

An Introduction to Large Scale Structure

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Abstract

With the development of more and more large scale surveys in the past few years, and the years to come, the importance of understanding the large scale structure of the universe is steadily increasing. Without an effective model to describe this structure we would not be able to glean all the information we can from the surveys, wasting their potential. In this work some of the dynamics behind the evolution of large scale structures will be explained, as well as methods to accurately study this in both the linear and the perturbative nonlinear regime, using Standard Perturbation Theory and the Effective Field Theory of Large Scale Structure. From this we will find that using either method we can compute the time dependence of the density contrast perturbatively. The EFToLSS will be more accurate but needs more information to determine important parameters like the speed of sound and equation of state. We will also look at an elaboration on the theory by considering some of the implications from General Relativity. Here the most important result will be a motivation why using a Newtonian approximation in the first chapters of the thesis is warranted. After this a simplification of the theory is considered in the form of Spherical Collapse, where it will be shown that the importance of the growing mode in linear SPT with respect to higher order modes depends highly on the begin conditions.

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1 Introduction

1.1 Cosmology

The goal of this work is to review some descriptions of the formation of the largest structures in the universe. But what does this even mean? There are several scales at which one can view the universe: at the smallest scale we have solar systems, like our own. One scale higher up are galaxies, which are essentially groups of solar systems. It stands to reason then that on an even larger scale we find clusters of galaxies. But what happens on even larger scales? Galaxy surveys, like the Sloan Digital Sky Survey in figure 1, have shown us that galaxies form “filaments” in space, with huge voids in between. These galaxies seem almost randomly distributed, and it will turn out that on the largest scales, they are. This follows from some fundamental observations that can be made when looking at the largest scales of the universe:

1. Non-gravitational interactions can be ignored. On the distances considered in cosmology short range forces like the weak and strong force become negligible. Electromagnetism is a long range force, but the universe is electrically neutral on large scales, and any magnets that have so far been encountered in the universe have too weak a field to affect the large scale structure.
2. There is no privileged position in the universe. This is sometimes called the “Copernican Principle” and it amounts to a having a homogeneous and translationally invariant universe (at very large scales).
3. Every direction in the universe looks the same. This means that the universe is isotropic (again, at very large scales).

These last two observations are usually combined in what is called “The Cosmological Principle”. As the name suggests this principle lies at the basis of many cosmological theories. The universe has more properties that will prove useful. For example, the universe consists of “things”. We can split the most abundant things into three different groups: (ordinary) matter, dark matter and dark energy. These groups make up roughly 4.9%, 26.8% and 68.3% of all the energy density in the universe, respectively. The energy density of each of these constituents contributes to the expansion of the universe (how exactly this works follows from the general relativistic Friedmann equation (1)), and the degree by which it has expanded with respect to its size at some reference time t_0 is given by the scale factor $a(t)$, so logically $a(t_0)$ equals one. The exact time dependence of the scale factor depends on which model of the universe we are using. Note that the scale factor describes the expansion of the “grid” on which we are working; so if two points started off with a very large separation, the expansion will cause them to move away from each other much faster than other, closer points. Thus, because the grid on which we work is expanding, it makes sense for us to define so called “comoving coordinates” \mathbf{x} . These are basically the physical coordinates at the reference time, so if $\mathbf{r}(t)$ denotes the physical coordinates at a time t then $\mathbf{r}(t) = a(t)\mathbf{x}$. Taking the time derivative of course gives us a corresponding velocity, called the “peculiar velocity” \mathbf{v} , which is defined as $\mathbf{v} = \dot{\mathbf{x}}$ (the dot denotes a time derivative).

We now have everything we need to describe points in space. The question that remains is, in the context of large scale structure, what are the points we are trying to describe? Remembering the constituents of the universe, we note that of these three dark energy is

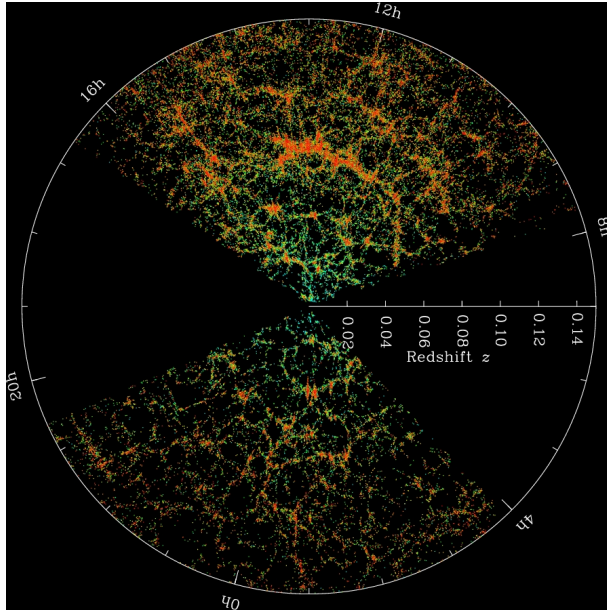


Figure 1: A slice out of the three dimensional map of sky coordinates vs. distance for galaxies as measured in the Sloan Digital Sky Survey [1]. Each of the points represents a galaxy, so you can clearly see the clustering and the forming of filaments on large scales. Note that using Hubbles law we can relate the distance of stars to their redshift due to the expansion of the universe. The outer circle is at a distance of two billion light years. The left and right parts are empty because dust prevented accurate measurements.

only active on Hubble scales, scales too large to influence structure formation. We then only need to describe regular matter and dark matter, but of these two dark matter is far more abundant, and regular matter is coupled to dark matter through the gravitational force. This means that regular matter will follow the motion of dark matter, and we can thus get away with only describing the motion of dark matter particles in the universe.

So, if we want to explain figure 1 we should attempt to describe the distribution of matter. Because dark matter is much more abundant than regular matter, and because regular matter tends to be drawn to dark matter by gravitational effects, it seems like it is enough to look at dark matter particles to be able to describe large scale structure. This is easier said than done, since we do not yet know what dark matter is, exactly. The best model we have for the dark matter particles we are looking at is the “Cold Dark Matter” (CDM) model, where cold simply implies that they move slowly. Now, all possible constituents of CDM have in common that they are extremely light in comparison to the observed galaxies. From this we can derive that the number density of CDM particles must be extremely high. So high even, that we are able to model it as a fluid, which is what we assume when we do Standard Perturbation Theory and when we develop the Effective Field Theory of Large Scale Structure later on in this work. Both theories give us a way to compute the density contrast $\delta(\mathbf{x}, t)$ defined through $\rho(\mathbf{x}, t) = (1 + \delta(\mathbf{x}, t))\rho_b$, which tells us how the density of dark matter $\rho(\mathbf{x}, t)$ is distributed with respect to the background density ρ_b . This is directly related to the distribution of dark matter in space, and therefore the evolution of large scale structure; the computation of the density contrast will thus be the main focus of this review.

1.2 Large Scale Structure

When the universe came into being, it did so with small overdensities and underdensities of dark matter. These contrasts started off small, but as the universe aged they grew and became the cosmological structures that we perceive today. Understanding what processes play a role in this evolution and their physical background is crucial to interpret new experimental findings from Large Scale Structure surveys. To do this however we need knowledge of the dynamics behind the evolution of large scale structures, so that we can develop methods to make analytical predictions. This has been difficult to do, because extracting information from large scale structures poses several problems. One of these problems arises from the fact that Large Scale Structures are heavily influenced by effects on small (also called Ultraviolet or UV) scales. These short scale modes¹ couple to other modes in order to affect physics on larger scales; all this means is that even though short scale modes are much smaller than the larger scale modes, they still affect them.

There are several ways to work with this problem. Before going into them we will mention the dynamics behind Large Scale Structures. After this Standard Perturbation Theory (SPT) is introduced mostly for introductory purposes; because of the aforementioned problem SPT is less than ideal to accurately model Large Scale Structure evolution. However, more success has been found using the Effective Field Theory of Large Scale Structure (EFToLSS). This is a field theory in the sense that it incorporates all the relevant degrees of freedom and describes all the relevant physics at a macroscopic scale. Conceptually this is very similar to for example statistical physics, where one can treat interaction of nearest neighbors in a lattice using an effective field. However now the “nearest neighbors” that we are approximating are our short scale modes. What makes this approach effective in statistical physics is the fact that the approximated interactions work on a very short scale. This is the case for EFToLSS as well; we can describe the coupling of UV physics to IR (that is, long-wavelength) physics using a coupling parameter that is small on large scales (which means that there is very little coupling here), and increases in size as we move closer to the non-linear scale. It would make sense to define such a parameter by taking the ratio of the wavenumber of the considered mode k and the wavenumber at the nonlinear scale k_{NL} ; k/k_{NL} . The fact that in the universe we can see a clear distinction between the Hubble scale, where perturbations are linear, and the non-linear scale (about the scale where gravitational collapse overtakes the expansion of the universe) makes this coupling parameter concrete. Both SPT and EFToLSS use this distinction, however in SPT it is used to neglect contributions of higher order modes, whereas in EFToLSS it is used to compute the coupling parameters. This makes EFToLSS a more rigorous theory than SPT.

After introducing EFToLSS we discuss the spherical collapse approximation, which literally assumes that the collapse of dark matter into structures is done in a spherically symmetric way, and its possible uses for EFToLSS.

An important note is that we do not make use of general relativity in the largest part of this paper. This is because, as explained in chapter 6 later on, all of the outcomes for both SPT and EFToLSS are valid even when using a Newtonian approximation. However, general relativity does play an important role in cosmology. Because of this a small part has been

¹The term “mode” is derived from Fourier transformations, where the transformed quantity now depends on the modes represented by wavenumber \mathbf{k} . When considering the Fourier transform of $\delta(\mathbf{x}, t)$, given by $\hat{\delta}(\mathbf{k}, t)$ the small, short scale or short wavelength modes correspond to large values of \mathbf{k} .

dedicated to the derivation of an important equation for spherical symmetry using the metric tensor.

2 Dynamics of Large Scale Structure

2.1 Notation

In this work we will mostly follow the notation introduced by [3], the relevant parts of which we will be repeated in the tables below. Throughout this paper we will make use of Einstein's summation convention.

Table 1: Abbreviations

SPT	Standard Perturbation Theory
SC	Spherical Collapse
CDM	Cold Dark Matter (model)
EFToLSS	Effective Field Theory of Large Scale Structure
EdS	Einstein-de Sitter (universe)

Table 2: Various Cosmological Variables

Ω_m	The total matter density in units of critical density
H	The Hubble constant, given by $H = \dot{a}/a$
a	The Cosmological scale factor
τ	The conformal time, $\tau = \int_0^t \frac{dt'}{a(t')}$, $d\tau = dt/a$
\mathcal{H}	The conformal expansion rate, $\mathcal{H} = aH = \dot{a}$

Table 3: Notation for Cosmic Fields

\hat{X}	The Fourier transform of a field X , given by $\hat{X}(\mathbf{k}) = (2\pi)^{-3} \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} X(\mathbf{x})$
x	The comoving position in real space
$\rho(\mathbf{x})$	The local cosmic density
$\delta(\mathbf{x})$	The local density contrast, $\delta = \rho/\rho_b - 1$
$\phi(\mathbf{x})$	The Newtonian gravitational field
$\mathbf{v}(\mathbf{x})$	The local peculiar velocity field
$\theta(\mathbf{x})$	The local velocity divergence

2.2 Equations of motion

2.2.1 Cosmology

In the introduction we mentioned how the Friedmann equation can affect the scale factor. The Friedmann equation does more than that however; it gives us a relation between the scale factor, energy density and curvature of space. It is a result from general relativity, and a derivation is therefore beyond the scope of this work, for more information view [10][11]. Some of our results will make use of it however, so for completeness' sake we give it here[11]:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}, \quad (1)$$

here κ denotes the curvature of the space and R_0 the radius of the space.

Another equation of note is the fluid equation, which relates the energy density evolution of a constituent of the universe to its pressure and the scale factor:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0, \quad (2)$$

where P is the pressure.

Together these two equations tell us a lot of important things about the physical aspects of our universe.

2.2.2 Cosmic fluid

The first step in deriving a mathematical description of the evolution is of course to determine the equations of motion governing our process. We already argued that because of their high number density we could describe the CDM particles as a fluid, which naturally obeys the so called “Vlasov” equation, or collisionless Boltzmann equation, given by equation (7) [4]. This equation directly follows from Liouville’s theorem.² Throughout we will assume that the vorticity of our fluid vanishes [3]. This is because upon working out the vorticity term we find that it scales with $1/a$, so as time passes it decays and we can neglect it.

We first introduce the total phase space density f . This phase space density is defined such that $f d^3\mathbf{x} d^3\mathbf{p}$ is equal to the number of particles in an infinitesimal volume $d^3\mathbf{x} d^3\mathbf{p}$. An intuitive definition for our phase space density “per particle” is then $f_n(\mathbf{x}, \mathbf{p}) \equiv \delta^{(3)}(\mathbf{x} - \mathbf{x}_n) \delta^{(3)}(\mathbf{p} - m\mathbf{v}_n)$, and so

$$f(\mathbf{x}, \mathbf{p}) = \sum_n f_n(\mathbf{x}, \mathbf{p}) = \sum_n \delta^{(3)}(\mathbf{x} - \mathbf{x}_n) \delta^{(3)}(\mathbf{p} - m\mathbf{v}_n). \quad (3)$$

Using our total phase space density we can then define several useful quantities that relate the phase space density to the mass, momentum and kinetics of our system. We thus define

²Liouville’s theorem tells us that any volume in phase space is conserved, which tells us something about the phase-space distribution of our particles.

the mass density ρ , momentum density π^i and the kinetic tensor σ^{ij} as follows:

$$\rho(\mathbf{x}, t) = \frac{m}{a^3} \int d^3\mathbf{p} f(\mathbf{x}, \mathbf{p}) = \frac{m}{a^3} \sum_n \delta^{(3)}(\mathbf{x} - \mathbf{x}_n) \quad (4)$$

$$\pi^i(\mathbf{x}, t) = \frac{1}{a^4} \int d^3\mathbf{p} p^i f(\mathbf{x}, \mathbf{p}) = \frac{m}{a^3} \sum_n v_n^i \delta^{(3)}(\mathbf{x} - \mathbf{x}_n) \quad (5)$$

$$\sigma^{ij}(\mathbf{x}, t) = \frac{1}{ma^5} \int d^3\mathbf{p} p^i p^j f(\mathbf{x}, \mathbf{p}) = \frac{m}{a^3} \sum_n v_n^i v_n^j \delta^{(3)}(\mathbf{x} - \mathbf{x}_n) \quad (6)$$

By our previous argument we know that the evolution of this phase space density is governed by the Vlasov equation,

$$\frac{Df_n}{Dt} = \frac{\partial f_n}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f_n}{\partial \mathbf{x}} - m \sum_{\bar{n} \neq n} \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}} \cdot \frac{\partial f_n}{\partial \mathbf{p}} = 0. \quad (7)$$

Here we have ϕ_n the Newtonian potential due to a single particle. The full Newtonian potential is defined through the Poisson equation;

$$\Delta \phi = 4\pi G a^2 \rho = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta. \quad (8)$$

Here we have $\Delta \equiv \nabla^2$ the laplacian. The full solution is then given by the following expression;

$$\phi = \sum_n \phi_n, \quad (9)$$

$$\phi_n(\mathbf{x}) = -\frac{Gm}{|\mathbf{x} - \mathbf{x}_n|}. \quad (10)$$

We will now define the full Vlasov equation, found by summing (7) over n :

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f}{\partial \mathbf{x}} - m \sum_{n, \bar{n}; \bar{n} \neq n} \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}} \cdot \frac{\partial f_n}{\partial \mathbf{p}} = 0. \quad (11)$$

Next we need a way to extract information on the physical system from this equation for the distribution, preferably in terms of the physical quantities defined in equations (4), (5) and (6). The way we do this is by taking the moments of the Vlasov equation. For any function $F(\mathbf{x}, \mathbf{p})$ the n 'th moment is defined as:

$$\int d^3\mathbf{p} p^{i_1} \dots p^{i_n} F(\mathbf{x}, \mathbf{p}). \quad (12)$$

To illustrate the process we will calculate the zeroth moment of the Vlasov equation, which will yield the continuity equation,

$$0 = \int d^3\mathbf{p} \left[\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f}{\partial \mathbf{x}} - m \sum_{n, \bar{n}; \bar{n} \neq n} \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}} \cdot \frac{\partial f_n}{\partial \mathbf{p}} \right] \quad (13)$$

$$= \frac{\partial}{\partial t} \left(\frac{a^3}{m} \rho \right) + \frac{\partial}{\partial \mathbf{x}} \cdot \int d^3\mathbf{p} \frac{\mathbf{p}}{ma^2} f - m \sum_{n, \bar{n}; \bar{n} \neq n} \int d^3\mathbf{p} \frac{\partial}{\partial \mathbf{p}} \cdot \left[\frac{\partial \phi}{\partial \mathbf{x}} f \right] \quad (14)$$

$$= \frac{a^3}{m} \frac{\partial \rho}{\partial t} + 3 \frac{a^2 \dot{a}}{m} \rho + \frac{a^2}{m} \frac{\partial}{\partial \mathbf{x}} \cdot (\pi), \quad (15)$$

and so we have,

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \frac{\partial}{\partial \mathbf{x}} \cdot (\rho \mathbf{v}) = 0. \quad (16)$$

Going from the first to the second line we used that \mathbf{p} , t and \mathbf{x} are independent, and we filled in equation (4). From the second to the third line we filled in equation (5) and used the boundary conditions, namely that f vanishes there, to get rid of the third term. We then defined the velocity field \mathbf{v} as $v^i = \pi^i / \rho$. So we have that the zeroth moment yields the continuity equation, while the first moment gives us the Euler equation, as in [4]:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \frac{\partial}{\partial \mathbf{x}} \cdot (\rho \mathbf{v}) = 0 \quad (17)$$

$$\frac{\partial \mathbf{v}}{\partial t} + H\mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}}) \mathbf{v} + \frac{1}{a} \frac{\partial \phi}{\partial \mathbf{x}} = -\frac{1}{a\rho} \frac{\partial \tau^{ij}}{\partial \mathbf{x}}. \quad (18)$$

Here we denoted the stress energy tensor by τ^{ij} , which also depends on σ^{ij} . We will consider it in more detail in chapter 4.

Note that in our derivation we have integrated out the information on the phase space distribution, to be left with more easily interpretable information on the physical observables of our system.

It is important to note that, at this point, we still do not have a closed system of equations, after all we do not know what σ^{ij} is. We could assume some relation between the stress energy tensor and the velocity, for example using the symmetries of the problem, however there is an easier solution. We know from hydrodynamics that for a single coherent flow $\sigma^{ij} = 0$, and luckily for us at early times, before structures had time to virialize and collapse, we can describe dark matter approximately with such a coherent flow. It turns out that this approximation remains useful to explore large scale structures at our current time.

After doing some rewriting we will use equations (17) and (18) for SPT, and (11) will be used to derive the similar equations for our EFToLSS.

3 Standard Perturbation Theory

Consider again the density function $\delta(\mathbf{x}, t)$, and the divergence of the velocity field $\theta(\mathbf{x}, t)$,

$$\rho = \rho_b(1 + \delta), \quad \theta \equiv \nabla \cdot \mathbf{v}(\mathbf{x}, t).$$

So δ tells us how much the density at a certain location differs from the background density. If we think about δ in this way, then it is certainly possible for us to have modes of waves in our dark matter, which consist of a wave of overdensities and underdensities succeeding each other.

For simplicity we rewrite our equations (17) and (18) from a standard time viewpoint, to so called ‘‘conformal time’’, given by $d\tau = dt/a$. Remembering also that the stress energy tensor can be approximated by zero, we find:

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot ((1 + \delta) \mathbf{v}) = 0 \quad (19)$$

$$\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H} \mathbf{v} + (\mathbf{v} \cdot \vec{\nabla}) \mathbf{v} + \vec{\nabla} \phi = 0. \quad (20)$$

We can now proceed to apply perturbation theory to our equations of motion, starting off with linear perturbation theory.

3.1 Linear perturbation theory

At the largest scales we expect the universe to become smooth, which is why the fluctuation fields δ and θ will become smaller. This means that terms of order two and higher can be neglected, and we can accurately work with the linearised versions of equations (19) and (20). After taking the divergence of equation (20) and filling in the Poisson equation (8) we find,

$$\frac{\partial \delta}{\partial \tau} + \theta = 0, \quad (21)$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = 0. \quad (22)$$

This gives us easy access to the equation governing the evolution of the density contrast, by substituting $-\partial \delta / \partial \tau$ for θ in equation (22);

$$\frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta}{\partial \tau} - \frac{3}{2}\Omega_m \mathcal{H}^2 \delta = 0. \quad (23)$$

Because of equation (21) knowing the solution to equation (23) directly gives us a solution for θ .

In principle we are now stuck, since we do not know the value of Ω_m (remember that Ω_m is a measure for the percentage of the energy in the universe consisting of matter). This value is determined by the model we use for our universe, so before you specify this you cannot continue.

As an example we will work out the density contrast in an Einstein-de Sitter universe, i.e. a flat universe in which all energy is made up of regular matter; $\Omega_m = 1$. Note that (23) is a second order differential equation, so if we manage to find two independent solutions then we have found all solutions.

To solve our differential equation we start off by considering the characteristics of the Einstein-de Sitter universe. In order to compute the density contrast we will need to know how $a(t)$ depends on time, which follows from solving equations (1) and (2) in an EdS universe. In this model for the universe we only have matter, so $P = 0$ (the pressure exerted outwards by matter is equal to zero), and we have $\kappa = 0$ since our universe is flat.

We will first solve the fluid equation for $\rho(a)$:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \implies \frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \implies \rho = \rho_0 a^{-3}. \quad (24)$$

Filling this into the Friedmann equation (1) we find,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\rho_0\pi G}{3a^{-3}c^2} \implies \dot{a}^2 = \frac{8\rho_0\pi G}{3c^2} \frac{1}{a} \implies a(t) \propto t^{2/3} \quad (25)$$

Note that t is the “real”, cosmological time, not the proper time. In order to solve our differential equation we then make the assumption that $\delta(\tau, \mathbf{x}) = \delta_0(\mathbf{x})a(\tau)^\alpha$, with $\delta_0(\mathbf{x}) =$

$\delta(0, \mathbf{x})$. Since we have a differential equation with respect to proper time and we know how all of the terms depend on time, our equation becomes easy to solve. To be able to use these time dependencies we rewrite equation (23) in terms of cosmological time using that $d\tau = dt/a$. Remembering also that $\mathcal{H} = \dot{a}$, combining everything and dividing out the space-dependent part we get:

$$(a\ddot{a}^2 + a^2\ddot{a})\alpha a^{\alpha-1} + a^2\dot{a}^2\alpha(\alpha-1)a^{\alpha-2} + a^2\dot{a}\alpha a^{\alpha-1} - \frac{3}{2}a^2\dot{a}^2a^\alpha = 0. \quad (26)$$

Our next step is realizing that we can express all the derivatives of a in terms of a itself, since

$$\dot{a} \propto \frac{2}{3}t^{-\frac{1}{3}} \propto \frac{2}{3}a^{\frac{1}{2}} \quad (27)$$

$$\ddot{a} \propto -\frac{2}{9}t^{-\frac{4}{3}} \propto -\frac{2}{9}a^{-2}. \quad (28)$$

Finally, since we assume $\delta \propto a^\alpha$ to be a valid solution at any a we can simply consider $a = 1$. Filling all of this in again we get the following quadratic equation for α ,

$$\left(\frac{4}{9} - \frac{2}{9}\right)\alpha + \frac{4}{9}\alpha(\alpha-1) + \frac{4}{9}\alpha - \frac{3}{2}\frac{4}{9} = 0 \implies 4\alpha^2 + 2\alpha - 6 = 0. \quad (29)$$

This equation has as two solutions $\alpha = 1$ and $\alpha = -3/2$, which means that in the linear approximation density fluctuations grow as the scale factor. Because the $\alpha = -3/2$ solution (called the decaying mode) quickly becomes negligible with respect to the $\alpha = 1$ solution (called the growing mode) we tend to ignore this decaying mode, and focus on the growing mode.

3.2 Non-linear perturbation theory

When considering the non-linear regime things get slightly more complicated. Our goal is again to determine δ , however now we try to incorporate the nonlinear terms in the previous equations. In order to solve this problem we will transform our equations into Fourier space, and rewrite them somewhat so that we can solve for $\delta(\mathbf{k}, \tau)$ perturbatively. In our treatment of this we will follow [3]. As before we will assume that there is no vorticity in the velocity field.

We start by transforming our problem into Fourier space. We will use the following convention for the Fourier transform:

$$\hat{f}(\mathbf{k}, \tau) = \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}, \tau). \quad (30)$$

To use perturbation theory we then assume that we can expand δ and θ around the linear solutions (for a motivation as to why this is possible, see appendix A). What this essentially means is that we can write our fields as a power series in the linear terms, i.e. $\delta^{(n)} \propto (\delta^{(1)})^n$ where $\delta^{(1)}$ is the linear density contrast. Because at large scales (linear scales if you will) the universe is approximately homogeneous we know that the linear density contrast will be

small, and this expansion therefore valid.

So we have,

$$\delta(\mathbf{x}, \tau) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x}, \tau), \quad \theta = \sum_{n=1}^{\infty} \theta^{(n)}(\mathbf{x}, \tau) \quad (31)$$

Here $\delta^{(1)}$ and $\theta^{(1)}$ are the linear solutions we found before, $\delta^{(2)}$ and $\theta^{(2)}$ are quadratic, and so on.

Now remember that the relevant equations from which we can solve δ are (19) and (20). We will proceed by computing the Fourier transform of equation (19) and leave the Fourier transform of the Euler equation as an exercise for the reader. We can write the continuity equation as:

$$\frac{\partial \delta}{\partial \tau} + \theta = -\delta \theta - \mathbf{v} \cdot \nabla \delta. \quad (32)$$

Fourier transforming both sides and using the convolution theorem³ we find,

$$\frac{\partial \hat{\delta}(\mathbf{k}, \tau)}{\partial \tau} + \hat{\theta}(\mathbf{k}, \tau) = - \int d^3 \mathbf{k}_1 \hat{\delta}(\mathbf{k} - \mathbf{k}_1) \hat{\theta}(\mathbf{k}_1) - \int \frac{d^3 \mathbf{x}}{(2\pi)^3} \mathbf{v}(\mathbf{x}) \cdot \nabla \delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (33)$$

The first part of the equation is already in the desired form, so we will focus on rewriting the second term on the right hand side. Since we are again trying to take the Fourier transform of a product, $(\mathbf{v}(\mathbf{x}) \cdot \nabla \delta(\mathbf{x}))$ in this case) we can again apply the convolution theorem to find that,

$$\int \frac{d^3 \mathbf{x}}{(2\pi)^3} \mathbf{v}(\mathbf{x}) \cdot \nabla \delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = \mathcal{F}(\mathbf{v}(\mathbf{x}) \cdot \nabla \delta(\mathbf{x})) = \mathcal{F}(\mathbf{v}(\mathbf{x})) * \mathcal{F}(\nabla \delta(\mathbf{x})), \quad (34)$$

so if we compute their individual Fourier transforms we are done. Starting with the Fourier transform of $\nabla \delta(\mathbf{x})$,

$$\int \frac{d^3 \mathbf{x}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \nabla \delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 \mathbf{x} \nabla [\delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}}] + \int \frac{d^3 \mathbf{x}}{(2\pi)^3} i\mathbf{k} \delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = i\mathbf{k} \hat{\delta}(\mathbf{k}), \quad (35)$$

where we used in the second step that δ is zero at the boundary. We will now compute the Fourier transform of \mathbf{v} ,

$$\int \frac{d^3 \mathbf{x}}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \mathbf{v}(\mathbf{x}) = \frac{-i}{(2\pi)^3} \int d^3 \mathbf{x} \nabla \left[\frac{\mathbf{k}}{(\mathbf{k})^2} e^{-i\mathbf{k} \cdot \mathbf{x}} \cdot \mathbf{v}(\mathbf{x}) \right] - i \frac{\mathbf{k}}{\mathbf{k}^2} \frac{1}{(2\pi)^3} \int d^3 \mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} \nabla \cdot \mathbf{v}(\mathbf{x}) \quad (36)$$

$$= -i \frac{\mathbf{k}}{(\mathbf{k})^2} \hat{\theta}(\mathbf{k}). \quad (37)$$

This is all the information we need to compute equation (34). So taking the convolution of our last two results we find,

$$\int \frac{d^3 \mathbf{x}}{(2\pi)^3} \mathbf{v}(\mathbf{x}) \cdot \nabla \delta(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} = \int d^3 \mathbf{k}_1 i(\mathbf{k} - \mathbf{k}_1) \hat{\delta}(\mathbf{k} - \mathbf{k}_1) (-i \frac{\mathbf{k}_1}{(\mathbf{k})^2} \hat{\theta}(\mathbf{k}_1)) \quad (38)$$

$$= \int d^3 \mathbf{k}_1 \frac{(\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{k}_1}{(\mathbf{k}_1)^2} \hat{\delta}(\mathbf{k} - \mathbf{k}_1) \hat{\theta}(\mathbf{k}_1). \quad (39)$$

³The convolution theorem tells us that the Fourier transform of a product is equal to the convolution of the Fourier transforms; $\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g)$.

We can then proceed to combine this with our original equation (33):

$$\frac{\partial \hat{\delta}(\mathbf{k}, \tau)}{\partial \tau} + \hat{\theta}(\mathbf{k}, \tau) = - \int d^3 \mathbf{k}_1 \hat{\delta}(\mathbf{k} - \mathbf{k}_1) \hat{\theta}(\mathbf{k}_1) - \int d^3 \mathbf{k}_1 \frac{(\mathbf{k} - \mathbf{k}_1) \cdot \mathbf{k}_1}{(\mathbf{k}_1)^2} \hat{\delta}(\mathbf{k} - \mathbf{k}_1) \hat{\theta}(\mathbf{k}_1) \quad (40)$$

$$= - \int d^3 \mathbf{k}_1 \frac{\mathbf{k} \cdot \mathbf{k}_1}{(\mathbf{k}_1)^2} \hat{\delta}(\mathbf{k} - \mathbf{k}_1) \hat{\theta}(\mathbf{k}_1). \quad (41)$$

We will now try to make more physical sense of the above equation by introducing a new variable, namely a second wavenumber \mathbf{k}_2 , which we write in a shorter notation by abbreviating $\mathbf{k}_1 + \mathbf{k}_2$ as $\mathbf{k}_{12} = \mathbf{k}_1 + \mathbf{k}_2$. This is merely a mathematical trick to view \mathbf{k} as the sum of two wavenumbers. We can rewrite our previous expression using a three-dimensional Dirac delta $\delta_D(\mathbf{k} - \mathbf{k}_{12})$ to find,

$$\frac{\partial \hat{\delta}(\mathbf{k}, \tau)}{\partial \tau} + \hat{\theta}(\mathbf{k}, \tau) = - \int d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \hat{\delta}(\mathbf{k}_2, \tau) \hat{\theta}(\mathbf{k}_1, \tau). \quad (42)$$

And in a similar way,

$$\frac{\partial \hat{\theta}(\mathbf{k}, \tau)}{\partial \tau} + \mathcal{H} \hat{\theta}(\mathbf{k}, \tau) + \frac{3}{2} \Omega_m \mathcal{H}^2(\tau) \hat{\delta}(\mathbf{k}, \tau) = - \int d^3 \mathbf{k}_1 d^3 \mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \beta(\mathbf{k}_1, \mathbf{k}_2) \hat{\theta}(\mathbf{k}_1, \tau) \hat{\theta}(\mathbf{k}_2, \tau), \quad (43)$$

where we have defined α, β as follows,

$$\alpha(\mathbf{k}_1, \mathbf{k}_2) \equiv \frac{\mathbf{k}_{12} \cdot \mathbf{k}_1}{(\mathbf{k}_1)^2}, \quad \beta(\mathbf{k}_1, \mathbf{k}_2) = \frac{\mathbf{k}_{12}^2 (\mathbf{k}_1 \cdot \mathbf{k}_2)}{2 \mathbf{k}_1^2 \mathbf{k}_2^2}. \quad (44)$$

Note that from equations (42) and (43) we can now view the evolution of $\hat{\delta}(\mathbf{k}, \tau)$ and $\hat{\theta}(\mathbf{k}, \tau)$ as depending on the coupling of the fields at all wavevectors $\mathbf{k}_1, \mathbf{k}_2$ where $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$.

Our next step is, of course, to solve these equations for δ and θ . In general this is tricky, so as an example we will again consider the Einstein-de Sitter universe and solve our equations there.

3.2.1 Solutions in Einstein-de Sitter Universe

As the title of this chapter gave away, we are going to attempt to use perturbation theory in order to compute δ and θ . Let us consider the EdS universe again; this is a flat universe in which all energy is made up of regular matter; so $\Omega_m = 1$. We can then make the following ansatz for our expansion, [6][7][8] (as cited in [3]):

$$\hat{\delta}(\mathbf{k}, \tau) = \sum_{n=1}^{\infty} a(\tau)^n \delta^{(n)}(\mathbf{k}), \quad \hat{\theta}(\mathbf{k}, \tau) = -\mathcal{H}(\tau) \sum_{n=1}^{\infty} a(\tau)^n \theta^{(n)}(\mathbf{k}). \quad (45)$$

In appendix A we will work out this example explicitly up to second order, and there we will also motivate this assumption. Here we will briefly note the general solution before moving on.

In the previous section we mentioned that we expand around the linear solutions, which is exactly what is happening in (45). Remember however that in EdS we had two possible solutions, a decaying one and a growing one. We are mainly interested in the growing one,

since the decaying term will vanish after a certain time. So choosing this linear solution, note that from filling in the expansion in the linear version of equation (22) we directly find that $\delta_1 = \theta_1$. Therefore the linear density field $\delta_1(\mathbf{k})$ completely determines the linear fluctuations, because through perturbation theory $\delta_1(\mathbf{k})$ and $\theta_1(\mathbf{k})$ also determine all the higher order terms.

By working out the perturbation theory in EdS, and finally seeing a pattern emerge in the answers, some relations have been found for the solutions to equations (42) and (43) up to n th order [3]:

$$\delta_n(\mathbf{k}) = \int d^3\mathbf{q}_1 \dots \int d^3\mathbf{q}_n \delta_D(\mathbf{k} - \mathbf{q}_{1\dots n}) F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n), \quad (46)$$

$$\theta_n(\mathbf{k}) = \int d^3\mathbf{q}_1 \dots \int d^3\mathbf{q}_n \delta_D(\mathbf{k} - \mathbf{q}_{1\dots n}) G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) \delta_1(\mathbf{q}_1) \dots \delta_1(\mathbf{q}_n) \quad (47)$$

here F_n and G_n are homogeneous functions of the wave vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$ of degree zero. These functions are found by filling in the expansion and working out the cross terms up to the desired degree. They are constructed from the coupling functions α and β using the following relations:

$$F_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} [(2n+1)\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) + 2\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n)], \quad (48)$$

$$G_n(\mathbf{q}_1, \dots, \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, \dots, \mathbf{q}_m)}{(2n+3)(n-1)} [3\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n) + 2n\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, \dots, \mathbf{q}_n)], \quad (49)$$

where $\mathbf{k}_1 \equiv \mathbf{q}_1 + \dots + \mathbf{q}_m$, $\mathbf{k}_2 \equiv \mathbf{q}_{m+1} + \dots + \mathbf{q}_n$, $\mathbf{k} \equiv \mathbf{k}_1 + \mathbf{k}_2$, and $F_1 = G_1 = 1$. A derivation of these recursive relations can be found in [6][7], and in appendix A the F_2 function will be derived from equations (43), (42) and (45).

4 Effective Field Theory of Large Scale Structure

SPT does a decent job of approximating the large scale structure evolutions, however in reality the short and long wavelength modes are coupled, which is why SPT is not as accurate as we would like. We therefore need a new theory which can incorporate this coupling, so that we can continue to describe the evolution of our density contrast.

It turns out that the way to do this is by modeling our dark matter, as we have before, by a viscous fluid that is coupled to gravity. However now we will not neglect the short wavelength modes, instead we will use a process called “integrating out” to “renormalize” the background. What this means is that we average all of our functions over a neighborhood around the point we are looking at, and the size of the neighborhood is determined by a parameter Λ . So if we apply this in \mathbf{k} space then it means that we average our function over all the modes that lie nearby, effectively incorporating the coupling of short wavelength modes into our function. The reason we want to do this is because short wavelength fluctuations can be very large (consider the density of a human with respect to the density in the voids

of space), and in order to accurately apply perturbation theory we cannot have these kinds of “spikes”.

What this means in practice is that we use a window function to assign weights to each value of our observable quantities (e.g. the density and velocity fields) in accordance with their corresponding wavelength.

Through this averaging process we can define long wavelength variations⁴ of for example the density and the velocity field. After applying this averaging to the relevant equations we can take the expectation value over the remaining terms which do depend on shortwave fluctuations to entirely eliminate them.

This smoothing out and subsequent averaging is the main method by which we seek to explain the effects of short wavelength modes on larger scales. This method introduces some new parameters like an equation of state, speed of sound and a viscosity parameter which then characterize these long wavelength effects of the shorter modes.

4.1 Smoothing

We will again start with the full Vlasov equation given by equation (11). However if we want to incorporate effects of short scale modes and be able to perform perturbation theory within the EFToLSS we need to smooth our equations. This means that we somehow average our function over an area. So in order to achieve this we will use the Gaussian smoothing function given by [5],

$$W_\Lambda(\mathbf{x}) = \left(\frac{\Lambda}{\sqrt{2\pi}}\right)^3 e^{-\frac{1}{2}\Lambda^2 x^2}, \quad W_\Lambda(\mathbf{k}) = e^{-\frac{1}{2}\frac{k^2}{\Lambda^2}} \quad (50)$$

with Λ^2 representing the cutoff scale, or size of our window, in comoving k -space. As we can see from the exponential this window function will smooth out any quantities with a wavenumber $k \geq \Lambda$, but because of the prefactor $(\Lambda/\sqrt{2})^3$ in the \mathbf{x} dependent definition $W_\Lambda(k) \rightarrow 1$ as $k \rightarrow 0$, and our quantities remain unchanged at the largest scale. The prefactor of $(\Lambda/\sqrt{2})^3$ is a normalisation factor which we need so that the smoothing function does not change the norm (or “average value” if you wish) of whatever we wanted to smooth. We can now define our regularized physical quantities $\mathcal{O}(\mathbf{x}, t)$, ρ , π , ϕ by convoluting them in real space with the window function;

$$\mathcal{O}(\mathbf{x}, t)_l = [\mathcal{O}]_\Lambda(\mathbf{x}, t) = \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \mathcal{O}(\mathbf{x}', t), \quad (51)$$

where \mathcal{O} is a placeholder which we can replace by any observable. So, again, the smoothed version of \mathcal{O} consists of an average of \mathcal{O} over all points that that lie in the neighborhood of \mathbf{x} , or for Fourier space, wave-lengths that lie in a neighborhood of \mathbf{k} .

When applying this to an entire equation we will end up with one part that is induced by the short wavelength modes. So, in order to develop an effective field theory we will attempt to separate the small and long scale effects by smoothing over the entire Vlasov equation (meaning that we convolute the entire thing with our window function), which then becomes:

⁴What is meant by “long wavelength variation” is that these functions only fluctuate in the long wavelength regime, since we have integrated out the short scales, by essentially averaging over them.

$$\left[\frac{Df}{Dt}\right]_{\Lambda} = \frac{\partial f_l}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f_l}{\partial \mathbf{x}} - m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}}. \quad (52)$$

Now just like before we take the moments of equation (52) in order to find a more manageable and useful set of differential equations to solve. In principle there are an infinite amount of moments that in some way contribute to the whole Vlasov equation, however as an approximation we can again consider a truncation of this hierarchy after the first moments [4]. These zeroth and first moment will give us the continuity equation and the momentum equations in addition to a stochastic term:

$$\frac{\partial \rho_l}{\partial t} + 3H\rho_l + \frac{1}{a} \frac{\partial}{\partial x^i} (\rho_l v_l^i) = 0, \quad (53)$$

$$\frac{\partial v_l^i}{\partial t} + H v_{\frac{1}{a}}^i v_l^j \partial v_l^i \partial x^j + \frac{1}{a} \frac{\partial \phi_l}{\partial x^i} = -\frac{1}{a \rho_l} \frac{\partial}{\partial x^j} [\tau^{ij}]_{\Lambda}. \quad (54)$$

Here we have introduced some new quantities, again following [4]. We have defined the long wavelength velocity field as the ratio of the momentum and the density fields;

$$v_l^i = \frac{\pi_l^i}{\rho_l}. \quad (55)$$

And on the right hand side of the momentum equation (54) we have the divergence of an effective stress tensor, which arises from the short wavelength contributions. It is defined as follows,

$$[\tau^{ij}]_{\Lambda} = \kappa_l^{ij} + \Phi_l^{ij}, \quad (56)$$

where κ_l^{ij} is the so called “kinetically-induced” part, and Φ_l^{ij} the gravitationally induced part;

$$\kappa_l^{ij} = \sigma_l^{ij} - \rho_l v_l^i v_l^j, \quad (57)$$

$$\Phi_l^{ij} = -\frac{1}{8\pi G a^2} [w_l^{kk} \delta^{ij} - 2w_l^{ij} - \frac{\partial}{\partial x^k} \phi_l \frac{\partial}{\partial x_k} \phi_l \delta^{ij} + 2 \frac{\partial}{\partial x^i} \phi_l \frac{\partial}{\partial x^j} \phi_l], \quad (58)$$

with

$$w_l^{ij} = \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left[\frac{\partial}{\partial x^i} \phi_l \frac{\partial}{\partial x^j} \phi_l \right] - \sum_n \frac{\partial}{\partial x^i} \phi_n \frac{\partial}{\partial x^j} \phi_n. \quad (59)$$

In deriving this expression we have used the Poisson equation (8) to arrive at our result. Since the actual calculations are tedious and not that insightful the entire derivation has been given a place in appendix B.

4.2 Integrating out small scale physics

What we have derived above is an equation for our fields in the long wavelength regime, which we can treat in our effective theory, since long range perturbations are generally small. The right hand side however contains terms that still depend on the short wavelength fluctuations. As mentioned before, these are generally large and strongly coupled, and can

therefore not be treated within the EFToLSS.

Another problem that we have is that the actual expression in equations (56) and (57) is hard to work with, so now we want to use the symmetries of our problem to derive the form of the stress energy tensor in a more manageable way. This does not mean that our previous derivation was done for nothing; it did give us information about the composition of the stress energy tensor, namely that it depends on both the long and short wavelength perturbations, and it told us what the long wavelength part of the Euler equation looks like. What remains for us now is to somehow remove the short wavelength dependency from the equation entirely. As before, we can take the expectation value of the small wavelength fluctuations, to end up with a function that only depends on the long wavelength modes. Note that the long wavelength modes can also couple to the short wavelength ones to affect our expectation value. However the long wavelength modes vary slowly, and we can incorporate their effect on the expectation value by considering the expectation value on a long wavelength background. We would then end up with a stress energy tensor only dependent on long wavelength fluctuations, and since these are perturbatively small we can consider the schematic Taylor expansion:

$$\langle [\tau^{ij}]_{\Lambda} \rangle_{\delta_l} = \langle [\tau^{ij}]_{\Lambda} \rangle_0 + \left. \frac{\partial \langle [\tau^{ij}]_{\Lambda} \rangle_{\delta_l}}{\partial \delta_l} \right|_0 \delta_l + \dots \quad (60)$$

For our introductory purposes we can afford to stop at first order in δ_l , which means that this is also the highest order in δ_l that we are going to encounter in our stress energy tensor.

We should now identify the symmetries that we can use in order to determine our stress energy tensor, there are five in total [13]:

1. The Cosmological Principle; that is, homogeneity and isotropy at large scales
2. Galilean invariance
3. Conservation of total number of dark matter particles
4. Conservation of total momentum
5. The equivalence principle (see 6.1 for an explanation).

The cosmological principle tells us that any expectation values can only depend on the norm of the distance between points, since, apart from the distance they have from each other, their location is irrelevant. Homogeneity also tells us that numerical coefficients can only depend on time.

Galilean invariance implies that our function should be invariant under transformations of the form $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{u}t$ for some velocity \mathbf{u} . We can then consider how our observables transform, so that we can incorporate the desired symmetry in the stress energy tensor: $\rho \rightarrow \rho$, $\mathbf{v} \rightarrow \mathbf{v} - \mathbf{u}$. Time derivatives are by themselves not Galilean invariant, because we have that $\partial_t \rightarrow \partial_t + \mathbf{u} \cdot \nabla$. So whenever we have a time derivative in the stress energy tensor it should be accompanied by a $\mathbf{v} \cdot \nabla$ term.

The conservation of dark matter particles tells us that the total number of dark matter particles stays constant, i.e. $\int d^3\mathbf{x} \partial_t \rho = 0$.

Conservation of momentum tells us something similar, namely that $\int d^3x \partial_t (\mathbf{v} \rho) = 0$.

And lastly the equivalence principle implies that the mass of a particle does not affect the way it accelerates in a gravitational field. This means that all bodies in the same gravitational field accelerate in the same way.

If we put all of these symmetries together then we can restrict our stress energy tensor to the following function[13][4], (for an in-depth derivation see [13]),

$$\langle [\tau^{ij}]_{\Lambda} \rangle_{\delta_i} = p_b \delta^{ij} + \rho_b \left[c_s^2 \delta_l^{ij} - \frac{c_{bv}^2}{Ha} \delta^{ij} \partial_k v_l^k - \frac{3}{4} \frac{c_{sv}^2}{Ha} \left(\partial^j v_l^i + \partial^i v_j^j - \frac{2}{3} \delta^{ij} \partial_k v_l^k \right) \right] + \Delta \tau^{ij}. \quad (61)$$

This is the entire stress energy tensor, as allowed by symmetries. As in [4] we have p_b the background pressure induced by short wavelength fluctuations. c_s^2 is the speed of sound of fluctuations. The parameters c_{bv} and c_{sv} are the coefficients for the bulk and shear viscosity respectively. Finally $\Delta \tau^{ij}$ is a stochastic term that incorporates the difference between the actual realisation of τ^{ij} and its expectation value. The parameters c_s^2 , c_{bv} and c_{sv} are parameters that are allowed based on the symmetries, but that can not be computed from within the theory. In order to find these parameters extra information is needed, for example by making a simplification or by fitting to simulations.

When these values have been determined however, then equations (54) and (53) can be used for perturbation theory in the same fashion as we have done in the previous section. The calculations do not enter this work, but more on this can be found in [4].

5 Spherical Collapse

In the previous section we saw that the EFToLSS depends on several parameters, which can be computed using simulations [4]. It might however be possible to compute them using an approximation to the theory, namely spherical collapse (SC). While we do not consider the uses of SC in this work we will evaluate its validity in EdS by comparing with SPT, and show some elementary calculations that can be done.

In spherical collapse we assume that an initial over- or underdensity collapses in a spherically symmetric way. So in this section we will be concerned with a spherical patch of (dark) matter with a radius of R that is located somewhere in the universe. If we assume that no shell crossing takes place during the collapse (this means that it is impossible for phase space curves of particles to overlap), the equation governing the collapse can be written as [2],

$$\frac{d^2 R}{dt^2} = -\frac{GM}{R^2}. \quad (62)$$

Here R is the distance from the center of collapse, and t is the time. This equation has an exact solution given by the parametric equations,

$$R(\theta) = A(1 - \cos(\theta)), \quad t(\theta) = B(\theta - \sin(\theta)) + t_0, \quad (63)$$

where A and B are two integration constants related by $A^3 = GMB^2$ (this follows from filling our solution back into equation (62)) and t_0 is another integration constant. These equations become useful in finding out how the overdensity develops if we also consider that $\rho_b(1 + \delta) = \rho = 3M/4\pi R^3$, and so

$$\delta = \frac{3M}{4\pi R^3 \rho_b} - 1. \quad (64)$$

If we can figure out what exactly ρ_b is then we are all set to look at the density evolution. Note that in our statement that $\rho = 3M/4\pi R^3$ we have already made an assumption about the composition of the universe we are working in. This is because, as the available volume changes as R^3 , different types of energy scale differently (as an example, dark energy remains constant). By assuming that the density scales with R^3 we have therefore implicitly assumed that we are working in a universe that only consists of matter; an EdS universe.

5.1 Comparison with SPT

Before we go into the density contrast in this situation we will first show that this model for collapse agrees with SPT. To achieve this we are going to derive the solution for δ that follows from spherical collapse and compare it with SPT.

Considering early times, where overdensities in the universe were small, which correspond to small θ by equation (63), we consider the first non-zero term in the expansion of equation (63). This gives us,

$$R(\theta) \approx A\theta^2/2 \quad (65)$$

$$t(\theta) \approx t_0 + B\theta^3/6. \quad (66)$$

Since these are the solutions at the lowest order, they must originate from the background density term, which means that we can use this approximation to calculate $\bar{\rho}$:

$$\bar{\rho}(t) = \frac{3M}{4\pi R^3} = \frac{24M}{4\pi A^3\theta^6} = \frac{B^2M}{6\pi A^3(t-t_0)^2} = \frac{1}{6\pi G(t-t_0)^2}, \quad (67)$$

where in the last step we used the relation between A and B . Noting that $\rho \propto t^{-2}$ in EdS [2], we see that this is exactly the density evolution in an EdS universe, where $t = t_0$ gives an infinite background density; so t_0 corresponds to the time at which the initial singularity took place.

Now we note that the two processes (evolution of the background density and spherical collapse) can be considered separately. This does not mean that the background density does not actually influence equation (62), since we just saw it determines the zeroth order term. It does mean that spherical collapse (in some sense due to Gauss' theorem), does not care about what happens outside of the collapsing sphere. So, we can take $t_0 = 0$ for the background density, without taking this parameter to zero in our terms that govern spherical collapse. We can now use this value for the background density to find the following equation for δ :

$$\delta = \frac{9GMt^2}{2R^3} - 1. \quad (68)$$

To consider how the spherical collapse model compares to perturbation theory we will now look at the next non-zero term in the expansion of equation (63), which gives us

$$R(\theta) \approx A\frac{\theta^2}{2}\left(1 - \frac{\theta^2}{12}\right), \quad t(\theta) \approx t_0 + B\frac{\theta^3}{6}\left(1 - \frac{\theta^2}{20}\right) = B\frac{\theta^3}{6}\left(1 - \frac{\theta^2}{20} + \frac{6t_0}{B\theta^3}\right). \quad (69)$$

Next we can fill this into our expression for δ (68) to find:

$$\delta = \frac{9GM}{2}t^2R^{-3} - 1 = \frac{9GM}{2}\left(B\frac{\theta^3}{6}\left(1 - \frac{\theta^2}{20} + \frac{6t_0}{B\theta^3}\right)\right)^2\left(A\frac{\theta^2}{2}\left(1 - \frac{\theta^2}{12}\right)\right)^{-3} - 1 \approx \left(1 - \frac{\theta^2}{10} + \frac{12t_0}{B\theta^3}\right)\left(1 + \frac{\theta^2}{4}\right) - 1. \quad (70)$$

Note that in the last step we used that $A^3 = GMB^2$, and we considered θ and t_0 to be small (consistent with the SPT approximation). We can then expand this, dropping terms of order higher than four in θ to find,

$$\delta = \frac{3\theta^2}{20} + \frac{3t_0}{B\theta} + \frac{12t_0}{B\theta^3} = \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} + 3t_0\left(\frac{1}{B^2 6t}\right)^{1/3} + \frac{2t_0}{t}, \quad (71)$$

where we used $t = B\theta^3/6$ (which holds for t_0 and θ very small) in the last step.

We now have δ as a linear combination of a term that scales with $t^{2/3}$, a term that scales with t^{-1} and one that scales with $t^{-1/3}$. The first two terms correspond exactly with the growing and decaying mode in linear perturbation theory [3]. In the limit $t_0 \rightarrow 0$ the cross term also goes to zero, leaving us with a pure growing mode. The same holds for a pure decaying mode when we let $B \rightarrow \infty$. The third term that shows up is a sort of “mixed mode” of the growing and decaying mode (this is clear when we interpret t_0 and B as the amplitude of the decaying and growing mode respectively) which is predicted by second order SPT. To see that it is predicted by SPT consider as mentioned before that every higher order term scales as a power of the linear solutions. So, at first order we had $\delta^{(1)} = a + a^{-3/2}$. It follows that $\delta^{(2)}$ has a term that scales with $a \times a^{-3/2} = a^{-1/2} = t^{-1/3}$.

So it is clear that, at least for small fluctuations, SPT and SC are entirely in agreement.

5.2 Radial evolution of the cosmological structure

With the expressions (68) and (71) that we found we can now consider the evolution of the radius of our spherical mass distribution. Combining these two equations we find,

$$R = (1 + \delta)^{-1/3} \left(\frac{9}{2}t^2\right)^{1/3} \approx (1 - \delta\frac{1}{3})\left(\frac{9t^2}{2}\right)^{1/3}. \quad (72)$$

Substituting δ with equation (71) and taking the relevant limits of B and t_0 we can find an expression for the growing and decaying mode of R .

Another thing that we can derive from equation (71) is that the different terms can “overtake” each other as time passes; a solution that starts out as a decaying solution (because the t^{-1} term dominates for small t) will eventually lose to the growing solution for any finite, nonzero choice of t_0 and B (see figure 2). To determine the time at which the growing solution becomes larger than the decaying solution we can consider the ratio of δ_g and δ_d (the growing and decaying mode respectively),

$$\left|\frac{\delta_g}{\delta_d}\right| = \frac{3}{40t_0}\left(\frac{6}{B}\right)^{2/3}t^{5/3}. \quad (73)$$

This clearly illustrates that for $t \ll 1$ the decaying mode dominates, while for later times the growing mode dominates the solution. In order to find out when the growing mode surpasses the decaying mode we take the fraction of the two equal to one. Solving for t gives us,

$$t_1 = \left(\frac{40t_0}{3}\right)^{3/5}\left(\frac{B}{6}\right)^{2/5} \approx 2.31B^{2/5}t_0^{3/5}. \quad (74)$$

As was to be expected the time at which the growing solution surpasses the decaying one is highly dependent on the amplitudes of the growing and decaying modes.

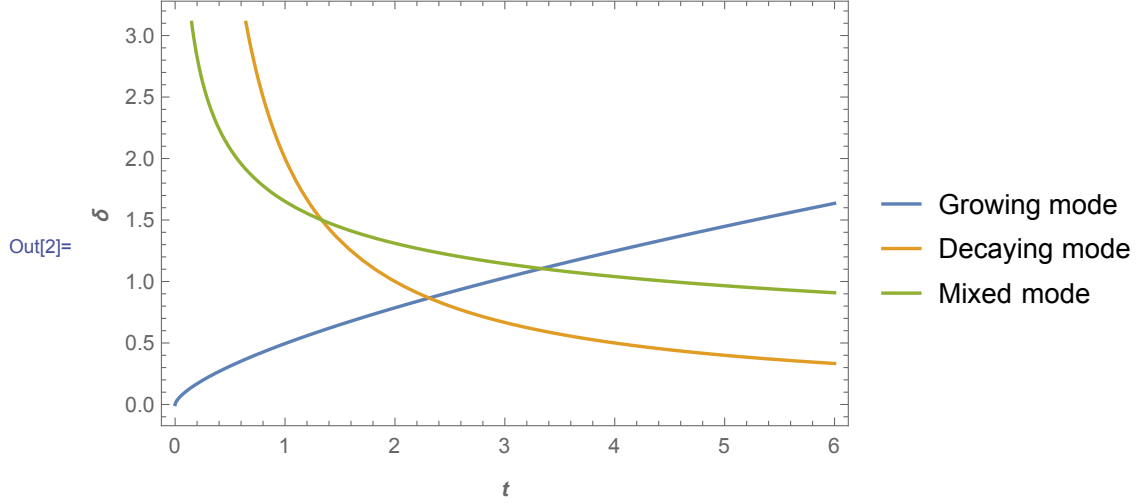


Figure 2: A plot of the three different modes for δ that arise in the spherical collapse approximation. To illustrate how the different modes overtake each other as time passes we took $B = t_0 = 1$. For this choice of parameters it seems that the higher order term becomes important for a short time, before the growing mode takes over.

One question that still remains open is whether the linear solutions will still be valid when the decaying mode ends and the growing mode begins. It might be possible that higher order terms have actually become more important than the linear term by this time. Note that figure 2 shows us that for some choice of parameters the decaying mode will actually go into the mixed mode; however in SPT this is just a higher order correction to the decaying mode, which will still become negligible as time passes.

Therefore we will consider the first higher order term for the growing solution, namely $-\theta^4/40$. Using that $t = B\theta^3/6$ we find,

$$\left| \frac{\delta_g^{(1)}}{\delta_d} \right| = \frac{1}{80t_0} \left(\frac{6}{B} \right)^{4/3} t^{7/3}. \quad (75)$$

And so,

$$t_2 = (80t_0)^{3/7} \left(\frac{B}{6} \right)^{4/7} \approx 2.35 B^{4/7} t_0^{3/7}, \quad \frac{t_1}{t_2} \approx 0.98 \left(\frac{t_0}{B} \right)^{6/35} \quad (76)$$

So interestingly enough it depends on the fraction of t_0 and B whether the system goes back to linear growth or to some higher order nonlinear one.

6 Relativity

As mentioned in the introduction the only force relevant for cosmology, on the scales that we are considering, is gravity. So far we have only considered the Newtonian view of gravity, which has suited us well since we have not encountered any deep potential minima (or strong spatial curvature, as it would be called in general relativity). The reason why in the weak curvature limit these two interpretations are the same is not trivial however, and is to some extent explained by the equivalence principle.

6.1 Equivalence principle

If we consider Newtonian theory there are two equations especially important. These are Newton's second law and Newton's law of gravitation;

$$\mathbf{F} = m_i \mathbf{a}, \quad (77)$$

$$\mathbf{F}_g = -m_g \vec{\nabla} \phi. \quad (78)$$

In principle we could have two different kinds of mass enter our equations, one “inertial” mass in the Newtonian equation of motion that governs how a body or particle resists against acceleration, and a “gravitational” mass that behaves as the gravitational charge of a particle. It is a common known fact that these two masses are the same, and experiments proving this go back as far as Galileo. In Newtonian theory this equality (called the equivalence principle) is no more than a coincidence; the equation for acceleration towards a mass M would still be consistent if $m_g \neq m_i$ and we had

$$a = -\frac{GM_g}{r^2} \frac{m_g}{m_i}. \quad (79)$$

It was Einstein who realised that this equality between the two types of mass has a much deeper cause. He called this insight “the happiest thought of my life” [9]:

*The gravitational field has only a relative existence...
Because for an observer freely falling from the roof of a house
- at least in his immediate surroundings -
there exists no gravitational field.*

This means that we can locally interchange a gravitational field for an accelerating frame of reference. To see why this is the case let us look at some thought experiments that Einstein himself devised, and have since become known as the “elevator thought experiments”. Consider yourself being locked up in a box, with two stones as your only companions. While this usually would be cause for panic, you instead stay calm and ponder the workings of nature. We will consider several situations [10].

In the first situation the box is located in outer space. The box is not being accelerated, and there are no forces acting on it; therefore you can see yourself and the two stones float (see figure 3).

Next, consider the situation in which the box is in fact being accelerated upwards with an acceleration $a \approx 9.81 \text{ m/s}^2$. You feel the upwards motion of the box pushing against you and observe the stones falling to the floor (see figure 4).

Now we place the box in a uniform gravitational field without securing it. The box will fall down at the same speed as you and the stones, and it will appear like you are floating as before (figure 5).

We can then consider holding the box up with a rope. This means that you and the stones fall down towards the floor, and you feel the force from the box pushing up against you. (figure 6).

It follows then that for the uniform gravitational field we discussed here (or, locally in a general gravitational field) the effects of gravity on an observer are indistinguishable from a

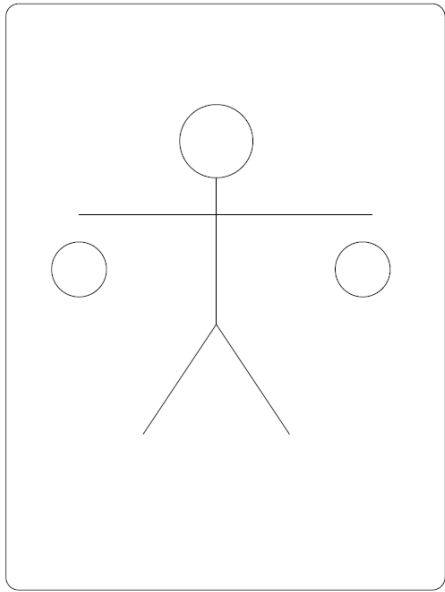


Figure 3: When no forces are present you, the box and the two stones will float. Reproduced from [10] with permission from the author.

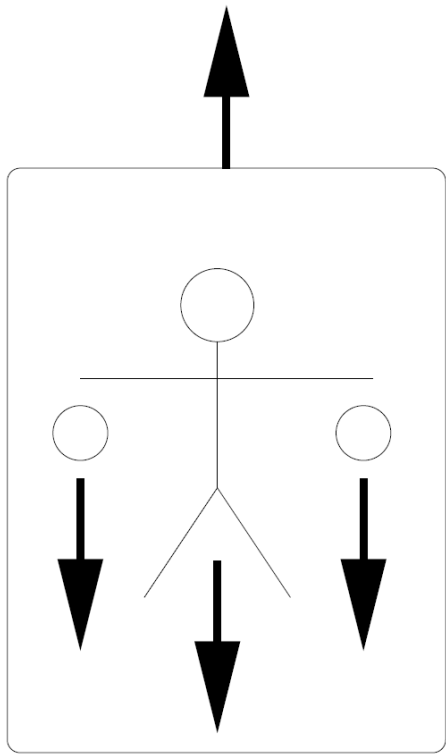


Figure 4: When the box is accelerated upwards, for you this looks just like you and the stones are pulled down by a gravitational field. Reproduced from [10] with permission from the author.

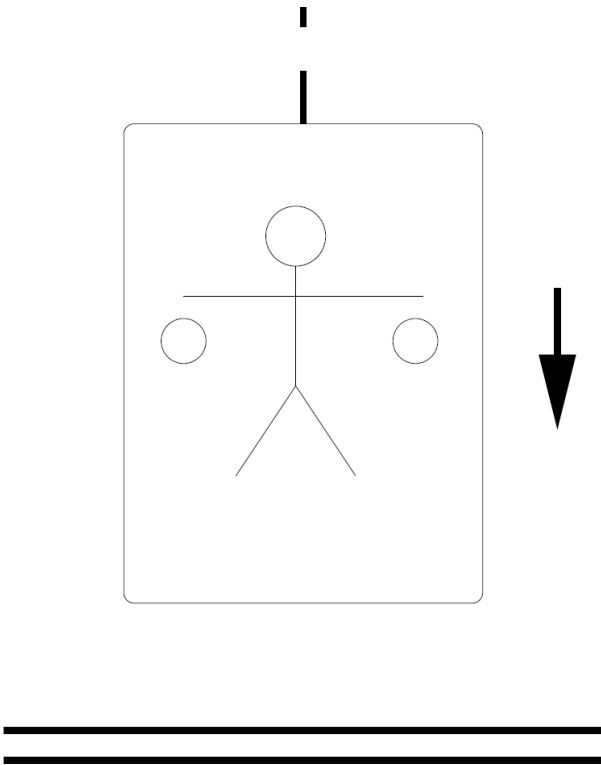


Figure 5: When the box and its contents are pulled down by a constant, uniform gravitational field this free fall is indistinguishable from floating. Reproduced from [10] with permission from the author.

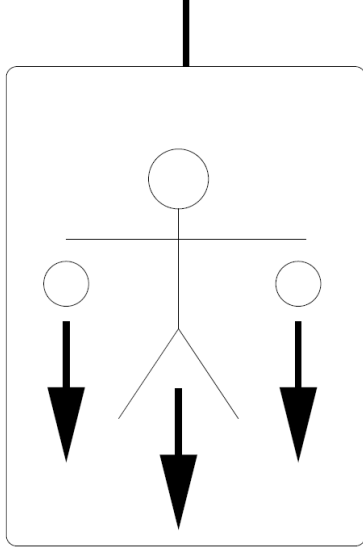


Figure 6: When the box is secured in a uniform gravitational field its contents are pulled down, as if the box was moving upwards in space. Reproduced from [10] with permission from the author.

uniform acceleration.

Now what would happen if we were to try sending a light beam from one part of the box to the other, as our box was falling down in a uniform gravitational field? From our previous argumentation we know that this situation is the same as the one described in figure 4. This means that when the light beam reaches the other side of the box, the box will have moved up, causing you to observe a bend in the path taken by the light (see figure 7). From the equivalence principle we can then conclude that gravitational fields, or masses, bend light. This in itself is already very noteworthy, but let us also consider the fact that light follows the shortest path between two points. This means that, in a gravitational field, the shortest path between two points is not a straight line. In Euclidean, or flat, space the shortest path between two points is always a straight line. This means something extraordinary, namely that space is not Euclidean.

A third way to view our problem with the box in the gravitational field is then to say that you, the stones and the light follow “geodesics” (shortest paths) in curved space-time.

6.1.1 Non-uniform gravitational fields

The previous thought experiment made use of uniform gravitational fields, however, a general mass would have a spherically symmetric gravitational field. How does the equivalence of a gravitational field and a uniform acceleration hold up for a spherically symmetric gravitational field? The answer is, it does not.

To see this let us mentally place our box, secured with a rope, into this gravitational field (for example the earth’s gravitational field). The box does not move, so you feel the pull

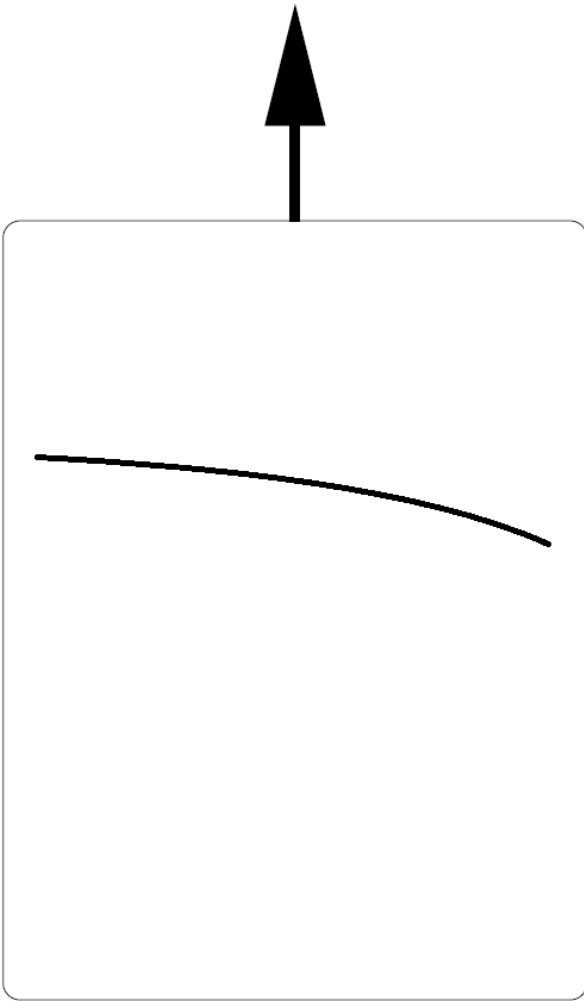


Figure 7: As the light travels from one side to the other the box moves upward, causing you to see the light curving off downwards.

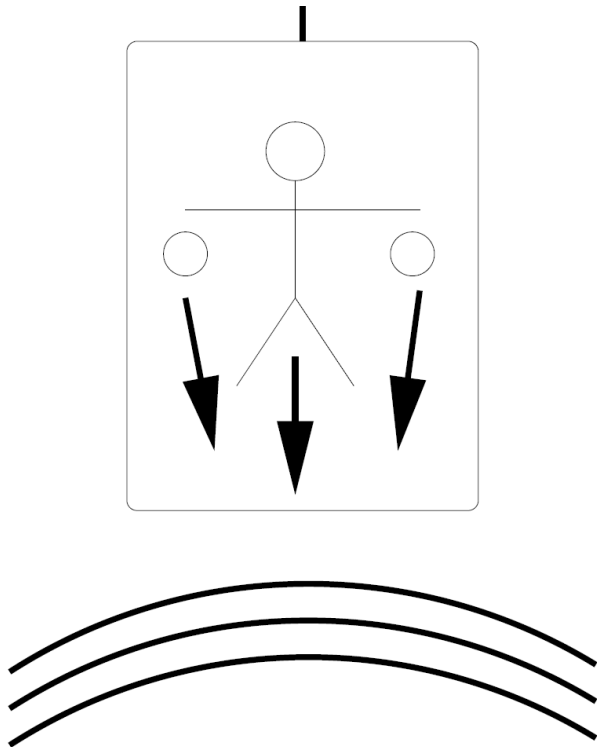


Figure 8: When the box is secured in a non-uniform gravitational field the stones approach each other as they fall towards the bottom of the elevator. Reproduced from [10] with permission from the author.

from gravity, and observe the stones moving towards each other as they fall towards the center of the earth (figure 8).

If we now cut the rope securing our box we will again be in a free fall. However now the stones will approach each other, and you must conclude that there is some external force at work here; this cannot be due to a uniform acceleration (figure 9).

From this we can indeed conclude that we cannot get rid of the effects of gravity in a non-uniform field by going to a free falling coordinate system. This can only be done in a uniform gravitational field, or locally, where the gravitational field is approximately constant.

6.2 The metric tensor

So now we have a good reason to assume that mass curves space-time, and it is clear that this space-time does not act in the same way Euclid thought it did. Since gravity seems to be the same as a curved space-time we should consider what we can learn about the effects of gravity by transforming our original system from inertial Cartesian coordinates to a different coordinate system, which is governed by a general metric.

The space that lies at the basis of (special) relativity is so called “Minkowski” space-time. It is the union of a Euclidean space with the time component; the space of events labelled by 3 Cartesian spatial coordinates x^k and a time-coordinate t . We will denote a point in Minkowski space-time with the coordinates

$$(x^\mu) = (x^0 = ct, x^k), \quad (80)$$

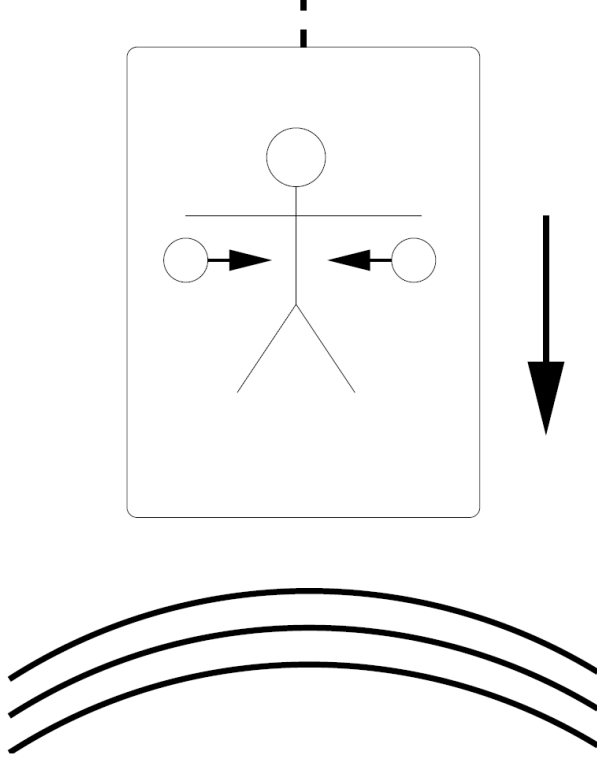


Figure 9: When the box is free falling in a non-uniform gravitational field the stones approach each other. Reproduced from [10] with permission from the author.

where c is the speed of light, and the superscript μ denotes that we are considering a vector with four components. Throughout this work Greek indices will be used for vectors with four components, of which the zeroth component is the time component, and Latin indices will indicate regular three dimensional vectors.

This union of space and time comes equipped with a way to measure a distance. An infinitesimal distance in Minkowski space is given by,

$$ds^2 = -(dx^0)^2 + \sum_k (dx^k)^2. \quad (81)$$

Contrast this to the distance in Euclidean space, $ds^2 = dx^2 + dy^2 + dz^2$ and we see that all we have added is a time component. If we introduce the following matrix,

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (82)$$

we can use Einstein's summation index to write our distance in a more compact way,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu. \quad (83)$$

The difference between upper and lower indices is an important part of the notation in general relativity, however for our purposes it is enough to note that we only contract (i.e., sum

over) an index in Einstein notation when we combine a lower index with an upper one. We can move from a lower to a higher index and vice versa by contracting with the standard metric tensor $\eta_{\mu\nu}$; i.e. for a general tensor $g_{\mu\nu}$ we have that $g^{\mu\nu} = \eta^{\lambda\mu} \eta^{\rho\nu} g_{\lambda\rho}$. Note that we need to use one $\eta_{\mu\nu}$ per index that we want to lower/raise.

This matrix $\eta_{\mu\nu}$ is what is called the Minkowski metric; it encodes all the basic properties of the space by showing us how to measure the distance. If we were to have a more complicated space the matrix elements of the metric will change accordingly. Consider now a general coordinate transformation from our Minkowski space to a new one, where we denote the coordinates by y^μ . Before we consider the implications of this transformation we first introduce a new concept, called the proper time. Consider a massive free particle. If we view everything from its frame of reference, the particle stays in one point, so $dx^i = 0$. However in this frame we can still measure the time, called the proper time τ , for which holds that,

$$ds^2 = \eta_{ij} dx^i dx^j - d\tau^2 = -d\tau^2 \implies d\tau^2 = -ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu. \quad (84)$$

Note that ds is still in our original coordinates. τ is the so called proper time; the time as measured by a clock following the path of the particle. The proper time is analogous to the arclength in Euclidean space, and can be used to parametrize paths in Minkowski space. Returning now to our general parametrisation, we can view the path of our particle in the new coordinates y^μ .

$$d\tau^2 = -ds^2 = -\eta_{\rho\lambda} dx^\rho dx^\lambda = -\eta_{\rho\lambda} \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\lambda}{\partial y^\nu} dy^\mu dy^\nu \quad (85)$$

So we can see that in our new coordinates the proper time and distance are not measured by the standard Minkowski metric $\eta_{\mu\nu}$, but by

$$d\tau^2 = -g_{\mu\nu} dy^\mu dy^\nu, \quad (86)$$

where

$$g_{\mu\nu} = \eta_{\rho\lambda} \frac{\partial x^\rho}{\partial y^\mu} \frac{\partial x^\lambda}{\partial y^\nu} \quad (87)$$

is the “metric tensor”, which, like the standard metric before, encodes all the information we have about our space in the new coordinates and gives us a way to measure distances via the line element $ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$. Note that the general metric is dependent on x , which is not surprising; if we make a transformation from Cartesian coordinates to spherical coordinates our new coordinates also depend on the original.

As an example of how the metric comes back in cosmology we will quickly mention an important example, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. These four scientists individually started from the assumed homogeneity and isotropy of the universe and derived a space-time metric compatible with the Cosmological Principle (for the derivation please see [10]),

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \quad (88)$$

Here we have written our metric in spherical coordinates, where k denotes the curvature of our space (we can have $k = 1$, $k = 0$ or $k = -1$ for a positively curved, flat and negatively curved space respectively), $a(t)$ is the scale factor and $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the volume

element on the 3-dimensional sphere.

Let us quickly recapitulate what has happened in this paragraph. We started off by considering how to measure distances in Euclidean and Minkowski space. By rewriting the equation (81) using the $\eta_{\mu\nu}$ matrix we saw how this standard metric tensor encoded all the information about measuring distances in this space. Finally we performed a coordinate transformation to end up with the general metric tensor $g_{\mu\nu}$.

In the previous paragraph we motivated why in general relativity gravity is no longer viewed as a force, but as the curvature of space-time which causes mass to move. It is this new paradigm which makes it useful to consider the metric tensor in connection to gravity. Because if gravity is the curvature of space-time, then the effects of gravity should be encoded in our metric tensor, since this tells us how exactly our space looks. This might then motivate us to think that the Newtonian potential in equation (78) will be replaced by the metric tensor when viewing gravity from a general relativistic viewpoint.

6.3 The geodesic equation

To make our previous statement about the role of the metric tensor more substantial we will consider the equations of motion of a massive free particle; also known as the geodesic equation. In Newtonian theory we have $\ddot{\mathbf{y}} = 0$. In our relativistic view of time this becomes

$$\frac{d^2}{d\tau^2} y^\rho(\tau) = 0, \quad (89)$$

where τ is the proper time as before. Equation (89) is from the point of view of our particle, however we would like to consider the motion from a general point of view. So we perform a coordinate transformation like before. Using the standard rules for a change of variables;

$$\frac{d}{d\tau} y^\rho = \frac{\partial y^\rho}{\partial x^\mu} \frac{dx^\mu}{d\tau}. \quad (90)$$

Taking the second derivative we find,

$$\begin{aligned} \frac{d^2}{d\tau^2} y^\rho &= \frac{\partial y^\rho}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 y^\rho}{\partial x^\gamma \partial x^\nu} \frac{dx^\gamma}{d\tau} \frac{dx^\nu}{d\tau} \\ &= \frac{\partial y^\rho}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} + \delta_\lambda^\rho \frac{\partial^2 y^\lambda}{\partial x^\gamma \partial x^\nu} \frac{dx^\gamma}{d\tau} \frac{dx^\nu}{d\tau} \\ &= \frac{\partial y^\rho}{\partial x^\mu} \left[\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial y^\lambda} \frac{\partial^2 y^\lambda}{\partial x^\gamma \partial x^\nu} \frac{dx^\gamma}{d\tau} \frac{dx^\nu}{d\tau} \right]. \end{aligned} \quad (91)$$

Because the term outside of the brackets is invertible our equation of motion (or, if you will, equation for a straight line in Minkowski space) can be written in terms of x^μ as,

$$\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial y^\lambda} \frac{\partial^2 y^\lambda}{\partial x^\gamma \partial x^\nu} \frac{dx^\gamma}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (92)$$

This we can rewrite as follows,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\gamma\nu}^\mu \frac{dx^\gamma}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (93)$$

with

$$\Gamma_{\nu\lambda}^{\mu} = \frac{\partial x^{\mu}}{\partial y^{\gamma}} \frac{\partial^2 y^{\gamma}}{\partial x^{\nu} \partial x^{\lambda}} \quad (94)$$

a pseudo-force, or fictitious gravitational force (like the centrifugal force, which also arises after a coordinate transformation). This is a tensor with three indices, the components of which are called “Christoffel symbols”. Note that equation (93) is simply a coordinate transformation, like a transformation into polar coordinates, however now it is a general one written down using a summation convention. What makes it useful to define our fictitious gravitational force is that we can express it entirely in terms of the metric (this is left as an exercise for the reader, a derivation can be found in [10]),

$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\rho}} \right). \quad (95)$$

Which means that the metric actually appears in our equations of motion through the fictitious force term. The components of the metric play the role of potentials, and each of the components has an effect on the force term. This means that a Newtonian theory, where gravitational potential is given by a scalar, is entirely insufficient.

6.4 The Newtonian limit

In the previous section we discussed how the geometry of our space should replace the gravitational potential when doing correct calculations with general relativity. However in the previous chapters we have often ignored relativistic effects, without mentioning our reasons for doing so. It turns out that in certain limits we can still use our Newtonian approximation.

There are several things that changed when we started to look at mechanics from a general relativistic viewpoint. We introduced a maximal velocity c through special relativity and we concluded that gravity is a consequence of the curvature of space-time. It is evident then that only in the case where these new aspects are not relevant, or not present, we can approximate reality using Newton’s theory. Our first two requirements are then;

1. Slow motion: our Newtonian approximation can only work when particles move at such speeds that we can ignore relativistic effects, so slow means small in comparison to the speed of light.
2. Weak fields: we need curvature to be negligible, from which it follows that gravitational effects will be small. What exactly we mean by small is not clear at this point, however we will make this exact.

It turns out that we need a third, less obvious condition; namely that our fields vary slowly in time. Apparently there are some relativistic effects that show up even when weak fields vary rapidly enough (for example oscillating fields). One very popular example of this effect are gravitational waves.

3. Stationary fields: the gravitational field varies slowly on the time scales experienced by the test particle we want to describe.

We would now like to use our approximation to find the Newtonian equations of motion that we started this chapter off with and which we used in chapter 5,

$$\frac{d^2}{dt^2}\mathbf{x} = -\vec{\nabla}\phi \quad (96)$$

with

$$\phi = -\frac{G_N M}{r}. \quad (97)$$

It is of course our hope that our new theory is consistent with the proven Newtonian limit, which would mean that equation (93) will reduce to equation (96) in this limit. Before we can attempt to prove this however we need to make our approximations more mathematically rigorous.

1. This one is fairly clear; we need that

$$\frac{dx^i}{dt} \ll c \quad (98)$$

or, in proper time,

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}. \quad (99)$$

2. We now need to define what we mean by a weak field. Our previous arguments showed that the gravitational force term, (95), is entirely dependent on the metric. So a condition for the strength of our field should also be a condition on our metric, and therefore our choice of coordinates. Thus we assume that the metric in the coordinates x^μ that we chose only differs slightly from the standard Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (100)$$

where a small deviation in this case means that we can drop all terms that are quadratic in $h_{\mu\nu}$ or its derivatives.

3. The condition that our fields do not vary too rapidly with time is incorporated simply by assuming that our fields are entirely time independent;

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0 \implies \frac{\partial h_{\mu\nu}}{\partial x^0} = 0. \quad (101)$$

We start off by considering the geodesic equation,

$$\ddot{x}^\mu + \Gamma_{\nu\lambda}^\mu \dot{x}^\nu \dot{x}^\lambda = 0, \quad (102)$$

where we took dots to mean derivatives with respect to the proper time τ . Then we apply the weak field approximation; from the decomposition $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we can conclude that $\Gamma_{\nu\lambda}^\mu$ is at least linear in $h_{\mu\nu}$, and our approximation tells us that we only need to keep the linear terms, so we can drop the rest. From our slow motion condition we can derive that $\dot{x}^\mu = c\dot{t} \gg \dot{x}^i$, since the speed of light is much larger than the speed of our particle itself.

Thus we can remove all but the leading term $c^2 \dot{t}^2$ in the product $\dot{x}^\nu \dot{x}^\lambda$. Applying this we find that the geodesic equation can be written as,

$$\ddot{x}^\mu + \Gamma_{00}^\mu \dot{t}^2 = 0. \quad (103)$$

Applying our third condition, stationarity, to the definition (94) we find that

$$\Gamma_{00}^\mu = -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^\nu} = -\frac{1}{2} g^{\mu i} \frac{\partial g_{00}}{\partial x^i}, \quad (104)$$

remember that an index i is only used for the three spatial coordinates. Since we only have a definition for the elements of $g_{\mu\nu}$ with a lower index, we will now consider what they look like with raised indices, again using our weak field approximation:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (105)$$

with

$$h^{\mu\nu} = \eta^{\mu\lambda} \eta^{\rho\lambda} h_{\lambda\rho}. \quad (106)$$

And plugging this in we find that

$$\Gamma_{00}^\mu = -\frac{1}{2} \eta^{\mu i} \frac{\partial h_{00}}{\partial x^i}. \quad (107)$$

Since the time derivatives of our field are zero, we are left with mostly i dependencies in our previous equations. This motivates us to split the geodesic equation in a spatial part (so with an i index) and a temporal part (with the index 0). We should therefore consider the relevant Christoffel symbols;

$$\Gamma_{00}^0 = 0, \quad \Gamma_{00}^i = -\frac{1}{2} \frac{\partial h_{00}}{\partial x^i}. \quad (108)$$

So our geodesic equations splits into two equations (remembering that $x^0 = ct$):

$$\ddot{t} = 0 \quad (109)$$

$$\ddot{x}^i = \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \dot{t}^2. \quad (110)$$

The first equation simply tells us that there is no difference between coordinate time and proper time in the Newtonian approximation (apart from a scale/chosen units);

$$t(\tau) = a\tau + b. \quad (111)$$

We are now left with a differential equation for x^i in terms of the proper time. Using the relation we found between proper time and regular time we can rewrite the τ derivatives in our geodesic equation,

$$\dot{t} = 0 \implies \frac{1}{\dot{t}^2} \frac{d^2}{d\tau^2} = \frac{1}{\dot{t}} \frac{d}{d\tau} \frac{1}{\dot{t}} \frac{d}{d\tau} = \frac{1}{c} \frac{d}{dt} \frac{1}{c} \frac{d}{dt} = \frac{1}{c^2} \frac{d^2}{dt^2}. \quad (112)$$

So, applying this to equation (110), we find that

$$\frac{d^2 x^i}{dt^2} = \frac{c^2}{2} \frac{\partial h_{00}}{\partial x^i}. \quad (113)$$

So in the Newtonian approximation the relativistic geodesic equation reduces to the equations (111) and (113). We can compare this second equation to the Newtonian equation,

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial \phi}{\partial x^i}. \quad (114)$$

It is easy to see then that if we pick $h_{00} = -2\phi$ we find exactly the Newtonian equation. Rewriting this in terms of the metric we get,

$$g_{00} = \eta_{00} + h_{00} = -(1 + 2\frac{\phi}{c^2}). \quad (115)$$

So this is the relation between the (00) component of the metric tensor and Newtonian gravity. It directly follows that Newtonian gravity can be viewed in a general relativistic way through a space-time metric of the form,

$$ds^2 = -(1 + 2\frac{\phi}{c^2}(\mathbf{x}))dt^2 + d\mathbf{x}^2. \quad (116)$$

7 Conclusion

The original goal of this thesis was to compute the parameters in the EFToLSS through use of the spherical collapse model. Having no previous experience with the subject it was necessary to first get familiarized with the basic theory behind Large Scale Structure. This proved more difficult than anticipated, and it soon became clear that getting an actual estimate was not going to happen. So the goal changed. Instead of determining the parameters, we wanted to write down and explain the theory needed in such a way that a likely successor could use it to go through the theory faster, and find the desired results. Therefore this work assumed only basic knowledge of cosmology and some hydrodynamics. Starting from the very beginning, the dynamics of large scale structures, we first introduced perturbation theory in cosmology by considering SPT, after which we moved on to the more rigorous EFToLSS. Since knowledge of SC is of course needed to use SC, after covering EFToLSS the premise of SC is explained and some of the implications from the assumption are shown. After following this fairly straight path we suddenly take a sharp turn and end up with the final chapter, relativity. It seems strange that in what is mostly a Newtonian theory there would be an entire chapter dedicated to the implications of general relativity. It is important however to know why this theory is mostly Newtonian, since cosmology relies heavily on general relativity through, for example, the Friedmann equation. We decided that this warranted giving it a place in this review. There were also topics we did not reach due to lack of time. In an expansion of this review more attention could be given to correlation functions and power spectra (mentioned in [3]), two very important concepts which have not once been mentioned. While complicated, it would also be good to consider the renormalization of the EFToLSS, which was mentioned in several sources [12][5][4].

So, essentially this work sketches the basics needed to understand Large Scale Structure. Once you know the basics, you should consider where to go from there. As mentioned before it is essential to be familiar with these topics to be able to compute the speed of sound, equation of state and viscosity parameter present in EFToLSS. The EFToLSS itself has great

merit in being a more accurate theory than SPT for computing the density fluctuations and divergence of the velocity field, but it also has important applications [5]. It might for example be used to solve “preheating”, this is the process where the energy of the homogeneous inflaton field loses its energy to excite other fields. As a nonlinear theory it could also give a better description of anisotropies in the CMB. To summarize, the most important part of this work was to give a clear and understandable overview of the basics of large scale structure theory, a goal which I believe it has achieved.

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A Perturbative second order solutions in an EdS universe

The goal now is to determine the time dependence of the density contrast $\hat{\delta}$ up to second order. If we also want to compute the \mathbf{k} dependence, we would need more information on the initial density field at $\tau = 0$. This can for example be gleaned from the CMB, but we will not go into this further.

In order to compute δ we are going to use perturbation theory, but first we are going to rewrite equations (42) and (43) into a single equation that only depends on δ , as we did before. For simplicity we first define the following terms,

$$S_\alpha \equiv - \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \hat{\delta}(\mathbf{k}_2, \tau) \hat{\theta}(\mathbf{k}_1, \tau) \quad (117)$$

$$S_\beta \equiv - \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \beta(\mathbf{k}_1, \mathbf{k}_2) \hat{\theta}(\mathbf{k}_1, \tau) \hat{\theta}(\mathbf{k}_2, \tau). \quad (118)$$

With this new notation the equations (42) and (43) become,

$$\frac{\partial \hat{\delta}(\mathbf{k}, \tau)}{\partial \tau} + \hat{\theta}(\mathbf{k}, \tau) = S_\alpha \quad (119)$$

$$\frac{\partial \hat{\theta}(\mathbf{k}, \tau)}{\partial \tau} + \mathcal{H} \hat{\theta}(\mathbf{k}, \tau) + \frac{3}{2} \Omega_m \mathcal{H}^2(\tau) \hat{\delta}(\mathbf{k}, \tau) = S_\beta. \quad (120)$$

Removing the first equation times \mathcal{H} and its derivative from the second equation we find,

$$-\frac{\partial^2 \hat{\delta}}{\partial \tau^2} - \mathcal{H} \frac{\partial \hat{\delta}}{\partial \tau} + \frac{3}{2} \Omega_m \mathcal{H}^2 \hat{\delta} = S_\beta - \left(\frac{\partial}{\partial \tau} + \mathcal{H} \right) S_\alpha \quad (121)$$

as our nonlinear differential equation for $\hat{\delta}$. Note that if we decide to go back to the linear limit S_α and S_β vanish, and we retrieve equation (23). Now we move this discussion into an EdS universe. Remember that this universe was made up entirely of ordinary matter ($\Omega_m = 1$), and that in this universe $a \propto t^{2/3}$ [11]. From this expression for a follows that $\partial \mathcal{H} / \partial a = -\mathcal{H} / 2a$. Next we use the relation between a , τ and t to rewrite our differential equation from a differential equation in τ to a differential equation in a . We have;

$$\frac{d}{d\tau} = \frac{dt}{d\tau} \frac{d}{dt} = \frac{dt}{d\tau} \frac{da}{dt} \frac{d}{da} = \mathcal{H} a \frac{d}{da}. \quad (122)$$

So putting all of this together we get,

$$\begin{aligned} -\mathcal{H}a \frac{\partial}{\partial a} \left(\mathcal{H}a \frac{\partial \hat{\delta}}{\partial a} \right) - \mathcal{H}^2 a \frac{\partial \hat{\delta}}{\partial a} + \frac{3}{2} \Omega_m \mathcal{H}^2 \hat{\delta} = \\ -\mathcal{H}^2 a^2 \frac{\partial^2 \hat{\delta}}{\partial a^2} - \frac{3\mathcal{H}^2 a}{2} \frac{\partial \hat{\delta}}{\partial a} + \frac{3}{2} \Omega_m \mathcal{H}^2 \hat{\delta} = S_\beta - \mathcal{H} \frac{\partial}{\partial a} (a S_\alpha). \end{aligned} \quad (123)$$

As it stands equation (123) looks very difficult to solve, however we can use the “Green’s function” method to derive our perturbative solutions.

What the Green’s function method means is that we consider the operator L on the left hand side of equation (23) such that $L\hat{\delta} = S_\beta - \mathcal{H}\partial(aS_\alpha)/\partial a$. So in this case we find

$$L = -\mathcal{H}^2 a^2 \frac{\partial^2}{\partial a^2} - \frac{3\mathcal{H}^2 a}{2} \frac{\partial}{\partial a} + \frac{3}{2} \Omega_m \mathcal{H}^2. \quad (124)$$

If we then determine a function $G(a, a')$, called the Green’s function, such that $LG(a, a') = \delta_D(a - a')$, with δ_D the Dirac delta function, then the solution to equation (123) will be given by,

$$\hat{\delta}(a, \mathbf{k}) = \int da' G(a, a') \left(S_\beta(a') - \mathcal{H}(a') \frac{\partial}{\partial a'} (a' S_\alpha(a')) \right), \quad (125)$$

which is easily verified by applying L on both sides and working out the integral using the properties of the Dirac delta function.

The next logical step is determining our Green’s function. To do this we note that (123) can be written in the form $\partial_a^2 \delta + p(a) \partial_a \delta + q(a) = f(a)$. It turns out that for this type of equation the Green’s function is entirely determined by the homogeneous solutions[14],

$$G(a, a') = \theta(a - a') \frac{y_1(a') y_2(a) - y_1(a) y_2(a')}{y_1(a) \partial'_a (y_2(a')) (a) - \partial'_a (y_1(a)) y'_2(a)}, \quad (126)$$

where y_1 and y_2 denote the solutions to the homogeneous equation and $\theta(a - a')$ is the Heaviside step function defined to be 1 for $a > a'$ and 0 for $a < a'$. Remember that we have already determined these solutions in chapter 3.1. So using equation (126) we find, by working out the derivatives and products, that,

$$G(a, a') = \theta(a - a') \frac{2}{5\mathcal{H}^2 a} \left[\left(\frac{a'}{a} \right)^{3/2} - \frac{a}{a'} \right]. \quad (127)$$

And so after some work we find the following equation for δ :

$$\hat{\delta}(a, \mathbf{k}) = \frac{2}{5\mathcal{H}^2 a} \int da' \theta(a - a') \left[\left(\frac{a'}{a} \right)^{3/2} - \frac{a}{a'} \right] \left(S_\beta(a', \mathbf{k}) - \mathcal{H}(a') \frac{\partial}{\partial a'} (a' S_\alpha(a', \mathbf{k})) \right). \quad (128)$$

We now want to want to expand δ in some way so that we can apply perturbation theory to it. In chapter 3.1 we showed that up to linear order $\hat{\delta} = a \hat{\delta}_i(\mathbf{k})$, with $\hat{\delta}_i(\mathbf{k})$ the initial density field. Because of (21) we then have that, up to linear order, $\hat{\theta} = -\mathcal{H} a \delta_i(\mathbf{k})$. This means that the first order corrections. denoted $\hat{\delta}^{(1)}$ and $\hat{\theta}^{(1)}$ scale linearly with the initial fields (with a time dependent factor).

In order to use perturbation we will now use this as motivation to assume that it is possible to expand the density and velocity fields about the initial density fields; in other words we assume that $\hat{\delta}^{(n)} \propto \hat{\delta}_i^n$. In this way we effectively treat the fluctuations in density and velocity fields as a small parameter. This is only true for \mathbf{k} very small; it is only on the largest scales that the universe is approximately homogenous, and therefore the linear density fluctuations small enough to do perturbation theory with. So if we then expand $\hat{\delta}$ and $\hat{\theta}$ around their linear solutions we have,

$$\hat{\delta}(\mathbf{k}, a) = \sum_{n=1}^{\infty} \hat{\delta}^{(n)}(\mathbf{k}, a) \quad (129)$$

$$\hat{\theta}(\mathbf{k}, a) = \sum_{n=1}^{\infty} \hat{\theta}^{(n)}(\mathbf{k}, a). \quad (130)$$

For our purposes we are only interested in the second order solutions, so we ignore all contributions of third order and higher. The next step is then to compute S_α and S_β up to second order. Filling our expansion into equations (117) and (118) and neglecting terms of third order and higher we find,

$$S_\alpha \equiv - \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \hat{\delta}(\mathbf{k}_2, \tau)^{(1)} \hat{\theta}(\mathbf{k}_1, \tau)^{(1)} \quad (131)$$

$$= a^2 \mathcal{H} \tilde{\alpha} \quad (132)$$

$$S_\beta \equiv - \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \beta(\mathbf{k}_1, \mathbf{k}_2) \hat{\theta}(\mathbf{k}_1, \tau)^{(1)} \hat{\theta}(\mathbf{k}_2, \tau)^{(1)} \quad (133)$$

$$= -a^2 \mathcal{H}^2 \tilde{\beta}. \quad (134)$$

Where, since we were only interested in the time dependence of $\hat{\delta}$, we defined,

$$\tilde{\alpha}(\mathbf{k}) = \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \alpha(\mathbf{k}_1, \mathbf{k}_2) \hat{\delta}_i(\mathbf{k}_1) \hat{\delta}_i(\mathbf{k}_2) \quad (135)$$

$$\tilde{\beta}(\mathbf{k}) = \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 \delta_D(\mathbf{k} - \mathbf{k}_{12}) \beta(\mathbf{k}_1, \mathbf{k}_2) \hat{\delta}_i(\mathbf{k}_1) \hat{\delta}_i(\mathbf{k}_2). \quad (136)$$

In order to compute $\tilde{\alpha}(\mathbf{k})$ and $\tilde{\beta}(\mathbf{k})$ more information about the initial conditions of the universe is needed.

Now all that is left is to fill in the expressions for S_α and S_β in equation (128) to find,

$$\begin{aligned} \hat{\delta}^{(2)}(a, \mathbf{k}) &= \frac{2}{5\mathcal{H}^2 a} \int da' \theta(a - a') \left[\left(\frac{a'}{a} \right)^{3/2} - \frac{a}{a'} \right] \left(-(a')^2 \mathcal{H}(a')^2 \tilde{\beta} \right. \\ &\quad \left. - \mathcal{H}(a') \left((a')^2 \mathcal{H}(a') \tilde{\alpha} + 2(a')^2 \mathcal{H}(a') \tilde{\alpha} - \frac{(a')^2}{2} \mathcal{H}(a') \tilde{\alpha} \right) \right) \\ &= \frac{2}{5\mathcal{H}^2 a} \int da' \theta(a - a') \left[\left(\frac{a'}{a} \right)^{3/2} - \frac{a}{a'} \right] \left(-(a')^2 \mathcal{H}(a')^2 \tilde{\beta} - (a')^2 \mathcal{H}(a')^2 \frac{5}{2} \tilde{\alpha} \right) \\ &= \frac{1}{\mathcal{H}^2 a} \left(\frac{2}{5} \tilde{\beta} + \tilde{\alpha} \right) \int da' \theta(a - a') (a')^2 \mathcal{H}(a')^2 \left[\left(\frac{a}{a'} - \frac{a'}{a} \right)^{3/2} \right]. \end{aligned} \quad (137)$$

In an EdS universe we previously saw that $\partial\mathcal{H}/\partial a = -\mathcal{H}/2a$. Taking $a_0 = a(0) = 1$ and solving this differential equation yields $\mathcal{H} = 1/\sqrt{a}$. We can fill this in to find,

$$\hat{\delta}^{(2)}(a, \mathbf{k}) = \left(\frac{2}{5}\tilde{\beta} + \tilde{\alpha} \right) \int da' \theta(a - a') \left[a - \frac{(a')^{5/2}}{(a)^{3/2}} \right]. \quad (138)$$

This integral is easily evaluated if we consider that $a > 0$, and that $\theta(a - a') = 0$ for $a' > a$. Therefore our solution for the second order correction to the density contrast in an EdS universe is given by,

$$\hat{\delta}^{(2)}(a, \mathbf{k}) = a^2 \left(\frac{2}{7}\tilde{\beta} + \frac{5}{7}\tilde{\alpha} \right). \quad (139)$$

Note that this second order correction scales with a^2 . Through our perturbative method the third order correction will be some combination of first and second order corrections, such that the total order is equal to three. This means that it in turn will scale with a^3 . Continuing this we see that $\hat{\delta}^{(n)} \propto a^n$. This is a direct result of the form of our Green's function, because through our perturbative method we find a correction of order n by applying the Green's function to a combination of terms of order $n - 1$. To illustrate this consider applying the Green's function to a correction of order n ;

$$\frac{2}{5} \int da' \theta(a - a') \left[\left(\frac{a'}{a} \right)^{3/2} - \frac{a}{a'} \right] (a')^n = \frac{2a^{n+1}}{5n + 2n^2}, \quad (140)$$

so the next order correction scales with 1 higher power of a .

The achieved result in equation (139) does not look all that useful. One can wonder what exactly this tells us about the dynamics of the universe. By itself, not so much. But in combination with correlation functions and observations of the universe we can use it to compute the evolution of density fluctuations, deduce how important corrections to the linear order solutions are, thereby finding information on the homogeneity of our universe. It also indirectly tells us about the type of universe we are in, and how closely it resembles a universe that only contains mass.

B Determining the smoothed continuity and Euler equations

In order to use perturbation theory within our effective field theory we need to somehow make sure we stay within the regime where this is possible, this is achieved by smoothing over the Vlasov equation given by (11). The resulting function is then given by (52),

$$\left[\frac{Df}{Dt} \right]_{\Lambda} = \frac{\partial f_l}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f_l}{\partial \mathbf{x}} - m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}} = 0.$$

By taking the first and second moments of this equation we find the smoothed versions of the continuity equation and the Euler equation respectively.

Remember the relations between the phase space density f and the density, momentum density and the kinetic tensor given by equations (4), (5) and (6). These relations will not do however, we need to find out what the relations are between the smoothed phase space density and each of the smoothed physical quantities. Notice that the smoothing only

depends on \mathbf{x} or \mathbf{k} , this means that we can take the smoothing inside the integral we used to define ρ to find,

$$\begin{aligned}\rho_l(\mathbf{x}, t) &\equiv \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \rho(\mathbf{x}', t) = \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{m}{a^3} \int d^3\mathbf{p} f(\mathbf{x}', \mathbf{p}) \\ &= \frac{m}{a^3} \int d^3\mathbf{p} \int d^3\mathbf{x} W_\Lambda(\mathbf{x} - \mathbf{x}') f(\mathbf{x}', \mathbf{p}) \equiv \frac{m}{a^3} \int d^3\mathbf{p} f_l(\mathbf{x}, \mathbf{p}).\end{aligned}\quad (141)$$

The derivations for π^i and σ^{ij} are done in exactly the same way; so the relations we had are the same for the smoothed equations.

B.1 The smoothed continuity equation

Taking the zeroth moment of (52) gives us;

$$\begin{aligned}&\int d^3\mathbf{p} p^i \left[\frac{\partial f_l}{\partial t} + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f_l}{\partial \mathbf{x}} - m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}} \right] \\ &= \frac{\partial}{\partial t} \left[\frac{a^3}{m} \rho_l \right] + \frac{1}{ma^2} \int d^3\mathbf{p} \left[\mathbf{p} \cdot \frac{\partial f_l}{\partial \mathbf{x}} \right] - m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \int d^3\mathbf{p} \frac{\partial}{\partial \mathbf{p}} \left[\frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot f_n \right],\end{aligned}\quad (142)$$

where we could take the derivative with respect to \mathbf{p} out of the integral because \mathbf{x} and \mathbf{p} are independent. We will now continue by considering each of the three terms appearing in (142) separately, starting with the first term:

$$\frac{\partial}{\partial t} \left[\frac{a^3}{m} \rho_l \right] = \frac{a^3}{m} \frac{\partial \rho_l}{\partial t} + 3H \frac{a^3}{m} \rho_l. \quad (143)$$

Now we will treat the second term. By taking out the gradient and using the product rule we get,

$$\frac{1}{ma^2} \int d^3\mathbf{p} \left[\mathbf{p} \cdot \frac{\partial f_l}{\partial \mathbf{x}} \right] = \frac{1}{ma^2} \frac{\partial}{\partial \mathbf{x}} \cdot \int d^3\mathbf{p} [\mathbf{p} f_l] - \frac{1}{ma^2} \int d^3\mathbf{p} \left[\left(\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{p} \right) f_l \right]. \quad (144)$$

Note that in our derivation \mathbf{p} and \mathbf{x} are two independent coordinates in phase space, which means that the gradient of \mathbf{p} must be equal to zero. So by also filling in our definition for π^i we get;

$$\frac{1}{ma^2} \int d^3\mathbf{p} \left[\mathbf{p} \cdot \frac{\partial f_l}{\partial \mathbf{x}} \right] = \frac{a^2}{m} \frac{\partial}{\partial \mathbf{x}} \cdot \pi_l = \frac{a^2}{m} \frac{\partial}{\partial \mathbf{x}} (\rho_l \mathbf{v}_l), \quad (145)$$

where we defined the long wavelength velocity field $v_l^i = \pi_l^i / \rho_l$.

Now for the third term notice that f_l defines a density function, which means that it should vanish at infinity if you want any kind of normalization. Using this we see that the last term must be equal to zero.

Putting all of this together and multiplying equation (142) by m/a^3 we find the smoothed continuity equation:

$$\frac{\partial \rho_l}{\partial t} + 3H \rho_l + \frac{1}{a} \frac{\partial}{\partial \mathbf{x}} (\rho_l \mathbf{v}_l) = 0. \quad (146)$$

B.2 The smoothed Euler equation

As is evident from the form of the stress energy tensor in equation (54) the derivation of this equation is going to be a lot more messy. We will however begin in exactly the same way as before by taking the first moment of equation (52),

$$\begin{aligned} & \int d^3\mathbf{p} p^i \frac{\partial f_l}{\partial t} + \int d^3\mathbf{p} p^i \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial f_l}{\partial \mathbf{x}} - m \int d^3\mathbf{p} p^i \left[\sum_{n,\bar{n};n \neq \bar{n}} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}} \right] \\ &= \frac{\partial}{\partial t} \int d^3\mathbf{p} p^i f_l + \frac{1}{ma^2} \frac{\partial}{\partial \mathbf{x}} \cdot \int d^3\mathbf{p} p^i \mathbf{p} f_l - m \int d^3\mathbf{p} d^3\mathbf{x}' p^i \left[\sum_{n,\bar{n};n \neq \bar{n}} W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}} \right]. \end{aligned} \quad (147)$$

Again we split our derivation up into the calculation of the three terms appearing in (147), starting with the first.

Recognizing the definition of the momentum density we plug it, rewrite it to the velocity field, and work out the derivatives to find,

$$\begin{aligned} \frac{\partial}{\partial t} \int d^3\mathbf{p} p^i f_l &= \frac{\partial}{\partial t} (a^4 v_l^i \rho_l) = 4 \frac{\partial a}{\partial t} a^3 v_l^i \rho_l + \frac{\partial v_l^i}{\partial t} \rho_l a^4 + \frac{\partial \rho_l}{\partial t} a^4 v_l^i \\ &= a^4 \rho_l \left(4H v_l^i + \frac{\partial v_l^i}{\partial t} + \frac{\partial v_l^i}{\partial t} + \frac{1}{\rho_l} \frac{\partial \rho_l}{\partial t} v_l^i \right) \\ &= a^4 \rho_l \left(4H v_l^i + \frac{\partial v_l^i}{\partial t} - 3H v_l^i - \frac{1}{a \rho_l} v_l^i \frac{(\rho_l v_l^j)}{\partial x^j} \right) \\ &= a^4 \rho_l \left(H v_l^i + \frac{\partial v_l^i}{\partial t} - \frac{1}{a \rho_l} \frac{(v_l^i v_l^j \rho_l)}{\partial x^j} + \frac{1}{a} v_l^i \frac{\partial v_l^i}{\partial x^j} \right), \end{aligned} \quad (148)$$

where we used the smoothed continuity equation (146) when going from the second to third line in the equation. The reason for stopping our rewriting here is that we ended up with an expression that contains several useful physical quantities, like the acceleration $\partial v_l^i / \partial t$ and the divergence of the velocity field $\partial v_l^i / \partial x^j$. When actually working with the stress energy tensor it will be useful to have it in terms of these quantities.

Next we will consider the second term,

$$\frac{1}{ma^2} \frac{\partial}{\partial \mathbf{x}} \cdot \int d^3\mathbf{p} p^i \mathbf{p} f_l = \frac{1}{ma^2} \frac{\partial}{\partial x^j} \int d^3\mathbf{p} p^i p^j f_l = a^3 \frac{\partial \sigma^{ij}}{\partial x^j}. \quad (149)$$

And last, but definitely not least, we consider the third term. We start by using partial integration with respect to \mathbf{p} to find,

$$\begin{aligned} & m \int d^3\mathbf{p} d^3\mathbf{x}' p^i \left[\sum_{n,\bar{n};n \neq \bar{n}} W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \frac{\partial f_n}{\partial \mathbf{p}} \right] \\ &= m \sum_{n,\bar{n};n \neq \bar{n}} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial \mathbf{x}'}(\mathbf{x}') \cdot \left(\int d^3\mathbf{p} \frac{\partial}{\partial \mathbf{p}} (p^i f_n) - \frac{\partial p^i}{\partial \mathbf{p}} \int d^3\mathbf{p} f_n \right) \\ &= m \sum_{n,\bar{n};n \neq \bar{n}} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial x^i}(\mathbf{x}') \int d^3\mathbf{p} f_n. \end{aligned}$$

In going from the second to the third line we used the boundary conditions for our phase space density, and we worked out the inner product.

We now rewrite our sum as the difference between one sum over all n, \bar{n} and the “self term” which is one sum over all n, \bar{n} such that $n = \bar{n}$:

$$\begin{aligned}
& m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial x^i}(\mathbf{x}') \int d^3 \mathbf{p} f_n \\
& = m \sum_{n, \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial x^i}(\mathbf{x}') \int d^3 \mathbf{p} f_n - m \sum_n \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_n}{\partial x^i}(\mathbf{x}') \int d^3 \mathbf{p} f_n
\end{aligned} \tag{150}$$

We will treat both sums separately, starting with the double summation term. We find,

$$\begin{aligned}
& m \sum_{n, \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial x^i}(\mathbf{x}') \int d^3 \mathbf{p} f_n \\
& = a^3 \sum_n \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_n}{\partial x^i}(\mathbf{x}') \rho.
\end{aligned}$$

Filling in the Poisson equation (8) for the density (note that we start using the convention $\partial/\partial x^i \equiv \partial_i = \partial^i$),

$$\begin{aligned}
& = a^3 \sum_n \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \partial_{i'} \phi_n(\mathbf{x}') \frac{\partial_{j'} \partial^{j'} \phi(\mathbf{x}')}{4\pi G a^2} \\
& = \frac{a}{8\pi G} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') 2\partial_{i'} \phi(\mathbf{x}') \partial_{j'} \partial^{j'} \phi(\mathbf{x}').
\end{aligned}$$

We will now show an identity for the derivatives of ϕ that will be used to write the previous equation in a more insightful way (in the end). Note that,

$$\begin{aligned}
2\partial_{i'} \phi \partial_{j'} \partial^{j'} \phi & = 2\partial_{j'} (\partial_{i'} \phi \partial^{j'} \phi) - 2\partial_{i'} \partial_{j'} \phi \partial^{j'} \phi = 2\partial_{j'} (\partial_{i'} \partial^{j'} \phi) - \partial_{i'} (\partial_{j'} \phi \partial^{j'} \phi) \\
& = 2\partial_{j'} (\partial_{i'} \phi \partial^{j'} \phi) - (\delta^{i'j'} \partial_{k'}) (\partial_{k'} \phi \partial^{k'} \phi) = \partial_{j'} (2\partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi)). \tag{151}
\end{aligned}$$

We will also need a property of the window function under derivatives. Remember that W_{Λ} is a Gaussian window function, as given by equation (50), so using the antisymmetries of the smoothing function we get,

$$\partial_{i'} W_{\Lambda}(\mathbf{x} - \mathbf{x}') = \partial_{i'} \left(\frac{\Lambda}{\sqrt{2\pi}} \right)^3 e^{-\frac{1}{2}\Lambda^2 x^2} = -\partial_i \left(\frac{\Lambda}{\sqrt{2\pi}} \right)^3 e^{-\frac{1}{2}\Lambda^2 x^2} = -\partial_i W_{\Lambda}(\mathbf{x} - \mathbf{x}'). \tag{152}$$

Putting these two relations together we have,

$$\begin{aligned}
& \frac{a}{8\pi G} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') 2\partial_{i'} \phi(\mathbf{x}') \partial_{j'} \partial^{j'} \phi(\mathbf{x}') \\
& = \frac{a}{8\pi G} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \partial_{j'} \left(2\partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi) \right). \tag{153}
\end{aligned}$$

We will now apply partial integration, remembering that for ϕ to be a physical quantity both it and its derivatives should vanish at the boundary (this being infinity):

$$= \frac{a}{8\pi G} \int d^3\mathbf{x}' \partial_{j'} \left[W_\Lambda(\mathbf{x} - \mathbf{x}') \left(2\partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi) \right) \right] - \quad (154)$$

$$\frac{a}{8\pi G} \int d^3\mathbf{x}' \partial_{j'} [W_\Lambda(\mathbf{x} - \mathbf{x}')] \left(2\partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi) \right) \quad (155)$$

$$= \frac{a}{8\pi G} \partial_j \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \left(2\partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi) \right) \quad (156)$$

where we end our rewriting of the sum over n and \bar{n} . Now we still need to compute the self term. This will go in exactly the same way, only we will use the following version of the Poisson equation. We define ρ_n such that

$$\rho = \sum_n \rho_n = \frac{m}{a^3} \sum_n \int d^3\mathbf{p} f_n(\mathbf{x}, \mathbf{p}) = \frac{\partial_{j'} \partial^{j'} \sum_n \phi_n}{8\pi G a^2}. \quad (157)$$

We can then take,

$$\rho_n = \frac{m}{a^3} \int d^3\mathbf{p} f_n(\mathbf{x}, \mathbf{p}) = \frac{\partial_{j'} \partial^{j'} \phi_n}{8\pi G a^2}. \quad (158)$$

It follows that we can compute the self term in exactly the same way as we found the double summation term, however now the ϕ get a subscript n extra. Thus we get,

$$\begin{aligned} & m \sum_n \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \sum_n \frac{\partial \phi_n}{\partial x'^i}(\mathbf{x}') \int d^3\mathbf{p} f_n \\ &= \frac{a}{8\pi G} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \sum_n 2\partial_{i'} \phi_n(\mathbf{x}') \partial_{j'} \partial^{j'} \phi_n(\mathbf{x}') \\ &= \frac{a}{8\pi G} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \sum_n 2\partial_{i'} \phi_n(\mathbf{x}') \partial_{j'} \partial^{j'} \phi_n(\mathbf{x}') \\ &= \frac{a}{8\pi G} \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \sum_n \partial_{j'} \left(2\partial_{i'} \phi_n \partial^{j'} \phi_n - \delta^{i'j'} (\partial_{k'} \phi_n \partial^{k'} \phi_n) \right) \\ &= \frac{a}{8\pi G} \partial_j \int d^3\mathbf{x}' W_\Lambda(\mathbf{x} - \mathbf{x}') \left(2 \sum_n \partial_{i'} \phi_n \partial^{j'} \phi_n - \delta^{i'j'} \sum_n (\partial_{k'} \phi_n \partial^{k'} \phi_n) \right). \end{aligned} \quad (159)$$

Putting our new expressions for the self term and the full summation together into equation (150) we get,

$$m \sum_{n, \bar{n}; n \neq \bar{n}} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \frac{\partial \phi_{\bar{n}}}{\partial x'^i}(\mathbf{x}') \int d^3 \mathbf{p} f_n$$

$$= \frac{a}{8\pi G} \partial_j \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left(2 \partial_{i'} \phi \partial^{j'} \phi - \delta^{i'j'} (\partial_{k'} \phi \partial^{k'} \phi) \right) \quad (160)$$

$$- \frac{a}{8\pi G} \partial_j \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left(2 \sum_n \partial_{i'} \phi_n \partial^{j'} \phi_n - \delta^{i'j'} \left(\sum_n \partial_{k'} \phi_n \partial^{k'} \phi_n \right) \right)$$

$$= \frac{a}{8\pi G} \partial_j \left[2 \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left(\partial_{i'} \phi \partial^{j'} \phi - \sum_n \partial_{i'} \phi_n \partial^{j'} \phi_n \right) \right] \quad (161)$$

$$- \delta^{i'j'} \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left(\partial_{k'} \phi \partial^{k'} \phi - \sum_n \partial_{k'} \phi_n \partial^{k'} \phi_n \right) \Big] \quad (162)$$

$$= \frac{a}{8\pi G} \partial_j [2w_l^{ij} - \delta^{ij} w_l^{kk}] \quad (163)$$

where we dropped the prime notation in the last step (this is just for notational purposes) and defined,

$$w_l^{ij} = \int d^3 \mathbf{x}' W_{\Lambda}(\mathbf{x} - \mathbf{x}') \left[\partial_i \phi \partial^j \phi - \sum_n \partial_i \phi_n \partial^j \phi_n \right]. \quad (164)$$

We are now ready to combine all of the three terms we started with into our full smoothed Euler equation (147) to get the following equation,

$$a^4 \rho_l \left(H v_l^i + \frac{\partial v_l^i}{\partial t} - \frac{1}{a \rho_l} \partial_j (v_l^i v_l^j \rho_l) + \frac{1}{a} v_l^i \partial_j v_l^j \right) + a^3 \partial_j \sigma^{ij} - \frac{a}{8\pi G} \partial_j [2w_l^{ij} - \delta^{ij} w_l^{kk}] = 0. \quad (165)$$

Again, our goal is to split our equation up into a purely longwavelength dependent part, and a part sourced by the short wavelength fluctuations. In order to achieve this (we will motivate this at the end) we rewrite our equation to

$$H v_l^i + \frac{\partial v_l^i}{\partial t} + \frac{1}{a} v_l^i \partial_j v_l^j + \frac{1}{a} \partial_i \phi_l = - \frac{1}{a \rho_l} \partial_j [\tau^{ij}]_{\Lambda}, \quad (166)$$

where we defined the effective stress energy tensor induced by short wavelength fluctuations as

$$[\tau^{ij}]_{\Lambda} = \kappa_l^{ij} + \Phi_l^{ij}. \quad (167)$$

Here κ_l^{ij} is the kinetically induced part of the stress energy tensor, and Φ_l^{ij} the gravitational part, defined as follows,

$$\kappa_l^{ij} = \sigma_l^{ij} - v_l^i v_l^j \rho_l, \quad (168)$$

$$\Phi_l^{ij} = - \frac{1}{8\pi G a^2} [\delta^{ij} w_l^{kk} - 2w_l^{ij} - \delta^{ij} \partial_k \phi_l \partial^k \phi_l + 2\partial^i \phi_l \partial^j \phi_l]. \quad (169)$$

There are several things to note about this equation. First of all, the presence of two new terms: $\partial_i \phi_l / a$ on the left hand side and $\partial_j [\delta^{ij} \partial_k \phi_l \partial^k \phi_l + 2 \partial^i \phi_l \partial^j \phi_l] / [8\pi G a^3 \rho_l]$ on the right hand side. These terms are exactly equal, and therefore cancel out when we rewrite them equation (151) and the Poisson equation (8). Then why did we add them? The reason is still that we want an effective stress energy tensor fueled solely by the short scale fluctuations, and depending on clear physical quantities. Note that in the limit where the short scale fluctuations vanish and $\Lambda \rightarrow \infty$ we find that $W_\Lambda(\mathbf{x}' - \mathbf{x}) \rightarrow \delta_D(\mathbf{x}' - \mathbf{x})$, causing the terms without index n to cancel. The terms with an index n are negligible in this limit, and so both κ_l^{ij} and Φ_l^{ij} go to zero while the left hand side remains[4].

References

- [1] M. Blanton and the SDSS. “Sloan Digital Sky Survey”, http://classic.sdss.org/includes/sideimages/sdss_pie2.html
- [2] V. Mukhanov. (2005). “Physical Foundations of Cosmology”
- [3] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro. (2001). “Large Scale Structure of the Universe and Cosmological Perturbation Theory”. arXiv:astro-ph/0112551
- [4] J.J.M. Carrasco, M.P. Hertzberg, L. Senatore. (2012). “The Effective Field Theory of Cosmological Large Scale Structures”. arXiv:1206.2926 [astro-ph.CO]
- [5] D. Baumann, A. Nicolis, L. Senatore, M. Zaldarriaga. (2010) “Cosmological Non-Linearities as an Effective Fluid”. arXiv:1004.2488 [astro-ph.CO]
- [6] M.H. Goroff, B. Grinstein, S.-J. Rey, M.B. Wise. (1986). “Coupling of Modes of Cosmological Mass Density Fluctuations”. ApJ, 311, p.614.
- [7] B. Jain, E. Bertschinger. (1994). “Gravitational Instability of Cold Dark Matter”. ApJ, 431, p.495505.
- [8] N. Makino, M. Sasaki, Y. Suto. (1992). “Analytic approach to the perturbative expansion of nonlinear gravitational fluctuations in cosmological density and velocity fields”. Phys. Rev. D, 46, p.585602.
- [9] J. Wudka. (1998). http://physics.ucr.edu/~wudka/Physics7/Notes_www/node85.html
- [10] M. Blau. (2015). “Lecture notes on General Relativity”. <http://www.blau.itp.unibe.ch/Lecturenotes.html>
- [11] B. Ryden. (2002). “Introduction to Cosmology”
- [12] E. Pajer, M. Zaldarriaga. (2013). “On the Renormalization of the Effective Field Theory of Large Scale Structures”. arXiv:1301.7182 [astro-ph.CO]
- [13] L. Mercolli, E. Pajer. (2013). “On the Velocity in the Effective Field Theory of Large Scale Structures”. arXiv:1307.3220 [astro-ph.CO]

- [14] J. Mathews, R. Walker. (1970). “Mathematical Methods of Physics”
(as found in [http://math.stackexchange.com/questions/442277/
general-solution-of-a-differential-equation-using-greens-function](http://math.stackexchange.com/questions/442277/general-solution-of-a-differential-equation-using-greens-function))