta m_x - \delta n_y \delta m_y \end{pmatrix} e^{i(k_x x - \omega t)}. \tag{42}$$

Where we made use of the constraints $\mathbf{m} \cdot \mathbf{n} = 0$ and $|\mathbf{n}| = 1$. Plugging these into the equations of motion 38 and 39 yields:

$$\begin{aligned} i\omega \cdot \delta m_x &= k^2 A\gamma \cdot \delta n_y, \\ i\omega \cdot \delta m_y &= -k^2 A\gamma \cdot \delta n_x, \\ i\omega \cdot \delta n_x &= a\gamma \cdot \delta m_y, \\ i\omega \cdot \delta n_y &= -a\gamma \cdot \delta m_x. \end{aligned} \tag{43}$$

These equation can then be written in a more convenient matrix form:

$$\begin{pmatrix} i\omega & 0 & 0 & -\gamma a \\ 0 & i\omega & \gamma a & 0 \\ 0 & k^2 A\gamma & i\omega & 0 \\ -k^2 A\gamma & 0 & 0 & i\omega \end{pmatrix} \begin{pmatrix} \delta n_x \\ \delta n_y \\ \delta m_x \\ \delta m_y \end{pmatrix} = 0. \tag{44}$$

As before the dispersion relation follows from finding the zeroes of the determinant which gives us the dispersion relation $\omega_k = \gamma k \sqrt{aA}$. As expected we found ω_k to be linear in k and therefore the phenomenological approach yields similar results as the semiclassical approach.

The eigenvectors of the characteristic matrix can be found by plugging this dispersion relation back in and setting $\delta m_x = \delta m_y = 1$ we find:

$$\nu_1 = \frac{1}{2}\begin{bmatrix} \frac{i\sqrt{a}}{\sqrt{A}k} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \nu_2 = \frac{1}{2}\begin{bmatrix} 0 \\ -\frac{i\sqrt{a}}{\sqrt{A}k} \\ 1 \\ 0 \end{bmatrix}, \quad \nu_3 = \frac{1}{2}\begin{bmatrix} -\frac{i\sqrt{a}}{\sqrt{A}k} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \nu_4 = \frac{1}{2}\begin{bmatrix} 0 \\ \frac{i\sqrt{a}}{\sqrt{A}k} \\ 0 \\ 1 \end{bmatrix}. \tag{45}$$

These can now be transformed back into the sublattice spin-operators $\mathbf{S}_A$ and $\mathbf{S}_B$ via:

$$\mathbf{S}_A^\alpha = \frac{\delta n_\alpha + \delta m_\alpha}{2}, \qquad -\mathbf{S}_B^\alpha = \frac{\delta n_\alpha - \delta m_\alpha}{2}. \tag{46}$$

From here we can construct new vectors $(\delta S_A^x, \delta S_A^y, -\delta S_B^x, -\delta S_B^y)$ when we are looking at deviations of $\mathbf{S}_A$ and $\mathbf{S}_B$ around the z-axis:

$$\chi_1 = \frac{1}{2}\begin{bmatrix} \frac{i\sqrt{a}}{\sqrt{A}k} \\ 1 \\ \frac{i\sqrt{a}}{\sqrt{A}k} \\ -1 \end{bmatrix}, \quad \chi_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -\frac{i\sqrt{a}}{\sqrt{A}k} \\ -1 \\ -\frac{i\sqrt{a}}{\sqrt{A}k} \end{bmatrix}, \quad \chi_3 = \frac{1}{2}\begin{bmatrix} -\frac{i\sqrt{a}}{\sqrt{A}k} \\ 1 \\ \frac{i\sqrt{a}}{\sqrt{A}k} \\ -1 \end{bmatrix}, \quad \chi_4 = \frac{1}{2}\begin{bmatrix} 1 \\ \frac{i\sqrt{a}}{\sqrt{A}k} \\ -1 \\ \frac{i\sqrt{a}}{\sqrt{A}k} \end{bmatrix}. \tag{47}$$

These eigenvectors correspond to elliptical precession around the z-axis, of which the superpositions restore the circular precession we found in 3.1. We have thus shown the equivalence of the phenomenological approach with the semiclassical approach.

3.3.3 Damping, external field and anisotropy

If we take into account Gilbert damping, an external field and easy-axis anisotropy, the equations of motion are given by equations 38 and 39. Again looking at small deviations of $\mathbf{n}$ around the z-axis using the planar wave equation 42, we find the following set of equations:

$$\begin{aligned}
\delta\dot{n}_x &= -\gamma a\delta m_y - \gamma H_z\delta n_y - G_1\delta\dot{m}_y, \\
\delta\dot{n}_y &= +\gamma a\delta m_x + \gamma H_z\delta n_x + G_1\delta\dot{m}_x, \\
\delta\dot{m}_x &= \gamma A\nabla^2\delta n_y - \gamma H_z\delta m_y - G_2\delta\dot{n}_y + \gamma\kappa_z\delta n_y, \\
\delta\dot{m}_y &= -\gamma A\nabla^2\delta n_x + \gamma H_z\delta m_x + G_2\delta\dot{n}_x - \gamma\kappa_z\delta n_x,
\end{aligned} \tag{48}$$

If we assume $G_1 \ll G_2$ we can already set $G_1 = 0$. Now the above set of equations translates into the following matrix equation:

$$\begin{pmatrix} i\omega & -\gamma H_z & 0 & -\gamma a \\ \gamma H_z & i\omega & \gamma a & 0 \\ 0 & i\omega G_2 - k^2\gamma A - \gamma\kappa_z & i\omega & -\gamma H_z \\ k^2\gamma A - i\omega G_2 + \gamma\kappa_z & 0 & \gamma H_z & i\omega \end{pmatrix} \begin{pmatrix} \delta n_x \\ \delta n_y \\ \delta m_x \\ \delta m_y \end{pmatrix} = 0 \tag{49}$$

The dispersion relation follows from the zero of the determinant and is given by:

$$\omega = \Omega - iG_2\frac{a}{2}\frac{1}{\sqrt{a(Ak^2+\kappa_z)}}\Omega, \tag{50}$$

where $\Omega = (H_z - \sqrt{a\gamma^2(Ak^2+\kappa_z)})$. We can clearly see the imaginary term added to the frequency due to the damping. The ground state of the antiferromagnet minimizes the free energy U in equation 37. Since the external field tends to align $\mathbf{m}$ in the z-direction and the easy-axis anisotropy tends to align $\mathbf{n}$ in the z-direction. For $H_z \ll \kappa_z$ the free energy will be minimized when $\mathbf{n}$ is parallel to the z-axis, whereas for $\kappa_z \ll H_z$ the energy will be minimized when $\mathbf{m}$ is parallel to the z-axis. The threshold for this can be determined by considering the dispersion relation in equation 50. Consider some small deviations of $\mathbf{n}$ around the z-direction, corresponding to a magnon for which $k \to 0$. When the ground state is given by $\mathbf{n}$ parallel to the z-axis and since the ground state must be stable, then the Gilbert damping will relax these deviations back to the ground state. However when the ground state is given by $\mathbf{m}$ parallel to the z-direction, the unstable state of $\mathbf{n}$ being parallel to the z-direction will be driven into the ground state by the Gilbert damping. This damping or driving corresponds to the sign of the Gilbert damping term. From this we can infer that there should be some sign difference in the Gilbert damping term for these different ground states. For $k \to 0$ the Gilbert damping term is given by:

$$\omega_{damp} = iG_2\frac{}{2\sqrt{a\kappa_z}}(H_z\gamma - \sqrt{a\gamma^2\kappa_z})$$

The sign of this term may flip at the threshold given by $H_z\gamma - \sqrt{a\gamma^2\kappa_z} = 0$, from which we see that the sign change occurs at $H_z = \sqrt{a\kappa_z}$.

4 Magnon interactions in antiferromagnetic spin configurations

In this section we will focus on relaxation of magnon interactions within the antiferromagnet. For this we will need their interaction amplitudes. In order to do this we return to the quantum mechanical approach using the Holstein Primakoff transformation followed by the Bogoliubov transformation introduced in section 3.1. This time we will add an anisotropy and external field term in the z-direction and keep the higher order terms which effectively describe the magnon interaction in the system in terms of creation and annihilation operators.

4.1 Holstein-Primakoff transformation with external field and anisotropy

We start of with the Hamiltonian:

$$H = J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j - B\sum_i S_i^z - \frac{\kappa}{2S}\sum_i (S_i^z)^2, \tag{51}$$

which can again be split up into two sublatices to yield:

$$\begin{aligned}H = J\sum_{j\in A}\sum_{\delta}\mathbf{S}_j^A\cdot\mathbf{S}_{j+\delta}^B + J\sum_{j\in B}\sum_{\delta}\mathbf{S}_j^B\cdot\mathbf{S}_{j+\delta}^A - B\sum_{j\in A}S_j^{A,z} + B\sum_{j\in B}S_j^{B,z}\\ - \frac{\kappa}{2S}\sum_{j\in A}(S_j^{A,z})^2 - \frac{\kappa}{2S}\sum_{j\in B}(S_j^{B,z})^2.\end{aligned} \tag{52}$$

We can write out the dot product so that our Hamiltonian reads:

$$\begin{aligned}H = J\sum_{j\in A}\sum_{\delta}\left[S_j^{A,x}S_{j+\delta}^{B,x} + S_j^{A,y}S_{j+\delta}^{B,y} + S_j^{A,z}S_{j+\delta}^{B,z}\right] + J\sum_{j\in B}\sum_{\delta}\left[S_j^{B,x}S_{j+\delta}^{A,x} + S_j^{B,y}S_{j+\delta}^{A,y} + S_j^{B,z}S_{j+\delta}^{A,z}\right]\\ - BJ\sum_{j\in A}S_j^{A,z} + BJ\sum_{j\in B}S_j^{B,z} - \frac{\kappa}{2S}\sum_{j\in A}(S_j^{A,z})^2 - \frac{\kappa}{2S}\sum_{j\in B}(S_j^{B,z})^2.\end{aligned} \tag{53}$$

If we now introduce the spin-operators $S^+ = S_x + iS_y$ and $S^- = S_x - iS_y$ and express the Hamiltonian in terms of these operators we find:

$$\begin{aligned}H = J\sum_{j\in A}\sum_{\delta}\left[\frac{1}{2}S_j^{A,+}S_{j+\delta}^{B,-} + \frac{1}{2}S_j^{A,-}S_{j+\delta}^{B,+} + S_j^{A,z}S_{j+\delta}^{B,z}\right] + J\sum_{j\in B}\sum_{\delta}\left[\frac{1}{2}S_j^{B,+}S_{j+\delta}^{A,-} + \frac{1}{2}S_j^{B,-}S_{j+\delta}^{A,+} + S_j^{B,z}S_{j+\delta}^{A,z}\right]\\ - BJ\sum_{j\in A}S_j^{A,z} + BJ\sum_{j\in B}S_j^{B,z} - \frac{\kappa}{2S}\sum_{j\in A}(S_j^{A,z})^2 - \frac{\kappa}{2S}\sum_{j\in B}(S_j^{B,z})^2.\end{aligned} \tag{54}$$

As before, we now introduce the Holstein-Primakoff transformations as given in 27:

$$\begin{aligned}H = J\hbar^2\sum_{j\in A}\sum_{\delta}\left[S(a_jb_{j+\delta} + a_j^\dagger b_{j+\delta}^\dagger) - S^2 + S(a_j^\dagger a_j + b_{j+\delta}^\dagger b_{j+\delta}) - a_j^\dagger a_j b_{j+\delta}^\dagger b_{j+\delta}\right]\\ + J\hbar^2\sum_{j\in B}\sum_{\delta}\left[S(b_ja_{j+\delta} + b_j^\dagger a_{j+\delta}^\dagger) - S^2 + S(b_j^\dagger b_j + a_{j+\delta}^\dagger a_{j+\delta}) - b_j^\dagger b_j a_{j+\delta}^\dagger a_{j+\delta}\right]\\ - \frac{\kappa\hbar^2}{2S}\sum_{j\in A}(S^2 - 2Sa_j^\dagger a_j + a_j^\dagger a_j a_j^\dagger a_j) - \frac{\kappa\hbar^2}{2S}\sum_{j\in B}(S^2 - 2Sb_j^\dagger b_j + b_j^\dagger b_j b_j^\dagger b_j)\\ - B\hbar\sum_{j\in A}(S - a_j^\dagger a_j) + B\hbar\sum_{j\in B}(S - b_j^{\dagger]b_j}).\end{aligned} \tag{55}$$

Next we want to perform the fourier transformation, yielding:

$$H = \underbrace{\frac{-J\hbar^2 NS^2 z}{2} - \kappa\hbar^2 SN - B\hbar NS}_{H_0} + J\hbar^2 Sz \sum_{\mathbf{k}} \left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right] + J\hbar^2 Sz \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \left[a_{\mathbf{k}} b_{-\mathbf{k}} + a^\dagger_{\mathbf{k}} b^\dagger_{-\mathbf{k}}\right]$$
$$+ \kappa\hbar^2 \sum_{\mathbf{k}} \left[a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}}\right] + B\hbar \sum_{\mathbf{k}} \left[b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\right]$$
$$- \frac{2J\hbar^2 z}{N} \sum_{\delta} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \mathbf{k}_3,\mathbf{k}_4}} \left[a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_2} b^\dagger_{\mathbf{k}_3} b_{\mathbf{k}_4} - b^\dagger_{\mathbf{k}_1} b_{\mathbf{k}_2} a^\dagger_{\mathbf{k}_3} a_{\mathbf{k}_4}\right] e^{i(\mathbf{k}_4 - \mathbf{k}_3)\cdot\delta} \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4}$$
$$- \frac{\kappa\hbar^2}{NS} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \mathbf{k}_3,\mathbf{k}_4}} \left[a^\dagger_{\mathbf{k}_1} a_{\mathbf{k}_2} a^\dagger_{\mathbf{k}_3} a_{\mathbf{k}_4} + b^\dagger_{\mathbf{k}_1} b_{\mathbf{k}_2} b^\dagger_{\mathbf{k}_3} b_{\mathbf{k}_4}\right] \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4}. \quad (56)$$

As we have seen the Hamiltonian up to quadratic order can be diagonalized by performing a Bogoliubov transformation:

$$a_{\mathbf{k}} = u_{\mathbf{k}} \alpha_{\mathbf{k}} + v_{\mathbf{k}} \beta^\dagger_{-\mathbf{k}}, \qquad b_{\mathbf{k}} = u_{\mathbf{k}} \beta_{\mathbf{k}} + v_{\mathbf{k}} \alpha^\dagger_{-\mathbf{k}}. \quad (57)$$

To quadratic order, the Hamiltonian then reads:

$$H = H_0 + J\hbar^2 Sz \sum_{\mathbf{k}} [(u^2_{\mathbf{k}} + v^2_{\mathbf{k}} + 2\gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}})(\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}}) + 2(v^2_{\mathbf{k}} + \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}})$$
$$+ (\gamma_{\mathbf{k}}(u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) + 2u_{\mathbf{k}} v_{\mathbf{k}})(\beta_{-\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{-\mathbf{k}} \alpha^\dagger_{\mathbf{k}})]$$
$$+ \kappa\hbar^2 \sum_{\mathbf{k}} \left[(u^2_{\mathbf{k}} + v^2_{\mathbf{k}})(\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}}) + 2v^2_{\mathbf{k}} + 2u_{\mathbf{k}} v_{\mathbf{k}}(\beta_{-\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{-\mathbf{k}} \alpha^\dagger_{\mathbf{k}})\right]$$
$$+ B\hbar \sum_{\mathbf{k}} \left[(u^2_{\mathbf{k}} - v^2_{\mathbf{k}})(\beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} - \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}})\right]. \quad (58)$$

After some rearranging we find:

$$H = H_0 + J\hbar^2 Sz \sum_{\mathbf{k}} \{\left(1 + \frac{\kappa}{JSz}\right)(u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) + 2\gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}\}(\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}})$$
$$+ 2\{\left(1 + \frac{\kappa}{JSz}\right) v^2_{\mathbf{k}} + \gamma_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}\} + \{\gamma_{\mathbf{k}}(u^2_{\mathbf{k}} + v^2_{\mathbf{k}}) + 2\left(1 + \frac{\kappa}{JSz}\right) u_{\mathbf{k}} v_{\mathbf{k}}\}(\alpha^\dagger_{\mathbf{k}} \beta^\dagger_{-\mathbf{k}} + \alpha_{\mathbf{k}} \beta_{-\mathbf{k}})$$
$$+ B\hbar \sum_{\mathbf{k}} \left[(u^2_{\mathbf{k}} - v^2_{\mathbf{k}})(\beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} - \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}})\right]. \quad (59)$$

We want the term that couples the α and β magnons to vanish therefore we find that $\gamma_{\mathbf{k}}$ is constraint to:

$$\gamma_{\mathbf{k}} = -(1 + \frac{\kappa}{JSz}) \tanh(2\theta_{\mathbf{k}}). \quad (60)$$

Which for $\kappa \ll J$ is approximated by $\gamma_{\mathbf{k}} = -\tanh(2\theta_{\mathbf{k}})$. The lower order terms now diagonalize to:

$$H = E_0 + \sum_{\mathbf{k}} \hbar\omega^-_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \sum_{\mathbf{k}} \hbar\omega^+_{\mathbf{k}} \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} + \sum_{\mathbf{k}} \hbar(\omega^+_{\mathbf{k}} - B). \quad (61)$$

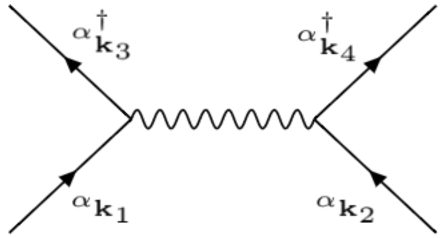

Figure 5: schematic representiation of the $\alpha\alpha\alpha^\dagger\alpha^\dagger$ interaction

Where $\omega^{\pm}_{\mathbf{k}} = \pm B + J\hbar Sz\sqrt{(1+\frac{\kappa}{JSz})^2 - \gamma^2_{\mathbf{k}}}$ and $E_0 = H_0 - \frac{J\hbar^2 SzN}{2}$. We see that for an external magnetic field the energies of the α and β magnons split, such that the β magnons have a higher energy.

4.2 Relaxation time approximation in the low-temperature regime

The higher order terms expand into pairs of four operators corresponding to magnon-magnon interactions and can be found in the Appendix (section 6). The magnon-magnon interactions (or scattering) can be drawn schematically using Feynman diagrams (see for example figure 5). We would like to find the rate of these *scattering* events. From quantum mechanical time-dependent perturbation theory we know that the rate at which an initial quantum state $|i\rangle$ turns into some other, final state $|f\rangle$ is given by Fermi's Golden Rule and reads:

$$\Gamma = \frac{2\pi}{\hbar}\sum_{i,f} W_i |\langle f|V_{int}|i\rangle|^2 \delta(\epsilon_f - \epsilon_i), \tag{62}$$

where the sum runs over all possible initial and final states, W_i is the Boltzman weight that gives the probability of being in some initial state $|i\rangle$, V_{int} is the matrix element of the Hamiltonian corresponding to the interaction and the delta function enforces conservation of energy. We will now derive the rate of the $\alpha\alpha\alpha^\dagger\alpha^\dagger$ interaction, which are most present at low temperature (due to energy splitting in case of an external magnetic field). The matrix element for this interaction is:

$$V_{int} = \alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2}\alpha^\dagger_{\mathbf{k}_3}\alpha^\dagger_{\mathbf{k}_4}\left[\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}\right] \times \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4}. \tag{63}$$

Where $\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}$ is the amplitude given in the appendix. Also we added a symmetric term by swapping out $\mathbf{k}_3$ and $\mathbf{k}_4$ since these correspond to the same interaction. Since we start off in some initial state and after the scattering event end up in the final state we readily see that $|f\rangle = \frac{1}{\sqrt{n_{\mathbf{k}_1}n_{\mathbf{k}_2}[1+n_{\mathbf{k}_3}][1+n_{\mathbf{k}_4}]}}\alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2}\alpha^\dagger_{\mathbf{k}_3}\alpha^\dagger_{\mathbf{k}_4}|i\rangle$, with the prefactors ensuring proper normalization. From this we see that summing over the final states becomes summing over $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\mathbf{k}_4$ and the scattering rate now reads:

$$\Gamma = \frac{2\pi}{\hbar}\sum_{\substack{\mathbf{k}_1,\mathbf{k}_2\\ \mathbf{k}_3,\mathbf{k}_4}}\sum_i W_i |\langle i|\frac{\alpha^\dagger_{\mathbf{k}_1}\alpha^\dagger_{\mathbf{k}_2}\alpha_{\mathbf{k}_3}\alpha_{\mathbf{k}_4}\alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2}\alpha^\dagger_{\mathbf{k}_3}\alpha^\dagger_{\mathbf{k}_4}}{\sqrt{n_{\mathbf{k}_1}n_{\mathbf{k}_2}[1+n_{\mathbf{k}_3}][1+n_{\mathbf{k}_4}]}}|i\rangle|^2 \times |\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2$$
$$\times\, \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4}\delta(\epsilon_{\mathbf{k}_1}+\epsilon_{\mathbf{k}_2}-\epsilon_{\mathbf{k}_3}-\epsilon_{\mathbf{k}_4}). \tag{64}$$

Using $a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} = n_{\mathbf{k}}$ where $n_{\mathbf{k}}$ gives the number of magnons of momentum $\mathbf{k}$ and $[\alpha_{\mathbf{k}}, \alpha^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$ so that we can rearrange the operators to find:

$$\Gamma = \frac{2\pi}{\hbar} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \\ \mathbf{k}_3,\mathbf{k}_4}} \sum_i W_i |\langle i| \frac{n_{\mathbf{k}_1} n_{\mathbf{k}_2} [1+n_{\mathbf{k}_3}][1+n_{\mathbf{k}_4}]}{\sqrt{n_{\mathbf{k}_1} n_{\mathbf{k}_2} [1+n_{\mathbf{k}_3}][1+n_{\mathbf{k}_4}]}} |i\rangle|^2 \times |\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2$$
$$\times \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4} \delta(\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2} - \epsilon_{\mathbf{k}_3} - \epsilon_{\mathbf{k}_4}). \quad (65)$$

Since $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and $\mathbf{k}_4$ are independent momenta we can write $\sum_i W_i = \sum_{\substack{i_{\mathbf{k}_1}, i_{\mathbf{k}_2} \\ i_{\mathbf{k}_3}, i_{\mathbf{k}_4}}} W_{i_{\mathbf{k}_1}} W_{i_{\mathbf{k}_2}} W_{i_{\mathbf{k}_3}} W_{i_{\mathbf{k}_4}}$ and introduce the Planck distribution as:

$$f(\mathbf{k}) = \sum_{i_{\mathbf{k}}} W_{i_{\mathbf{k}}} n_{\mathbf{k}} = \frac{1}{e^{\beta \epsilon_{\mathbf{k}}} - 1}. \quad (66)$$

Then we find for the rate of scattering into this state:

$$\Gamma = \frac{2\pi}{\hbar} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2 \\ \mathbf{k}_3,\mathbf{k}_4}} |\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2 \times \delta_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4} \delta(\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2} - \epsilon_{\mathbf{k}_3} - \epsilon_{\mathbf{k}_4})$$
$$\times f(\mathbf{k}_1) f(\mathbf{k}_2) [1 + f(\mathbf{k}_3)][1 + f(\mathbf{k}_4)]. \quad (67)$$

Now we can look at the scattering rate for some $\mathbf{k}$ by fixing one of the four $\mathbf{k}$'s that we have. Also we can go to a continuous system by replacing sums by integrals yielding:

$$\Gamma = \frac{2\pi V^2}{\hbar} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \int \frac{d^3\mathbf{k}_3}{(2\pi)^3} \int \frac{d^3\mathbf{k}_4}{(2\pi)^3} |\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2$$
$$\times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2} - \epsilon_{\mathbf{k}_3} - \epsilon_{\mathbf{k}_4}) \times f(\mathbf{k}_1) f(\mathbf{k}_2) [1 + f(\mathbf{k}_3)][1 + f(\mathbf{k}_4)]. \quad (68)$$

We have now found an expression to calculate the rate of magnon interactions in the antiferromagnet. Next we will derive from this the relaxation time of an out-of-equilibrium antiferromagnet. We use the *relaxation time approximation*:

$$\Gamma = \frac{\partial f}{\partial t} = -\frac{1}{\tau} [f(\mathbf{k}_1) - f_{eq,\mathbf{k}_1}], \quad (69)$$

which describes how fast some non-equilibrium distribution relaxes to equilibrium. We can use this to approximate the relaxation time by noting that we scatter out of the state with momentum $\mathbf{k}_1$ so that $f_{eq,\mathbf{k}_1} = 0$ and we find:

$$\Gamma = -\frac{f(\mathbf{k}_1)}{\tau}. \quad (70)$$

Combining this with the rate found by using Fermi's Golden Rule we obtain an expression for the relaxation time:

$$\frac{1}{\tau} = -\frac{2\pi V^2}{\hbar} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \int \frac{d^3\mathbf{k}_3}{(2\pi)^3} \int \frac{d^3\mathbf{k}_4}{(2\pi)^3} |\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2 \times f(\mathbf{k}_2) [1 + f(\mathbf{k}_3)][1 + f(\mathbf{k}_4)]$$
$$\times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\epsilon_{\mathbf{k}_1} + \epsilon_{\mathbf{k}_2} - \epsilon_{\mathbf{k}_3} - \epsilon_{\mathbf{k}_4}). \quad (71)$$

One can evaluate this integral numerically for some desired interaction. We are mostly interested in its general behaviour and would like to know how the relaxation time depends on temperature. In order to do this we make the integrals dimensionless. We use the dimensionless quantity:

$$\beta\epsilon_{\mathbf{k}} = -B\beta\hbar + \beta JS\hbar^2 z\sqrt{(1+\frac{\kappa}{JSz})^2 - \gamma_{\mathbf{k}}^2}, \tag{72}$$

wWhich can be rewritten and expanded for $ka \ll 1$ to read:

$$\beta\epsilon_{\mathbf{x}} = -y + \sqrt{\nu + \mathbf{x}^2} \approx -y + \sqrt{\nu} + \frac{\mathbf{x}^2}{2\sqrt{\nu}}, \tag{73}$$

with dimensionless quantities $y = \beta B\hbar$, $\nu = \beta^2 J^2 S^2 \hbar^4 z^2 \left(\frac{\kappa^2}{J^2S^2z^2} + \frac{2\kappa}{JSz}\right)$ and $\mathbf{x} = JS\hbar^2\beta z\mathbf{k}a = \beta c\mathbf{k}$. From this we find:

$$d\mathbf{k} = \frac{d\mathbf{x}}{\beta c}. \tag{74}$$

The integral becomes dimensionless by:

$$\frac{1}{\tau} = -\frac{2\pi V^2}{\hbar}\int\frac{d^3\mathbf{x}_2}{(2\beta c\pi)^3}\int\frac{d^3\mathbf{x}_3}{(2\beta c\pi)^3}\int\frac{d^3\mathbf{x}_4}{(2\beta c\pi)^3}A|\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4}+\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_4,\mathbf{x}_3}|^2\times f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)]$$
$$\times\frac{\delta(\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4)}{(d^3\mathbf{k}/d^3\mathbf{x})_{\mathbf{x}=\mathbf{x}_1}}\frac{\delta(|\mathbf{x}_1|+|\mathbf{x}_2|-|\mathbf{x}_3|-|\mathbf{x}_4|)}{(d\epsilon/dx)_{\mathbf{x}=\mathbf{x}_1}}, \tag{75}$$

where A is some constant that makes the scattering amplitude dimensionless, in order to find it we want to express $|\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}+\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2$ explicitly in terms of $\mathbf{k}$. For this type of scattering it is given by:

$$|\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}+\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2 = \frac{1}{N^2}\left|\frac{\hbar^2\kappa}{S}(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4}+u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})+2\hbar^2J(v_{\mathbf{k}_1}v_{\mathbf{k}_3}u_{\mathbf{k}_2}u_{\mathbf{k}_4}+v_{\mathbf{k}_1}v_{\mathbf{k}_4}u_{\mathbf{k}_2}u_{\mathbf{k}_3})\right|^2, \tag{76}$$

where we recall $u_{\mathbf{k}} = \cosh\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \sinh\theta_{\mathbf{k}}$. If we now express $v_{\mathbf{k}}$ and $u\mathbf{k}$ in terms of $\gamma_{\mathbf{k}} = -\tanh(2\theta_{\mathbf{k}}) = coska \approx 1 - \frac{k^2a^2}{2}$, where we now also assume the long wavelength limit, we find:

$$\cosh(\theta_{\mathbf{k}}) = \sqrt{\frac{1+\frac{\gamma_{\mathbf{k}}}{\sqrt{\gamma_{\mathbf{k}}^2-1}}}{2}} \approx -\frac{(-1)^{3/4}}{\sqrt{2|\mathbf{k}|a}} \approx \sinh\theta_{\mathbf{k}}. \tag{77}$$

Since the scattering amplitudes always have pairs of four $\cosh\theta_{\mathbf{k}}$ and/or $\sinh\theta_{\mathbf{k}}$, $|\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2$ becomes dimensionless as:

$$|\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}+\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3}|^2 = \left(\frac{\beta c}{a}\right)^4|\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4}+\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_4,\mathbf{x}_3}|^2, \tag{78}$$

where due to the approximation we also find (see Appendix for other processes):

$$|\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4}+\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_4,\mathbf{x}_3}|^2 = \frac{4}{N^2}\left(\frac{2\hbar^2\kappa}{S}+4\hbar^2J\right)^2\times\frac{1}{16|\mathbf{x}_1||\mathbf{x}_2||\mathbf{x}_3||\mathbf{x}_4|}. \tag{79}$$

And we see that $A = \left(\frac{\beta c}{a}\right)^4$. Since $V^2 = N^2 a^6$ we find the dimensionless integral:

$$\begin{aligned}\frac{1}{\tau} &= \frac{4\pi k_b T}{8\hbar c^2}\left(\frac{2\hbar^2\kappa}{S} + 4\hbar^2 J\right)^2 \frac{\sqrt{\nu}}{\mathbf{x}_1}\int\frac{d^3\mathbf{x}_2}{(2\pi)^3}\int\frac{d^3\mathbf{x}_3}{(2\pi)^3}\int\frac{d^3\mathbf{x}_4}{(2\pi)^3}\frac{1}{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} \\ &\quad\times f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)] \times \delta(\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4)\delta(|\mathbf{x}_1|+|\mathbf{x}_2|-|\mathbf{x}_3|-|\mathbf{x}_4|) \\ &= \frac{4\pi k_b^2 T^2}{8\hbar^7 J^3 S^3 z^3 a^2 \mathbf{k}_1^2}\left(\frac{2\hbar^2\kappa}{S} + 4\hbar^2 J\right)^2\left(\frac{\kappa^2}{J^2S^2z^2} + \frac{2\kappa}{JSz}\right)^{1/2}\int\frac{d^3\mathbf{x}_2}{(2\pi)^3}\int\frac{d^3\mathbf{x}_3}{(2\pi)^3}\int\frac{d^3\mathbf{x}_4}{(2\pi)^3}\frac{1}{\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4} \\ &\quad\times f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)] \times \delta(\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4)\delta(|\mathbf{x}_1|+|\mathbf{x}_2|-|\mathbf{x}_3|-|\mathbf{x}_4|). \quad (80)\end{aligned}$$

Lastly the distribution functions are given by:

$$f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)] = \frac{1}{e^{-y+\sqrt{\nu}+\frac{\mathbf{x}_2^2}{2\sqrt{\nu}}}-1}\left[1+\frac{1}{e^{-y+\sqrt{\nu}+\frac{\mathbf{x}_3^2}{2\sqrt{\nu}}}-1}\right]\left[1+\frac{1}{e^{-y+\sqrt{\nu}+\frac{\mathbf{x}_4^2}{2\sqrt{\nu}}}-1}\right] \quad (81)$$

In the low temperature regime these reduce to:

$$f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)] \approx \frac{e^{y-\sqrt{\nu}}}{e^{\frac{\mathbf{x}_2^2}{\sqrt{\nu}}}} \times 1 \times 1, \quad (82)$$

from which we find for the relaxation time:

$$\frac{1}{\tau} \propto \frac{4\pi k_b^2 T^2}{8\hbar^7 J^3 S^3 z^3 a^2 \mathbf{k}_1^2}\left(\frac{2\hbar^2\kappa}{S} + 4\hbar^2 J\right)^2\left(\frac{\kappa^2}{J^2S^2z^2} + \frac{2\kappa}{JSz}\right)^{1/2} \times e^{\beta B\hbar - \sqrt{\nu}}. \quad (83)$$

Which we can simplify for $\kappa \ll J$ to find:

$$\frac{1}{\tau} \propto \frac{8\pi k_b^2 T^2}{\hbar^3 J^2 S^4 z^4 a^2 \mathbf{k}_1^2}(2\kappa JSz)^{1/2} \times e^{\beta B\hbar - \sqrt{\nu}}. \quad (84)$$

Since antiferromagnetic ordering disappears above the so called Néel Temperature, which depends on the exchange constant like $T_N = JSz$, it is convenient to write the relaxation time as:

$$\frac{1}{\tau} \propto \frac{8\sqrt{2\kappa J} k_b^2}{\hbar^3\sqrt{Sz} a^2 \mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2 \times e^{\beta B\hbar - \sqrt{\nu}}. \quad (85)$$

In a similar way all relaxation times can be determined and are given in the Appendix.

4.3 High temperature regime

The relaxation time approximation above assumed the low temperature regime ($k_b T \ll \epsilon_{\mathbf{k}}$) allowing for approximating the distributions with Boltzmann distributions. However, in the high energy regime we are allowed to neglect the anisotropy and external field since these are small compared to the high thermal energy and we can approximate:

$$\beta\epsilon_{\mathbf{k}} = \beta\hbar\omega_{\mathbf{k}} = \beta JS\hbar^2 z|\mathbf{k}|a = x. \quad (86)$$

Again looking at the process of $\alpha\alpha\alpha^\dagger\alpha^\dagger$, we find the relaxation time to be given by the integral:

$$\frac{1}{\tau} = \frac{4\pi k_b T}{8\hbar c^2 a}\left(\frac{2\hbar^2\kappa}{S} + 4\hbar^2 J\right)^2 \int \frac{d^3\mathbf{x}_2}{(2\pi)^3}\int \frac{d^3\mathbf{x}_3}{(2\pi)^3}\int \frac{d^3\mathbf{x}_4}{(2\pi)^3}\frac{1}{\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4}$$
$$\times f(\mathbf{x}_2)[1+f(\mathbf{x}_3)][1+f(\mathbf{x}_4)] \times \delta(\mathbf{x}_1+\mathbf{x}_2-\mathbf{x}_3-\mathbf{x}_4)\delta(|\mathbf{x}_1|+|\mathbf{x}_2|-|\mathbf{x}_3|-|\mathbf{x}_4|). \quad (87)$$

Here we can neglect the κ term since it is small compared to J and we find:

$$\frac{1}{\tau} \propto \frac{8\pi k_b^2 T^2}{\hbar^3 J S^3 z^3 a|\mathbf{k}_1|}, \quad (88)$$

which in terms of T_N reads:

$$\frac{1}{\tau} \propto \frac{8\pi k_b^2 J}{\hbar^3 S z a|\mathbf{k}_1|}\left(\frac{T}{T_N}\right)^2. \quad (89)$$

So we see that in the high temperature regime the external field and anisotropy can be completely neglected and therefore don't affect the relaxation time, such that there is no longer any exponential suppression due to field and anisotropy. For *thermal magnons* we get from $\epsilon = k_b T$ that $|\mathbf{k}_1| = \frac{k_b T}{JS\hbar^2 za} = \frac{k_b T}{T_N \hbar^2 a}$. In this way the relaxation time reduces to:

$$\frac{1}{\tau} \propto \frac{8\pi k_b J}{\hbar S z}\left(\frac{T}{T_N}\right). \quad (90)$$

4.4 Gilbert damping

Another way for an out-of-equilibrium antiferromagnet to relax to equilibrium is by means of Gilbert damping. In this case there is no interaction/scattering with other magnons involved. From Landau-Lifshitz-Gilbert phenomenology we find that relaxation due to Gilbert damping comes into the Boltzmann equation as:

$$\Gamma = \frac{\partial f}{\partial t} = -\frac{1}{\tau_G}[f_{\mathbf{k}_1} - f_{eq}] = -2\alpha\omega_{\mathbf{k}_1}[f_{\mathbf{k}_1} - f_{eq}] = -\frac{2\alpha\epsilon_{\mathbf{k}_1}}{\hbar}[f_{\mathbf{k}_1} - f_{eq}], \quad (91)$$

where α is the Gilbert damping coefficient. For $\epsilon_{\mathbf{k}_1} = k_b T$ (thermal magnons) this reduces to:

$$\frac{1}{\tau_G} = \frac{2\alpha k_b T}{\hbar}. \quad (92)$$

So we find that in the thermal magnon regime the system relaxes to equilibrium due to Gilbert damping at a rate that is linear in temperature.

5 Conclusion

In this thesis we performed an analysis of the behaviour of ferromagnets as well as antiferromagnets. It is found that perturbations of the ground state result in a propagating spin wave within the magnetic material. The ground state of the ferromagnet turns out to be an eigenstate of the Heisenberg exchange Hamiltonian and thus has a constant magnetization. However for the antiferromagnet we found that the ground state is *not* an eigenstate of the Heisenberg exchange Hamiltonian, which results in corrections due to quantum fluctuations. These results were confirmed using semiclassical, quantum mechanical and phenomenological approaches. We went on to look at the antiferromagnet including an external field and easy-axis anisotropy and performed the Holstein-Primakoff transformation, followed by the Bogoliubov transformation to diagonalize the Hamiltonion, yielding a dispersion relation for spin waves in such a system. This time we also included higher order terms in bosonic operators, effectively describing magnon-magnon interactions/scattering. We performed quantum mechanical, time-dependent perturbation theory in the form of Fermi's Golden Rule to do a relaxation time approximation of these interactions. For the different temperature regimes we found different relaxation times. Finally we compared the results with the relaxation time approximation for Gilbert damping and noted that for higher temperatures, interactions become more and more important. Previous research confirms T^2 dependence of relaxation rates [11] [12].

The results contribute to a better understanding of the dynamics and the transport of spin waves within the antiferromagnet, with the goal of obtaining a complete theory of transport. The results found in this thesis can be used for kinetic theories by making use of the Boltzmann equation:

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \left(\frac{\partial}{\partial t} + \mathbf{v}(\mathbf{k})\vec{\nabla} + \dot{\mathbf{k}}\vec{\nabla}_{\mathbf{k}}\right) f(\mathbf{k}) = -\frac{1}{\tau}[f(\mathbf{k}) - f_{eq}]. \tag{93}$$

For future studies we suggest numerical calculations of the relaxation time, by calculating the dimensionless integrals. Also it might be interesting to look into the three-magnon interactions that are obtained when not expanding the Holstein-Primakoff spin operators and various other possible magnon interactions, to extract a relaxation time. These relaxation times can then be used for various calculations on transport within the antiferromagnet.

6 Appendix

6.1 Constants

J	Exchange coefficient/Spin stiffness
S	Spin number
a	Lattice constant/spacing
z	Number of nearest neighbours
k_b	Boltzmann constant
T	Temperature
$\hbar$	Planck's constant
κ	Anisotropy coefficient
B/H	Magnetic field coefficients
β	$\frac{1}{k_b T}$
$\mathbf{k}$	Wave vector
τ	Relaxation time
T_N	Curie temperature, $T_N = JSz$

6.2 Scattering amplitudes

Here we give a list of the symmetrized scattering amplitudes $\frac{N}{2}(\Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} + \Gamma_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_3})$ as used in calculating the scattering rates.

$\alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2}\alpha^\dagger_{\mathbf{k}_3}\alpha^\dagger_{\mathbf{k}_4}$:

$$-\frac{\hbar^2\kappa}{S}(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}) - 2\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_3}u_{\mathbf{k}_2}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_4}u_{\mathbf{k}_2}u_{\mathbf{k}_3})$$

$\alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2}\beta_{\mathbf{k}_3}\beta_{\mathbf{k}_4}$:

$$-\frac{\hbar^2\kappa}{S}(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}) - 4\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})$$

$\alpha_{\mathbf{k}_1}\beta_{\mathbf{k}_2}\beta_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_4}$:

$$-\frac{2\hbar^2\kappa}{S}(u_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + v_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}) - 2\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + 2u_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})$$

$\beta_{\mathbf{k}_1}\beta_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_4}$:

$$-\frac{\hbar^2\kappa}{S}(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}) - 2\hbar^2 J(v_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4})$$

$\alpha^\dagger_{\mathbf{k}_1}\alpha^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_4}$:

$$-\frac{\hbar^2\kappa}{S}(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}) - 4\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})$$

$\beta_{\mathbf{k}_1}\alpha^\dagger_{\mathbf{k}_2}\beta^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_4}$:

$$-\frac{2\hbar^2\kappa}{S}(v_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})$$
$$- 2\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4})$$

$\alpha_{\mathbf{k}_1}\alpha^\dagger_{\mathbf{k}_2}\alpha^\dagger_{\mathbf{k}_3}\beta^\dagger_{\mathbf{k}_4}$:

$$-\frac{\hbar^2\kappa}{S}(u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4} + u_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4})$$
$$- 2\hbar^2 J(v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4} + v_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4} + 2u_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4})$$

$\alpha_{\mathbf{k}_1}\beta_{\mathbf{k}_2}\alpha^{\dagger}_{\mathbf{k}_3}\beta^{\dagger}_{\mathbf{k}_4}$:

$$-\frac{2\hbar^2\kappa}{S}(u_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4}+u_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4}+v_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4}+v_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4})$$
$$-2\hbar^2 J(2v_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}v_{\mathbf{k}_4}+2u_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}u_{\mathbf{k}_4}+v_{\mathbf{k}_1}u_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4}+v_{\mathbf{k}_1}u_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4}+u_{\mathbf{k}_1}v_{\mathbf{k}_2}v_{\mathbf{k}_3}u_{\mathbf{k}_4}+u_{\mathbf{k}_1}v_{\mathbf{k}_2}u_{\mathbf{k}_3}v_{\mathbf{k}_4})$$

6.3 Relaxation times for magnon-magnon scattering processes

Table 1: Relaxation times for different scattering processes.

Scattering ↓	$\lvert\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4}+\Gamma_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_4,\mathbf{x}_3}\rvert^2$	Low Temperature (T): $\frac{1}{\tau}\propto$	High T: $\frac{1}{\tau}\propto$
$\alpha\to\alpha\alpha^\dagger\alpha^\dagger$	$\frac{4}{N^2}\left(\frac{2\hbar^2\kappa}{S}+4\hbar^2J\right)^2$	$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha^\dagger\to\alpha\alpha\alpha^\dagger$		$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha\to\alpha\beta\beta$	$\frac{4}{N^2}\left(\frac{2\hbar^2\kappa}{S}+4\hbar^2J\right)^2$	$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta\to\alpha\alpha\beta$		$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha\to\beta\beta\beta^\dagger$	$\frac{4}{N^2}\left(\frac{4\hbar^2\kappa}{S}+8\hbar^2J\right)^2$	$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta\to\alpha\beta\beta^\dagger$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\alpha\beta\beta$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta\to\beta\beta\beta^\dagger$	$\frac{4}{N^2}\left(\frac{2\hbar^2\kappa}{S}+4\hbar^2J\right)^2$	$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\beta\beta\beta^\dagger$		$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha^\dagger\to\alpha^\dagger\beta^\dagger\beta^\dagger$	$\frac{4}{N^2}\left(\frac{2\hbar^2\kappa}{S}+4\hbar^2J\right)^2$	$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\alpha^\dagger\alpha^\dagger\beta^\dagger$		$\frac{8\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{8\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta\to\alpha^\dagger\beta^\dagger\beta^\dagger$	$\frac{4}{N^2}\left(\frac{4\hbar^2\kappa}{S}+8\hbar^2J\right)^2$	$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha^\dagger\to\beta\beta^\dagger\beta^\dagger$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\alpha^\dagger\beta\beta^\dagger$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha\to\alpha^\dagger\alpha^\dagger\beta^\dagger$	$\frac{4}{N^2}\left(\frac{4\hbar^2\kappa}{S}+8\hbar^2J\right)^2$	$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha^\dagger\to\alpha\alpha^\dagger\beta^\dagger$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\alpha\alpha^\dagger\alpha^\dagger$		$\frac{32\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{32\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha\to\beta\alpha^\dagger\beta^\dagger$	$\frac{4}{N^2}\left(\frac{8\hbar^2\kappa}{S}+16\hbar^2J\right)^2$	$\frac{128\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{128\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta\to\alpha\alpha^\dagger\beta^\dagger$		$\frac{128\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2\times e^{\beta B\hbar-\sqrt{\nu}}.$	$\frac{128\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\beta^\dagger\to\alpha\beta\alpha^\dagger$		$\frac{128\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{128\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$
$\alpha^\dagger\to\alpha\beta\beta^\dagger$		$\frac{128\pi\sqrt{2\kappa J}V^2k_b^2}{\hbar^3\sqrt{Sz}a^8\mathbf{k}_1^2}\left(\frac{T}{T_N}\right)^2.$	$\frac{128\pi V^2k_b^2J}{\hbar^3Sza^7\lvert\mathbf{k}_1\rvert}\left(\frac{T}{T_N}\right)^2.$

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